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AGUJEROS NEGROS CUÁNTICO-RELATIVISTAS MASIVOS O SUPERMASIVOS

MASSIVE OR SUPERMASSIVE QUANTUM-RELATIVISTIC BLACK HOLES

Manuel Ignacio Albuja Bustamante
Investigador Independiente

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Agujeros Negros Cuántico-Relativistas Masivos o Supermasivos

Manuel Ignacio Albuja Bustamante

ignaciomanuelalbujabustamante@gmail.com

Investigador Independiente

RESUMEN

En este trabajo, abordaremos los agujeros negros causados a escala microscópica. El modelamiento matemático, si bien es cierto que parte de la métrica cosmológica, pues, refleja una demostración cuantitativamente significa. Asimismo, se abordará al agujero negro cuántico masivo y supermasivo respectivamente, los cuales tienen algunas diferencias esenciales, muy especialmente, en el aspecto morfológico. El agujero negro cuántico, es el resultado de la aniquilación o colapso de una partícula supermasiva, en el caso de una partícula supermasiva, o de la aniquilación o colapso de una partícula de tipo estrella u oscura, a la luz de la Teoría Cuántica de Campos Relativistas o Curvos formulada por este investigador en trabajos anteriores. Se abordarán escenarios perturbativos y no perturbativos en relación a los agujeros negros cuánticos, incluyendo aquellos que superan dimensiones múltiples o distintas. Los agujeros negros cuánticos, surgen en entornos de gravedad o supergravedad cuánticas.

Palabras clave: agujero negro cuántico, agujero negro cuántico supermasivo, agujero negro cuántico masivo, supergravedad cuántic2

Massive or Supermassive Quantum-Relativistic Black Holes

ABSTRACT

In this paper, we will address caused black holes on a microscopic scale. Mathematical modeling, although it is true that it is based on cosmological metrics, therefore, reflects a quantitative demonstration. Likewise, the massive and supermassive quantum black holes respectively will be addressed, which have some essential differences, especially in the morphological aspect. The quantum black hole is the result of the annihilation or collapse of a supermassive particle, in the case of a supermassive particle, or the annihilation or collapse of a star or dark particle, in the light of the Quantum Theory of Relativistic or Curved Fields formulated by this researcher in previous works. Perturbative and non-perturbative scenarios in relation to quantum black holes will be addressed, including those that overcome multiple or different dimensions. Quantum black holes arise in environments of quantum gravity or supergravity.

Keywords: quantum black hole, supermassive quantum black hole, massive quantum black hole, quantum supergravity

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INTRODUCCIÓN

Los agujeros negros cuánticos, ocurren a escala microscópica, muy al contrario del agujero negro cosmológico, aunque gozan de similares características físicas. El agujero negro cuántico, surge a propósito de la aniquilación o en su defecto, de la interacción de una partícula supermasiva, respecto de un espacio – tiempo cuántico específico. Surge por aniquilación o colapso, en tratándose de una partícula supermasiva, de tipo estrella u oscura, en cuyo caso, se forma un agujero negro cuántico supermasivo, esto es, exponencialmente denso, dotado de radiación, de un radio orbital creciente y con horizonte de eventos relativamente pequeño, lo que implica, que cualquier partícula que supere dicho horizonte, entrará al punto de no retorno y en consecuencia, será engullida por el agujero negro hasta fundirse con la singularidad. Surge por interacción de una partícula supermasiva, esto es, cuando una partícula supermasiva, dotada de gravedad endógena o exógena, sufre de aniquilación, deformando así, el espacio – tiempo cuántico, y por ende, provocando un agujero negro masivo, dotado de radiación menor, de un radio orbital creciente y con un horizonte de eventos relativamente extenso, lo que implica que cualquier partícula, puede propagarse en cuanto a sus orbitales, alrededor del agujero negro cuántico, sin que sea engullida, a menos que supere el horizonte de eventos. En ambos casos, las hiperpartículas, no escapan al diámetro de absorción del agujero negro, sea masivo o supermasivo, según sea el caso. Es importante precisar en este punto que, un agujero negro masivo, es susceptible de convertirse en un agujero negro supermasivo, muy especialmente cuando engulle una partícula de tipo estrella u oscura. En tratándose del agujero negro masivo o en su defecto, del agujero negro supermasivo, la singularidad, está constituida, en primer término, por un cúmulo contraído de materia oscura, es decir, de masa negra, ultramente pesada y densa, generadora de energía oscura y capaz de estirar el espacio – tiempo cuántico, en el que, el tiempo, es una dimensión física más, susceptible de alteración, hacia adelante y hacia atrás. Llegado a esta singularidad primaria, la materia y la energía entrantes se licúan en el núcleo duro del agujero negro cuántico, aumentando su capacidad gravitacional. Como se notará, la materia y la energía, se transforman, más sin embargo, su información se destruye, pues pasa a configurarse a la morfología propia del agujero negro. Superada esta fase de singularidad primaria, existe una singularidad secundaria o final, en la que, si un objeto dotado de materia y energía lo atraviesa, conservará su información y por ende, se transformará en la medida en que, conservará su estado inicial, pero interactuante en una nueva dimensión, en la que, su naturaleza morfológica y fenomenológica es arbitraria, y por ende, engranada en la nueva dimensión, cuyas leyes son en contrario o no, a la dimensión de origen. En este trabajo, se desarrolla un modelo matemático que explica tanto al agujero negro masivo como al agujero negro cuántico supermasivo, desde su formación. El agujero negro cuántico, no es susceptible de extinción, más si de mutación, incluso en agujero blanco cuántico, que es todo lo opuesto al aquí teorizado. Es preciso indicar, que el agujero negro cuántico, en cuanto a su tamaño, es directamente proporcional al espacio – tiempo cuántico en el que se ha originado.

RESULTADOS Y DISCUSIÓN.

Agujeros negros cuánticos masivos. Modelo Matemático aplicable a Campos Cuánticos Relativistas o Curvos. Modelo Ramond - Eddington-Finkelstein.

$$I_{NS} = \frac{1}{2}\int d^{10}x\sqrt{g}e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}H^2\right)$$

$$I_R = -\int d^{10}x\sqrt{g}\left(\frac{1}{2\cdot 2!}F^2 + \frac{1}{2\cdot 4!}F_4'^2\right) - \frac{1}{4}\int F_4 \wedge F_4 \wedge B$$

$$\{Q_\alpha, Q'_{\dot\alpha}\} \sim \delta_{\alpha\dot\alpha}W$$

$$M \geq c_0|W|,$$

$$M \geq \frac{c_1}{\lambda}|W|$$

$$\frac{1}{4e^2}\int d^n x\sqrt{g}F^2$$

$$\frac{1}{4}\int d^n x\sqrt{g}e^{\gamma\phi}F^2$$

$$M = \frac{c|n|}{\lambda}$$

$$I = \frac{1}{2}\int d^{11}x\sqrt{G}(R + |dA_3|^2) + \int A_3 \wedge dA_3 \wedge dA_3$$

$$I = \frac{1}{2}\int d^{10}x\sqrt{G^{10}}(e^\gamma(R + |\nabla\gamma|^2 + |dA_3|^2) + e^{3\gamma}|dA|^2 + e^{-\gamma}|dB|^2) + \cdots$$

$$I = \frac{1}{2}\int d^{10}x\sqrt{g}(e^{-3\gamma}(R + |\nabla\gamma|^2 + |dB|^2) + |dA|^2 + |dA_3|^2 + \cdots)$$

$$e^{-2\phi} = e^{-3\gamma}$$

$$r(\lambda) = e^{\frac{2\phi}{3}} = \lambda^{2/3}$$

$$\frac{e^{-\gamma/2}}{r(\lambda)} \sim \lambda^{-1}$$

$$\int d^d x \sqrt{g} e^{-2\phi} R$$

$$w_d = \lambda^{2/(d-2)}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} e^{\lambda_i} & 0 \\ 0 & e^{-\lambda_i} \end{pmatrix}$$

$$(12,2) \oplus (32,1)$$

$$\left\{Q_\alpha^i, Q_\beta^j\right\} = \epsilon_{\alpha\beta} Z^{ij}$$

$$T_g = T g^{-1}$$

$$\psi \to Z(\psi) = T_g \psi$$

$$\psi \to g'\psi$$
$$g \to g'g.$$

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$g_t = \begin{pmatrix} e^{a_1 t} & 0 & 0 & 0 & 0 \\ 0 & e^{a_2 t} & 0 & 0 & 0 \\ 0 & 0 & e^{a_3 t} & 0 & 0 \\ 0 & 0 & 0 & e^{a_4 t} & 0 \\ 0 & 0 & 0 & 0 & e^{a_5 t} \end{pmatrix}$$

$$a_1 \geq a_2 \geq \cdots \geq a_5$$

$$M(\psi_{ij}) \sim e^{-t(a_i + a_j)}$$

$$g_t = \begin{pmatrix} e^t & 0 & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 & 0 \\ 0 & 0 & e^t & 0 & 0 \\ 0 & 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & 0 & e^{-4t} \end{pmatrix}$$

$$g_t = \begin{pmatrix} e^{4t} & 0 & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 & 0 \\ 0 & 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & e^{-t} \end{pmatrix}$$

$$g_t = \begin{pmatrix} e^{3t} & 0 & 0 & 0 & 0 \\ 0 & e^{3t} & 0 & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & 0 & e^{-2t} \end{pmatrix}$$

$$g_t = \begin{pmatrix} e^{2t} & 0 & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \\ 0 & 0 & 0 & e^{-3t} & 0 \\ 0 & 0 & 0 & 0 & e^{-3t} \end{pmatrix}$$

$$(\psi, \psi) = 2\psi_{12}\psi_{34}$$

$$(p, p) = |p_L|^2 - |p_R|^2$$

$$x_i \to e^{c_i t} x_i$$

$$M_\rho \sim \exp\left(-\sum_i c_i e_i t\right)$$

$$M_{\rho'} \sim \exp\left(-\sum_i c_i f_i t\right)$$

$$c_i = 0 \text{ whenever } f_i < e_i.$$

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$dH = \mathrm{tr}F \wedge F - \mathrm{tr}R \wedge R$$

$$J =* \mathrm{tr}F \wedge F -* \mathrm{tr}R \wedge R$$

$$D^m G_{mn} = J_n$$

$$M = \frac{16\pi^2 |n|}{\lambda^2}$$

$$I = \int d^6x \sqrt{g} e^{-2\phi} (R + |\nabla\phi|^2 + |dB|^2 + |dC|^2)$$

$$I' = \int d^6x \sqrt{g}' \left(e^{-2\phi'} (R' + |\nabla\phi'|^2 + |dB'|^2) + |dC'|^2\right)$$

$$I' = \int d^6x \sqrt{g}'' \left(e^{2\phi'} (R'' + |\nabla\phi'|^2) + e^{-2\phi'} |dB'|^2 + e^{2\phi'} |dC'|^2\right)$$

$$e^{-2\phi'} dB' =* dB''$$

$$I' = \int d^6x \sqrt{g}'' e^{2\phi'} (R'' + |\nabla\phi'|^2 + |dB''|^2 + |dC'|^2)$$

$$\phi = -\phi'$$
$$g = e^{2\phi} g' = e^{-2\phi'} g'$$
$$dB = e^{-2\phi'} * dB'$$
$$C = C'$$

$$\lambda' = \lambda^{-1}$$
$$r' = \lambda^{-1} r.$$

$$r'' = \frac{1}{r'} = \frac{\lambda}{r}$$

$$\lambda'' = \frac{\lambda'}{r'}$$

$$\frac{r'}{(\lambda')^2} = \frac{r''}{(\lambda'')^2}$$

$$\lambda'' = r^{-1}$$
$$r'' = \frac{\lambda}{r}$$

$$\mathcal{N} = SO(21,5;\mathbf{Z}) \setminus SO(21,5;\mathbf{R})/(SO(21) \times SO(5))$$

$$\mathcal{M} = \mathcal{N} \times \mathbf{R}^+$$

$$g_{m6}: \frac{\lambda}{r}$$
$$B_{m6}: \lambda r$$
$$A_m: \frac{r}{\lambda}$$

$$g'_{m6}: \frac{1}{r'}$$
$$B'_{m6}: r'$$
$$A'_m: \frac{r'}{(\lambda')^2}$$

$$\lambda_4 = \lambda \rho^{-1/2}$$

$$\mathcal{M} = SO(20,4;\mathbf{Z}) \setminus SO(20,4;\mathbf{R})/(SO(20) \times SO(4))$$

$$\mathcal{M} = \mathcal{M}_1 \times \mathbf{R}^+$$

$$\mathcal{M}_1 = SO(19,3;\mathbf{Z}) \setminus SO(19,3;\mathbf{R})/SO(19) \times SO(3)$$

$$I = \frac{1}{2}\int \; d^{11}x\sqrt{G}(R + |dA_3|^2) + \int \; A_3 \wedge dA_3 \wedge dA_3$$

$$\int \; d^7x\sqrt{\tilde{g}}\left(e^{4\gamma}\left(\tilde{R} + |d\gamma|^2 + |da_3|^2\right) + |dA|^2\right)$$

$$\int \; d^7x\sqrt{g}e^{-6\gamma}(R + |d\gamma|^2 + |dB|^2 + |dA|^2)$$

$$e^{\gamma} = e^{\phi/3} = \lambda^{1/3}$$

$$\frac{1}{\lambda_6^2} = \frac{r_1}{\lambda_7^2}$$

$$\lambda_6' = \frac{1}{\lambda_6} = \frac{r_1^{1/2}}{\lambda_7}$$

$$g = e^{2\phi} g' = \lambda_6^2 g' = \frac{\lambda_7^2}{r_1} g'$$

$$M \sim \frac{1}{r_1}.$$

$$\int d^{10}x \sqrt{g_{10}'} \left(e^{-2\phi_{10}'} R_{10}' + |dA|^2 + |dA_3|^2 + \cdots\right)$$

$$\int d^6x \sqrt{g'} \left(\frac{1}{(\lambda_6')^2} R' + V|da|^2 + V|da_3|^2 + |dC_I|^2\right)$$

$$\int d^6x \sqrt{g'} \left(\frac{1}{(\lambda_6')^2} R' + V|da|^2 + \frac{1}{V}|db|^2 + |dC_I|^2\right)$$

$$\int d^6x \frac{1}{4e_{\text{eff}}^2} |dA|^2$$

$$M' = \frac{1}{V^{1/2}\lambda_6'}$$

$$V = r_1^2$$

$$\frac{V}{(\lambda_{10}')^2} = \frac{1}{(\lambda_6')^2}$$

$$\lambda_{10}' = \frac{r_1^{3/2}}{\lambda_7}$$

$$r_{11} = (\lambda_{10}')^{2/3} = \frac{r_1}{\lambda_7^{2/3}}$$

$$g' = (\lambda_{10}')^{2/3} G^{10}$$

$$V_{11} = (\lambda_{10}')^{-4/3} V = \lambda_7^{4/3} r_1^{-2} V = \lambda_7^{4/3}$$

$$\int d^{10}x \sqrt{g} e^{-2\phi} (R + |\nabla\phi|^2 + F^2 + |dB|^2)$$

$$\int d^{10}x \sqrt{g}' \left(e^{-2\phi'}(R' + |\nabla\phi'|^2) + e^{-\phi'} F^2 + |dB|^2\right)$$

$$\frac{(r')^{10-d}}{(\lambda')^2} = \frac{(r'')^{10-d}}{(\lambda'')^2}$$

$$\lambda'' = \lambda' \left(\frac{r''}{r'}\right)^{(10-d)/2} = \frac{\lambda^{(8-d)/2}}{r^{10-d}}$$

$$\Phi = \sum_s \phi_s |s\rangle + \sum_s \phi_s^* |\hat{s}\rangle.$$

$$\langle\langle | s\rangle, | \hat{r}\rangle\rangle\rangle = \delta_{rs},$$

$$S(\phi_s, \phi_s^*) = S[\Phi] = -\frac{1}{g_s^2}\left(\frac{1}{2}\langle\langle\Phi, Q\Phi\rangle\rangle + \sum_{g=0}^{\infty} g_s^{2g} \mathcal{V}^{(g)}(\Phi)\right)$$

$$\mathcal{V}^{(g)}(\Phi) = \sum_{k=1}^{\infty} \mathcal{V}_k^{(g)} \underbrace{(\Phi, \cdots, \Phi)}_{k \text{ times}}$$

$$\frac{1}{2}\{S[\Phi], S[\Phi]\}_{\text{BV}} + \Delta_{\text{BV}} S[\Phi] = 0$$

$$\{S, S\}_{\text{BV}} = \sum_s S\left(\frac{\overleftarrow{\partial}}{\partial\phi_s}\frac{\vec{\partial}}{\partial\phi_s^*} - \frac{\overleftarrow{\partial}}{\partial\phi_s^*}\frac{\vec{\partial}}{\partial\phi_s}\right) S$$

$$\Delta_{\text{BV}} S = \sum_s (-1)^{d(\phi_s)} \frac{\vec{\partial}}{\partial\phi_s}\frac{\vec{\partial}}{\partial\phi_s^*} S$$

$$\frac{b_0 + \bar{b}_0}{L_0 + \bar{L}_0} = (b_0 + \bar{b}_0)\int_0^{\infty} dt e^{-t(L_0 + \bar{L}_0)}$$

$$\partial\mathcal{V} + \frac{1}{2}\{\mathcal{V}, \mathcal{V}\} + \Delta\mathcal{V} = 0$$

$$\mathcal{V} = \sum_g g_s^{2g} \mathcal{V}^{(g)},$$

$$\mathcal{V}^{(g)} = \sum_k \mathcal{V}_k^{(g)}.$$

$$\partial\mathcal{V} + \frac{1}{2}\{\mathcal{V}, \mathcal{V}\} + \Delta\mathcal{V} = 0 \ \rightarrow \ \frac{1}{2}\{S[\Phi], S[\Phi]\}_{\text{BV}} + \Delta_{\text{BV}} S[\Phi] = 0$$

$$\langle \cdot, \cdot \rangle : \mathcal{H}_0 \otimes \mathcal{H}_0 \rightarrow \mathbb{C},$$

$$S_{\text{free}}[\Psi] = -\frac{1}{2g_s}\langle\Psi, Q_B\Psi\rangle,$$

$$Q_B\Psi = 0$$

$$\Psi \sim \Psi + Q_B\Lambda$$

$$S[\Psi] = -\frac{1}{g_s}(\frac{1}{2}\langle\Psi, Q_B\Psi\rangle + \sum_{n=2}^{\infty}\frac{1}{n+1}\langle\Psi, m_n(\underbrace{\Psi,\cdots,\Psi}_{n\text{ times}})\rangle)$$

$$m_n : \mathcal{H}_o^{\otimes n} \to \mathcal{H}_o$$

$$\{S[\Psi], S[\Psi]\}_{\mathrm{BV}} = 0$$

$$\sum_{k=1}^{n-1} m_k m_{n-k} = 0$$

$$m_k m_p : \mathcal{H}_o^{\otimes k+p-1} \to \mathcal{H}_o$$

$$\begin{aligned} m_k m_p(\Psi_1,\cdots,\Psi_{k+p-1}) &:= m_k\big(m_p(\Psi_1,\cdots,\Psi_p),\Psi_{p+1},\cdots\Psi_{k+p-1}\big) \\ &+(-1)^{d_1} m_k\big(\Psi_1, m_p(\Psi_2\cdots,\Psi_{p+1}),\Psi_{p+2}\cdots\Psi_{k+p-1}\big) \\ &\vdots \\ &+(-1)^{d_1+\cdots d_{k-1}} m_k\left(\Psi_1,\cdots\Psi_{k-1}, m_p(\Psi_k\cdots,\Psi_{k+p-1})\right) \end{aligned}$$

$$S_{\text{Witten}}[\Psi] = -\frac{1}{g_s}\left(\frac{1}{2}\langle\Psi, Q_B\Psi\rangle + \frac{1}{3}\langle\Psi, m_2(\Psi,\Psi)\rangle\right)$$

$$S_{\text{Witten}}^{(\mu)}[\Psi] = -\frac{1}{g_s}\left(\frac{1}{2}\langle\Psi, Q_B\Psi\rangle + \frac{1}{3}\langle\Psi, m_2(\Psi,\Psi)\rangle\right) + \sum_k \mu_k\langle V^k(i,-i),\tilde{\Psi}\rangle,$$

$$\begin{aligned} (b_0 - \bar{b}_0)\Phi &:= b_0^-\Phi = 0 \\ (L_0 - \bar{L}_0)\Phi &:= L_0^-\Phi = 0 \end{aligned}$$

$$S_{\text{free}}[\Phi] = -\frac{1}{2g_s^2}\langle\Phi, c_0^- Q_B\Phi\rangle$$

$$Q_B\Phi = 0$$

$$\Phi \sim \Phi + Q_B\Lambda$$

$$S_{\text{class}}[\Phi] = -\frac{1}{g_s^2}\left(\frac{1}{2}\langle\Phi, c_0^- Q_B\Phi\rangle + \sum_{k=2}^{\infty}\frac{1}{(k+1)!}\langle\Phi, c_0^- l_k(\Phi^{\wedge k})\rangle\right)$$

$$l_n : \mathcal{H}_c^{\wedge n} \to \mathcal{H}_c$$

$$\{S_{\text{class}}[\Phi], S_{\text{class}}[\Phi]\}_{\mathrm{BV}} = 0$$

$$\sum_{k=1}^{n-1} l_k l_{n-k} = 0$$

$$l_n \to l_n^{(g)}, g = 0,1,\cdots$$

$$S_{\text{quant}}[\Phi] = -\sum_{g=0}^{\infty} g_s^{-2+2g}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\langle\Phi, c_0^- l_k^{(g)}(\Phi^{\wedge k})\rangle$$

$$\frac{1}{2}\{S_{\text{quant}}[\Phi], S_{\text{quant}}[\Phi]\}_{\text{BV}} + \Delta_{\text{BV}} S_{\text{quant}}[\Phi] = 0$$

$$S^{oc}_{\text{free}}[\Phi, \Psi] = -\frac{1}{2g_s^2}\langle \Phi, c_0^- Q_B \Phi\rangle - \frac{1}{2g_s}\langle \Psi, Q_B \Psi\rangle.$$

$$S^{oc}_{\text{quant}}[\Phi, \Psi] = \sum_{g,b=0}^{\infty} g_s^{2g+b-2} \sum_{k=0}^{\infty} \sum_{\{l_1,\ldots,l_b\}=0}^{\infty} \mathcal{A}^{g,b}_{k;\{l_1,\ldots,l_b\}}\left(\Phi^{\wedge k} \otimes' \Psi^{\odot l_1} \wedge' \ldots \wedge' \Psi^{\odot l_b}\right)$$

$$\{S^{oc}_{\text{quant}}, S^{oc}_{\text{quant}}\}_{\text{c}} + \{S^{oc}_{\text{quant}}, S^{oc}_{\text{quant}}\}_{\text{o}} + 2\Delta_{\text{c}} S^{oc}_{\text{quant}} + 2\Delta_{\text{o}} S^{oc}_{\text{quant}} = 0$$

$$\Phi_{\text{v}} = 0 \rightarrow \Phi_{\text{v}} = \Phi_{\text{v}}(g_s)$$

$$l_k^{(1PI)} = \sum_{g=0}^{\infty} g_s{}^{2g} l_k^{(1PI,g)}$$

$$S^{(1PI)}(\Phi) = \frac{1}{g_s^2}\sum_{k=0}^{\infty} \frac{1}{(k+1)!}\left\langle \Phi, c_0^- l_k^{(1PI)}(\Phi, \cdots, \Phi)\right\rangle,$$

$$\{S^{(1PI)}(\Phi)), S^{(1PI)}(\Phi)\}_{\text{BV}} = 0$$

$$l_0^{(1PI)} = \sum_{g=1}^{\infty} g_s^{2g} l_0^{(1PI,g)}$$

$$\sum_{k=0}^{n} l_k^{(1PI)} l_{n-k}^{(1PI)} = 0$$

$$\left(l_1^{(1PI)}\right)^2 = -l_2^{(1PI)} l_0^{(1PI)} \neq 0$$

$$\sum_{k=1}^{\infty} \frac{1}{k!} l_k^{(1PI)}\left(\Phi^{\wedge k}\right) = -l_0^{(1PI)}$$

$$\Phi_{\text{v}}(g_s) = \sum_{g=1}^{\infty} (g_s)^{2g} \Phi_g$$

$$\begin{aligned} O(g_2^2)&: Q\Phi_1 = -l_0^{(1PI,1)} \\ O(g_2^4)&: Q\Phi_2 = -l_0^{(1PI,2)} - l_1^{(1PI,1)}(\Phi_1) - \frac{1}{2} l_2^{(1PI,0)}(\Phi_1, \Phi_1) \\ &\vdots \end{aligned}$$

$$\Phi = \Phi_{\text{v}} + \hat{\Phi}$$

$$S_2[\hat{\Phi}] = \frac{1}{2g_s^2}\langle \hat{\Phi}, l_1(g_s)\hat{\Phi}\rangle$$

$$P:\mathcal{H} \rightarrow P\mathcal{H}$$
$$[Q_B, P] = 0.$$

$$l_k^{(g)} \rightarrow \tilde{l}_k^{(g)}$$

$$S_{\text{eff}}[\phi] = -\sum_{g=0}^{\infty} g_s^{-2+2g} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left\langle \phi, c_0^- \tilde{l}_k^{(g)}(\phi, \cdots, \phi) \right\rangle.$$

$$h := \frac{b_0^+}{L_0^+} \bar{P}$$

$$\frac{1}{2}\{S_{\text{eff}}(\phi), S_{\text{eff}}(\phi)\}_{\text{BV}} + \Delta_{\text{BV}} S_{\text{eff}}(\phi) = 0$$

$$Q\Phi + \sum_{k=1}^{\infty} \frac{1}{k!} l_k(\Phi^{\wedge k}) = 0$$

$$Q\Psi + \Psi * \Psi = 0$$

$$|T\rangle = tc_1|0\rangle_{SL(2.\mathbb{R})}$$

$$S[\Psi_{\text{tv}}] = \frac{1}{2\pi^2 g_s} Z_{\text{disk}}^{\text{BCFT}_0}$$

$$Q_{\text{tv}} \equiv Q + [\Psi_{\text{tv}}, \cdot]$$

$$Q_{\text{tv}} A = 1$$

$$\Psi_* = \Psi_{\text{tv}}^{(0)} - \Sigma \Psi_{\text{tv}}^{(*)} \bar{\Sigma}$$

$$Q_{\text{tv}} \Sigma = 0$$
$$Q_{\text{tv}} \bar{\Sigma} = 0$$
$$\bar{\Sigma}\Sigma = 1.$$

$$\bar{\sigma}(x)\sigma(0) = x^{-2h} \mathbf{1}_{\text{BCFT}_*} + \text{ less singular}$$

$$S[\Psi_*] = \frac{1}{2\pi^2 g_s} \left(Z_{\text{disk}}^{\text{BCFT}_0} - Z_{\text{disk}}^{\text{BCFT}_*} \right)$$

$$f: \mathcal{H}_{\text{BCFT}_*} \rightarrow \mathcal{H}_{\text{BCFT}_0}$$
$$f(\phi) := \Sigma \phi \bar{\Sigma}$$

$$S[\Psi_* + \Sigma\phi\bar{\Sigma}] = \frac{1}{2\pi^2 g_s} \left(Z_{\text{disk}}^{\text{BCFT}_0} - Z_{\text{disk}}^{\text{BCFT}_*} \right) + S^{(*)}[\phi]$$

$$\Lambda = \frac{1}{2\pi^2} \left(\frac{1}{g_s^{(0)}} Z_{\text{disk}}^{\text{BCFT}_0} - \frac{1}{g_s^{(*)}} Z_{\text{disk}}^{\text{BCFT}_*} \right)$$

Agujero negro cuántico supermasivo. Modelo Matemático aplicable a un campo cuántico relativista o curvo. Métrica Bekenstein-Hawking.

$$S_{bh} = \frac{A_h c^3}{4\hbar G_N}$$

$$S_{bh} = \frac{A}{4A_{Pl}}$$

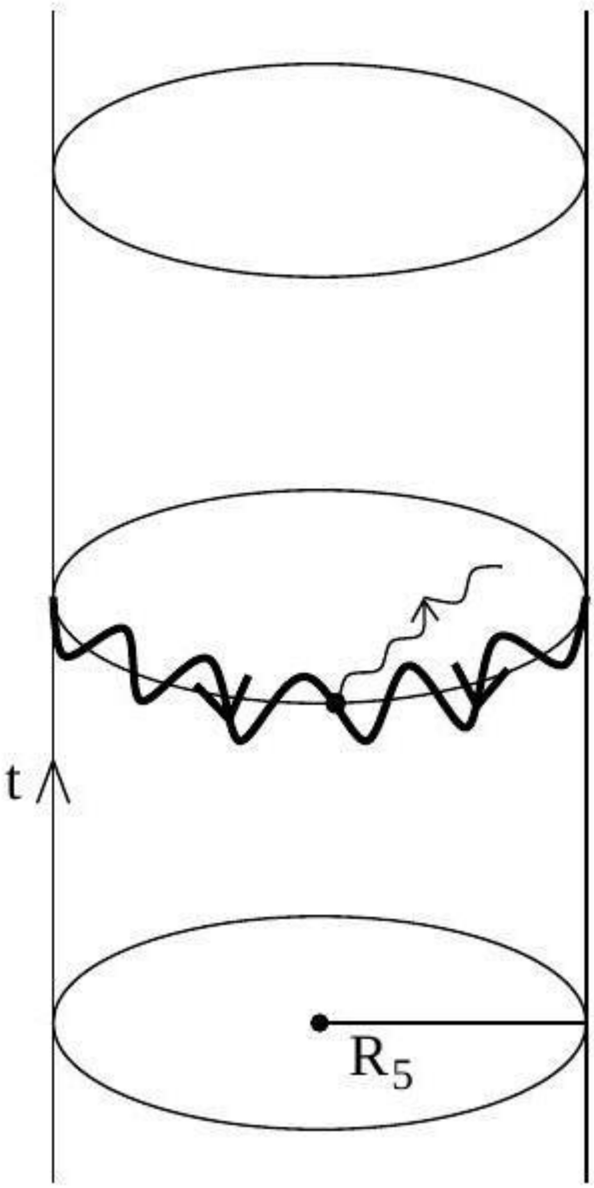
$$S = \frac{1}{\ell_s^2} \int d\sigma dt \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right)$$

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \delta\psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon.$$

$$G_N^{(10)} = 8\pi^6 \ell_s^8 g_s^2$$

$$\Delta x \geq \frac{\hbar}{\Delta p} + \ell_s^2 \frac{\Delta p}{\hbar}$$

$$E_n = \frac{c\hbar|n|}{R} + \frac{c\hbar|m|R}{\ell_s^2}$$



$$\ln \Omega(E,P) = 2\pi\sqrt{Q_1Q_5}\left(\sqrt{\frac{(E+cP)R_5}{2\hbar c}}+\sqrt{\frac{(E-cP)R_5}{2\hbar c}}\right)+o\left(\sqrt{Q_1Q_5}\right)$$

$$S = 2\pi\sqrt{Q_1Q_5n}$$

$$S = \frac{A_h c^3}{4G_N\hbar}$$

$$A_h = 2\pi^2 r_h^3, r_h^2 = \left(gQ_1gQ_5\frac{g^2n}{R_5^2}\right)^{1/3}$$

$$G_N^{-1} = \frac{2\pi R_5(2\pi)^4}{8\pi^6 g_s^2}$$

$$P_{abs}(i \to f) = \frac{1}{\Omega}\sum_{i,f} |\langle f|S|i\rangle \square|^2$$

$$P_{\text{decay}}(i \to f) = \frac{1}{\Omega'}\sum_{i,f} |\langle f|S|i\rangle \square|^2$$

$$\Gamma_H = \frac{\sigma_{abs}(\omega)}{e^{\omega/T_H} - 1}\frac{d^4k}{(2\pi)^4}$$

$$ds^2 = H^{-1/2}(-dt^2 + d\vec{x}\cdot d\vec{x}) + H^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right)$$
$$H = \left(1 + \frac{R^4}{r^4}\right), \left(\frac{R}{\ell_s}\right)^4 = 4\pi g_s N$$

$$\frac{E_\infty}{\ell_s E_r} \cong \frac{r}{\ell_s^2} = U$$

$$ds^2 = \ell_s^2\left[\frac{U^2}{\sqrt{4\pi\lambda}}(-dt^2 + d\vec{x}\cdot d\vec{x}) + 4\sqrt{4\pi\lambda}\frac{dU^2}{U^2} + \sqrt{4\pi\lambda}d\Omega_5^2\right]$$

$$ds^2 = \frac{R^2}{u^2}(-dt^2 + d\vec{x}\cdot d\vec{x} + du^2) + R^2 d\Omega_5^2$$

$$ds^2 = R^2\left[-\left(\frac{1+r^2}{1-r^2}\right)^2 dt^2 + \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\Omega^2)\right]$$

$$S = \frac{\text{Area}}{4G_N} \cong \frac{R^8\delta^{-3}}{4G_N} \sim \frac{R^8\delta^{-3}}{g_s^2\ell_s^8} \sim N^2\delta^{-3}$$

$$ds^2 = \frac{R^2}{\delta^2}(-dt^2 + d\vec{x}\cdot d\vec{x})$$

$$S_I = \int d\vec{x}dt\phi(\vec{x}, t, z = \delta)\mathcal{O}(\vec{x}, t)\delta^{\Delta-4}$$

$$\left\langle \exp\, i\int d\vec{x}dt\delta^{\Delta-4}\mathcal{O}(\vec{x}, t)\phi_0(\vec{x}, t)\right\rangle_{\text{SCFT}} = \mathcal{Z}_{\text{string}}\left(\phi(\vec{x}, t, \delta \to 0) = \phi_0(\vec{x}, t)\right)$$

$$W(C) = \text{Tr}\left[P\exp\left(\oint \left(iA_\mu\frac{dx^\mu}{d\tau} + y_I\Phi^I\sqrt{\dot{x}^2}\right)d\tau\right)\right]$$

$$E(L) = -\frac{4\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}\frac{\sqrt{2\lambda}}{L}$$

$$R_{ij} + \frac{4}{R^2}g_{ij} = 0$$

$$ds^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2 d\Omega_3^2$$
$$V(r) = 1 + \frac{r^2}{R^2} - \frac{\mu}{r^2}$$

$$\frac{1}{TR} = \frac{2\pi R r_+}{2r_+^2 + R^2}$$

$$I(X_2) - I(X_1) = \frac{\pi^2 r_+^3 (R^2 - r_+^2)}{4G_5(2r_+^2 + R^2)}$$

$$I(X_2) - I(X_1) = -\frac{\pi^5}{8}(TR)^3 \frac{1}{R^{-3}G_5} = -\frac{\pi^5}{8} N^2 (RT)^3$$

$$P(\vec{x},\beta) = \mathrm{Tr}\left[P\exp\left(\int_0^\beta d\tau A_0(\vec{x},\tau)\right)\right]$$

$$Z_N = \left\{e^{\frac{i2\pi k}{N}}, k = 1, \cdots, N\right\}$$

$$P'(\vec{x},\beta) = gP(\vec{x},\beta), g \in Z_N.$$

$$F \sim N^2(RT)^3$$

$$T_c = \frac{3}{2\pi R}$$

$$F(T_c) = N^2 \frac{9\pi^2}{64}$$

$$F(T) = N^2 \frac{\pi^5}{8}(RT)^3$$

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu} - \eta\left(P^{\mu\alpha}P^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \frac{1}{3}P^{\mu\nu}\partial_\alpha u^\alpha\right) + \cdots$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$ds^2 = -2u_\mu dx^\mu dv - r^2 f(br)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu$$

$$f(r) = 1 - \frac{1}{r^4}$$
$$u^v = \frac{1}{\sqrt{1-\beta_i^2}}, u^i = \frac{\beta^i}{\sqrt{1-\beta_i^2}}$$

$$P = (\pi T)^4 \text{ and } \eta = 2(\pi T)^3$$

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

$$Ng^2(L\wedge) = -\frac{1}{\beta_0 \ln\ L\wedge}, \beta_0 = \frac{11}{24\pi^2}$$

Agujero negro cuántico cromodinámico. Modelo Matemático aplicable al espacio – tiempo cuántico relativista o curvo

$$\mathcal{L}_{\text{ferm}} = \sum_{i}^{n} \bar{f}_i (i\,\partial\!\!\!/ - m_i) f_i,$$

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$$

$$f_i \to f_i' = \exp{(ie_i\theta)} f_i$$

$$f_i(x) \to f_i'(x) = \exp{(ie_i\theta(x))} f_i(x)$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{i}{e}\big(\partial_\mu \exp{(i\theta(x))}\big)\exp{(-i\theta(x))},$$

$$D_\mu = \partial_\mu + ie\hat{Q}A_\mu$$

$$\hat{Q} f_i = e_i f_i$$

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_{\text{classical}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{i}^{n} \bar{f}_i (i D\!\!\!\!/ - m_i) f_i$$

$$S = i \int d^4x \mathcal{L}$$

$$\Delta_{\gamma,\mu\nu}(p) \times i[p^2 g^{\nu\sigma} - p^\nu p^\sigma] = \delta_\mu^\sigma$$

$$\partial^\mu A_\mu = 0$$

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\lambda}\left(\partial^\mu A_\mu\right)^2$$

$$\Delta_{\gamma,\mu\nu}(p) \times i\left[p^2 g^{\nu\sigma} - \left(1 - \frac{1}{\lambda}\right) p^\nu p^\sigma\right] = \delta_\mu^\sigma$$

$$\Delta_{\gamma,\mu\nu} = \frac{i}{p^2}\left(-g_{\mu\nu} + (1-\lambda)\frac{p_\mu p_\nu}{p^2}\right)$$

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\lambda}\left(n^\mu A_\mu\right)^2$$

$$\Delta_{\gamma,\mu\nu} = \frac{i}{p^2}\left(-g_{\mu\nu} + \frac{n_\mu p_\nu + p_\mu n_\nu}{n \cdot p} - \frac{(n^2 + \lambda p^2) p_\mu p_\nu}{(n \cdot p)^2}\right)$$

$$\Delta_i = \frac{i}{\not{p} - m_i} = i\frac{\not{p} + m_i}{p^2 - m_i^2}$$
$$\Delta_{\gamma,\mu\nu} = i\frac{-g_{\mu\nu}}{p^2}$$
$$\Gamma^{\mu}_{\gamma f_i \bar{f}_i} = -ie_i e\gamma^{\mu}$$

$$\sigma = \frac{1}{S}\frac{1}{2s}\int d\Gamma \sum |\mathcal{M}|^2$$

$$d\Gamma = \prod_{i=1}^{n}\left(\frac{d^4 p_i}{(2\pi)^4}(2\pi)\delta\left(p_i^2 - m_i^2\right)\right)(2\pi)^4\delta^4\left(p_{\mathrm{tot}} - \sum_i p_i\right)$$
$$= \prod_{i=1}^{n}\left(\frac{d^3 p_i}{(2\pi)^3 2E_i}\right)(2\pi)^4\delta^4\left(p_{\mathrm{tot}} - \sum_i p_i\right)$$

$$i\mathcal{M} = \bar{v}(p_{e^+})(ie)\gamma^{\mu}u(p_{e^-})i\frac{-g_{\mu\nu}}{(p_{e^+} + p_{e^-})^2}\bar{u}\left(p_{\mu^-}\right)(ie)\gamma^{\nu}v\left(p_{\mu^+}\right)$$
$$= \frac{-ie^2}{(p_{e^+} + p_{e^-})^2}\bar{v}(p_{e^+})\gamma^{\mu}u(p_{e^-})\bar{u}\left(p_{\mu^-}\right)\gamma_{\mu}v\left(p_{\mu^+}\right)$$

$$\sum |\mathcal{M}|^2 = \frac{(4\pi\alpha)^2}{s^2}\mathrm{Tr}\{\not{p}_{e^+}\gamma^{\mu}\not{p}_{e^-}\gamma^{\nu}\}\mathrm{Tr}\left\{\not{p}_{\mu^-}\gamma_{\mu}\not{p}_{\mu^+}\gamma_{\nu}\right\},$$

$$\sum |\mathcal{M}|^2 = \frac{16(4\pi\alpha)^2}{s^2}\left(p^{\mu}_{e^+}p^{\nu}_{e^-} + p^{\mu}_{e^-}p^{\nu}_{e^+} - p_{e^+}\cdot p_{e^-}g^{\mu\nu}\right)\left(p_{\mu^-,\mu}p_{\mu^+,\nu} + p_{\mu^+,\mu}p_{\mu^-,\nu} - p_{\mu^+}\right.$$

$$\left.\cdot\, p_{\mu^-}g_{\mu\nu}\right) = 8(4\pi\alpha)^2\frac{t^2 + u^2}{s^2}$$

$$\sigma = \frac{1}{4}\frac{1}{2s}\int_{-s}^{0}\frac{dt}{8\pi s}8(4\pi\alpha)^2\frac{t^2 + u^2}{s^2} = \frac{4\pi\alpha^2}{3s}$$

$$U = 1 + iG,$$

$$G = \sum_{A}^{N^2-1}\epsilon^A t^A$$

$$[t^A, t^B] \equiv if^{ABC}t^C$$

$$(T^A)_{BC} \equiv -if^{ABC}$$
$$[T^A, T^B] = if^{ABC}T^C$$

$$U = \lim_{N\to\infty}(1 + i\theta^A t^A/N)^N = \exp\left(i\theta^A t^A\right) \equiv \exp\left(it\cdot\theta\right)$$

$$U^{-1} = \exp\left(-it\cdot\theta\right)$$

$$
\begin{aligned}
\mathrm{Tr}(t^A t^B) &= \frac{1}{2}\delta^{AB} \equiv T_R \delta^{AB} \\
\sum_A t^A_{ab} t^A_{bc} &= \frac{N^2-1}{2N}\delta_{ac} \equiv C_F \delta_{ac} \\
\mathrm{Tr}(T^C T^D) &= \sum_{A,B} f^{ABC} f^{ABD} = N\delta^{CD} \equiv C_A \delta^{CD}
\end{aligned}
$$

$$
\begin{aligned}
T_R &= \frac{1}{2}, \\
C_F &= \frac{4}{3}, \\
C_A &= 3,
\end{aligned}
$$

$$
\mathcal{L}_{\text{quarks}} = \sum_i^n \bar{q}_i^a (i\,\partial\!\!\!/ - m_i)_{ab} q_i^b,
$$

$$
q_a \to q'_a = \exp\,(it\cdot\theta)_{ab} q_b
$$

$$
q_a(x) \to q'_a(x) = \exp\,(it\cdot\theta(x))_{ab} q_b(x)
$$

$$
D_{\mu,ab} = \partial_\mu 1_{ab} + ig_s \left(t\cdot A_\mu\right)_{ab}
$$

$$
D'_{\mu,ab} q'_b(x) = \exp\,(it\cdot\theta(x))_{ab} D_{\mu,bc} q_c(x)
$$

$$
t\cdot A'_\mu = \exp\,(it\cdot\theta(x)) t\cdot A_\mu \exp\,(-it\cdot\theta(x)) + \frac{i}{g_s}\left(\partial_\mu \exp\,(it\cdot\theta(x))\right)\exp\,(-it\cdot\theta(x)).
$$

$$
\mathcal{L}_{\text{kin}} = -\frac{1}{4} F^A_{\mu\nu} F_A^{\mu\nu}
$$

$$
F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g_s f^{ABC} A^B_\mu A^C_\nu,
$$

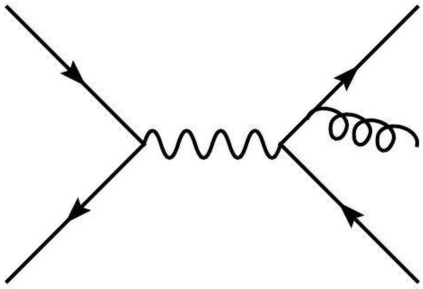
$$
\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\lambda}\left(\partial^\mu A^A_\mu\right)^2
$$

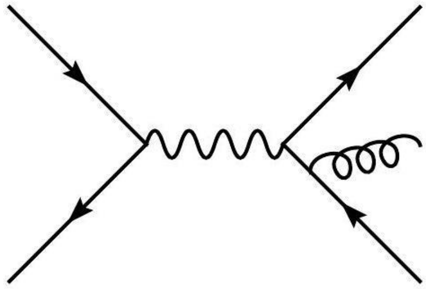
$$
\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F^A_{\mu\nu} F_A^{\mu\nu} + \sum_i^n \bar{q}_i^a (i D\!\!\!\!/ - m_i)_{ab} q_i^b - \frac{1}{2\lambda}\left(\partial^\mu A^A_\mu\right)^2 + \mathcal{L}_{\text{ghost}}
$$

$$
\begin{aligned}
\Delta_i^{ab} &= \delta^{ab} \frac{i}{p\!\!\!/ - m_i} = \delta^{ab} i \frac{p\!\!\!/ + m_i}{p^2 - m_i^2} \\
\Delta_{g,\mu\nu}^{AB} &= \delta^{AB} i \frac{-g_{\mu\nu}}{p^2} \\
\Gamma^\mu_{gq\bar{q}} &= -ig_s t^A \gamma^\mu \\
\Gamma_{ggg} &= -g_s f^{ABC}\left[(p-q)^\lambda g^{\mu\nu} + (q-r)^\mu g^{\nu\lambda} + (r-p)^\nu g^{\lambda\mu}\right]
\end{aligned}
$$

$$
\alpha_S \equiv \frac{g_s^2}{4\pi}
$$

$$\sigma(e^+e^- \to q\bar{q}) = \sigma(e^+e^- \to \mu^+\mu^-) \times e_q^2 \times \sum_{a,b} \delta^{ab}\delta^{ba}$$





$$\sum_{a,b} \delta^{ab}\delta^{ba} = \sum_{a} \delta^{aa} = N_c,$$

$$R_{\text{had}} \equiv \frac{\sigma(e^+e^- \to \text{ hadrons })}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{q} e_q^2 N_c$$

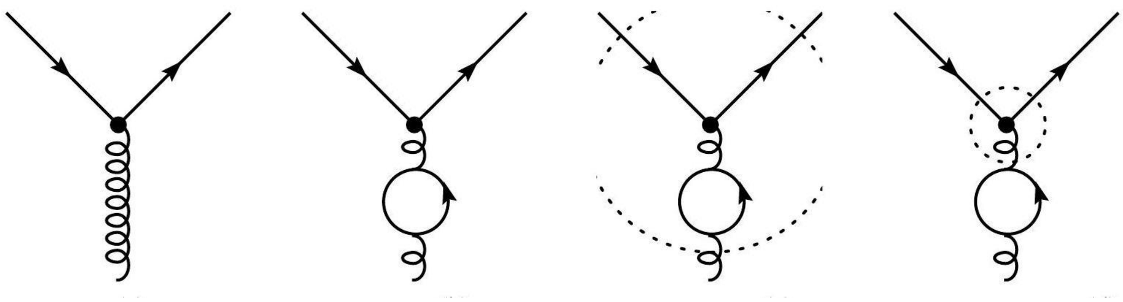
$$i\mathcal{M} = \bar{v}(p_+)(ie)\gamma^\mu u(p_-) i \frac{-g_{\mu\nu}}{s} \varepsilon_A^{*\lambda}$$

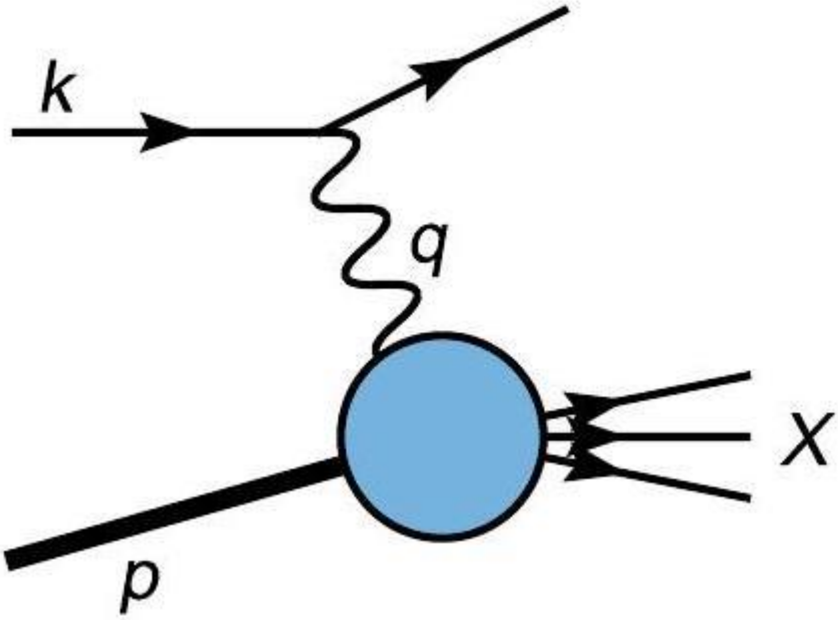
$$\bar{u}_a(p_1)\left\{(-ig_s)t_{ab}^A\gamma^\lambda \frac{\not{p}_1 + \not{p}_3}{(p_1+p_3)^2}(-iee_q)\gamma^\nu + (-iee_q)\gamma^\nu \frac{-\not{p}_2 - \not{p}_3}{(p_2+p_3)^2}(-ig_s)t_{ab}^A\gamma^\lambda\right\} v_b(p_2)$$

$$\sum_{\text{spin}} \varepsilon_A^{*\mu}\varepsilon_B^{\nu} = -g^{\mu\nu}\delta_{AB}$$

$$\sum |\mathcal{M}|^2 \propto \sum_{a,b,A} t_{ba}^A\left(t_{ba}^A\right)^* = \sum_{a,b,A} t_{ba}^A t_{ab}^A = \sum_{A} \text{Tr}(t^A t^A) = C_F \text{Tr}(1) = C_F N_c,$$

$$\sum |\mathcal{M}|^2 = \frac{16 C_F N_c e^4 e_q^2 g_s^2}{s p_1 \cdot p_3 p_2 \cdot p_3}\left((p_1 \cdot p_+)^2 + (p_2 \cdot p_+)^2 + (p_1 \cdot p_-)^2 + (p_2 \cdot p_-)^2\right)$$





Figuras 1 y 2. Modalidades de Aniquilación de una partícula, tipo estrella u oscura y emisión de radiación.

$$R = R[Q^2/\mu^2, \alpha_S(\mu^2)].$$

$$\mu^2 \frac{d}{d\mu^2} R(Q^2/\mu^2, \alpha_S) = 0 \ = \left[\mu^2 \frac{\partial}{\partial\mu^2} + \mu^2 \frac{\partial\alpha_S}{\partial\mu^2} \frac{\partial}{\partial\alpha_S}\right] R$$

$$\equiv \left[\mu^2 \frac{\partial}{\partial\mu^2} + \beta(\alpha_S) \frac{\partial}{\partial\alpha_S}\right] R$$

$$\beta(\alpha_S) = -\frac{\mu^2 \frac{\partial R}{\partial\mu^2}}{\frac{\partial R}{\partial\alpha_S}}$$

$$\beta(\alpha_S) = -\alpha_S^2 (\beta_0 + \beta_1 \alpha_S + \beta_2 \alpha_S^2 + \beta_3 \alpha_S^2 + \cdots)$$

$$\beta_0 = \frac{11C_A - 4T_R N_f}{12\pi}$$

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\beta_0 \ln \frac{Q^2}{\mu^2}}.$$

$$R = R_1 \alpha_s + \cdots,$$

$$R \approx R_1 \alpha_S$$

$$R\left(1, \alpha_S(Q^2)\right) \approx R_1 \alpha_S(Q^2)$$

$$= R_1 \alpha_S(\mu^2)\left[1 - \beta_0 \alpha_S(\mu^2) \ln \frac{Q^2}{\mu^2} + \beta_0^2 \alpha_S^2(\mu^2) \ln^2 \frac{Q^2}{\mu^2} + \cdots\right]$$

$$\Lambda^2 = \mu^2 \exp \int_{\alpha_S(\mu^2)}^{\infty} \frac{dx}{\beta(x)} \approx \mu^2 \mathrm{e}^{-1/\beta_0 \alpha_S(\mu^2)}$$

$$\alpha_S^{(d)} = \alpha_S \mu^{2\epsilon}$$

$$\alpha_S(\mu_R) = \alpha_S + \beta_0 F(\epsilon) SPp, \left(\frac{\mu^2}{\mu_R^2}\right)^{\epsilon} \frac{1}{\epsilon} \alpha_S^2.$$

$$F_{\mathrm{MS}}(\epsilon) = 1$$

$$F_{\overline{\mathrm{MS}}}(\epsilon) = \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} = 1 + (\ln\ 4\pi - \gamma_E)\epsilon$$

$$m(\mu^2) = M[\alpha_S(\mu^2)]^{\frac{1}{\pi \beta_0}},$$

$$\mathcal{M} = \left\{\bar{v}(q_2) e \gamma_\mu u(q_1)\right\} \frac{-g^{\mu\nu}}{q^2} T_\nu(n, q, \{p_1 \dots p_n\})$$

$$\sigma = \frac{1}{2s} \frac{1}{4} \frac{e^2}{s^2} \mathrm{Tr}\left(\not{q}_2 \gamma^\mu \not{q}_1 \gamma^\nu\right)$$

$$\times \sum_n \int dPS_n T_\mu(n, q, \{p_1 \dots p_n\}) T_\nu^*(n, q, \{p_1 \dots p_n\})$$

$$H_{\mu\nu}(q) \equiv \sum_n \int dPS_n T_\mu T_\nu^*$$

$$H_{\mu\nu} = A(q^2) g_{\mu\nu} + B(q^2) q_\mu q_\nu$$

$$q^\mu H_{\mu\nu} = q^\nu H_{\mu\nu} = 0$$

$$A = -q^2 B$$

$$R(e^+e^-) \equiv \frac{\sigma(e^+e^- \to \text{ hadrons })}{\sigma(e^+e^- \to \mu^+\mu^-)} = \text{constant}$$

$$\sigma(e^+e^- \to \text{ hadrons }) \approx \sigma(e^+e^- \to \text{ quarks }),$$

$$\sigma(e^+e^- \to \text{ hadrons }) = \sigma(e^+e^- \to \text{ quarks }) \times \left(1 + \mathcal{O}\left(\frac{m_{\text{had}}}{\sqrt{s}}\right)^n\right).$$

$$R_{e^+e^-} \equiv \frac{\sigma(\text{ hadrons })}{\sigma(\text{ muons })} = N_c \sum_q e_q^2,$$

$$R \equiv \frac{\sigma(e^+e^- \to \text{ hadrons })}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

$$R \equiv \frac{\sigma(e^+e^- \rightarrow \text{ hadrons })}{\frac{4\pi\alpha^2}{3s}}$$

$$R_{had} = N_c \frac{\sum_q \, v_q^2 + a_q^2}{v_\mu^2 + a_\mu^2} = 20.095,$$

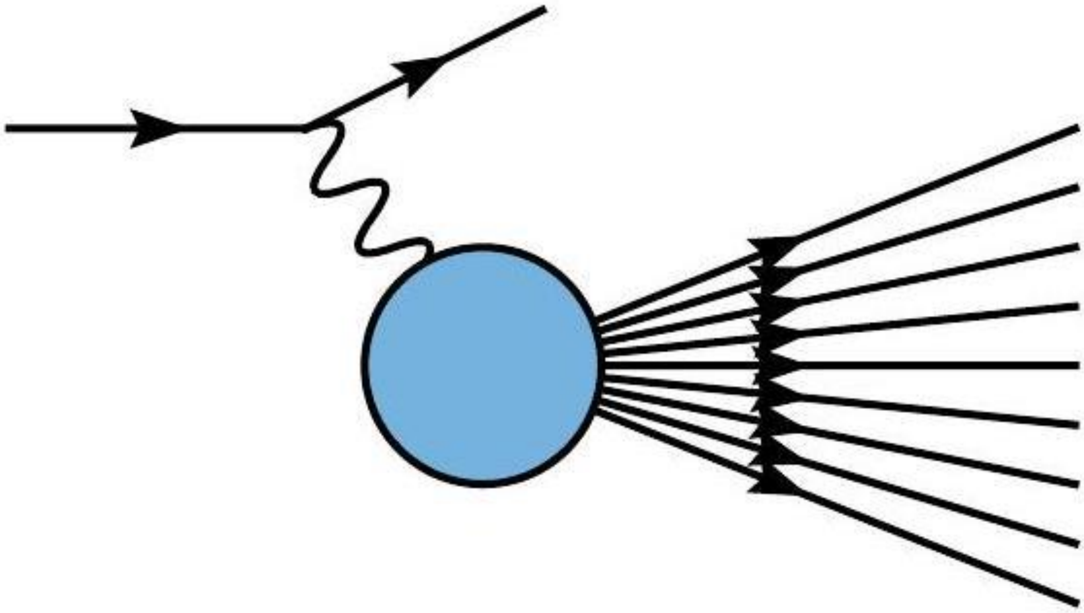


Figura 3. Emisión de radicación de un agujero negro supermasivo.

$$R_\tau \equiv \frac{\mathcal{B}(\tau \rightarrow \text{ hadrons })}{\mathcal{B}(\tau \rightarrow \mu)} = N_c \sum_{i,j} |V_{ij}|^2 \approx N_c$$

$$s \ = (k+p)^2,$$
$$Q^2 \ = -q^2,$$
$$x \ = \frac{Q^2}{2p \cdot q},$$

$$W^2 = (p+q)^2 = Q^2 \frac{1-x}{x}, \text{ and } y = \frac{p \cdot q}{p \cdot k} = \frac{Q^2}{xs}.$$

$$Q^2 < s, \text{ and } x > \frac{Q^2}{s}$$

$$eT_\mu(p,q;\{p_X\})$$

$$\frac{1}{4}|\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{Q^4} \mathrm{Tr}\{\not k \gamma^\mu \not k' \gamma^\nu\} T_\mu(p,q;\{p_X\}) T_\nu^*(p,q;\{p_X\})$$

$$L^{\mu\nu} = \mathrm{Tr}\{\not k \gamma^\mu \not k' \gamma^\nu\}$$

$$dPS = \frac{Q^2}{16\pi^2 s x^2} dQ^2 dx dPS_X$$

$$\sum_X \int dPS_X \frac{1}{4} |\mathcal{M}|^2 \equiv \frac{e^4}{Q^4} L^{\mu\nu} H_{\mu\nu}$$

$$\sum_X \int dPS_X T_\mu(p,q;\{p_X\}) T_\nu^*(p,q;\{p_X\}) = H_{\mu\nu}$$

$$H_{\mu\nu} = -H_1 g_{\mu\nu} + H_2 \frac{p_\mu p_\nu}{Q^2} + H_4 \frac{q_\mu q_\nu}{Q^2} + H_5 \frac{p_\mu q_\nu + q_\mu p_\nu}{Q^2},$$

$$L^{\mu\nu} H_{\mu\nu} = 4k \cdot k' H_1 + 4 \frac{p \cdot kp \cdot k'}{Q^2} H_2.$$

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{xQ^4} [y^2 x F_1(x,Q^2) + (1-y) F_2(x,Q^2)].$$

$$F_T(x,Q^2) = 2xF_1(x,Q^2)$$
$$F_L(x,Q^2) = F_2(x,Q^2) - 2xF_1(x,Q^2)$$

$$\frac{d^2\sigma}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4} [(1 + (1-y)^2) F_T(x,Q^2) + 2(1-y) F_L(x,Q^2)].$$

Figura 4. Morfología de un agujero negro cuántico.

$$\frac{d^2\sigma}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4} [(1 + (1-y)^2) F_2(x,Q^2) - y^2 F_L(x,Q^2)]$$

$$\frac{d^2\sigma}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4} [(1 + (1-y)^2) F_2(x) - y^2 F_L(x)]$$

$$\frac{d^2\sigma(e + p(p))}{dxdQ^2} = \sum_q \int_0^1 d\eta f_q(\eta) \frac{d^2\sigma(e + q(\eta p))}{dxdQ^2}$$

$$(q + \eta p)^2 = 2\eta p \cdot q - Q^2 = 0$$

$$\eta = x$$

$$\sum |\mathcal{M}|^2 = 8(4\pi\alpha)^2 e_q^2 N_c \frac{(p_e \cdot p_q)^2 + (p_e \cdot p_q')^2}{(p_e \cdot p_e')^2}$$

$$\sum |\mathcal{M}|^2 = 8(4\pi\alpha)^2 e_q^2 N_c \frac{1+(1-y)^2}{y^2}.$$

$$dPS = \frac{Q^2}{16\pi^2 s x^2} dQ^2 dx dPS_X$$

$$\begin{aligned} dPS_X &= \frac{d^4 p_X}{(2\pi)^3} \delta(p_X^2)(2\pi)^4 \delta^4(\eta p + q - p_X) \\ &= (2\pi)\delta((\eta p + q)^2) \\ &= \frac{2\pi x}{Q^2} \delta(\eta - x) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma}{dx dQ^2} &= \frac{1}{4N_c} \frac{1}{2\hat{s}} \frac{Q^2}{16\pi^2 s x^2} \frac{2\pi x}{Q^2} \delta(x-\eta) \sum |\mathcal{M}|^2 \\ &= \frac{1}{4N_c} \frac{y^2}{16\pi Q^4} \delta(x-\eta) \sum |\mathcal{M}|^2 \end{aligned}$$

$$\frac{d\sigma(e+q)}{dx dQ^2} = \frac{2\pi\alpha^2}{Q^4} \delta(x-\eta) e_q^2 (1+(1-y)^2)$$

$$\frac{d\sigma(e+p)}{dx dQ^2} = \frac{2\pi\alpha^2}{x Q^4} (1+(1-y)^2) \sum_q e_q^2 x f_q(x)$$

$$F_2(x,Q^2) = \sum_q e_q^2 x f_q(x)$$
$$F_L(x,Q^2) = 0$$

$$\begin{aligned} L_{\mu\nu}^{\substack{\nu \\ \bar{\nu}}} &= L_{\mu\nu}^e \pm 2i\epsilon_{\mu\nu\rho\sigma} k^\rho k'^\sigma \\ H^{\mu\nu} &= -H_1 g^{\mu\nu} + H_2 \frac{p^\mu p^\nu}{Q^2} - \frac{i}{Q^2} \epsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma H_3 \\ \Rightarrow L_{\mu\nu}^\nu H^{\mu\nu} &= 2Q^2 H_1 + Q^2 \frac{1-y}{x^2 y^2} H_2 \pm \frac{Q^2}{xy} H_3 (1-y/2). \end{aligned}$$

$$\frac{d^2\sigma(\substack{\nu \\ \bar{\nu}} + p)}{dx dQ^2} = \frac{G_F^2}{4\pi x} \left(\frac{M_W^2}{Q^2 + M_W^2}\right)^2 [(1+(1-y)^2) F_2^\nu - y^2 F_L^\nu \pm (1-(1-y)^2) x F_3^\nu]$$

$$\begin{aligned} F_2^\nu(x,Q^2) &= \sum_q 2x f_q(x) + \sum_{\bar{q}} 2x f_{\bar{q}}(x) \\ x F_3^\nu(x,Q^2) &= \sum_q 2x f_q(x) - \sum_{\bar{q}} 2x f_{\bar{q}}(x) \end{aligned}$$

$$\begin{aligned} f_{u/n}(x) &= f_{d/p}(x) \\ f_{\bar{u}/n}(x) &= f_{\bar{d}/p}(x) \\ f_{d/n}(x) &= f_{u/p}(x) \\ f_{s/n}(x) &= f_{s/p}(x) \end{aligned}$$

$$F_2^{ep} = \frac{1}{9}xf_d + \frac{4}{9}xf_u + \frac{1}{9}xf_{\bar{d}} + \frac{4}{9}xf_{\bar{u}} + \frac{1}{9}xf_s + \frac{1}{9}xf_{\bar{s}} + \frac{4}{9}xf_c + \frac{4}{9}xf_{\bar{c}}$$

$$F_2^{en} = \frac{4}{9}xf_d + \frac{1}{9}xf_u + \frac{4}{9}xf_{\bar{d}} + \frac{1}{9}xf_{\bar{u}} + \frac{1}{9}xf_s + \frac{1}{9}xf_{\bar{s}} + \frac{4}{9}xf_c + \frac{4}{9}xf_{\bar{c}}$$

$$F_2^{\nu p} = 2xf_d + 2xf_{\bar{u}} + 2xf_s + 2xf_{\bar{c}}$$

$$xF_3^{\nu p} = 2xf_d - 2xf_{\bar{u}} + 2xf_s - 2xf_{\bar{c}}$$

$$F_2^{\bar{\nu} p} = 2xf_u + 2xf_{\bar{d}} + 2xf_c + 2xf_{\bar{s}}$$

$$xF_3^{\bar{\nu} p} = 2xf_u - 2xf_{\bar{d}} + 2xf_c - 2xf_{\bar{s}}$$

$$f_{u_v} \equiv f_u - f_{\bar{u}}$$

$$f_{d_v} \equiv f_d - f_{\bar{d}}$$

$$\int_0^1 dx f_{u_v}(x) = 2$$

$$\int_0^1 dx f_{d_v}(x) = 1$$

$$\frac{1}{2}\int_0^1 dx\left(F_3^{\nu p} + F_3^{\bar{\nu} p}\right) = 3$$

$$\frac{1}{2}\int_0^1 \frac{dx}{x}\left(F_2^{\bar{\nu} p} - F_2^{\nu p}\right) = 1$$

$$\int_0^1 \frac{dx}{x}\left(F_2^{ep} - F_2^{en}\right) \approx 0.23$$

$$\frac{1}{2}\int_0^1 dx\left(F_2^{\nu p} + F_2^{\bar{\nu} p}\right) \approx 0.5$$

$$h_1 + h_2 \to \mu^+ + \mu^- + X$$

$$q + \bar{q} \to \mu^+ + \mu^-$$

$$\frac{d\sigma(h_1(p_1) + h_2(p_2) \to \mu^+\mu^-)}{dM^2}$$

$$= \sum_q \int_0^1 d\eta_1 f_{q/h_1}(\eta_1) \int_0^1 d\eta_2 f_{\bar{q}/h_2}(\eta_2) \frac{d\sigma(q(\eta_1 p_1) + \bar{q}(\eta_2 p_2) \to \mu^+\mu^-)}{dM^2}$$

$$y \equiv \frac{1}{2}\ln\frac{E_{\mu^+\mu^-} + p_{z,\mu^+\mu^-}}{E_{\mu^+\mu^-} - p_{z,\mu^+\mu^-}}$$

$$\frac{d^2\sigma}{dM^2 dy} = \frac{4\pi\alpha^2}{3N_c M^2 s}\sum_q e_q^2 f_{q/h_1}\left(e^y M/\sqrt{\square s}\right) f_{\bar{q}/h_2}\left(e^{-y} M/\sqrt{\square s}\right)$$

$$h_1 + h_2 \to \gamma + X$$

$$h_1 + h_2 \rightarrow q + q + X$$
$$h_1 + h_2 \rightarrow q + \bar{q} + X$$
$$h_1 + h_2 \rightarrow q + g + X$$
$$h_1 + h_2 \rightarrow g + g + X, \text{ etc.}$$

$$q + \bar{q} \rightarrow \gamma + g$$
$$q + g \rightarrow \gamma + q$$

$$\sigma = \sigma_0 C_F \frac{\alpha_s}{2\pi} \int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)},$$

$$\frac{1}{(p+k)^2} = \frac{1}{2p \cdot k} = \frac{1}{2E\omega(1 - \cos\,\theta)} \approx \frac{1}{E\omega\theta^2},$$

$$|\mathcal{M}|^2 \sim \frac{1}{\theta^2}$$

$$|\mathcal{M}|^2 \sim \frac{p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k} \sim \frac{1}{\omega^2}.$$

$$\frac{d^3k}{2\omega} = \frac{1}{2}\omega d\omega \sin\,\theta d\theta d\phi \sim \omega d\omega \theta d\theta$$

$$\sigma_{q\bar{q}g} = \sigma_0 C_F \frac{\alpha_s}{2\pi}\left(\log^2 \frac{1}{\epsilon} - 3\log\frac{1}{\epsilon} + 7 - \frac{\pi^2}{3} + \mathcal{O}(\epsilon)\right).$$

$$d^d k \delta_+(k^2) = \frac{d^{d-1}k}{2\omega} = \frac{1}{2}\omega^{d-3} d\omega d\Omega_{d-2}$$

$$k = \omega(1; \sin\,\phi \sin\,\theta, \cos\,\phi \sin\,\theta, \cos\,\theta) \text{ (4 dimensions)}$$

$$k = \omega(1; \sin\,\psi \sin\,\phi \sin\,\theta, \cos\,\psi \sin\,\phi \sin\,\theta, \cos\,\phi \sin\,\theta, \cos\,\theta) \text{ (5 dimensions)}$$

$$k = \omega(1; \ldots, \cos\,\phi \sin\,\theta, \cos\,\theta) \text{ (}d\text{ dimensions)}$$

$$k = \omega(1; \ldots, \cos\,\theta) \text{ (}d\text{ dimensions)}$$

$$\int d\Omega_1 = \int d\phi = 2\pi$$

$$\int d\Omega_2 = \int d\phi \sin\,\theta d\theta = 4\pi$$
$$\int d\Omega_3 = \int d\psi \sin\,\phi d\phi \sin^2\,\theta d\theta = 2\pi^2$$

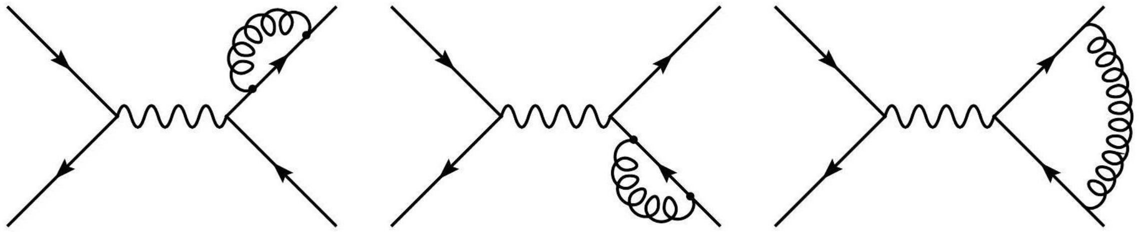
$$\int d\Omega_n = \int d\Omega_{n-1} \sin^{n-1}\,\theta d\theta$$

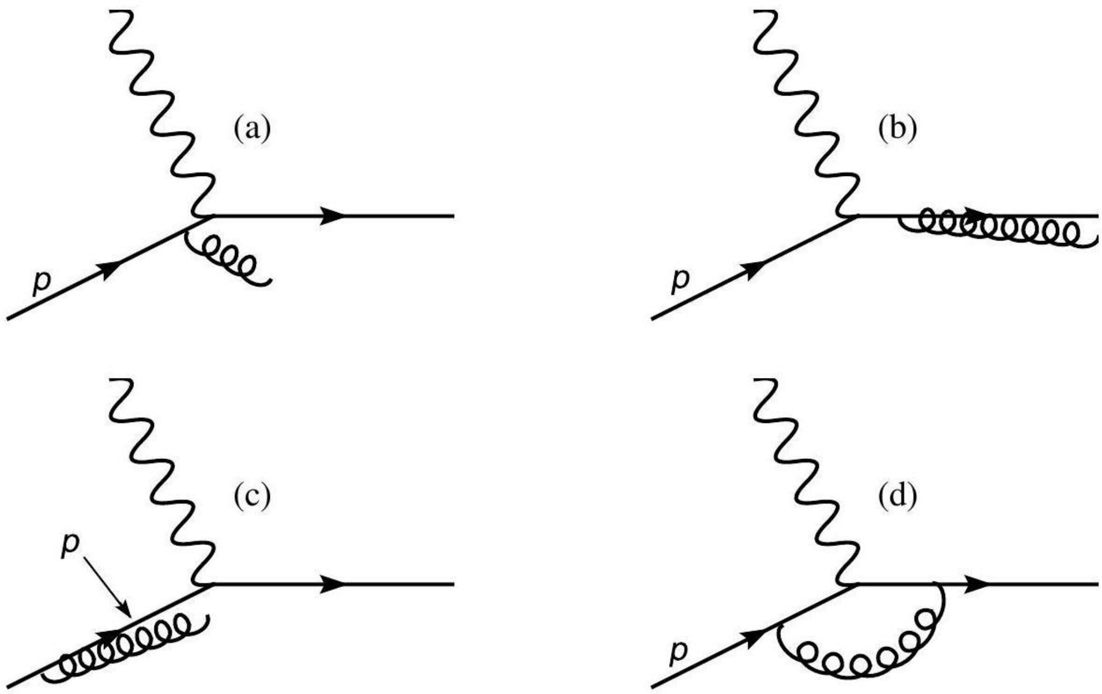
$$\Omega_n \equiv \int d\Omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma[(n+1)/2]}$$

$$\int_0 \omega^{1-2\epsilon} d\omega \frac{1}{\omega^2} = \int_0 \frac{d\omega}{\omega^{1+2\epsilon}} \sim -\frac{1}{2\epsilon}, \epsilon < 0$$

$$\int_0 \sin^{1-2\epsilon}\ \theta d\theta \frac{1}{\theta^2} \sim \int_0 \frac{d\theta}{\theta^{1-2\epsilon}} \sim -\frac{1}{2\epsilon}, \epsilon < 0$$

$$\sigma_{q\bar{q}g} = \sigma_0 C_F \frac{\alpha_s}{2\pi} H(\epsilon)\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon)\right),$$





Figuras 5 y 6. Diagramas de Feynman relativos a la formación de agujeros negros cuánticos.

$$\mathcal{M}_1 \propto \alpha_S \mathcal{M}_0$$

$$\int d^d k$$

$$\sigma_{q\bar{q}} = \sigma_0 C_F \frac{\alpha_s}{2\pi} H(\epsilon)\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon)\right)$$

$$\sigma_{q\bar{q}} = \sigma_0 C_F \frac{\alpha_s}{2\pi} H(\epsilon)\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon)\right)$$
$$\sigma_{q\bar{q}g} = \sigma_0 C_F \frac{\alpha_s}{2\pi} H(\epsilon)\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon)\right)$$

$$\sigma_{e^+e^-\to\text{ hadrons}} = \sigma_0\left(1 + C_F\frac{\alpha_S}{2\pi}\frac{3}{2}\right)$$
$$= \sigma_0\left(1 + \frac{\alpha_S}{\pi}\right)$$

$$\sigma_{q\bar{q}} = \sigma_0 C_F \frac{\alpha_S}{2\pi}\left[-\log^2\frac{1}{\epsilon} + 3\log\frac{1}{\epsilon} - \frac{11}{2} + \frac{\pi^2}{3} + \mathcal{O}(\epsilon)\right],$$
$$\sigma_{q\bar{q}g} = \sigma_0 C_F \frac{\alpha_S}{2\pi}\left[\log^2\frac{1}{\epsilon} - 3\log\frac{1}{\epsilon} + 7 - \frac{\pi^2}{3} + \mathcal{O}(\epsilon)\right],$$
$$\sigma_{\text{had}} = \sigma_0\left[1 + \frac{\alpha_S}{\pi}\right].$$

$$\sigma_{e^+e^-\to\text{ hadrons}} = \sigma_0\left(1 + \frac{\alpha_S(\mu)}{\pi} + C_2\left(\frac{\mu^2}{s}\right)\left(\frac{\alpha_S(\mu)}{\pi}\right)^2 + C_3\left(\frac{\mu^2}{s}\right)\left(\frac{\alpha_S(\mu)}{\pi}\right)^3\right).$$

$$R(\text{LEP}) = 20.767 \pm 0.025,$$

$$R_0(M_z) = 19.984$$

$$\alpha_S(M_z) = 0.124 \pm 0.004$$

$$R(\text{PETRA}) = 3.88 \pm 0.03$$

$$R_0(34\text{GeV}) = 3.69$$

$$\alpha_S(34\text{GeV}) = 0.162 \pm 0.026$$

$$\alpha_S(M_z) = 0.134 \pm 0.018.$$

$$\alpha_S(M_\tau = 1.77\text{GeV}) = 0.33 \pm 0.3.$$

$$\alpha_S^{(\text{one-loop})}(M_z) = 0.126 \pm 0.004,$$

$$\alpha_S(M_z) = 0.118 \pm 0.004,$$

$$\alpha_S^{(\text{world average})}(M_z) = 0.1183 \pm 0.0027.$$

$$e(k) + q(\eta p) \to e(k') + q(p_1) + g(p_2),$$

$$\sum |\mathcal{M}|^2 = \frac{8C_F N_c e^4 e_q^2 g_s^2}{k\cdot k' p_1\cdot p_2 \eta p\cdot p_2}\left((p_1\cdot k)^2 + (\eta p\cdot k)^2 + (p_1\cdot k')^2 + (\eta p\cdot k')^2\right)$$

$$dPS = \frac{Q^2}{16\pi^2 s x^2} dQ^2 dx dPS_X$$

$$dPS_X = \frac{d\cos\,\theta d\phi}{32\pi^2}$$

$$z \equiv \frac{p_1\cdot p}{q\cdot p} = \frac{1}{2}(1 - \cos\,\theta)$$

$$dPS_X = \frac{dzd\phi}{16\pi^2}$$

$$k_\perp^2 = Q^2\left(\frac{\eta}{x} - 1\right)z(1-z)$$

$$\frac{d\sigma^2(e+q)}{dxdQ^2} = \frac{1}{4N_c}\frac{1}{2\hat{s}}\frac{Q^2}{16\pi^2 sx^2}\int \frac{dzd\phi}{16\pi^2}\frac{8C_F N_c e^4 e_q^2 g_s^2}{k\cdot k' p_1\cdot p_2 \eta p\cdot p_2}((p_1\cdot k)^2 + (\eta p\cdot k)^2 + (p_1\cdot k')^2 + (\eta p\cdot k')^2)$$

$$\left\langle\frac{(p_1\cdot k)^2 + (\eta p\cdot k)^2 + (p_1\cdot k')^2 + (\eta p\cdot k')^2}{k\cdot k' p_1\cdot p_2 \eta p\cdot p_2}\right\rangle_\phi = \frac{1}{y^2Q^2}\left((1+(1-y)^2)\left[\frac{1+x_p^2}{1-x_p}\frac{1+z^2}{1-z} + 3 - z - x_p + 11x_p z\right] - y^2[8zx_p]\right)$$

$$\frac{d\sigma^2(e+q)}{dxdQ^2} = \frac{C_F\alpha^2 e_q^2\alpha_S}{2\eta x^2 y^2 s^2}\int_0^1 dz\left((1+(1-y)^2)\left[\frac{1+x_p^2}{1-x_p}\frac{1+z^2}{1-z} + 3 - z - x_p + 11x_p z\right] - y^2[8zx_p]\right)$$

$$F_2(x,Q^2) = \sum_q \int_x^1 dx_p e_q^2 \frac{x}{x_p} f_q\left(\frac{x}{x_p}\right)\frac{C_F\alpha_S}{2\pi}\int_0^1 dz\left(\frac{1+x_p^2}{1-x_p}\frac{1+z^2}{1-z} + 3 - z - x_p + 11x_p z\right)$$

$$F_2(x,Q^2) = \sum_q \int_x^1 dx_p e_q^2 \frac{x}{x_p} f_q\left(\frac{x}{x_p}\right)\frac{\alpha_S}{2\pi}\left(\hat{P}(x_p)\log\frac{Q^2}{\mu^2} + R(x_p)\right)$$

$$\hat{P}(x) = C_F\frac{1+x^2}{1-x}$$

$$P(x) = \hat{P}(x) + P_{\text{virtual}}(x)$$

$$f(x)_+ = f(x) - \delta(1-x)\int_0^1 dx' f(x')$$

$$\int_0^1 dx f(x)_+ g(x) = \int_0^1 dx f(x)(g(x) - g(1))$$

$$P(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$

$$\mathcal{P}(x) \equiv \delta(1-x) + \frac{\alpha_S}{2\pi}\log\frac{Q^2}{Q_0^2}P(x) + \mathcal{O}(\alpha_S^2\log^2)$$

$$F_2(x,Q^2) = \sum_q e_q^2 \int_x^1 dx_p \frac{x}{x_p} f_q\left(\frac{x}{x_p},\mu^2\right)\left\{\delta(1-x_p) + \frac{\alpha_S}{2\pi}\left(P(x_p)\log\frac{Q^2}{\mu^2} + R(x_p)\right) + \mathcal{O}(\alpha_S^2)\right\}$$

$$\sigma_h(p_h) = \sum_q \int d\eta f_q(\eta,\mu^2)\left\{\sigma_q(\eta p_h) + \frac{\alpha_S}{2\pi}\log\frac{Q^2}{\mu^2}\int dz P(z)\sigma_q(z\eta p_h)\right\}$$

$$\mu^2 \frac{dF_2(x,Q^2)}{d\mu^2} = 0,$$

$$\mu^2 \frac{dF_2(x,Q^2)}{d\mu^2} = \mathcal{O}(\alpha_S^2).$$

$$\mu^2 \frac{d}{d\mu^2} f_q(x,\mu^2) = \frac{\alpha_S}{2\pi}\int_x^1 \frac{dx_p}{x_p} f_q\left(\frac{x}{x_p},\mu^2\right)P(x_p) + \mathcal{O}(\alpha_S^2).$$

$$P(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right] = C_F\left(\frac{1+x^2}{1-x}\right)_+$$

$$\mu^2 \frac{d}{d\mu^2} f_q(x,\mu^2) = C_F\frac{\alpha_S}{2\pi}\int_x^1 \frac{dx_p}{x_p} f_q\left(\frac{x}{x_p},\mu^2\right)\frac{1+x_p^2}{1-x_p} - C_F\frac{\alpha_S}{2\pi} f_q(x,\mu^2)\int_0^1 dx_p \frac{1+x_p^2}{1-x_p}$$

$$f_N = \int_0^1 dx x^{N-1} f(x)$$

$$\begin{aligned}\mu^2 \frac{d}{d\mu^2} f_{qN}(\mu^2) &= \frac{\alpha_S}{2\pi}\int_0^1 dx x^{N-1}\int_x^1 \frac{dx_p}{x_p} f_q\left(\frac{x}{x_p},\mu^2\right)P(x_p) + \mathcal{O}(\alpha_S^2)\\ &= \frac{\alpha_S}{2\pi} P_N f_{qN}(\mu^2)\end{aligned}$$

$$\gamma_N(\alpha_S) = \frac{\alpha_S}{2\pi} P_N + \mathcal{O}(\alpha_S^2)$$

$$f_{qN}(\mu^2) = f_{qN}(\mu_0^2)\left(\frac{\mu^2}{\mu_0^2}\right)^{\gamma_N(\alpha_S)}$$

$$\mu^2 \frac{d}{d\mu^2}\alpha_S(\mu^2) = \beta(\alpha_S(\mu^2)) = -\frac{\beta_0}{2\pi}\alpha_S^2(\mu^2) + \mathcal{O}(\alpha_S^3)$$

$$f_{qN}(\mu^2) = f_{qN}(\mu_0^2)\left(\frac{\alpha_S(\mu_0)}{\alpha_S(\mu)}\right)^{\frac{P_N}{\beta_0}}$$

$$f_q(x) = \frac{1}{2\pi i}\int_C dN f_{qN} x^{-N}$$

$$F_2(x,Q^2) = \sum_q e_q^2 \int_x^1 dx_p \frac{x}{x_p}\bar{f}_q\left(\frac{x}{x_p}\right)\left\{\delta(1-x_p) + \frac{\alpha_S}{2\pi}\left(\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\frac{-1}{\epsilon}P(x_p) + R(x_p)\right) + \mathcal{O}(\epsilon)\right\}$$

$$x\bar{f}_q(x) \equiv \int_x^1 dx_p \frac{x}{x_p} f_q\left(\frac{x}{x_p}, \mu_F^2\right)\left\{\delta(1-x_p) - \frac{\alpha_S}{2\pi}\left(\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}\frac{-1}{\epsilon}P(x_p) + K(x_p)\right)\right\}$$

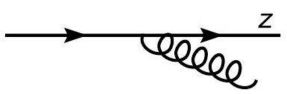
$$F_2(x,Q^2) = \sum_q e_q^2 \int_x^1 dx_p \frac{x}{x_p} f_q\left(\frac{x}{x_p}, \mu_F^2\right)\left\{\delta(1-x_p) + \frac{\alpha_S}{2\pi}\left(P(x_p)\log\frac{Q^2}{\mu_F^2} + R(x_p) - K(x_p)\right) + \mathcal{O}(\alpha_S^2)\right\}$$

$$F_2(x,Q^2) = \sum_q e_q^2 \int_x^1 dx_p \frac{x}{x_p} f_g\left(\frac{x}{x_p}, \mu^2\right)\left\{\frac{\alpha_S}{2\pi}\left(P_{qg}(x_p)\log\frac{Q^2}{\mu^2} + R_g(x_p) - K_{qg}(x_p)\right) + \mathcal{O}(\alpha_S^2)\right\}$$

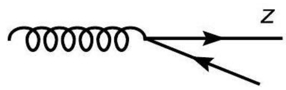
$$\mu^2\frac{d}{d\mu^2}f_a(x,\mu^2) = \sum_b \frac{\alpha_S}{2\pi}\int_x^1 \frac{dx_p}{x_p} f_b\left(\frac{x}{x_p},\mu^2\right)P_{ab}(x_p) + \mathcal{O}(\alpha_S^2).$$

$$\mu^2\frac{d}{d\mu^2}\begin{pmatrix} f_{qN} \\ f_{\bar{q}N} \\ f_{gN}\end{pmatrix} = \begin{pmatrix} \gamma_{qqN}(\alpha_S(\mu)) & 0 & \gamma_{qgN}(\alpha_S(\mu)) \\ 0 & \gamma_{qqN}(\alpha_S(\mu)) & \gamma_{qgN}(\alpha_S(\mu)) \\ \gamma_{gqN}(\alpha_S(\mu)) & \gamma_{gqN}(\alpha_S(\mu)) & \gamma_{ggN}(\alpha_S(\mu))\end{pmatrix}\begin{pmatrix} f_{qN} \\ f_{\bar{q}N} \\ f_{gN}\end{pmatrix}.$$

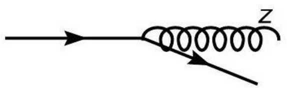
$$\begin{pmatrix} f_{qN}(\mu^2) \\ f_{\bar{q}N}(\mu^2) \\ f_{gN}(\mu^2)\end{pmatrix} = \exp\int_{\mu_0^2}^{\mu^2}\frac{d\mu'^2}{\mu'^2}\begin{pmatrix} \gamma_{qqN}(\alpha_S(\mu')) & 0 & \gamma_{qgN}(\alpha_S(\mu')) \\ 0 & \gamma_{qqN}(\alpha_S(\mu')) & \gamma_{qgN}(\alpha_S(\mu')) \\ \gamma_{gqN}(\alpha_S(\mu')) & \gamma_{gqN}(\alpha_S(\mu')) & \gamma_{ggN}(\alpha_S(\mu'))\end{pmatrix}\begin{pmatrix} f_{qN}(\mu_0^2) \\ f_{\bar{q}N}(\mu_0^2) \\ f_{gN}(\mu_0^2)\end{pmatrix}$$



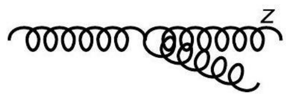
$$P_{qq}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$



$$P_{qg}(x) = T_R\left[x^2 + (1-x)^2\right]$$



$$P_{gq}(x) = C_F\left[\frac{1+(1-x)^2}{x}\right]$$



$$P_{gg}(x) = C_A\left[\frac{2x}{(1-x)_+} + 2\frac{1-x}{x} + 2x(1-x)\right] + \beta_0\delta(1-x)$$

CONCLUSIONES

Queda claro, que un agujero negro cuántico, es una fisura producida en el espacio – tiempo cuántico, sea por aniquilación o colapso de una partícula supermasiva, tipo estrella u oscura o en su defecto por aniquilación pura de una partícula supermasiva, esto último, habitualmente por colisión con otra partícula de similares o distintas características de masa y energía, incluso de momento angular. La relatividad general, sugiere la existencia de este fenómeno, a nivel cosmológico, volviéndose las ecuaciones de campo de Einstein, esenciales para su descifrado, sin embargo, al existir un modelo matemático, como es el caso, en el que, se describe la existencia y comportamiento de un agujero negro a escala cuántica, es un intento no despreciable de unificación, que amerita ser estudiado más a detalle.

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Anexo A

Agujeros negros cuánticos y espacio – tiempo cuántico – relativista. Modelo Matemático Unitario. Métrica de Baker-Campbell-Hausdorff-Majorana-Grassmann-Nielsen-Ninomiya-Ginsparg-Wilson-Becchi-Stora-Tyutin-Aharonov-Bohm-Goldstone-Mermin-Wagner-Coleman-Polyakov-Cabibbo-Kobayashi-Maskawa-Pontecorvo-Maki-Nakagawa-Sakata-Wess-Zumino-Novikov-Witten-Nambu.

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$$

$$x^{\mu} = (x^0,\vec{x}), x_{\mu} = g_{\mu\nu}x^{\nu} = (x_0,-\vec{x}), x^0 = x_0 = ct$$

$$\partial_{\mu} = \left(\frac{\partial}{\partial x^0},\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\partial_t,\vec{\nabla}\right), \partial^{\mu} = \left(\frac{1}{c}\partial_t,-\vec{\nabla}\right).$$

$$[T^a,T^b] = \mathrm{i}f_{abc}T^c, \mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta_{ab}$$

$$\vec{\tau} = (\tau^1,\tau^2,\tau^3) = \left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right),$$

$$\phi(\vec{x}) = \frac{e}{4\pi|\vec{x}|}$$

$$A'_{\mu}(x) = A_{\mu}(x) - \partial_{\mu}\alpha(x), \Phi'(x) = \exp\left(\mathrm{i}Qe\alpha(x)\right)\Phi(x)$$

$$B'_{\mu}(x) = B_{\mu}(x) - \partial_{\mu}\varphi(x), \Phi'(x) = \exp\left(\mathrm{i}Yg'\varphi(x)\right)\Phi(x)$$

$$W_{\mu}(x) = \mathrm{i}gW_{\mu}^{a}(x)\frac{\tau^a}{2}, \quad W_{\mu}^{a}(x) \in \mathbb{R}, \quad W'_{\mu}(x) = L(x)\big(W_{\mu}(x) + \partial_{\mu}\big)L(x)^{\dagger}$$

$$G_{\mu}(x) = \mathrm{i}g_sG_{\mu}^{a}(x)\frac{\lambda^a}{2}, \quad G_{\mu}^{a}(x) \in \mathbb{R}, \quad G'_{\mu}(x) = \Omega(x)\big(G_{\mu}(x) + \partial_{\mu}\big)\Omega(x)^{\dagger}$$

$$X_{\mu}(x) = \mathrm{i}g'X_{\mu}^{a}(x)\frac{\tau^a}{2}, \quad X_{\mu}^{3}(x) = B_{\mu}(x), X'_{\mu}(x) = R(x)\big(X_{\mu}(x) + \partial_{\mu}\big)R(x)^{\dagger},$$

$$V_{\mu}(x) = \mathrm{i}g_5V_{\mu}^{a}(x)\frac{\eta^a}{2}, \quad V_{\mu}^{a}(x) \in \mathbb{R}, V'_{\mu}(x) = \Upsilon(x)\big(V_{\mu}(x) + \partial_{\mu}\big)\Upsilon(x)^{\dagger}.$$

$$\vec{\sigma} = (\sigma^1,\sigma^2,\sigma^3) = \left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right).$$

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}, \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3, \{\gamma^{\mu},\gamma^5\} = 0$$

$$\gamma^0 = \begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}, \gamma^i = \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}, \gamma^5 = \begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}.$$

$$\psi(x)=\begin{pmatrix}\psi_{\rm L}(x)\\ \psi_{\rm R}(x)\end{pmatrix},\bar{\psi}(x)=(\bar{\psi}_{\rm R}(x),\bar{\psi}_{\rm L}(x)),P_{\rm L}=\frac{1}{2}(1-\gamma^5),P_{\rm R}=\frac{1}{2}(1+\gamma^5)$$
$$\begin{pmatrix}\psi_{\rm L}(x)\\ 0\end{pmatrix}=P_L\psi(x),\begin{pmatrix}0\\ \psi_{\rm R}(x)\end{pmatrix}=P_R\psi(x)$$

$$\gamma^\mu=\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix},\sigma^\mu=(\mathbb{1},\vec{\sigma}),\bar{\sigma}^\mu=(\mathbb{1},-\vec{\sigma}).$$

$$\psi_{\rm L}(x)=-{\rm i}\sigma^2\bar{\psi}_{\rm R}(x)^\top,\bar{\psi}_{\rm L}(x)=\psi_{\rm R}(x)^\top{\rm i}\sigma^2$$

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu},\gamma_5=-\gamma_1\gamma_2\gamma_3\gamma_4,\{\gamma_\mu,\gamma_5\}=0.$$

$$\gamma_i=\begin{pmatrix}0 & -{\rm i}\sigma^i\\ {\rm i}\sigma^i & 0\end{pmatrix},\gamma_4=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix},\gamma_5=\begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}.$$

$$\gamma_\mu=\begin{pmatrix}0 & \sigma_\mu\\ \bar{\sigma}_\mu & 0\end{pmatrix},\sigma_\mu=(-{\rm i}\vec{\sigma},\mathbb{1}),\bar{\sigma}_\mu=({\rm i}\vec{\sigma},\mathbb{1})$$

$$\begin{aligned}
&{}^{\rm C}\psi_{\rm R}(x)={\rm i}\sigma^2\bar{\psi}_{\rm L}(x)^{\rm T}={}^{\rm c}\psi_{\rm L}(x),\ {}^{\rm C}\bar{\psi}_{\rm R}(x)=-\psi_{\rm L}(x)^{\rm T}{\rm i}\sigma^2={}^{\rm c}\bar{\psi}_{\rm L}(x),\\
&{}^{\rm C}\psi_{\rm L}(x)=-{\rm i}\sigma^2\bar{\psi}_{\rm R}(x)^{\rm T}={}^{\rm c}\psi_{\rm R}(x),\ {}^{\rm C}\bar{\psi}_{\rm L}(x)=\psi_{\rm R}(x)^{\rm T}{\rm i}\sigma^2={}^{\rm c}\bar{\psi}_{\rm R}(x),\\
&{}^{\rm P}\psi_{\rm R}(x)=\psi_{\rm L}(-\vec{x},x_4),\ {}^{\rm P}\bar{\psi}_{\rm R}(x)=\bar{\psi}_{\rm L}(-\vec{x},x_4),\\
&{}^{\rm P}\psi_{\rm L}(x)=\psi_{\rm R}(-\vec{x},x_4),\ {}^{\rm P}\bar{\psi}_{\rm L}(x)=\bar{\psi}_{\rm R}(-\vec{x},x_4),\\
&{}^{\rm T}\psi_{\rm R}(x)={\rm i}\sigma^2\bar{\psi}_{\rm R}(\vec{x},-x_4)^{\rm T},\ {}^{\rm T}\bar{\psi}_{\rm R}(x)=\psi_{\rm R}(\vec{x},-x_4)^{\rm T}{\rm i}\sigma^2,\\
&{}^{\rm T}\psi_{\rm L}(x)={\rm i}\sigma^2\bar{\psi}_{\rm L}(\vec{x},-x_4)^{\rm T},\ {}^{\rm T}\bar{\psi}_{\rm L}(x)=\psi_{\rm L}(\vec{x},-x_4)^{\rm T}{\rm i}\sigma^2.
\end{aligned}$$

$$\begin{aligned}
&{}^{\rm C}\psi(x)=C\bar{\psi}(x)^\top,\ {}^{\rm C}\bar{\psi}(x)=-\psi(x)^\top C^{-1},C=\begin{pmatrix}-{\rm i}\sigma^2 & 0\\ 0 & {\rm i}\sigma^2\end{pmatrix},\\
&{}^{\rm P}\psi(x)=P\psi(-\vec{x},x_4),\ {}^{\rm P}\bar{\psi}(x)=\bar{\psi}(-\vec{x},x_4)P^{-1},P=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix},\\
&{}^{\rm T}\psi(x)=T\bar{\psi}(\vec{x},-x_4)^\top,\ {}^{\rm T}\bar{\psi}(x)=-\psi(\vec{x},-x_4)^\top T^{-1},T=\begin{pmatrix}0 & {\rm i}\sigma^2\\ {\rm i}\sigma^2 & 0\end{pmatrix}.
\end{aligned}$$

$$\Phi(x)=\begin{pmatrix}\Phi^+(x)\\ \Phi^0(x)\end{pmatrix},\Phi^+(x),\Phi^0(x)\in\mathbb{C}$$

$$\begin{aligned}
&\vec{\phi}(x)=(\phi_1(x),\phi_2(x),\phi_3(x),\phi_4(x))\in\mathbb{R}^4\\
&\Phi^+(x)=\phi_2(x)+{\rm i}\phi_1(x),\Phi^0(x)=\phi_4(x)-{\rm i}\phi_3(x)
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Phi}(x)&=\begin{pmatrix}\Phi^0(x)^* & \Phi^+(x)\\ -\Phi^+(x)^* & \Phi^0(x)\end{pmatrix}\\
&=\phi_4(x)\mathbb{1}+{\rm i}[\phi_1(x)\tau^1+\phi_2(x)\tau^2+\phi_3(x)\tau^3]
\end{aligned}$$

$$M\partial_t^2x=F(x)=-\frac{dV(x)}{dx}$$

$$S[x]=\int dtL(x,\partial_tx)$$

$$L(x,\partial_tx)=\frac{M}{2}(\partial_tx)^2-V(x)$$

$$\partial_t \frac{\delta L}{\delta \partial_t x} - \frac{\delta L}{\delta x} = 0$$

$$\Box \phi = \partial_\mu \partial^\mu \phi = -\frac{dV(\phi)}{d\phi}$$

$$S[\phi] = \int d^4 x \mathcal{L}(\phi, \partial_\mu \phi), d^4 x = d(ct) d^3 x = cdtd^3 x$$

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)$$

$$V(\phi) = \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0$$

$$\Psi(x,t) = \langle x \mid \Psi(t) \rangle, \Psi(p,t) = \langle p \mid \Psi(t) \rangle,$$

$$\int dx |x\rangle\langle x| = \frac{1}{2\pi\hbar} \int dp |p\rangle\langle p| = \sum_n |n\rangle\langle n| = \hat{\mathbb{1}}$$

$$\langle \Psi' \mid \Psi \rangle = \int dx \langle \Psi' \mid x \rangle \langle x \mid \Psi \rangle = \int dx \Psi'(x)^* \Psi(x)$$

$$\Psi(p,t) = \int dx \langle p \mid x \rangle \langle x \mid \Psi(t) \rangle = \int dx \exp(-\mathrm{i}px/\hbar) \Psi(x,t)$$
$$\Psi(x,t) = \frac{1}{2\pi\hbar} \int dp \langle x \mid p \rangle \langle p \mid \Psi(t) \rangle = \frac{1}{2\pi\hbar} \int dp \exp(\mathrm{i}px/\hbar) \Psi(p,t)$$

$$\langle \Psi | \hat{O} | \Psi \rangle = \int dx \Psi(x)^* \hat{O} \Psi(x)$$

$$\mathrm{i}\hbar \partial_t |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle.$$

$$|\Psi(t')\rangle = \hat{U}(t',t) |\Psi(t)\rangle, t' \geq t$$

$$\hat{U}(t',t) = \exp\left(-\frac{\mathrm{i}}{\hbar} \hat{H}(t'-t)\right)$$

$$\Psi(x',t') = \int dx \langle x' | \hat{U}(t',t) | x \rangle \Psi(x,t)$$

$$\langle x | \hat{U}(t',t) | x \rangle = \langle x | \exp\left(-\frac{\mathrm{i}}{\hbar} \hat{H}(t'-t)\right) | x \rangle = \sum_n |\langle x \mid n \rangle|^2 \exp\left(-\frac{\mathrm{i}}{\hbar} E_n (t'-t)\right)$$

$$\langle x'|\hat{U}(t',t)|x\rangle = \langle x'|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(t'-t_1)\right)\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(t_1-t)\right)|x\rangle$$

$$= \int dx_1 \langle x'|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(t'-t_1)\right)|x_1\rangle\langle x_1|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(t_1-t)\right)|x\rangle$$

$$= \int dx_1 \langle x'|\hat{U}(t',t_1)|x_1\rangle\langle x_1|\hat{U}(t_1,t)|x\rangle$$

$$t'-t = N\varepsilon.$$

$$\langle x'|\hat{U}(t',t)|x\rangle = \int dx_1 \int dx_2 \dots \int dx_{N-1} \langle x'|\hat{U}(t',t_{N-1})|x_{N-1}\rangle \dots \times \dots \langle x_2|\hat{U}(t_2,t_1)|x_1\rangle\langle x_1|\hat{U}(t_1,t)|x\rangle$$

$$\hat{H} = \frac{\hat{p}^2}{2M} + \hat{V}(\hat{x})$$

$$\langle x_{n+1}|\hat{U}(t_{n+1},t_n)|x_n\rangle = \langle x_{n+1}|\exp\left(-\frac{\mathrm{i}\varepsilon}{2\hbar}\hat{V}(\hat{x})\right)\exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar}\frac{\hat{p}^2}{2M}\right)\exp\left(-\frac{\mathrm{i}\varepsilon}{2\hbar}\hat{V}(\hat{x})\right)|x_n\rangle$$

$$= \frac{1}{2\pi\hbar}\int dp \langle x_{n+1}|\exp\left(-\frac{\mathrm{i}\varepsilon}{2\hbar}\hat{V}(\hat{x})\right)$$

$$\times \exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar}\frac{\hat{p}^2}{2M}\right)|p\rangle\langle p|\exp\left(-\frac{\mathrm{i}\varepsilon}{2\hbar}\hat{V}(\hat{x})\right)|x_n\rangle$$

$$= \frac{1}{2\pi\hbar}\int dp \exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar}\frac{p^2}{2M}\right)\exp\left(-\frac{\mathrm{i}}{\hbar}p(x_{n+1}-x_n)\right)$$

$$\times \exp\left(-\frac{\mathrm{i}\varepsilon}{2\hbar}[V(x_{n+1})+V(x_n)]\right)$$

$$\langle x_{n+1}|\hat{U}(t_{n+1},t_n)|x_i\rangle = \left(\frac{M}{2\pi\mathrm{i}\hbar\varepsilon}\right)^{1/2}\exp\left(\frac{\mathrm{i}}{\hbar}\varepsilon\left[\frac{M}{2}\left(\frac{x_{n+1}-x_n}{\varepsilon}\right)^2 - \frac{1}{2}\big(V(x_n)+V(x_{n+1})\big)\right]\right)$$

$$\langle x'|\hat{U}(t',t)|x\rangle = \int \mathcal{D}x \exp\left(\frac{\mathrm{i}}{\hbar}S[x]\right)$$

$$S[x] = \lim_{\varepsilon\to 0}\varepsilon\sum_n \left[\frac{M}{2}\left(\frac{x_{n+1}-x_n}{\varepsilon}\right)^2 - \frac{1}{2}\big(V(x_{n+1})+V(x_n)\big)\right] = \int dt\left[\frac{M}{2}(\partial_t x)^2 - V(x)\right]$$

$$\int \mathcal{D}x = \lim_{\varepsilon\to 0}\left(\frac{M}{2\pi\mathrm{i}\hbar\varepsilon}\right)^{N/2}\int dx_1\int dx_2 \dots \int dx_{N-1}$$

$$Z = \mathrm{Tr}\exp(-\beta\hat{H}),$$

$$\beta = \frac{\mathrm{i}}{\hbar}(t' - t).$$

$$t_{\mathrm{E}} = \mathrm{i}t.$$

$$Z = \mathrm{Tr}\exp\left(-\beta\hat{H}\right) = \int \mathcal{D}x\exp\left(-\frac{1}{\hbar}S_{\mathrm{E}}[x]\right)$$

$$S_{\mathrm{E}}[x] = \lim_{a\to 0}\sum_{n} a\left[\frac{M}{2}\left(\frac{x_{n+1} - x_n}{a}\right)^2 + V(x_n)\right] = \int_0^{\beta} dt_{\mathrm{E}}\left[\frac{M}{2}\left(\partial_{t_{\mathrm{E}}}x\right)^2 + V(x)\right]$$

$$\int \mathcal{D}x = \lim_{a\to 0}\left(\frac{M}{2\pi\hbar a}\right)^{N/2}\int dx_1\int dx_2 \ldots \int dx_N$$

$$\langle\hat{O}(\hat{x})\rangle = \frac{1}{Z}\mathrm{Tr}[\hat{O}(\hat{x})\exp\left(-\beta\hat{H}\right)] = \frac{1}{Z}\int \mathcal{D}x\mathcal{O}(x(0))\exp\left(-\frac{1}{\hbar}S_{\mathrm{E}}[x]\right)$$

$$\langle 0|\hat{O}(\hat{x})|0\rangle = \lim_{\beta\to\infty}\frac{1}{Z}\int \mathcal{D}x\mathcal{O}(x(0))\exp\left(-\frac{1}{\hbar}S_{\mathrm{E}}[x]\right)$$

$$\left\langle\hat{O}(\hat{x}(0))\hat{O}\left(\hat{x}(t_{\mathrm{E}})\right)\right\rangle = \frac{1}{Z}\mathrm{Tr}\left[\exp\left(t_{\mathrm{E}}\hat{H}/\hbar\right)\hat{O}(\hat{x}(0))\exp\left(-t_{\mathrm{E}}\hat{H}/\hbar\right)\hat{O}(\hat{x}(0))\exp\left(-\beta\hat{H}\right)\right]$$

$$= \frac{1}{Z}\sum_{n,m}\langle n|\exp\left(t_{\mathrm{E}}\hat{H}/\hbar\right)\hat{O}(\hat{x}(0))\exp\left(-t_{\mathrm{E}}\hat{H}/\hbar\right)|m\rangle\langle m|\hat{O}(\hat{x}(0))\exp\left(-\beta\hat{H}\right)|n\rangle$$

$$= \frac{1}{Z}\sum_{n,m}|\langle n|\hat{O}(\hat{x}(0))|m\rangle|^2\exp\left(-E_n\beta\right.$$

$$\left.- t_{\mathrm{E}}[E_m - E_n]/\hbar\right) \stackrel{!}{=} \frac{1}{Z}\int \mathcal{D}x\mathcal{O}\left(x(t_{\mathrm{E}})\right)\mathcal{O}(x(0))\exp\left(-\frac{1}{\hbar}S_{\mathrm{E}}[x]\right)$$

$$\lim_{\beta,t_{\mathrm{E}}\to\infty}\left\langle\hat{O}\left(x(t_{\mathrm{E}})\right)\hat{O}(x(0))\right\rangle - |\langle\hat{O}(x)\rangle|^2\Big|_{\beta\gg t_{\mathrm{E}}} = |\langle 1|\hat{O}(x)|0\rangle|^2\exp\left(-t_{\mathrm{E}}[E_1 - E_0]/\hbar\right)$$

$$\mathcal{H}[s] = -J\sum_{\langle xy\rangle}\vec{s}_x\cdot\vec{s}_y - \vec{B}\cdot\sum_{x}\vec{s}_x$$

$$Z = \prod_{x}\sum_{s_x=\pm 1}\exp\left(-\mathcal{H}[s]/T\right) = \int \mathcal{D}s\exp\left(-\mathcal{H}[s]/T\right)$$

$$\mathcal{D}s = \prod_{x}\int_{S^{N-1}} d^{N-1}s_x = \prod_{x}\int_{-1}^{1} ds_x^1 \ldots \int_{-1}^{1} ds_x^N\delta(|\vec{s}_x| - 1)$$

$$\langle\vec{s}_x\rangle = \frac{1}{Z}\int \mathcal{D}s\vec{s}_x\exp\left(-\mathcal{H}[s]/T\right)$$

$$\left\langle\vec{s}_x\cdot\vec{s}_y\right\rangle = \frac{1}{Z}\int \mathcal{D}s\vec{s}_x\cdot\vec{s}_y\exp\left(-\mathcal{H}[s]/T\right)$$

$$\langle\vec{s}_x\cdot\vec{s}_y\rangle_c=\langle\vec{s}_x\cdot\vec{s}_y\rangle-\langle\vec{s}_x\rangle\cdot\langle\vec{s}_y\rangle=\langle\vec{s}_x\cdot\vec{s}_y\rangle-\langle\vec{s}\rangle^2\sim\exp\left(-\frac{|x-y|}{\xi}\right),$$

$$M=\sum_x s_x$$

$$\langle M\rangle=-\left.\frac{\partial F}{\partial B}\right|_{T=\text{const}}$$

$$F=-T\log Z$$

$$\lim_{T\nearrow T_c}\langle M\rangle\propto(T_c-T)^{\beta}$$

$$\chi=\frac{1}{V}\left.\frac{\partial\langle M\rangle}{\partial B}\right|_{B=0}=-\frac{1}{V}\left.\frac{\partial^2F}{\partial B^2}\right|_{B=0}=\frac{1}{V}(\langle M^2\rangle-\langle M\rangle^2)\propto|T-T_c|^{-\gamma}$$

$$C=\frac{T}{V}\left.\frac{\partial S}{\partial T}\right|_{B=\text{ const.}}=-\frac{T}{V}\left.\frac{\partial^2F}{\partial T^2}\right|_{B=\text{ const.}}=\frac{1}{VT^2}(\langle\mathcal{H}^2\rangle-\langle\mathcal{H}\rangle^2)\propto|T-T_c|^{-\alpha}$$

$$\hat{\mathcal{T}}=\exp(-a\hat{H})\Rightarrow Z=\text{Tr}\exp(-\beta\hat{H})=\text{Tr}\hat{\mathcal{T}}^N,\beta=Na$$

$$Z=\prod_x\int_{S^{N-1}}d^{N-1}s_x\exp(-\mathcal{H}[s]/T)=\text{Tr}\hat{\mathcal{T}}^N$$

$$\langle s|\hat{\mathcal{T}}|s'\rangle=\exp\left(-\frac{1}{2}\mathcal{H}_{x_d}[s]/T\right)\exp\left(-\mathcal{H}_{x_d,x_d'}[s,s']/T\right)\exp\left(-\frac{1}{2}\mathcal{H}_{x_d'}[s']/T\right)$$

$$\mathcal{H}_{x_d}[s]=-J\sum_{\langle xy\rangle,x,y\in\Lambda_{x_d}}\vec{s}_x\cdot\vec{s}_y,\mathcal{H}_{x_d'}[s']=-J\sum_{\langle xy\rangle,x,y\in\Lambda_{x_d'}}\vec{s}_x'\cdot\vec{s}_y'$$

$$\mathcal{H}_{x_d,x_d'}[s,s']=-J\sum_{\langle xy\rangle,x\in\Lambda_{x_d},y\in\Lambda_{x_d'}}\vec{s}_x\cdot\vec{s}_y'$$

$$\langle s|\hat{\mathcal{T}}^2|s''\rangle=\prod_{x\in\Lambda_{x_d'}}\int_{S^{N-1}}d^{N-1}s_x'\langle s|\hat{\mathcal{T}}|s'\rangle\langle s'|\hat{\mathcal{T}}|s''\rangle$$

$$S_E[\phi]=\int d^4x\mathcal{L}_E(\phi,\partial_\mu\phi),\mathcal{L}_E(\phi,\partial_\mu\phi)=\frac{1}{2}\partial_\mu\phi(x)\partial_\mu\phi(x)+\frac{m^2}{2}\phi(x)^2$$

$$Z=\int\mathcal{D}\phi\exp(-S_E[\phi])$$

$$\hbar = c = 1,$$

$$S_{\mathrm{E}}[\phi] = \frac{1}{2} a^d \sum_x \left[\sum_{\mu=1}^{d} \left(\frac{\phi_{x+\hat{\mu}} - \phi_x}{a} \right)^2 + m^2 \phi_x^2 \right]$$

$$Z = \prod_x \sqrt{\frac{a^{d-2}}{2\pi}} \int_{-\infty}^{\infty} d\phi_x \exp\left(-S_{\mathrm{E}}[\phi]\right)$$

$$S_{\mathrm{E}}[\phi] = \frac{1}{2} a^d \sum_{x,y} \phi_x \Delta_{xy} \phi_y$$

$$\Delta_{xy} = \frac{1}{a^2} \left[\sum_{\mu} \left(2\delta_{xy} - \delta_{x,y-\hat{\mu}} - \delta_{x,y+\hat{\mu}} \right) + (ma)^2 \delta_{xy} \right],$$

$$\Delta = \frac{1}{a^2} \begin{pmatrix} 2+(ma)^2 & -1 & 0 & \cdot\ \cdot & 0 & -1 \\ -1 & 2+(ma)^2 & -1 & \cdot\ \cdot & 0 & 0 \\ 0 & -1 & 2+(ma)^2 & \cdot\ \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot\ \cdot & 2+(ma)^2 & -1 \\ -1 & 0 & 0 & \cdot\ \cdot & -1 & 2+(ma)^2 \end{pmatrix}$$

$$\Delta = \Omega^{\top} D \Omega, D = \mathrm{diag}(D_1, D_2, \ldots, D_N)$$

$$\phi'_x = \Omega_{xy} \phi_y$$

$$Z = \prod_x \sqrt{\frac{a^{d-2}}{2\pi}} \int_{-\infty}^{\infty} d\phi'_x \exp\left(-\frac{a^d}{2} \sum_x \phi'_x D_{xx} \phi'_x \right) = \prod_x \left(a^2 D_{xx}\right)^{-1/2} = \frac{1}{\sqrt{\det(a^2 D)}}$$

$$= \frac{1}{\sqrt{\det(a^2 \Delta)}}$$

$$\langle \phi_x \phi_y \rangle = \frac{1}{Z} \int \mathcal{D}\phi \phi_x \phi_y \exp\left(-S_{\mathrm{E}}[\phi]\right)$$

$$Z[j] = \int \mathcal{D}\phi \exp\left(-S_{\mathrm{E}}[\phi] + a^d j^{\top} \phi\right)$$

$$\langle \phi_x \phi_y \rangle = \frac{1}{Z} \frac{\delta^2}{\delta j_x \delta j_y} Z[j] \bigg|_{j=0}$$

$$\langle \phi_x \phi_y \rangle_{\mathrm{c}} = \langle \phi_x \phi_y \rangle - \langle \phi_x \rangle \langle \phi_y \rangle = \frac{\delta^2}{\delta j_x \delta j_y} \log Z[j] \bigg|_{j=0}$$

$$\frac{\delta}{\delta j_x} = \frac{1}{a^d} \frac{\partial}{\partial j_x}.$$

$$\phi' = \phi - \Delta^{-1} j,$$

$$-\frac{1}{2}\phi^{\top}\Delta\phi + j^{\top}\phi = -\frac{1}{2}\phi'^{\top}\Delta\phi' + \frac{1}{2}j^{\top}\Delta^{-1}j,$$

$$Z[j] = \frac{1}{\sqrt{\det(a^2\Delta)}}\exp\left(\frac{a^d}{2}j^{\top}\Delta^{-1}j\right)$$

$$\langle\phi_x\phi_y\rangle_{\mathrm{c}} = \frac{\delta}{\delta j_x}\frac{\delta}{\delta j_y}\frac{a^d}{2}j_u(\Delta^{-1})_{uv}j_v = \frac{1}{2}\frac{\delta}{\delta j_x}\left[\delta_{yu}(\Delta^{-1})_{uv}j_v + j_u(\Delta^{-1})_{uv}\delta_{vy}\right]$$
$$= \frac{a^{-d}}{2}\left[(\Delta^{-1})_{yv}\delta_{vx} + \delta_{xu}(\Delta^{-1})_{uy}\right] = a^{-d}(\Delta^{-1})_{xy}$$

$$\phi(p) = a^d\sum_x \phi_x \exp(-\mathrm{i}px), \phi_x = \frac{1}{(2\pi)^d}\int_B d^d p\phi(p)\exp(\mathrm{i}px)$$

$$B = \left]-\frac{\pi}{a},\frac{\pi}{a}\right]^d.$$

$$S_{\mathrm{E}}[\phi] = \frac{1}{2}\frac{1}{(2\pi)^d}\int_B d^d p\phi(-p)\Delta(p)\phi(p)$$
$$\Delta(p) = \sum_{\mu=1}^{d}\left(\frac{2}{a}\sin\left(\frac{p_\mu a}{2}\right)\right)^2 + m^2$$

$$\langle\phi(k)\phi(p)\rangle_{\mathrm{c}} = \frac{\delta^2}{\delta j(k)\delta j(p)}\frac{1}{2}\frac{1}{(2\pi)^d}\int_{B^2} d^d q d^d s j(q)\Delta(s)^{-1}j(s)\delta(q+s)$$

$$= \frac{\delta}{\delta j(k)}\frac{1}{2}\left[\int_B d^d s\Delta(s)^{-1}j(s)\delta(p+s) + \int_B d^d q j(q)\Delta(p)^{-1}\delta(p+q)\right]$$

$$= \Delta(p)^{-1}(2\pi)^d\delta(k+p) \Rightarrow \frac{1}{(2\pi)^d}\int d^d k\langle\phi(k)\phi(p)\rangle_{\mathrm{c}}$$

$$= \left[\sum_\mu\left(\frac{2}{a}\sin(p_\mu a/2)\right)^2 + m^2\right]^{-1} \underset{p^2+m^2}{\overset{a\to 0}{r}}$$

$$\langle\phi(x)\phi(y)\rangle_{\mathrm{c}} = \frac{1}{(2\pi)^d}\int d^d p\frac{\exp(\mathrm{i}p(x-y))}{p^2+m^2}$$

$$(-\partial^2 + m^2)\langle\phi(0)\phi(x)\rangle_{\mathrm{c}} = \delta(x).$$

$$\phi(\vec{p})_{x_d} = a^{d-1}\sum_{\vec{x}}\phi_{\vec{x},x_d}\exp(-\mathrm{i}\vec{p}\cdot\vec{x}) = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dp_d\phi(p)\exp(\mathrm{i}p_d x_d)$$

$$\langle\phi(-\vec{p})_0\phi(\vec{p})_{x_d}\rangle_{\mathrm{c}} = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dp_d\langle\phi(-p)\phi(p)\rangle\exp(\mathrm{i}p_d x_d)$$

$$\left[\frac{2}{a}\sinh\left(\frac{Ea}{2}\right)\right]^2 = \sum_{i=1}^{d-1}\left(\frac{2}{a}\sin\left(\frac{p_i a}{2}\right)\right)^2 + m^2$$

$$\left\langle \phi(-\vec{p})_0 \phi(\vec{p})_{x_d} \right\rangle_{\mathrm{c}} \propto \exp\left(-E x_d\right)$$

$$E = \sqrt{\vec{p}^2 + m^2}.$$

$$\left\langle \phi_x\phi_y\phi_z\phi_w \right\rangle = \frac{1}{Z[j]}\frac{\delta^4}{\delta j_x \delta j_y \delta j_z \delta j_w} Z[j]\bigg|_{j=0} = \frac{\delta}{\delta j_x}\frac{\delta}{\delta j_y}\frac{\delta}{\delta j_z}\frac{\delta}{\delta j_w}\exp\left(\frac{a^d}{2} j^{\mathrm{T}}\Delta^{-1} j\right)\bigg|_{j=0}$$

$$= a^{-2d}\left[(\Delta^{-1})_{xy}(\Delta^{-1})_{zw} + (\Delta^{-1})_{xz}(\Delta^{-1})_{yw} + (\Delta^{-1})_{xw}(\Delta^{-1})_{yz}\right]$$

$$\frac{\exp\left(\mathrm{i}p(y-x)\right)}{p^2+m^2} \to \exp\left(\mathrm{i}p(y-x)\right)\left[\frac{1}{p^2+m^2} - \frac{1}{p^2+M_{\mathrm{PV}}^2}\right]$$

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi^2(x) - \frac{\lambda}{4!}\phi^4(x)$$

$$\exp\left(\varepsilon\hat{A}\right)\exp\left(\varepsilon\hat{B}\right) = \exp\left(\varepsilon\hat{X} + \varepsilon^2\hat{Y} + \varepsilon^3\hat{Z} + \mathcal{O}(\varepsilon^4)\right)$$

$$\int_{-\infty}^{\infty}\exp\left(-\mathrm{i}\alpha x^2\right)dx, \alpha > 0$$

$$\sum_{n\in\mathbb{Z}}\delta(x-n) = \sum_{n\in\mathbb{Z}}\exp\left(2\pi\mathrm{i}nx\right)$$

$$\mathcal{H}[s] = -J\sum_{\langle xy\rangle} s_x s_y.$$

$$Z = \int \mathcal{D}s\exp\left(-\mathcal{H}[s]/T\right)$$

$$\langle s|\hat{\mathcal{T}}|s'\rangle = \exp\left(Jss'/T\right)$$

$$\mathcal{H}[s] = -J\sum_{\langle xy\rangle}\vec{s}_x\cdot\vec{s}_y$$

$$\langle s|\hat{\mathcal{T}}|s'\rangle = \exp\left(J\vec{s}\cdot\vec{s}'/T\right)$$

$$\begin{aligned}\langle\phi_x\rangle_{\mathrm{c}} &= \langle\phi_x\rangle\\ \left\langle\phi_x\phi_y\right\rangle_{\mathrm{c}} &= \left\langle\phi_x\phi_y\right\rangle - \langle\phi_x\rangle\left\langle\phi_y\right\rangle\\ \left\langle\phi_x\phi_y\phi_z\right\rangle_{\mathrm{c}} &= \left\langle\phi_x\phi_y\phi_z\right\rangle - \left\langle\phi_x\phi_y\right\rangle_{\mathrm{c}}\langle\phi_z\rangle - \langle\phi_x\phi_z\rangle_{\mathrm{c}}\left\langle\phi_y\right\rangle - \left\langle\phi_y\phi_z\right\rangle_{\mathrm{c}}\langle\phi_x\rangle - \langle\phi_x\rangle\left\langle\phi_y\right\rangle\langle\phi_z\rangle\end{aligned}$$

$$Z[j] = \int \mathcal{D}\phi\exp\left(-S[\phi] + a^d j^{\mathrm{T}}\phi\right)$$

$$\left\langle\phi_x\phi_y\phi_z\right\rangle_{\mathrm{c}} = \frac{\delta}{\delta j_x}\frac{\delta}{\delta j_y}\frac{\delta}{\delta j_z}\log Z[j]\bigg|_{j=0}$$

$$Z[j]=\int D\phi\exp\left(-\frac{a^d}{2}\phi^\top\Delta\phi+a^d j^\top\phi\right)=\int D\phi'\exp\left(-\frac{a^d}{2}\phi^\top\Delta\phi'+\frac{a^d}{2}j^\top\Delta^{-1}j\right)$$

($\phi'=\phi-\Delta^{-1}j$)

$$\langle\phi_{x_1}\phi_{x_2}\dots\phi_{x_n}\rangle=\frac{1}{Z[j]}\frac{\delta^n}{\delta j_{x_1}\delta j_{x_2}\dots\delta j_{x_n}}Z[j]\bigg|_{j=0}$$

$$\phi'(x')=\phi(x)=\phi(\Lambda^{-1}x'),\Lambda_\mu^\rho\Lambda_\nu^\sigma g_{\rho\sigma}=g_{\mu\nu}$$

$$\vec{\phi}(x)=(\phi_1(x),\phi_2(x),\dots,\phi_N(x))\in\mathbb{R}^N$$

$$\mathcal{L}(\vec{\phi},\partial_\mu\vec{\phi})=\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}-V(\vec{\phi})$$

$$V(\vec{\phi})=\frac{m^2}{2}\vec{\phi}\cdot\vec{\phi}+\frac{\lambda}{4!}(\vec{\phi}\cdot\vec{\phi})^2$$

$$\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}-\frac{\delta\mathcal{L}}{\delta\phi_i}=\partial_\mu\partial^\mu\phi_i+\frac{\partial V(\vec{\phi})}{\partial\phi_i}=\left[\partial_\mu\partial^\mu+m^2+\frac{\lambda}{3!}\phi^2\right]\phi_i=0,i\in\{1,2,\dots,N\}.$$

$$\partial_\mu\partial^\mu\phi_i+m^2\phi_i=0$$

$$\vec{\phi}(x)=\vec{\phi}_0\cos(\vec{k}\cdot\vec{x}-\omega t+\varphi)$$

$$\omega^2=\vec{k}^2+m^2$$

$$E^2=\vec{p}^2c^2+(mc^2)^2$$

$$\vec{\phi}'(x)=\Omega\vec{\phi}(x),\Omega\in O(N)$$

$$\vec{\phi}'(x)=\Omega(x)\vec{\phi}(x)$$

$$\left|\vec{\phi}'(x)\right|^2=\vec{\phi}(x)\Omega(x)^\top\cdot\Omega(x)\vec{\phi}(x)=|\vec{\phi}(x)|^2$$

$$\Omega(x)=\exp(\mathrm{i}\omega(x))\approx\mathbb{1}+\mathrm{i}\omega(x)$$

$$\mathbb{1}=\Omega(x)^\top\Omega(x)\approx[\mathbb{1}+\mathrm{i}\omega(x)^\top][\mathbb{1}+\mathrm{i}\omega(x)]\approx\mathbb{1}+\mathrm{i}[\omega(x)+\omega(x)^\top]\Rightarrow$$
$$\omega(x)^\top=-\omega(x)$$

$$\vec{\phi}'(x)\approx[\mathbb{1}+\mathrm{i}\omega(x)]\vec{\phi}(x)$$

$$\partial^\mu\vec{\phi}'(x)\approx\partial^\mu\vec{\phi}(x)+\mathrm{i}\partial^\mu\omega(x)\vec{\phi}(x)+\mathrm{i}\omega(x)\partial^\mu\vec{\phi}(x)$$

$$\partial_\mu\vec{\phi}'\cdot\partial^\mu\vec{\phi}'\approx\left(\partial_\mu\vec{\phi}-\mathrm{i}\vec{\phi}\partial_\mu\omega-\mathrm{i}\partial_\mu\vec{\phi}\omega\right)\cdot\left(\partial^\mu\vec{\phi}+\mathrm{i}\partial^\mu\omega\vec{\phi}+\mathrm{i}\omega\partial^\mu\vec{\phi}(x)\right)$$
$$=\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}+\mathrm{i}\partial_\mu\vec{\phi}\cdot\partial^\mu\omega\vec{\phi}-\mathrm{i}\vec{\phi}\partial_\mu\omega\cdot\partial^\mu\vec{\phi}$$

$$S[\phi'] - S[\phi] = \int d^4x[\mathcal{L}(\phi', \partial_\mu\phi') - \mathcal{L}(\phi, \partial_\mu\phi)] = -\mathrm{i}\int d^4x\partial_\mu\omega_{ij}j^\mu_{ij}$$
$$= \mathrm{i}\int d^4x\omega_{ij}\partial_\mu j^\mu_{ij}$$

$$j^\mu_{ij}(x) = \phi_i(x)\partial^\mu\phi_j(x) - \phi_j(x)\partial^\mu\phi_i(x)$$

$$\partial_\mu j^\mu_{ij} = \partial_\mu[\phi_i\partial^\mu\phi_j - \phi_j\partial^\mu\phi_i] = \phi_i\partial_\mu\partial^\mu\phi_j - \phi_j\partial_\mu\partial^\mu\phi_i$$
$$= -\phi_i\frac{\partial V(\vec{\phi})}{\partial\phi_j} + \phi_j\frac{\partial V(\vec{\phi})}{\partial\phi_i}$$
$$= \phi_i\left[m^2\phi_j + \frac{\lambda}{3!}\phi^2\phi_j\right] - \phi_j\left[m^2\phi_i + \frac{\lambda}{3!}\phi^2\phi_i\right] = 0$$

$$\mathcal{L}(\phi, \partial_\mu\phi) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2$$

$$\Pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\phi(x)} = \partial^0\phi(x)$$

$$\mathcal{H}(\phi, \Pi) = \Pi(x)\partial^0\phi(x) - \mathcal{L} = \frac{1}{2}\Pi(x)^2 + \frac{1}{2}\partial_i\phi(x)\partial_i\phi(x) + \frac{m^2}{2}\phi(x)^2$$

$$H[\phi, \Pi] = \int d^3x\mathcal{H}(\phi, \Pi) = \int d^3x\frac{1}{2}[\Pi(\vec{x})^2 + \partial_i\phi(\vec{x})\partial_i\phi(\vec{x}) + m^2\phi(\vec{x})^2]$$

$$[\hat{x}_i, \hat{p}_j] = \mathrm{i}\delta_{ij}, [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$$

$$[\hat{\phi}(\vec{x}), \hat{\Pi}(\vec{y})] = \mathrm{i}\delta(\vec{x} - \vec{y}), [\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\Pi}(\vec{x}), \hat{\Pi}(\vec{y})] = 0.$$

$$\hat{p}_i = -\mathrm{i}\frac{\partial}{\partial x_i}$$

$$\hat{\Pi}(\vec{x}) = -\mathrm{i}\frac{\delta}{\delta\phi(\vec{x})},$$

$$\hat{H} = \int d^3x\frac{1}{2}[\hat{\Pi}(\vec{x})^2 + \partial_i\hat{\phi}(\vec{x})\partial_i\hat{\phi}(\vec{x}) + m^2\hat{\phi}(\vec{x})^2]$$

$$\hat{\phi}(\vec{p}) = \int d^3x\hat{\phi}(\vec{x})\exp(-\mathrm{i}\vec{p}\cdot\vec{x}), \hat{\Pi}(\vec{p}) = \int d^3x\hat{\Pi}(\vec{x})\exp(-\mathrm{i}\vec{p}\cdot\vec{x})$$

$$\hat{\phi}(\vec{p})^\dagger = \hat{\phi}(-\vec{p}), \hat{\Pi}(\vec{p})^\dagger = \hat{\Pi}(-\vec{p}).$$

$$[\hat{\phi}(\vec{p}), \hat{\Pi}(\vec{q})] = \mathrm{i}(2\pi)^3\delta(\vec{p} + \vec{q}), [\hat{\phi}(\vec{p}), \hat{\phi}(\vec{q})] = [\hat{\Pi}(\vec{p}), \hat{\Pi}(\vec{q})] = 0$$

$$\hat{H} = \frac{1}{(2\pi)^3}\int d^3p\frac{1}{2}[\hat{\Pi}(\vec{p})^\dagger\hat{\Pi}(\vec{p}) + (\vec{p}^2 + m^2)\hat{\phi}(\vec{p})^\dagger\hat{\phi}(\vec{p})]$$

$$\hat{a}(\vec{p}) = \frac{1}{\sqrt{2}}\left[\sqrt{\omega}\hat{\phi}(\vec{p}) + \frac{\mathrm{i}}{\sqrt{\omega}}\hat{\Pi}(\vec{p})\right], \hat{a}(\vec{p})^{\dagger} = \frac{1}{\sqrt{2}}\left[\sqrt{\omega}\hat{\phi}(-\vec{p}) - \frac{\mathrm{i}}{\sqrt{\omega}}\hat{\Pi}(-\vec{p})\right],$$

$$\left[\hat{a}(\vec{p}), \hat{a}(\vec{q})^{\dagger}\right] = \frac{\mathrm{i}}{2}[\hat{\Pi}(\vec{p}), \hat{\phi}(-\vec{q})] - \frac{\mathrm{i}}{2}[\hat{\phi}(\vec{p}), \hat{\Pi}(-\vec{q})] = (2\pi)^3\delta(\vec{p} - \vec{q}),$$
$$[\hat{a}(\vec{p}), \hat{a}(\vec{q})] = \left[\hat{a}(\vec{p})^{\dagger}, \hat{a}(\vec{q})^{\dagger}\right] = 0.$$

$$\hat{H} = \frac{1}{(2\pi)^3}\int d^3p\sqrt{\vec{p}^2 + m^2}\left(\hat{a}(\vec{p})^{\dagger}\hat{a}(\vec{p}) + \frac{V}{2}\right)$$

$$\delta(\vec{p}) = \frac{1}{(2\pi)^3}\int d^3x\exp(-\mathrm{i}\vec{p}\cdot\vec{x}) \Rightarrow (2\pi)^3\delta(\vec{0}) = \int d^3x 1 = V$$

$$\hat{a}(\vec{p})|0\rangle = 0,$$

$$E_0 = \frac{1}{(2\pi)^3}\frac{V}{2}\int d^3p\sqrt{\vec{p}^2 + m^2}$$

$$\rho = \frac{E_0}{V} = \frac{1}{(2\pi)^3}\frac{1}{2}\int d^3p\sqrt{\vec{p}^2 + m^2} = \frac{1}{4\pi^2}\int_0^{\infty} d|\vec{p}|\vec{p}^2\sqrt{\vec{p}^2 + m^2}$$

$$\rho = \frac{1}{4\pi^2}\int_0^{\Lambda} d|\vec{p}|\vec{p}^2\sqrt{\vec{p}^2 + m^2} \sim \mathcal{O}(\Lambda^4)$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda_c g_{\mu\nu}$$

$$M_{\text{Planck}} = \frac{1}{\sqrt{G}} \approx 1.2 \times 10^{19}\text{GeV}$$

$$\rho_{\text{naive}} \approx M_{\text{Planck}}^4.$$

$$\rho_{\text{observation}} \approx (2 \times 10^{-3}\text{eV})^4$$

$$\frac{\rho_{\text{naive}}}{\rho_{\text{observation}}} \approx \mathcal{O}(10^{120}).$$

$$|\vec{p}\rangle = \hat{a}(\vec{p})^{\dagger}|0\rangle,$$

$$E(\vec{p}) - E_0 = \omega = \sqrt{\vec{p}^2 + m^2}.$$

$$|\vec{p}_1; \vec{p}_2\rangle = \hat{a}(\vec{p}_1)^{\dagger}\hat{a}(\vec{p}_2)^{\dagger}|0\rangle.$$

$$|\vec{p}_2; \vec{p}_1\rangle = |\vec{p}_1; \vec{p}_2\rangle,$$

$$\mathcal{T}_{\mu\nu}(x) = \partial_\mu\phi(x)\partial_\nu\phi(x) - g_{\mu\nu}\mathcal{L}.$$

$$\partial^\mu\mathcal{T}_{\mu\nu}(x) = 0$$

$$\mathcal{H}(x) = \mathcal{T}_{00}(x) = \partial_0\phi(x)\partial_0\phi(x) - g_{00}\mathcal{L} = \Pi(x)^2 - \mathcal{L}$$

$$\mathcal{P}_i(x) = \mathcal{T}_{0i}(x) = \partial_0\phi(x)\partial_i\phi(x) - g_{0i}\mathcal{L} = \Pi(x)\partial_i\phi(x)$$

$$\hat{P}_i = \int d^3x \frac{1}{2}[\hat{\Pi}(\vec{x})\partial_i\hat{\phi}(\vec{x}) + \partial_i\hat{\phi}(\vec{x})\hat{\Pi}(\vec{x})]$$
$$= \frac{1}{(2\pi)^3}\int d^3p \frac{\mathrm{i}p_i}{2}[\hat{\Pi}(\vec{p})\hat{\phi}(-\vec{p}) + \hat{\phi}(-\vec{p})\hat{\Pi}(\vec{p})]$$
$$= \frac{1}{(2\pi)^3}\int d^3p p_i \hat{a}(\vec{p})^\dagger \hat{a}(\vec{p})$$

$$\hat{P}_i|\vec{p}\rangle = p_i|\vec{p}\rangle$$

$$V(\Phi) = \frac{m^2}{2}|\Phi|^2 + \frac{\lambda}{4!}|\Phi|^4$$

$$\hat{H} = \sum_{n=1}^{N}\left[\frac{\hat{p}_n^2}{2M} + \frac{M\omega_0^2}{2}(\hat{x}_{n+1} - \hat{x}_n - a)^2\right]$$

$$\hat{x}_{N+1} = \hat{x}_1 + Na = \hat{x}_1 + L$$

$$\hat{y}_n = \hat{x}_n - na$$

$$\hat{H} = \sum_{n=1}^{N}\left[\frac{\hat{p}_n^2}{2M} + \frac{M\omega_0^2}{2}(\hat{y}_{n+1} - \hat{y}_n)^2\right]$$

$$k = \frac{2\pi l}{L} = \frac{2\pi l}{Na} \in B = \left]-\frac{\pi}{a}, \frac{\pi}{a}\right]$$

$$\hat{y}(k) = \sum_{n=1}^{N} \hat{y}_n \exp(-\mathrm{i}kna).$$

$$\hat{y}_n = \frac{1}{N}\sum_{k\in B} \hat{y}(k)\exp(\mathrm{i}kna)$$

$$\hat{P} = \sum_{n=1}^{N} \hat{p}_n$$

$$\hat{V} = \frac{M\omega_0^2}{2}\sum_{n=1}^{N}(\hat{y}_{n+1} - \hat{y}_n)^2 = \frac{M\omega_0^2}{2}\frac{1}{N^2}\sum_{k,k'\in B}\hat{y}(k')^\dagger\hat{y}(k)$$
$$\times \sum_{n=1}^{N}[\exp(-\mathrm{i}k'(n+1)a) - \exp(-\mathrm{i}k'na)][\exp(\mathrm{i}k(n+1)a) - \exp(\mathrm{i}kna)]$$
$$= \frac{1}{N}\frac{M}{2}\sum_{k\in B}\omega(k)^2\hat{y}(k)^\dagger\hat{y}(k)$$

$$\omega(k) = 2\omega_0\left|\sin\frac{ka}{2}\right|$$

$$\frac{1}{N}\sum_{n=1}^{N} \exp\left(\mathrm{i}(k-k')na\right) = \delta_{kk'},$$

$$2-\exp(\mathrm{i}ka)-\exp(-\mathrm{i}ka) = 2(1-\cos(ka)) = \left(2\sin\frac{ka}{2}\right)^2,$$

$$\hat{p}(k) = \sum_{n=1}^{N} \hat{p}_n \exp(-\mathrm{i}kna), \hat{p}(k)^\dagger = \hat{p}(-k),$$

$$\hat{p}_n = \frac{1}{N}\sum_{k\in B} \hat{p}(k)\exp(\mathrm{i}kna)$$

$$\hat{T} = \sum_{n=1}^{N} \frac{\hat{p}_n^2}{2M} = \frac{1}{N^2}\sum_{k,k'\in B} \frac{\hat{p}(k')\hat{p}(k)}{2M}\sum_{n=1}^{N} \exp\left(\mathrm{i}(k'+k)na\right)$$
$$= \frac{1}{N}\sum_{k\in B} \frac{\hat{p}(k)^\dagger \hat{p}(k)}{2M}$$

$$\hat{H} = \hat{T}+\hat{V} = \frac{1}{N}\sum_{k\in B}\left[\frac{\hat{p}(k)^\dagger\hat{p}(k)}{2M} + \frac{M\omega(k)^2}{2}\hat{y}(k)^\dagger\hat{y}(k)\right]$$
$$= \frac{\hat{P}^2}{2NM} + \frac{1}{N}\sum_{k\in B, k\neq 0}\left[\frac{\hat{p}(k)^\dagger\hat{p}(k)}{2M} + \frac{M\omega(k)^2}{2}\hat{y}(k)^\dagger\hat{y}(k)\right]$$

$$\hat{P} = \sum_{n=1}^{N} \hat{p}_n = \hat{p}(k=0)$$

$$\hat{a}(k) = \frac{1}{\sqrt{2N}}\left(\alpha(k)\hat{y}(k) + \frac{\mathrm{i}\hat{p}(k)}{\alpha(k)\hbar}\right)$$
$$\hat{a}(k)^\dagger = \frac{1}{\sqrt{2N}}\left(\alpha(k)\hat{y}(k)^\dagger - \frac{\mathrm{i}\hat{p}(k)^\dagger}{\alpha(k)\hbar}\right)$$

$$\alpha(k) = \sqrt{\frac{M\omega(k)}{\hbar}}$$

$$\left[\hat{a}(k), \hat{a}(k')^\dagger\right] = -\frac{\mathrm{i}\alpha(k)}{2N\alpha(k')\hbar}[\hat{y}(k), \hat{p}(-k')] + \frac{\mathrm{i}\alpha(k')}{2N\alpha(k)\hbar}[\hat{p}(k), \hat{y}(-k')]$$

$$[\hat{y}_n, \hat{y}_{n'}] = 0, [\hat{p}_n, \hat{p}_{n'}] = 0, [\hat{y}_n, \hat{p}_{n'}] = \mathrm{i}\hbar\delta_{nn'}$$

$$[\hat{y}(k), \hat{p}(-k')] = \sum_{n,n'=1}^{N} [\hat{y}_n, \hat{p}_{n'}]\exp(-\mathrm{i}kna + \mathrm{i}k'n'a)$$
$$= \mathrm{i}\hbar\sum_{n=1}^{N} \exp\left(\mathrm{i}(k'-k)na\right) = \mathrm{i}\hbar N\delta_{kk'}$$

$$\left[\hat{a}(k),\hat{a}(k')^{\dagger}\right]=\delta_{kk'}.$$

$$[\hat{a}(k),\hat{a}(k')]=0,\left[\hat{a}(k)^{\dagger},\hat{a}(k')^{\dagger}\right]=0$$

$$\begin{aligned}\hat{a}(k)^{\dagger}\hat{a}(k)&=\frac{1}{2N}\left(\alpha(k)\hat{y}(k)^{\dagger}-\frac{\mathrm{i}\hat{p}(k)^{\dagger}}{\alpha(k)\hbar}\right)\left(\alpha(k)\hat{y}(k)+\frac{\mathrm{i}\hat{p}(k)}{\alpha(k)\hbar}\right)\\&=\frac{1}{2N}\left(\alpha(k)^{2}\hat{y}(k)^{\dagger}\hat{y}(k)+\frac{\hat{p}(k)^{\dagger}\hat{p}(k)}{\alpha(k)^{2}\hbar^{2}}\right)+\frac{\mathrm{i}}{2N\hbar}[\hat{y}(-k)\hat{p}(k)-\hat{p}(-k)\hat{y}(k)]\end{aligned}$$

$$\begin{aligned}\hat{H}&=\frac{\hat{P}^{2}}{2NM}+\sum_{k\in B,k\neq 0}\hbar\omega(k)\left(\hat{a}(k)^{\dagger}\hat{a}(k)-\frac{\mathrm{i}}{2N\hbar}[\hat{y}(k),\hat{p}(-k)]\right)\\&=\frac{\hat{P}^{2}}{2NM}+\sum_{k\in B,k\neq 0}\hbar\omega(k)\left(\hat{v}(k)+\frac{1}{2}\right).\end{aligned}$$

$$\hat{v}(k)=\hat{a}(k)^{\dagger}\hat{a}(k),$$

$$\hat{P}|0\rangle=0,\hat{a}(k)|0\rangle=0$$

$$E_{0}=\langle 0|\hat{H}|0\rangle=\sum_{k\in B}\frac{1}{2}\hbar\omega(k)$$

$$\rho_{0}=\frac{E_{0}}{L}=\frac{1}{2\pi}\int_{B}dk\frac{\hbar\omega(k)}{2}=\frac{\hbar}{\pi}\int_{0}^{\pi/a}dk\omega_{0}\sin\frac{ka}{2}=\frac{2\hbar\omega_{0}}{\pi a}$$

$$|k\rangle=\hat{a}(k)^{\dagger}|0\rangle$$

$$E(k)=\hbar\omega(k)=2\hbar\omega_{0}\left|\sin\frac{ka}{2}\right|$$

$$c=\omega_{0}a$$

$$c(k)=\left|\frac{dE}{dp}\right|=\omega_{0}a\cos\frac{ka}{2}$$

$$|k_{1};k_{2}\rangle=\hat{a}(k_{1})^{\dagger}\hat{a}(k_{2})^{\dagger}|0\rangle.$$

$$|k_{1};k_{2}\rangle=|k_{2};k_{1}\rangle$$

$$|n\rangle=\frac{a}{2\pi}\int_{B}dk|k\rangle\exp{(\mathrm{i}kna)}$$

$$\hat{a}_{n}^{\dagger}=\frac{a}{2\pi}\int_{B}dk\hat{a}(k)^{\dagger}\exp{(\mathrm{i}kna)}$$

$$|n\rangle=\hat{a}_{n}^{\dagger}|0\rangle$$

$$\hat{a}_{n}=\frac{a}{2\pi}\int_{B}dk\hat{a}(k)\exp{(-\mathrm{i}kna)}$$

$$[\hat{a}_n, \hat{a}_{n'}^\dagger] = \left(\frac{a}{2\pi}\right)^2 \int_B dk \int_B dk' [\hat{a}(k), \hat{a}(k')^\dagger] \exp\left(\mathrm{i}(k'n' - kn)a\right)$$
$$= \frac{a}{2\pi} \int_B dk \exp\left(\mathrm{i}k(n' - n)a\right) = \delta_{nn'}$$

$$[\hat{a}_n, \hat{a}_{n'}] = 0, [\hat{a}_n^\dagger, \hat{a}_{n'}^\dagger] = 0$$

$$\hat{a}_n^\dagger = \frac{a}{2\pi} \int_B dk \frac{1}{\sqrt{2N}} \left(\alpha(k) \hat{y}(k)^\dagger - \frac{\mathrm{i}\hat{p}(k)^\dagger}{\alpha(k)\hbar}\right) \exp\left(\mathrm{i}kna\right)$$

$$\hat{a}_n^\dagger = \frac{1}{\sqrt{N}} \sum_{n' \in \mathbb{Z}} \left(f_{n-n'} \hat{y}_{n'} - g_{n-n'} \frac{\mathrm{i}}{\hbar} \hat{p}_{n'}\right)$$

$$f_n = f_{-n} = \frac{a}{2\pi} \int_B dk \frac{\alpha(k)}{\sqrt{2}} \exp\left(\mathrm{i}kna\right) = \frac{a}{\pi} \sqrt{\frac{M\omega_0}{2\hbar}} \int_0^{\pi/a} dk \sqrt{\sin(ka/2)} \cos(kna)$$

$$g_n = g_{-n} = \frac{a}{2\pi} \int_B dk \frac{1}{\sqrt{2}\alpha(k)} \exp\left(\mathrm{i}kna\right) = \frac{a}{\pi} \sqrt{\frac{\hbar}{2M\omega_0}} \int_0^{\pi/a} dk \frac{\cos(kna)}{\sqrt{\sin(ka/2)}}$$

$$f_n \sim \frac{1}{\sqrt{|n|}^3}, g_n \sim \frac{1}{\sqrt{|n|}},$$

$$\hat{V}' = \sum_{n=1}^{N} \frac{M\omega_0'^2}{2} (\hat{x}_n - na)^2 = \sum_{n=1}^{N} \frac{M\omega_0'^2}{2} \hat{y}_n^2 = \frac{1}{N} \sum_{k \in B} \frac{M\omega_0'^2}{2} \hat{y}(k)^\dagger \hat{y}(k)$$

$$\hat{H} = \hat{T} + \hat{V} + \hat{V}' = \frac{1}{N} \sum_{k \in B} \left[\frac{\hat{p}(k)^\dagger \hat{p}(k)}{2M} + \frac{M}{2} \left(\left(2\omega_0 \sin\frac{ka}{2}\right)^2 + \omega_0'^2\right) \hat{y}(k)^\dagger \hat{y}(k)\right]$$
$$= \sum_{k \in B} \hbar\omega(k) \left(\hat{v}(k) + \frac{1}{2}\right)$$

$$\omega(k) = \sqrt{\left(2\omega_0 \sin\frac{ka}{2}\right)^2 + \omega_0'^2}$$

$$E(k) = \hbar\omega(k) = \sqrt{(\hbar kc)^2 + (\hbar\omega_0')^2} = \sqrt{(p_{\mathrm{dB}} c)^2 + (mc^2)^2},$$

$$m = \frac{\hbar\omega_0'}{c^2} = \frac{\hbar\omega_0'}{\omega_0^2 a^2}$$

$$f_n, g_n \sim \exp\left(-\frac{mc|n|a}{\hbar}\right) = \exp\left(-\frac{|x|}{\lambda_\phi}\right).$$

$$\lambda_\phi = \frac{\hbar}{mc}$$

$$\hat{\phi}(x=na)=\sqrt{M\omega_0^2 a}\hat{y}_n$$

$$\hat{\Pi}(x=na)=\frac{\hat{p}_n}{\sqrt{Mc^2 a}}=-\frac{\mathrm{i}\hbar}{a\sqrt{M\omega_0^2 a}}\frac{\partial}{\partial y_n}=-\frac{\mathrm{i}\hbar}{a}\frac{\delta}{\delta\phi(x)}$$

$$\left[\hat{\phi}(x),\hat{\Pi}(x')\right]=\frac{\mathrm{i}\hbar}{a}\delta_{nn'}\longrightarrow \mathrm{i}\hbar\delta(x-x')$$

$$\hat{H}=\sum_{n=1}^{N}\left[\frac{\hat{p}_n^2}{2M}+\frac{M\omega_0^2}{2}(\hat{y}_{n+1}-\hat{y}_n)^2+\frac{M\omega_0'^2}{2}\hat{y}_n^2\right]$$

$$=a\sum_x\frac{1}{2}\left[c^2\hat{\Pi}(x)^2+\left(\frac{\hat{\phi}(x+a)-\hat{\phi}(x)}{a}\right)^2+\left(\frac{mc}{\hbar}\right)^2\hat{\phi}(x)^2\right]$$

$$\rightarrow\int_0^L dx\frac{1}{2}\left[c^2\hat{\Pi}(x)^2+\partial_x\hat{\phi}(x)\partial_x\hat{\phi}(x)+\left(\frac{mc}{\hbar}\right)^2\hat{\phi}(x)^2\right]$$

$$\mathcal{L}(\phi,\partial_\mu\phi)=\frac{1}{2}\left[\frac{1}{c^2}\partial_t\phi(x)\partial_t\phi(x)-\partial_x\phi(x)\partial_x\phi(x)-\left(\frac{mc}{\hbar}\right)^2\phi(x)^2\right]$$

$$\lambda_e=\frac{\hbar}{m_e c}\simeq 3.9\times 10^{-13}\text{ m},$$

$$r_{\mathrm{B}}=\frac{4\pi\hbar^2}{e^2 m_e}=\frac{\hbar}{\alpha m_e c}=\frac{\lambda_e}{\alpha}\simeq 5.3\cdot 10^{-11}\text{ m}=0.53\text{Å},\alpha=\frac{e^2}{4\pi\hbar c}\simeq\frac{1}{137.036}$$

$$E(k)=\hbar\omega(k)=\sqrt{\left(\frac{2\hbar c}{a}\sin\frac{ka}{2}\right)^2+(mc^2)^2}$$

$$\Phi_n(k)=A\exp\left(\mathrm{i}kna\right)$$

$$\hat{\mathbf{H}}\Phi_n(k)=E(k)\Phi_n(k)$$

$$\Psi_n(t=0)=\frac{1}{2\pi}\int_B dkA(k)\exp\left(\mathrm{i}kna\right)$$

$$\mathrm{i}\hbar\partial_t\Psi_n(t)=\hat{\mathbf{H}}\Psi_n(t)$$

$$\Psi_n(t)=\frac{1}{2\pi}\int_B dkA(k)\exp\left(\mathrm{i}kna-\mathrm{i}\omega(k)t\right)$$

$$\lambda_{\mathrm{dB}}=\frac{2\pi}{k}=\frac{2\pi\hbar}{p_{\mathrm{dB}}}$$

$$\lambda_{\mathrm{dB}}\gg\lambda_\phi\ \Rightarrow\ \frac{2\pi\hbar}{p_{\mathrm{dB}}}\gg\frac{\hbar}{mc}\ \Rightarrow\ p_{\mathrm{dB}}\ll mc$$

$$\hat{H}=\frac{\hat{p}_1^2}{2M}+\frac{\hat{p}_2^2}{2M}+\frac{1}{2}M\omega_0^2(\hat{x}_2-\hat{x}_1-a)^2$$

$$\mathcal{L}_{\mathrm{M}}(\phi,\partial_\mu\phi)=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)=\frac{1}{2}(\partial_t\phi\partial_t\phi-\partial_i\phi\partial_i\phi)-V(\phi),$$

$$S_{\mathrm{M}}[\phi]=\int\, dtd^3x\mathcal{L}_{\mathrm{M}}(\phi,\partial_\mu\phi)$$

$$Z_{\mathrm{M}}=\int\,\mathcal{D}\phi\exp\left(\mathrm{i}S_{\mathrm{M}}[\phi]\right)$$

$$x_4=\mathrm{i}t$$

$$-\left[\frac{1}{2}(\partial_4\phi\partial_4\phi+\partial_i\phi\partial_i\phi)+V(\phi)\right]=-\left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi+V(\phi)\right]=-\mathcal{L}(\phi,\partial_\mu\phi)$$

$$dtd^3x=-\mathrm{i}d^3xdx_4=-\mathrm{i}d^4x$$

$$Z=\int\,\mathcal{D}\phi\exp\left(-S[\phi]\right)$$

$$S[\phi]=\int\, d^4x\mathcal{L}(\phi,\partial_\mu\phi)=\int\, d^3x\int_0^\beta dx_4\left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi+V(\phi)\right]$$

$$S[\phi]=\int\, d^dx\left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi+V(\phi)\right]$$

$$Z[j]=\int\,\mathcal{D}\phi\exp\left(-S[\phi]+\int\, d^dxj\phi\right)$$

$$\langle\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle=\frac{1}{Z}\int\,\mathcal{D}\phi\phi(x_1)\phi(x_2)\dots\phi(x_n)\exp\left(-S[\phi]\right)$$

$$\frac{1}{Z}\int\,\mathcal{D}\phi\phi(x_1)\phi(x_2)\exp\left(-S[\phi]\right)=\frac{1}{Z}\frac{\delta^2Z[j]}{\delta j(x_1)\delta j(x_2)}\bigg|_{j=0}$$

$$Z[j]=Z\exp\left[\frac{1}{2}\int\, d^dxd^dyj(x)G(x-y)j(y)\right]$$

$$\tilde{G}(p)=\int\, d^dxG(x)\exp\left(-\mathrm{i}px\right)=\frac{1}{p^2+m^2}$$

$$G(x)=\frac{1}{(2\pi)^d}\int\, d^dp\frac{\exp\left(\mathrm{i}px\right)}{p^2+m^2}$$

$$(-\partial_\mu\partial_\mu+m^2)G(x)=\delta^d(x)$$

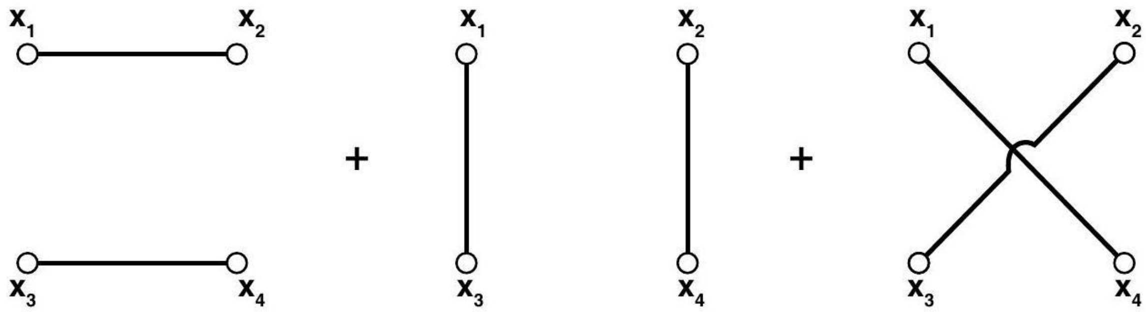


Figura 1. Interacciones entre una partícula supermasiva y varias partículas repercutidas en D – Dimensiones.

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{Z}\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\exp(-S[\phi])$$
$$= \frac{1}{Z}\frac{\delta^4 Z[j]}{\delta j(x_1)\delta j(x_2)\delta j(x_3)\delta j(x_4)}\bigg|_{j=0}$$

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = G(x_1-x_2)G(x_3-x_4)+G(x_1-x_3)G(x_2-x_4)+G(x_1-x_4)G(x_2-x_3)$$

$$\langle\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \sum_{\text{contractions}} G\left(x_{i_1}-x_{i_2}\right)G\left(x_{i_3}-x_{i_4}\right)\ldots G\left(x_{i_{n-1}}-x_{i_n}\right).$$

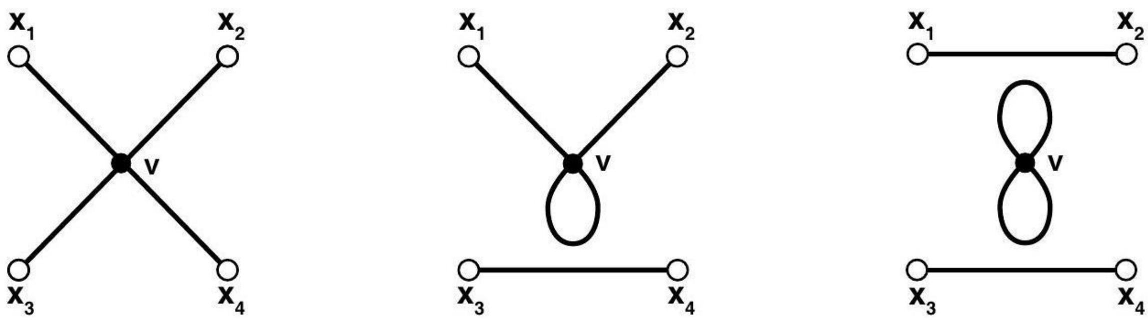


Figura 2. Interacciones entre una partícula supermasiva y varias partículas repercutidas en D – Dimensiones.

$$S[\phi] = S_\text{f}[\phi] + S_\text{i}[\phi]$$
$$S_\text{f}[\phi] = \int d^d x \frac{1}{2}(\partial_\mu\phi\partial_\mu\phi + m^2\phi^2), S_\text{i}[\phi] = \int d^d x \frac{\lambda}{4!}\phi^4$$

$$\exp(-S[\phi]) = \exp(-S_\text{f}[\phi])\left(1 - S_\text{i}[\phi] + \frac{1}{2}S_\text{i}[\phi]^2 + \cdots\right)$$

$$\langle\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle$$

$$=\frac{1}{Z}\int\mathcal{D}\phi\phi(x_1)\phi(x_2)\dots\phi(x_n)$$

$$\times\left[\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(4!)^k k!}\int d^d\mathrm{v}_1 d^d\mathrm{v}_2\dots d^d\mathrm{v}_k\phi(\mathrm{v}_1)^4\phi(\mathrm{v}_2)^4\dots\phi(\mathrm{v}_k)^4\right]\exp\left(-S_{\mathrm{f}}[\phi]\right)$$

$$G(x)=\frac{1}{(2\pi)^d}\int d^d p\frac{\exp\left(ipx\right)}{p^2+m^2}$$

$$\frac{1}{p^2+m^2}=\int_0^{\infty}dt\exp\left(-t(p^2+m^2)\right)$$

$$\begin{aligned}G(x)&=\frac{1}{(2\pi)^d}\int_0^{\infty}dt\exp\left(-tm^2\right)\int d^d p\exp\left(ipx-tp^2\right)\\&=\frac{1}{(2\pi)^d}\int_0^{\infty}dt\exp\left(-tm^2-\frac{x^2}{4t}\right)\int d^d q\exp\left(-tq^2\right)\\&=\frac{1}{(4\pi)^{d/2}}\int_0^{\infty}dt t^{-d/2}\exp\left(-tm^2-\frac{x^2}{4t}\right)\\&=\frac{1}{(2\pi)^{d/2}}m^{d-2}(m|x|)^{1-d/2}K_{1-d/2}(m|x|)\end{aligned}$$

$$G(0)=\frac{1}{(4\pi)^{d/2}}\int_0^{\infty}dt t^{-d/2}\exp\left(-tm^2\right)=\frac{1}{(4\pi)^{d/2}}m^{d-2}\Gamma\left(1-\frac{d}{2}\right)$$

$$\Gamma\left(1-\frac{d}{2}\right)=\frac{2}{d-4}+\gamma-1+\mathcal{O}(d-4)$$

$$\langle\phi(x_1)\phi(x_2)\rangle=\frac{1}{Z}\int\mathcal{D}\phi\phi(x_1)\phi(x_2)\left[1-\frac{\lambda}{4!}\int d^d\mathrm{v}\phi(\mathrm{v})^4\right]\exp\left(-S_{\mathrm{f}}[\phi]\right)$$

$$Z=\int\mathcal{D}\phi\left[1-\frac{\lambda}{4!}\int d^d\mathrm{v}\phi(\mathrm{v})^4\right]\exp\left(-S_{\mathrm{f}}[\phi]\right),$$

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\rangle&=\frac{1}{Z_{\mathrm{f}}}\int\mathcal{D}\phi\phi(x_1)\phi(x_2)\exp\left(-S_{\mathrm{f}}[\phi]\right)\\&-\frac{1}{Z_{\mathrm{f}}}\int\mathcal{D}\phi\phi(x_1)\phi(x_2)\frac{\lambda}{4!}\int d^d\mathrm{v}\phi(\mathrm{v})^4\exp\left(-S_{\mathrm{f}}[\phi]\right)\\&+\frac{1}{Z_{\mathrm{f}}}\int\mathcal{D}\phi\frac{\lambda}{4!}\int d^d\mathrm{v}\phi(\mathrm{v})^4\exp\left(-S_{\mathrm{f}}[\phi]\right)\\&\times\frac{1}{Z_{\mathrm{f}}}\int\mathcal{D}\phi\phi(x_1)\phi(x_2)\exp\left(-S_{\mathrm{f}}[\phi]\right)\end{aligned}$$

$$Z_{\mathrm{f}}=\int\mathcal{D}\phi\exp\left(-S_{\mathrm{f}}[\phi]\right)$$

$$\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\int d^d v\phi(v)^4 \exp(-S_f[\phi])$$
$$= G(x_1-x_2)\int \mathcal{D}\phi\int d^d v\phi(v)^4 \exp(-S_f[\phi])$$
$$+4\int d^d vG(x_1-v)\int \mathcal{D}\phi\phi(x_2)\phi(v)^3 \exp(-S_f[\phi])$$

$$\frac{1}{Z_f}\int \mathcal{D}\phi\phi(x_2)\phi(v)^3 \exp(-S_f[\phi]) = 3G(v-x_2)G(0)$$

$$\frac{1}{Z_f}\int \mathcal{D}\phi\phi(v)^4 \exp(-S_f[\phi]) = 3G(0)^2$$

$$\frac{1}{Z_f}\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\int d^d v\phi(v)^4 \exp(-S_f[\phi]) =$$
$$3G(x_1-x_2)G(0)^2\int d^d v 1 + 12G(0)\int d^d vG(x_1-v)G(v-x_2)$$

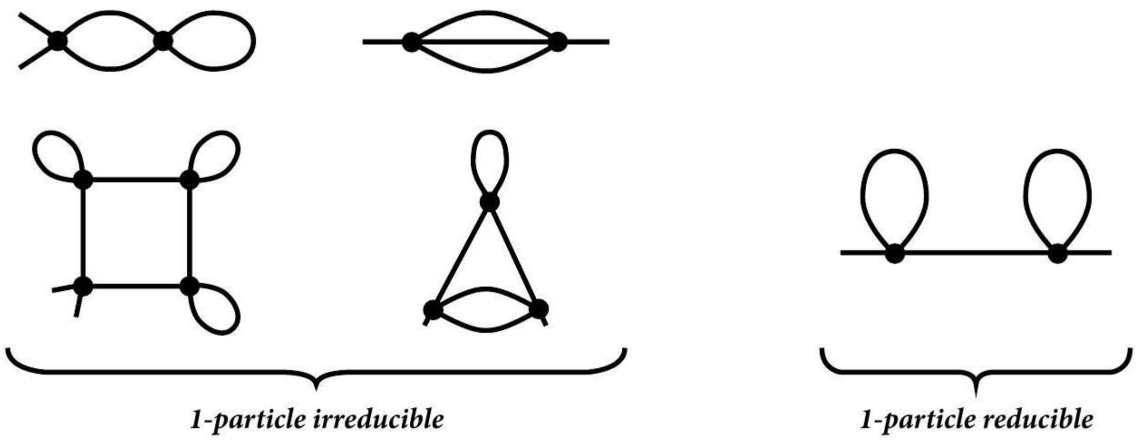
$$\frac{1}{Z_f}\int \mathcal{D}\phi\int d^d v\phi(v)^4 \exp(-S_f[\phi])\frac{1}{Z_f}\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\exp(-S_f[\phi])$$
$$= 3G(x_1-x_2)G(0)^2\int d^d v 1$$

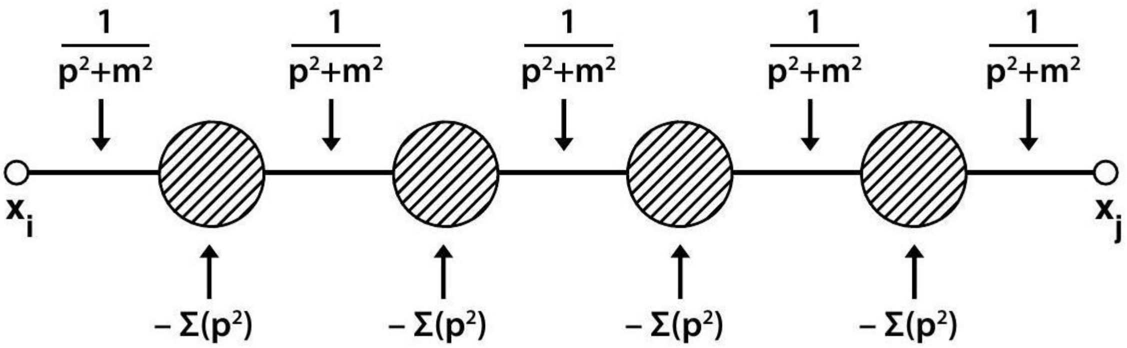
$$\langle\phi(x_1)\phi(x_2)\rangle = G(x_1-x_2) - \frac{\lambda}{2}G(0)\int d^d vG(x_1-v)G(v-x_2)$$

$$\tilde{G}_r(p) = \frac{1}{p^2+m^2} - \frac{\lambda}{2}G(0)\frac{1}{(p^2+m^2)^2} + \mathcal{O}(\lambda^2) = \frac{1}{p^2+m_r^2} + \mathcal{O}(\lambda^2)$$

$$m_r^2 = m^2 + \frac{\lambda}{2}G(0) + \mathcal{O}(\lambda^2) = m^2 + \frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}m^{d-2}\Gamma\left(1-\frac{d}{2}\right) + \mathcal{O}(\lambda^2)$$

$$m^2 = m_r^2 - \frac{\lambda}{2}G(0) = m_r^2 - \frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}m_r^{d-2}\Gamma\left(1-\frac{d}{2}\right) + \mathcal{O}(\lambda^2)$$
$$= m_r^2 - \frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}m_r^{d-2}\left[\frac{2}{d-4} + \gamma - 1 + \mathcal{O}(d-4)\right] + \mathcal{O}(\lambda^2)$$





$$\tilde{G}_{\mathrm{r}}(p) = \frac{1}{p^2+m^2} - \frac{1}{p^2+m^2}\Sigma(p^2)\frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Sigma(p^2)\frac{1}{p^2+m^2}\Sigma(p^2)\frac{1}{p^2+m^2} - \cdots = \frac{1}{p^2+m^2}\frac{1}{1+\Sigma(p^2)/(p^2+m^2)} = \frac{1}{p^2+m^2+\Sigma(p^2)}$$

$$m_{\mathrm{r}}^2 = m^2 + \Sigma(-m_{\mathrm{r}}^2)$$

$$\Sigma(p^2) = \frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}} m^{d-2}\Gamma\left(1-\frac{d}{2}\right) + \mathcal{O}(\lambda^2)$$

$$\tilde{G}_{\mathrm{r}}(p) = \frac{Z_\phi}{p^2+m_{\mathrm{r}}^2},$$

$$\Sigma(p^2) = \Sigma(-m_{\mathrm{r}}^2) + (p^2+m_{\mathrm{r}}^2)\Sigma'(-m_{\mathrm{r}}^2) + \cdots$$

$$Z_\phi = \frac{1}{1+\Sigma'(-m_{\mathrm{r}}^2)}$$

$$\int d^d x_1 d^d x_2 \ldots d^d x_n \langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle \exp\left[-\mathrm{i}(p_1x_1+p_2x_2+\cdots+p_nx_n)\right] = (2\pi)^d\delta(p_1+p_2+\cdots+p_n)\Gamma(p_1,p_2,\ldots,p_n)$$

$$(2\pi)^d\delta(p_1+p_2+\cdots+p_n)\Gamma(p_1,p_2,\ldots,p_n) = \frac{1}{Z_{\mathrm{f}}}\int \mathcal{D}\phi\tilde{\phi}(p_1)\tilde{\phi}(p_2)\ldots\tilde{\phi}(p_n)\exp\left(-S_{\mathrm{f}}[\phi]\right) \times \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(4!)^k k!}\int d^d\mathrm{v}_1 d^d\mathrm{v}_2\ldots d^d\mathrm{v}_k \phi(\mathrm{v}_1)^4\phi(\mathrm{v}_2)^4\ldots\phi(\mathrm{v}_k)^4$$

$$\frac{1}{Z_{\mathrm{f}}}\int \mathcal{D}\phi\tilde{\phi}(p_i)\phi(\mathrm{v}_j)\exp\left(-S_{\mathrm{f}}[\phi]\right) = \frac{\exp\left(-\mathrm{i}p_i\mathrm{v}_j\right)}{p_i^2+m^2}$$

$$\frac{1}{Z_{\mathrm{f}}}\int \mathcal{D}\phi\phi(\mathrm{v}_i)\phi(\mathrm{v}_j)\exp\left(-S_{\mathrm{f}}[\phi]\right) = \frac{1}{(2\pi)^d}\int d^dq\frac{\exp\left(-\mathrm{i}q(\mathrm{v}_i-\mathrm{v}_j)\right)}{q^2+m^2}$$

$$\tilde{\phi}(p_1)\tilde{\phi}(p_2)\ldots\tilde{\phi}(p_n)\phi(\mathrm{v}_1)^4\phi(\mathrm{v}_2)^4\ldots\phi(\mathrm{v}_k)^4$$

$$\int d^d q_1 \ldots d^d q_I/(2\pi)^{Id}$$

$$I = \frac{1}{2}(4k - n)$$

$$l = I - (k-1) = k + 1 - \frac{n}{2}$$

$$(2\pi)^d \delta(p_1 + p_2 + p_3 + p_4)\Gamma(p_1, p_2, p_3, p_4) =$$
$$4! \frac{1}{(p_1^2 + m^2)(p_2^2 + m^2)(p_3^2 + m^2)(p_4^2 + m^2)} \frac{(-\lambda)}{4!} (2\pi)^d \delta(p_1 + p_2 + p_3 + p_4)$$

$$\Gamma(p_1, p_2, p_3, p_4) = -\frac{\lambda}{(p_1^2 + m^2)(p_2^2 + m^2)(p_3^2 + m^2)(p_4^2 + m^2)}.$$

$$\frac{1}{(2\pi)^{2d}} \int d^d q_1 d^d q_2 (4!)^2 \frac{1}{2!} \frac{1}{(p_1^2 + m^2)(p_2^2 + m^2)(p_3^2 + m^2)(p_4^2 + m^2)}$$
$$\times \frac{1}{(q_1^2 + m^2)(q_2^2 + m^2)} \left(\frac{-\lambda}{4!}\right)^2 (2\pi)^{2d} \delta(p_1 - q_1)\delta(q_1 + p_2 + p_3 + p_4)$$
$$= \frac{\lambda^2 G(0)/2}{(p_1^2 + m^2)^2(p_2^2 + m^2)(p_3^2 + m^2)(p_4^2 + m^2)} (2\pi)^d \delta(p_1 + p_2 + p_3 + p_4)$$

$$\Gamma(p_1, p_2, p_3, p_4) = -\frac{\lambda}{(p_1^2 + m_r^2)(p_2^2 + m_r^2)(p_3^2 + m_r^2)(p_4^2 + m_r^2)},$$

$$s = (p_1 + p_2)^2, t = (p_1 + p_3)^2, u = (p_1 + p_4)^2.$$

$$\frac{1}{(2\pi)^{2d}} \int d^d q_1 d^d q_2 (4!)^2 \frac{1}{2!} \frac{1}{(p_1^2 + m^2)(p_2^2 + m^2)(p_3^2 + m^2)(p_4^2 + m^2)}$$

$$\times \frac{1}{(q_1^2 + m^2)(q_2^2 + m^2)} \left(\frac{-\lambda}{4!}\right)^2 (2\pi)^{2d} \delta(p_1 + p_2 - q_1 - q_2)\delta(q_1 + q_2 + p_3 + p_4)$$

$$= \frac{\lambda^2 J(s)/2}{(p_1^2 + m^2)(p_2^2 + m^2)(p_3^2 + m^2)(p_4^2 + m^2)} (2\pi)^d \delta(p_1 + p_2 + p_3 + p_4)$$

$$J(p^2) = \frac{1}{(2\pi)^d} \int d^d q \frac{1}{(q^2 + m^2)((p - q)^2 + m^2)}$$

$$\Gamma(p_1, p_2, p_3, p_4) = -\frac{\lambda - \lambda^2 [J(s) + J(t) + J(u)]/2}{(p_1^2 + m_r^2)(p_2^2 + m_r^2)(p_3^2 + m_r^2)(p_4^2 + m_r^2)}.$$

$$\frac{1}{AB} = \int_0^1 d\tau \frac{1}{[(1 - \tau)A + \tau B]^2}$$

$$J(p^2)=\frac{1}{(2\pi)^d}\int_0^1 d\tau\int d^dq\frac{1}{[(1-\tau)(q^2+m^2)+\tau((p-q)^2+m^2)]^2}$$
$$=\frac{1}{(2\pi)^d}\int_0^1 d\tau\int d^d\bar{q}\frac{1}{(\bar{q}^2+\bar{m}^2)^2}$$

$$\frac{1}{(\bar{q}^2+\bar{m}^2)^2}=\int_0^\infty dt t\exp\left(-t(\bar{q}^2+\bar{m}^2)\right)$$

$$\frac{1}{(2\pi)^d}\int d^d\bar{q}\exp\left(-t\bar{q}^2\right)=(4\pi t)^{-d/2}$$

$$J(p^2)=\int_0^1 d\tau\int_0^\infty dt(4\pi t)^{-d/2}t\exp\left(-t\bar{m}^2\right)$$
$$=\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}}\int_0^1 d\tau\bar{m}^{d-4}=-\frac{1}{8\pi^2(d-4)}+\cdots$$

$$s=t=u=\frac{4}{3}m_{\mathrm{r}}^2$$

$$\bar{J}(p^2)=J(p^2)-J(4m_{\mathrm{r}}^2/3),$$

$$\Gamma(p_1,p_2,p_3,p_4)=-\frac{\lambda_{\mathrm{r}}-\lambda_{\mathrm{r}}^2[\bar{J}(s)+\bar{J}(t)+\bar{J}(u)]/2}{(p_1^2+m_{\mathrm{r}}^2)(p_2^2+m_{\mathrm{r}}^2)(p_3^2+m_{\mathrm{r}}^2)(p_4^2+m_{\mathrm{r}}^2)},$$

$$\lambda_{\mathrm{r}}=\lambda-\frac{3}{2}J(4m_{\mathrm{r}}^2/3)\lambda^2+\mathcal{O}(\lambda^3)$$

$$V(\phi)=\sum_\nu g_\nu\phi^\nu$$

$$S[\phi]=\int d^dx\left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi+V(\phi)\right]$$

$$d_\phi=\frac{d-2}{2}$$

$$d_{g_v}=d-vd_\phi=d\left(1-\frac{v}{2}\right)+v$$

$$d\left(1-\frac{v}{2}\right)+v\geq 0\ \Rightarrow\ v\leq\frac{2d}{d-2}$$

$$I=\frac{1}{2}\left(\sum_v k_v v-n\right)$$

$$l=I-k+1$$

$$\delta=dl-2I=\frac{d-2}{2}\left(\sum_v k_v v-n\right)-d\sum_v k_v+d$$

$$\sum_{v}\left[k_v\left(\frac{d-2}{2}v-d\right)\right]<\frac{d-2}{2}n-d$$

$$\frac{d-2}{2}v-d\leq 0\ \Rightarrow\ v\leq\frac{2d}{d-2}$$

$$n\leq\frac{2d}{d-2}$$

$$\Gamma(\varepsilon)=\frac{1}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon),$$

$$\langle\tilde{\phi}(p_i)\phi(\mathrm{v}_j)\rangle=\frac{\exp\left(-\mathrm{i}p_i\mathrm{v}_j\right)}{p_i^2+m^2}$$

$$J(q)=\int\ d^dx\exp\left(\mathrm{i}qx\right)G(x)^2=\int\ \frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m^2}\frac{1}{(p-q)^2+m^2}$$

$$J(q)=-\frac{1}{8\pi^2(d-4)}+\aleph_{\text{finite terms}}$$

$$\frac{1}{A_1A_2\ldots A_n}=(n-1)!\int_0^1 d\tau_1\ldots\int_0^1 d\tau_n\delta\left(1-\sum_{k=1}^{n}\tau_k\right)\left[\sum_{k=1}^{n}\tau_kA_k\right]^{-n}$$

$$\mathcal{L}(\phi,\partial_\mu\phi)=\frac{1}{2}\partial_\mu\phi\partial_\mu\phi+\frac{m^2}{2}\phi^2+g\phi^3$$

$$\mathcal{L}(\phi,\partial_\mu\phi)=\frac{1}{2}\partial_\mu\phi\partial_\mu\phi+\frac{m^2}{2}\phi^2+\frac{\lambda}{6!}\phi^6.$$

$$Z=\int\ \mathcal{D}\Phi\exp\left(-S[\Phi]\right)=\prod_{x\in\Lambda}\prod_{a=1}^{N}\sqrt{\frac{a^{d-2}}{2\pi}}\int_{-\infty}^{\infty}d\Phi_x^a\exp\left(-S[\Phi]\right)$$

$$S[\Phi]=\sum_{x\in\Lambda}c_x^a\Phi_x^a+\sum_{x,y\in\Lambda}c_{xy}^{ab}\Phi_x^a\Phi_y^b+\sum_{x,y,z\in\Lambda}c_{xyz}^{abc}\Phi_x^a\Phi_y^b\Phi_z^c+\sum_{x,y,z,w\in\Lambda}c_{xyzw}^{abcd}\Phi_x^a\Phi_y^b\Phi_z^c\Phi_w^d+\cdots$$

$$\langle\Phi_x^a\Phi_y^b\rangle=\frac{1}{Z}\int\ \mathcal{D}\Phi\Phi_x^a\Phi_y^b\exp\left(-S[\Phi]\right)$$

$$\exp\left(-T[\Phi',\Phi]\right)=\prod_{x'\in\Lambda'}\frac{a'}{\sqrt{\alpha_2}}\exp\left(-\frac{a'^d}{2\alpha_2}\left(\Phi_{x'}'^a-\frac{\beta_2}{2^d}\sum_{x\in c_{x'}}\Phi_x^a\right)^2\right)$$

$$\exp\left(-S'[\Phi']\right)=\int\ \mathcal{D}\Phi\exp\left(-T[\Phi',\Phi]\right)\exp\left(-S[\Phi]\right)$$

$$\int \mathcal{D}\Phi'\exp\left(-T[\Phi',\Phi]\right) =$$

$$\prod_{x'\in\Lambda'}\prod_{a=1}^{N}\sqrt{\frac{a'^{d-2}}{2\pi}}\int_{-\infty}^{\infty} d\Phi'^{a}_{x'}\frac{a'}{\sqrt{\alpha_2}}\exp\left(-\frac{a'^{d}}{2\alpha_2}\left(\Phi'^{a}_{x'}-\frac{\beta_2}{2^{d}}\sum_{x\in c_{x'}}\Phi^{a}_{x}\right)^{2}\right)$$

$$Z' = \int \mathcal{D}\Phi'\exp\left(-S'[\Phi']\right) = \int \mathcal{D}\Phi\int \mathcal{D}\Phi'\exp\left(-T[\Phi',\Phi]\right)\exp\left(-S[\Phi]\right)$$
$$= \int \mathcal{D}\Phi\exp\left(-S[\Phi]\right) = Z$$

$$S[\Phi] \to S'[\Phi'] \to \cdots \to S^{(\nu)}\left[\Phi^{(\nu)}\right] \to \cdots \to S^{*}[\Phi^{*}].$$

$$S[\Phi] = S^{*}[\Phi] + \delta S[\Phi] \;\Rightarrow\; S'[\Phi'] = S^{*}[\Phi'] + 2^{\Delta}\delta S[\Phi'].$$

$$S[\Phi] = a^{d}\sum_{x,\mu}\frac{1}{2a^{2}}\left(\Phi_{x+\hat{\mu}}-\Phi_{x}\right)^{2}$$

$$S[\Phi] = \frac{a^{d}}{2}\sum_{x,y}\Phi_{x}\Delta(x-y)\Phi_{y}$$

$$\Delta(z) = \frac{1}{a^{2}}\sum_{\mu=1}^{d}\left(\delta_{z-\hat{\mu},0}-2\delta_{z,0}+\delta_{z+\hat{\mu},0}\right)$$

$$\Phi(p) = a^{d}\sum_{x}\Phi_{x}\exp\left(-ipx\right), \Phi_{x} = \frac{1}{(2\pi)^{d}}\int_{B} d^{d}p\Phi(p)\exp\left(ipx\right)$$

$$S[\Phi] = \frac{1}{(2\pi)^{d}}\int_{B} d^{d}p\Phi(-p)\Delta(p)\Phi(p), \Delta(p) = \sum_{\mu=1}^{d}\left(\frac{2}{a}\sin\frac{p_{\mu}a}{2}\right)^{2}$$

$$\Pi_{2}(pa) = \prod_{\mu=1}^{d}\frac{\sin\left(p_{\mu}a\right)}{2\sin\left(p_{\mu}a/2\right)} = \prod_{\mu=1}^{d}\cos\left(p_{\mu}a/2\right)$$

$$\exp\left(-S'[\Phi']\right) = \int \mathcal{D}\Phi\mathcal{D}\Omega'\exp\left\{-\frac{1}{(2\pi)^{d}}\int_{B} d^{d}p\frac{1}{2}\Phi(-p)\Delta(p)\Phi(p)\right.$$
$$\left.-\frac{1}{(2\pi)^{d}}\int_{B'} d^{d}p'\left[i\Omega'(-p')\left(\Phi'(p')-\beta_{2}\sum_{l}\Phi\left(p'+\frac{2\pi l}{a'}\right)\Pi_{2}(p'a+\pi l)\right)+\frac{\alpha_{2}}{2}\Omega'(-p')\Omega'(p')\right]\right\}$$
$$= \int \mathcal{D}\Phi\mathcal{D}\Omega'\exp\left\{-\frac{1}{(2\pi)^{d}}\int_{B} d^{d}p\left[\frac{1}{2}\Phi(-p)\Delta(p)\Phi(p)-i\Omega'(-p)\beta_{2}\Phi(p)\Pi_{2}(pa)\right]\right.$$
$$\left.-\frac{1}{(2\pi)^{d}}\int_{B'} d^{d}p'\left[\frac{\alpha_{2}}{2}\Omega'(-p')\Omega'(p')+i\Omega'(-p')\Phi'(p')\right]\right\}$$

$$\Phi_{c}(p) = i\beta_{2}\Delta(p)^{-1}\Pi_{2}(pa)\Omega'(p)$$

$$\exp\left(-S'[\Phi']\right) = \int \mathcal{D}\Omega' \exp\left\{-\frac{1}{(2\pi)^d}\int_{B'} d^d p' [\mathrm{i}\Omega'(-p')\Phi'(p') + \frac{1}{2}\Omega'(-p')\left(\beta_2^2 \sum_l \Delta\left(p' + \frac{2\pi l}{a'}\right)^{-1} \Pi_2(p'a+\pi l)^2 + \alpha_2\right)\Omega'(p')]\right\}$$

$$\Omega_c'(p') = -\mathrm{i}\Delta'(p')\Phi'(p'), \Delta'(p')^{-1} = \beta_2^2 \sum_l \Delta\left(p' + \frac{2\pi l}{a'}\right)^{-1} \Pi_2(p'a+\pi l)^2 + \alpha_2.$$

$$S'[\Phi'] = \frac{1}{(2\pi)^d}\int_{B'} d^d p' \frac{1}{2}\Phi'(-p')\Delta'(p')\Phi'(p')$$

$$\Delta^{(v)}(p)^{-1} = \left(\frac{\beta_2^2}{2^d}\right)^v \sum_l \Delta\left(\frac{p+2\pi l/a}{2^v}\right)^{-1} \prod_{\mu=1}^{d}\left(\frac{\sin(p_\mu a/2)}{2^v \sin\left((p_\mu a + 2\pi l_\mu)/2^{v+1}\right)}\right)^2 + \alpha_2 \frac{1-\left(\beta_2^2/2^d\right)^v}{1-\beta_2^2/2^d}$$

$$\Delta^*(p)^{-1} = \lim_{v\to\infty}\left(\frac{\beta_2^2}{2^d}\right)^v \sum_l \frac{2^{2v}}{(p+2\pi l/a)^2}\prod_{\mu=1}^{d}\left(\frac{\sin(p_\mu a/2)}{p_\mu a/2 + \pi l_\mu}\right)^2 + \alpha_2 \frac{1-\left(\beta_2^2/2^d\right)^v}{1-\beta_2^2/2^d}$$

$$\left(\frac{\beta_2^2}{2^d}\right)^v 2^{2v} = 1 \;\Rightarrow\; \beta_2 = 2^{(d-2)/2}$$

$$\Delta^*(p)^{-1} = \sum_{l\in\mathbb{Z}^d} \frac{1}{(p+2\pi l/a)^2}\prod_{\mu=1}^{d}\left(\frac{\sin(p_\mu a/2)}{p_\mu a/2 + \pi l_\mu}\right)^2 + \frac{4\alpha_2}{3}.$$

$$\Phi_x = \frac{1}{a^d}\int_{c_x} d^d y \phi(y)$$

$$\Phi(p) = \sum_{l\in\mathbb{Z}^d} \phi(p+2\pi l/a)\Pi(pa+2\pi l), \Pi(pa) = \prod_{\mu=1}^{d} \frac{2\sin(p_\mu a/2)}{p_\mu a}.$$

$$\langle\Phi(-p)\Phi(p)\rangle = \sum_{l\in\mathbb{Z}^d} \langle\phi(-p-2\pi l/a)\phi(p+2\pi l/a)\rangle \Pi(pa+2\pi l)^2 = \sum_{l\in\mathbb{Z}^d} \frac{\Pi(pa+2\pi l)^2}{(p+2\pi l/a)^2} = \Delta^*(p)^{-1}$$

$$\exp\left(-S_{\text{perfect}}[\Phi]\right) = \int \mathcal{D}\phi\mathcal{D}\Omega \exp\left\{-\frac{1}{(2\pi)^d}\int d^dp \frac{1}{2}\phi(-p)(p^2+m^2)\phi(p)\right\}$$

$$\times \exp\left\{-\frac{1}{(2\pi)^d}\int_B d^dp\left[\mathrm{i}\Omega(-p)\left(\Phi(p)-\sum_{l\in\mathbb{Z}^d}\phi(p+2\pi l/a)\Pi(pa+2\pi l)\right)+\frac{\alpha}{2}\Omega(-p)\Omega(p)\right]\right\}$$

$$= \int \mathcal{D}\phi\mathcal{D}\Omega \exp\left\{-\frac{1}{(2\pi)^d}\int d^dp\left[\frac{1}{2}\phi(-p)(p^2+m^2)\phi(p)-\mathrm{i}\Omega(-p)\phi(p)\Pi(pa)\right]\right\}$$

$$\times \exp\left\{-\frac{1}{(2\pi)^d}\int_B d^dp\left[\frac{\alpha}{2}\Omega(-p)\Omega(p)+\mathrm{i}\Omega(-p)\Phi(p)\right]\right\}$$

$$\phi_c(p) = \mathrm{i}(p^2+m^2)^{-1}\Pi(pa)\Omega(p),$$

$$\exp\left(-S_{\text{perfect}}[\Phi]\right) = \int \mathcal{D}\Omega \exp\left\{-\frac{1}{(2\pi)^d}\int d^dp\left[\mathrm{i}\Omega(-p)\Phi(p) + \frac{1}{2}\Omega(-p)\left(\sum_{l\in\mathbb{Z}^d}\frac{\Pi(pa+2\pi l)^2}{(p+2\pi l/a)^2+m^2}+\alpha\right)\Omega(p)\right]\right\}$$

$$\Omega_c(p) = -\mathrm{i}\Delta_{\text{perfect}}(p)\Phi(p).$$

$$S_{\text{perfect}}[\Phi] = \frac{1}{(2\pi)^d}\int d^dp\frac{1}{2}\Phi(-p)\Delta_{\text{perfect}}(p)\Phi(p)$$

$$\Delta_{\text{perfect}}(p)^{-1} = \langle\Phi(-p)\Phi(p)\rangle = \sum_{l\in\mathbb{Z}^d}\frac{\Pi(pa+2\pi l)^2}{(p+2\pi l/a)^2+m^2}+\alpha$$

$$\Phi(\vec{p})_{x_d} = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dp_d\Phi(p)\exp(\mathrm{i}p_dx_d)$$

$$\langle\Phi_{\vec{x},0}\Phi(\vec{p})_{x_d}\rangle = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dp_d\Delta_{\text{perfect}}(p)^{-1}\exp(\mathrm{i}p_dx_d)$$

$$= \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dp_d\left[\sum_{l\in\mathbb{Z}^d}\frac{\Pi(pa+2\pi l)^2}{(p+2\pi l/a)^2+m^2}+\alpha\right]\exp(\mathrm{i}p_dx_d)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp_d\sum_{\vec{l}\in\mathbb{Z}^{d-1}}\frac{1}{(\vec{p}+2\pi\vec{l}/a)^2+p_d^2+m^2}$$

$$\times\prod_{j=1}^{d-1}\left(\frac{2\sin(p_ja/2)}{p_ja+2\pi l_j}\right)^2\left(\frac{2\sin(p_da/2)}{p_da}\right)^2\exp(\mathrm{i}p_dx_d)+\alpha\frac{\delta_{x_d,0}}{a}$$

$$= \sum_{\vec{l}\in\mathbb{Z}^{d-1}} C(\vec{p}+2\pi\vec{l}/a)\exp\left(-E(\vec{p}+2\pi\vec{l}/a)x_d\right)+\alpha\frac{\delta_{x_d,0}}{a}$$

$$E(\vec{p}+2\pi\vec{l}/a)^2 = -p_d^2 = (\vec{p}+2\pi\vec{l}/a)^2+m^2$$

$$\Delta_{\text{perfect}}(p)^{-1} = \sum_{l\in\mathbb{Z}}\frac{\Pi(pa+2\pi l)^2}{(p+2\pi l/a)^2+m^2}+\alpha$$

$$= \frac{1}{m^2}-\frac{2}{m^3a}\frac{\coth(ma/2)}{\cot^2(pa/2)+\coth^2(ma/2)}+\alpha$$

$$\alpha = \frac{\tilde{m} - m}{m^3}, \tilde{m} = \frac{1}{a}\sinh{(ma)}, \tilde{m} = \frac{2}{a}\sinh{(ma/2)}$$

$$\Delta_{\text{perfect}}(p) = \frac{m^3}{\tilde{m}\tilde{m}^2}\left[\left(\frac{2}{a}\sin{(pa/2)}\right)^2 + \tilde{m}^2\right]$$

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial_\mu \phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$\tilde{G}(p) = \frac{1}{p^2 + m^2},$$

$$\tilde{G}_{\mathrm{r}}(p) = \frac{1}{p^2 + m^2} - \frac{\lambda}{2}G(0)\frac{1}{(p^2 + m^2)^2} + \mathcal{O}(\lambda^2) = \frac{1}{p^2 + m_{\mathrm{r}}^2} + \mathcal{O}(\lambda^2)$$

$$m_{\mathrm{r}}^2 = m^2 + \frac{\lambda}{2}G(0) + \mathcal{O}(\lambda^2)$$

$$G(0) = \frac{1}{(2\pi)^4}\int d^4p \frac{1}{p^2 + m^2}$$

$$= \frac{1}{8\pi^2}\int_0^\Lambda dp \frac{p^3}{p^2 + m^2} = \frac{1}{16\pi^2}\left(\Lambda^2 - m^2 \log\left(1 + \frac{\Lambda^2}{m^2}\right)\right)$$

$$m_{\mathrm{r}}^2 = m^2 + \frac{\lambda}{2}\frac{1}{16\pi^2}\left(\Lambda^2 - m^2 \log\left(1 + \frac{\Lambda^2}{m^2}\right)\right) = 0 \Rightarrow m^2 = -\frac{\lambda}{32\pi^2}\Lambda^2 + \mathcal{O}(\lambda^2)$$

$$\Gamma(p_1, p_2, p_3, p_4) = -\frac{\lambda}{(p_1^2 + m^2)(p_2^2 + m^2)(p_3^2 + m^2)(p_4^2 + m^2)}$$

$$\Gamma'(p_1, p_2, p_3, p_4) = -\frac{\lambda}{(p_1^2 + m_{\mathrm{r}}^2)(p_2^2 + m_{\mathrm{r}}^2)(p_3^2 + m_{\mathrm{r}}^2)(p_4^2 + m_{\mathrm{r}}^2)} = -\frac{\lambda}{p_1^2 p_2^2 p_3^2 p_4^2}$$

$$\Gamma(p_1, p_2, p_3, p_4) = -\frac{\lambda - \lambda^2(J(s) + J(t) + J(u))/2}{p_1^2 p_2^2 p_3^2 p_4^2}$$

$$s = (p_1 + p_2)^2, t = (p_1 + p_3)^2, u = (p_1 + p_4)^2$$

$$\phi_{\mathrm{r}}(x) = \sqrt{Z_\phi}\phi(x), \lambda = Z_\lambda \lambda_{\mathrm{r}}.$$

$$\tilde{G}_{\mathrm{r}}(p)\big|_{p^2 = \mu^2} = \frac{1}{\mu^2}$$

$$k_i k_j = \mu^2\left(\delta_{ij} - \frac{1}{4}\right) \Rightarrow k_i^2 = \frac{3}{4}\mu^2, k_i k_j = -\frac{1}{4}\mu^2, i \neq j$$

$$\Gamma(k_1, k_2, k_3, k_4) = -\frac{\lambda_{\mathrm{r}}}{k_1^2 k_2^2 k_3^2 k_4^2}$$

$$\Gamma_{\mathrm{r}}^{(n)}(p_1, p_2, \dots, p_n; \lambda_r, \mu) = Z_\phi^{n/2}(\lambda, \Lambda/\mu)\Gamma^{(n)}(p_1, p_2, \dots, p_n; \lambda, \Lambda).$$

$$\mu \frac{d}{d\mu} \Gamma^{(n)}(p_1, p_2, \dots, p_n; \lambda, \Lambda) = 0 \Rightarrow$$
$$\mu \frac{d}{d\mu} \left[Z_\phi^{-n/2}(\lambda, \Lambda/\mu) \Gamma_\mathrm{r}^{(n)}(p_1, p_2, \dots, p_n; \lambda_\mathrm{r}, \mu) \right] = 0$$

$$\left[\mu \frac{\partial}{\partial\mu} + \beta(\lambda_\mathrm{r}) \frac{\partial}{\partial\lambda_\mathrm{r}} - n\gamma(\lambda_\mathrm{r}) \right] \Gamma_\mathrm{r}^{(n)}(p_1, p_2, \dots, p_n; \lambda_\mathrm{r}, \mu) = 0,$$
$$\beta(\lambda_\mathrm{r}) = \mu \frac{\partial}{\partial\mu} \lambda_\mathrm{r}(\mu),$$
$$\gamma(\lambda_\mathrm{r}) = \frac{1}{2} \mu \frac{\partial}{\partial\mu} \log\, Z_\phi(\lambda, \Lambda/\mu),$$

$$\mu \frac{d}{d\mu} Z_\phi^{-n/2}(\lambda, \Lambda/\mu) = -\mu \frac{n}{2} Z_\phi^{-n/2-1}(\lambda, \Lambda/\mu) \frac{\partial}{\partial\mu} Z_\phi(\lambda, \Lambda/\mu)$$
$$= -n Z_\phi^{-n/2}(\lambda, \Lambda/\mu) \frac{1}{2} \mu \frac{\partial}{\partial\mu} \log\, Z_\phi(\lambda, \Lambda/\mu)$$

$$\lambda_\mathrm{r} = \lambda - \frac{\lambda^2}{2}(J(s) + J(t) + J(u))$$

$$s = (k_1 + k_2)^2 = t = (k_1 + k_3)^2 = u = (k_1 + k_4)^2 = \mu^2 \Rightarrow \lambda_\mathrm{r} = \lambda - \frac{3\lambda^2}{2} J(\mu^2)$$

$$J(\mu^2) = \frac{1}{(2\pi)^4} \int_0^1 d\tau \int d^4\bar{q} \frac{1}{(\bar{q}^2 + \bar{\mu}^2)^2}, \bar{\mu}^2 = \tau(1-\tau)\mu^2, \bar{q} = q - \tau\mu$$

$$\frac{1}{(2\pi)^4} \int d^4\bar{q} \frac{1}{(\bar{q}^2 + \bar{\mu}^2)^2} = \frac{2\pi^2}{(2\pi)^4} \int_0^\Lambda d\bar{q} \frac{\bar{q}^3}{(\bar{q}^2 + \bar{\mu}^2)^2} \sim \frac{1}{16\pi^2} \log\left(\frac{\Lambda^2}{\bar{\mu}^2}\right)$$

$$J(\mu^2) = \int_0^1 d\tau \frac{1}{16\pi^2} \log\left(\frac{\Lambda^2}{\tau(1-\tau)\mu^2}\right) \sim \frac{1}{8\pi^2} \log\left(\frac{\Lambda}{\mu}\right)$$

$$\lambda_\mathrm{r} = \lambda + \frac{3\lambda^2}{16\pi^2} \log\left(\frac{\mu}{\Lambda}\right)$$

$$\beta(\lambda_\mathrm{r}) = \mu \frac{\partial}{\partial\mu} \lambda_\mathrm{r} = \frac{3}{16\pi^2} \lambda^2 = \frac{3}{16\pi^2} \lambda_\mathrm{r}^2 + \mathcal{O}(\lambda_\mathrm{r}^3)$$

$$\beta(\lambda_\mathrm{r}) = \beta_0 \lambda_\mathrm{r}^2 + \beta_1 \lambda_\mathrm{r}^3 + \mathcal{O}(\lambda_\mathrm{r}^4), \beta_0 = \frac{3}{(4\pi)^2}, \beta_1 = -\frac{17}{3(4\pi)^4}$$

$$\mu \frac{\partial}{\partial\mu} \lambda_\mathrm{r} = \beta(\lambda_\mathrm{r}) = \beta_0 \lambda_\mathrm{r}^2 + \mathcal{O}(\lambda_\mathrm{r}^3) \Rightarrow \int_{\lambda_\mathrm{r}(\mu_0)}^{\lambda_\mathrm{r}(\mu)} \frac{d\lambda_\mathrm{r}}{\beta_0 \lambda_\mathrm{r}^2} = \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} + \mathcal{O}(\lambda_\mathrm{r}^3) \Rightarrow$$
$$\lambda_\mathrm{r}(\mu) = \frac{\lambda_\mathrm{r}(\mu_0)}{1 - \lambda_\mathrm{r}(\mu_0)\beta_0 \log(\mu/\mu_0)} + \mathcal{O}(\lambda_\mathrm{r}^3)$$

$$\lambda_\mathrm{r}(\mu) = \lambda_\mathrm{r}(\mu_0) + \lambda_\mathrm{r}(\mu_0)^2 \beta_0 \log(\mu/\mu_0) + \mathcal{O}(\lambda_\mathrm{r}^3).$$

$$\int_{\lambda_{\mathrm{r}}(\mu_0)}^{\lambda_{\mathrm{r}}(\mu)} \frac{d\lambda_{\mathrm{r}}}{\beta_0\lambda_{\mathrm{r}}^2+\beta_1\lambda_{\mathrm{r}}^3}+\mathcal{O}(\lambda_{\mathrm{r}}^4)=\int_{\mu_0}^{\mu}\frac{d\mu}{\mu}\Rightarrow$$

$$\lambda_{\mathrm{r}}(\mu)=\frac{\lambda_{\mathrm{r}}(\mu_0)}{1-\lambda_{\mathrm{r}}(\mu_0)\beta_0\log(\mu/\mu_0)+\lambda_{\mathrm{r}}(\mu_0)(\beta_1/\beta_0)\log[1-\lambda_{\mathrm{r}}(\mu_0)\beta_0\log(\mu/\mu_0)]}+\mathcal{O}(\lambda_{\mathrm{r}}^4)$$

$$\lambda_{\mathrm{r}}(\mu)\to\frac{1}{-\beta_0\log(\mu/\mu_0)+(\beta_1/\beta_0)\log[-\lambda_{\mathrm{r}}(\mu_0)\beta_0\log(\mu/\mu_0)]}.$$

$$\beta(g_{\mathrm{r}})=\beta_0 g_{\mathrm{r}}^3+\beta_1 g_{\mathrm{r}}^5+\mathcal{O}(g_{\mathrm{r}}^6),\beta_0=-\frac{11N}{3(4\pi)^2},\beta_1=-\frac{34N^2}{3(4\pi)^4}$$

$$\vec{\nabla}\times\vec{E}(\vec{x},t)+\frac{1}{c}\partial_t\vec{B}(\vec{x},t)=\vec{0},\vec{\nabla}\cdot\vec{B}(\vec{x})=0$$

$$\vec{E}(\vec{x},t)=-\vec{\nabla}\phi(\vec{x},t)-\frac{1}{c}\partial_t\vec{A}(\vec{x},t),\vec{B}(\vec{x},t)=\vec{\nabla}\times\vec{A}(\vec{x},t)$$

$$\phi'(\vec{x},t)=\phi(\vec{x},t)-\frac{1}{c}\partial_t\alpha(\vec{x},t),\vec{A}'(\vec{x},t)=\vec{A}(\vec{x},t)+\vec{\nabla}\alpha(\vec{x},t),$$

$$\begin{aligned}\vec{E}'(\vec{x},t)&=-\vec{\nabla}\phi'(\vec{x},t)-\frac{1}{c}\partial_t\vec{A}'(\vec{x},t)\\&=-\vec{\nabla}\phi(\vec{x},t)-\frac{1}{c}\vec{\nabla}\partial_t\alpha(\vec{x},t)-\frac{1}{c}\partial_t\vec{A}(\vec{x},t)+\frac{1}{c}\partial_t\vec{\nabla}\alpha(\vec{x},t)=\vec{E}(\vec{x},t),\end{aligned}$$

$$\vec{B}'(\vec{x},t)=\vec{\nabla}\times\vec{A}'(\vec{x},t)=\vec{\nabla}\times\vec{A}(\vec{x},t)+\vec{\nabla}\times\vec{\nabla}\alpha(\vec{x},t)=\vec{B}(\vec{x},t)$$

$$x^0=ct,x^\mu=(x^0,\vec{x}),\partial_\mu=\left(\frac{1}{c}\partial_t,\vec{\nabla}\right),A^\mu(x)=(\phi(\vec{x},t),\vec{A}(\vec{x},t)).$$

$$F^{\mu\nu}(x)=\partial^\mu A^\nu(x)-\partial^\nu A^\mu(x)=\begin{pmatrix}0&-E_x(\vec{x},t)&-E_y(\vec{x},t)&-E_z(\vec{x},t)\\E_x(\vec{x},t)&0&-B_z(\vec{x},t)&B_y(\vec{x},t)\\E_y(\vec{x},t)&B_z(\vec{x},t)&0&-B_x(\vec{x},t)\\E_z(\vec{x},t)&-B_y(\vec{x},t)&B_x(\vec{x},t)&0\end{pmatrix}.$$

$$A'^\mu(x)=A^\mu(x)-\partial^\mu\alpha(x),F'^{\mu\nu}(x)=F^{\mu\nu}(x).$$

$$\mathcal{L}(\partial^\mu A^\nu)=-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)=\frac{1}{2}\left(\vec{E}(x)^2-\vec{B}(x)^2\right)$$

$$A^0(x)=\phi(\vec{x},t)=0.$$

$$A'^0(x)=A^0(x)-\partial^0\alpha(x)=A^0(x)-\partial^0\alpha(\vec{x})=A^0(x)=0$$

$$\Pi_i(x)=\frac{\delta\mathcal{L}}{\delta\partial^0A^i(x)}=\partial^0A^i(x)=-E_i(x)$$

$$\vec{A}(\vec{x},t)=\left(A_x(\vec{x},t),A_y(\vec{x},t),A_z(\vec{x},t)\right)=(A^1(x),A^2(x),A^3(x))=-(A_1(x),A_2(x),A_3(x)).$$

$$\vec{\nabla} = (\partial_x, \partial_y, \partial_z) = (\partial_1, \partial_2, \partial_3) = -(\partial^1, \partial^2, \partial^3)$$

$$\mathcal{H}(A^i, \Pi_i) = \Pi_i(x)\partial^0 A^i(x) - \mathcal{L} = \frac{1}{2}(\Pi_i(x)\Pi_i(x) + B_i(x)B_i(x))$$
$$= \frac{1}{2}(E_i(x)E_i(x) + B_i(x)B_i(x))$$

$$H[A^i, \Pi_i] = \int d^3x\mathcal{H} = \int d^3x\frac{1}{2}[\Pi_i(\vec{x})\Pi_i(\vec{x}) + \epsilon_{ijk}\partial_j A^k(\vec{x})\epsilon_{ilm}\partial_l A^m(\vec{x})]$$

$$[\hat{A}^i(\vec{x}), \hat{\Pi}_j(\vec{y})] = \mathrm{i}\delta_{ij}\delta(\vec{x} - \vec{y})$$
$$[\hat{A}^i(\vec{x}), \hat{A}^j(\vec{y})] = [\hat{\Pi}_i(\vec{x}), \hat{\Pi}_j(\vec{y})] = 0.$$

$$\hat{\Pi}_i(\vec{x}) = -\mathrm{i}\frac{\delta}{\delta A^i(\vec{x})}$$

$$\hat{H} = \int d^3x\frac{1}{2}[\hat{\Pi}_i(\vec{x})\hat{\Pi}_i(\vec{x}) + \epsilon_{ijk}\partial_j\hat{A}^k(\vec{x})\epsilon_{ilm}\partial_l\hat{A}^m(\vec{x})]$$

$$\hat{A}^i(\vec{p}) = \int d^3x\hat{A}^i(\vec{x})\exp(-\mathrm{i}\vec{p}\cdot\vec{x}), \hat{\Pi}_i(\vec{p}) = \int d^3x\hat{\Pi}_i(\vec{x})\exp(-\mathrm{i}\vec{p}\cdot\vec{x})$$

$$\hat{A}^i(\vec{p})^\dagger = \hat{A}^i(-\vec{p}), \hat{\Pi}_i(\vec{p})^\dagger = \hat{\Pi}_i(-\vec{p}).$$

$$[\hat{A}^i(\vec{p}), \hat{\Pi}_j(\vec{q})] = \mathrm{i}(2\pi)^3\delta_{ij}\delta(\vec{p} + \vec{q}), [\hat{A}^i(\vec{p}), \hat{A}^j(\vec{q})] = [\hat{\Pi}_i(\vec{p}), \hat{\Pi}_j(\vec{q})] = 0,$$

$$\hat{H} = \frac{1}{(2\pi)^3}\int d^3p\frac{1}{2}[\hat{\Pi}_i(\vec{p})^\dagger\hat{\Pi}_i(\vec{p}) + \epsilon_{ijk}p_j\hat{A}^k(\vec{p})^\dagger\epsilon_{ilm}p_l\hat{A}^m(\vec{p})]$$

$$\vec{\nabla}\cdot\hat{\vec{E}}(\vec{x})|\Psi\rangle = 0 \Rightarrow p_i\hat{\Pi}_i(\vec{p})|\Psi\rangle = 0$$

$$\epsilon_{ijk}p_j\hat{A}^k(\vec{p})^\dagger\epsilon_{ilm}p_l\hat{A}^m(\vec{p}) = \hat{A}^i(\vec{p})^\dagger\mathcal{M}(\vec{p})_{ij}\hat{A}^j(\vec{p}) =$$
$$(\hat{A}^1(\vec{p})^\dagger, \hat{A}^2(\vec{p})^\dagger, \hat{A}^3(\vec{p})^\dagger)\begin{pmatrix} \vec{p}^2 - p_1^2 & -p_1p_2 & -p_1p_3 \\ -p_2p_1 & \vec{p}^2 - p_2^2 & -p_2p_3 \\ -p_3p_1 & -p_3p_2 & \vec{p}^2 - p_3^2 \end{pmatrix}\begin{pmatrix} \hat{A}^1(\vec{p}) \\ \hat{A}^2(\vec{p}) \\ \hat{A}^3(\vec{p}) \end{pmatrix}$$

$$\mathcal{M}(\vec{p})_{ij} = \vec{p}^2\left(\delta_{ij} - e_{p_i}e_{p_j}\right), \vec{e}_p = \frac{\vec{p}}{|\vec{p}|}$$

$$\vec{e}_1\cdot\vec{e}_p = \vec{e}_2\cdot\vec{e}_p = 0, \vec{e}_1\cdot\vec{e}_2 = 0, \vec{e}_1\times\vec{e}_2 = \vec{e}_p$$

$$\vec{e}_\pm = \frac{1}{\sqrt{2}}(\vec{e}_1 \pm \mathrm{i}\vec{e}_2), \vec{e}_\pm^*\cdot\vec{e}_\pm = 1, \vec{e}_\pm\cdot\vec{e}_p = 0, \vec{e}_-^*\cdot\vec{e}_+ = 0, \vec{e}_-\times\vec{e}_+ = \mathrm{i}\vec{e}_p$$

$$U(\vec{p}) = \begin{pmatrix} e_{+_1} & e_{+_2} & e_{+_3} \\ e_{p_1} & e_{p_2} & e_{p_3} \\ e_{-_1} & e_{-_2} & e_{-_3} \end{pmatrix}, U(\vec{p})\mathcal{M}(\vec{p})U(\vec{p})^\dagger = \vec{p}^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \hat{A}_+(\vec{p}) \\ \hat{A}_p(\vec{p}) \\ \hat{A}_-(\vec{p}) \end{pmatrix} = U(\vec{p})\begin{pmatrix} \hat{A}^1(\vec{p}) \\ \hat{A}^2(\vec{p}) \\ \hat{A}^3(\vec{p}) \end{pmatrix}, \begin{pmatrix} \hat{\Pi}_+(\vec{p}) \\ \hat{\Pi}_p(\vec{p}) \\ \hat{\Pi}_-(\vec{p}) \end{pmatrix} = U(\vec{p})\begin{pmatrix} \hat{\Pi}_1(\vec{p}) \\ \hat{\Pi}_2(\vec{p}) \\ \hat{\Pi}_3(\vec{p}) \end{pmatrix}$$

$$\hat{H} = \frac{1}{(2\pi)^3} \int d^3p \frac{1}{2} [\hat{\Pi}_+(\vec{p})^\dagger \hat{\Pi}_+(\vec{p}) + \vec{p}^2 \hat{A}_+(\vec{p})^\dagger \hat{A}_+(\vec{p})$$
$$+ \hat{\Pi}_-(\vec{p})^\dagger \hat{\Pi}_-(\vec{p}) + \vec{p}^2 \hat{A}_-(\vec{p})^\dagger \hat{A}_-(\vec{p}) + \hat{\Pi}_p(\vec{p})^\dagger \hat{\Pi}_p(\vec{p})].$$

$$\hat{\Pi}_p(\vec{p}) = U(\vec{p})_{pi} \hat{\Pi}_i(\vec{p}) = -e_{p_i} \hat{E}_i(\vec{p}) = -\frac{\vec{p}}{|\vec{p}|} \cdot \hat{\vec{E}}(\vec{p})$$

$$\hat{a}_\pm(\vec{p}) = \frac{1}{\sqrt{2}} \left[\sqrt{\omega} \hat{A}_\pm(\vec{p}) + \frac{\mathrm{i}}{\sqrt{\omega}} \hat{\Pi}_\pm(\vec{p}) \right]$$
$$\hat{a}_\pm(\vec{p})^\dagger = \frac{1}{\sqrt{2}} \left[\sqrt{\omega} \hat{A}_\pm(-\vec{p}) - \frac{\mathrm{i}}{\sqrt{\omega}} \hat{\Pi}_\pm(-\vec{p}) \right] (6.32)$$

$$[\hat{a}_\pm(\vec{p}), \hat{a}_\pm(\vec{q})^\dagger] = \frac{\mathrm{i}}{2} [\hat{\Pi}_\pm(\vec{p}), \hat{A}_\pm(-\vec{q})] - \frac{\mathrm{i}}{2} [\hat{A}_\pm(\vec{p}), \hat{\Pi}_\pm(-\vec{q})] = (2\pi)^3 \delta(\vec{p} - \vec{q})$$
$$[\hat{a}_+(\vec{p}), \hat{a}_-(\vec{q})^\dagger] = [\hat{a}_-(\vec{p}), \hat{a}_+(\vec{q})^\dagger] = 0$$
$$[\hat{a}_\pm(\vec{p}), \hat{a}_\pm(\vec{q})] = [\hat{a}_\pm(\vec{p}), \hat{a}_\mp(\vec{q})] = [\hat{a}_\pm(\vec{p})^\dagger, \hat{a}_\pm(\vec{q})^\dagger] = [\hat{a}_\pm(\vec{p})^\dagger, \hat{a}_\mp(\vec{q})^\dagger] = 0.$$

$$\hat{H} = \frac{1}{(2\pi)^3} \int d^3p |\vec{p}| \left(\hat{a}_+(\vec{p})^\dagger \hat{a}_+(\vec{p}) + \hat{a}_-(\vec{p})^\dagger \hat{a}_-(\vec{p}) + V \right)$$

$$\hat{a}_\pm(\vec{p}) |0\rangle = 0$$

$$E_0 = \frac{V}{(2\pi)^3} \int d^3p |\vec{p}|$$

$$\rho = \frac{E_0}{V} = \frac{1}{(2\pi)^3} \int d^3p |\vec{p}| = \frac{1}{2\pi^2} \int_0^\infty dp p^3$$

$$|\vec{p}, \pm\rangle = \hat{a}_\pm(\vec{p})^\dagger |0\rangle$$

$$E(\vec{p}) - E_0 = \omega = |\vec{p}|.$$

$$|\vec{p}_1, \pm; \vec{p}_2, \pm\rangle = \hat{a}_\pm(\vec{p}_1)^\dagger \hat{a}_\pm(\vec{p}_2)^\dagger |0\rangle$$

$$|\vec{p}_2, \pm; \vec{p}_1, \pm\rangle = |\vec{p}_1, \pm; \vec{p}_2, \pm\rangle$$

$$\mathcal{T}_{\mu\nu}(x) = -F_\mu{}^\rho(x) F_{\nu\rho}(x) - g_{\mu\nu} \mathcal{L}$$

$$\mathcal{H} = \mathcal{T}_{00}(x) = -F_0{}^i(x) F_{0i}(x) - g_{00} \mathcal{L} = E_i(x) E_i(x) - \mathcal{L}$$
$$= \frac{1}{2} (E_i(x) E_i(x) + B_i(x) B_i(x))$$

$$\mathcal{P}_i(x) = \mathcal{T}_{0i}(x) = -F_0{}^\rho(x) F_{i\rho}(x) - g_{0i} \mathcal{L} = -F_0{}^j(x) F_{ij}(x) = \epsilon_{ijk} E_j(x) B_k(x)$$

$$\hat{P}_i = \int d^3x \epsilon_{ijk} \frac{1}{2} \left(\hat{E}_j(\vec{x}) \hat{B}_k(\vec{x}) + \hat{B}_k(\vec{x}) \hat{E}_j(\vec{x}) \right)$$
$$= \frac{1}{(2\pi)^3} \int d^3p p_i \left(\hat{a}_+(\vec{p})^\dagger \hat{a}_+(\vec{p}) + \hat{a}_-(\vec{p})^\dagger \hat{a}_-(\vec{p}) \right)$$

$$[\hat{P}_i, \hat{a}_\pm(\vec{p})^\dagger] = p_i \hat{a}_\pm(\vec{p})^\dagger$$

$$\hat{P}_i|\vec{p},\pm\rangle = p_i|\vec{p},\pm\rangle$$

$$\vec{J} = \int d^3x\vec{x} \times (\vec{E}(\vec{x}) \times \vec{B}(\vec{x}))$$

$$\hat{\vec{J}} = \int d^3x\vec{x} \times \frac{1}{2}(\hat{\vec{E}}(\vec{x}) \times \hat{\vec{B}}(\vec{x}) - \hat{\vec{B}}(\vec{x}) \times \hat{\vec{E}}(\vec{x}))$$

$$[\hat{P}_i, \hat{P}_j] = 0, [\hat{P}_i, \hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{P}_k, [\hat{J}_i, \hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{J}_k$$

$$[\hat{P}_i, \hat{P}_j\hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{P}_j\hat{P}_k = 0$$

$$\left[\hat{\vec{J}} \cdot \vec{e}_p, \hat{a}_\pm(\vec{p})^\dagger\right] = \pm\hat{a}_\pm(\vec{p})^\dagger.$$

$$\hat{\vec{J}} \cdot \vec{e}_p|\vec{p},\pm\rangle = \hat{\vec{J}} \cdot \vec{e}_p\hat{a}_\pm(\vec{p})^\dagger|0\rangle = \left[\hat{\vec{J}} \cdot \vec{e}_p, \hat{a}_\pm(\vec{p})^\dagger\right]|0\rangle = \pm\hat{a}_\pm(\vec{p})^\dagger|0\rangle = \pm|\vec{p},\pm\rangle$$

$$\vec{p} = \frac{2\pi}{L}\vec{m}, m_i \in \mathbb{Z}$$

$$Z_\pm(\vec{p}) = \sum_{n_\pm(\vec{p})=0}^{\infty} \exp(-\beta n_\pm(\vec{p})|\vec{p}|) = \frac{1}{1 - \exp(-\beta|\vec{p}|)}$$

$$\begin{aligned}\langle n_\pm(\vec{p})|\vec{p}|\rangle &= \frac{1}{Z_\pm(\vec{p})}\sum_{n_\pm(\vec{p})=0}^{\infty} n_\pm(\vec{p})|\vec{p}|\exp(-\beta n_\pm(\vec{p})|\vec{p}|) \\ &= -\frac{1}{Z_\pm(\vec{p})}\frac{\partial Z_\pm(\vec{p})}{\partial\beta} = -\frac{\partial \log Z_\pm(\vec{p})}{\partial\beta} = \frac{|\vec{p}|}{\exp(\beta|\vec{p}|) - 1}\end{aligned}$$

$$\langle\hat{H}\rangle = \sum_{\vec{p},\pm}\langle n_\pm(\vec{p})|\vec{p}|\rangle \to 2\left(\frac{L}{2\pi}\right)^3 \int d^3p\langle n_\pm(\vec{p})|\vec{p}|\rangle.$$

$$u = \frac{1}{\pi^2}\int_0^\infty d\omega\frac{\omega^3}{\exp(\beta\omega) - 1}$$

$$\frac{du(\omega)}{d\omega} = \frac{1}{\pi^2}\frac{\omega^3}{\exp(\beta\omega) - 1}.$$

$$\frac{du(\omega)}{d\omega} = \frac{\omega^2}{\pi^2\beta}$$

$$u = \frac{\pi^2}{15\beta^4}$$

$$S[A] = \int d^dx\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

$${}^{\alpha}A_\mu(x) = A_\mu(x) - \partial_\mu\alpha(x),$$

$$Z=\int\mathcal{D}A\exp\left(-S[A]\right)$$

$$\mathcal{L}(A)=\frac{1}{2}A_\mu(x)\left[-\partial_\rho\partial_\rho\delta_{\mu\nu}+\partial_\mu\partial_\nu\right]A_\nu(x)$$

$$G_{\mu\nu}(p)=\frac{1}{p^2\delta_{\mu\nu}-p_\mu p_\nu}$$

$$\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)p_\mu=0$$

$$\partial_\mu{}^{\alpha}A_\mu(x)=\partial_\mu\left(A_\mu(x)-\partial_\mu\alpha(x)\right)=0\ \Rightarrow\ \alpha(x)=\square^{-1}\,\partial_\mu A_\mu(x),\square=\partial_\mu\partial_\mu$$

$$C(x)=\partial_\mu{}^{\alpha}A_\mu(x),$$

$$\int\mathcal{D}C\exp\left(-S_{\rm gf}[C]\right)=\int\mathcal{D}C\exp\left(-\int d^dx\frac{1}{2\xi}C^2\right),S_{\rm gf}[C]=\int d^dx\frac{1}{2\xi}\left(\partial_\mu{}^{\alpha}A_\mu\right)^2$$

$$\mathcal{D}C=\mathcal{D}\alpha\det\left(\frac{\delta C}{\delta\alpha}\right)=\mathcal{D}\alpha\det\left(\frac{\delta\partial_\mu{}^{\alpha}A_\mu}{\delta\alpha}\right)$$

$$\delta C(x)=\delta\partial_\mu{}^{\alpha}A_\mu(x)=\partial_\mu{}^{\alpha}A_\mu(x)-\partial_\mu A_\mu(x)=-\partial_\mu\partial_\mu\alpha(x)$$

$$\det\left(\frac{\delta C}{\delta\alpha}\right)=\det\left(\frac{\delta\partial_\mu{}^{\alpha}A_\mu}{\delta\alpha}\right)=\det\left(-\partial_\mu\partial_\mu\right)$$

$$\begin{aligned}Z&=\int\mathcal{D}C\int\mathcal{D}A\exp\left(-S[A]-S_{\rm gf}[{}^{\alpha}A]\right)\\&=\int\mathcal{D}\alpha\det\left(-\partial_\mu\partial_\mu\right)\int\mathcal{D}A\exp\left(-S[A]-S_{\rm gf}[{}^{\alpha}A]\right)\\&=\int\mathcal{D}\alpha\det\left(-\partial_\mu\partial_\mu\right)\int\mathcal{D}^{\alpha}A\exp\left(-S[{}^{\alpha}A]-S_{\rm gf}[{}^{\alpha}A]\right)\\&=\int\mathcal{D}\alpha\det\left(-\partial_\mu\partial_\mu\right)\int\mathcal{D}A\exp\left(-S[A]-S_{\rm gf}[A]\right)\\&\propto\int\mathcal{D}A\exp\left(-S[A]-S_{\rm gf}[A]\right)\end{aligned}$$

$$S[A]+S_{\rm gf}[A]=\int d^dx\left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{1}{2\xi}\left(\partial_\mu A_\mu\right)^2\right]$$

$$S[A]+S_{\rm gf}[A]=\frac{1}{(2\pi)^d}\int d^dp\frac{1}{2}A_\mu(-p)G_{\mu\nu}(p)^{-1}A_\nu(p)$$

$$G_{\mu\nu}(p)=\frac{1}{p^2\delta_{\mu\nu}-p_\mu p_\nu(1-1/\xi)}$$

$$\hat{\vec{P}}=\int d^3x\frac{1}{2}\left(\hat{\vec{E}}(\vec{x})\times\hat{\vec{B}}(\vec{x})-\hat{\vec{B}}(\vec{x})\times\hat{\vec{E}}(\vec{x})\right)$$

$$\hat{\vec{J}}=\int d^3x\vec{x}\times\frac{1}{2}\left(\hat{\vec{E}}(\vec{x})\times\hat{\vec{B}}(\vec{x})-\hat{\vec{B}}(\vec{x})\times\hat{\vec{E}}(\vec{x})\right)$$

$$\left[\hat{P}_i,\hat{P}_j\right]=0,\left[\hat{P}_i,\hat{J}_j\right]=i\epsilon_{ijk}\hat{P}_k,\left[\hat{J}_i,\hat{J}_j\right]=i\epsilon_{ijk}\hat{J}_k$$

$$Z=\int \mathcal{D}\Phi\exp(-S[\Phi]), S[\Phi]=\int d^4x\mathcal{L}(\Phi,\partial_\mu\Phi)$$

$$\mathcal{L}(\Phi,\partial_\mu\Phi)=\frac{1}{2}\partial_\mu\Phi^*\partial_\mu\Phi+V(\Phi)=\frac{1}{2}\partial_\mu\phi_1\partial_\mu\phi_1+\frac{1}{2}\partial_\mu\phi_2\partial_\mu\phi_2+V(\phi_1,\phi_2)$$

$$V(\Phi)=\frac{m^2}{2}|\Phi|^2+\frac{\lambda}{4!}|\Phi|^4, |\Phi|^2=\Phi^*\Phi=\phi_1^2+\phi_2^2$$

$$\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}-\frac{\delta\mathcal{L}}{\delta\Phi}=(\partial_\mu\partial_\mu-m^2)\Phi=0$$

$$\Phi'(x)=\exp(\mathrm{i}e\alpha)\Phi(x) \Leftrightarrow \Phi'(x)^*=\exp(-\mathrm{i}e\alpha)\Phi(x)^*,$$

$$^{\mathrm{C}}\Phi(x)=\Phi(x)^*, S[^{\mathrm{C}}\Phi]=S[\Phi].$$

$$\Phi(\vec{p},x_4)=\int d^3x\Phi(\vec{x},x_4)\exp(-\mathrm{i}\vec{p}\cdot\vec{x})$$
$$\langle\Phi(\vec{p},0)\Phi(\vec{p},x_4)^*\rangle\sim\exp(-E(\vec{p})x_4)$$

$$\Phi'(x)=\exp(\mathrm{i}e\alpha(x))\Phi(x)$$

$$\partial_\mu\Phi'(x)=\exp(\mathrm{i}e\alpha(x))[\partial_\mu\Phi(x)+\mathrm{i}e\partial_\mu\alpha(x)\Phi(x)]$$

$$D_\mu\Phi(x)=[\partial_\mu+\mathrm{i}eA_\mu(x)]\Phi(x)$$

$$A'_\mu(x)=A_\mu(x)-\partial_\mu\alpha(x)\Rightarrow$$
$$(D_\mu\Phi)'(x)=[\partial_\mu+\mathrm{i}eA'_\mu(x)]\Phi'(x)=\exp(\mathrm{i}e\alpha(x))D_\mu\Phi(x)$$

$$\mathcal{L}(\Phi,\partial_\mu\Phi,A_\mu)=\frac{1}{2}(D_\mu\Phi)^*D_\mu\Phi+V(\Phi)$$

$$(D_\mu\Phi)^{*\prime}(x)=\exp(-\mathrm{i}e\alpha(x))D_\mu\Phi^*(x)$$

$$F_{\mu\nu}(x)=\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)$$

$$F'_{\mu\nu}(x)=\partial_\mu A'_\nu(x)-\partial_\nu A'_\mu(x)=\partial_\mu A_\nu(x)-\partial_\mu\partial_\nu\alpha(x)-\partial_\nu A_\mu(x)+\partial_\nu\partial_\mu\alpha(x)=F_{\mu\nu}(x)$$

$$\mathcal{L}(\Phi,\partial_\mu\Phi,A_\mu,\partial_\mu A_\nu)=\frac{1}{2}(D_\mu\Phi)^*D_\mu\Phi+V(\Phi)+\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

$$^{\mathrm{C}}\Phi(x)=\Phi(x)^*, {}^{\mathrm{C}}A_\mu(x)=-A_\mu(x)\Rightarrow$$
$$^{\mathrm{C}}(D_\mu\Phi(x))=[\partial_\mu+\mathrm{i}e{}^{\mathrm{C}}A_\mu(x)]^{\mathrm{C}}\Phi(x)=[\partial_\mu-\mathrm{i}eA_\mu(x)]\Phi(x)^*=(D_\mu\Phi(x))^*$$

$$\partial_\mu F_{\mu\nu}(x)=j_\nu(x)=\frac{\mathrm{i}e}{2}(D_\nu\Phi^*(x)\Phi(x)-\Phi^*(x)D_\nu\Phi(x))$$

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}(x)=\epsilon_{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma(x)=0$$

$$\partial_i A'_i(\vec{x})=\partial_i(A_i(\vec{x})-\partial_i\alpha_{\mathrm{C}}(\vec{x}))=0\Rightarrow \alpha_{\mathrm{C}}(\vec{x})=\Delta^{-1}\partial_i A_i(\vec{x})$$

$$\alpha_C'(\vec{x}) = \Delta^{-1}\partial_i A_i'(\vec{x}) = \Delta^{-1}\partial_i(A_i(\vec{x}) - \partial_i\alpha(\vec{x})) = \alpha_C(\vec{x}) - \alpha(\vec{x})$$

$$\Phi_C(\vec{x}) = \exp\,(\mathrm{i}e\alpha_C(\vec{x}))\Phi(\vec{x}) = \exp\,(\mathrm{i}e\Delta^{-1}\partial_i A_i(\vec{x}))\Phi(\vec{x})$$

$$\Phi'(\vec{x}) = \exp\,(\mathrm{i}e\alpha)\Phi(\vec{x}), A_i'(\vec{x}) = A_i(\vec{x}) \;\Rightarrow\; \Phi_C'(\vec{x}) = \exp\,(\mathrm{i}e\alpha)\Phi_C(\vec{x})$$

$$\Phi_C(\vec{p}, x_4) = \int\; d^3x\Phi_C(\vec{x}, x_4)\exp\,(-\mathrm{i}\vec{p}\cdot\vec{x})$$
$$\langle\Phi_C(\vec{p}, 0)\Phi_C(\vec{p}, x_4)^*\rangle \sim \exp\,(-E(\vec{p})x_4)$$

$$Q = \int_{T^3} d^3x\partial_i E_i = \int_{\partial T^3} d^2\sigma_i E_i = 0$$

$$\alpha(\vec{x}) = \frac{2\pi}{eL}\vec{n}\cdot\vec{x}, n_i \in \mathbb{Z}$$

$$A_i'(\vec{x}) = A_i(\vec{x}) - \partial_i\alpha(\vec{x}) = A_i(\vec{x}) - \frac{2\pi}{eL}n_i$$

$$\prod_{a=1}^{N}\Phi_C(\vec{x}_a)\Phi_C(\vec{y}_a)^* = \prod_{a=1}^{N}\exp\left(\mathrm{i}e\Delta^{-1}\partial_i A_i(\vec{x}_a)\right)\Phi(\vec{x}_a)\exp\left(-\mathrm{i}e\Delta^{-1}\partial_i A_i(\vec{y}_a)\right)\Phi(\vec{y}_a)^*$$

$$\prod_{a=1}^{N}\Phi_C'(\vec{x}_a)\Phi_C'(\vec{y}_a)^* = \prod_{a=1}^{N}\exp\left(\mathrm{i}\frac{2\pi}{L}\vec{n}\cdot(\vec{x}_a - \vec{y}_a)\right)\Phi_C(\vec{x}_a)\Phi_C(\vec{y}_a)^*$$

$$\vec{\theta} = \frac{2\pi}{L}\sum_{a=1}^{N}(\vec{x}_a - \vec{y}_a)$$

$$W_{\mathcal{C}} = \exp\left(\mathrm{i}\int_{\mathcal{C}} dl_i A_i(\vec{x})\right)$$

$$W_{\mathcal{C}}' = \exp\left(\mathrm{i}\frac{2\pi}{e}\vec{m}\cdot\vec{n}\right)W_{\mathcal{C}}$$

$$\Phi(\vec{x} + L\vec{e}_i) = {}^{C}\Phi(\vec{x}) = \Phi(\vec{x})^*$$
$$A_\mu(\vec{x} + L\vec{e}_i) = {}^{C}A_\mu(\vec{x}) = -A_\mu(\vec{x}), \alpha(\vec{x} + L\vec{e}_i) = -\alpha(\vec{x})$$

$$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3) = \left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right).$$

$$\hat{H}_R = \int\; d^3x\hat{\psi}_R^\dagger(\vec{x})(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_R(\vec{x})$$

$$\hat{\psi}_R(\vec{x}) = \begin{pmatrix}\hat{\psi}_R^1(\vec{x})\\ \hat{\psi}_R^2(\vec{x})\end{pmatrix}, \hat{\psi}_R^\dagger(\vec{x}) = \left(\hat{\psi}_R^{1\dagger}(\vec{x}), \hat{\psi}_R^{2\dagger}(\vec{x})\right)$$

$$\{\hat{\psi}_R^a(\vec{x}), \hat{\psi}_R^{b\dagger}(\vec{y})\} = \delta_{ab}\delta(\vec{x} - \vec{y})$$
$$\{\hat{\psi}_R^a(\vec{x}), \hat{\psi}_R^b(\vec{y})\} = \{\hat{\psi}_R^{a\dagger}(\vec{x}), \hat{\psi}_R^{b\dagger}(\vec{y})\} = 0$$

$$\hat{\psi}_{\mathrm{R}}(\vec{p}) = \int d^3x \hat{\psi}_{\mathrm{R}}(\vec{x}) \exp(-\mathrm{i}\vec{p}\cdot\vec{x}), \hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{p}) = \int d^3x \hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{x}) \exp(\mathrm{i}\vec{p}\cdot\vec{x})$$

$$\{\hat{\psi}_{\mathrm{R}}^{a}(\vec{p}), \hat{\psi}_{\mathrm{R}}^{b\dagger}(\vec{q})\} = (2\pi)^3 \delta_{ab}\delta(\vec{p}-\vec{q})$$
$$\{\hat{\psi}_{\mathrm{R}}^{a}(\vec{p}), \hat{\psi}_{\mathrm{R}}^{b}(\vec{q})\} = \{\hat{\psi}_{\mathrm{R}}^{a\dagger}(\vec{p}), \hat{\psi}_{\mathrm{R}}^{b\dagger}(\vec{q})\} = 0$$

$$\hat{H}_{\mathrm{R}} = \frac{1}{(2\pi)^3}\int d^3p \hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{p})\vec{\sigma}\cdot\vec{p}\hat{\psi}_{\mathrm{R}}(\vec{p})$$

$$U(\vec{p})(\vec{\sigma}\cdot\vec{p})U(\vec{p})^{\dagger} = |\vec{p}|\sigma^3$$

$$\vec{p} = |\vec{p}|\vec{e}_p, \vec{e}_p = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

$$U(\vec{p}) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)\exp(-\mathrm{i}\varphi) \\ -\sin(\theta/2)\exp(\mathrm{i}\varphi) & \cos(\theta/2) \end{pmatrix}$$

$$\hat{\psi}_{\mathrm{R}}(\vec{p}) = \begin{pmatrix} \hat{\psi}_{\mathrm{R}}^{1}(\vec{p}) \\ \hat{\psi}_{\mathrm{R}}^{2}(\vec{p}) \end{pmatrix} = U(\vec{p})^{\dagger}\begin{pmatrix} \hat{c}_{\mathrm{R}}(\vec{p}) \\ \hat{d}_{\mathrm{R}}^{\dagger}(-\vec{p}) \end{pmatrix}$$

$$\{\hat{c}_{\mathrm{R}}(\vec{p}), \hat{c}_{\mathrm{R}}^{\dagger}(\vec{q})\} = (2\pi)^3\delta(\vec{p}-\vec{q}), \{\hat{d}_{\mathrm{R}}(\vec{p}), \hat{d}_{\mathrm{R}}^{\dagger}(\vec{q})\} = (2\pi)^3\delta(\vec{p}-\vec{q})$$
$$\{\hat{c}_{\mathrm{R}}(\vec{p}), \hat{c}_{\mathrm{R}}(\vec{q})\} = \{\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p}), \hat{c}_{\mathrm{R}}^{\dagger}(\vec{q})\} = 0, \{\hat{d}_{\mathrm{R}}(\vec{p}), \hat{d}_{\mathrm{R}}(\vec{q})\} = \{\hat{d}_{\mathrm{R}}^{\dagger}(\vec{p}), \hat{d}_{\mathrm{R}}^{\dagger}(\vec{q})\} = 0$$
$$\{\hat{c}_{\mathrm{R}}(\vec{p}), \hat{d}_{\mathrm{R}}(\vec{q})\} = \{\hat{c}_{\mathrm{R}}(\vec{p}), \hat{d}_{\mathrm{R}}^{\dagger}(\vec{q})\} = \{\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p}), \hat{d}_{\mathrm{R}}(\vec{q})\} = \{\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p}), \hat{d}_{\mathrm{R}}^{\dagger}(\vec{q})\} = 0$$

$$\begin{aligned} \hat{H}_{\mathrm{R}} &= \frac{1}{(2\pi)^3}\int d^3p|\vec{p}|\left[\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\hat{c}_{\mathrm{R}}(\vec{p}) - \hat{d}_{\mathrm{R}}(\vec{p})\hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})\right] \\ &= \frac{1}{(2\pi)^3}\int d^3p|\vec{p}|\left[\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\hat{c}_{\mathrm{R}}(\vec{p}) + \hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})\hat{d}_{\mathrm{R}}(\vec{p}) - V\right] \end{aligned}$$

$$\hat{c}_{\mathrm{R}}(\vec{p})|0\rangle_{\mathrm{R}} = \hat{d}_{\mathrm{R}}(\vec{p})|0\rangle_{\mathrm{R}} = 0$$

$$\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{R}} = \left|\vec{p}, \vec{\sigma}\cdot\vec{e}_p = 1\right\rangle_{\mathrm{R}}, \hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{R}} = \left|\vec{p}, \vec{\sigma}\cdot\vec{e}_p = -1\right\rangle_{\mathrm{R}}$$

$$\begin{aligned} \left|\vec{p}_1, \vec{\sigma}\cdot\vec{e}_{p_1} = 1; \vec{p}_2, \vec{\sigma}\cdot\vec{e}_{p_2} = 1\right\rangle_{\mathrm{R}} &= c_{\mathrm{R}}^{\dagger}(\vec{p}_1)c_{\mathrm{R}}^{\dagger}(\vec{p}_2)|0\rangle_{\mathrm{R}} = -c_{\mathrm{R}}^{\dagger}(\vec{p}_2)c_{\mathrm{R}}^{\dagger}(\vec{p}_1)|0\rangle_{\mathrm{R}} \\ &= -\left|\vec{p}_2, \vec{\sigma}\cdot\vec{e}_{p_2} = 1; \vec{p}_1, \vec{\sigma}\cdot\vec{e}_{p_1} = 1\right\rangle_{\mathrm{R}} \end{aligned}$$

$$\rho = \frac{E_0}{V} = -\frac{1}{(2\pi)^3}\int d^3p|\vec{p}|$$

$$\hat{H}_{\mathrm{L}} = \int d^3x \hat{\psi}_{\mathrm{L}}^{\dagger}(\vec{x})(\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_{\mathrm{L}}(\vec{x}) = \frac{1}{(2\pi)^3}\int d^3p|\vec{p}|\left[\hat{c}_{\mathrm{L}}^{\dagger}(\vec{p})\hat{c}_{\mathrm{L}}(\vec{p}) + \hat{d}_{\mathrm{L}}^{\dagger}(\vec{p})\hat{d}_{\mathrm{L}}(\vec{p}) - V\right]$$

$$\hat{\psi}_{\mathrm{L}}(\vec{p}) = \begin{pmatrix} \hat{\psi}_{\mathrm{L}}^{1}(\vec{p}) \\ \hat{\psi}_{\mathrm{L}}^{2}(\vec{p}) \end{pmatrix} = U(-\vec{p})^{\dagger}\begin{pmatrix} \hat{c}_{\mathrm{L}}(\vec{p}) \\ \hat{d}_{\mathrm{L}}^{\dagger}(-\vec{p}) \end{pmatrix}, U(-\vec{p})(-\vec{\sigma}\cdot\vec{p})U(-\vec{p})^{\dagger} = |\vec{p}|\sigma^3,$$

$$\hat{c}_{\mathrm{L}}(\vec{p})|0\rangle_{\mathrm{L}} = \hat{d}_{\mathrm{L}}(\vec{p})|0\rangle_{\mathrm{L}} = 0$$

$$\hat{c}_{\mathrm{L}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{L}} = \left|\vec{p}, \vec{\sigma}\cdot\vec{e}_p = -1\right\rangle_{\mathrm{L}}, \hat{d}_{\mathrm{L}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{L}} = \left|\vec{p}, \vec{\sigma}\cdot\vec{e}_p = 1\right\rangle_{\mathrm{L}}$$

$$\hat{\vec{P}}_{\mathrm{R}} = \int d^3x \hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{x})(-\mathrm{i}\vec{\nabla})\hat{\psi}_{\mathrm{R}}(\vec{x})$$
$$\hat{\vec{J}}_{\mathrm{R}} = \int d^3x \hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{x})\left(\vec{x} \times (-\mathrm{i}\vec{\nabla}) + \frac{1}{2}\vec{\sigma}\right)\hat{\psi}_{\mathrm{R}}(\vec{x})$$
$$\hat{\vec{K}}_{\mathrm{R}} = \int d^3x \hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{x})\frac{1}{2}(\vec{x}(-\mathrm{i}\vec{\nabla} \cdot \vec{\sigma}) + (-\mathrm{i}\vec{\nabla} \cdot \vec{\sigma})\vec{x})\hat{\psi}_{\mathrm{R}}(\vec{x})$$

$$\hat{\vec{P}}_{\mathrm{R}} = \frac{1}{(2\pi)^3}\int d^3p\vec{p}[\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\hat{c}_{\mathrm{R}}(\vec{p}) + \hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})\hat{d}_{\mathrm{R}}(\vec{p})]$$

$$\left[\hat{\vec{P}}_{\mathrm{R}}, \hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\right] = \vec{p}\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p}), \left[\hat{\vec{P}}_{\mathrm{R}}, \hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})\right] = \vec{p}\hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})$$

$$\hat{\vec{P}}_{\mathrm{R}}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{R}} = \hat{\vec{P}}_{\mathrm{R}}\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{R}} = \left(\left[\hat{\vec{P}}_{\mathrm{R}}, \hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\right] + \hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\hat{\vec{P}}_{\mathrm{R}}\right)|0\rangle_{\mathrm{R}}$$
$$= \vec{p}\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{R}} = \vec{p}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{R}}$$

$$\left[\hat{\vec{J}}_{\mathrm{R}} \cdot \vec{e}_p, \hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\right] = \frac{1}{2}\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p}), \left[\hat{\vec{J}}_{\mathrm{R}} \cdot \vec{e}_p, \hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})\right] = -\frac{1}{2}\hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})$$

$$\hat{\vec{J}}_{\mathrm{R}} \cdot \vec{e}_p\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{R}} = \left[\hat{\vec{J}}_{\mathrm{R}} \cdot \vec{e}_p, \hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\right]|0\rangle_{\mathrm{R}} = \frac{1}{2}\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{R}} = \frac{1}{2}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{R}}$$
$$\hat{\vec{J}}_{\mathrm{R}} \cdot \vec{e}_p\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = -1\right\rangle_{\mathrm{R}} = \left[\hat{\vec{J}}_{\mathrm{R}} \cdot \vec{e}_p, \hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})\right]|0\rangle_{\mathrm{R}} = -\frac{1}{2}\hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{R}} = -\frac{1}{2}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = -1\right\rangle_{\mathrm{R}}$$

$$\left[\hat{\vec{J}}_{\mathrm{L}} \cdot \vec{e}_p, \hat{c}_{\mathrm{L}}^{\dagger}(\vec{p})\right] = -\frac{1}{2}\hat{c}_{\mathrm{L}}^{\dagger}(\vec{p}), \left[\hat{\vec{J}}_{\mathrm{L}} \cdot \vec{e}_p, \hat{d}_{\mathrm{L}}^{\dagger}(\vec{p})\right] = \frac{1}{2}\hat{d}_{\mathrm{L}}^{\dagger}(\vec{p})$$
$$\hat{\vec{J}}_{\mathrm{L}} \cdot \vec{e}_p\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = -1\right\rangle_{\mathrm{L}} = \left[\hat{\vec{J}}_{\mathrm{L}} \cdot \vec{e}_p, \hat{c}_{\mathrm{L}}^{\dagger}(\vec{p})\right]|0\rangle_{\mathrm{L}} = -\frac{1}{2}\hat{c}_{\mathrm{L}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{L}} = -\frac{1}{2}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = -1\right\rangle_{\mathrm{L}}$$
$$\hat{\vec{J}}_{\mathrm{L}} \cdot \vec{e}_p\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{L}} = \left[\hat{\vec{J}}_{\mathrm{L}} \cdot \vec{e}_p, \hat{d}_{\mathrm{L}}^{\dagger}(\vec{p})\right]|0\rangle_{\mathrm{L}} = \frac{1}{2}\hat{d}_{\mathrm{L}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{L}} = \frac{1}{2}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{L}}$$

$$\hat{F}_{\mathrm{R}} = \frac{1}{(2\pi)^3}\int d^3p[\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\hat{c}_{\mathrm{R}}(\vec{p}) - \hat{d}_{\mathrm{R}}^{\dagger}(\vec{p})\hat{d}_{\mathrm{R}}(\vec{p})]$$

$$\hat{U}_{\mathrm{R}}(\chi_{\mathrm{R}}) = \exp\left(\mathrm{i}\chi_{\mathrm{R}}\hat{F}_{\mathrm{R}}\right), \chi_{\mathrm{R}} \in \mathbb{R}$$

$$\hat{U}_{\mathrm{R}}(\chi_{\mathrm{R}})\hat{\psi}_{\mathrm{R}}(\vec{x})\hat{U}_{\mathrm{R}}(\chi_{\mathrm{R}})^{\dagger} = \exp(\mathrm{i}\chi_{\mathrm{R}})\hat{\psi}_{\mathrm{R}}(\vec{x})$$
$$\hat{U}_{\mathrm{R}}(\chi_{\mathrm{R}})\hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{x})\hat{U}_{\mathrm{R}}(\chi_{\mathrm{R}})^{\dagger} = \hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{x})\exp(-\mathrm{i}\chi_{\mathrm{R}})$$

$$\hat{F}_{\mathrm{L}} = \frac{1}{(2\pi)^3}\int d^3p[\hat{c}_{\mathrm{L}}^{\dagger}(\vec{p})\hat{c}_{\mathrm{L}}(\vec{p}) - \hat{d}_{\mathrm{L}}^{\dagger}(\vec{p})\hat{d}_{\mathrm{L}}(\vec{p})]$$

$$\hat{F}_{\mathrm{R}}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{R}} = \left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{R}}\hat{F}_{\mathrm{R}}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = -1\right\rangle_{\mathrm{R}} = -\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = -1\right\rangle_{\mathrm{R}}$$
$$\hat{F}_{\mathrm{L}}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = -1\right\rangle_{\mathrm{L}} = \left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = -1\right\rangle_{\mathrm{L}}, \hat{F}_{\mathrm{L}}\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{L}} = -\left|\vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{L}}$$

$$^{\mathrm{P}}\hat{\psi}_{\mathrm{R}}(\vec{x}) = \hat{U}_{\mathrm{P}}\hat{\psi}_{\mathrm{R}}(\vec{x})\hat{U}_{\mathrm{P}}^{\dagger} = \hat{\psi}_{\mathrm{L}}(-\vec{x})$$
$$^{\mathrm{P}}\hat{\psi}_{\mathrm{L}}(\vec{x}) = \hat{U}_{\mathrm{P}}\hat{\psi}_{\mathrm{L}}(\vec{x})\hat{U}_{\mathrm{P}}^{\dagger} = \hat{\psi}_{\mathrm{R}}(-\vec{x})$$

$$
{}^{\mathrm{P}}\hat{H}_{\mathrm{R}} = \hat{U}_{\mathrm{P}}\hat{H}_{\mathrm{R}}\hat{U}_{\mathrm{P}}^{\dagger} = \int d^3x\,{}^{\mathrm{P}}\hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{x})(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\,{}^{\mathrm{P}}\hat{\psi}_{\mathrm{R}}(\vec{x})
$$

$$
= \int d^3x\hat{\psi}_{\mathrm{L}}^{\dagger}(-\vec{x})(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_{\mathrm{L}}(-\vec{x}) = \int d^3x\hat{\psi}_{\mathrm{L}}^{\dagger}(\vec{x})(\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_{\mathrm{L}}(\vec{x}) = \hat{H}_{\mathrm{L}}
$$

$$
[\hat{H}_{\mathrm{R}} + \hat{H}_{\mathrm{L}}, \hat{U}_{\mathrm{P}}] = 0
$$

$$
{}^{\mathrm{C}}\hat{\psi}_{\mathrm{R}}(\vec{x}) = \hat{U}_{\mathrm{C}}\hat{\psi}_{\mathrm{R}}(\vec{x})\hat{U}_{\mathrm{C}}^{\dagger} = \mathrm{i}\sigma^2\hat{\psi}_{\mathrm{L}}^{\dagger}(\vec{x})^{\mathsf{T}}
$$
$$
{}^{\mathrm{C}}\hat{\psi}_{\mathrm{L}}(\vec{x}) = \hat{U}_{\mathrm{C}}\hat{\psi}_{\mathrm{L}}(\vec{x})\hat{U}_{\mathrm{C}}^{\dagger} = -\mathrm{i}\sigma^2\hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{x})^{\mathsf{T}}
$$

$$
{}^{\mathrm{C}}\hat{H}_{\mathrm{R}} = \hat{U}_{\mathrm{C}}\hat{H}_{\mathrm{R}}\hat{U}_{\mathrm{C}}^{\dagger} = \int d^3x\,{}^{\mathrm{C}}\hat{\psi}_{\mathrm{R}}^{\dagger}(\vec{x})(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\,{}^{\mathrm{C}}\hat{\psi}_{\mathrm{R}}(\vec{x})
$$
$$
= \int d^3x\hat{\psi}_{\mathrm{L}}(\vec{x})^{\mathsf{T}}(\mathrm{i}\sigma^2)^{\dagger}(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\mathrm{i}\sigma^2\hat{\psi}_{\mathrm{L}}^{\dagger}(\vec{x})^{\mathsf{T}}
$$
$$
= \int d^3x\hat{\psi}_{\mathrm{L}}(\vec{x})^{\mathsf{T}}\left(\mathrm{i}\vec{\sigma}^{\mathsf{T}}\cdot\vec{\nabla}\right)\hat{\psi}_{\mathrm{L}}^{\dagger}(\vec{x})^{\mathsf{T}} = \int d^3x\left(-\vec{\nabla}\hat{\psi}_{\mathrm{L}}^{\dagger}(\vec{x})\cdot\mathrm{i}\vec{\sigma}\right)\hat{\psi}_{\mathrm{L}}(\vec{x})
$$
$$
= \int d^3x\hat{\psi}_{\mathrm{L}}^{\dagger}(\vec{x})(\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_{\mathrm{L}}(\vec{x}) = \hat{H}_{\mathrm{L}}
$$

$$
\hat{\psi}_{\mathrm{L}}(\vec{x})^{\mathsf{T}}\left(\mathrm{i}\vec{\sigma}^{\mathsf{T}}\cdot\vec{\nabla}\right)\hat{\psi}_{\mathrm{L}}^{\dagger}(\vec{x})^{\mathsf{T}} = -\left(\vec{\nabla}\hat{\psi}_{\mathrm{L}}^{\dagger}(\vec{x})\cdot\mathrm{i}\vec{\sigma}\right)\hat{\psi}_{\mathrm{L}}(\vec{x})
$$

$$
\begin{pmatrix} {}^{\mathrm{e}}\hat{c}_{\mathrm{R}}(\vec{p}) \\ {}^{\mathrm{P}}\hat{c}_{\mathrm{L}}(\vec{p}) \end{pmatrix} = \sigma^1\begin{pmatrix} \hat{c}_{\mathrm{R}}(-\vec{p}) \\ \hat{c}_{\mathrm{L}}(-\vec{p}) \end{pmatrix}, \begin{pmatrix} {}^{\mathrm{P}}\hat{d}_{\mathrm{R}}(\vec{p}) \\ {}^{\mathrm{P}}\hat{d}_{\mathrm{L}}(\vec{p}) \end{pmatrix} = \sigma^1\begin{pmatrix} \hat{d}_{\mathrm{R}}(-\vec{p}) \\ \hat{d}_{\mathrm{L}}(-\vec{p}) \end{pmatrix}
$$
$$
\begin{pmatrix} {}^{\mathrm{C}}\hat{c}_{\mathrm{R}}(\vec{p}) \\ {}^{\mathrm{C}}\hat{c}_{\mathrm{L}}(\vec{p}) \end{pmatrix} = \mathrm{i}\sigma^2\begin{pmatrix} \hat{d}_{\mathrm{R}}(\vec{p}) \\ \hat{d}_{\mathrm{L}}(\vec{p}) \end{pmatrix}, \begin{pmatrix} {}^{\mathrm{C}}\hat{d}_{\mathrm{R}}(\vec{p}) \\ {}^{\mathrm{C}}\hat{d}_{\mathrm{L}}(\vec{p}) \end{pmatrix} = -\mathrm{i}\sigma^2\begin{pmatrix} \hat{c}_{\mathrm{R}}(\vec{p}) \\ \hat{c}_{\mathrm{L}}(\vec{p}) \end{pmatrix}
$$
$$
\begin{pmatrix} {}^{\mathrm{CP}}\hat{c}_{\mathrm{R}}(\vec{p}) \\ {}^{\mathrm{CP}}\hat{c}_{\mathrm{L}}(\vec{p}) \end{pmatrix} = -\sigma^3\begin{pmatrix} \hat{d}_{\mathrm{R}}(-\vec{p}) \\ \hat{d}_{\mathrm{L}}(-\vec{p}) \end{pmatrix}, \begin{pmatrix} {}^{\mathrm{CP}}\hat{d}_{\mathrm{R}}(\vec{p}) \\ {}^{\mathrm{CP}}\hat{d}_{\mathrm{L}}(\vec{p}) \end{pmatrix} = \sigma^3\begin{pmatrix} \hat{c}_{\mathrm{R}}(-\vec{p}) \\ \hat{c}_{\mathrm{L}}(-\vec{p}) \end{pmatrix}
$$

$$
\hat{U}_{\mathrm{CP}}\left|\vec{p},\vec{\sigma}\cdot\vec{e}_p = 1\right\rangle_{\mathrm{R}} = \hat{U}_{\mathrm{CP}}\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})|0\rangle_{\mathrm{R}} = {}^{\mathrm{CP}}\hat{c}_{\mathrm{R}}^{\dagger}(\vec{p})\hat{U}_{\mathrm{CP}}|0\rangle_{\mathrm{R}} = -\hat{d}_{\mathrm{R}}^{\dagger}(-\vec{p})|0\rangle_{\mathrm{R}} = -\left|-\vec{p},-\vec{\sigma}\cdot\vec{e}_p = -1\right\rangle_{\mathrm{R}}
$$

.

$$
\hat{U}_{\mathrm{CP}}\left|\vec{p},\vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{R}} = \mp\left|-\vec{p},-\vec{\sigma}\cdot\vec{e}_p = \mp 1\right\rangle_{\mathrm{R}},
$$
$$
\hat{U}_{\mathrm{CP}}\left|\vec{p},\vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{L}} = \mp\left|-\vec{p},-\vec{\sigma}\cdot\vec{e}_p = \mp 1\right\rangle_{\mathrm{L}}.
$$

$$
Z = \mathrm{Tr}\exp\left(-\beta(\hat{H}_{\mathrm{L}} - \mu\hat{F}_{\mathrm{L}})\right)
$$

$$
Z_{\pm}(\vec{p}) = \sum_{n=0}^{1}\exp\left(-\beta n(|\vec{p}| \pm \mu)\right) = 1 + \exp\left(-\beta(|\vec{p}| \pm \mu)\right)
$$

$$
\left\langle\hat{H}_{\mathrm{L}}\right\rangle_{\pm}(\vec{p}) = -\frac{\partial\log Z_{\pm}(\vec{p})}{\partial\beta} = \frac{|\vec{p}|}{\exp\left(\beta(|\vec{p}| \pm \mu)\right) + 1}
$$

$$
\rho = \frac{1}{L^3}\left\langle\hat{H}_{\mathrm{L}}\right\rangle = \frac{1}{L^3}\sum_{\vec{p}}\left(\left\langle\hat{H}_{\mathrm{L}}\right\rangle_{+}(\vec{p}) + \left\langle\hat{H}_{\mathrm{L}}\right\rangle_{-}(\vec{p})\right)
$$
$$
\to \frac{1}{(2\pi)^3}\int d^3p\left(\frac{|\vec{p}|}{\exp\left(\beta(|\vec{p}| + \mu)\right) + 1} + \frac{|\vec{p}|}{\exp\left(\beta(|\vec{p}| - \mu)\right) + 1}\right)
$$

$$\frac{d\rho(\omega)}{d\omega} = \frac{\omega^3}{2\pi^2}\left(\frac{1}{\exp\left(\beta(\omega+\mu)\right)+1} + \frac{1}{\exp\left(\beta(\omega-\mu)\right)+1}\right)$$

$$\rho = \int_0^\infty d\omega \frac{d\rho(\omega)}{d\omega} = \frac{1}{\pi^2}\int_0^\infty d\omega \frac{\omega^3}{\exp(\beta\omega)+1} = \frac{7\pi^2}{120\beta^4} = \frac{7\pi^2 T^4}{120}$$

$$\langle \hat{F}_{\mathrm{L}} \rangle_{\pm}(\vec{p}) = \frac{\partial \log Z_{\pm}(\vec{p})}{\partial(\beta\mu)} = \mp \frac{1}{\exp\left(\beta(|\vec{p}| \pm \mu)\right)+1}$$

$$f_{\mathrm{L}} = \frac{1}{L^3}\langle \hat{F}_{\mathrm{L}} \rangle \to \frac{1}{2\pi^2}\int_0^\infty d\omega \left(\frac{\omega^2}{\exp\left(\beta(|\vec{p}|-\mu)\right)+1} - \frac{\omega^2}{\exp\left(\beta(|\vec{p}|+\mu)\right)+1}\right)$$

$$\hat{\psi}(\vec{x}) = \begin{pmatrix} \hat{\psi}_{\mathrm{L}}(\vec{x}) \\ \hat{\psi}_{\mathrm{R}}(\vec{x}) \end{pmatrix} = \begin{pmatrix} \hat{\psi}_{\mathrm{L}}^1(\vec{x}) \\ \hat{\psi}_{\mathrm{L}}^2(\vec{x}) \\ \hat{\psi}_{\mathrm{R}}^1(\vec{x}) \\ \hat{\psi}_{\mathrm{R}}^2(\vec{x}) \end{pmatrix}$$

$$\hat{\psi}^\dagger(\vec{x}) = \left(\hat{\psi}_{\mathrm{L}}^\dagger(\vec{x}), \hat{\psi}_{\mathrm{R}}^\dagger(\vec{x})\right) = \left(\hat{\psi}_{\mathrm{L}}^{1\dagger}(\vec{x}), \hat{\psi}_{\mathrm{L}}^{2\dagger}(\vec{x}), \hat{\psi}_{\mathrm{R}}^{1\dagger}(\vec{x}), \hat{\psi}_{\mathrm{R}}^{2\dagger}(\vec{x})\right).$$

$$P_{\mathrm{L}} = \frac{1}{2}(1-\gamma^5) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, P_{\mathrm{R}} = \frac{1}{2}(1+\gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$${}^{\mathrm{P}}\hat{\psi}(\vec{x}) = \begin{pmatrix} {}^{\mathrm{P}}\hat{\psi}_{\mathrm{L}}(\vec{x}) \\ {}^{\mathrm{P}}\hat{\psi}_{\mathrm{R}}(\vec{x}) \end{pmatrix} = \begin{pmatrix} \hat{\psi}_{\mathrm{R}}(-\vec{x}) \\ \hat{\psi}_{\mathrm{L}}(-\vec{x}) \end{pmatrix} = \gamma^0 \hat{\psi}(-\vec{x})$$

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3) = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3, \{\gamma^\mu, \gamma^5\} = 0$$

$${}^{\mathrm{C}}\hat{\psi}(\vec{x}) = \begin{pmatrix} {}^{\mathrm{C}}\hat{\psi}_{\mathrm{L}}(\vec{x}) \\ {}^{\mathrm{C}}\hat{\psi}_{\mathrm{R}}(\vec{x}) \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\sigma^2 \hat{\psi}_{\mathrm{R}}^\dagger(\vec{x})^\top \\ \mathrm{i}\sigma^2 \hat{\psi}_{\mathrm{L}}^\dagger(\vec{x})^\top \end{pmatrix} = C\gamma^0 \hat{\psi}^\dagger(\vec{x})^\top, C = \begin{pmatrix} -\mathrm{i}\sigma^2 & 0 \\ 0 & \mathrm{i}\sigma^2 \end{pmatrix}$$

$$\hat{H}_{\mathrm{D}} = \int d^3x \left[\hat{\psi}_{\mathrm{R}}^\dagger(\vec{x})(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_{\mathrm{R}}(\vec{x}) + \hat{\psi}_{\mathrm{L}}^\dagger(\vec{x})(\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_{\mathrm{L}}(\vec{x}) + m\left(\hat{\psi}_{\mathrm{R}}^\dagger(\vec{x})\hat{\psi}_{\mathrm{L}}(\vec{x}) + \hat{\psi}_{\mathrm{L}}^\dagger(\vec{x})\hat{\psi}_{\mathrm{R}}(\vec{x})\right)\right] = \int d^3x \hat{\psi}^\dagger(\vec{x})(-\mathrm{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m)\hat{\psi}(\vec{x})$$

$$\vec{\alpha} = \gamma^0\vec{\gamma} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \beta = \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

$$\begin{aligned} \hat{H}_{\mathrm{D}} = \frac{1}{(2\pi)^3}\int d^3p \Bigg[&\left(\hat{c}_{\mathrm{R}}^\dagger(\vec{p}), \hat{d}_{\mathrm{L}}(-\vec{p})\right)\begin{pmatrix} |\vec{p}| & m\exp(-\mathrm{i}\varphi) \\ m\exp(\mathrm{i}\varphi) & -|\vec{p}| \end{pmatrix}\begin{pmatrix} \hat{c}_{\mathrm{R}}(\vec{p}) \\ \hat{d}_{\mathrm{L}}^\dagger(-\vec{p}) \end{pmatrix} \\ &+\left(\hat{c}_{\mathrm{L}}^\dagger(\vec{p}), \hat{d}_{\mathrm{R}}(-\vec{p})\right)\begin{pmatrix} |\vec{p}| & -m\exp(-\mathrm{i}\varphi) \\ -m\exp(\mathrm{i}\varphi) & -|\vec{p}| \end{pmatrix}\begin{pmatrix} \hat{c}_{\mathrm{L}}(\vec{p}) \\ \hat{d}_{\mathrm{R}}^\dagger(-\vec{p}) \end{pmatrix}\Bigg] \\ = \frac{1}{(2\pi)^3}\int d^3p \sqrt{\vec{p}^2 + m^2}\Big[&\hat{c}_+^\dagger(\vec{p})\hat{c}_+(\vec{p}) + \hat{d}_+^\dagger(\vec{p})\hat{d}_+(\vec{p}) \\ &+\hat{c}_-^\dagger(\vec{p})\hat{c}_-(\vec{p}) + \hat{d}_-^\dagger(\vec{p})\hat{d}_-(\vec{p}) - 2V\Big] \end{aligned}$$

$$\begin{pmatrix} \hat{c}_{\mathrm{R}}(\vec{p}) \\ \hat{d}^{\dagger}_{\mathrm{L}}(-\vec{p}) \end{pmatrix} = V(\vec{p})^{\dagger} \begin{pmatrix} \hat{c}_{+}(\vec{p}) \\ \hat{d}^{\dagger}_{+}(-\vec{p}) \end{pmatrix}, \begin{pmatrix} \hat{c}_{\mathrm{L}}(\vec{p}) \\ \hat{d}^{\dagger}_{\mathrm{R}}(-\vec{p}) \end{pmatrix} = V(-\vec{p})^{\dagger} \begin{pmatrix} \hat{c}_{-}(\vec{p}) \\ \hat{d}^{\dagger}_{-}(-\vec{p}) \end{pmatrix}$$

$$V(\vec{p}) \begin{pmatrix} |\vec{p}| & m\exp(-\mathrm{i}\varphi) \\ m\exp(\mathrm{i}\varphi) & -|\vec{p}| \end{pmatrix} V(\vec{p})^{\dagger} = \begin{pmatrix} \sqrt{\vec{p}^2 + m^2} & 0 \\ 0 & -\sqrt{\vec{p}^2 + m^2} \end{pmatrix}$$

$$V(\vec{p}) = \begin{pmatrix} \cos(\chi/2) & \sin(\chi/2)\exp(-\mathrm{i}\varphi) \\ -\sin(\chi/2)\exp(\mathrm{i}\varphi) & \cos(\chi/2) \end{pmatrix}, \cos\chi = \frac{|\vec{p}|}{\sqrt{\vec{p}^2 + m^2}}$$

$$V(-\vec{p}) \begin{pmatrix} |\vec{p}| & -m\exp(-\mathrm{i}\varphi) \\ -m\exp(\mathrm{i}\varphi) & -|\vec{p}| \end{pmatrix} V(-\vec{p})^{\dagger} = \begin{pmatrix} \sqrt{\vec{p}^2 + m^2} & 0 \\ 0 & -\sqrt{\vec{p}^2 + m^2} \end{pmatrix}$$

$$V(-\vec{p}) = \begin{pmatrix} \cos(\chi/2) & -\sin(\chi/2)\exp(-\mathrm{i}\varphi) \\ \sin(\chi/2)\exp(\mathrm{i}\varphi) & \cos(\chi/2) \end{pmatrix}$$

$$\hat{c}_{\pm}(\vec{p})|0\rangle_{\mathrm{D}} = \hat{d}_{\pm}(\vec{p})|0\rangle_{\mathrm{D}} = 0$$

$$\hat{c}^{\dagger}_{\pm}(\vec{p})|0\rangle_{\mathrm{D}} = \left|F = 1, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}}$$
$$\hat{d}^{\dagger}_{\pm}(\vec{p})|0\rangle_{\mathrm{D}} = \left|F = -1, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}}$$

$$\hat{\vec{P}}_{\mathrm{D}} = \hat{\vec{P}}_{\mathrm{R}} + \hat{\vec{P}}_{\mathrm{L}} = \frac{1}{(2\pi)^3}\int d^3p\vec{p}\left[\hat{c}^{\dagger}_{\mathrm{R}}(\vec{p})\hat{c}_{\mathrm{R}}(\vec{p}) + \hat{d}^{\dagger}_{\mathrm{R}}(\vec{p})\hat{d}_{\mathrm{R}}(\vec{p}) + \hat{c}^{\dagger}_{\mathrm{L}}(\vec{p})\hat{c}_{\mathrm{L}}(\vec{p}) + \hat{d}^{\dagger}_{\mathrm{L}}(\vec{p})\hat{d}_{\mathrm{L}}(\vec{p})\right]$$
$$= \frac{1}{(2\pi)^3}\int d^3p\vec{p}\left[\hat{c}^{\dagger}_{+}(\vec{p})\hat{c}_{+}(\vec{p}) + \hat{d}^{\dagger}_{+}(\vec{p})\hat{d}_{+}(\vec{p}) + \hat{c}^{\dagger}_{-}(\vec{p})\hat{c}_{-}(\vec{p}) + \hat{d}^{\dagger}_{-}(\vec{p})\hat{d}_{-}(\vec{p})\right]$$

$$\left[\hat{\vec{P}}_{\mathrm{D}}, \hat{c}^{\dagger}_{\pm}(\vec{p})\right] = \vec{p}\hat{c}^{\dagger}_{\pm}(\vec{p}), \left[\hat{\vec{P}}_{\mathrm{D}}, \hat{d}^{\dagger}_{\pm}(\vec{p})\right] = \vec{p}\hat{d}^{\dagger}_{\pm}(\vec{p})$$

$$\hat{\vec{P}}_{\mathrm{D}}\left|F, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}} = \vec{p}\left|F, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}}$$

$$\left[\hat{\vec{J}}_{\mathrm{D}}\cdot\vec{e}_p, \hat{c}^{\dagger}_{\pm}(\vec{p})\right] = \pm\frac{1}{2}\hat{c}^{\dagger}_{\pm}(\vec{p}), \left[\hat{\vec{J}}_{\mathrm{D}}\cdot\vec{e}_p, \hat{d}^{\dagger}_{\pm}(\vec{p})\right] = \pm\frac{1}{2}\hat{d}^{\dagger}_{\pm}(\vec{p})$$

$$\hat{\vec{J}}_{\mathrm{D}}\cdot\vec{e}_p\left|F, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}} = \pm\frac{1}{2}\left|F, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}}$$

$${}^{\mathrm{P}}\hat{c}_{\pm}(\vec{p}) = \hat{c}_{\mp}(-\vec{p}), \quad {}^{\mathrm{P}}\hat{d}_{\pm}(\vec{p}) = \hat{d}_{\mp}(-\vec{p}),$$
$${}^{\mathrm{C}}\hat{c}_{\pm}(\vec{p}) = \pm\hat{d}_{\pm}(\vec{p}), \quad {}^{\mathrm{C}}\hat{d}_{\pm}(\vec{p}) = \pm\hat{c}_{\pm}(\vec{p}).$$

$$\hat{U}_{\mathrm{P}}\left|F = 1, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}} = \hat{U}_{\mathrm{P}}\hat{c}^{\dagger}_{\pm}(\vec{p})|0\rangle_{\mathrm{D}} = {}^{\mathrm{P}}\hat{c}^{\dagger}_{\pm}(\vec{p})\hat{U}_{\mathrm{P}}|0\rangle_{\mathrm{D}} = \hat{c}^{\dagger}_{\mp}(-\vec{p})|0\rangle_{\mathrm{D}}$$
$$= \left|F = 1, -\vec{p}, -\vec{\sigma}\cdot\vec{e}_p = \mp 1\right\rangle_{\mathrm{D}}$$

$$\hat{U}_{\mathrm{P}}\left|F, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}} = \left|F, -\vec{p}, -\vec{\sigma}\cdot\vec{e}_p = \mp 1\right\rangle_{\mathrm{D}}$$

$$\hat{U}_{\mathrm{C}}\left|F, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}} = \pm\left|-F, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}}$$

$$\hat{U}_{\mathrm{CP}}\left|F, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_{\mathrm{D}} = \mp\left|-F, -\vec{p}, -\vec{\sigma}\cdot\vec{e}_p = \mp 1\right\rangle_{\mathrm{D}}$$

$${}^{\mathrm{C}}\hat{\psi}(\vec{x}) = \hat{\psi}(\vec{x}) \Rightarrow {}^{\mathrm{C}}\hat{\psi}_{\mathrm{R}}(\vec{x}) = \mathrm{i}\sigma^2\hat{\psi}^{\dagger}_{\mathrm{L}}(\vec{x})^{\top} = \hat{\psi}_{\mathrm{R}}(\vec{x}) \Rightarrow \hat{\psi}^{a}_{\mathrm{R}}(\vec{x}) = \epsilon_{ab}\hat{\psi}^{b\dagger}_{\mathrm{L}}(\vec{x})$$

$$\hat{\psi}_{\mathrm{L}}(\vec{x})' = \exp(\mathrm{i}\chi_{\mathrm{L}})\hat{\psi}_{\mathrm{L}}(\vec{x}), \hat{\psi}_{\mathrm{R}}(\vec{x})' = \exp(\mathrm{i}\chi_{\mathrm{R}})\hat{\psi}_{\mathrm{R}}(\vec{x})$$

$$\mathrm{i}\sigma^2\hat{\psi}_\mathrm{L}^\dagger(\vec{x})'^\top = \exp(-\mathrm{i}\chi_\mathrm{L})\mathrm{i}\sigma^2\hat{\psi}_\mathrm{L}^\dagger(\vec{x})^\top = \exp(-\mathrm{i}\chi_\mathrm{L})\hat{\psi}_\mathrm{R}(\vec{x}) = \exp(-\mathrm{i}\chi_\mathrm{L} - \mathrm{i}\chi_\mathrm{R})\hat{\psi}_\mathrm{R}(\vec{x})'$$

$$\mathrm{i}\sigma^{2\mathrm{P}}\hat{\psi}_\mathrm{L}^\dagger(\vec{x})^\top = \mathrm{i}\sigma^2\hat{\psi}_\mathrm{R}^\dagger(-\vec{x})^\top = \mathrm{i}\sigma^2(-\mathrm{i}\sigma^2)^\top\hat{\psi}_\mathrm{L}(-\vec{x}) = -{}^\mathrm{P}\hat{\psi}_\mathrm{R}(\vec{x})$$

$$\begin{aligned}
{}^\mathrm{P}\hat{\psi}_\mathrm{L}(\vec{x})' &= \exp(\mathrm{i}\chi_\mathrm{L})^\mathrm{P}\hat{\psi}_\mathrm{L}(\vec{x}) = \exp(\mathrm{i}\chi_\mathrm{L})\hat{\psi}_\mathrm{R}(-\vec{x})\\
{}^\mathrm{P}\hat{\psi}_\mathrm{R}(\vec{x})' &= \exp(\mathrm{i}\chi_\mathrm{R})^\mathrm{P}\hat{\psi}_\mathrm{R}(\vec{x}) = \exp(\mathrm{i}\chi_\mathrm{R})\hat{\psi}_\mathrm{L}(-\vec{x})\\
\mathrm{i}\sigma^{2\mathrm{P}}\hat{\psi}_\mathrm{L}^\dagger(\vec{x})'^\top &= \exp(-\mathrm{i}\chi_\mathrm{L})\mathrm{i}\sigma^2\hat{\psi}_\mathrm{R}^\dagger(-\vec{x})^\top = -\exp(-\mathrm{i}\chi_\mathrm{L})\hat{\psi}_\mathrm{L}(-\vec{x})\\
&= -\exp(-\mathrm{i}\chi_\mathrm{L} - \mathrm{i}\chi_\mathrm{R})^\mathrm{P}\hat{\psi}_\mathrm{R}(\vec{x})'
\end{aligned}$$

$${}^{\mathrm{P}'}\hat{\psi}_\mathrm{L}(\vec{x}) = \mathrm{i}\hat{\psi}_\mathrm{R}(-\vec{x}),\ {}^{\mathrm{P}'}\hat{\psi}_\mathrm{R}(\vec{x}) = \mathrm{i}\hat{\psi}_\mathrm{L}(-\vec{x})$$

$$\begin{aligned}
\hat{H}_\mathrm{M} &= \int d^3x\left[\hat{\psi}_\mathrm{R}^\dagger(\vec{x})(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_\mathrm{R}(\vec{x}) + \frac{m}{2}\left(\hat{\psi}_\mathrm{R}(\vec{x})^\top\mathrm{i}\sigma^2\hat{\psi}_\mathrm{R}(\vec{x}) - \hat{\psi}_\mathrm{R}^\dagger(\vec{x})\mathrm{i}\sigma^2\hat{\psi}_\mathrm{R}^\dagger(\vec{x})^\top\right)\right]\\
&= \int d^3x\left[\hat{\psi}_\mathrm{L}^\dagger(\vec{x})(\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_\mathrm{L}(\vec{x}) + \frac{m}{2}\left(-\hat{\psi}_\mathrm{L}(\vec{x})^\top\mathrm{i}\sigma^2\hat{\psi}_\mathrm{L}(\vec{x}) + \hat{\psi}_\mathrm{L}^\dagger(\vec{x})\mathrm{i}\sigma^2\hat{\psi}_\mathrm{L}^\dagger(\vec{x})^\top\right)\right]
\end{aligned}$$

$$\begin{aligned}
\hat{\psi}_\mathrm{R}(\vec{p})^\top\mathrm{i}\sigma^2\hat{\psi}_\mathrm{R}(-\vec{p}) &= \left(\hat{c}_\mathrm{R}(\vec{p}), \hat{d}_\mathrm{R}^\dagger(-\vec{p})\right)U(\vec{p})^*\mathrm{i}\sigma^2 U(-\vec{p})^\dagger\begin{pmatrix}\hat{c}_\mathrm{R}(-\vec{p})\\ \hat{d}_\mathrm{R}^\dagger(\vec{p})\end{pmatrix}\\
&= -\exp(\mathrm{i}\varphi)\hat{c}_\mathrm{R}(\vec{p})\hat{c}_\mathrm{R}(-\vec{p}) - \exp(-\mathrm{i}\varphi)\hat{d}_\mathrm{R}^\dagger(-\vec{p})\hat{d}_\mathrm{R}^\dagger(\vec{p})
\end{aligned}$$

$$\begin{aligned}
\hat{H}_\mathrm{M} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3/2} d^3p\Bigg[&\left(\hat{c}_\mathrm{R}^\dagger(\vec{p}), \hat{c}_\mathrm{R}(-\vec{p})\right)\begin{pmatrix}|\vec{p}| & m\exp(-\mathrm{i}\varphi)\\ m\exp(\mathrm{i}\varphi) & -|\vec{p}|\end{pmatrix}\begin{pmatrix}\hat{c}_\mathrm{R}(\vec{p})\\ \hat{c}_\mathrm{R}^\dagger(-\vec{p})\end{pmatrix}\\
&+\left(-\hat{d}_\mathrm{R}^\dagger(\vec{p}), \hat{d}_\mathrm{R}(-\vec{p})\right)\begin{pmatrix}|\vec{p}| & -m\exp(-\mathrm{i}\varphi)\\ -m\exp(\mathrm{i}\varphi) & -|\vec{p}|\end{pmatrix}\begin{pmatrix}-\hat{d}_\mathrm{R}(\vec{p})\\ \hat{d}_\mathrm{R}^\dagger(-\vec{p})\end{pmatrix}\Bigg].
\end{aligned}$$

$$\begin{pmatrix}\hat{c}_\mathrm{R}(\vec{p})\\ \hat{c}_\mathrm{R}^\dagger(-\vec{p})\end{pmatrix} = V(\vec{p})^\dagger\begin{pmatrix}\hat{c}_+(\vec{p})\\ \hat{c}_+^\dagger(-\vec{p})\end{pmatrix}, \begin{pmatrix}-\hat{d}_\mathrm{R}(\vec{p})\\ \hat{d}_\mathrm{R}^\dagger(-\vec{p})\end{pmatrix} = V(-\vec{p})^\dagger\begin{pmatrix}\hat{c}_-(\vec{p})\\ -\hat{c}_-^\dagger(-\vec{p})\end{pmatrix}$$

$$\begin{aligned}
\hat{c}_\mathrm{R}(\vec{p}) &= \cos(\chi/2)\hat{c}_+(\vec{p}) - \sin(\chi/2)\exp(-\mathrm{i}\varphi)\hat{c}_+^\dagger(-\vec{p})\\
\hat{c}_\mathrm{R}^\dagger(-\vec{p}) &= \sin(\chi/2)\exp(\mathrm{i}\varphi)\hat{c}_+(\vec{p}) + \cos(\chi/2)\hat{c}_+^\dagger(-\vec{p})
\end{aligned}$$

$$\begin{aligned}
\hat{H}_\mathrm{M} &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3/2} d^3p\sqrt{\vec{p}^2 + m^2}\left[\hat{c}_+^\dagger(\vec{p})\hat{c}_+(\vec{p}) - \hat{c}_+(-\vec{p})\hat{c}_+^\dagger(-\vec{p})\right.\\
&\qquad\left. + \hat{c}_-^\dagger(\vec{p})\hat{c}_-(\vec{p}) - \hat{c}_-(-\vec{p})\hat{c}_-^\dagger(-\vec{p})\right]\\
&= \frac{1}{(2\pi)^3}\int d^3p\sqrt{\vec{p}^2 + m^2}\left[\hat{c}_+^\dagger(\vec{p})\hat{c}_+(\vec{p}) + \hat{c}_-^\dagger(\vec{p})\hat{c}_-(\vec{p}) - V\right]
\end{aligned}$$

$$\hat{c}_\pm^\dagger(\vec{p})|0\rangle_\mathrm{M} = \left|(-1)^F = -1, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_\mathrm{M}$$

$$\begin{aligned}
\hat{U}_{\mathrm{P}'}\left|(-1)^F = -1, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_\mathrm{M} &= {}^{\mathrm{P}'}\hat{c}_\pm^\dagger(\vec{p})|0\rangle_\mathrm{M} = \hat{c}_\mp^\dagger(-\vec{p})|0\rangle_\mathrm{M}\\
&= \left|(-1)^F = -1, -\vec{p}, -\vec{\sigma}\cdot\vec{e}_p = \mp 1\right\rangle_\mathrm{M}
\end{aligned}$$

$$\hat{U}_\mathrm{C}\left|(-1)^F = -1, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_\mathrm{M} = \left|(-1)^F = -1, \vec{p}, \vec{\sigma}\cdot\vec{e}_p = \pm 1\right\rangle_\mathrm{M}$$

$$\hat{H}_\mathrm{W} = \int d^3x\left[\hat{\psi}_\mathrm{R}^\dagger(\vec{x})(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\psi}_\mathrm{R}(\vec{x}) + \frac{m}{2}\left(\hat{\psi}_\mathrm{R}(\vec{x})^\top\mathrm{i}\sigma^2\hat{\psi}_\mathrm{R}(\vec{x}) - \hat{\psi}_\mathrm{R}^\dagger(\vec{x})\mathrm{i}\sigma^2\hat{\psi}_\mathrm{R}^\dagger(\vec{x})^\top\right)\right]$$

$$\begin{pmatrix} \hat{c}_{\mathrm{R}}(\vec{p}) \\ c_{\mathrm{R}}^{\dagger}(-\vec{p}) \end{pmatrix} = V(\vec{p})^{\dagger} \begin{pmatrix} \hat{c}_{+}(\vec{p}) \\ \hat{c}_{+}^{\dagger}(-\vec{p}) \end{pmatrix}, \begin{pmatrix} -\hat{d}_{\mathrm{R}}(\vec{p}) \\ \hat{d}_{\mathrm{R}}^{\dagger}(-\vec{p}) \end{pmatrix} = V(-\vec{p})^{\dagger} \begin{pmatrix} -\hat{d}_{-}(\vec{p}) \\ \hat{d}_{-}^{\dagger}(-\vec{p}) \end{pmatrix}$$

$$\hat{H}_{\mathrm{W}} = \frac{1}{(2\pi)^3} \int d^3p \sqrt{\vec{p}^2 + m^2} \left[\hat{c}_{+}^{\dagger}(\vec{p}) \hat{c}_{+}(\vec{p}) + \hat{d}_{-}^{\dagger}(\vec{p}) \hat{d}_{-}(\vec{p}) - V \right]$$

$$\rho = \frac{E_0}{V} = -\frac{1}{(2\pi)^3} \int d^3p \sqrt{\vec{p}^2 + m^2}$$

$$\hat{c}_{+}^{\dagger}|0\rangle_{\mathrm{R}} = \left|(-1)^{F_{\mathrm{R}}} = -1, \vec{p}, \vec{\sigma} \cdot \vec{e}_p = 1\right\rangle_{\mathrm{R}}$$
$$\hat{d}_{-}^{\dagger}|0\rangle_{\mathrm{R}} = \left|(-1)^{F_{\mathrm{R}}} = -1, \vec{p}, \vec{\sigma} \cdot \vec{e}_p = -1\right\rangle_{\mathrm{R}}$$

$$\hat{U}_{\mathrm{CP}'} \left|(-1)^{F_{\mathrm{R}}} = -1, \vec{p}, \vec{\sigma} \cdot \vec{e}_p = \pm 1\right\rangle_{\mathrm{R}} = \mp \left|(-1)^{F_{\mathrm{R}}} = -1, -\vec{p}, -\vec{\sigma} \cdot \vec{e}_p = \mp 1\right\rangle_{\mathrm{R}};$$

$$\hat{H} = \sum_{a=1}^{N} \hat{\vec{p}}_a{}^2/2m + \sum_{a<b} \hat{V}\left(\left|\hat{\vec{x}}_a - \hat{\vec{x}}_b\right|\right)$$

$$\hat{P}_{12} \Psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N, S_1^3, S_2^3, \dots, S_N^3) = \Psi(\vec{x}_2, \vec{x}_1, \dots, \vec{x}_N, S_2^3, S_1^3, \dots, S_N^3).$$

$$\hat{P} \Psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N, S_1^3, S_2^3, \dots, S_N^3) = \mathrm{sign}(P) \Psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N, S_1^3, S_2^3, \dots, S_N^3).$$

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3) = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} \mathrm{i}\sigma^3 & 0 \\ 0 & \mathrm{i}\sigma^3 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} -\mathrm{i}\sigma^1 & 0 \\ 0 & -\mathrm{i}\sigma^1 \end{pmatrix}.$$

$$\Gamma^c \in \{\mathbb{1}, \gamma^\mu, \gamma^\mu \gamma^5, \sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/(2\mathrm{i}), \mathrm{i}\gamma^5\}.$$

$$\eta_i \eta_j = -\eta_j \eta_i$$

$$f(\eta) = f + \sum_i f_i \eta_i + \sum_{ij} f_{ij} \eta_i \eta_j + \sum_{ijk} f_{ijk} \eta_i \eta_j \eta_k + \cdots.$$

$$\frac{\delta}{\delta \eta_i} \eta_j = \delta_{ij}, \frac{\delta}{\delta \eta_i} \eta_i \eta_j = \eta_j \Rightarrow \frac{\delta}{\delta \eta_i} \eta_j \eta_i = -\eta_j \ (i \neq j)$$

$$\int d\eta_i (a + b\eta_i) = a \int d\eta_i + b \int d\eta_i \eta_i, a, b \in \mathbb{C}$$

$$\int d\eta_i (a + b\eta_i) = \int d\eta_i \left(a + b\left(\eta_i + c + d\eta_j\right)\right) = \int d\eta_i \left(a + b\eta_i + bc + bd\eta_j\right) \ (i \neq j)$$

$$\int d\eta_i = 0, \int d\eta_i \eta_j = \delta_{ij}, \int d\eta_i d\eta_j \eta_i \eta_j = -1 \ (i \neq j)$$

$$\int d\eta_1 d\eta_2 \dots d\eta_N \eta_N \dots \eta_2 \eta_1 = 1$$

$$\int d\eta_1 d\eta_2 \exp(-\eta_1 A_{12}\eta_2) = \int d\eta_1 d\eta_2 (1-\eta_1 A_{12}\eta_2) = A_{12}$$

$$\int d\eta_1 d\eta_2 d\eta_3 \exp(-\eta_1 A_{12}\eta_2 - \eta_1 A_{13}\eta_3 - \eta_2 A_{23}\eta_3) =$$
$$\int d\eta_1 d\eta_2 d\eta_3 (1-\eta_1 A_{12}\eta_2 - \eta_1 A_{13}\eta_3 - \eta_2 A_{23}\eta_3) = 0$$

$$\int \mathcal{D}\eta \exp\left(-\frac{1}{2}\eta A \eta\right) = \int d\eta_1 d\eta_2 \dots d\eta_N \exp\left(-\frac{1}{2}\eta_i A_{ij}\eta_j\right)$$

$$\int \mathcal{D}\eta \exp\left(-\frac{1}{2}\eta A \eta\right) = \int d\eta_1 d\eta_2 d\eta_3 d\eta_4 \frac{1}{2}\left(\frac{1}{2}\eta_i A_{ij}\eta_j\right)^2 = A_{12}A_{34} - A_{13}A_{24} + A_{23}A_{14}$$

$$\int \mathcal{D}\eta \exp\left(-\frac{1}{2}\eta A \eta\right) = \mathrm{Pf}(A)$$

$$\mathrm{Pf}(A)^2 = \det(A)$$

$$\mathrm{Pf}(A) = \frac{1}{2^{N/2}(N/2)!}\sum_{P\in S_N} \mathrm{sign}(P)\prod_{i=1}^{N/2} A_{P(2i-1),P(2i)}.$$

$$\int d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2 \exp\left(-(\bar{\eta}_1,\bar{\eta}_2)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix}\right) = M_{11}M_{22} - M_{12}M_{21}.$$

$$\int \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp(-\bar{\eta}M\eta) = \int d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2 \dots d\bar{\eta}_N d\eta_N \exp(-\bar{\eta}_i M_{ij}\eta_j)$$
$$= \det(M)$$

$$\int \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp(-\bar{\eta}M\eta) = \int \mathcal{D}\bar{\eta}\mathcal{D}\eta' \det(M)\exp(-\bar{\eta}\eta') =$$
$$\det(M)\prod_i \int d\bar{\eta}_i d\eta'_i \exp(-\bar{\eta}_i\eta'_i) = \det(M)\prod_i \int d\bar{\eta}_i d\eta'_i(-\bar{\eta}_i\eta'_i) = \det(M)$$

$$\det(M) = \sum_{P\in S_N}\mathrm{sign}(P)\prod_{i=1}^{N} A_{i,P(i)}$$

$$\int \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp(-\bar{\eta}M\eta) = \int \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp\left(-\frac{1}{2}\bar{\eta}_i M_{ij}\eta_j + \frac{1}{2}\eta_j M^{\top}_{ji}\bar{\eta}_i\right)$$
$$= \int \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp\left(-\frac{1}{2}(\eta,\bar{\eta})\begin{pmatrix} 0 & -M^{\top} \\ M & 0\end{pmatrix}\begin{pmatrix}\eta \\ \bar{\eta}\end{pmatrix}\right) = \mathrm{Pf}(A),$$

$$A = \begin{pmatrix} 0 & -M^{\top} \\ M & 0\end{pmatrix}, \mathrm{Pf}(A)^2 = \det(A) = \det(M)\det(M^{\top}) = \det(M)^2$$

$$\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\phi A\phi\right) = \int_{\mathbb{R}} d\phi_1 \int_{\mathbb{R}} d\phi_2 \dots \int_{\mathbb{R}} d\phi_N \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{1}{2}\phi_i A_{ij}\phi_j\right)$$

$$\Omega A \Omega^{\top} = D = \mathrm{diag}(a_1, a_2, \dots, a_N),$$

$$\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\phi A\phi\right) = \int \mathcal{D}\phi' \exp\left(-\frac{1}{2}\phi' D\phi'\right) =$$
$$\int_{\mathbb{R}} d\phi_1' \int_{\mathbb{R}} d\phi_2' \ldots \int_{\mathbb{R}} d\phi_N' \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}a_i\phi_i'^2\right) = \prod_{i=1}^{N} \frac{1}{\sqrt{a_i}} = \frac{1}{\sqrt{\det(A)}},$$

$$\int \mathcal{D}\Phi \exp\left(-\frac{1}{2}\Phi^\dagger M\Phi\right) = \int_{\mathbb{C}} d\Phi_1 \int_{\mathbb{C}} d\Phi_2 \cdots \int_{\mathbb{C}} d\Phi_N \frac{1}{(2\pi)^N} \exp\left(-\frac{1}{2}\Phi_i^* M_{ij}\Phi_j\right)$$

$$UMU^\dagger = D = \operatorname{diag}(m_1, m_2, \ldots, m_N), m_i \in \mathbb{R}$$

$$\int \mathcal{D}\Phi \exp\left(-\frac{1}{2}\Phi^\dagger M\Phi\right) = \int \mathcal{D}\Phi' \exp\left(-\frac{1}{2}\Phi'^\dagger D\Phi'\right) =$$
$$\int_{\mathbb{C}} d\Phi_1' \int_{\mathbb{C}} d\Phi_2' \cdots \int_{\mathbb{C}} d\Phi_N' \prod_{i=1}^{N} \frac{1}{2\pi} \exp\left(-\frac{1}{2}m_i|\Phi_i'|^2\right) = \prod_{i=1}^{N} \frac{1}{m_i} = \frac{1}{\det(M)}$$

$$\int d\eta_1 d\eta_2 \eta_1\eta_2 \exp(-\eta_1 A_{12}\eta_2) = -1$$

$$\langle\eta_1\eta_2\rangle = \frac{\int d\eta_1 d\eta_2 \eta_1\eta_2 \exp(-\eta_1 A_{12}\eta_2)}{\int d\eta_1 d\eta_2 \exp(-\eta_1 A_{12}\eta_2)} = -\frac{1}{A_{12}} = (A^{-1})_{12}$$
$$A^{-1} = \begin{pmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1/A_{12} \\ 1/A_{12} & 0 \end{pmatrix}$$

$$\langle\eta_i\eta_j\rangle = \frac{\int \mathcal{D}\eta \eta_i\eta_j \exp\left(-\frac{1}{2}\eta A\eta\right)}{\int \mathcal{D}\eta \exp\left(-\frac{1}{2}\eta A\eta\right)} = (A^{-1})_{ij}$$

$$\int d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2 \eta_1\bar{\eta}_2 \exp\left(-(\bar{\eta}_1, \bar{\eta}_2)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}\right) =$$
$$\int d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2 \eta_1\bar{\eta}_2(-\bar{\eta}_1 M_{12}\eta_2) = -M_{12}$$

$$\langle\eta_1\bar{\eta}_2\rangle = (M^{-1})_{12}, M^{-1} = \frac{1}{M_{11}M_{22} - M_{12}M_{21}}\begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}.$$

$$\langle\eta_i\bar{\eta}_j\rangle = \frac{\int \mathcal{D}\bar{\eta}\mathcal{D}\eta \eta_i\bar{\eta}_j \exp(-\bar{\eta}M\eta)}{\int \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp(-\bar{\eta}M\eta)} = (M^{-1})_{ij}$$

$$A^{-1} = \begin{pmatrix} 0 & -M^\top \\ M & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & M^{-1} \\ -(M^{-1})^\top & 0 \end{pmatrix} \Rightarrow$$
$$(M^{-1})_{ij} = (A^{-1})_{i,j+N} \Rightarrow \langle\eta_i\bar{\eta}_j\rangle = \langle\eta_i\eta_{j+N}\rangle$$

$$Z[\bar{\xi}, \xi] = \int \mathcal{D}\bar{\eta}\mathcal{D}\eta \left(-\bar{\eta}M\eta + \bar{\xi}\eta_i + \bar{\eta}_j\xi\right)$$

$$\frac{\delta}{\delta\bar{\xi}}\exp\left(\bar{\xi}\eta_i\right)=\frac{\delta}{\delta\bar{\xi}}\left(1+\bar{\xi}\eta_i\right)=\eta_i,\frac{\delta}{\delta\xi}\exp\left(\bar{\eta}_j\xi\right)=\frac{\delta}{\delta\xi}\left(1+\bar{\eta}_j\xi\right)=-\bar{\eta}_j\Rightarrow$$
$$\frac{\delta}{\delta\xi}\frac{\delta}{\delta\bar{\xi}}Z[\bar{\xi},\xi]=\int\mathcal{D}\bar{\eta}\mathcal{D}\eta\eta_i\bar{\eta}_j\exp\left(-\bar{\eta}M\eta\right)$$

$$-\bar{\eta}_kM_{kl}\eta_l+\bar{\xi}\eta_i+\bar{\eta}_j\xi=-\left(\bar{\eta}'_k+\bar{\xi}M_{ik}^{-1}\right)M_{kl}\left(\eta'_l+M_{lj}^{-1}\xi\right)+\bar{\xi}\left(\eta'_i+M_{ij}^{-1}\xi\right)+\left(\bar{\eta}'_j+\bar{\xi}M_{ij}^{-1}\right)\xi$$

$$Z[\bar{\xi},\xi]=-\bar{\eta}'_kM_{kl}\eta'_l+\bar{\xi}M_{ij}^{-1}\xi\Rightarrow\det(M)\exp\left(\bar{\xi}M_{ij}^{-1}\xi\right)$$
$$\frac{\delta}{\delta\xi}\frac{\delta}{\delta\bar{\xi}}\exp\left(\bar{\xi}M_{ij}^{-1}\xi\right)=\frac{\delta}{\delta\xi}\frac{\delta}{\delta\bar{\xi}}\left(1+\bar{\xi}M_{ij}^{-1}\xi\right)=M_{ij}^{-1}\Rightarrow$$
$$\frac{\delta}{\delta\xi}\frac{\delta}{\delta\bar{\xi}}Z[\bar{\xi},\xi]=\det(M)M_{ij}^{-1}$$

$$Z[\bar{\xi},\xi]=\int\mathcal{D}\bar{\eta}\mathcal{D}\eta\exp\left(-S[\bar{\eta},\eta]\right),S[\bar{\eta},\eta]=\bar{\eta}M\eta-\bar{\xi}\eta_i-\bar{\eta}_j\xi$$

$$\frac{\delta S[\bar{\eta},\eta]}{\delta\bar{\eta}_k}=M_{kl}\eta_l-\delta_{kj}\xi=0\;\Rightarrow\;\eta_l=M_{lj}^{-1}\xi$$
$$\frac{\delta S[\bar{\eta},\eta]}{\delta\eta_l}=-\bar{\eta}_kM_{kl}+\bar{\xi}\delta_{il}=0\;\Rightarrow\;\bar{\eta}_k=\bar{\xi}M_{ik}^{-1}$$

$$S_0[\bar{\eta},\eta]=\bar{\xi}M_{ik}^{-1}M_{kl}M_{lj}^{-1}\xi-\bar{\xi}M_{ij}^{-1}\xi-\bar{\xi}M_{ij}^{-1}\xi=-\bar{\xi}M_{ij}^{-1}\xi$$

$$Z[\bar{\xi},\xi]=\int\mathcal{D}\bar{\eta}\mathcal{D}\eta\exp\left(-S[\bar{\eta},\eta]\right)=\det(M)\exp\left(-S_0[\bar{\eta},\eta]\right)$$
$$=\det(M)\exp\left(\bar{\xi}M_{ij}^{-1}\xi\right)$$

$$\left(\mathrm{i}\gamma^\mu\partial_\mu-m\right)\psi(x)=0,\psi(x)=\begin{pmatrix}\psi_{\mathrm{L}}(x)\\\psi_{\mathrm{R}}(x)\end{pmatrix}$$

$$\gamma^0=\begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix},\gamma^i=\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix},\gamma^5=\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}-\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix}$$

$$\mathrm{i}\partial_0\psi(x)=-\mathrm{i}\gamma^0\gamma^i\partial_i\psi(x)+\gamma^0m\psi(x)=(-\mathrm{i}\vec{\alpha}\cdot\vec{\nabla}+\beta m)\psi(x)$$

$$\hat{H}_{\mathrm{D}}=\int d^3x\hat{\psi}^\dagger(\vec{x})(-\mathrm{i}\vec{\alpha}\cdot\vec{\nabla}+\beta m)\hat{\psi}(\vec{x})$$

$$\hat{\psi}(x)=\hat{\psi}(x^0,\vec{x})=\exp\left(\mathrm{i}\hat{H}_{\mathrm{D}}x^0\right)\hat{\psi}(\vec{x})\exp\left(-\mathrm{i}\hat{H}_{\mathrm{D}}x^0\right)$$

$$\mathrm{i}\partial_0\hat{\psi}(x)=\left[\hat{\psi}(x),\hat{H}_{\mathrm{D}}\right]$$

$$\left[\hat{\psi}(x),\hat{H}_{\mathrm{D}}\right]=\exp\left(\mathrm{i}\hat{H}_{\mathrm{D}}x^0\right)\left[\hat{\psi}(\vec{x}),\hat{H}_{\mathrm{D}}\right]\exp\left(-\mathrm{i}\hat{H}_{\mathrm{D}}x^0\right)$$
$$=\exp\left(\mathrm{i}\hat{H}_{\mathrm{D}}x^0\right)(-\mathrm{i}\vec{\alpha}\cdot\vec{\nabla}+\beta m)\hat{\psi}(\vec{x})\exp\left(-\mathrm{i}\hat{H}_{\mathrm{D}}x^0\right)$$
$$=(-\mathrm{i}\vec{\alpha}\cdot\vec{\nabla}+\beta m)\hat{\psi}(x)$$

$$\left(\mathrm{i}\gamma^\mu\partial_\mu-m\right)\hat{\psi}(x)=0$$

$$-\mathrm{i}\partial_\mu\hat{\bar{\psi}}(x)\gamma^\mu-m\hat{\bar{\psi}}(x)=0$$

$$\mathcal{L}_D(\bar\psi,\psi) = \bar\psi(x)\big(\mathrm{i}\gamma^\mu\partial_\mu - m\big)\psi(x),$$
$$\psi(x) = \begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix}, \bar\psi(x) = (\bar\psi_R(x), \bar\psi_L(x)).$$

$$\frac{\delta\mathcal{L}_D(\bar\psi,\psi)}{\delta\bar\psi} - \partial_\mu\frac{\delta\mathcal{L}_D(\bar\psi,\psi)}{\delta\partial_\mu\bar\psi} = \big(\mathrm{i}\gamma^\mu\partial_\mu - m\big)\psi(x) = 0,$$

$$\frac{\delta\mathcal{L}_D(\bar\psi,\psi)}{\delta\psi} - \partial_\mu\frac{\delta\mathcal{L}_D(\bar\psi,\psi)}{\delta\partial_\mu\psi} = \mathrm{i}\partial_\mu\bar\psi(x)\gamma^\mu + m\bar\psi(x) = 0.$$

$$j^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x), \partial_\mu j^\mu(x) = 0$$

$$\Pi_\psi(x) = \frac{\delta\mathcal{L}_D(\bar\psi,\psi)}{\delta\partial_0\psi} = -\mathrm{i}\bar\psi(x)\gamma^0, \Pi_{\bar\psi}(x) = \frac{\delta\mathcal{L}_D(\bar\psi,\psi)}{\delta\partial_0\bar\psi} = 0$$

$$\begin{aligned}\mathcal{H}_D(\bar\psi,\psi) &= \partial_0\bar\psi(x)\Pi_{\bar\psi}(x) - \Pi_\psi(x)\partial_0\psi(x) - \mathcal{L}_D(\bar\psi,\psi)\\ &= \mathrm{i}\bar\psi(x)\gamma^0\partial_0\psi(x) - \bar\psi(x)\big(\mathrm{i}\gamma^\mu\partial_\mu - m\big)\psi(x)\\ &= \bar\psi(x)\big(-\mathrm{i}\gamma^i\partial_i + m\big)\psi(x)\end{aligned}$$

$$\begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix} \leftrightarrow \begin{pmatrix}\hat\psi_L(\vec{x})\\ \hat\psi_R(\vec{x})\end{pmatrix}, (\bar\psi_R(x), \bar\psi_L(x))\gamma^0 = (\bar\psi_L(x), \bar\psi_R(x)) \leftrightarrow \big(\hat\psi_L^\dagger(\vec{x}), \hat\psi_R^\dagger(\vec{x})\big)$$

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}, \sigma^\mu = (\mathbb{1}, \vec\sigma), \bar\sigma^\mu = (\mathbb{1}, -\vec\sigma).$$

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}$$

$$P_L = \frac{1}{2}(1-\gamma^5) = P_L^2, P_R = \frac{1}{2}(1+\gamma^5) = P_R^2$$

$$P_L\psi = \begin{pmatrix}\psi_L\\ 0\end{pmatrix}, P_R\psi = \begin{pmatrix}0\\ \psi_R\end{pmatrix};\ \bar\psi P_L = (\bar\psi_R, 0), \bar\psi P_R = (0, \bar\psi_L)$$

$$\begin{aligned}\mathrm{i}\sigma^\mu\partial_\mu\psi_R(x) - m\psi_L(x) &= 0, \quad \mathrm{i}\partial_\mu\bar\psi_R(x)\sigma^\mu + m\bar\psi_L(x) = 0\\ \mathrm{i}\bar\sigma^\mu\partial_\mu\psi_L(x) - m\psi_R(x) &= 0, \quad \mathrm{i}\partial_\mu\bar\psi_L(x)\bar\sigma^\mu + m\bar\psi_R(x) = 0\end{aligned}$$

$$\begin{aligned}\mathrm{i}\sigma^\mu\partial_\mu\psi_R(x) &= 0, \quad \mathrm{i}\partial_\mu\bar\psi_R(x)\sigma^\mu = 0\\ \mathrm{i}\bar\sigma^\mu\partial_\mu\psi_L(x) &= 0, \quad \mathrm{i}\partial_\mu\bar\psi_L(x)\bar\sigma^\mu = 0\end{aligned}$$

$$\hat\psi_L(\vec{x}) = -\mathrm{i}\sigma^2\hat\psi_R^\dagger(\vec{x})^\top \Rightarrow \hat\psi_L^\dagger(\vec{x}) = \hat\psi_R(\vec{x})^\top\mathrm{i}\sigma^2$$

$$\psi_L(x) = -\mathrm{i}\sigma^2\bar\psi_R(x)^\top, \bar\psi_L(x) = \psi_R(x)^\top\mathrm{i}\sigma^2$$

$$\begin{aligned}\mathcal{L}_M(\bar\psi_R,\psi_R) =& \frac{1}{2}\bar\psi_R(x)\mathrm{i}\sigma^\mu\partial_\mu\psi_R(x) - \frac{1}{2}\partial_\mu\bar\psi_R(x)\mathrm{i}\sigma^\mu\psi_R(x)\\ &- \frac{m}{2}\big(\psi_R(x)^\mathrm{T}\mathrm{i}\sigma^2\psi_R(x) - \bar\psi_R(x)\mathrm{i}\sigma^2\bar\psi_R(x)^\mathrm{T}\big)\end{aligned}$$

$$\mathrm{i}\sigma^\mu\partial_\mu\psi_\mathrm{R}(x)+\mathrm{i}\sigma^2 m\bar\psi_\mathrm{R}(x)^\top=0$$
$$\mathrm{i}\partial_\mu\bar\psi_\mathrm{R}(x)\sigma^\mu+m\psi_\mathrm{R}(x)^\top\mathrm{i}\sigma^2=0$$

$$\Pi_{\psi_\mathrm{R}}(x)=\frac{\delta\mathcal{L}_\mathrm{M}(\bar\psi_\mathrm{R},\psi_\mathrm{R})}{\delta\partial_0\psi_\mathrm{R}}=-\frac{\mathrm{i}}{2}\bar\psi_\mathrm{R}(x),\Pi_{\bar\psi_\mathrm{R}}(x)=\frac{\delta\mathcal{L}_\mathrm{M}(\bar\psi_\mathrm{R},\psi_\mathrm{R})}{\delta\partial_0\bar\psi_\mathrm{R}}=-\frac{\mathrm{i}}{2}\psi_\mathrm{R}(x),$$

$$\begin{aligned}\mathcal{H}_\mathrm{M}(\bar\psi_\mathrm{R},\psi_\mathrm{R})&=\partial_0\bar\psi_\mathrm{R}(x)\Pi_{\bar\psi_\mathrm{R}}(x)-\Pi_{\psi_\mathrm{R}}(x)\partial_0\psi_\mathrm{R}(x)-\mathcal{L}_\mathrm{M}(\bar\psi_\mathrm{R},\psi_\mathrm{R})\\&=\frac{1}{2}\partial_i\bar\psi_\mathrm{R}(x)\mathrm{i}\sigma^i\psi_\mathrm{R}(x)-\frac{1}{2}\bar\psi_\mathrm{R}(x)\mathrm{i}\sigma^i\partial_i\psi_\mathrm{R}(x)\\&+\frac{m}{2}\left(\psi_\mathrm{R}(x)^\top\mathrm{i}\sigma^2\psi_\mathrm{R}(x)-\bar\psi_\mathrm{R}(x)\mathrm{i}\sigma^2\bar\psi_\mathrm{R}(x)^\top\right)\end{aligned}$$

$$\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left(\mathrm{i}S[\bar\psi,\psi]\right)=\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left(\mathrm{i}\int dtd^3x\bar\psi\left(\mathrm{i}\gamma^\mu\partial_\mu-m\right)\psi\right)$$

$$\begin{aligned}\bar\psi(x)\left(\mathrm{i}\gamma^\mu\partial_\mu-m\right)\psi(x)&=-\bar\psi(x)\left(\gamma_4\partial_4-\mathrm{i}\gamma^i\partial_i+m\right)\psi(x)\\&=-\bar\psi(x)\left(\gamma_\mu\partial_\mu+m\right)\psi(x)=-\mathcal{L}_\mathrm{D}(\bar\psi,\psi)\end{aligned}$$

$$\gamma_i=-\mathrm{i}\gamma^i=\begin{pmatrix}0&-\mathrm{i}\sigma^i\\\mathrm{i}\sigma^i&0\end{pmatrix},\gamma_4=\gamma^0=\begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix},$$
$$\gamma_5=-\gamma_1\gamma_2\gamma_3\gamma_4=\begin{pmatrix}-\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix}=\gamma^5.$$

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu},\gamma_\mu^\dagger=\gamma_\mu.$$

$$\begin{aligned}Z&=\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left(-S_\mathrm{D}[\bar\psi,\psi]\right)=\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left(-\int d^4x\mathcal{L}_\mathrm{D}(\bar\psi,\psi)\right)\\&=\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left(-\int d^4x\bar\psi\left(\gamma_\mu\partial_\mu+m\right)\psi\right)\end{aligned}$$

$$\mathcal{D}\bar\psi\mathcal{D}\psi=\prod_{x\in\Lambda}\prod_a d\bar\psi_x^a d\psi_x^a$$

$$Z=\mathrm{Tr}\exp\left(-\beta\hat{H}_\mathrm{D}\right)$$

$$\psi(\vec{x},x_4+\beta)=-\psi(\vec{x},x_4),\bar\psi(\vec{x},x_4+\beta)=-\bar\psi(\vec{x},x_4).$$

$$\begin{aligned}Z&=\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left(-\int d^4x\bar\psi(x)\left(\gamma_\mu\partial_\mu+m\right)\psi(x)\right)=\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left(-\bar\psi D\psi\right)\\&=\det(D)\end{aligned}$$

$$\bar\psi D\psi=\int d^4xd^4y\bar\psi(x)D(x,y)\psi(y),D(x,y)=\delta(x-y)\left(\gamma_\mu\partial_\mu+m\right)$$

PARTE II.

$$\gamma_\mu=\begin{pmatrix}0&\sigma_\mu\\\bar\sigma_\mu&0\end{pmatrix},\sigma_\mu=(-\mathrm{i}\vec\sigma,\mathbb{1}),\bar\sigma_\mu=(\mathrm{i}\vec\sigma,\mathbb{1})$$

$$\mathcal{L}_\mathrm{R}(\bar\psi_\mathrm{R},\psi_\mathrm{R})=\bar\psi_\mathrm{R}(x)\sigma_\mu\partial_\mu\psi_\mathrm{R}(x),\mathcal{L}_\mathrm{L}(\bar\psi_\mathrm{L},\psi_\mathrm{L})=\bar\psi_\mathrm{L}(x)\bar\sigma_\mu\partial_\mu\psi_\mathrm{L}(x)$$

$$W_\mathrm{R}(x,y)=\delta(x-y)\sigma_\mu\partial_\mu,W_\mathrm{L}(x,y)=\delta(x-y)\bar\sigma_\mu\partial_\mu$$

$$D = \begin{pmatrix} m\delta(x-y) & W_{\mathrm{R}} \\ W_{\mathrm{L}} & m\delta(x-y) \end{pmatrix}$$

$$\mathcal{L}_{\mathrm{M}}(\bar{\psi}_{\mathrm{R}}, \psi_{\mathrm{R}}) = \frac{1}{2}\bar{\psi}_{\mathrm{R}}(x)\sigma_\mu\partial_\mu\psi_{\mathrm{R}}(x) - \frac{1}{2}\partial_\mu\bar{\psi}_{\mathrm{R}}(x)\sigma_\mu\psi_{\mathrm{R}}(x) + \frac{m}{2}\left(\psi_{\mathrm{R}}(x)^{\top}\mathrm{i}\sigma^2\psi_{\mathrm{R}}(x) - \bar{\psi}_{\mathrm{R}}(x)\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(x)^{\top}\right)$$

$$Z = \int \mathcal{D}\bar{\psi}_{\mathrm{R}}\mathcal{D}\psi_{\mathrm{R}}\exp\left(-\int d^4x\mathcal{L}_{\mathrm{M}}(\bar{\psi}_{\mathrm{R}}, \psi_{\mathrm{R}})\right) = \int \mathcal{D}\bar{\psi}_{\mathrm{R}}\mathcal{D}\psi_{\mathrm{R}}\exp\left(-\frac{1}{2}(\psi_{\mathrm{R}}^{\top}, \bar{\psi}_{\mathrm{R}})A_{\mathrm{R}}\begin{pmatrix}\psi_{\mathrm{R}} \\ \bar{\psi}_{\mathrm{R}}^{\top}\end{pmatrix}\right) = \mathrm{Pf}(A_{\mathrm{R}})$$

$$A_{\mathrm{R}} = \begin{pmatrix} \mathrm{i}\sigma^2 m\delta(x-y) & -W_{\mathrm{R}}^{\top} \\ W_{\mathrm{R}} & -\mathrm{i}\sigma^2 m\delta(x-y) \end{pmatrix}, A_{\mathrm{R}}^{\top} = -A_{\mathrm{R}}$$

$$\mathrm{Spin}(4) = \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$$

$$\sigma_{\mu\nu} = \frac{1}{2\mathrm{i}}[\gamma_\mu, \gamma_\nu] \Rightarrow \sigma_{4i} = \begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix}, \sigma_{ij} = \epsilon_{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$$

$$L_i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & 0\end{pmatrix}, R_i = \frac{1}{2}\begin{pmatrix}0 & 0 \\ 0 & \sigma^i\end{pmatrix}.$$

$$J_i = R_i + L_i = \frac{1}{2}\epsilon_{ijk}\sigma_{jk}, K_i = R_i - L_i = \frac{1}{2}\sigma_{i4}$$

$$x_\nu = \Lambda^{-1}_{\nu\rho}x'_\rho = \Lambda^{\top}_{\nu\rho}x'_\rho = \Lambda_{\rho\nu}x'_\rho \Rightarrow \partial'_\mu x_\nu = \frac{\partial x_\nu}{\partial x'_\mu} = \Lambda_{\rho\nu}\delta_{\mu\rho} = \Lambda_{\mu\nu}$$

$$\psi'_{\mathrm{R}}(x') = \Lambda_{\mathrm{R}}\psi_{\mathrm{R}}(\Lambda^{-1}x'), \quad \bar{\psi}'_{\mathrm{R}}(x') = \bar{\psi}_{\mathrm{R}}(\Lambda^{-1}x')\Lambda_{\mathrm{L}}^{\dagger}, \quad \Lambda_{\mathrm{R}} \in \mathrm{SU}(2)_{\mathrm{R}}$$
$$\psi'_{\mathrm{L}}(x') = \Lambda_{\mathrm{L}}\psi_{\mathrm{L}}(\Lambda^{-1}x'), \quad \bar{\psi}'_{\mathrm{L}}(x') = \bar{\psi}_{\mathrm{L}}(\Lambda^{-1}x')\Lambda_{\mathrm{R}}^{\dagger}, \quad \Lambda_{\mathrm{L}} \in \mathrm{SU}(2)_{\mathrm{L}}$$

$$\Lambda_{\mu\nu} = \frac{1}{2}\mathrm{ReTr}\left(\Lambda_{\mathrm{L}}^{\dagger}\sigma_\mu\Lambda_{\mathrm{R}}\bar{\sigma}_\nu\right) \Rightarrow$$
$$\Lambda_{\mathrm{L}}^{\dagger}\sigma_\mu\Lambda_{\mathrm{R}} = \Lambda_{\mu\nu}\sigma_\nu = \sigma_\nu\Lambda^{-1}_{\nu\mu}, \quad \Lambda_{\mathrm{R}}^{\dagger}\bar{\sigma}_\mu\Lambda_{\mathrm{L}} = \Lambda_{\mu\nu}\bar{\sigma}_\nu = \bar{\sigma}_\nu\Lambda^{-1}_{\nu\mu}$$

$$\Lambda_{ij} = \frac{1}{2}\mathrm{ReTr}\left(\Lambda_{\mathrm{V}}^{\dagger}\sigma_i\Lambda_{\mathrm{V}}\sigma_j\right) = O_{ij}, O \in \mathrm{SO}(3),$$
$$\Lambda_{i4} = 0, \Lambda_{44} = 1, \Lambda_{\mathrm{V}}^{\dagger}\sigma_i\Lambda_{\mathrm{V}} = O_{ij}\sigma_j.$$

$$\partial'_\mu\psi'_{\mathrm{R}}(x') = \Lambda_{\mathrm{R}}\partial'_\mu\psi_{\mathrm{R}}(\Lambda^{-1}x') = \Lambda_{\mathrm{R}}\frac{\partial x_\nu}{\partial x'_\mu}\partial_\nu\psi_{\mathrm{R}}(x) = \Lambda_{\mathrm{R}}\Lambda_{\mu\nu}\partial_\nu\psi_{\mathrm{R}}(x),$$
$$\partial'_\mu\psi'_{\mathrm{L}}(x') = \Lambda_{\mathrm{L}}\partial'_\mu\psi_{\mathrm{L}}(\Lambda^{-1}x') = \Lambda_{\mathrm{L}}\frac{\partial x_\nu}{\partial x'_\mu}\partial_\nu\psi_{\mathrm{L}}(x) = \Lambda_{\mathrm{L}}\Lambda_{\mu\nu}\partial_\nu\psi_{\mathrm{L}}(x).$$

$$\bar{\psi}'_{\mathrm{R}}(x')\sigma_\mu\partial'_\mu\psi'_{\mathrm{R}}(x')=\bar{\psi}_{\mathrm{R}}(x)\Lambda_{\mathrm{L}}^\dagger\sigma_\mu\Lambda_{\mathrm{R}}\Lambda_{\mu\nu}\partial_\nu\psi_{\mathrm{R}}(x)=\bar{\psi}_{\mathrm{R}}(x)\sigma_\rho\Lambda_{\rho\mu}^{-1}\Lambda_{\mu\nu}\partial_\nu\psi_{\mathrm{R}}(x)$$
$$=\bar{\psi}_{\mathrm{L}}(x)\Lambda_{\mathrm{R}}^\dagger\bar{\sigma}_\mu\Lambda_{\mathrm{L}}\Lambda_{\mu\nu}\partial_\nu\psi_{\mathrm{L}}(x)=\bar{\psi}_{\mathrm{L}}(x)\bar{\sigma}_\rho\Lambda_{\rho\mu}^{-1}\Lambda_{\mu\nu}\partial_\nu\psi_{\mathrm{L}}(x)$$
$$\bar{\psi}'_{\mathrm{L}}(x')\bar{\sigma}_\mu\partial'_\mu\psi'_{\mathrm{L}}(x')=\bar{\psi}_{\mathrm{R}}(x)\sigma_\rho\delta_{\rho\nu}\partial_\nu\psi_{\mathrm{R}}(x)=\bar{\psi}_{\mathrm{R}}(x)\sigma_\nu\partial_\nu\psi_{\mathrm{R}}(x)$$
$$=\bar{\psi}_{\mathrm{L}}(x)\bar{\sigma}_\rho\delta_{\rho\nu}\partial_\nu\psi_{\mathrm{L}}(x)=\bar{\psi}_{\mathrm{L}}(x)\bar{\sigma}_\nu\partial_\nu\psi_{\mathrm{L}}(x)$$

$$\psi'(x')=\Lambda_{\mathrm{D}}\psi(\Lambda^{-1}x'),\bar{\psi}'(x')=\bar{\psi}(\Lambda^{-1}x')\Lambda_{\mathrm{D}}^\dagger,\Lambda_{\mathrm{D}}=\begin{pmatrix}\Lambda_{\mathrm{L}} & 0\\ 0 & \Lambda_{\mathrm{R}}\end{pmatrix}$$

$$\begin{pmatrix}\Lambda_{\mathrm{L}}^\dagger & 0\\ 0 & \Lambda_{\mathrm{R}}^\dagger\end{pmatrix}\begin{pmatrix}0 & \sigma_\mu\\ \bar{\sigma}_\mu & 0\end{pmatrix}\begin{pmatrix}\Lambda_{\mathrm{L}} & 0\\ 0 & \Lambda_{\mathrm{R}}\end{pmatrix}=\Lambda_{\mu\nu}\begin{pmatrix}0 & \sigma_\nu\\ \bar{\sigma}_\nu & 0\end{pmatrix}\Rightarrow\Lambda_{\mathrm{D}}^\dagger\gamma_\mu\Lambda_{\mathrm{D}}=\Lambda_{\mu\nu}\gamma_\nu$$

$$\bar{\psi}'(x')\psi'(x')=\bar{\psi}(\Lambda^{-1}x')\Lambda_{\mathrm{D}}^\dagger\Lambda_{\mathrm{D}}\psi(\Lambda^{-1}x')=\bar{\psi}(x)\psi(x).$$

$$\bar{\psi}'(x')\gamma_\mu\psi'(x')=\bar{\psi}(\Lambda^{-1}x')\Lambda_{\mathrm{D}}^\dagger\gamma_\mu\Lambda_{\mathrm{D}}\psi(\Lambda^{-1}x')=\Lambda_{\mu\nu}\bar{\psi}(x)\gamma_\nu\psi(x),$$

$$\bar{\psi}'(x')\sigma_{\mu\nu}\psi'(x')=\bar{\psi}(\Lambda^{-1}x')\Lambda_{\mathrm{D}}^\dagger\frac{1}{2\mathrm{i}}[\gamma_\mu,\gamma_\nu]\Lambda_{\mathrm{D}}\psi(\Lambda^{-1}x')$$
$$=\bar{\psi}(x)\frac{1}{2\mathrm{i}}[\Lambda_{\mathrm{D}}^\dagger\gamma_\mu\Lambda_{\mathrm{D}},\Lambda_{\mathrm{D}}^\dagger\gamma_\nu\Lambda_{\mathrm{D}}]\psi(x)$$
$$=\Lambda_{\mu\rho}\Lambda_{\nu\sigma}\bar{\psi}(x)\frac{1}{2\mathrm{i}}[\gamma_\rho,\gamma_\sigma]\psi(x)=\Lambda_{\mu\rho}\Lambda_{\nu\sigma}\bar{\psi}(x)\sigma_{\rho\sigma}\psi(x).$$

$$\psi'_{\mathrm{L}}(x')=\Lambda_{\mathrm{L}}\psi_{\mathrm{L}}(\Lambda^{-1}x')=-\Lambda_{\mathrm{L}}\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(\Lambda^{-1}x')^\top=-\Lambda_{\mathrm{L}}\mathrm{i}\sigma^2\Lambda_{\mathrm{L}}^\top\bar{\psi}'_{\mathrm{R}}(x')^\top=-\mathrm{i}\sigma^2\bar{\psi}'_{\mathrm{R}}(x')^\top$$

$$\bar{\psi}'_{\mathrm{L}}(x')=\bar{\psi}_{\mathrm{L}}(\Lambda^{-1}x')\Lambda_{\mathrm{R}}^\dagger=\psi_{\mathrm{R}}(\Lambda^{-1}x')^\top\mathrm{i}\sigma^2\Lambda_{\mathrm{R}}^\dagger=\psi'_{\mathrm{R}}(x')^\top\Lambda_{\mathrm{R}}^*\mathrm{i}\sigma^2\Lambda_{\mathrm{R}}^\dagger=\psi'_{\mathrm{R}}(x')^\top\mathrm{i}\sigma^2$$

$$\psi'_{\mathrm{R}}(x')^\top\mathrm{i}\sigma^2\psi'_{\mathrm{R}}(x')=\psi_{\mathrm{R}}(\Lambda^{-1}x')^\top\Lambda_{\mathrm{R}}^\top\mathrm{i}\sigma^2\Lambda_{\mathrm{R}}\psi_{\mathrm{R}}(\Lambda^{-1}x')=\psi_{\mathrm{R}}(x)^\top\mathrm{i}\sigma^2\psi_{\mathrm{R}}(x),$$
$$\bar{\psi}'_{\mathrm{R}}(x')\mathrm{i}\sigma^2\bar{\psi}'_{\mathrm{R}}(x')^\top=\bar{\psi}_{\mathrm{R}}(\Lambda^{-1}x')\Lambda_{\mathrm{L}}^\dagger\mathrm{i}\sigma^2\Lambda_{\mathrm{L}}^*\bar{\psi}_{\mathrm{R}}(\Lambda^{-1}x')^\top=\bar{\psi}_{\mathrm{R}}(x)\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(x)^\top,$$
$$\psi'_{\mathrm{L}}(x')^\top\mathrm{i}\sigma^2\psi'_{\mathrm{L}}(x')=\psi_{\mathrm{L}}(\Lambda^{-1}x')^\top\Lambda_{\mathrm{L}}^\top\mathrm{i}\sigma^2\Lambda_{\mathrm{L}}\psi_{\mathrm{L}}(\Lambda^{-1}x')=\psi_{\mathrm{L}}(x)^\top\mathrm{i}\sigma^2\psi_{\mathrm{L}}(x),$$
$$\bar{\psi}'_{\mathrm{L}}(x')\mathrm{i}\sigma^2\bar{\psi}'_{\mathrm{L}}(x')^\top=\bar{\psi}_{\mathrm{L}}(\Lambda^{-1}x')\Lambda_{\mathrm{R}}^\dagger\mathrm{i}\sigma^2\Lambda_{\mathrm{R}}^*\bar{\psi}_{\mathrm{L}}(\Lambda^{-1}x')^\top=\bar{\psi}_{\mathrm{L}}(x)\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{L}}(x)^\top.$$

$$^{\mathrm{C}}\psi_{\mathrm{R}}(x)=\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{L}}(x)^{\mathrm{T}},\quad ^{\mathrm{C}}\bar{\psi}_{\mathrm{R}}(x)=-\psi_{\mathrm{L}}(x)^{\mathrm{T}}\mathrm{i}\sigma^2$$
$$^{\mathrm{C}}\psi_{\mathrm{L}}(x)=-\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(x)^{\mathrm{T}},\quad ^{\mathrm{C}}\bar{\psi}_{\mathrm{L}}(x)=\psi_{\mathrm{R}}(x)^{\mathrm{T}}\mathrm{i}\sigma^2$$

$$S_{\mathrm{R}}[{}^{\mathrm{C}}\bar{\psi}_{\mathrm{R}},{}^{\mathrm{C}}\psi_{\mathrm{R}}]=\int d^4x\,{}^{\mathrm{C}}\bar{\psi}_{\mathrm{R}}(x)\sigma_\mu\partial_\mu\,{}^{\mathrm{C}}\psi_{\mathrm{R}}(x)=\int d^4x\psi_{\mathrm{L}}(x)^\top(-\mathrm{i}\sigma^2)\sigma_\mu\partial_\mu\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{L}}(x)^\top$$
$$=\int d^4x\psi_{\mathrm{L}}(x)^\top\bar{\sigma}_\mu^\top\partial_\mu\bar{\psi}_{\mathrm{L}}(x)^\top=\int d^4x\bar{\psi}_{\mathrm{L}}(x)\bar{\sigma}_\mu\partial_\mu\psi_{\mathrm{L}}(x)$$
$$=S_{\mathrm{L}}[\bar{\psi}_{\mathrm{L}},\psi_{\mathrm{L}}]$$

$$^{\mathrm{P}}\psi_{\mathrm{R}}(x)=\psi_{\mathrm{L}}(-\vec{x},x_4),\quad ^{\mathrm{P}}\bar{\psi}_{\mathrm{R}}(x)=\bar{\psi}_{\mathrm{L}}(-\vec{x},x_4),$$
$$^{\mathrm{P}}\psi_{\mathrm{L}}(x)=\psi_{\mathrm{R}}(-\vec{x},x_4),\quad ^{\mathrm{P}}\bar{\psi}_{\mathrm{L}}(x)=\bar{\psi}_{\mathrm{R}}(-\vec{x},x_4).$$

$$
\begin{aligned}
S_{\mathrm{R}}[{}^{\mathrm{P}}\bar{\psi}_{\mathrm{R}}, {}^{\mathrm{P}}\psi_{\mathrm{R}}] &= \int d^4x\, {}^{\mathrm{P}}\bar{\psi}_{\mathrm{R}}(x)\sigma_\mu\partial_\mu\, {}^{\mathrm{P}}\psi_{\mathrm{R}}(x) \\
&= \int d^4x\bar{\psi}_{\mathrm{L}}(-\vec{x},x_4)(-\mathrm{i}\sigma_i\partial_i+\partial_4)\psi_{\mathrm{L}}(-\vec{x},x_4) \\
&= \int d^4x\bar{\psi}_{\mathrm{L}}(\vec{x},x_4)(\mathrm{i}\sigma_i\partial_i+\partial_4)\psi_{\mathrm{L}}(\vec{x},x_4) \\
&= \int d^4x\bar{\psi}_{\mathrm{L}}(x)\bar{\sigma}_\mu\partial_\mu\psi_{\mathrm{L}}(x) = S_{\mathrm{L}}[\bar{\psi}_{\mathrm{L}},\psi_{\mathrm{L}}]
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{CP}}\psi_{\mathrm{R}}(x) &= {}^{\mathrm{C}}\psi_{\mathrm{L}}(-\vec{x},x_4) = -\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(-\vec{x},x_4)^{\mathrm{T}}, \\
{}^{\mathrm{CP}}\bar{\psi}_{\mathrm{R}}(x) &= {}^{\mathrm{C}}\bar{\psi}_{\mathrm{L}}(-\vec{x},x_4) = \psi_{\mathrm{R}}(-\vec{x},x_4)^{\mathrm{T}}\mathrm{i}\sigma^2, \\
{}^{\mathrm{CP}}\psi_{\mathrm{L}}(x) &= {}^{\mathrm{C}}\psi_{\mathrm{R}}(-\vec{x},x_4) = \mathrm{i}\sigma^2\bar{\psi}_{\mathrm{L}}(-\vec{x},x_4)^{\mathrm{T}}, \\
{}^{\mathrm{CP}}\bar{\psi}_{\mathrm{L}}(x) &= {}^{\mathrm{C}}\bar{\psi}_{\mathrm{R}}(-\vec{x},x_4) = -\psi_{\mathrm{L}}(-\vec{x},x_4)^{\mathrm{T}}\mathrm{i}\sigma^2.
\end{aligned}
$$

$$
\begin{aligned}
\psi'_{\mathrm{L}}(x) &= \exp(\mathrm{i}\chi_{\mathrm{L}})\psi_{\mathrm{L}}(x), \quad \bar{\psi}'_{\mathrm{L}}(x) = \bar{\psi}_{\mathrm{L}}(x)\exp(-\mathrm{i}\chi_{\mathrm{L}}) \\
\psi'_{\mathrm{R}}(x) &= \exp(\mathrm{i}\chi_{\mathrm{R}})\psi_{\mathrm{R}}(x), \quad \bar{\psi}'_{\mathrm{R}}(x) = \bar{\psi}_{\mathrm{R}}(x)\exp(-\mathrm{i}\chi_{\mathrm{R}})
\end{aligned}
$$

$$
\begin{aligned}
&-\mathrm{i}\sigma^2\bar{\psi}'_{\mathrm{R}}(x)^{\mathrm{T}} = -\exp(-\mathrm{i}\chi_{\mathrm{R}})\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(x)^{\mathrm{T}} = \exp(-\mathrm{i}\chi_{\mathrm{R}})\psi_{\mathrm{L}}(x) = \exp(-\mathrm{i}\chi_{\mathrm{R}}-\mathrm{i}\chi_{\mathrm{L}})\psi'_{\mathrm{L}}(x) \\
&\psi'_{\mathrm{R}}(x)^{\mathrm{T}}\mathrm{i}\sigma^2 = \psi_{\mathrm{R}}(x)^{\mathrm{T}}\mathrm{i}\sigma^2\exp(\mathrm{i}\chi_{\mathrm{R}}) = \bar{\psi}_{\mathrm{L}}(x)\exp(\mathrm{i}\chi_{\mathrm{R}}) = \bar{\psi}'_{\mathrm{L}}(x)\exp(\mathrm{i}\chi_{\mathrm{R}}+\mathrm{i}\chi_{\mathrm{L}})
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{P}'}\psi_{\mathrm{R}}(x) &= \mathrm{i}\psi_{\mathrm{L}}(-\vec{x},x_4), \; {}^{\mathrm{P}'}\bar{\psi}_{\mathrm{R}}(x) = -\mathrm{i}\bar{\psi}_{\mathrm{L}}(-\vec{x},x_4), \\
{}^{\mathrm{P}'}\psi_{\mathrm{L}}(x) &= \mathrm{i}\psi_{\mathrm{R}}(-\vec{x},x_4), \; {}^{\mathrm{P}'}\bar{\psi}_{\mathrm{L}}(x) = -\mathrm{i}\bar{\psi}_{\mathrm{R}}(-\vec{x},x_4),
\end{aligned}
$$

$$
\begin{aligned}
&-\mathrm{i}\sigma^2\,{}^{\mathrm{P}'}\bar{\psi}_{\mathrm{R}}(x)^{\mathrm{T}} = -\sigma^2\bar{\psi}_{\mathrm{L}}(-\vec{x},x_4)^{\mathrm{T}} = -\sigma^2(\mathrm{i}\sigma^2)^{\mathrm{T}}\psi_{\mathrm{R}}(-\vec{x},x_4) = \mathrm{i}\psi_{\mathrm{R}}(-\vec{x},x_4) = {}^{\mathrm{P}'}\psi_{\mathrm{L}}(x), \\
&{}^{\mathrm{P}'}\psi_{\mathrm{R}}(x)^{\mathrm{T}}\mathrm{i}\sigma^2 = -\psi_{\mathrm{L}}(-\vec{x},x_4)^{\mathrm{T}}\sigma^2 = -\bar{\psi}_{\mathrm{R}}(-\vec{x},x_4)^{\mathrm{T}}(-\mathrm{i}\sigma^2)^{\mathrm{T}}\sigma^2 = -\mathrm{i}\bar{\psi}_{\mathrm{R}}(-\vec{x},x_4)^{\mathrm{T}} = {}^{\mathrm{P}'}\bar{\psi}_{\mathrm{L}}(x).
\end{aligned}
$$

$$
\begin{aligned}
&{}^{\mathrm{CP}}\psi_{\mathrm{R}}(x)^{\mathrm{T}}\mathrm{i}\sigma^2\,{}^{\mathrm{CP}}\psi_{\mathrm{R}}(x) - {}^{\mathrm{CP}}\bar{\psi}_{\mathrm{R}}(x)\mathrm{i}\sigma^2\,{}^{\mathrm{CP}}\bar{\psi}_{\mathrm{R}}(x)^{\mathrm{T}} = \\
&\bar{\psi}_{\mathrm{R}}(-\vec{x},x_4)\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(-\vec{x},x_4)^{\mathrm{T}} - \psi_{\mathrm{R}}(-\vec{x},x_4)^{\mathrm{T}}\mathrm{i}\sigma^2\psi_{\mathrm{R}}(-\vec{x},x_4).
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{T}}\psi_{\mathrm{R}}(x) &= \mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(\vec{x},-x_4)^{\mathrm{T}}, \quad {}^{\mathrm{T}}\bar{\psi}_{\mathrm{R}}(x) = \psi_{\mathrm{R}}(\vec{x},-x_4)^{\mathrm{T}}\mathrm{i}\sigma^2, \\
{}^{\mathrm{T}}\psi_{\mathrm{L}}(x) &= \mathrm{i}\sigma^2\bar{\psi}_{\mathrm{L}}(\vec{x},-x_4)^{\mathrm{T}}, \quad {}^{\mathrm{T}}\bar{\psi}_{\mathrm{L}}(x) = \psi_{\mathrm{L}}(\vec{x},-x_4)^{\mathrm{T}}\mathrm{i}\sigma^2.
\end{aligned}
$$

$$
\begin{aligned}
S_{\mathrm{R}}[{}^{\mathrm{T}}\bar{\psi}_{\mathrm{R}}, {}^{\mathrm{T}}\psi_{\mathrm{R}}] &= \int d^4x\,{}^{\mathrm{T}}\bar{\psi}_{\mathrm{R}}(x)\sigma_\mu\partial_\mu\,{}^{\mathrm{T}}\psi_{\mathrm{R}}(x) \\
&= \int d^4x\psi_{\mathrm{R}}(\vec{x},-x_4)^{\mathrm{T}}\mathrm{i}\sigma^2(-\mathrm{i}\sigma_i\partial_i+\partial_4)\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(\vec{x},-x_4)^{\mathrm{T}} \\
&= \int d^4x\psi_{\mathrm{R}}(x)^{\mathrm{T}}\left(-\mathrm{i}\sigma_i^{\mathrm{T}}\partial_i+\partial_4\right)\bar{\psi}_{\mathrm{R}}(x)^{\mathrm{T}} \\
&= \int d^4x\bar{\psi}_{\mathrm{R}}(x)(-\mathrm{i}\sigma_i\partial_i+\partial_4)\psi_{\mathrm{R}}(x) \\
&= \int d^4x\bar{\psi}_{\mathrm{R}}(x)\sigma_\mu\partial_\mu\psi_{\mathrm{R}}(x) \\
&= S_{\mathrm{R}}[\bar{\psi}_{\mathrm{R}},\psi_{\mathrm{R}}]
\end{aligned}
$$

$$
\mathrm{P}^2 = \mathrm{C}^2 = \mathrm{T}^2 = 1,
$$

$$
\mathrm{PC} = -\mathrm{CP}, \mathrm{C\,T} = -\mathrm{TC}, \; \mathrm{TP} = \mathrm{P\,T}.
$$

$$
\begin{aligned}
{}^{\mathrm{C}}\psi(x) &= \begin{pmatrix} {}^{\mathrm{C}}\psi_{\mathrm{L}}(x) \\ {}^{\mathrm{C}}\psi_{\mathrm{R}}(x) \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(x)^{\mathrm{T}} \\ \mathrm{i}\sigma^2\bar{\psi}_{\mathrm{L}}(x)^{\mathrm{T}} \end{pmatrix} = C\bar{\psi}(x)^{\mathrm{T}}, \\
{}^{\mathrm{C}}\bar{\psi}(x) &= \left({}^{\mathrm{C}}\bar{\psi}_{\mathrm{R}}(x),\ {}^{\mathrm{C}}\bar{\psi}_{\mathrm{L}}(x)\right) = \left(-\psi_{\mathrm{L}}(x)^{\mathrm{T}}\mathrm{i}\sigma^2, \psi_{\mathrm{R}}(x)^{\mathrm{T}}\mathrm{i}\sigma^2\right) = -\psi(x)^{\mathrm{T}}C^{-1} \quad , \\
{}^{\mathrm{P}}\psi(x) &= \begin{pmatrix} {}^{\mathrm{P}}\psi_{\mathrm{L}}(x) \\ {}^{\mathrm{P}}\psi_{\mathrm{R}}(x) \end{pmatrix} = \begin{pmatrix} \psi_{\mathrm{R}}(-\vec{x},x_4) \\ \psi_{\mathrm{L}}(-\vec{x},x_4) \end{pmatrix} = P\psi(-\vec{x},x_4), \\
{}^{\mathrm{P}}\bar{\psi}(x) &= \left({}^{\mathrm{P}}\bar{\psi}_{\mathrm{R}}(x),\ {}^{\mathrm{P}}\bar{\psi}_{\mathrm{L}}(x)\right) = \left(\bar{\psi}_{\mathrm{L}}(-\vec{x},x_4), \bar{\psi}_{\mathrm{R}}(-\vec{x},x_4)\right) = \bar{\psi}(-\vec{x},x_4)P^{-1}, \\
{}^{\mathrm{T}}\psi(x) &= \begin{pmatrix} {}^{\mathrm{T}}\psi_{\mathrm{L}}(x) \\ {}^{\mathrm{T}}\psi_{\mathrm{R}}(x) \end{pmatrix} = \begin{pmatrix} \mathrm{i}\sigma^2\bar{\psi}_{\mathrm{L}}(\vec{x},-x_4)^{\mathrm{T}} \\ \mathrm{i}\sigma^2\bar{\psi}_{\mathrm{R}}(\vec{x},-x_4)^{\mathrm{T}} \end{pmatrix} = T\bar{\psi}(\vec{x},-x_4)^{\mathrm{T}}, \\
{}^{\mathrm{T}}\bar{\psi}(x) &= \left({}^{\mathrm{T}}\bar{\psi}_{\mathrm{R}}(x),\ {}^{\mathrm{T}}\bar{\psi}_{\mathrm{L}}(x)\right) = \left(\psi_{\mathrm{R}}(\vec{x},-x_4)^{\mathrm{T}}\mathrm{i}\sigma^2, \psi_{\mathrm{L}}(\vec{x},-x_4)^{\mathrm{T}}\mathrm{i}\sigma^2\right) \\
&= -\psi(\vec{x},-x_4)^{\mathrm{T}}T^{-1}.
\end{aligned}
$$

$$
\begin{aligned}
C &= -C^{-1} = \begin{pmatrix} -\mathrm{i}\sigma^2 & 0 \\ 0 & \mathrm{i}\sigma^2 \end{pmatrix} = -\sigma^3 \otimes \mathrm{i}\sigma^2 = \gamma_2\gamma_4, C^{-1}\gamma_\mu C = -\gamma_\mu^{\top} \\
P &= P^{-1} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \sigma_1 \otimes \mathbb{1} = \gamma_4, P^{-1}\gamma_i P = -\gamma_i, P^{-1}\gamma_4 P = \gamma_4 \\
T &= -T^{-1} = \begin{pmatrix} 0 & \mathrm{i}\sigma^2 \\ \mathrm{i}\sigma^2 & 0 \end{pmatrix} = \sigma^1 \otimes \mathrm{i}\sigma^2 = \gamma_5\gamma_2, T^{-1}\gamma_i T = -\gamma_i^{\top}, T^{-1}\gamma_4 T = \gamma_4^{\top}
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{CPT}}\psi_{\mathrm{R}}(x) &= \mathrm{i}\sigma^2\ {}^{\mathrm{CP}}\bar{\psi}_{\mathrm{R}}(\vec{x},-x_4)^{\top} = \mathrm{i}\sigma^2(\mathrm{i}\sigma^2)^{\top}\psi_{\mathrm{R}}(-\vec{x},-x_4) = \psi_{\mathrm{R}}(-x) \\
{}^{\mathrm{CPT}}\bar{\psi}_{\mathrm{R}}(x) &= {}^{\mathrm{CP}}\psi_{\mathrm{R}}(\vec{x},-x_4)^{\top}\mathrm{i}\sigma^2 = \bar{\psi}_{\mathrm{R}}(-\vec{x},-x_4)(-\mathrm{i}\sigma^2)^{\top}\mathrm{i}\sigma^2 = -\bar{\psi}_{\mathrm{R}}(-x) \\
{}^{\mathrm{CPT}}\psi_{\mathrm{L}}(x) &= \mathrm{i}\sigma^2\ {}^{\mathrm{CP}}\bar{\psi}_{\mathrm{L}}(\vec{x},-x_4)^{\top} = -\mathrm{i}\sigma^2(\mathrm{i}\sigma^2)^{\top}\psi_{\mathrm{L}}(-\vec{x},-x_4) = -\psi_{\mathrm{L}}(-x) \\
{}^{\mathrm{CPT}}\bar{\psi}_{\mathrm{L}}(x) &= {}^{\mathrm{CP}}\psi_{\mathrm{L}}(\vec{x},-x_4)^{\top}\mathrm{i}\sigma^2 = \bar{\psi}_{\mathrm{L}}(-\vec{x},-x_4)(\mathrm{i}\sigma^2)^{\top}\mathrm{i}\sigma^2 = \bar{\psi}_{\mathrm{L}}(-x)
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{CPT}}\psi(x) &= \begin{pmatrix} {}^{\mathrm{CPT}}\psi_{\mathrm{L}}(x) \\ {}^{\mathrm{CPT}}\psi_{\mathrm{R}}(x) \end{pmatrix} = \begin{pmatrix} -\psi_{\mathrm{L}}(-x) \\ \psi_{\mathrm{R}}(-x) \end{pmatrix} = \gamma_5\psi(-x), \\
{}^{\mathrm{CPT}}\bar{\psi}(x) &= \left({}^{\mathrm{CPT}}\bar{\psi}_{\mathrm{R}}(x),\ {}^{\mathrm{CPT}}\bar{\psi}_{\mathrm{L}}(x)\right) = (-\bar{\psi}_{\mathrm{R}}(-x), \bar{\psi}_{\mathrm{L}}(-x)) = \bar{\psi}(-x)\gamma_5
\end{aligned}
$$

$$
\Lambda_{\mu\nu} = \frac{1}{2}\mathrm{ReTr}\left(\Lambda_{\mathrm{L}}^{\dagger}\sigma_\mu\Lambda_{\mathrm{R}}\bar{\sigma}_\nu\right) = -\frac{1}{2}\mathrm{ReTr}\left(\sigma_\mu\bar{\sigma}_\nu\right) = -\delta_{\mu\nu}, \Lambda_{\mathrm{D}} = \begin{pmatrix} \Lambda_{\mathrm{L}} & 0 \\ 0 & \Lambda_{\mathrm{R}} \end{pmatrix} = \gamma_5.
$$

$$
{}^{\mathrm{CPT}}\Phi(x) = \Phi(-x),\ {}^{\mathrm{CPT}}A_\mu(x) = -A_\mu(-x),\ {}^{\mathrm{CPT}}G_\mu(x) = -G_\mu(-x)
$$

$$
\phi'_{\mathrm{R}}(x') = \Lambda_{\mathrm{R}}\phi_{\mathrm{R}}(\Lambda^{-1}x')
$$

$$
(0,1/2) \times (0,1/2) = (0,0) + (0,1)
$$

$$
\epsilon_{ab}\phi_{\mathrm{R}}^a(x)\phi_{\mathrm{R}}^b(x) = \phi_{\mathrm{R}}(x)^{\top}\mathrm{i}\sigma^2\phi_{\mathrm{R}}(x) = \left(\phi_{\mathrm{R}}^1(x), \phi_{\mathrm{R}}^2(x)\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \phi_{\mathrm{R}}^1(x) \\ \phi_{\mathrm{R}}^2(x) \end{pmatrix}
$$

$$
\bar{\phi}'_{\mathrm{R}}(x') = \bar{\phi}_{\mathrm{R}}(\Lambda^{-1}x')\Lambda_{\mathrm{L}}^{\dagger}
$$

$$
\begin{aligned}
\Pi_{\phi_{\mathrm{R}}}(\vec{x}) &= \frac{\delta\mathcal{L}(\bar{\phi}_{\mathrm{R}},\phi_{\mathrm{R}})}{\delta\partial_0\phi_{\mathrm{R}}(\vec{x})} = \bar{\phi}_{\mathrm{R}}(\vec{x}), \Pi_{\bar{\phi}_{\mathrm{R}}}(\vec{x}) = \frac{\delta\mathcal{L}(\bar{\phi}_{\mathrm{R}},\phi_{\mathrm{R}})}{\delta\partial_0\bar{\phi}_{\mathrm{R}}(\vec{x})} = 0 \ \Rightarrow \\
\mathcal{H}(\bar{\phi}_{\mathrm{R}},\phi_{\mathrm{R}}) &= \partial_0\phi_{\mathrm{R}}(\vec{x})\Pi_{\phi_{\mathrm{R}}}(\vec{x}) + \partial_0\bar{\phi}_{\mathrm{R}}(\vec{x})\Pi_{\bar{\phi}_{\mathrm{R}}}(\vec{x}) - \mathcal{L}(\bar{\phi}_{\mathrm{R}},\phi_{\mathrm{R}}) \\
&= \bar{\phi}_{\mathrm{R}}(\vec{x})(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\phi_{\mathrm{R}}(\vec{x}) \ \Rightarrow \\
\hat{H} &= \int d^3x\hat{\bar{\phi}}_{\mathrm{R}}(\vec{x})(-\mathrm{i}\vec{\sigma}\cdot\vec{\nabla})\hat{\phi}_{\mathrm{R}}(\vec{x})
\end{aligned}
$$

$$Z = \mathrm{Tr}\exp\left(-\beta\left(\hat{H}_{\mathrm{L}} - \mu\hat{F}_{\mathrm{L}}\right)\right) = \prod_{\vec{p}} Z(\vec{p})$$

$$Z = \int \mathcal{D}\bar{\psi}_{\mathrm{L}}\mathcal{D}\psi_{\mathrm{L}}\exp\left(-S_{\mathrm{L}}[\bar{\psi}_{\mathrm{L}},\psi_{\mathrm{L}}]\right)$$

$$S_{\mathrm{L}}[\bar{\psi}_{\mathrm{L}},\psi_{\mathrm{L}}] = \int_0^\beta dx_4 \int d^3x\left(\bar{\psi}_{\mathrm{L}}(x)\bar{\sigma}_\mu\partial_\mu\psi_{\mathrm{L}}(x) - \mu\bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{L}}(x)\right)$$

$$\bar{\psi}_{\mathrm{L}}(x)\bar{\sigma}_4 D_4\psi_{\mathrm{L}}(x) = \bar{\psi}_{\mathrm{L}}(x)\partial_4\psi_{\mathrm{L}}(x) - \mu\bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{L}}(x)$$

$$\psi_{\mathrm{L}}(\vec{x},x_4+\beta) = -\psi_{\mathrm{L}}(\vec{x},x_4), \bar{\psi}_{\mathrm{L}}(\vec{x},x_4+\beta) = -\bar{\psi}_{\mathrm{L}}(\vec{x},x_4)$$

$$\psi_{\mathrm{L}}(\vec{p},x_4) = \int d^3x\psi_{\mathrm{L}}(\vec{x},x_4)\exp\left(-\mathrm{i}\vec{p}\cdot\vec{x}\right)$$

$$\bar{\psi}_{\mathrm{L}}(\vec{p},x_4) = \int d^3x\bar{\psi}_{\mathrm{L}}(\vec{x},x_4)\exp\left(\mathrm{i}\vec{p}\cdot\vec{x}\right)$$

$$Z(\vec{p}) = \mathrm{Tr}\exp\left(-\beta(\hat{H} - \mu\hat{F})\right) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left(-S[\bar{\psi},\psi]\right)$$

$$S[\bar{\psi},\psi] = \int_0^\beta dx_4\bar{\psi}(x_4)(\partial_4 + \vec{\sigma}\cdot\vec{p} - \mu)\psi(x_4)$$

$$\left\{\hat{\psi}^a,\hat{\psi}^{b\dagger}\right\} = \delta_{ab}, \left\{\hat{\psi}^a,\hat{\psi}^b\right\} = \left\{\hat{\psi}^{a\dagger},\hat{\psi}^{b\dagger}\right\} = 0, a,b \in \{1,2\}$$

$$\hat{\psi}^1|0\rangle = \hat{\psi}^2|0\rangle = 0, \hat{\psi}^{1\dagger}|0\rangle = |1\rangle, \hat{\psi}^{2\dagger}|0\rangle = |2\rangle, \hat{\psi}^{2\dagger}\hat{\psi}^{1\dagger}|0\rangle = |12\rangle.$$

$$\hat{\psi}^a|\psi\rangle = \psi^a|\psi\rangle, |\psi\rangle = |0\rangle - \psi^1|1\rangle - \psi^2|2\rangle + \psi^1\psi^2|12\rangle.$$

$$\hat{\psi}^1|\psi\rangle = \hat{\psi}^1|0\rangle + \psi^1\hat{\psi}^1|1\rangle + \psi^2\hat{\psi}^1|2\rangle + \psi^1\psi^2\hat{\psi}^1|12\rangle = \psi^1|0\rangle - \psi^1\psi^2|2\rangle = \psi^1|\psi\rangle.$$

$$\langle\bar{\psi}|\hat{\psi}^{a\dagger} = \langle\bar{\psi}|\bar{\psi}^a, \langle\bar{\psi}| = \langle 0| - \langle 1|\bar{\psi}^1 - \langle 2|\bar{\psi}^2 + \langle 12|\bar{\psi}^2\bar{\psi}^1.$$

$$\langle\bar{\psi} \mid \psi\rangle = \langle 0 \mid 0\rangle + \langle 1 \mid 1\rangle\bar{\psi}^1\psi^1 + \langle 2 \mid 2\rangle\bar{\psi}^2\psi^2 + \langle 12 \mid 12\rangle\bar{\psi}^2\bar{\psi}^1\psi^1\psi^2 = \exp\left(\bar{\psi}^1\psi^1 + \bar{\psi}^2\psi^2\right)$$

$$\int d\bar{\psi}^1 d\psi^1 d\bar{\psi}^2 d\psi^2|\psi\rangle\langle\bar{\psi}|\exp\left(-\bar{\psi}^1\psi^1 - \bar{\psi}^2\psi^2\right) = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |12\rangle\langle 12| = \mathbb{1}$$

$$\mathrm{Tr}\hat{A} = \int d\bar{\psi}^1 d\psi^1 d\bar{\psi}^2 d\psi^2\exp\left(-\bar{\psi}^1\psi^1 - \bar{\psi}^2\psi^2\right)\langle\bar{\psi}|\hat{A}| - \psi\rangle$$

$$\hat{A} = \exp\left(\hat{\psi}^\dagger\Lambda\hat{\psi}\right) \Rightarrow \langle\bar{\psi}|\hat{A}|\psi\rangle = \exp\left(\bar{\psi}\mathrm{e}^\Lambda\psi\right)$$

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_{n=1}^N \prod_{a=1,2} d\bar{\psi}_n^a d\psi_n^a$$

$$S[\bar{\psi},\psi] = a\sum_n \left(\frac{1}{2a}[\bar{\psi}_n(\psi_{n+1} - \psi_n) - (\bar{\psi}_{n+1} - \bar{\psi}_n)\psi_{n+1}] - \bar{\psi}_n(\vec{\sigma}\cdot\vec{p} + \mu)\psi_n\right)$$

$$Z(\vec{p}) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-S[\bar{\psi},\psi]\right) = \mathrm{Tr}\hat{T}^N, -\lim_{a\to 0}\frac{1}{a}\log\left(\hat{T}\right) = \hat{H} - \mu\hat{F}$$

$$\mathrm{Tr}\hat{T}^N = \int d\bar{\psi}_N^1 d\psi_1^1 d\bar{\psi}_N^2 d\psi_1^2 \exp\left(-\bar{\psi}_N\psi_1\right)\langle\bar{\psi}_N|\hat{T}^N|-\psi_1\rangle$$

$$\int d\bar{\psi}_n^1 d\psi_{n+1}^1 d\bar{\psi}_n^2 d\psi_{n+1}^2 |\psi_{n+1}\rangle\langle\bar{\psi}_n|\exp\left(-\bar{\psi}_n\psi_{n+1}\right) = \mathbb{1}$$

$$\langle\bar{\psi}_n|\hat{T}|\psi_n\rangle = \exp\left(\bar{\psi}_n\psi_n + a\bar{\psi}_n(\vec{\sigma}\cdot\vec{p}+\mu)\psi_n\right)$$

$$\hat{T} = \exp\left(\hat{\psi}^\dagger\Lambda\hat{\psi}\right) = \exp\left(\hat{\psi}^\dagger\log\left(\mathbb{1}+a(\vec{\sigma}\cdot\vec{p}+\mu)\right)\hat{\psi}\right) \Rightarrow$$
$$\lim_{a\to 0}\hat{T} = \exp\left(a\hat{\psi}^\dagger(\vec{\sigma}\cdot\vec{p}+\mu)\hat{\psi}\right) = \exp\left(-a(\hat{H}-\mu\hat{F})\right)$$

$$\gamma^0 = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}, \gamma^1 = \begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}, \gamma^2 = \begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix}, \gamma^3 = \begin{pmatrix}0 & \sigma^3\\ -\sigma^3 & 0\end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix}0 & \sigma^2\\ \sigma^2 & 0\end{pmatrix}, \gamma^1 = \begin{pmatrix}\mathrm{i}\sigma^3 & 0\\ 0 & \mathrm{i}\sigma^3\end{pmatrix}, \gamma^2 = \begin{pmatrix}0 & -\sigma^2\\ \sigma^2 & 0\end{pmatrix}, \gamma^3 = \begin{pmatrix}-\mathrm{i}\sigma^1 & 0\\ 0 & -\mathrm{i}\sigma^1\end{pmatrix}.$$

$$Z = \int \mathcal{D}\bar{\psi}_{\mathrm{L}}\mathcal{D}\psi_{\mathrm{L}} \exp\left(-\frac{1}{2}\left(\psi_{\mathrm{L}}^{\mathrm{T}},\bar{\psi}_{\mathrm{L}}\right)A_{\mathrm{L}}\begin{pmatrix}\psi_{\mathrm{L}}\\ \bar{\psi}_{\mathrm{L}}^{\mathrm{T}}\end{pmatrix}\right) = \mathrm{Pf}(A_{\mathrm{L}})$$
$$A_{\mathrm{L}} = \begin{pmatrix}0 & -\mathrm{i}\sigma^2\\ -\mathrm{i}\sigma^2 & 0\end{pmatrix}A_{\mathrm{R}}\begin{pmatrix}0 & \mathrm{i}\sigma^2\\ \mathrm{i}\sigma^2 & 0\end{pmatrix} = \begin{pmatrix}-\mathrm{i}\sigma^2 m\delta(x-y) & W_{\mathrm{L}}\\ -W_{\mathrm{L}}^{\mathrm{T}} & \mathrm{i}\sigma^2 m\delta(x-y)\end{pmatrix}$$

$$S[\bar{\psi},\psi] = \bar{\psi}_x\psi_y + \bar{\psi}_y\psi_x + m\left(\bar{\psi}_x\psi_x + \bar{\psi}_y\psi_y\right)$$

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-S[\bar{\psi},\psi]\right)$$

$$\begin{aligned}\mathcal{L}(\bar{\psi},\psi) &= \bar{\psi}(x)\left(\gamma_\mu\partial_\mu + m\right)\psi(x)\\ &= \bar{\psi}_{\mathrm{R}}(x)\sigma_\mu\partial_\mu\psi_{\mathrm{R}}(x) + \bar{\psi}_{\mathrm{L}}(x)\bar{\sigma}_\mu\partial_\mu\psi_{\mathrm{L}}(x) + m[\bar{\psi}_{\mathrm{R}}(x)\psi_{\mathrm{L}}(x) + \bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{R}}(x)].\end{aligned}$$

$$\psi_{\mathrm{L}}(x) \to \exp(\mathrm{i}\chi_{\mathrm{L}})\psi_{\mathrm{L}}(x), \quad \bar{\psi}_{\mathrm{L}}(x) \to \bar{\psi}_{\mathrm{L}}(x)\exp(-\mathrm{i}\chi_{\mathrm{L}})$$
$$\psi_{\mathrm{R}}(x) \to \exp(\mathrm{i}\chi_{\mathrm{R}})\psi_{\mathrm{R}}(x), \quad \bar{\psi}_{\mathrm{R}}(x) \to \bar{\psi}_{\mathrm{R}}(x)\exp(-\mathrm{i}\chi_{\mathrm{R}})$$

$$\psi(x) \to \exp(\mathrm{i}\chi_{\mathrm{v}})\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)\exp(-\mathrm{i}\chi_{\mathrm{v}})$$
$$\psi(x) \to \exp(\mathrm{i}\chi_{\mathrm{a}}\gamma_5)\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)\exp(\mathrm{i}\chi_{\mathrm{a}}\gamma_5)$$

$$j_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x), j_\mu^5(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)$$

$$F(x_4) = \int d^3x j_4(\vec{x},x_4), \partial_\mu j_\mu(x) = 0$$

$$\partial_\mu j_\mu^5(x) = 2m\bar{\psi}(x)\gamma_5\psi(x)$$

$$S[\bar{\psi},\psi] = \int d^dx\bar{\psi}\left(\gamma_\mu\partial_\mu + m\right)\psi$$

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-S[\bar{\psi},\psi]\right)$$

$$S[\bar{\Psi},\Psi] = a^d \sum_{x,\mu} \frac{1}{2a}\left(\bar{\Psi}_x\gamma_\mu\Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}}\gamma_\mu\Psi_x\right) + a^d \sum_x m\bar{\Psi}_x\Psi_x$$

$$S[\bar{\Psi},\Psi]= \sum_{xy} \bar{\Psi}_x D_{xy}\Psi_y \equiv \bar{\Psi}D\Psi$$

$$\frac{1}{a^d}D_{xy} = \frac{1}{2a}\sum_{\mu} \gamma_\mu\left(\delta_{x+\hat{\mu},y} - \delta_{x-\hat{\mu},y}\right) + m\delta_{xy}$$

$$Z = \int \prod_x d\bar{\Psi}_x d\Psi_x \exp\left(-S[\bar{\Psi},\Psi]\right) = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\left(-\bar{\Psi}D\Psi\right)$$

$$\langle\bar{\Psi}(-p)\Psi(p)\rangle = \left[\mathrm{i}\sum_{\mu} \gamma_\mu \frac{1}{a}\sin\left(p_\mu a\right) + m\right]^{-1}$$

$$\langle\bar{\Psi}(-\vec{p},0)\Psi(\vec{p},x_d)\rangle = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dp_d\langle\bar{\Psi}(-p)\Psi(p)\rangle\exp\left(\mathrm{i}p_d x_d\right) \sim \exp\left(-E(\vec{p})x_d\right)$$

$$\sinh^2\left(E(\vec{p})a\right) = \sum_i \sin^2\left(p_i a\right) + (ma)^2$$

$$S[\bar{\Psi},\Psi] = a^d \sum_{x,y,\mu} \bar{\Psi}_x\gamma_\mu\rho_\mu(x-y)\Psi_y$$

$$\langle\bar{\Psi}(-p)\Psi(p)\rangle = \left[\mathrm{i}\sum_{\mu} \gamma_\mu\rho_\mu(p)\right]^{-1}$$

$$\langle\bar{\Psi}_{\mathrm{L}}(-p)\Psi_{\mathrm{L}}(p)\rangle = \left[\mathrm{i}\sum_{\mu} \bar{\sigma}_\mu \frac{1}{a}\sin\left(p_\mu a\right)\right]^{-1}$$

$$(-\mathrm{i}\sigma^1,\mathrm{i}\sigma^2,\mathrm{i}\sigma^3,\mathbb{1}) = (\mathrm{i}\sigma^1)(-\mathrm{i}\sigma^1,-\mathrm{i}\sigma^2,-\mathrm{i}\sigma^3,\mathbb{1})(-\mathrm{i}\sigma^1) = (\mathrm{i}\sigma^1)\sigma_\mu(\mathrm{i}\sigma^1)^\dagger$$

$$(-\mathrm{i}\sigma^1,-\mathrm{i}\sigma^2,\mathrm{i}\sigma^3,\mathbb{1}) = (\mathrm{i}\sigma^3)(\mathrm{i}\sigma^1,\mathrm{i}\sigma^2,\mathrm{i}\sigma^3,\mathbb{1})(-\mathrm{i}\sigma^3) = (\mathrm{i}\sigma^3)\bar{\sigma}_\mu(\mathrm{i}\sigma^3)^\dagger$$

$$S[\bar{\Psi},\Psi]= \sum_{x,y} \bar{\Psi}_x D_{\mathrm{W},xy}\Psi_y = a^d \sum_{x,\mu} \frac{1}{2a}\left(\bar{\Psi}_x\gamma_\mu\Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}}\gamma_\mu\Psi_x\right) + a^d \sum_x m\bar{\Psi}_x\Psi_x$$
$$+a^d \sum_{x,\mu} \frac{1}{2a}\left(2\bar{\Psi}_x\Psi_x - \bar{\Psi}_x\Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}}\Psi_x\right)$$

$$\Psi_x = \frac{1}{a^d}\int_{c_x} d^d y\psi(y), \bar{\Psi}_x = \frac{1}{a^d}\int_{c_x} d^d y\bar{\psi}(y)$$

$$\Psi(p)=\sum_{l\in\mathbb{Z}^d}\psi(p+2\pi l/a)\Pi(p+2\pi l/a)$$

$$\bar{\Psi}(-p)=\sum_{n\in\mathbb{Z}^d}\bar{\psi}(-p-2\pi n/a)\Pi(p+2\pi n/a)$$

$$\Pi(p)=\prod_{\mu=1}^{d}\frac{2\sin\left(p_\mu a/2\right)}{p_\mu a}$$

$$\begin{aligned}\langle\bar{\Psi}(-p)\Psi(p)\rangle&=\sum_{l\in\mathbb{Z}^d}\langle\bar{\psi}(-p-2\pi l/a)\psi(p+2\pi l/a)\rangle\Pi(p+2\pi l/a)^2\\&=\sum_{l\in\mathbb{Z}^d}\frac{\Pi(p+2\pi l/a)^2}{\mathrm{i}\gamma_\mu\left(p_\mu+2\pi l_\mu/a\right)+m}=D_{\text{perfect}}(p)^{-1}\end{aligned}$$

$$S[\bar{\Psi},\Psi]=a^d\sum_{x,y}\bar{\Psi}_x\gamma_\mu\rho_\mu(x-y)\Psi_y=\sum_{x,y}\bar{\Psi}_x D_{\text{perfect},xy}\Psi_y$$

$$\left|\rho_\mu(x-y)\right|\propto|x-y|^{1-d}$$

$$\langle\bar{\Psi}(-p)\Psi(p)\rangle=\sum_{l\in\mathbb{Z}}\frac{\Pi(p+2\pi l/a)^2}{\mathrm{i}(p+2\pi l/a)}=\frac{a}{2\mathrm{i}}\cot\left(\frac{pa}{2}\right),$$

$$D_{\text{perfect}}(p)=\rho_1(p)=\frac{2}{a}\tan\left(\frac{pa}{2}\right)$$

$$\rho_1(x-y)=\frac{1}{a}(-1)^{(x-y)/a}$$

$$\begin{aligned}\exp\left(-S[\bar{\Psi},\Psi]\right)=&\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left\{-\frac{1}{(2\pi)^d}\int d^dp\bar{\psi}(-p)\left(\mathrm{i}\gamma_\mu p_\mu+m\right)\psi(p)\right\}\\&\times\exp\left\{-\frac{1}{\alpha}\frac{1}{(2\pi)^d}\int_B d^dp\left[\bar{\Psi}(-p)-\sum_{n\in\mathbb{Z}^d}\bar{\psi}(-p-2\pi n/a)\Pi(p+2\pi n/a)\right]\right.\\&\left.\times\left[\Psi(p)-\sum_{l\in\mathbb{Z}^d}\psi(p+2\pi l/a)\Pi(p+2\pi l/a)\right]\right\}\end{aligned}$$

$$\langle\bar{\Psi}(-p)\Psi(p)\rangle=\sum_{l\in\mathbb{Z}^d}\frac{\Pi(p+2\pi l/a)^2}{\mathrm{i}\gamma_\mu\left(p_\mu+2\pi l_\mu/a\right)+m}+\alpha$$

$$\langle\bar{\Psi}(-p)\Psi(p)\rangle=\frac{1}{m}-\frac{2}{m^2a}\left[\coth\left(\frac{ma}{2}\right)-\mathrm{i}\cot\left(\frac{pa}{2}\right)\right]^{-1}+\alpha$$

$$\alpha=\frac{\exp(ma)-1-ma}{m^2a},$$

$$\langle\bar{\Psi}(-p)\Psi(p)\rangle=\left(\frac{\exp(ma)-1}{ma}\right)^2\left[\frac{\mathrm{i}}{a}\sin(pa)+\frac{\exp(ma)-1}{a}+\frac{2}{a}\sin^2\left(\frac{pa}{2}\right)\right]^{-1}.$$

$$
\begin{aligned}
\langle\bar{\Psi}(-\vec{p},0)\Psi(\vec{p},x_d)\rangle =& \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dp_d \langle\bar{\Psi}(-p)\Psi(p)\rangle \exp\,(\mathrm{i}p_d x_d) \\
=& \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dp_d \left[\sum_{l\in\mathbb{Z}^d} \frac{\Pi(p+2\pi l/a)^2}{\mathrm{i}\gamma_\mu(p_\mu+2\pi l_\mu/a)+m} + \alpha\right] \exp\,(\mathrm{i}p_d x_d) \\
=& \frac{1}{2\pi}\int_{-\infty}^{\infty} dp_d \sum_{\vec{l}\in\mathbb{Z}^{d-1}} \frac{m}{(\vec{p}+2\pi\vec{l}/a)^2+p_d^2+m^2} \\
&\times \prod_{i=1}^{d-1}\left(\frac{2\sin\,(p_i a/2)}{p_i a+2\pi l_i}\right)^2 \left(\frac{2\sin\,(p_d a/2)}{p_d a}\right)^2 \exp\,(\mathrm{i}p_d x_d) + \alpha\delta_{x_d,0} \\
=& \sum_{\vec{l}\in\mathbb{Z}^{d-1}} C(\vec{p}+2\pi\vec{l}/a)\exp\left(-E(\vec{p}+2\pi\vec{l}/a)x_d\right) + \alpha\delta_{x_d,0}
\end{aligned}
$$

$$E(\vec{p}+2\pi\vec{l}/a)^2 = -p_d^2 = (\vec{p}+2\pi\vec{l}/a)^2 + m^2$$

$$\{D^{-1},\gamma_5\} = 2\alpha\gamma_5$$

$$\{D,\gamma_5\} = aD\gamma_5 D$$

$$\Psi \to \left(1+\mathrm{i}\gamma_5\varepsilon\left[1-\frac{a}{2}D\right]\right)\Psi, \bar{\Psi} \to \bar{\Psi}\left(1+\mathrm{i}\left[1-\frac{a}{2}D\right]\varepsilon\gamma_5\right).$$

$$
\begin{aligned}
\mathcal{L} = \bar{\Psi}D\Psi \to & \bar{\Psi}\left(1+\mathrm{i}\left[1-\frac{a}{2}D\right]\varepsilon\gamma_5\right)D\left(1+\mathrm{i}\gamma_5\varepsilon\left[1-\frac{a}{2}D\right]\right)\Psi \\
=& \bar{\Psi}D\Psi + \mathrm{i}\varepsilon\bar{\Psi}\left(\left[1-\frac{a}{2}D\right]\gamma_5 D + D\gamma_5\left[1-\frac{a}{2}D\right]\right)\Psi + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

$$D_{\mathrm{W}}^\dagger = \gamma_5 D_{\mathrm{W}}\gamma_5.$$

$$D + D^\dagger = aD^\dagger D$$

$$A^\dagger A = a^2 D^\dagger D - aD - aD^\dagger + \mathbb{1}.$$

$$A_{\text{overlap}} = A_{\mathrm{W}}\left(A_{\mathrm{W}}^\dagger A_{\mathrm{W}}\right)^{-1/2} \Rightarrow A_{\text{overlap}}^\dagger A_{\text{overlap}} = \mathbb{1}.$$

$$D_{\text{overlap}} = \frac{1}{a}\left(A_{\text{overlap}} + \mathbb{1}\right) = (D_{\mathrm{W}} - \mathbb{1}/a)\left[(aD_{\mathrm{W}}^\dagger - \mathbb{1})(aD_{\mathrm{W}} - \mathbb{1})\right]^{-1/2} + \frac{\mathbb{1}}{a}$$

$$G_\mu(x) = \mathrm{i}g_s G_\mu^a(x)T^a$$

$$[T^a,T^b] = \mathrm{i}f_{abc}T^c, \mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta_{ab}$$

$$G'_\mu(x) = \Omega(x)\left(G_\mu(x) + \partial_\mu\right)\Omega(x)^\dagger.$$

$$G_{\mu\nu}(x) = \partial_\mu G_\nu(x) - \partial_\nu G_\mu(x) + \left[G_\mu(x), G_\nu(x)\right]$$

$$G'_{\mu\nu}(x) = \Omega(x)G_{\mu\nu}(x)\Omega(x)^\dagger.$$

$$\mathcal{L}(G) = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} = \frac{1}{2g_s^2}\mathrm{Tr}\left[G_{\mu\nu}G^{\mu\nu}\right], G_{\mu\nu}(x) = \mathrm{i}g_s G^a_{\mu\nu}(x)T^a$$

$$E_i(x) = G^{i0}(x) = \partial^i G^0(x) - \partial^0 G^i(x) + \left[G^i(x), G^0(x)\right] = -\partial^0 G^i(x).$$

$$\Pi_i^a(x) = \frac{\delta\mathcal{L}}{\delta\partial^0 G^{ai}(x)} = \partial^0 G^i(x) = -E_i^a(x)$$

$$\mathcal{H}\left(G^{ai}, \Pi_i^a\right) = \Pi_i^a(x)\partial^0 G^{ai}(x) - \mathcal{L} = \frac{1}{2}\left(E_i^a(x)E_i^a(x) + B_i^a(x)B_i^a(x)\right).$$

$$B_i(x) = -\frac{1}{2}\epsilon_{ijk}G^{jk}(x) = \mathrm{i}g_s B_i^a(x)T^a$$

$$Z = \int \mathcal{D}G\exp\left(-S[G]\right), S[G] = -\int d^d x\frac{1}{2g_s^2}\mathrm{Tr}\left[G_{\mu\nu}G_{\mu\nu}\right]$$

$$\Omega(x) = \exp\left(\mathrm{i}\omega^a(x)T^a\right) \approx \mathbb{1} + \omega(x), \omega(x) = \mathrm{i}\omega^a(x)T^a,$$

$$\begin{aligned} {}^{\omega}G_\mu(x) &= \Omega(x)\left(G_\mu(x) + \partial_\mu\right)\Omega(x)^\dagger \approx \left(\mathbb{1} + \omega(x)\right)\left(G_\mu(x) + \partial_\mu\right)\left(\mathbb{1} - \omega(x)\right) \\ &= G_\mu(x) + \left[\omega(x), G_\mu(x)\right] - \partial_\mu\omega(x) + \mathcal{O}(\omega^2) \\ &= G_\mu(x) - D_\mu\omega(x) + \mathcal{O}(\omega^2) \end{aligned}$$

$$D_\mu\omega(x) = \partial_\mu\omega(x) + \left[G_\mu(x), \omega(x)\right].$$

$$C^a(x) = \partial_\mu {}^{\omega}G_\mu^a(x), C(x) = g_s C^a(x)T^a = -\mathrm{i}\partial_\mu {}^{\omega}G_\mu(x),$$

$$\int \mathcal{D}C\exp\left(-S_{\mathrm{gf}}[C]\right) = \int \mathcal{D}C\exp\left(-\int d^d x\frac{1}{2\xi}C^a C^a\right), \xi > 0$$

$$S_{\mathrm{gf}}[C] = \int d^d x\frac{1}{2\xi}C^a C^a = \int d^d x\frac{1}{\xi g_s^2}\mathrm{Tr}[C^2] = -\int d^d x\frac{1}{\xi g_s^2}\mathrm{Tr}\left[\left(\partial_\mu {}^{\omega}G_\mu\right)^2\right] = S_{\mathrm{gf}}[G]$$

$$\mathcal{D}C = \mathcal{D}\omega\det\left(\frac{\delta C}{\delta\omega}\right) = \mathcal{D}\omega\det\left(-\mathrm{i}\frac{\delta\partial_\mu {}^{\omega}G_\mu}{\delta\omega}\right).$$

$$\delta C(x) = -\mathrm{i}\delta\partial_\mu {}^{\omega}G_\mu(x) = -\mathrm{i}\left(\partial_\mu {}^{\omega}G_\mu(x) - \partial_\mu G_\mu(x)\right) = -\partial_\mu D_\mu\omega(x).$$

$$\det\left(\frac{\delta C}{\delta\omega}\right) = \det\left(-\mathrm{i}\frac{\delta\partial_\mu {}^{\omega}G_\mu}{\delta\omega}\right) = \det\left(-\partial_\mu D_\mu\right).$$

$$\begin{aligned} \det\left(-\partial_\mu D_\mu\right) &= \int \mathcal{D}\bar{c}\mathcal{D}c\exp\left(-S_{\mathrm{gh}}[\bar{c}, c, G]\right) \\ S_{\mathrm{gh}}[\bar{c}, c, G] &= \int d^d x\mathrm{Tr}\left[\bar{c}\partial_\mu D_\mu c\right], D_\mu c(x) = \partial_\mu c(x) + \left[G_\mu(x), c(x)\right] \end{aligned}$$

$$c(x) = c^a(x)T^a, \bar{c}(x) = \bar{c}^a(x)T^a$$

$$
\begin{aligned}
Z=&\int \mathcal{D}C\int \mathcal{D}G\exp\left(-S[G]-S_{\mathrm{gf}}[C]\right)\\
=&\int \mathcal{D}\omega\int \mathcal{D}G\det\left(-\partial_\mu D_\mu\right)\exp\left(-S[G]-S_{\mathrm{gf}}[{}^{\omega}G]\right)\\
=&\int \mathcal{D}\omega\int \mathcal{D}^{\omega}G\det\left(-\partial_\mu D_\mu\right)\exp\left(-S[{}^{\omega}G]-S_{\mathrm{gf}}[{}^{\omega}G]\right)\\
=&\int \mathcal{D}\omega\int \mathcal{D}G\det\left(-\partial_\mu D_\mu\right)\exp\left(-S[G]-S_{\mathrm{gf}}[G]\right)\\
=&\int \mathcal{D}G\mathcal{D}\bar{c}\mathcal{D}c\exp\left(-S[G]-S_{\mathrm{gf}}[G]-S_{\mathrm{gh}}[\bar{c},c,G]\right)
\end{aligned}
$$

$$
\begin{aligned}
\exp\left(\mathrm{i}\theta(x)\right)=\exp\left(\mathrm{i}\bar{\varepsilon}c(x)\right)&=\mathbb{1}+\mathrm{i}\bar{\varepsilon}c(x)=\mathbb{1}+\mathrm{i}\bar{\varepsilon}c^{a}(x)T^{a},\\
{}^{\theta}G_\mu(x)&=\exp\left(\mathrm{i}\theta(x)\right)\left(G_\mu(x)+\partial_\mu\right)\exp\left(-\mathrm{i}\theta(x)\right)\\
&=\left(\mathbb{1}+\mathrm{i}\theta(x)\right)\left(G_\mu(x)+\partial_\mu\right)\left(\mathbb{1}-\mathrm{i}\theta(x)\right)\\
&=G_\mu(x)+\mathrm{i}\left[\theta(x),G_\mu(x)\right]-\mathrm{i}\partial_\mu\theta(x).
\end{aligned}
$$

$$
\begin{aligned}
\delta G_\mu(x)=&\ {}^{\theta}G_\mu(x)-G_\mu(x)=\mathrm{i}\left[\theta(x),G_\mu(x)\right]-\mathrm{i}\partial_\mu\theta(x)\\
&=\mathrm{i}\bar{\varepsilon}\left[c(x),G_\mu(x)\right]-\mathrm{i}\bar{\varepsilon}\partial_\mu c(x).
\end{aligned}
$$

$$
\delta c(x)=\mathrm{i}\bar{\varepsilon}c(x)c(x)
$$

$$
\delta\bar{c}(x)=\frac{\mathrm{i}}{\xi g_{\mathrm{s}}^{2}}\bar{\varepsilon}\partial_\mu G_\mu(x)
$$

$$
\delta\mathcal{L}_{\mathrm{gf}}=-\frac{1}{\xi g_{\mathrm{s}}^{2}}\mathrm{Tr}\left[\partial_\mu\delta G_\mu\partial_\nu G_\nu\right]=-\frac{\mathrm{i}}{\xi g_{\mathrm{s}}^{2}}\mathrm{Tr}\left[\bar{\varepsilon}\partial_\mu\left(\left[c,G_\mu\right]-\partial_\mu c\right)\partial_\nu G_\nu\right].
$$

$$
\begin{aligned}
\delta\mathcal{L}_{\mathrm{gh}}=&-\mathrm{Tr}\left\{\delta\bar{c}\partial_\mu\left(\partial_\mu c+\left[G_\mu,c\right]\right)+\bar{c}\partial_\mu\left(\partial_\mu\delta c+\left[\delta G_\mu,c\right]+\left[G_\mu,\delta c\right]\right)\right\}\\
=&-\mathrm{Tr}\left\{\frac{\mathrm{i}}{\xi g_{\mathrm{s}}^{2}}\bar{\varepsilon}\partial_\nu G_\nu\partial_\mu\left(\partial_\mu c+\left[G_\mu,c\right]\right)\right.\\
&\left.+\mathrm{i}\bar{c}\partial_\mu\left(\bar{\varepsilon}\partial_\mu cc+\bar{\varepsilon}c\partial_\mu c+\left[\bar{\varepsilon}\left[c,G_\mu\right],c\right]-\left[\bar{\varepsilon}\partial_\mu c,c\right]+\left[G_\mu,\bar{\varepsilon}cc\right]\right)\right\}
\end{aligned}
$$

$$
\delta\left(\mathcal{L}_{\mathrm{gf}}+\mathcal{L}_{\mathrm{gh}}\right)=0
$$

$$
\begin{aligned}
\delta G_\mu(x)=\bar{\varepsilon}{}^{Q}G_\mu(x)&\Rightarrow {}^{Q}G_\mu(x)=\mathrm{i}\left[c(x),G_\mu(x)\right]-\mathrm{i}\partial_\mu c(x),\\
\delta c(x)=\bar{\varepsilon}{}^{Q}c(x)&\Rightarrow Q_c(x)=\mathrm{i}c(x)c(x),\\
\delta\bar{c}(x)=\bar{\varepsilon}{}^{Q}\bar{c}(x)&\Rightarrow Q_{\bar{c}}(x)=\frac{\mathrm{i}}{\xi g_{\mathrm{s}}^{2}}\partial_\mu G_\mu(x).
\end{aligned}
$$

$$
\begin{aligned}
Q^{2}G_\mu(x)=&\ {}^{Q}\left\{\mathrm{i}\left[c(x),G_\mu(x)\right]-\mathrm{i}\partial_\mu c(x)\right\}\\
&=\mathrm{i}\left\{{}^{Q}c(x)G_\mu(x)-c(x){}^{Q}G_\mu(x)-{}^{Q}G_\mu(x)c(x)-G_\mu(x){}^{Q}c(x)-\partial_\mu{}^{Q}c(x)\right\}=0
\end{aligned}
$$

$$
Q^{2}c(x)=\ {}^{Q}\{\mathrm{i}c(x)c(x)\}=\mathrm{i}^{Q}c(x)c(x)-\mathrm{i}c(x)^{Q}c(x)=-c(x)c(x)c(x)+c(x)c(x)c(x)=0
$$

$$
Q^{2}\bar{c}(x)=Q\left\{\frac{1}{\xi g_{\mathrm{s}}^{2}}\partial_\mu G_\mu(x)\right\}=\frac{\mathrm{i}}{\xi g_{\mathrm{s}}^{2}}\partial_\mu\left\{\left[c(x),G_\mu(x)\right]-\partial_\mu c(x)\right\}=0
$$

$$|\psi\rangle = \hat{Q}|\chi\rangle \;\Rightarrow\; \hat{Q}|\psi\rangle = \hat{Q}^2|\chi\rangle = 0$$

$$|\psi\rangle = \hat{Q}|\chi\rangle \;\Rightarrow\; \langle\psi \mid \psi\rangle = \langle\chi|\hat{Q}^2|\chi\rangle = 0.$$

$$\langle\psi \mid \psi'\rangle = \langle\chi|\hat{Q}^2|\chi'\rangle = 0$$

$$\mathcal{H} = \mathcal{H}_> + \mathcal{H}_< + \mathcal{H}_0$$
$$\mathcal{H}_> = \mathcal{H} \setminus \ker(\hat{Q}), \mathcal{H}_< = \operatorname{im}(\hat{Q}), \mathcal{H}_0 = \operatorname{cohom}(\hat{Q}) = \ker(\hat{Q}) \setminus \operatorname{im}(\hat{Q})$$

$$|\psi'\rangle = |\psi\rangle - \hat{Q}|\eta\rangle;$$

$$\langle\psi' \mid \psi'\rangle = \langle\psi \mid \psi\rangle - \langle\eta|\hat{Q}|\psi\rangle - \langle\psi|\hat{Q}|\eta\rangle + \langle\eta|\hat{Q}^2|\eta\rangle = \langle\psi \mid \psi\rangle.$$

$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} - \vec{\nabla} \times \vec{\nabla}\alpha = \vec{B}$$

$$A' = A - d\alpha \;\Rightarrow\; {}^\star B' = dA' = dA - d^2\alpha = {}^\star B$$

$$d^\star B = d^2 A = 0$$

$$U_{x,i} = \exp\left(\mathrm{ie}\int_x^{x+\hat{\imath}} dy_i A_i(y)\right) \in \mathrm{U}(1)$$

$$U'_{x,i} = \exp\left(\mathrm{ie}\int_x^{x+\hat{\imath}} dy_i A'_i(y)\right) = \exp\left(\mathrm{ie}\int_x^{x+\hat{\imath}} dy_i[A_i(y) - \partial_i\alpha(y)]\right)$$
$$= \exp\left(\mathrm{ie}\left[\int_x^{x+\hat{\imath}} dy_i A_i(y) + \alpha(x) - \alpha(x+\hat{\imath})\right]\right) = \Omega_x U_{x,i}\Omega^\dagger_{x+\hat{\imath}}$$
$$\Omega_x = \exp\left(\mathrm{ie}\alpha(x)\right) \in \mathrm{U}(1)$$

$$\hat{J} = -\mathrm{i}\partial_\varphi, \hat{J}|m\rangle = m|m\rangle, m \in \mathbb{Z}, \langle\varphi \mid m\rangle = \frac{1}{\sqrt{2\pi}}\exp\left(\mathrm{i}m\varphi\right)$$

$$[\hat{J}_{x,i}, \hat{U}_{y,j}] = \delta_{xy}\delta_{ij}\hat{U}_{x,i}, [\hat{J}_{x,i}, \hat{U}^\dagger_{y,j}] = -\delta_{xy}\delta_{ij}\hat{U}^\dagger_{x,i},$$
$$[\hat{J}_{x,i}, \hat{J}_{y,j}] = [\hat{U}_{x,i}, \hat{U}_{y,j}] = [\hat{U}^\dagger_{x,i}, \hat{U}^\dagger_{y,j}] = [\hat{U}_{x,i}, \hat{U}^\dagger_{y,j}] = 0.$$

$$a\hat{H} = \frac{e^2}{2}\sum_{x,i} \hat{J}^2_{x,i} + \frac{1}{2e^2}\sum_{x,i>j} \left(2 - \hat{U}_{x,i}\hat{U}_{x+\hat{\imath},j}\hat{U}^\dagger_{x+\hat{\jmath},i}\hat{U}^\dagger_{x,j} - \hat{U}_{x,j}\hat{U}_{x+\hat{\jmath},i}\hat{U}^\dagger_{x+\hat{\imath},j}\hat{U}^\dagger_{x,i}\right).$$

$$U_{x,i}U_{x+\hat{\imath},j}U^*_{x+\hat{\jmath},i}U^*_{x,j} =$$
$$\exp\left(\mathrm{i}ea\left[A_i(x+\hat{\imath}/2) + A_j(x+\hat{\imath}+\hat{\jmath}/2) - A_i(x+\hat{\jmath}+\hat{\imath}/2) - A_j(x+\hat{\jmath}/2)\right]\right) \to$$
$$\exp\left(\mathrm{i}ea^2 F_{ij}(x+\hat{\imath}/2+\hat{\jmath}/2)\right) \Rightarrow$$
$$\frac{1}{a^4}\left(2 - U_{x,i}U_{x+\hat{\imath},j}U^*_{x+\hat{\jmath},i}U^*_{x,j} - U_{x,j}U_{x+\hat{\jmath},i}U^*_{x+\hat{\imath},j}U^*_{x,i}\right) =$$
$$\frac{2}{a^4}\left[1 - \cos\left(ea^2 F_{ij}(x+\hat{\imath}/2+\hat{\jmath}/2)\right)\right] \to e^2 F_{ij}(x+\hat{\imath}/2+\hat{\jmath}/2)^2 \Rightarrow$$
$$H \to \int d^3x \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$$

$$\hat{G}_x = \sum_i \left(\hat{J}_{x,i} - \hat{J}_{x-\hat{\imath},i}\right) = \frac{a^2}{e}\sum_i \left(\hat{E}_{x,i} - \hat{E}_{x-\hat{\imath},i}\right), [\hat{G}_x, \hat{G}_y] = 0, [\hat{H}, \hat{G}_x] = 0$$

$$\hat{V}\hat{U}_{x,i}\hat{V}^{\dagger} = \Omega_x \hat{U}_{x,i}\Omega_{x+i}^{\dagger}, \hat{V} = \prod_x \exp\left(\mathrm{i}e\alpha_x \hat{G}_x\right)$$

$$\hat{G}_x|\psi, Q\rangle = Q_x|\psi, Q\rangle$$

$$Z_Q = \mathrm{Tr}\left[\exp\left(-\beta\hat{H}\right)\hat{P}_Q\right]$$

$$\frac{Z_Q}{Z} = \exp\left(-\beta V(x-y)\right), V(x-y) \sim \sigma|x-y|$$

$$\begin{aligned} U_{x,i} &= \mathcal{P}\exp\left(\int_x^{x+\hat{\imath}} dy_i G_i(y)\right) \\ &= \lim_{N\to\infty} \exp\left(\frac{a}{N}G_i(x+\hat{\imath}/2N)\right)\exp\left(\frac{a}{N}G_i(x+3\hat{\imath}/2N)\right)\exp\left(\frac{a}{N}G_i(x+5\hat{\imath}/2N)\right)\ldots \\ &\ldots\exp\left(\frac{a}{N}G_i(x+(2N-3)\hat{\imath}/2N)\right)\exp\left(\frac{a}{N}G_i(x+(2N-1)\hat{\imath}/2N)\right) \end{aligned}$$

$$U'_{x,i} = \Omega_x U_{x,i}\Omega_{x+\hat{\imath}}^{\dagger}$$

$$\hat{U} = \cos\,\alpha\mathbb{1} + \mathrm{i}\sin\,\alpha\vec{e}_\alpha\cdot\vec{\tau}, \vec{e}_\alpha = (\sin\,\theta\cos\,\varphi, \sin\,\theta\sin\,\varphi, \cos\,\theta).$$

$$\begin{aligned} &\hat{R}^a = \frac{1}{2}\left(\hat{J}^a + \hat{K}^a\right), \hat{L}^a = \frac{1}{2}\left(\hat{J}^a - \hat{K}^a\right), \hat{J}^{\pm} = \exp\left(\pm\mathrm{i}\varphi\right)\left(\pm\partial_\theta + \mathrm{i}\cot\,\theta\partial_\varphi\right), \hat{J}^3 = -\mathrm{i}\partial_\varphi \\ &\hat{K}^{\pm} = \exp\left(\pm\mathrm{i}\varphi\right)\left(\mathrm{i}\sin\,\theta\partial_\alpha + \mathrm{i}\cot\,\alpha\cos\,\theta\partial_\theta \mp \frac{\cot\,\alpha}{\sin\,\theta}\partial_\varphi\right) \\ &\hat{K}^3 = \mathrm{i}\left(\cos\,\theta\partial_\alpha - \cot\,\alpha\sin\,\theta\partial_\theta\right) \end{aligned}$$

$$\begin{aligned} &\left[\hat{R}^a, \hat{U}\right] = \hat{U}\frac{\tau^a}{2}, \left[\hat{L}^a, \hat{U}\right] = -\frac{\tau^a}{2}\hat{U} \\ &\left[\hat{R}^a, \hat{R}^b\right] = \mathrm{i}\epsilon_{abc}\hat{R}^c, \left[\hat{L}^a, \hat{L}^b\right] = \mathrm{i}\epsilon_{abc}\hat{L}^c, \left[\hat{R}^a, \hat{L}^b\right] = 0 \end{aligned}$$

$$\hat{R}^a\hat{R}^a + \hat{L}^a\hat{L}^a = \frac{1}{2}\left(\hat{J}^a\hat{J}^a + \hat{K}^a\hat{K}^a\right) = \frac{a^4}{2g_s^2}\hat{E}^a\hat{E}^a$$

$$\begin{aligned} &\left[\hat{R}_{x,i}^a, \hat{U}_{y,j}\right] = \delta_{xy}\delta_{ij}\hat{U}_{x,i}T^a, \left[\hat{L}_{x,i}^a, \hat{U}_{y,j}\right] = -\delta_{xy}\delta_{ij}T^a\hat{U}_{x,i}, \left[\hat{U}_{x,i}, \hat{U}_{y,j}\right] = 0 \\ &\left[\hat{R}_{x,i}^a, \hat{R}_{y,j}^b\right] = \mathrm{i}\delta_{xy}\delta_{ij}f_{abc}\hat{R}_{x,i}^c, \left[\hat{L}_{x,i}^a, \hat{L}_{y,j}^b\right] = \mathrm{i}\delta_{xy}\delta_{ij}f_{abc}\hat{L}_{x,i}^c, \left[\hat{R}_{x,i}^a, \hat{L}_{y,j}^b\right] = 0 \end{aligned}$$

$$\begin{aligned} a\hat{H} = {} & g_s^2\sum_{x,i}\left(\hat{R}_{x,i}^a\hat{R}_{x,i}^a + \hat{L}_{x,i}^a\hat{L}_{x,i}^a\right) \\ & + \frac{2}{g_s^2}\sum_{x,i>j}\mathrm{Tr}\left(\mathbb{1} - \frac{1}{2}\hat{U}_{x,i}\hat{U}_{x+\hat{\imath},j}\hat{U}_{x+\hat{\jmath},i}^{\dagger}\hat{U}_{x,j}^{\dagger} - \frac{1}{2}\hat{U}_{x,j}\hat{U}_{x+\hat{\jmath},i}\hat{U}_{x+\hat{\imath},j}^{\dagger}\hat{U}_{x,i}^{\dagger}\right) \end{aligned}$$

$$\hat{G}_x^a = \sum_i\left(\hat{L}_{x,i}^a + \hat{R}_{x-\hat{\imath},i}^a\right), \left[\hat{H}, \hat{G}_x^a\right] = 0, \left[\hat{G}_x^a, \hat{G}_y^b\right] = \mathrm{i}\delta_{xy}f_{abc}\hat{G}_x^c$$

$$\hat{V} = \prod_x \exp\left(\mathrm{i}\omega_x^a\hat{G}_x^a\right), \hat{V}\hat{U}_{x,i}\hat{V}^{\dagger} = \Omega_x\hat{U}_{x,i}\Omega_{x+\hat{\imath}}^{\dagger}$$

$$\hat{G}_x^a\hat{G}_x^a|\Psi, Q, Q^3\rangle = Q_x(Q_x+1)|\Psi, Q, Q^3\rangle, \hat{G}_x^3|\Psi, Q, Q^3\rangle = Q_x^3|\Psi, Q, Q^3\rangle$$

$$\frac{Z_Q}{Z} = \exp\left(-\beta V(x-y)\right), Z_Q = \mathrm{Tr}\left[\exp\left(-\beta \hat{H}\right)\hat{P}_Q\right]$$

$$\hat{\mathcal{T}} \approx \exp\left(-\frac{1}{2}a\hat{H}_B\right)\exp\left(-a\hat{H}_E\right)\exp\left(-\frac{1}{2}a\hat{H}_B\right).$$

$$\mathbb{1} = \int \mathcal{D}U|U\rangle\langle U| = \prod_{x,i}\int_{\mathrm{U}(1)} dU_{x,i}|U_{x,i}\rangle\langle U_{x,i}| = \prod_{x,i}\int_{-\pi}^{\pi} d\varphi_{x,i}|\varphi_{x,i}\rangle\langle\varphi_{x,i}|$$

$$\begin{aligned}\langle U|\exp\left(-a\hat{H}_B\right)|U\rangle &= \prod_{x,i>j}\exp\left(-\frac{1}{2e^2}\left(2-U_{x,i}U_{x+\hat{\imath},j}U^*_{x+\hat{\jmath},i}U^*_{x,j}-U_{x,j}U_{x+\hat{\jmath},i}U^*_{x+\hat{\imath},j}U^*_{x,i}\right)\right)\\ &= \prod_{x,i>j}\exp\left(-\frac{1}{e^2}\left(1-\cos\left(\varphi_{x,i}+\varphi_{x+\hat{\imath},j}-\varphi_{x+\hat{\jmath},i}-\varphi_{x,j}\right)\right)\right)\end{aligned}$$

$$\begin{aligned}\langle U|\exp\left(-a\hat{H}_E\right)|U'\rangle &= \prod_{x,i}\langle U_{x,i}|\exp\left(-\frac{e^2}{2}\hat{J}^2_{x,i}\right)|U'_{x,i}\rangle\\ &= \prod_{x,i}\sum_{m_{x,i}\in\mathbb{Z}}\langle\varphi_{x,i}\mid m_{x,i}\rangle\exp\left(-\frac{e^2}{2}m^2_{x,i}\right)\langle m_{x,i}\mid\varphi'_{x,i}\rangle\\ &= \prod_{x,i}\sum_{m_{x,i}\in\mathbb{Z}}\exp\left(-\frac{e^2}{2}m^2_{x,i}\right)\frac{1}{2\pi}\exp\left(\mathrm{i}m_{x,i}\left(\varphi_{x,i}-\varphi'_{x,i}\right)\right)\\ &= \prod_{x,i}\frac{1}{\sqrt{2\pi}e}\sum_{n_{x,i}\in\mathbb{Z}}\exp\left(-\frac{1}{2e^2}\left(\varphi_{x,i}-\varphi'_{x,i}+2\pi n_{x,i}\right)^2\right)\end{aligned}$$

$$\frac{1}{a}\sum_i\left(\hat{E}_{x,i}-\hat{E}_{x-\hat{\imath},i}\right)|\Psi,Q\rangle = \frac{e}{a^3}\sum_i\left(\hat{J}_{x,i}-\hat{J}_{x-\hat{\imath},i}\right)|\Psi,Q\rangle = \frac{e}{a^3}Q_x|\Psi,Q\rangle = \rho_x|\Psi,Q\rangle$$

$$\langle m|\hat{P}_Q|m\rangle = \prod_x\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi_x\exp\left(\mathrm{i}\left(Q_x-\sum_i\left(m_{x,i}-m_{x-\hat{\imath},i}\right)\right)\varphi_x\right)$$

$$\begin{aligned}\langle U|\exp\left(-a\hat{H}_E\right)\hat{P}_Q|U'\rangle &= \prod_x\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi_x\exp\left(\mathrm{i}Q_x\varphi_x\right)\prod_{x,i}\sum_{m_{x,i}\in\mathbb{Z}}\\ &\exp\left(-\frac{e^2}{2}m^2_{x,i}\right)\frac{1}{2\pi}\exp\left(\mathrm{i}m_{x,i}\left(\varphi_{x,i}-\varphi'_{x,i}-\varphi_x+\varphi_{x+\hat{\imath}}\right)\right)\\ &= \prod_x\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi_x\Phi_{Q_x}\prod_{x,i}\frac{1}{\sqrt{2\pi}e}\sum_{n_{x,i}\in\mathbb{Z}}\exp\left(-\frac{1}{2e^2}\left(\varphi_{x,i}-\varphi'_{x,i}-\varphi_x+\varphi_{x+\hat{\imath}}+2\pi n_{x,i}\right)^2\right)\end{aligned}$$

$$\Phi_{Q_x} = \prod_{n=0}^{N-1}U^{Q_x}_{x+n\hat{4},4'}$$

$$Z_Q = \int\mathcal{D}U\exp\left(-S[U]\right)\prod_x\Phi_{Q_x} = \prod_{x,\mu}\int_{\mathrm{U}(1)}dU_{x,\mu}\exp\left(-S[U]\right)\prod_x\Phi_{Q_x}$$

$$\exp(-S[U]) = \prod_{x,i>j} \exp\left(-\frac{1}{e^2}\left(1-\cos\left(\varphi_{x,i}+\varphi_{x+\hat{\imath},j}-\varphi_{x+\hat{\jmath},i}-\varphi_{x,j}\right)\right)\right)$$
$$\times \prod_{x,i} \sum_{n_{x,i}\in\mathbb{Z}} \exp\left(-\frac{1}{2e^2}\left(\varphi_{x,i}-\varphi'_{x,i}-\varphi_x+\varphi_{x+\hat{\imath}}+2\pi n_{x,i}\right)^2\right)$$

$$Z_Q = \int \mathcal{D}U \exp(-S[U]) \prod_x \Phi_{Q_x} = \prod_{x,\mu} \int_G dU_{x,\mu} \exp(-S[U]) \prod_x \Phi_{Q_x}$$

$$S[U] = \frac{2}{g_s^2} \sum_{x,\mu>\nu} \mathrm{Tr}\left(\mathbb{1} - \frac{1}{2} U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu} - \frac{1}{2} U_{x,\nu} U_{x+\hat{\nu},\mu} U^\dagger_{x+\hat{\mu},\nu} U^\dagger_{x,\mu}\right)$$

$$\langle U|\hat{\mathcal{T}}|U'\rangle = \exp\left(-\frac{1}{g_s^2} \sum_{x,i>j} \mathrm{Tr}\left(\mathbb{1} - \frac{1}{2} U_{x,i} U_{x+\hat{\imath},j} U^\dagger_{x+\hat{\jmath},i} U^\dagger_{x,j} - \frac{1}{2} U_{x,j} U_{x+\hat{\jmath},i} U^\dagger_{x+\hat{\imath},j} U^\dagger_{x,i}\right)\right)$$
$$\times \exp\left(-\frac{2}{g_s^2} \sum_{x,i} \mathrm{Tr}\left(\mathbb{1} - \frac{1}{2} U_{x,i} U'_{x,i} + \frac{1}{2} U'_{x,i} U^\dagger_{x,i}\right)\right)$$
$$\times \exp\left(-\frac{1}{g_s^2} \sum_{x,i>j} \mathrm{Tr}\left(\mathbb{1} - \frac{1}{2} U'_{x,i} U'_{x+\hat{\imath},j} U'_{x+\hat{\jmath},i} U'^\dagger_{x,j} - \frac{1}{2} U'_{x,j} U'_{x+\hat{\jmath},i} U'^\dagger_{x+\hat{\imath},j} U'^\dagger_{x,i}\right)\right)$$

$$G_{\mu\nu}(x) = \partial_\mu G_\nu(x) - \partial_\nu G_\mu(x) + \left[G_\mu(x), G_\nu(x)\right]$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx \;\leftrightarrow\; \mathcal{P}_a^b f(x)^{dx} \cdot \mathcal{P}_b^c f(x)^{dx} = \mathcal{P}_a^c f(x)^{dx}$$
$$\int_b^a f(x)dx = -\int_a^b f(x)dx \;\leftrightarrow\; \mathcal{P}_b^a f(x)^{dx} = [\mathcal{P}_a^b f(x)^{dx}]^{-1}$$
$$F(b) - F(a) = \int_a^b f(x)dx, \lim_{dx\to 0} \frac{F(x+dx)-F(x)}{dx} = f(x) \;\leftrightarrow$$
$$\frac{F(b)}{F(a)} = \mathcal{P}_a^b f(x)^{dx}, \lim_{dx\to 0} \left[\frac{F(x+dx)}{F(x)}\right]^{1/dx} = f(x)$$

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$
$$\mathcal{P}_a^b [f(x)g(x)]^{dx} = \mathcal{P}_a^b f(x)^{dx} \cdot \mathcal{P}_a^b g(x)^{dx}$$

$$\mathcal{L}(\Phi) = \frac{1}{2}\partial_\mu \Phi^* \partial_\mu \Phi + V(\Phi) = \frac{1}{2}\partial_\mu \phi_1 \partial_\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial_\mu \phi_2 + V(\phi_1, \phi_2)$$
$$V(\Phi) = \frac{m^2}{2}|\Phi|^2 + \frac{\lambda}{4!}|\Phi|^4, |\Phi|^2 = \Phi^*\Phi = \phi_1^2 + \phi_2^2$$

$$\Phi'(x) = \exp(\mathrm{i}Qe\alpha)\Phi(x) \;\Leftrightarrow\; \Phi'(x)^* = \exp(-\mathrm{i}Qe\alpha)\Phi^*(x)$$

$$\frac{\partial V}{\partial \Phi} = m^2\Phi + \frac{\lambda}{3!}|\Phi|^2\Phi = 0, \Phi \neq 0 \;\Rightarrow\; |\Phi|^2 = -\frac{6m^2}{\lambda}$$

$$\Phi(x) = \mathrm{v}\exp(\mathrm{i}\varphi), \mathrm{v} = \sqrt{-\frac{6m^2}{\lambda}}$$

$$\Phi(x) = \mathrm{v} + \sigma(x) + \mathrm{i}\pi(x) \Rightarrow \Phi(x)^* = \mathrm{v} + \sigma(x) - \mathrm{i}\pi(x)$$
$$|\Phi(x)|^2 = [\mathrm{v} + \sigma(x)]^2 + \pi(x)^2$$
$$\partial_\mu\Phi(x) = \partial_\mu\sigma(x) + \mathrm{i}\partial_\mu\pi(x) \Rightarrow \partial_\mu\Phi(x)^* = \partial_\mu\sigma(x) - \mathrm{i}\partial_\mu\pi(x)$$

$$\frac{1}{2}\partial_\mu\Phi^*\partial_\mu\Phi = \frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma + \frac{1}{2}\partial_\mu\pi\partial_\mu\pi$$
$$V(\Phi) = \frac{m^2}{2}(\mathrm{v}+\sigma)^2 + \frac{m^2}{2}\pi^2 + \frac{\lambda}{4!}[(\mathrm{v}+\sigma)^2 + \pi^2]^2$$
$$\approx \frac{m^2}{2}\mathrm{v}^2 + m^2\mathrm{v}\sigma + \frac{m^2}{2}\sigma^2 + \frac{m^2}{2}\pi^2 + \frac{\lambda}{4!}(\mathrm{v}^4 + 4\mathrm{v}^3\sigma + 6\mathrm{v}^2\sigma^2 + 2\mathrm{v}^2\pi^2)$$
$$= \frac{1}{2}\left(m^2 + \frac{\lambda}{2}\mathrm{v}^2\right)\sigma^2 + C$$

$$m_\sigma^2 = m^2 + \frac{\lambda}{2}\mathrm{v}^2 = \frac{\lambda}{3}\mathrm{v}^2 = -2m^2 > 0$$

$$\Phi(x) = \begin{pmatrix}\Phi^+(x)\\ \Phi^0(x)\end{pmatrix}, \Phi^+(x), \Phi^0(x) \in \mathbb{C}$$

$$\mathcal{L}(\Phi) = \frac{1}{2}\partial_\mu\Phi^\dagger\partial_\mu\Phi + V(\Phi)$$
$$V(\Phi) = \frac{m^2}{2}|\Phi|^2 + \frac{\lambda}{4!}|\Phi|^4, |\Phi|^2 = \Phi^\dagger\Phi = \Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0$$

$$\Phi'(x) = L\Phi(x), L \in \mathrm{SU}(2)_L$$

$$L^\dagger = L^{-1}, \det L = 1$$

$$L = \begin{pmatrix}z_1 & -z_2^*\\ z_2 & z_1^*\end{pmatrix} \Rightarrow$$
$$L^\dagger = \begin{pmatrix}z_1^* & z_2^*\\ -z_2 & z_1\end{pmatrix}, L^\dagger L = \mathbb{1}, \det L = |z_1|^2 + |z_2|^2 = 1.$$

$$|\Phi'(x)|^2 = \Phi'(x)^\dagger\Phi'(x) = [L\Phi(x)]^\dagger L\Phi(x) = \Phi(x)^\dagger L^\dagger L\Phi(x) = |\Phi(x)|^2$$
$$\partial_\mu\Phi'(x)^\dagger\partial_\mu\Phi'(x) = \partial_\mu\Phi(x)^\dagger L^\dagger L\partial_\mu\Phi(x) = \partial_\mu\Phi(x)^\dagger\partial_\mu\Phi(x)$$

$$\Phi'(x) = \exp\left(\mathrm{i}\frac{1}{2}g'\varphi\right)\Phi(x)$$

$$\mathbf{\Phi}(x) = \begin{pmatrix}\Phi^0(x)^* & \Phi^+(x)\\ -\Phi^+(x)^* & \Phi^0(x)\end{pmatrix}$$

$$\mathcal{L}(\mathbf{\Phi}) = \frac{1}{4}\mathrm{Tr}[\partial_\mu\mathbf{\Phi}^\dagger\partial_\mu\mathbf{\Phi}] + \frac{m^2}{4}\mathrm{Tr}[\mathbf{\Phi}^\dagger\mathbf{\Phi}] + \frac{\lambda}{4!}\left(\frac{1}{2}\mathrm{Tr}[\mathbf{\Phi}^\dagger\mathbf{\Phi}]\right)^2$$

$$\mathbf{\Phi}(x)' = L\mathbf{\Phi}(x)R^\dagger, L \in \mathrm{SU}(2)_L, R \in \mathrm{SU}(2)_R$$

$$R=\begin{pmatrix}\exp(\mathrm{i}g'\varphi/2) & 0\\ 0 & \exp(-\mathrm{i}g'\varphi/2)\end{pmatrix}$$

$$\vec{\phi}(x)=(\phi_1(x),\phi_2(x),\phi_3(x),\phi_4(x))\in\mathbb{R}^4$$
$$\Phi^+(x)=\phi_2(x)+\mathrm{i}\phi_1(x),\Phi^0(x)=\phi_4(x)-\mathrm{i}\phi_3(x)$$
$$\mathbf{\Phi}(x)=\phi_4(x)\mathbb{1}+\mathrm{i}[\phi_1(x)\tau^1+\phi_2(x)\tau^2+\phi_3(x)\tau^3]$$

$$\tau^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\tau^2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\tau^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

$$[\tau^a,\tau^b]=2\mathrm{i}\epsilon_{abc}\tau^c,\{\tau^a,\tau^b\}=2\delta_{ab}\mathbb{1}$$

$$\mathcal{L}(\vec{\phi})=\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi}+\frac{m^2}{2}\vec{\phi}\cdot\vec{\phi}+\frac{\lambda}{4!}(\vec{\phi}\cdot\vec{\phi})^2$$

$$\Phi(x)=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$|\Phi|^2\equiv|\vec{\phi}|^2\equiv\frac{1}{2}\mathrm{Tr}\left[\mathbf{\Phi}^\dagger\mathbf{\Phi}\right]=-\frac{6m^2}{\lambda}$$

$$\Phi(x)=\begin{pmatrix}0\\ v\end{pmatrix},v=\sqrt{-\frac{6m^2}{\lambda}}\in\mathbb{R}_+$$

$$\mathbf{\Phi}(x)=\mathrm{v}\mathbb{1},\vec{\phi}(x)=(0,0,0,\mathrm{v})$$

$$\Phi(x)=\begin{pmatrix}\pi_1(x)+\mathrm{i}\pi_2(x)\\ \mathrm{v}+\sigma(x)+\mathrm{i}\pi_3(x)\end{pmatrix},\sigma(x),\pi_i(x)\in\mathbb{R}$$

$$\begin{aligned}\frac{1}{2}\partial_\mu\Phi^\dagger\partial_\mu\Phi&=\frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma+\frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}\\ V(\Phi)&=\frac{m^2}{2}[(\mathrm{v}+\sigma)^2+\vec{\pi}^2]+\frac{\lambda}{4!}[(\mathrm{v}+\sigma)^2+\vec{\pi}^2]^2\\ &\approx\frac{m^2}{2}[\mathrm{v}^2+2\mathrm{v}\sigma+\sigma^2+\vec{\pi}^2]+\frac{\lambda}{4!}[\mathrm{v}^4+4\mathrm{v}^3\sigma+6\mathrm{v}^2\sigma^2+2\mathrm{v}^2\vec{\pi}^2]\\ &=\frac{1}{2}\left(m^2+\frac{\lambda}{2}\mathrm{v}^2\right)\sigma^2+C\end{aligned}$$

$$m_{\mathrm{H}}^2=m^2+\frac{\lambda}{2}\mathrm{v}^2=\frac{\lambda}{3}\mathrm{v}^2$$

$$m_{\mathrm{H}}=125.18(16)\mathrm{GeV}$$

$$\vec{\phi}(x)=(\phi_1(x),\phi_2(x),\dots,\phi_N(x))$$

$$\vec{\phi}'(x)=\exp(\mathrm{i}\omega_aT^a)\vec{\phi}(x)\approx(\mathbb{1}+\mathrm{i}\omega_aT^a)\vec{\phi}(x)$$

$$\left.\frac{\partial V}{\partial\phi_i}\right|_{\vec{\phi}=\vec{\mathrm{v}}}=0,i\in\{1,2,\dots,N\}.$$

$$M_{ij} = \left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right|_{\vec{\phi}=\vec{\mathrm{v}}}$$

$$\left(1 + \mathrm{i}\omega_b T^b\right)\vec{\mathrm{v}} = \vec{\mathrm{v}} \;\Rightarrow\; T^b\vec{\mathrm{v}} = 0$$

$$0 = V\left(\vec{\phi}'\right) - V\left(\vec{\phi}\right) = \mathrm{i}\frac{\partial V}{\partial\phi_i}\omega_a T^a_{ij}\phi_j.$$

$$0 = \left.\frac{\partial^2 V}{\partial\phi_k \partial\phi_i}\right|_{\vec{\phi}=\vec{\mathrm{v}}} \omega_a T^a_{ij}\mathrm{v}_j + \left.\frac{\partial V}{\partial\phi_i}\right|_{\vec{\phi}=\vec{\mathrm{v}}} \omega_a T^a_{ki} \;\Rightarrow\; M_{ki}(T^a\vec{\mathrm{v}})_i = 0$$

$$n_G - n_H = \frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1$$

$$S[\vec{s}] = a^d \sum_x \sum_{\mu=1}^{d} \frac{\mathrm{v}^2}{2a^2}\left(\vec{s}_{x+\hat{\mu}} - \vec{s}_x\right)^2 = \kappa \sum_{x,\mu}\left(1 - \vec{s}_x \cdot \vec{s}_{x+\hat{\mu}}\right),$$

$$\vec{s}_{\vec{x},x_d}(\vec{n}) = (\cos{(\vec{p}\cdot\vec{x})}, \sin{(\vec{p}\cdot\vec{x})}, 0, \dots, 0)$$

$$\vec{p}\cdot\vec{x} = \sum_{i=1}^{d_s} p_i x_i, \vec{p} = \frac{2\pi}{L}\vec{n}, \vec{n} \in \mathbb{Z}^{d_s}$$

$$\frac{\exp{(-S[\vec{s}(\vec{n} \neq \vec{0})])}}{\exp{(-S[\vec{s}(\vec{n} = \vec{0})])}} = \exp{\left(-2\pi^2 n^2 \kappa\beta L^{d_s-2}[1 + \mathcal{O}(1/L^2)]\right)}$$

$$G/H = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R/\mathrm{SU}(2)_{L=R} = \mathrm{SU}(2) = S^3$$

$$\mathbf{\Phi}(x) = \begin{pmatrix} \Phi^0(x)^* & \Phi^+(x) \\ -\Phi^+(x)^* & \Phi^0(x) \end{pmatrix} = |\Phi(x)|U(x), U(x) \in \mathrm{SU}(2)$$

$$|\Phi(x)|^2 = |\Phi^+(x)|^2 + |\Phi^0(x)|^2 = \det\mathbf{\Phi}(x)$$

$$U'(x) = LU(x)R^\dagger, L \in \mathrm{SU}(2)_L, R \in \mathrm{SU}(2)_R$$

$$\mathcal{L}(U) = \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right]$$

$$\mathcal{L}(\vec{\phi}) = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi} + \frac{m^2}{2}|\vec{\phi}|^2 + \frac{\lambda}{4!}|\vec{\phi}|^4 = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi} + \frac{\lambda}{4!}\left(|\vec{\phi}|^2 - \mathrm{v}^2\right)^2 + C$$

$$\mathcal{L}(\vec{s}) = \frac{\mathrm{V}^2}{2}\partial_\mu\vec{s}\cdot\partial_\mu\vec{s}, \vec{s}(x) = (s_1(x), s_2(x), s_3(x), s_4(x)), |\vec{s}(x)| = 1$$

$$U(x) = s_4(x)\mathbb{1} + \mathrm{i}[s_1(x)\tau^1 + s_2(x)\tau^2 + s_3(x)\tau^3]$$

$$\mathrm{v} = \sqrt{-\frac{6m^2}{\lambda}},$$

$$M_{\mathrm{Planck}} = \frac{1}{\sqrt{G}} \simeq 1.22 \times 10^{19}\mathrm{GeV},$$

$$\frac{\mathrm{v}}{M_{\text{Planck}}} = \mathcal{O}(10^{-17}),$$

$$Z = \int \mathcal{D}\vec{\phi} \exp\left(-S[\vec{\phi}]\right) = \prod_{x} \left(\frac{a}{2\pi}\right)^2 \int_{\mathbb{R}^4} d\vec{\phi}_x \exp\left(-S[\vec{\phi}]\right)$$

$$S[\vec{\phi}] = a^4 \sum_x \left[\frac{1}{2}\sum_\mu \left(\frac{\vec{\phi}_{x+\hat{\mu}} - \vec{\phi}_x}{a}\right)^2 + V(\vec{\phi}_x)\right], \vec{\phi}_x \in \mathbb{R}^4$$

$$V(\vec{\phi}_x) = \frac{m^2}{2}\left|\vec{\phi}_x\right|^2 + \frac{\lambda}{4!}\left|\vec{\phi}_x\right|^4$$

$$m_{\mathrm{H}} a = m_{\mathrm{H}} \frac{\pi}{\Lambda} \propto \left|\frac{\kappa - \kappa_{\mathrm{c}}(\lambda)}{\kappa_{\mathrm{c}}(\lambda)}\right|^{\nu}.$$

$$m_{\mathrm{H}} = \sqrt{-2m^2} = \sqrt{\frac{\lambda}{3}}\mathrm{v}, \mathrm{v} = \sqrt{-\frac{6m^2}{\lambda}}.$$

$$\Phi(x) = \begin{pmatrix}\Phi^+(x)\\ \Phi^0(x)\end{pmatrix}, \tilde{\Phi}(x) = \begin{pmatrix}\tilde{\Phi}^0(x)\\ \tilde{\Phi}^-(x)\end{pmatrix}$$

$$\Phi'(x) = L\Phi(x), \tilde{\Phi}'(x) = L\tilde{\Phi}(x), L \in \mathrm{SU}(2)_L$$

$$\Phi'(x) = \exp\left(\mathrm{i}\frac{g'}{2}\varphi\right)\Phi(x), \tilde{\Phi}'(x) = \exp\left(-\mathrm{i}\frac{g'}{2}\varphi\right)\tilde{\Phi}(x)$$

$$\Phi'(x) = \exp(\mathrm{i}\gamma)\Phi(x), \tilde{\Phi}'(x) = \exp(\mathrm{i}\gamma)\tilde{\Phi}(x)$$

$$\mathcal{L}(\Phi,\tilde{\Phi}) = \frac{1}{2}\partial_\mu\Phi^\dagger\partial_\mu\Phi + \frac{1}{2}\partial_\mu\tilde{\Phi}^\dagger\partial_\mu\tilde{\Phi} + V(\Phi,\tilde{\Phi})$$

$$V(\Phi,\tilde{\Phi}) = \frac{m^2}{2}|\Phi|^2 + \frac{\lambda}{4!}|\Phi|^4 + \frac{\tilde{m}^2}{2}|\tilde{\Phi}|^2 + \frac{\tilde{\lambda}}{4!}|\tilde{\Phi}|^4 + \frac{\varkappa}{2}|\Phi|^2|\tilde{\Phi}|^2 + \frac{\varkappa'}{2}\left|\Phi^\dagger\tilde{\Phi}\right|^2$$
$$|\Phi|^2 = \Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0, |\tilde{\Phi}|^2 = \tilde{\Phi}^{0*}\tilde{\Phi}^0 + \tilde{\Phi}^{-*}\tilde{\Phi}^-$$

$$\Phi(x) = \begin{pmatrix}0\\ \mathrm{v}\end{pmatrix}, \mathrm{v} = \sqrt{\frac{\tilde{\lambda}m^2 - 6\varkappa\tilde{m}^2}{6\varkappa^2 - \lambda\tilde{\lambda}/6}}$$
$$\tilde{\Phi}(x) = \begin{pmatrix}\tilde{\mathrm{v}}\\ 0\end{pmatrix}, \tilde{\mathrm{v}} = \sqrt{\frac{\lambda\tilde{m}^2 - 6\varkappa m^2}{6\varkappa^2 - \lambda\tilde{\lambda}/6}}$$

$$\Phi'(x) = \begin{pmatrix}\exp(\mathrm{i}e\alpha) & 0\\ 0 & 0\end{pmatrix}\Phi(x), \tilde{\Phi}'(x) = \begin{pmatrix}0 & 0\\ 0 & \exp(-\mathrm{i}e\alpha)\end{pmatrix}\tilde{\Phi}(x)$$

$$\Phi(x) = \begin{pmatrix}\pi_1(x) + \mathrm{i}\pi_2(x)\\ \mathrm{v} + \sigma(x) + \mathrm{i}\pi_3(x)\end{pmatrix}, \tilde{\Phi}(x) = \begin{pmatrix}\tilde{\mathrm{v}} + \tilde{\sigma}(x) - \mathrm{i}\tilde{\pi}_3(x)\\ -\tilde{\pi}_1(x) - \mathrm{i}\tilde{\pi}_2(x)\end{pmatrix}$$

$$V(\Phi,\widetilde{\Phi})=\frac{m^2}{2}[(\mathrm{v}+\sigma)^2+\pi_1^2+\pi_2^2+\pi_3^2]+\frac{\lambda}{4!}[(\mathrm{v}+\sigma)^2+\pi_1^2+\pi_2^2+\pi_3^2]^2$$
$$+\frac{\widetilde{m}^2}{2}[(\tilde{\mathrm{v}}+\tilde{\sigma})^2+\tilde{\pi}_1^2+\tilde{\pi}_2^2+\tilde{\pi}_3^2]+\frac{\tilde{\lambda}}{4!}[(\tilde{\mathrm{v}}+\tilde{\sigma})^2+\tilde{\pi}_1^2+\tilde{\pi}_2^2+\tilde{\pi}_3^2]^2$$
$$+\frac{\varkappa}{2}[(\mathrm{v}+\sigma)^2+\pi_1^2+\pi_2^2+\pi_3^2][(\tilde{\mathrm{v}}+\tilde{\sigma})^2+\tilde{\pi}_1^2+\tilde{\pi}_2^2+\tilde{\pi}_3^2]$$
$$+\frac{\varkappa'}{2}|(\pi_1-\mathrm{i}\pi_2)(\tilde{\mathrm{v}}+\tilde{\sigma}-\mathrm{i}\tilde{\pi}_3)+(\mathrm{v}+\sigma-\mathrm{i}\pi_3)(-\tilde{\pi}_1+\mathrm{i}\tilde{\pi}_2)|^2$$

$$\approx\frac{1}{2}\left(m^2+\frac{\lambda}{2}\mathrm{v}^2+\varkappa\tilde{\mathrm{v}}^2\right)\sigma^2+\frac{1}{2}\left(\widetilde{m}^2+\frac{\tilde{\lambda}}{2}\tilde{\mathrm{v}}^2+\varkappa\mathrm{v}^2\right)\tilde{\sigma}^2$$
$$+\frac{\varkappa'}{2}(\mathrm{v}^2+\tilde{\mathrm{v}}^2)\left[\left(\frac{\tilde{\mathrm{v}}\pi_1-\mathrm{v}\tilde{\pi}_1}{\sqrt{\mathrm{v}^2+\tilde{\mathrm{v}}^2}}\right)^2+\left(\frac{\tilde{\mathrm{v}}\pi_2-\mathrm{v}\tilde{\pi}_2}{\sqrt{\mathrm{v}^2+\tilde{\mathrm{v}}^2}}\right)^2\right]+C$$

$$\sigma(x),\tilde{\sigma}(x),\rho_1(x)=\frac{\tilde{\mathrm{v}}\pi_1(x)-\mathrm{v}\tilde{\pi}_1(x)}{\sqrt{\mathrm{v}^2+\tilde{\mathrm{v}}^2}},\rho_2(x)=\frac{\tilde{\mathrm{v}}\pi_2(x)-\mathrm{v}\tilde{\pi}_2(x)}{\sqrt{\mathrm{v}^2+\tilde{\mathrm{v}}^2}}$$

$$m_\sigma^2=\frac{\lambda}{3}\mathrm{v}^2,m_{\tilde{\sigma}}^2=\frac{\tilde{\lambda}^2}{3}\tilde{\mathrm{v}}^2,m_{\rho_1}^2=m_{\rho_2}^2=\varkappa'(\mathrm{v}^2+\tilde{\mathrm{v}}^2)$$

$$\pi_3(x)\,\tilde{\pi}_3(x),\zeta_1(x)=\frac{\mathrm{v}\pi_1(x)+\tilde{\mathrm{v}}\tilde{\pi}_1(x)}{\sqrt{\mathrm{v}^2+\tilde{\mathrm{v}}^2}},\zeta_2(x)=\frac{\mathrm{v}\pi_2(x)+\tilde{\mathrm{v}}\tilde{\pi}_2(x)}{\sqrt{\mathrm{v}^2+\tilde{\mathrm{v}}^2}}$$

$$G/H=\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y\times\mathrm{U}(1)_{\mathrm{PQ}}/\mathrm{U}(1)_{\mathrm{em}}=\mathrm{SU}(2)\times\mathrm{U}(1)$$

$$\mathcal{L}(V,a)=\frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu V^\dagger\partial_\mu V\right]+K\mathrm{Tr}\left[\partial_\mu V^\dagger\partial_\mu V\tau^3\right]+\frac{1}{2}\partial_\mu a\partial_\mu a$$
$$=\frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu V^\dagger\partial_\mu V\right]+\frac{1}{2}\partial_\mu a\partial_\mu a$$

$$M=\begin{pmatrix}a & b\\ c & d\end{pmatrix},a,b,c,d\in\mathbb{C}$$

$$d=a^*,c=-b^*,|a|^2+|b|^2=1.$$

$$\mathbf{\Phi}=\begin{pmatrix}\Phi^{0*} & \Phi^+\\ -\Phi^{+*} & \Phi^0\end{pmatrix},\Phi^+,\Phi^0\in\mathbb{C}.$$

$$\mathbf{\Phi}\to L\mathbf{\Phi}\ \text{ and }\ \mathbf{\Phi}\to\mathbf{\Phi}R^\dagger,$$

$$\mathrm{SU}(N_\mathrm{f})_L\times\mathrm{SU}(N_\mathrm{f})_R\to\mathrm{SU}(N_\mathrm{f}).$$

$$\mathcal{L}(\vec{\phi})=\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi}+\frac{m^2}{2}\vec{\phi}\cdot\vec{\phi}+\frac{\lambda}{4!}(\vec{\phi}\cdot\vec{\phi})^2$$

$$\Phi(x)=\begin{pmatrix}\Phi^+(x)\\ \Phi^0(x)\end{pmatrix},\widetilde{\Phi}(x)=\begin{pmatrix}\widetilde{\Phi}^0(x)\\ \widetilde{\Phi}^-(x)\end{pmatrix}$$

$$\mathcal{L}(\Phi,\partial_\mu\Phi,\widetilde{\Phi},\partial_\mu\widetilde{\Phi})=\frac{1}{2}\partial_\mu\Phi^\dagger\partial_\mu\Phi+\frac{1}{2}\partial_\mu\widetilde{\Phi}^\dagger\partial_\mu\widetilde{\Phi}+V(\Phi,\widetilde{\Phi})$$

$$V(\Phi,\widetilde{\Phi}) = \frac{m^2}{2}|\Phi|^2 + \frac{\lambda}{4!}|\Phi|^4 + \frac{\widetilde{m}^2}{2}|\widetilde{\Phi}|^2 + \frac{\tilde{\lambda}}{4!}|\widetilde{\Phi}|^4 + \frac{\varkappa}{2}|\Phi|^2|\widetilde{\Phi}|^2$$
$$|\Phi|^2 = \Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0, |\widetilde{\Phi}|^2 = \widetilde{\Phi}^{0*}\widetilde{\Phi}^0 + \widetilde{\Phi}^{-*}\widetilde{\Phi}^-$$

$$\Xi(x) = \Xi_a(x)\lambda^a, \Xi_a(x) \in \mathbb{R}, a \in \{1,2,\dots,8\}$$

$$\Xi'(x) = \Upsilon\Xi(x)\Upsilon^\dagger$$

$$V(\Xi) = \frac{M^2}{2}\mathrm{Tr}[\Xi^2] + \frac{\Lambda}{4!}\mathrm{Tr}[\Xi^4]$$

$$\mathcal{L}(\Phi,A) = \frac{1}{2}\left(D_\mu\Phi\right)^* D_\mu\Phi + V(\Phi) + \frac{1}{4}F_{\mu\nu}F_{\mu\nu}, V(\Phi) = \frac{m^2}{2}|\Phi|^2 + \frac{\lambda}{4!}|\Phi|^2$$

$$\Phi'(x) = \exp\left(\mathrm{i}Qe\alpha(x)\right)\Phi(x), A'_\mu(x) = A_\mu(x) - \partial_\mu\alpha(x)$$

$$D_\mu\Phi(x) = \left[\partial_\mu + \mathrm{i}QeA_\mu(x)\right]\Phi(x)$$

$$\mathrm{Re}\Phi(x) = \phi_1(x) \geq 0, \mathrm{Im}\Phi(x) = \phi_2(x) = 0$$

$$\Phi(x) = \mathrm{v} + \sigma(x);$$

$$\begin{aligned} V(\Phi) &= \frac{m^2}{2}(\mathrm{v}+\sigma)^2 + \frac{\lambda}{4!}(\mathrm{v}+\sigma)^4 \\ &\approx \frac{m^2}{2}\mathrm{v}^2 + m^2\mathrm{v}\sigma + \frac{m^2}{2}\sigma^2 + \frac{\lambda}{4!}(\mathrm{v}^4 + 4\mathrm{v}^3\sigma + 6\mathrm{v}^2\sigma^2) \\ &= \frac{1}{2}\left(m^2 + \frac{\lambda}{2}\mathrm{v}^2\right)\sigma^2 + C = \frac{\lambda}{6}\mathrm{v}^2\sigma^2 + C = \frac{1}{2}m_\sigma^2 + C \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\left(D_\mu\Phi\right)^* D_\mu\Phi &= \frac{1}{2}\left[\left(\partial_\mu - \mathrm{i}QeA_\mu\right)(\mathrm{v}+\sigma)\right]\left[\left(\partial_\mu + \mathrm{i}QeA_\mu\right)(\mathrm{v}+\sigma)\right] \\ &= \frac{1}{2}\left(\partial_\mu\sigma - \mathrm{i}QeA_\mu\mathrm{v} - \mathrm{i}QeA_\mu\sigma\right)\left(\partial_\mu\sigma + \mathrm{i}QeA_\mu\mathrm{v} + \mathrm{i}QeA_\mu\sigma\right) \\ &\approx \frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma + \frac{1}{2}Q^2e^2\mathrm{v}^2A_\mu A_\mu \end{aligned}$$

$$M_\gamma = |Q|e\mathrm{v}$$

$$\Phi'(x) = L(x)\Phi(x), \Phi(x) = \begin{pmatrix}\Phi^+(x)\\ \Phi^0(x)\end{pmatrix} \in \mathbb{C}^2$$

$$\partial_\mu\Phi'(x) = L(x)\partial_\mu\Phi(x) + \partial_\mu L(x)\Phi(x) = L(x)\left[\partial_\mu + L(x)^\dagger\partial_\mu L(x)\right]\Phi(x)$$

$$D_\mu\Phi(x) = \left[\partial_\mu + W_\mu(x)\right]\Phi(x)$$

$$\left(D_\mu\Phi(x)\right)^\dagger = \partial_\mu\Phi(x)^\dagger + \Phi(x)^\dagger W_\mu(x)^\dagger = \partial_\mu\Phi(x)^\dagger - \Phi(x)^\dagger W_\mu(x).$$

$$W_\mu(x)^\dagger = -W_\mu(x).$$

$$W_\mu(x) = \mathrm{i}gW_\mu^a(x)\frac{\tau^a}{2}, a = 1,2,3$$

$$W_\mu'(x) = L(x)\left[W_\mu(x) + \partial_\mu\right]L(x)^\dagger, L(x) \in \mathrm{SU}(2)_L$$

$$\begin{aligned} D_\mu'\Phi'(x) &= \left[\partial_\mu + W_\mu'(x)\right]\Phi'(x) \\ &= L(x)\left[\partial_\mu\Phi(x) + L(x)^\dagger\partial_\mu L(x)\Phi(x) + W_\mu(x)L(x)^\dagger L(x)\Phi(x) + \partial_\mu L(x)^\dagger L(x)\Phi(x)\right] \\ &= L(x)\left[\partial_\mu + W_\mu(x)\right]\Phi(x) = L(x)D_\mu\Phi(x) \end{aligned}$$

$$\left(D_\mu\Phi'(x)\right)^\dagger = \left(D_\mu\Phi(x)\right)^\dagger L(x)^\dagger$$

$$\mathcal{L}(\Phi, W) = \frac{1}{2}\left(D_\mu\Phi\right)^\dagger D_\mu\Phi + V(\Phi)$$

$$\begin{aligned} W_{\mu\nu}(x) &= \left[D_\mu, D_\nu\right] = \left[\partial_\mu + W_\nu(x), \partial_\nu + W_\nu(x)\right] \\ &= \partial_\mu W_\nu(x) - \partial_\nu W_\mu(x) + \left[W_\mu(x), W_\nu(x)\right] \end{aligned}$$

$$W_{\mu\nu}'(x) = L(x)W_{\mu\nu}(x)L(x)^\dagger$$

$$\mathcal{L}(W) = \frac{1}{4}W_{\mu\nu}^a W_{\mu\nu}^a = -\frac{1}{2g^2}\mathrm{Tr}\left[W_{\mu\nu}W_{\mu\nu}\right], W_{\mu\nu}(x) = \mathrm{i}gW_{\mu\nu}^a(x)\frac{\tau^a}{2},$$

$$\Phi'(x) = \exp\left(\frac{1}{2}\mathrm{i}'\varphi(x)\right)\Phi(x)$$

$$R(x) = \begin{pmatrix} \exp\left(\mathrm{i}g'\varphi(x)/2\right) & 0 \\ 0 & \exp\left(-\mathrm{i}g'\varphi(x)/2\right) \end{pmatrix}.$$

$$B_\mu'(x) = B_\mu(x) - \partial_\mu\varphi(x)$$

$$\begin{aligned} D_\mu\Phi(x) &= \left[\partial_\mu + W_\mu(x) + \mathrm{i}\frac{g'}{2}B_\mu(x)\right]\Phi(x) \\ &= \left[\partial_\mu + \mathrm{i}gW_\mu^a(x)\frac{\tau^a}{2} + \mathrm{i}\frac{g'}{2}B_\mu(x)\right]\begin{pmatrix}\Phi^+(x) \\ \Phi^0(x)\end{pmatrix} \end{aligned}$$

$$\boldsymbol{B}_{\mu\nu}(x) = \partial_\mu\boldsymbol{B}_\nu(x) - \partial_\nu\boldsymbol{B}_\mu(x)$$

$$\mathcal{L}(\Phi, W, B) = \frac{1}{2}\left(D_\mu\Phi\right)^\dagger D_\mu\Phi + V(\Phi) - \frac{1}{2g^2}\mathrm{Tr}\left[W_{\mu\nu}W_{\mu\nu}\right] + \frac{1}{4}B_{\mu\nu}B_{\mu\nu}$$

$$\Phi(x) = \begin{pmatrix}0 \\ \mathrm{v}\end{pmatrix}, \mathrm{v} \in \mathbb{R}_+$$

$$\Phi'(x) = \begin{pmatrix}\exp\left(\mathrm{i}e\alpha(x)\right) & 0 \\ 0 & 1\end{pmatrix}\Phi(x)$$

$$\begin{pmatrix}\exp\left(\mathrm{i}e\alpha(x)\right) & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix}\exp\left(\mathrm{i}e\alpha(x)/2\right) & 0 \\ 0 & \exp\left(\mathrm{i}e\alpha(x)/2\right)\end{pmatrix}\begin{pmatrix}\exp\left(\mathrm{i}e\alpha(x)/2\right) & 0 \\ 0 & \exp\left(-\mathrm{i}e\alpha(x)/2\right)\end{pmatrix}$$

$$\Phi(x) = \begin{pmatrix}0 \\ \mathrm{v} + \sigma(x)\end{pmatrix}, \sigma(x) \in \mathbb{R}$$

$$V(\Phi)=\frac{m^2}{2}(\mathrm{v}+\sigma)^2+\frac{\lambda}{4!}(\mathrm{v}+\sigma)^4=-m^2\sigma^2+C+\mathcal{O}(\sigma^4),\mathrm{v}^2=-\frac{6m^2}{\lambda}$$

$$M_{\mathrm{H}}^2=-2m^2>0$$

$$\frac{1}{2}\left(D_\mu\Phi\right)^\dagger D_\mu\Phi=\frac{1}{2}\left|\left(\partial_\mu+\mathrm{i}gW_\mu^a\frac{\tau^a}{2}+\mathrm{i}\frac{g'}{2}B_\mu\right)\begin{pmatrix}0\\ \mathrm{v}+\sigma\end{pmatrix}\right|^2$$
$$=\frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma+\frac{(\mathrm{v}+\sigma)^2}{2}(0,1)\left[\left(gW_\mu^a\frac{\tau^a}{2}+\frac{g'}{2}B_\mu\right)\left(gW_\mu^b\frac{\tau^b}{2}+\frac{g'}{2}B_\mu\right)\right]\begin{pmatrix}0\\1\end{pmatrix}$$
$$=\frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma+\frac{1}{8}(\mathrm{v}+\sigma)^2\left[g^2W_\mu^1W_\mu^1+g^2W_\mu^2W_\mu^2+\left(gW_\mu^3-g'B_\mu\right)\left(gW_\mu^3-g'B_\mu\right)\right].$$

$$M_W=\frac{1}{2}gv$$

$$Z_\mu(x)=\frac{gW_\mu^3(x)-g'B_\mu(x)}{\sqrt{g^2+g'^2}}$$

$$M_Z=\frac{1}{2}\sqrt{g^2+g'^2}\mathrm{v}$$

$$A_\mu(x)=\frac{g'W_\mu^3(x)+gB_\mu(x)}{\sqrt{g^2+g'^2}}$$

$$\begin{pmatrix}A_\mu(x)\\ Z_\mu(x)\end{pmatrix}=\begin{pmatrix}\cos\ \theta_{\mathrm{W}} & \sin\ \theta_{\mathrm{W}}\\ -\sin\ \theta_{\mathrm{W}} & \cos\ \theta_{\mathrm{W}}\end{pmatrix}\begin{pmatrix}B_\mu(x)\\ W_\mu^3(x)\end{pmatrix}$$

$$\frac{g}{\sqrt{g^2+g'^2}}=\cos\ \theta_{\mathrm{W}},\frac{g'}{\sqrt{g^2+g'^2}}=\sin\ \theta_{\mathrm{W}}$$

$$\frac{M_W}{M_Z}=\frac{g}{\sqrt{g^2+g'^2}}=\cos\ \theta_{\mathrm{W}}$$

$$M_W=80.377(12)\mathrm{GeV},M_Z=91.1876(21)\mathrm{GeV}$$

$$\sin^2\ \theta_{\mathrm{W}}=0.23119(14)$$

$$D_\mu\Phi=\left[\partial_\mu+\mathrm{i}gW_\mu^a\frac{\tau^a}{2}+\mathrm{i}\frac{g'}{2}B_\mu\right]\begin{pmatrix}\Phi^+\\ \Phi^0\end{pmatrix}$$

$$=\left[\partial_\mu+\mathrm{i}gW_\mu^1\frac{\tau^1}{2}+\mathrm{i}gW_\mu^2\frac{\tau^2}{2}+\frac{\mathrm{i}}{2}\begin{pmatrix}gW_\mu^3+g'B_\mu & 0\\ 0 & -gW_\mu^3+g'B_\mu\end{pmatrix}\right]\begin{pmatrix}\Phi^+\\ \Phi^0\end{pmatrix}$$

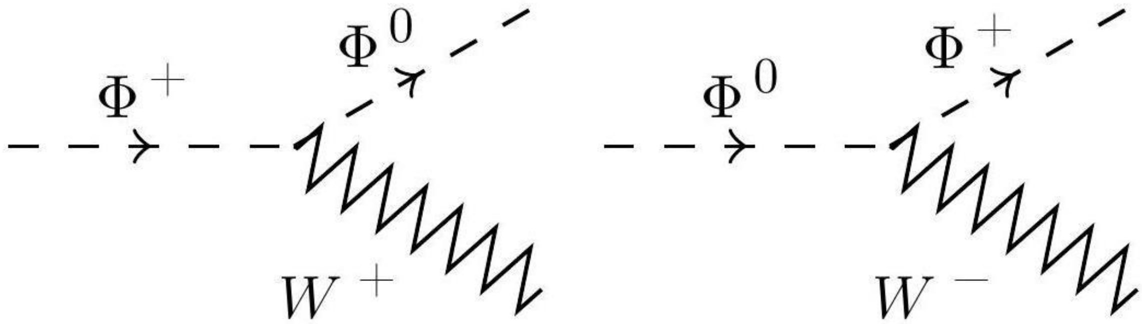
$$=\left[\partial_\mu+\mathrm{i}gW_\mu^1\frac{\tau^1}{2}+\mathrm{i}gW_\mu^2\frac{\tau^2}{2}\right.$$

$$\left.+\mathrm{i}\begin{pmatrix}\left[\frac{1}{2}(g^2-g'^2)Z_\mu+gg'A_\mu\right]/\sqrt{g^2+g'^2} & 0\\ 0 & -\frac{1}{2}\sqrt{g^2+g'^2}Z_\mu\end{pmatrix}\right]\begin{pmatrix}\Phi^+\\ \Phi^0\end{pmatrix}$$

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}$$

$$W_\mu^{\pm}(x) = \frac{1}{\sqrt{2}}\left(W_\mu^1(x) \mp \mathrm{i}W_\mu^2(x)\right)$$

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x), Z'_\mu(x) = Z_\mu(x), W_\mu^{\pm'}(x) = \exp\left(\pm \mathrm{i}e\alpha(x)\right)W_\mu^{\pm}(x)$$



$$\mathbf{\Phi}(x) = \begin{pmatrix} \Phi^0(x)^* & \Phi^+(x) \\ -\Phi^+(x)^* & \Phi^0(x) \end{pmatrix}$$

$$\mathbf{\Phi}'(x) = L(x)\mathbf{\Phi}(x)R(x)^\dagger, L(x) \in \mathrm{SU}(2)_L$$
$$R(x) = \begin{pmatrix} \exp\left(\mathrm{i}g'\varphi(x)/2\right) & 0 \\ 0 & \exp\left(-\mathrm{i}g'\varphi(x)/2\right) \end{pmatrix} \in \mathrm{U}(1)_Y$$

$$D_\mu \mathbf{\Phi}(x) = \partial_\mu \mathbf{\Phi}(x) + W_\mu(x)\mathbf{\Phi}(x) - \mathbf{\Phi}(x)\mathrm{i}g'\boldsymbol{B}_\mu(x)\frac{\tau^3}{2}$$

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\,\theta_\mathrm{W}} = 1$$

$$\mathbf{\Phi}(x)^\dagger \mathbf{\Phi}(x) = \mathbf{\Phi}(x)\mathbf{\Phi}(x)^\dagger = (|\Phi^0(x)|^2 + |\Phi^+(x)|^2)\mathbb{1}$$

$$\mathrm{Tr}\left[\mathbf{\Phi}(x)^\dagger \mathbf{\Phi}(x)\tau^3 \mathbf{\Phi}(x)^\dagger \mathbf{\Phi}(x)\tau^3\right] = 2(|\Phi^0(x)|^2 + |\Phi^+(x)|^2)^2$$

$$D_\mu \mathbf{\Phi}(x) = \partial_\mu \mathbf{\Phi}(x) + W_\mu(x)\mathbf{\Phi}(x) - \mathbf{\Phi}(x)X_\mu(x)$$

$$X'_\mu(x) = R(x)\left[X_\mu(x) + \partial_\mu\right]R(x)^\dagger$$

$$X_\mu(x) = \mathrm{i}g'B_\mu(x)\frac{\tau^3}{2}.$$

$$Z_\mu^a(x) = \frac{gW_\mu^a(x) - g'X_\mu^a(x)}{\sqrt{g^2 + g'^2}}, A_\mu^a(x) = \frac{g'W_\mu^a(x) + gX_\mu^a(x)}{\sqrt{g^2 + g'^2}}.$$

$$S[\Phi] = a^4 \sum_x V(\Phi_x) = a^4 \sum_x \frac{\lambda}{4!}(|\Phi_x|^2 - \mathrm{v}^2)^2$$

$$\Phi'_x = \exp\left(\mathrm{i}Qe\alpha_x\right)\Phi_x$$

$$A'_{x,\mu} = A_{x,\mu} - \frac{1}{a}(\alpha_{x+\hat{\mu}} - \alpha_x),$$

$$U^Q_{x,\mu} = \exp\left(\mathrm{i}QeaA_{x,\mu}\right) \Rightarrow U^{Q'}_{x,\mu} = \exp\left(\mathrm{i}Qe\alpha_x\right)U^Q_{x,\mu}\exp\left(-\mathrm{i}Qe\alpha_{x+\hat{\mu}}\right)$$

$$\left(D^{\mathrm{f}}\Phi\right)_{x,\mu} = \frac{1}{a}\left(U^Q_{x,\mu}\Phi_{x+\hat{\mu}} - \Phi_x\right) \Rightarrow \left(D^{\mathrm{f}}\Phi\right)'_{x,\mu} = \exp\left(\mathrm{i}Qe\alpha_x\right)\left(D^{\mathrm{f}}\Phi\right)_{x,\mu}$$
$$\left(D^{\mathrm{b}}\Phi\right)_{x,\mu} = \frac{1}{a}\left(\Phi_{x+\hat{\mu}} - U^{Q*}_{x,\mu}\Phi_x\right) \Rightarrow \left(D^{\mathrm{b}}\Phi\right)'_{x,\mu} = \exp\left(\mathrm{i}Qe\alpha_{x+\hat{\mu}}\right)\left(D^{\mathrm{b}}\Phi\right)_{x,\mu}$$

$$\frac{1}{2a^2}\left|\left(D^{\mathrm{f}}\Phi\right)_{x,\mu}\right|^2 = \frac{1}{2a^2}\left|\left(D^{\mathrm{b}}\Phi\right)_{x,\mu}\right|^2 = \frac{1}{2a^2}\left(|\Phi_x|^2 + \left|\Phi_{x+\hat{\mu}}\right|^2 - \Phi^*_x U^Q_{x,\mu}\Phi_{x+\hat{\mu}} - \Phi^*_{x+\hat{\mu}}U^{Q*}_{x,\mu}\Phi_x\right)$$

$$S[\Phi, A] = a^4\sum_{x,\mu}\frac{1}{2a^2}\left(|\Phi_x|^2 + \left|\Phi_{x+\hat{\mu}}\right|^2 - \Phi^*_x U^Q_{x,\mu}\Phi_{x+\hat{\mu}} - \Phi^*_{x+\hat{\mu}}U^{Q*}_{x,\mu}\Phi_x\right)$$

$$F_{x,\mu\nu} = \frac{1}{a}\left(A_{x+\hat{\mu},\nu} - A_{x,\nu} - A_{x+\hat{\nu},\mu} + A_{x,\mu}\right),$$

$$S[A] = a^4\sum_{x,\mu,\nu}\frac{1}{4}F^2_{x,\mu\nu} = a^4\sum_{x,\mu,\nu}\frac{1}{4a^2}\left(A_{x,\mu} + A_{x+\hat{\mu},\nu} - A_{x+\hat{\nu},\mu} - A_{x,\nu}\right)^2$$

$$\int \mathcal{D}\Phi\mathcal{D}A = \prod_x \frac{a}{2\pi}\int_{\mathbb{C}} d\Phi_x \prod_{y,\mu}\sqrt{\frac{a}{2\pi}}\int_{\mathbb{R}} dA_{y,\mu}$$

$$S_{\mathrm{gf}}[A] = a^4\sum_x \frac{1}{2\xi}\left[\sum_\mu \frac{1}{a}\left(A_{x+\hat{\mu},\mu} - A_{x,\mu}\right)\right]^2$$

$$Z = \int \mathcal{D}\Phi\mathcal{D}A\exp\left(-S[\Phi, A] - S[\Phi] - S[A] - S_{\mathrm{gf}}[A]\right)$$

$$\tilde{\Phi}_x = \frac{\Phi_x}{\mathrm{v}}, \tilde{A}_{x,\mu} = eaA_{x,\mu}$$

$$S[\Phi, A] + S[\Phi] \to S[\tilde{\Phi}, \tilde{A}] = -\sum_{x,\mu}\kappa\mathrm{Re}\left[\tilde{\Phi}^*_x U^Q_{x,\mu}\tilde{\Phi}_{x+\hat{\mu}}\right], U^Q_{x,\mu} = \exp\left(\mathrm{i}Q\tilde{A}_{x,\mu}\right)$$

$$S[A] \to S[\tilde{A}] = \sum_{x,\mu,\nu}\frac{1}{4e^2}\left(\tilde{A}_{x,\mu} + \tilde{A}_{x+\hat{\mu},\nu} - \tilde{A}_{x+\hat{\nu},\mu} - \tilde{A}_{x,\nu}\right)^2$$

$$\mathbf{\Phi}_x = \begin{pmatrix} \Phi^{0*}_x & \Phi^+_x \\ -\Phi^{+*}_x & \Phi^0_x \end{pmatrix}, \mathbf{\Phi}'_x = L_x\mathbf{\Phi}_x R^\dagger.$$

$$S[\mathbf{\Phi}] = a^4\sum_x V(\mathbf{\Phi}_x) = a^4\sum_x \frac{\lambda}{4!}\left(\left(\frac{1}{2}\mathrm{Tr}\left[\mathbf{\Phi}^\dagger_x\mathbf{\Phi}_x\right]\right)^2 - \mathrm{v}^2\right)^2$$
$$\frac{1}{2}\mathrm{Tr}\left[\mathbf{\Phi}^\dagger_x\mathbf{\Phi}_x\right] = |\Phi^+_x|^2 + |\Phi^0_x|^2.$$

$$U'_{x,\mu} = L_x U_{x,\mu} L^\dagger_{x+\hat{\mu}}$$

$$S[U] = \frac{2}{g^2} \sum_{x,\mu>\nu} \mathrm{Tr}\left[\mathbb{1} - \frac{1}{2} U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu} - \frac{1}{2} U_{x,\nu} U_{x+\hat{\nu},\mu} U^\dagger_{x+\hat{\mu},\nu} U^\dagger_{x,\mu}\right]$$

$$\left(D^{\mathrm{f}}\boldsymbol{\Phi}\right)_{x,\mu} = \frac{1}{a}\left(U_{x,\mu}\boldsymbol{\Phi}_{x+\hat{\mu}} - \boldsymbol{\Phi}_x\right) \Rightarrow \left(D^{\mathrm{f}}\boldsymbol{\Phi}\right)'_{x,\mu} = L_x\left(D^{\mathrm{f}}\boldsymbol{\Phi}\right)_{x,\mu} R^\dagger$$

$$\left(D^{\mathrm{b}}\boldsymbol{\Phi}\right)_{x,\mu} = \frac{1}{a}\left(\boldsymbol{\Phi}_{x+\hat{\mu}} - U^\dagger_{x,\mu}\boldsymbol{\Phi}_x\right) \Rightarrow \left(D^{\mathrm{b}}\boldsymbol{\Phi}\right)'_{x,\mu} = L_{x+\hat{\mu}}\left(D^{\mathrm{b}}\boldsymbol{\Phi}\right)_{x,\mu} R^\dagger$$

$$\frac{1}{4a^2}\mathrm{Tr}\left[\left(D^{\mathrm{f}}\boldsymbol{\Phi}\right)^\dagger_{x,\mu}\left(D^{\mathrm{f}}\boldsymbol{\Phi}\right)_{x,\mu}\right] = \frac{1}{4a^2}\mathrm{Tr}\left[\left(D^{\mathrm{b}}\boldsymbol{\Phi}\right)^\dagger_{x,\mu}\left(D^{\mathrm{b}}\boldsymbol{\Phi}\right)_{x,\mu}\right]$$
$$= \frac{1}{4a^2}\mathrm{Tr}\left[\boldsymbol{\Phi}^\dagger_x\boldsymbol{\Phi}_x + \boldsymbol{\Phi}^\dagger_{x+\hat{\mu}}\boldsymbol{\Phi}_{x+\hat{\mu}} - \boldsymbol{\Phi}^\dagger_x U_{x,\mu}\boldsymbol{\Phi}_{x+\hat{\mu}} - \boldsymbol{\Phi}^\dagger_{x+\hat{\mu}} U^\dagger_{x,\mu}\boldsymbol{\Phi}_x\right].$$

$$S[\boldsymbol{\Phi}, U] = a^4 \sum_{x,\mu} \frac{1}{4a^2}\mathrm{Tr}\left[\boldsymbol{\Phi}^\dagger_x\boldsymbol{\Phi}_x + \boldsymbol{\Phi}^\dagger_{x+\hat{\mu}}\boldsymbol{\Phi}_{x+\hat{\mu}} - \boldsymbol{\Phi}^\dagger_x U_{x,\mu}\boldsymbol{\Phi}_{x+\hat{\mu}} - \boldsymbol{\Phi}^\dagger_{x+\hat{\mu}} U^\dagger_{x,\mu}\boldsymbol{\Phi}_x\right].$$

$$Z = \int \mathcal{D}\Phi\mathcal{D}U \exp\left(-S[\boldsymbol{\Phi}, U] - S[\boldsymbol{\Phi}] - S[U]\right)$$
$$= \prod_x \left(\frac{a}{2\pi}\right)^2 \int_{\mathbb{C}^2} d\boldsymbol{\Phi}_x \prod_{y,\mu} \int_{\mathrm{SU}(2)} dU_{y,\mu} \exp\left(-S[\boldsymbol{\Phi}, U] - S[\boldsymbol{\Phi}] - S[U]\right)$$

$$S[\boldsymbol{\Phi}, U] + S[\boldsymbol{\Phi}] \to S[\widetilde{\boldsymbol{\Phi}}, U] = -\sum_{x,\mu} \frac{\kappa}{2}\mathrm{ReTr}\left[\widetilde{\boldsymbol{\Phi}}^\dagger_x U_{x,\mu}\widetilde{\boldsymbol{\Phi}}_{x+\hat{\mu}}\right]$$

$$\Xi(x) = \Xi_a(x)\lambda^a, \Xi_a(x) \in \mathbb{R}, a = 1,2,\dots,8$$

$$\Xi'(x) = \Upsilon(x)\Xi(x)\Upsilon(x)^\dagger, \Upsilon(x) \in \mathrm{SU}(3)$$

$$V(\Xi) = \frac{M^2}{2}\mathrm{Tr}[\Xi^2] + \frac{\kappa}{3!}\mathrm{Tr}[\Xi^3] + \frac{\Lambda}{4!}\mathrm{Tr}[\Xi^4]$$

$$\Xi(x) = \begin{pmatrix} \xi_1(x) & 0 & 0 \\ 0 & \xi_2(x) & 0 \\ 0 & 0 & -\xi_1(x) - \xi_2(x) \end{pmatrix}$$

$$V(\Xi) = \frac{M^2}{2}\left(\xi_1^2 + \xi_2^2 + (\xi_1 + \xi_2)^2\right)$$
$$+ \frac{\kappa}{3!}\left(\xi_1^3 + \xi_2^3 - (\xi_1 + \xi_2)^3\right) + \frac{\Lambda}{4!}\left(\xi_1^4 + \xi_2^4 + (\xi_1 + \xi_2)^4\right)$$

$$= M^2\left(\xi_1^2 + \xi_2^2 + \xi_1\xi_2\right) - \frac{\kappa}{2}\xi_1\xi_2(\xi_1 + \xi_2)$$
$$+ \frac{\Lambda}{4!}\left(2\xi_1^4 + 4\xi_1^3\xi_2 + 6\xi_1^2\xi_2^2 + 4\xi_1\xi_2^3 + 2\xi_2^4\right)$$

$$\det(\Xi) = -\xi_1\xi_2(\xi_1 + \xi_2) = \frac{1}{3}\mathrm{Tr}[\Xi^3]$$

$$(\mathrm{Tr}[\Xi^2])^2 = 4(\xi_1^2 + \xi_2^2 + \xi_1\xi_2)^2$$
$$= 4(\xi_1^4 + 2\xi_1^3\xi_2 + 3\xi_1^2\xi_2^2 + 2\xi_1\xi_2^3 + \xi_2^4) = 2\mathrm{Tr}[\Xi^4]$$

$$\frac{\partial V(\Xi)}{\partial \xi_1} = M^2(2\xi_1 + \xi_2) - \frac{\kappa}{2}(2\xi_1\xi_2 + \xi_2^2) + \frac{\Lambda}{3!}(\xi_1^2 + \xi_2^2 + \xi_1\xi_2)(2\xi_1 + \xi_2) = 0$$
$$\frac{\partial V(\Xi)}{\partial \xi_2} = M^2(2\xi_2 + \xi_1) - \frac{\kappa}{2}(2\xi_1\xi_2 + \xi_1^2) + \frac{\Lambda}{3!}(\xi_1^2 + \xi_2^2 + \xi_1\xi_2)(2\xi_2 + \xi_1) = 0$$

$$3M^2(\xi_1 + \xi_2) - \frac{\kappa}{2}(\xi_1^2 + \xi_2^2 + 4\xi_1\xi_2) + \frac{\Lambda}{2}(\xi_1^2 + \xi_2^2 + \xi_1\xi_2)(\xi_1 + \xi_2) = 0$$

$$M^2(\xi_1 - \xi_2) + \frac{\kappa}{2}(\xi_1^2 - \xi_2^2) + \frac{\Lambda}{3!}(\xi_1^2 + \xi_2^2 + \xi_1\xi_2)(\xi_1 - \xi_2) = 0$$

$$M^2 + \frac{\kappa}{2}(\xi_1 + \xi_2) + \frac{\Lambda}{3!}(\xi_1^2 + \xi_2^2 + \xi_1\xi_2) = 0$$

$$(\xi_1 + \xi_2)^2 + \frac{1}{2}\xi_1\xi_2 = 0$$

$$\xi_\pm = \frac{1}{2\Lambda}\left(\kappa \pm \sqrt{\kappa^2 - 8M^2\Lambda}\right)$$

$$V(\Xi) = \xi_\pm^2\left(\frac{3M^2}{2} - \frac{\kappa\xi_\pm}{4}\right)$$

$$\xi_+ = \frac{1}{2\Lambda}\left(\kappa + \sqrt{\kappa^2 - 8M^2\Lambda}\right) = \frac{\mathcal{V}}{\sqrt{3}},$$

$$\Xi(x) = \Xi_0 = \frac{\mathcal{V}}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \mathcal{V}\lambda^8.$$

$$V_\mu(x) = \mathrm{i}g_3 V_\mu^a(x)\frac{\lambda^a}{2}$$

$$V_\mu'(x) = \Upsilon(x)\big(V_\mu(x) + \partial_\mu\big)\Upsilon(x)^\dagger, \Upsilon(x) \in \mathrm{SU}(3)$$

$$W_\mu^1(x) = V_\mu^1(x), \quad W_\mu^2(x) = V_\mu^2(x), W_\mu^3(x) = V_\mu^3(x), B_\mu(x) = V_\mu^8(x),$$
$$X_\mu^1(x) = V_\mu^4(x), \quad X_\mu^2(x) = V_\mu^5(x), Y_\mu^1(x) = V_\mu^6(x), Y_\mu^2(x) = V_\mu^7(x).$$

$$X_\mu(x) = X_\mu^1(x) + \mathrm{i}X_\mu^2(x), \quad \bar{X}_\mu(x) = X_\mu^1(x) - \mathrm{i}X_\mu^2(x),$$
$$Y_\mu(x) = Y_\mu^1(x) + \mathrm{i}Y_\mu^2(x), \quad \bar{Y}_\mu(x) = Y_\mu^1(x) - \mathrm{i}Y_\mu^2(x),$$

$$V_\mu(x) = \mathrm{i}\frac{g_3}{2}\begin{pmatrix} W_\mu^3(x) + B_\mu(x)/\sqrt{3} & W_\mu^-(x) & \bar{X}_\mu(x) \\ W_\mu^+(x) & -W_\mu^3(x) + B_\mu(x)/\sqrt{3} & \bar{Y}_\mu(x) \\ X_\mu(x) & Y_\mu(x) & -2B_\mu(x)/\sqrt{3} \end{pmatrix}.$$

$$\Upsilon(x) = \begin{pmatrix} \exp\left(\mathrm{i}g'\varphi(x)/2\right)L(x) & 0_{2\times 1} \\ 0_{1\times 2} & \exp\left(-\mathrm{i}g'\varphi(x)\right) \end{pmatrix} \in \mathrm{SU}(3), L(x) \in \mathrm{SU}(2)_L$$

$$W'_\mu(x) = L(x)\big(W_\mu(x) + \partial_\mu\big)L(x)^\dagger, B'_\mu(x) = B_\mu(x) - \partial_\mu\varphi(x)$$

$$\big(X'_\mu(x), Y'_\mu(x)\big) = \exp\left(-\mathrm{i}g'\frac{3}{2}\varphi(x)\right)\big(X'_\mu(x), Y'_\mu(x)\big)L(x)^\dagger$$

$$\begin{pmatrix}\bar{X}'_\mu(x)\\ \bar{Y}'_\mu(x)\end{pmatrix} = \exp\left(\mathrm{i}g'\frac{3}{2}\varphi(x)\right)L(x)\begin{pmatrix}\bar{X}'_\mu(x)\\ \bar{Y}'_\mu(x)\end{pmatrix}$$

$$Q_X = -\frac{1}{2} - \frac{3}{2} = -2, Q_Y = \frac{1}{2} - \frac{3}{2} = -1, Q_{\bar{X}} = 2, Q_{\bar{Y}} = 1$$

$$\{8\} = \{3\}_0 + \{1\}_0 + \{2\}_{2/3} + \{2\}_{-2/3}$$

$$g = g_3, g' = \frac{1}{\sqrt{3}}g_3 \;\Rightarrow\; e = \frac{gg'}{\sqrt{g^2 + g'^2}} = \frac{g_3}{2}, \sin^2\,\theta_{\mathrm{W}} = \frac{g'^2}{g^2 + g'^2} = \frac{1}{4}$$

$$D_\mu\Xi(x) = \partial_\mu\Xi(x) + \big[V_\mu(x), \Xi(x)\big].$$

$$V_{\mu\nu}(x) = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) + \big[V_\mu(x), V_\nu(x)\big],$$

$$\mathcal{L}(\Xi, V) = \frac{1}{4}\mathrm{Tr}\big[D_\mu\Xi D_\mu\Xi\big] + V(\Xi) - \frac{1}{2g_3^2}\mathrm{Tr}\big[V_{\mu\nu}V_{\mu\nu}\big]$$

$$\frac{1}{4}\mathrm{Tr}\big[D_\mu\Xi D_\mu\Xi\big] = \frac{1}{4}\mathrm{Tr}\Big[\big[V_\mu, \Xi_0\big]\big[V_\mu, \Xi_0\big]\Big] = \frac{3}{8}g_3^2\mathcal{V}^2\big(\bar{X}_\mu X_\mu + \bar{Y}_\mu Y_\mu\big)$$

$$\big[V_\mu, \Xi_0\big] = \mathrm{i}\frac{\sqrt{3}}{2}g_3\mathcal{V}\begin{pmatrix}0 & 0 & -\bar{X}_\mu\\ 0 & 0 & -\bar{Y}_\mu\\ X_\mu & Y_\mu & 0\end{pmatrix}$$

$$M_X = M_Y = \frac{\sqrt{3}}{2}g_3\mathcal{V}$$

$$\Phi(x) = \begin{pmatrix}\Phi^+(x)\\ \Phi^0(x)\\ \phi^-(x)\end{pmatrix} \in \mathbb{C}^3$$

$$\{3\} = \{2\}_{1/2} + \{1\}_{-1}$$

$$V(\Phi, \Xi) = \frac{m'^2}{2}\Phi^\dagger\Phi + \frac{\kappa'}{3!}\Phi^\dagger\Xi\Phi + \frac{\lambda'}{4!}\Phi^\dagger\Xi^2\Phi + \frac{\lambda}{4!}\big(\Phi^\dagger\Phi\big)^2$$

$$\begin{aligned}V(\Phi, \Xi_0) = {} & \frac{m'^2}{2}(\Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0 + \phi^{-*}\phi^-) + \frac{\lambda}{4!}(\Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0 + \phi^{-*}\phi^-)^2\\ & + \frac{\kappa'}{3!}\frac{\mathcal{V}}{\sqrt{3}}(\Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0 - 2\phi^{-*}\phi^-) + \frac{\lambda'}{4!}\frac{\mathcal{V}^2}{3}(\Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0 + 4\phi^{-*}\phi^-)\end{aligned}$$

$$\frac{\partial V(\Phi,\Xi_0)}{\partial \Phi^{+*}} = \left(\frac{m'^2}{2} + \frac{\kappa'}{3!}\frac{\mathcal{V}}{\sqrt{3}} + \frac{\lambda'}{4!}\frac{\mathcal{V}^2}{3}\right)\Phi^+ + \frac{2\lambda}{4!}(\Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0 + \phi^{-*}\phi^-)\Phi^+ = 0$$

$$\frac{\partial V(\Phi,\Xi_0)}{\partial \Phi^{0*}} = \left(\frac{m'^2}{2} + \frac{\kappa'}{3!}\frac{\mathcal{V}}{\sqrt{3}} + \frac{\lambda'}{4!}\frac{\mathcal{V}^2}{3}\right)\Phi^0 + \frac{2\lambda}{4!}(\Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0 + \phi^{-*}\phi^-)\Phi^0 = 0$$

$$\frac{\partial V(\Phi,\Xi_0)}{\partial \phi^{-*}} = \left(\frac{m'^2}{2} - \frac{2\kappa'}{3!}\frac{\mathcal{V}}{\sqrt{3}} + \frac{\lambda'}{3!}\frac{\mathcal{V}^2}{3}\right)\phi^- + \frac{2\lambda}{4!}(\Phi^{+*}\Phi^+ + \Phi^{0*}\Phi^0 + \phi^{-*}\phi^-)\phi^- = 0$$

$$\frac{m'^2}{2} + \frac{\kappa'}{3!}\frac{\mathcal{V}}{\sqrt{3}} + \frac{\lambda'}{4!}\frac{\mathcal{V}^2}{3} + \frac{2\lambda}{4!}\mathrm{v}^2 = 0$$

$$M_{\mathrm{H}}^2 = m'^2 + \frac{\kappa'}{3\sqrt{3}}\mathcal{V} + \frac{\lambda'}{36}\mathcal{V}^2 + \frac{\lambda}{2}\mathrm{v}^2 = \frac{\lambda}{3}\mathrm{v}^2$$

$$M_{\phi}^2 = m'^2 - \frac{2\kappa'}{3\sqrt{3}}\mathcal{V} + \frac{\lambda'}{9}\mathcal{V}^2 + \frac{\lambda}{6}\mathrm{v}^2 = -\frac{\kappa'}{\sqrt{3}}\mathcal{V} + \frac{\lambda'}{12}\mathcal{V}^2$$

$$\mathcal{L}(\Phi,\Xi,V) = \frac{1}{2}\left(D_\mu\Phi\right)^\dagger D_\mu\Phi + V(\Phi,\Xi)$$

$$D_\mu\Phi(x) = \left(\partial_\mu + V_\mu(x)\right)\Phi(x)$$

$$\Omega(x) = \exp\,(\mathrm{i}\omega_a(x)T^a) \in \mathrm{SU}(N).$$

$$G_\mu(x) \to \Omega(x)\left(G_\mu(x) + \partial_\mu\right)\Omega(x)^\dagger$$

$$\Phi(x) = \begin{pmatrix}\Phi_1(x)\\ \Phi_2(x)\\ \cdot\\ \Phi_N(x)\end{pmatrix} \in \mathbb{C}^N$$

$$\Phi(x) \to \Omega(x)\Phi(x).$$

$$\mathcal{L}(\Phi,G) = \frac{1}{2}\left(D_\mu\Phi\right)^\dagger D_\mu\Phi + \frac{m^2}{2}\Phi^\dagger\Phi + \frac{\lambda}{4!}\left(\Phi^\dagger\Phi\right)^2$$

$$D_\mu = \partial_\mu + cG_\mu(x)$$

$$\mathcal{L}(\Phi,A) = \frac{1}{2}D_\mu\Phi^\dagger D_\mu\Phi + \frac{m^2}{2}\Phi^\dagger\Phi + \frac{\lambda}{4!}\left(\Phi^\dagger\Phi\right)^2 + \frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

$$\mathcal{L}(\Xi,V) = \frac{1}{4}\mathrm{Tr}\left[D_\mu\Xi D_\mu\Xi\right] + V(\Xi) - \frac{1}{2g_3^2}\mathrm{Tr}\left[V_{\mu\nu}V_{\mu\nu}\right]$$

$$V(\Xi) = \frac{M^2}{2}\mathrm{Tr}[\Xi^2] + \frac{\Lambda}{4!}\mathrm{Tr}[\Xi^4].$$

$$\Phi(x) = \begin{pmatrix}\Phi^+(x)\\ \Phi^0(x)\\ \phi^-(x)\end{pmatrix} \in \mathbb{C}^3$$

$$G_\mu(x) = \mathrm{i}g_s G_\mu^a(x)T^a, T^a = \frac{1}{2}\lambda^a$$

$$G'_\mu(x) = \Omega(x)\big(G_\mu(x) + \partial_\mu\big)\Omega(x)^\dagger, \Omega(x) \in \mathrm{SU}(3)_\mathrm{c}$$

$$G_{\mu\nu}(x) = \partial_\mu G_\nu(x) - \partial_\nu G_\mu(x) + \big[G_\mu(x), G_\nu(x)\big], G'_{\mu\nu}(x) = \Omega(x)G_{\mu\nu}(x)\Omega(x)^\dagger$$

$$\mathcal{L}(G) = -\frac{1}{2g_s^2}\mathrm{Tr}\big[G_{\mu\nu}G_{\mu\nu}\big] + \frac{\mathrm{i}\theta}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[G_{\mu\nu}G_{\rho\sigma}\big]$$

$$U_{x,\mu} = \exp\big(aG_\mu(x)\big) = \exp\big(\mathrm{i}ag_s G_\mu^a(x)T^a\big)$$

$$U'_{x,\mu} = \Omega_x U_{x,\mu}\Omega_{x+\hat{\mu}}^\dagger$$

$$U_\mathcal{C} = \mathcal{P}\prod_{(x,\mu)\in\mathcal{C}} U_{x,\mu}, U'_\mathcal{C} = \Omega_x U_\mathcal{C}\Omega_y^\dagger$$

$$S[U] = \frac{1}{g_s^2}\sum_{x,\mu>\nu}\mathrm{Tr}\big[2\mathbb{1} - U_{x,\mu\nu} - U_{x,\mu\nu}^\dagger\big]$$

$$U_{x,\mu\nu} = U_{x,\mu}U_{x+\hat{\mu},\nu}U_{x+\hat{\nu},\mu}^\dagger U_{x,\nu}^\dagger$$

$$Z = \prod_{x,\mu}\int_{\mathrm{SU}(N_\mathrm{c})} dU_{x,\mu}\exp\left(-S[U]\right)$$

$$\langle W_\mathcal{C}\rangle = \langle \mathrm{Tr}U_\mathcal{C}\rangle = \frac{1}{Z}\prod_{x,\mu}\int_{\mathrm{SU}(N_\mathrm{c})} dU_{x,\mu}\mathrm{Tr}\left[\mathcal{P}\prod_{(x,\mu)\in\mathcal{C}} U_{x,\mu}\right]\exp\left(-S[U]\right)$$

$$\lim_{T\to\infty}\langle W_\mathcal{C}\rangle \sim \exp\left(-V(R)T\right)$$

$$V(R) \sim \sigma R.$$

$$-\lim_{R,T\to\infty}\log\langle W_\mathcal{C}\rangle \sim \sigma A, A = RT$$

$$\lim_{R,T\to\infty}\langle W_\mathcal{C}\rangle \sim \exp\left(-\gamma(R+T)\right)$$

$$\int_{\mathrm{SU}(N)} dUf(\Omega_\mathrm{L}U) = \int_{\mathrm{SU}(N)} dUf\big(U\Omega_\mathrm{R}^\dagger\big) = \int_{\mathrm{SU}(N)} dUf(U)$$

$$\int_{\mathrm{SU}(N)} dU = 1$$

$$\int_{\mathrm{SU}(N)} dUU^{ij} = 0, \int_{\mathrm{SU}(N)} dUU^{ij}\big(U^\dagger\big)^{kl} = \frac{1}{N}\delta_{jk}\delta_{il}$$

$$U_{\Gamma=\{N^2-1\}}^{ab} = \frac{1}{2}\mathrm{Tr}\big[\lambda^a U\lambda^b U^\dagger\big].$$

$$U^{ab}_{\Gamma=\{N^2-1\}} = \frac{1}{2}\mathrm{Tr}[\lambda^a \mathbb{1} \lambda^b \mathbb{1}] = \delta_{ab}$$

$$\frac{1}{2}\mathrm{Tr}\left[\lambda^a U^\dagger \lambda^b U\right] = \frac{1}{2}\mathrm{Tr}\left[\lambda^b U \lambda^a U^\dagger\right] = U^{ba}_{\Gamma=\{N^2-1\}}$$

$$UV = W \Rightarrow U_\Gamma V_\Gamma = W_\Gamma$$

$$\chi_\Gamma(U) = \mathrm{Tr} U_\Gamma.$$

$$\chi_{\Gamma=\{N^2-1\}}(U) = \mathrm{Tr}U\mathrm{Tr}U^\dagger - 1 = |\mathrm{Tr}U|^2 - 1.$$

$$\langle U \mid \Gamma, ab\rangle = \sqrt{d_\Gamma} U^{ab}_\Gamma \;\Rightarrow\; \chi_\Gamma(U) = \mathrm{Tr}[U_\Gamma] = U^{aa}_\Gamma = \frac{1}{\sqrt{d_\Gamma}} \sum_{a=1}^{d_\Gamma} \langle U \mid \Gamma, aa\rangle.$$

$$\int_{\mathrm{SU}(N)} dU |U\rangle\langle U| = \mathbb{1}$$

$$\delta_{\Gamma,\Gamma'}\delta_{aa'}\delta_{bb'} = \langle \Gamma, ab \mid \Gamma', a'b'\rangle = \int_{\mathrm{SU}(N)} dU \langle \Gamma, ab \mid U\rangle\langle U \mid \Gamma', a'b'\rangle = \int_{\mathrm{SU}(N)} dU \sqrt{d_\Gamma d_{\Gamma'}} U^{ab*}_\Gamma U^{a'b'}_{\Gamma'}$$
$$\Rightarrow \int_{\mathrm{SU}(N)} dU \chi_\Gamma(U)^* \chi_{\Gamma'}(U) = \int_{\mathrm{SU}(N)} dU U^{aa*}_\Gamma U^{a'a'}_{\Gamma'} = \frac{1}{\sqrt{d_\Gamma d_{\Gamma'}}} \delta_{\Gamma,\Gamma'}\delta_{aa'}\delta_{aa'} = \delta_{\Gamma,\Gamma'}$$

$$\sum_{\Gamma,ab} |\Gamma, ab\rangle\langle \Gamma, ab| = \mathbb{1}$$

$$\langle U \mid V\rangle = \sum_{\Gamma,ab} \langle U \mid \Gamma, ab\rangle\langle \Gamma, ab \mid V\rangle = \sum_\Gamma d_\Gamma U^{ab}_\Gamma V^{ab*}_\Gamma = \sum_\Gamma d_\Gamma U^{ab}_\Gamma \left(V^\dagger_\Gamma\right)^{ba}$$
$$= \sum_\Gamma d_\Gamma \left(UV^\dagger\right)^{aa}_\Gamma = \sum_\Gamma d_\Gamma \chi_\Gamma\left(UV^\dagger\right)$$

$$\int_{\mathrm{SU}(N)} dU \chi_\Gamma(VU) \chi_{\Gamma'}\left(U^\dagger W\right) = \frac{1}{d_\Gamma} \chi_\Gamma(VW) \delta_{\Gamma,\Gamma'}$$

$$|f\rangle = \sum_{\Gamma,ab} |\Gamma, ab\rangle\langle \Gamma, ab \mid f\rangle \Rightarrow$$
$$\langle U \mid f\rangle = \sum_{\Gamma,ab} \langle U \mid \Gamma, ab\rangle\langle \Gamma, ab \mid f\rangle = \sum_{\Gamma,ab} U^{ab}_\Gamma \langle \Gamma, ab \mid f\rangle$$

$$\langle \Gamma, ab \mid f\rangle = \int_{\mathrm{SU}(N)} dU \langle \Gamma, ab \mid U\rangle\langle U \mid f\rangle = \int_{\mathrm{SU}(N)} dU U^{ab*}_\Gamma \langle U \mid f\rangle$$

$$f_\Gamma = \sum_{a=1}^{d_\Gamma} \langle \Gamma, aa \mid f\rangle = \int_{\mathrm{SU}(N)} dU U^{aa*}_\Gamma \langle U \mid f\rangle = \int_{\mathrm{SU}(N)} dU \chi_\Gamma(U)^* f(U)$$
$$f(U) = \sum_{\Gamma,ab} U^{ab}_\Gamma \langle \Gamma, ab \mid f\rangle = \sum_\Gamma U^{ab}_\Gamma f_\Gamma \delta_{ab} = \sum_\Gamma U^{aa}_\Gamma f_\Gamma = \sum_\Gamma \chi_\Gamma(U) f_\Gamma$$

$$\exp\left(\frac{1}{g_s^2}\mathrm{Tr}U_{x,\mu\nu}\right)=\sum_{\Gamma}\chi_\Gamma\big(U_{x,\mu\nu}\big)f_\Gamma,f_\Gamma=\int_{\mathrm{SU}(N_c)}dU\chi_\Gamma(U)^*\exp\left(\frac{1}{g_s^2}\mathrm{Tr}U_{x,\mu\nu}\right)$$

$$\langle W_{\mathcal{C}}\rangle=\frac{1}{Z}\prod_{x,\mu}\int_{\mathrm{SU}(N_c)}dU_{x,\mu}\mathrm{Tr}\left[\mathcal{P}\prod_{(x,\mu)\in\mathcal{C}}U_{x,\mu}\right]=0$$

$$\langle W_{\mathcal{C}}\rangle=\frac{1}{N_c^{A/a^2}}\frac{1}{(g_s^2)^{A/a^2}}=\exp\left(-A\log\left(N_c g_s^2\right)/a^2\right)\;\Rightarrow\;\sigma=\frac{1}{a^2}\log\left(N_c g_s^2\right)$$

$$\xi/a\sim\left(\beta_0 g_s^2\right)^{\beta_1/(2\beta_0^2)}\exp\left(\frac{1}{2\beta_0 g_s^2}\right),\beta_0=\frac{11N_c}{3(4\pi)^2},\beta_1=\frac{34N_c^2}{3(4\pi)^4}$$

$$V(R)=\sigma R+\mu-\frac{\pi}{12R}$$

$$R_0^2|F(R_0)|=1.65.$$

$$\sigma=Mg\;\Rightarrow\;M=\frac{\sigma}{g}\approx 1.39\frac{\hbar c}{gR_0^2}\approx 2\cdot 10^4\ \mathrm{kg}.$$

$$\mathcal{H}[h]=\frac{\kappa}{2}\sum_{x,i=1,2}(h_x-h_{x+i})^2,Z=\prod_x\sum_{h_x\in\mathbb{Z}}\exp\left(-\beta\mathcal{H}[h]\right),\kappa>0.$$

$$X^a(x_{d-1},x_d)=\left(X^1(x_{d-1},x_d),X^2(x_{d-1},x_d),\dots,X^{d-2}(x_{d-1},x_d)\right),a\in\{1,2,\dots,d-2\}$$

$$X^a(0,x_d)=X^a(R,x_d)=0$$

$$\begin{aligned}S_0[X]&=\int_0^\beta dx_d\int_0^R dx_{d-1}\sigma\left(1+\frac{1}{2}\partial_\nu X^a\partial_\nu X^a\right)+\int_0^\beta dx_d\mu\\&=\sigma R\beta+\mu\beta+\sigma\int_0^\beta dx_d\int_0^R dx_{d-1}\frac{1}{2}\partial_\nu X^a\partial_\nu X^a\end{aligned}$$

$$X^a(x_{d-1},x_d+\beta)=X^a(x_{d-1},x_d)$$

$$\hat{H}=\mu+\int_0^R dx_{d-1}\left(\sigma+\frac{1}{2\sigma}\hat{\Pi}^a\hat{\Pi}^a+\frac{\sigma}{2}\partial_{d-1}\hat{X}^a\partial_{d-1}\hat{X}^a\right)$$

$$\begin{aligned}\left[\hat{X}^a(x_{d-1}),\hat{\Pi}^b(y_{d-1})\right]&=\mathrm{i}\delta_{ab}\delta(x_{d-1}-y_{d-1})\\\left[\hat{X}^a(x_{d-1}),\hat{X}^b(y_{d-1})\right]&=\left[\hat{\Pi}^a(x_{d-1}),\hat{\Pi}^b(y_{d-1})\right]=0\end{aligned}$$

$$\hat{X}^a(x_{d-1})=\frac{2}{R}\sum_{n=1}^{\infty}\hat{X}_n^a\sin\left(\pi n x_{d-1}/R\right),\quad \hat{X}_n^a=\int_0^R dx_{d-1}\hat{X}^a(x_{d-1})\sin\left(\pi n x_{d-1}/R\right)$$

$$\hat{\Pi}^a(x_{d-1})=\frac{2}{R}\sum_{n=1}^{\infty}\hat{\Pi}_n^a\sin\left(\pi n x_{d-1}/R\right),\quad \hat{\Pi}_n^a=\int_0^R dx_{d-1}\hat{\Pi}^a(x_{d-1})\sin\left(\pi n x_{d-1}/R\right)$$

$$\left[\hat{X}_n^a, \hat{\Pi}_m^b\right] = \frac{1}{2}\mathrm{i}R\delta_{ab}\delta_{nm}, \left[\hat{X}_n^a, \hat{X}_m^b\right] = \left[\hat{\Pi}_n^a, \hat{\Pi}_m^b\right] = 0$$

$$\hat{H} = \sigma R + \mu + \frac{1}{R}\sum_{n=1}^{\infty}\left[\frac{1}{\sigma}\hat{\Pi}_n^a\hat{\Pi}_n^a + \sigma\left(\frac{\pi n}{R}\right)^2 \hat{X}_n^a\hat{X}_n^a\right].$$

$$\hat{a}_n^a = \frac{1}{\sqrt{R}}\left(\sqrt{\frac{\pi n\sigma}{R}}\hat{X}_n^a + \mathrm{i}\sqrt{\frac{R}{\pi n\sigma}}\hat{\Pi}_n^a\right), \hat{a}_n^{a\dagger} = \frac{1}{\sqrt{R}}\left(\sqrt{\frac{\pi n\sigma}{R}}\hat{X}_n^a - \mathrm{i}\sqrt{\frac{R}{\pi n\sigma}}\hat{\Pi}_n^a\right)$$

$$\hat{H} = \sigma R + \mu + \sum_{n=1}^{\infty}\frac{\pi n}{R}\left(\hat{a}_n^{a\dagger}\hat{a}_n^a + \frac{d-2}{2}\right), \left[\hat{a}_n^a, \hat{a}_m^{b\dagger}\right] = \delta_{ab}\delta_{nm}.$$

$$V(R) = \sigma R + \mu + \sum_{n=1}^{\infty}\frac{\pi n}{R}\frac{d-2}{2} \triangleq \sigma R + \mu - \frac{\pi(d-2)}{24R}$$

$$F(R) = -\frac{dV(R)}{dR} = -\sigma - \frac{\pi(d-2)}{24R^2}.$$

$$\sigma R_0^2 + \frac{\pi}{12} \approx 1.65 \Rightarrow \sigma R_0^2 \approx 1.39 \Rightarrow \sigma = (0.495(5)\mathrm{GeV})^2.$$

$$\sum_{n=1}^{\infty} n \triangleq -\frac{1}{12}$$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$\zeta(s) = \frac{(2\pi)^s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

$$\int_0^{\infty} dpp\exp(-tp) = -\frac{d}{dt}\int_0^{\infty} dp\exp(-tp) = -\frac{d}{dt}\frac{1}{t} = \frac{1}{t^2}$$

$$\frac{\pi}{R}\sum_{n=1}^{\infty} p_n\exp(-tp_n) = -\frac{\pi}{R}\frac{d}{dt}\sum_{n=1}^{\infty}\exp(-t\pi n/R) = -\frac{\pi}{R}\frac{d}{dt}\left(\frac{1}{1-\exp(-t\pi/R)} - 1\right)$$
$$= \frac{\pi^2}{R^2}\frac{\exp(-t\pi/R)}{[1-\exp(-t\pi/R)]^2} \overset{t\to 0}{\to} \frac{1}{t^2} - \frac{\pi^2}{12R^2} + \cdots$$

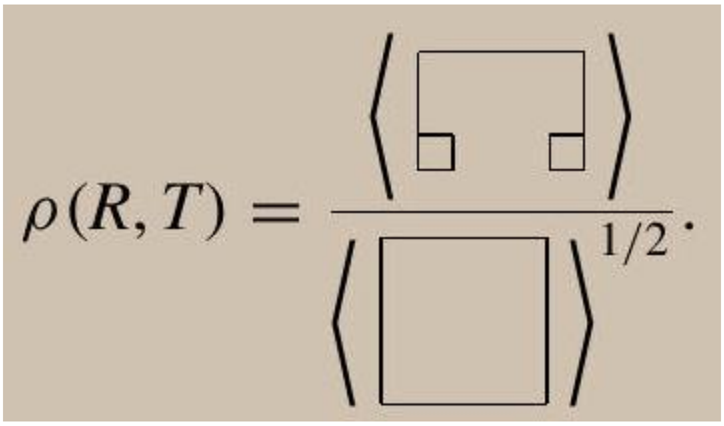
$$\sum_{n=1}^{\infty} p_n\exp(-tp_n) - \frac{R}{\pi}\int_0^{\infty} dpp\exp(-tp) \underset{t\to 0}{\longrightarrow} \frac{R}{\pi}\left(\frac{1}{t^2} - \frac{\pi^2}{12R^2} - \frac{1}{t^2}\right) = -\frac{\pi}{12R}$$

$$S_1[X] = \frac{b}{4}\int_0^{\beta} dx_d\left(\partial_{d-1}X^a\partial_{d-1}X^a|_{x_{d-1}=0} + \partial_{d-1}X^a\partial_{d-1}X^a|_{x_{d-1}=R}\right)$$

$$S_2[X] = \frac{c_2}{4}\int_0^\beta dx_d \int_0^R dx_{d-1}\partial_\nu X^a \partial_\nu X^a \partial_\rho X^b \partial_\rho X^b$$

$$S_3[X] = \frac{c_3}{4}\int_0^\beta dx_d \int_0^R dx_{d-1}\partial_\nu X^a \partial_\nu X^b \partial_\rho X^a \partial_\rho X^b$$

$$b = 0, (d-2)c_2 + c_3 = \frac{d-4}{2}\sigma$$



$$\square = \mathrm{Tr}\left[U_{x,ij}\lambda^a\right]\mathrm{Tr}\left[\lambda^a U_{\mathcal{C}_{xy}}\lambda^b U_{\mathcal{C}_{xy}}^\dagger\right]\mathrm{Tr}\left[U_{y,ij}^\dagger \lambda^b\right], U_{\mathcal{C}_{xy}} = \mathcal{P}\prod_{(z,\mu)\in\mathcal{C}_{xy}} U_{z,\mu}$$

$$\langle\mathcal{O}(0)\mathcal{O}(x_4)\rangle = \frac{1}{Z}\prod_{x,\mu}\int_{\mathrm{SU}(N_c)} dU_{x,\mu}\mathcal{O}(0)\mathcal{O}(x_4)\exp\left(-S[U]\right)$$
$$\sim |\langle 0|\mathcal{O}|0\rangle\square|^2 + |\langle J^{PC}|\mathcal{O}|0\rangle\square|^2 \exp\left(-Mx_4\right)$$

$$\mathcal{O}(x_4) = \sum_{\vec{x},i,j}\mathrm{Tr}U_{ij,\vec{x},x_4}.$$

$$\langle\mathcal{O}(0)\mathcal{O}(x_4)\rangle_c = \langle\mathcal{O}(0)\mathcal{O}(x_4)\rangle - |\langle\mathcal{O}\rangle|^2 = \frac{1}{(N_c g_s^2)^{4x_4/a}} \sim \exp\left(-Mx_4\right),$$

$$M = \frac{4}{a}\log\left(N_c g_s^2\right).$$

$$\Phi_{\vec{x}} = \mathrm{Tr}\left[\mathcal{P}\prod_{x_4} U_{4,\vec{x},x_4}\right]$$

$$\langle\Phi_{\vec{x}}\rangle = \exp\left(-\beta F\right),$$

$$z = \exp\left(2\pi i n/N_c\right) \in \mathbb{Z}(N_c), n \in \{1,2,\ldots,N_c\}$$

$$\Omega_{ab}\Omega_{bc}^\top = \Omega_{ab}\Omega_{cb} = \delta_{ac}.$$

$$T_{abc} = T_{def}\Omega_{da}\Omega_{eb}\Omega_{fc},$$

$$T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1.$$

$$\{7\} = \{1\} + \{3\} + \{\overline{3}\}, \{14\} = \{3\} + \{\overline{3}\} + \{8\}$$

$$\begin{aligned}\{14\} \times \{14\} \times \{14\} =& \{1\} + \{7\} + 5\{14\} + 3\{27\} + 2\{64\} + 4\{77\} \\ &+3\{77'\} + \{182\} + 3\{189\} + \{273\} + 2\{448\}\end{aligned}$$

$$\square = \left(U^{ab}_{x,ij}T_{abc}\right)U^{cd}_{\mathcal{C}xy}\left(T_{def}U^{ef}_{y,ij}\right).$$

$$\mathcal{L}_0(\bar{v}, v, \bar{e}, e) = \bar{v}_{\mathrm{L}}\bar{\sigma}_\mu\partial_\mu v_{\mathrm{L}} + \bar{e}_{\mathrm{L}}\bar{\sigma}_\mu\partial_\mu e_{\mathrm{L}} + \bar{e}_{\mathrm{R}}\sigma_\mu\partial_\mu e_{\mathrm{R}}$$

$$\begin{aligned} v'_{\mathrm{L}}(x) &= \exp(\mathrm{i}\chi)v_{\mathrm{L}}(x), \quad & \bar{v}'_{\mathrm{L}}(x) &= \bar{v}_{\mathrm{L}}(x)\exp(-\mathrm{i}\chi), \\ e'_{\mathrm{L}}(x) &= \exp(\mathrm{i}\chi)e_{\mathrm{L}}(x), \quad & \bar{e}'_{\mathrm{L}}(x) &= \bar{e}_{\mathrm{L}}(x)\exp(-\mathrm{i}\chi), \\ e'_{\mathrm{R}}(x) &= \exp(\mathrm{i}\chi)e_{\mathrm{R}}(x), \quad & \bar{e}'_{\mathrm{R}}(x) &= \bar{e}_{\mathrm{R}}(x)\exp(-\mathrm{i}\chi). \end{aligned}$$

$$l_{\mathrm{L}}(x) = \begin{pmatrix} v_{\mathrm{L}}(x) \\ e_{\mathrm{L}}(x) \end{pmatrix}, \bar{l}_{\mathrm{L}}(x) = (\bar{v}_{\mathrm{L}}(x), \bar{e}_{\mathrm{L}}(x))$$

$$\mathcal{L}_0(\bar{v}, v, \bar{e}, e) = \bar{l}_{\mathrm{L}}\bar{\sigma}_\mu\partial_\mu l_{\mathrm{L}} + \bar{e}_{\mathrm{R}}\sigma_\mu\partial_\mu e_{\mathrm{R}}$$

$$\begin{aligned} l'_{\mathrm{L}}(x) &= \begin{pmatrix} v'_{\mathrm{L}}(x) \\ e'_{\mathrm{L}}(x) \end{pmatrix} = L(x)\begin{pmatrix} v_{\mathrm{L}}(x) \\ e_{\mathrm{L}}(x) \end{pmatrix} = L(x)l_{\mathrm{L}}(x) \\ \bar{l}'_{\mathrm{L}}(x) &= (\bar{v}'_{\mathrm{L}}(x), \bar{e}'_{\mathrm{L}}(x)) = (\bar{v}_{\mathrm{L}}(x), \bar{e}_{\mathrm{L}}(x))L(x)^\dagger = \bar{l}_{\mathrm{L}}(x)L(x)^\dagger, L(x) \in \mathrm{SU}(2)_L \end{aligned}$$

$$e'_{\mathrm{R}}(x) = e_{\mathrm{R}}(x)$$

$$\begin{aligned} v'_{\mathrm{L}}(x) &= \exp\left(\mathrm{i}Y_{l_{\mathrm{L}}}g'\varphi(x)\right)v_{\mathrm{L}}(x), \quad & \bar{v}'_{\mathrm{L}}(x) &= \bar{v}_{\mathrm{L}}(x)\exp\left(-\mathrm{i}Y_{l_{\mathrm{L}}}g'\varphi(x)\right) \\ e'_{\mathrm{L}}(x) &= \exp\left(\mathrm{i}Y_{l_{\mathrm{L}}}g'\varphi(x)\right)e_{\mathrm{L}}(x), \quad & \bar{e}'_{\mathrm{L}}(x) &= \bar{e}_{\mathrm{L}}(x)\exp\left(-\mathrm{i}Y_{l_{\mathrm{L}}}g'\varphi(x)\right) \\ e'_{\mathrm{R}}(x) &= \exp\left(\mathrm{i}Y_{e_{\mathrm{R}}}g'\varphi(x)\right)e_{\mathrm{R}}(x), \quad & \bar{e}'_{\mathrm{R}}(x) &= \bar{e}_{\mathrm{R}}(x)\exp\left(-\mathrm{i}Y_{e_{\mathrm{R}}}g'\varphi(x)\right) \end{aligned}$$

$$T^3_{\mathrm{L}v_{\mathrm{L}}} = \frac{1}{2}, T^3_{\mathrm{L}e_{\mathrm{L}}} = -\frac{1}{2}, T^3_{\mathrm{L}e_{\mathrm{R}}} = 0$$

$$T^3_{\mathrm{R}v_{\mathrm{L}}} = 0, T^3_{\mathrm{R}e_{\mathrm{L}}} = 0, T^3_{\mathrm{R}e_{\mathrm{R}}} = -\frac{1}{2}.$$

$$T^3_{\mathrm{L}\nu_{\mathrm{R}}} = 0, T^3_{\mathrm{R}\nu_{\mathrm{R}}} = \frac{1}{2}.$$

$$D_\mu\begin{pmatrix} v_{\mathrm{L}}(x) \\ e_{\mathrm{L}}(x) \end{pmatrix} = \left[\partial_\mu + \mathrm{i}Y_{l_{\mathrm{L}}}g'B_\mu(x) + \mathrm{i}gW^a_\mu(x)\frac{\tau^a}{2}\right]\begin{pmatrix} v_{\mathrm{L}}(x) \\ e_{\mathrm{L}}(x) \end{pmatrix}$$

$$W_\mu(x) = \mathrm{i}gW^a_\mu(x)\frac{\tau^a}{2}$$

$$D_\mu l_{\mathrm{L}}(x) = \left[\partial_\mu + \mathrm{i}Y_{l_{\mathrm{L}}}g'B_\mu(x) + W_\mu(x)\right]l_{\mathrm{L}}(x).$$

$$D_\mu e_{\mathrm{R}}(x) = \left[\partial_\mu + \mathrm{i}Y_{e_{\mathrm{R}}} g' B_\mu(x)\right] e_{\mathrm{R}}(x)$$

$$\begin{aligned}\mathcal{L}(\bar{v}, v, \bar{e}, e, W, B) &= \bar{l}_{\mathrm{L}} \bar{\sigma}_\mu D_\mu l_{\mathrm{L}} + \bar{e}_{\mathrm{R}} \sigma_\mu D_\mu e_{\mathrm{R}} \\ &= (\bar{v}_{\mathrm{L}}, \bar{e}_{\mathrm{L}}) \bar{\sigma}_\mu D_\mu \begin{pmatrix} v_{\mathrm{L}} \\ e_{\mathrm{L}} \end{pmatrix} + \bar{e}_{\mathrm{R}} \sigma_\mu D_\mu e_{\mathrm{R}}\end{aligned}$$

$$\begin{aligned}&{}^{\mathrm{C}}B_\mu(x) = -B_\mu(x), \\ &{}^{\mathrm{P}}B_i(\vec{x}, x_4) = -B_i(-\vec{x}, x_4),\ {}^{\mathrm{P}}B_4(\vec{x}, x_4) = B_4(-\vec{x}, x_4), \\ &{}^{\mathrm{T}}B_i(\vec{x}, x_4) = -B_i(\vec{x}, -x_4),\ {}^{\mathrm{T}}B_4(\vec{x}, x_4) = B_4(\vec{x}, -x_4).\end{aligned}$$

$$\begin{aligned}&{}^{\mathrm{CP}}B_i(\vec{x}, x_4) = B_i(-\vec{x}, x_4),\ {}^{\mathrm{CP}}B_4(\vec{x}, x_4) = -B_4(-\vec{x}, x_4), \\ &{}^{\mathrm{CPT}}B_\mu(x) = -B_\mu(-x).\end{aligned}$$

$$\begin{aligned}&{}^{\mathrm{C}}W_\mu(x) = W_\mu(x)^*, \\ &{}^{\mathrm{P}}W_i(\vec{x}, x_4) = -W_i(-\vec{x}, x_4),\ {}^{\mathrm{P}}W_4(\vec{x}, x_4) = W_4(-\vec{x}, x_4), \\ &{}^{\mathrm{T}}W_i(\vec{x}, x_4) = W_i(\vec{x}, -x_4)^*,\ {}^{\mathrm{T}}W_4(\vec{x}, x_4) = -W_4(\vec{x}, -x_4)^*,\end{aligned}$$

$$\begin{aligned}&{}^{\mathrm{CP}}W_i(\vec{x}, x_4) = -W_i(-\vec{x}, x_4)^*,\ {}^{\mathrm{CP}}W_4(\vec{x}, x_4) = W_4(-\vec{x}, x_4)^*, \\ &{}^{\mathrm{CPT}}W_\mu(x) = -W_\mu(-x).\end{aligned}$$

$$\begin{aligned}S\left[{}^{\mathrm{CP}}\bar{e}_{\mathrm{R}}, {}^{\mathrm{CP}}e_{\mathrm{R}}, {}^{\mathrm{CP}}B\right] &= \int d^4x\, {}^{\mathrm{CP}}\bar{e}_{\mathrm{R}}(\vec{x}, x_4) \sigma_\mu \mathrm{i}Y_{l_{\mathrm{L}}} g' {}^{\mathrm{CP}^{\mathrm{P}}}B_\mu(\vec{x}, x_4) {}^{\mathrm{CP}}e_{\mathrm{R}}(\vec{x}, x_4) \\ &= -\int d^4x e_{\mathrm{R}}(-\vec{x}, x_4)^{\mathsf{T}} P^{\mathsf{T}} C^{-1} \left[-\sigma_i \mathrm{i}Y_{l_{\mathrm{L}}} g' B_i(-\vec{x}, x_4) + \sigma_4 \mathrm{i}Y_{l_{\mathrm{L}}} g' B_4(-\vec{x}, x_4)\right] CP \bar{e}_{\mathrm{R}}(-\vec{x}, x_4)^{\mathsf{T}} \\ &= \int d^4x e_{\mathrm{R}}(-\vec{x}, x_4)^{\mathsf{T}} \left[\sigma_i^{\mathsf{T}} \mathrm{i}Y_{l_{\mathrm{L}}} g' B_i(-\vec{x}, x_4) + \sigma_4^{\mathsf{T}} \mathrm{i}Y_{l_{\mathrm{L}}} g' B_4(-\vec{x}, x_4)\right] \bar{e}_{\mathrm{R}}(-\vec{x}, x_4)^{\mathsf{T}} \\ &= \int d^4x \bar{e}_{\mathrm{R}}(-\vec{x}, x_4) \sigma_\mu \mathrm{i}Y_{l_{\mathrm{L}}} g' B_\mu(-\vec{x}, x_4) e_{\mathrm{R}}(-\vec{x}, x_4) \\ &= S[\bar{e}_{\mathrm{R}}, e_{\mathrm{R}}, B]\end{aligned}$$

$$\begin{aligned}&{}^{\mathrm{C}}\Phi(x) = \Phi(x)^* \\ &{}^{\mathrm{P}}\Phi(\vec{x}, x_4) = \Phi(-\vec{x}, x_4) \\ &{}^{\mathrm{T}}\Phi(\vec{x}, x_4) = \Phi(\vec{x}, -x_4)^*\end{aligned}$$

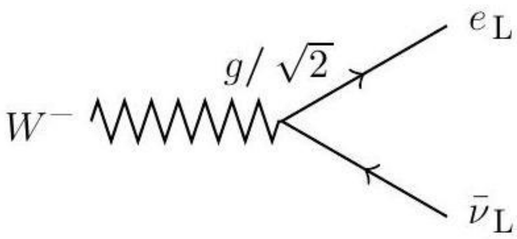
$$\begin{aligned}&{}^{\mathrm{CP}}\Phi(\vec{x}, x_4) = \Phi(-\vec{x}, x_4)^*, \\ &{}^{\mathrm{CPT}}\Phi(x) = \Phi(-x).\end{aligned}$$

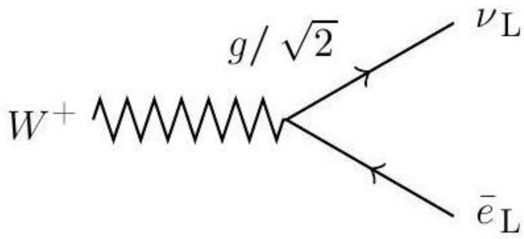
$$S[\Phi, W, B] = \int d^4x \left[\frac{1}{2} D_\mu \Phi^\dagger D_\mu \Phi + V(\Phi) - \frac{1}{2g^2} \mathrm{Tr}\left(W_{\mu\nu} W_{\mu\nu}\right) + \frac{1}{4} B_{\mu\nu} B_{\mu\nu}\right]$$

$$W_\mu^\pm(x) = \frac{1}{\sqrt{2}}\left(W_\mu^1(x) \mp \mathrm{i}W_\mu^2(x)\right)$$

$$W_\mu^1(x) \frac{\tau^1}{2} + W_\mu^2(x) \frac{\tau^2}{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & W_\mu^+(x) \\ W_\mu^-(x) & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(W_\mu^+(x)\tau^+ + W_\mu^-(x)\tau^-\right)$$

$$A_\mu(x) = \frac{g' W_\mu^3(x) + g B_\mu(x)}{\sqrt{g^2 + g'^2}}, Z_\mu(x) = \frac{g W_\mu^3(x) - g' B_\mu(x)}{\sqrt{g^2 + g'^2}}.$$





$$W_\mu^3(x) = \frac{g' A_\mu(x) + g Z_\mu(x)}{\sqrt{g^2 + g'^2}}, B_\mu(x) = \frac{g A_\mu(x) - g' Z_\mu(x)}{\sqrt{g^2 + g'^2}},$$

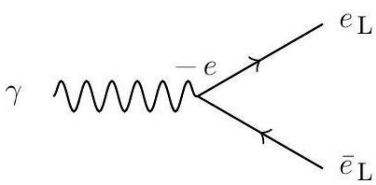
$$\mathcal{L}(\bar{v}, v, \bar{e}, e, A, Z) = (\bar{v}_\mathrm{L}, \bar{e}_\mathrm{L}) \bar{\sigma}_\mu \left[\partial_\mu + \mathrm{i} \begin{pmatrix} X_\mu^1 & g W_\mu^+ / \sqrt{2} \\ g W_\mu^- / \sqrt{2} & X_\mu^2 \end{pmatrix} \right] \begin{pmatrix} v_\mathrm{L} \\ e_\mathrm{L} \end{pmatrix}$$
$$+ \bar{e}_\mathrm{R} \sigma_\mu \left[\partial_\mu + \mathrm{i} \frac{Y_{e_\mathrm{R}} g'}{\sqrt{g^2 + g'^2}} (g A_\mu - g' Z_\mu) \right] e_\mathrm{R}$$

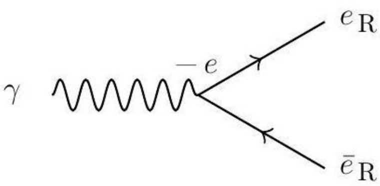
$$X_\mu^1(x) = \frac{1}{\sqrt{g^2 + g'^2}} \left[g g' \left(1/2 + Y_{l_\mathrm{L}} \right) A_\mu(x) + \left(g^2/2 - Y_{l_\mathrm{L}} g'^2 \right) Z_\mu(x) \right]$$
$$X_\mu^2(x) = \frac{1}{\sqrt{g^2 + g'^2}} \left[g g' \left(-1/2 + Y_{l_\mathrm{L}} \right) A_\mu(x) + \left(-g^2/2 - Y_{l_\mathrm{L}} g'^2 \right) Z_\mu(x) \right]$$

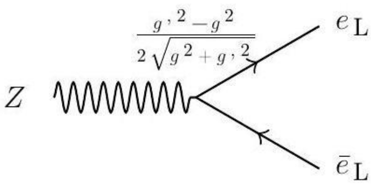
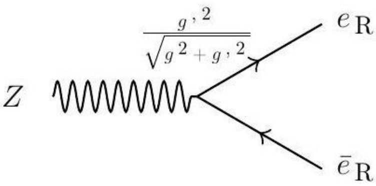
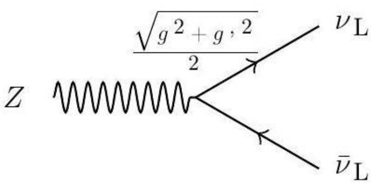
$$X_\mu^1(x) = \frac{\sqrt{g^2 + g'^2}}{2} Z_\mu(x),$$
$$X_\mu^2(x) = \frac{1}{\sqrt{g^2 + g'^2}} \left[\frac{g'^2 - g^2}{2} Z_\mu(x) - g g' A_\mu(x) \right]$$
$$= -\frac{\sqrt{g^2 + g'^2}}{2} \left[\cos(2\theta_\mathrm{W}) Z_\mu(x) + \sin(2\theta_\mathrm{W}) A_\mu(x) \right]$$

$$e = \frac{g g'}{\sqrt{g^2 + g'^2}}$$

$$Y = T_\mathrm{R}^3 - \frac{1}{2} L$$





$$Q = T_\mathrm{L}^3 + Y = T_\mathrm{L}^3 + T_\mathrm{R}^3 - \frac{1}{2} L$$

$$j_{\mu}^{\mathrm{em}}(x) = j_{\mu,\mathrm{L}}^{\mathrm{em}}(x) + j_{\mu,\mathrm{R}}^{\mathrm{em}}(x) = -e\big(\bar{e}_{\mathrm{L}}(x)\bar{\sigma}_{\mu}e_{\mathrm{L}}(x) + \bar{e}_{\mathrm{R}}(x)\sigma_{\mu}e_{\mathrm{R}}(x)\big)$$

$$j_{\mu}^{0}(x) = \frac{\sqrt{g^2+g'^2}}{2}\bar{\nu}_{\mathrm{L}}(x)\bar{\sigma}_{\mu}\nu_{\mathrm{L}}(x) + \frac{g'^2-g^2}{2\sqrt{g^2+g'^2}}\bar{e}_{\mathrm{L}}(x)\bar{\sigma}_{\mu}e_{\mathrm{L}}(x) + \frac{g'^2}{\sqrt{g^2+g'^2}}\bar{e}_{\mathrm{R}}(x)\sigma_{\mu}e_{\mathrm{R}}(x)$$

$$= \frac{\sqrt{g^2+g'^2}}{2}\big(\bar{\nu}_{\mathrm{L}}(x)\bar{\sigma}_{\mu}\nu_{\mathrm{L}}(x) - \cos(2\theta_{\mathrm{W}})\bar{e}_{\mathrm{L}}(x)\bar{\sigma}_{\mu}e_{\mathrm{L}}(x) + 2\sin^2\theta_{\mathrm{W}}\bar{e}_{\mathrm{R}}(x)\sigma_{\mu}e_{\mathrm{R}}(x)\big)$$

$$j_{\mu}^{+}(x) = \frac{g}{\sqrt{2}}\bar{\nu}_{\mathrm{L}}(x)\bar{\sigma}_{\mu}e_{\mathrm{L}}(x), j_{\mu}^{-}(x) = \frac{g}{\sqrt{2}}\bar{e}_{\mathrm{L}}(x)\bar{\sigma}_{\mu}\nu_{\mathrm{L}}(x)$$

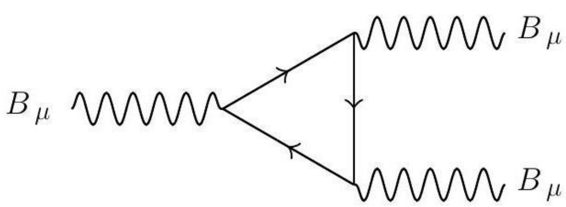
$$A = \sum_{\mathrm{L}} Y^3 - \sum_{\mathrm{R}} Y^3$$

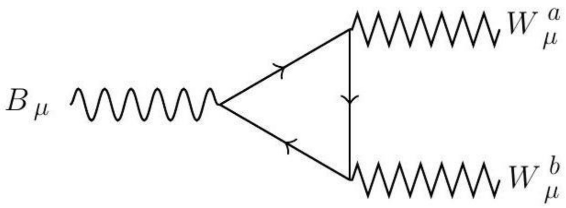
$$A_l = 2Y_{l_{\mathrm{L}}}^3 - Y_{e_{\mathrm{R}}}^3 = 2\left(-\frac{1}{2}\right)^3 - (-1)^3 = \frac{3}{4} \neq 0$$

$$A^{abc} = \mathrm{Tr}_{\mathrm{L}}\big[\big(T^aT^b + T^bT^a\big)T^c\big] - \mathrm{Tr}_{\mathrm{R}}\big[\big(T^aT^b + T^bT^a\big)T^c\big]$$

$$\mathrm{Tr}\big[\big(\tau^a\tau^b + \tau^b\tau^a\big)\tau^c\big] = 2\delta_{ab}\mathrm{Tr}\tau^c = 0$$

$$A_l^{ab4} = \mathrm{Tr}_{\mathrm{L}}\left[\frac{1}{4}\big(\tau^a\tau^b + \tau^b\tau^a\big)Y\right] = \frac{1}{2}\delta_{ab}\mathrm{Tr}_{\mathrm{L}}Y\mathbb{1}.$$





$$A_l^{ab4} = \delta_{ab}Y_{l_{\mathrm{L}}} = -\frac{1}{2}\delta_{ab} \neq 0$$

$$\Pi_4[\mathrm{SU}(2)] = \Pi_4[S^3] = \mathbb{Z}(2);$$

$$q_{\mathrm{L}}^{c}(x) = \begin{pmatrix} u_{\mathrm{L}}^{c}(x) \\ d_{\mathrm{L}}^{c}(x) \end{pmatrix}, \bar{q}_{\mathrm{L}}^{c}(x) = \big(\bar{u}_{\mathrm{L}}^{c}(x), \bar{d}_{\mathrm{L}}^{c}(x)\big), c \in \{1,2,\ldots,N_{\mathrm{c}}\}$$

$$T_{\mathrm{L}u_{\mathrm{L}}}^3 = \frac{1}{2}, T_{\mathrm{L}d_{\mathrm{L}}}^3 = -\frac{1}{2}, T_{\mathrm{L}u_{\mathrm{R}}}^3 = 0, T_{\mathrm{L}d_{\mathrm{R}}}^3 = 0$$
$$T_{\mathrm{R}u_{\mathrm{L}}}^3 = 0, T_{\mathrm{R}d_{\mathrm{L}}}^3 = 0, T_{\mathrm{R}u_{\mathrm{R}}}^3 = \frac{1}{2}, T_{\mathrm{R}d_{\mathrm{R}}}^3 = -\frac{1}{2}$$

$$\mathcal{L}_0(\bar{u},u,\bar{d},d) = \bar{u}_{\mathrm{L}}^{c}\bar{\sigma}_{\mu}\partial_{\mu}u_{\mathrm{L}}^{c} + \bar{u}_{\mathrm{R}}^{c}\sigma_{\mu}\partial_{\mu}u_{\mathrm{R}}^{c} + \bar{d}_{\mathrm{L}}^{c}\bar{\sigma}_{\mu}\partial_{\mu}d_{\mathrm{L}}^{c} + \bar{d}_{\mathrm{R}}^{c}\sigma_{\mu}\partial_{\mu}d_{\mathrm{R}}^{c}$$

$$\begin{aligned}
u_{\mathrm{L}}^{c\prime}(x) &= \exp(\mathrm{i}\rho/N_{\mathrm{c}})u_{\mathrm{L}}^{c}(x), & \bar{u}_{\mathrm{L}}^{c\prime}(x) &= \bar{u}_{\mathrm{L}}^{c}(x)\exp(-\mathrm{i}\rho/N_{\mathrm{c}}),\\
u_{\mathrm{R}}^{c\prime}(x) &= \exp(\mathrm{i}\rho/N_{\mathrm{c}})u_{\mathrm{R}}^{c}(x), & \bar{u}_{\mathrm{R}}^{c\prime}(x) &= \bar{u}_{\mathrm{R}}^{c}(x)\exp(-\mathrm{i}\rho/N_{\mathrm{c}}),\\
d_{\mathrm{L}}^{c\prime}(x) &= \exp(\mathrm{i}\rho/N_{\mathrm{c}})d_{\mathrm{L}}^{c}(x), & \bar{d}_{\mathrm{L}}^{c\prime}(x) &= \bar{d}_{\mathrm{L}}^{c}(x)\exp(-\mathrm{i}\rho/N_{\mathrm{c}}),\\
d_{\mathrm{R}}^{c\prime}(x) &= \exp(\mathrm{i}\rho/N_{\mathrm{c}})d_{\mathrm{R}}^{c}(x), & \bar{d}_{\mathrm{R}}^{c\prime}(x) &= \bar{d}_{\mathrm{R}}^{c}(x)\exp(-\mathrm{i}\rho/N_{\mathrm{c}}).
\end{aligned}$$

$$u_{\mathrm{L}}^{c\prime}(x)=\exp\left(\mathrm{i}Y_{q_{\mathrm{L}}}g'\varphi(x)\right)\quad u_{\mathrm{L}}^{c}(x),\quad \bar{u}_{\mathrm{L}}^{c'}(x)=\bar{u}_{\mathrm{L}}^{c}(x)\exp\left(-\mathrm{i}Y_{q_{\mathrm{L}}}g'\varphi(x)\right),$$
$$u_{\mathrm{R}}^{c'}(x)=\exp\left(\mathrm{i}Y_{u_{\mathrm{R}}}g'\varphi(x)\right)\quad u_{\mathrm{R}}^{c}(x),\quad \bar{u}_{\mathrm{R}}^{c'}(x)=\bar{u}_{\mathrm{R}}^{c}(x)\exp\left(-\mathrm{i}Y_{u_{\mathrm{R}}}g'\varphi(x)\right),$$
$$d_{\mathrm{L}}^{c'}(x)=\exp\left(\mathrm{i}Y_{q_{\mathrm{L}}}g'\varphi(x)\right)\quad d_{\mathrm{L}}^{c}(x),\quad \bar{d}_{\mathrm{L}}^{c'}(x)=\bar{d}_{\mathrm{L}}^{c}(x)\exp\left(-\mathrm{i}Y_{q_{\mathrm{L}}}g'\varphi(x)\right),$$
$$d_{\mathrm{R}}^{c'}(x)=\exp\left(\mathrm{i}Y_{d_{\mathrm{R}}}g'\varphi(x)\right)\quad d_{\mathrm{R}}^{c}(x),\quad \bar{d}_{\mathrm{R}}^{c'}(x)=\bar{d}_{\mathrm{R}}^{c}(x)\exp\left(-\mathrm{i}Y_{d_{\mathrm{R}}}g'\varphi(x)\right).$$

$$q_{\mathrm{L}}^{c\prime}(x)=\begin{pmatrix}u_{\mathrm{L}}^{c\prime}(x)\\ d_{\mathrm{L}}^{c\prime}(x)\end{pmatrix}=L(x)\begin{pmatrix}u_{\mathrm{L}}^{c}(x)\\ d_{\mathrm{L}}^{c}(x)\end{pmatrix}=L(x)q_{\mathrm{L}}^{c}(x),$$
$$\bar{q}_{\mathrm{L}}^{c\prime}(x)=\left(\bar{u}_{\mathrm{L}}^{c\prime}(x),\bar{d}_{\mathrm{L}}^{c\prime}(x)\right)=\left(\bar{u}_{\mathrm{L}}^{c}(x),\bar{d}_{\mathrm{L}}^{c}(x)\right)L(x)^{\dagger}=\bar{q}_{\mathrm{L}}^{c}(x)L(x)^{\dagger}$$
$$u_{\mathrm{R}}^{c\prime}(x)=u_{\mathrm{R}}^{c}(x),d_{\mathrm{R}}^{c\prime}(x)=d_{\mathrm{R}}^{c}(x),\bar{u}_{\mathrm{R}}^{c\prime}(x)=\bar{u}_{\mathrm{R}}^{c}(x),\bar{d}_{\mathrm{R}}^{c\prime}(x)=\bar{d}_{\mathrm{R}}^{c}(x)$$

$$q_{\mathrm{L}}'(x)=\Omega(x)q_{\mathrm{L}}(x),\quad \bar{q}_{\mathrm{L}}'(x)=\bar{q}_{\mathrm{L}}(x)\Omega(x)^{\dagger}$$
$$u_{\mathrm{R}}'(x)=\Omega(x)u_{\mathrm{R}}(x),\quad \bar{u}_{\mathrm{R}}'(x)=\bar{u}_{\mathrm{R}}(x)\Omega(x)^{\dagger}$$
$$d_{\mathrm{R}}'(x)=\Omega(x)d_{\mathrm{R}}(x),\quad \bar{d}_{\mathrm{R}}'(x)=\bar{d}_{\mathrm{R}}(x)\Omega(x)^{\dagger},\Omega(x)\in \mathrm{SU}(N_{\mathrm{c}})$$

$$D_{\mu}q_{\mathrm{L}}^{c}(x)=\left[\left(\partial_{\mu}+\mathrm{i}Y_{q_{\mathrm{L}}}g'B_{\mu}(x)+\mathrm{i}gW_{\mu}^{a}(x)\frac{\tau^{a}}{2}\right)\delta_{cc'}+\mathrm{i}g_{s}G_{\mu}^{a}(x)\frac{\lambda_{cc'}^{a}}{2}\right]q_{\mathrm{L}}^{c'}(x)$$

$$D_{\mu}q_{\mathrm{L}}(x)=\left[\partial_{\mu}+\mathrm{i}Y_{q_{\mathrm{L}}}g'B_{\mu}(x)+W_{\mu}(x)+G_{\mu}(x)\right]q_{\mathrm{L}}(x)$$

$$D_{\mu}u_{\mathrm{R}}^{c}(x)=\left[\left(\partial_{\mu}+\mathrm{i}Y_{u_{\mathrm{R}}}g'B_{\mu}(x)\right)\delta_{cc'}+\mathrm{i}g_{s}G_{\mu}^{a}(x)\frac{\lambda_{cc'}^{a}}{2}\right]u_{\mathrm{R}}^{c'}(x),$$
$$D_{\mu}d_{\mathrm{R}}^{c}(x)=\left[\left(\partial_{\mu}+\mathrm{i}Y_{d_{\mathrm{R}}}g'B_{\mu}(x)\right)\delta_{cc'}+\mathrm{i}g_{s}G_{\mu}^{a}(x)\frac{\lambda_{cc'}^{a}}{2}\right]d_{\mathrm{R}}^{c'}(x),$$

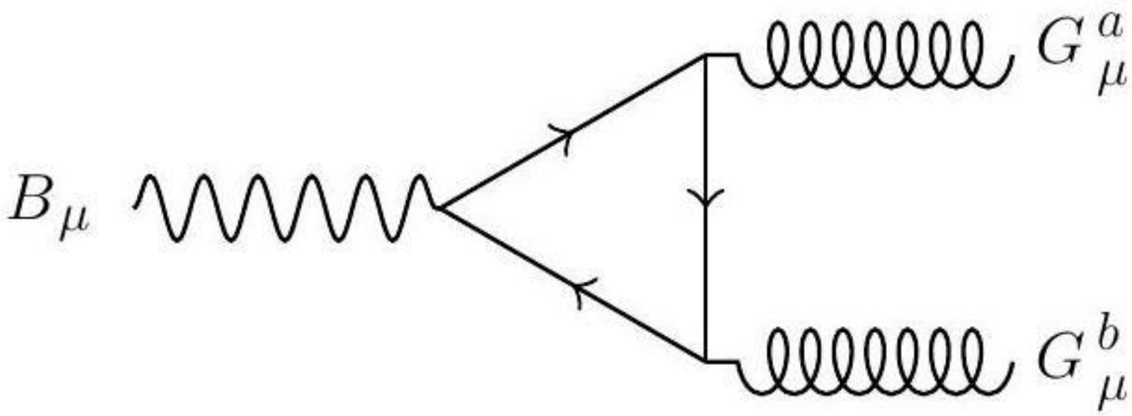
$$D_{\mu}u_{\mathrm{R}}(x)=\left[\partial_{\mu}+\mathrm{i}Y_{u_{\mathrm{R}}}g'B_{\mu}(x)+G_{\mu}(x)\right]u_{\mathrm{R}}(x)$$
$$D_{\mu}d_{\mathrm{R}}(x)=\left[\partial_{\mu}+\mathrm{i}Y_{d_{\mathrm{R}}}g'B_{\mu}(x)+G_{\mu}(x)\right]d_{\mathrm{R}}(x)$$

$$\mathcal{L}(\bar{u},u,\bar{d},d,G,W,B)=\bar{q}_{\mathrm{L}}\bar{\sigma}_{\mu}D_{\mu}q_{\mathrm{L}}+\bar{u}_{\mathrm{R}}\sigma_{\mu}D_{\mu}u_{\mathrm{R}}+\bar{d}_{\mathrm{R}}\sigma_{\mu}D_{\mu}d_{\mathrm{R}}$$
$$=\left(\bar{u}_{\mathrm{L}},\bar{d}_{\mathrm{L}}\right)\bar{\sigma}_{\mu}D_{\mu}\begin{pmatrix}u_{\mathrm{L}}\\ d_{\mathrm{L}}\end{pmatrix}+\bar{u}_{\mathrm{R}}\sigma_{\mu}D_{\mu}u_{\mathrm{R}}+\bar{d}_{\mathrm{R}}\sigma_{\mu}D_{\mu}d_{\mathrm{R}}$$

$$A_{q}^{444}=N_{\mathrm{c}}\left(2Y_{q\,\mathrm{L}}^{3}-Y_{u_{\mathrm{R}}}^{3}-Y_{d_{\mathrm{R}}}^{3}\right)$$

$$A_{q}^{ab4}=\delta_{ab}N_{\mathrm{c}}Y_{q_{\mathrm{L}}},a,b\in\{1,2,3\}$$

$$A_{l}^{ab4}+A_{q}^{ab4}=0\ \Rightarrow\ Y_{q_{\mathrm{L}}}=\frac{1}{2N_{\mathrm{c}}}$$
$$A_{l}^{444}+A_{q}^{444}=0\ \Rightarrow\ 2Y_{q_{\mathrm{L}}}^{3}-Y_{u_{\mathrm{R}}}^{3}-Y_{d_{\mathrm{R}}}^{3}=-\frac{3}{4N_{\mathrm{c}}}\ \Rightarrow$$
$$Y_{u_{\mathrm{R}}}^{3}+Y_{d_{\mathrm{R}}}^{3}=\frac{1}{4N_{\mathrm{c}}^{3}}+\frac{3}{4N_{\mathrm{c}}}$$



$$A_q^{ab4} = \delta_{ab}\left(2Y_{q_\mathrm{L}} - Y_{u_\mathrm{R}} - Y_{d_\mathrm{R}}\right), a-4, b-4 \in \{1,2,\dots,N_\mathrm{c}^2-1\}$$

$$Y_{u_\mathrm{R}} + Y_{d_\mathrm{R}} = 2Y_{q_\mathrm{L}} = \frac{1}{N_\mathrm{c}},$$

$$Y_{q_\mathrm{L}} = \frac{1}{2N_\mathrm{c}}, Y_{u_\mathrm{R}} = \frac{1}{2}\left(\frac{1}{N_\mathrm{c}}+1\right), Y_{d_\mathrm{R}} = \frac{1}{2}\left(\frac{1}{N_\mathrm{c}}-1\right);$$

$$Y = T_\mathrm{R}^3 + \frac{1}{2}B$$

$$Y_{q_\mathrm{L}} = \frac{1}{6}, Y_{u_\mathrm{R}} = \frac{2}{3}, Y_{d_\mathrm{R}} = -\frac{1}{3}$$

$$Q = T_\mathrm{L}^3 + Y = T_\mathrm{L}^3 + T_\mathrm{R}^3 + \frac{1}{2}B$$

$$Q_{u_\mathrm{L}} = T_{\mathrm{L}u_\mathrm{L}}^3 + Y_{q_\mathrm{L}} = \frac{1}{2} + \frac{1}{2N_\mathrm{c}} = \frac{1}{2}\left(\frac{1}{N_\mathrm{c}}+1\right)$$
$$Q_{d_\mathrm{L}} = T_{\mathrm{L}d_\mathrm{L}}^3 + Y_{q_\mathrm{L}} = -\frac{1}{2} + \frac{1}{2N_\mathrm{c}} = \frac{1}{2}\left(\frac{1}{N_\mathrm{c}}-1\right)$$

$$Q_{u_\mathrm{R}} = T_{\mathrm{L}u_\mathrm{R}}^3 + Y_{u_\mathrm{R}} = 0 + \frac{1}{2}\left(\frac{1}{N_\mathrm{c}}+1\right)$$
$$Q_{d_\mathrm{R}} = T_{\mathrm{L}d_\mathrm{R}}^3 + Y_{d_\mathrm{R}} = 0 + \frac{1}{2}\left(\frac{1}{N_\mathrm{c}}-1\right)$$

$$Q_u = \frac{1}{2}\left(\frac{1}{N_\mathrm{c}}+1\right) \overset{N_\mathrm{c}=3}{=} \frac{2}{3}, Q_d = \frac{1}{2}\left(\frac{1}{N_\mathrm{c}}-1\right) \overset{N_\mathrm{c}=3}{=} -\frac{1}{3}.$$

$$Q = T_\mathrm{L}^3 + Y = T_\mathrm{L}^3 + T_\mathrm{R}^3 + \frac{1}{2}(B-L)$$

$$Q_p = 2Q_u + Q_d = 2\frac{2}{3} - \frac{1}{3} = 1$$
$$Q_n = Q_u + 2Q_d = \frac{2}{3} - 2\frac{1}{3} = 0$$

$$Q_p = \frac{N_c+1}{2} Q_u + \frac{N_c-1}{2} Q_d = \frac{N_c+1}{4}\left(\frac{1}{N_c}+1\right) + \frac{N_c-1}{4}\left(\frac{1}{N_c}-1\right) = 1$$
$$Q_n = \frac{N_c-1}{2} Q_u + \frac{N_c+1}{2} Q_d = \frac{N_c-1}{4}\left(\frac{1}{N_c}+1\right) + \frac{N_c+1}{4}\left(\frac{1}{N_c}-1\right) = 0$$

$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, e_R, N_L = \begin{pmatrix} p_L \\ n_L \end{pmatrix}, p_R, n_R$$

$$Y_{N_L} = \frac{1}{2}, Y_{p_R} = 1, Y_{n_R} = 0$$

$$Q_{p_L} = T^3_{Lp_L} + Y_{N_L} = \frac{1}{2} + \frac{1}{2} = 1$$
$$Q_{n_L} = T^3_{Ln_L} + Y_{N_L} = -\frac{1}{2} + \frac{1}{2} = 0$$

$$Q_{p_R} = T^3_{Lp_R} + Y_{p_R} = 0 + 1 = 1,$$
$$Q_{n_R} = T^3_{Ln_R} + Y_{n_R} = 0 + 0 = 0.$$

$$A_N^{444} = 2Y^3_{N_L} - Y^3_{p_R} - Y^3_{n_R} = 2\left(\frac{1}{2}\right)^3 - 1^3 - 0^3 = -\frac{3}{4}$$
$$A_N^{ab4} = \delta_{ab} Y_{N_L} = \frac{1}{2}\delta_{ab}$$

$$\nu'_R(x) = \exp(i\chi)\nu_R(x), \bar{\nu}'_R(x) = \bar{\nu}_R(x)\exp(-i\chi),$$

$$T^3_{L\nu_R} = 0, T^3_{R\nu_R} = \frac{1}{2}.$$

$$Y_{\nu_R} = T^3_{R\nu_R} - \frac{1}{2} = 0, Q_{\nu_R} = T^3_{L\nu_R} + Y_{\nu_R} = 0$$

$$A^{4LL} = 2[\mathrm{Tr}_L L^2 Y - \mathrm{Tr}_R L^2 Y] = 2[2Y_{l_L} - Y_{\nu_R} - Y_{e_R}] = 2\left[2\left(-\frac{1}{2}\right) - 0 - (-1)\right] = 0$$

$$A^{44L} = 2[\mathrm{Tr}_L Y^2 L - \mathrm{Tr}_R Y^2 L] = 2[2Y^2_{l_L} - Y^2_{\nu_R} - Y^2_{e_R}] = 2\left[2\left(-\frac{1}{2}\right)^2 - 0^2 - (-1)^2\right] = -1$$

$$A^{abL} = \mathrm{Tr}_L[(T^aT^b + T^bT^a)L] - \mathrm{Tr}_R[(T^aT^b + T^bT^a)L]$$
$$= \mathrm{Tr}_L\left[\frac{1}{4}(\tau^a\tau^b + \tau^b\tau^a)L\right] = \frac{1}{2}\delta_{ab}\mathrm{Tr}_L L = \delta_{ab}$$

$$A^{44B} = 2[\mathrm{Tr}_L Y^2 B - \mathrm{Tr}_R Y^2 B] = 2N_c[2Y^2_{q_L} - Y^2_{u_R} - Y^2_{d_R}]\frac{1}{N_c}$$
$$= 2\left[2\left(\frac{1}{2N_c}\right)^2 - \frac{1}{4}\left(\frac{1}{N_c}+1\right)^2 - \frac{1}{4}\left(\frac{1}{N_c}-1\right)^2\right] = -1$$
$$A^{abB} = \mathrm{Tr}_L[(T^aT^b + T^bT^a)B] - \mathrm{Tr}_R[(T^aT^b + T^bT^a)B]$$
$$= \mathrm{Tr}_L\left[\frac{1}{4}(\tau^a\tau^b + \tau^b\tau^a)B\right] = \frac{1}{2}\delta_{ab}\mathrm{Tr}_L B = N_c\delta_{ab}\frac{1}{N_c} = \delta_{ab}$$

$$A^{44B} = A^{44L}, A^{abB} = A^{abL}, a, b \in \{1,2,3\},$$

$$2Y_{Q_L} = Y_{U_R} + Y_{D_R}$$

$$Y_{Q_L} = 0$$

$$2Y_{Q_L}^3 = Y_{U_R}^3 + Y_{D_R}^3$$

$$Y_{Q_L} = 0, Y_{U_R} + Y_{D_R} = 0$$

$$Q_{U_L} = \frac{1}{2}, Q_{D_L} = -\frac{1}{2}, Q_{U_R} = Y_{U_R}, Q_{D_R} = Y_{D_R}$$

$$Y_{U_R} = -Y_{D_R} = \frac{1}{2}$$

$$\mathcal{L}(\bar{e}_L, e_L, \bar{e}_R, e_R) = m_e[\bar{e}_R e_L + \bar{e}_L e_R].$$

$$\begin{aligned}\mathcal{L}(\bar{l}_L, l_L, \bar{e}_R, e_R, \Phi) &= f_e \bar{l}_L \Phi e_R + f_e^* \bar{e}_R \Phi^\dagger l_L \\ &= f_e (\bar{\nu}_L, \bar{e}_L) \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix} e_R + f_e^* \bar{e}_R (\Phi^{+*}, \Phi^{0*}) \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}\end{aligned}$$

$$-Y_{l_L} + Y_\Phi + Y_{e_R} = \frac{1}{2} + \frac{1}{2} - 1 = 0$$

$$S[{}^{\mathrm{CP}}\bar{l}_L, {}^{\mathrm{CP}}l_L, {}^{\mathrm{CP}}\bar{e}_R, {}^{\mathrm{CP}}e_R, {}^{\mathrm{CP}}\Phi]$$

$$= \int d^4x \left[-f_e l_L(-\vec{x}, x_4)^\top P^\top C^{-1} \Phi(-\vec{x}, x_4)^* C P \bar{e}_R(-\vec{x}, x_4)^\top - f_e^* e_R(-\vec{x}, x_4)^\top P^\top C^{-1} \Phi(-\vec{x}, x_4)^\top C P \bar{l}_L(-\vec{x}, x_4)^\top\right]$$

$$= \int d^4x [f_e \bar{e}_R(-\vec{x}, x_4) \Phi(-\vec{x}, x_4)^\dagger l_L(-\vec{x}, x_4) + f_e^* \bar{l}_L(-\vec{x}, x_4) \Phi(-\vec{x}, x_4) e_R(-\vec{x}, x_4)]$$

$$= \int d^4x [f_e \bar{e}_R(\vec{x}, x_4) \Phi(\vec{x}, x_4)^\dagger l_L(\vec{x}, x_4) + f_e^* \bar{l}_L(\vec{x}, x_4) \Phi(\vec{x}, x_4) e_R(\vec{x}, x_4)]$$

$$e_R'(x) = e_R(x) \exp(\mathrm{i}\theta), \bar{e}_R'(x) = \bar{e}_R(x) \exp(-\mathrm{i}\theta)$$

$$\mathcal{L}(\bar{l}_L, l_L, \bar{e}_R, e_R, \mathrm{v}) = f_e \left[(\bar{\nu}_L, \bar{e}_L) \begin{pmatrix} 0 \\ \mathrm{v} \end{pmatrix} e_R + \bar{e}_R (0, \mathrm{v}) \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}\right] = f_e \mathrm{v} [\bar{e}_L e_R + \bar{e}_R e_L].$$

$$m_e = f_e \mathrm{v}$$

$$-Y_{q_L} + Y_\Phi + Y_{d_R} = -\frac{1}{2N_c} + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{N_c} - 1\right) = 0$$

$$\begin{aligned}\mathcal{L}(\bar{q}_L, q_L, \bar{d}_R, d_R, \Phi) &= f_d [\bar{q}_L \Phi d_R + \bar{d}_R \Phi^\dagger q_L] \\ &= f_d \left[(\bar{u}_L, \bar{d}_L) \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix} d_R + \bar{d}_R (\Phi^{+*}, \Phi^{0*}) \begin{pmatrix} u_L \\ d_L \end{pmatrix}\right]\end{aligned}$$

$$\tilde{\Phi}(x) = \begin{pmatrix} \tilde{\Phi}^0(x) \\ \tilde{\Phi}^-(x) \end{pmatrix}$$

$$\tilde{\Phi}(x) = \begin{pmatrix} \tilde{\text{V}} \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathcal{L}(\bar{q}_\text{L}, q_\text{L}, \bar{u}_\text{R}, u_\text{R}, \tilde{\Phi}) &= f_u[\bar{q}_\text{L}\tilde{\Phi}u_\text{R} + \bar{u}_\text{R}\tilde{\Phi}^\dagger q_\text{L}] \\ &= f_u\left[(\bar{u}_\text{L}, \bar{d}_\text{L})\begin{pmatrix} \tilde{\Phi}^0 \\ \tilde{\Phi}^- \end{pmatrix} u_\text{R} + \bar{u}_\text{R}(\tilde{\Phi}^{0*}, \tilde{\Phi}^{-*})\begin{pmatrix} u_\text{L} \\ d_\text{L} \end{pmatrix}\right] \end{aligned}$$

$$-Y_{q_\text{L}} + Y_{\tilde{\Phi}} + Y_{u_\text{R}} = \frac{1}{2N_\text{c}} + Y_{\tilde{\Phi}} + \frac{1}{2}\left(\frac{1}{N_\text{c}} + 1\right) = 0 \ \Rightarrow \ Y_{\tilde{\Phi}} = -\frac{1}{2}$$

$$\tilde{\Phi}(x) = \begin{pmatrix} \tilde{\Phi}^0(x) \\ \tilde{\Phi}^-(x) \end{pmatrix} = \begin{pmatrix} \Phi^0(x)^* \\ -\Phi^+(x)^* \end{pmatrix} = \text{i}\tau^2\Phi(x)^*$$

$$\boldsymbol{\Phi}(x) = \begin{pmatrix} \Phi^0(x)^* & \Phi^+(x) \\ -\Phi^+(x)^* & \Phi^0(x) \end{pmatrix}$$

$$\mathcal{L}(\bar{q}_\text{L}, q_\text{L}, \bar{u}_\text{R}, u_\text{R}, \bar{d}_\text{R}, d_\text{R}, \boldsymbol{\Phi}) = (\bar{u}_\text{L}, \bar{d}_\text{L})\boldsymbol{\Phi}\mathcal{F}\begin{pmatrix} u_\text{R} \\ d_\text{R} \end{pmatrix} + (\bar{u}_\text{R}, \bar{d}_\text{R})\mathcal{F}^\dagger\boldsymbol{\Phi}^\dagger\begin{pmatrix} u_\text{L} \\ d_\text{L} \end{pmatrix}$$

$$\mathcal{F} = \begin{pmatrix} f_u & 0 \\ 0 & f_d \end{pmatrix} = \mathcal{F}^\dagger$$

$$\mathcal{M} = \boldsymbol{\Phi}\mathcal{F} = \begin{pmatrix} \text{v} & 0 \\ 0 & \text{v} \end{pmatrix}\begin{pmatrix} f_u & 0 \\ 0 & f_d \end{pmatrix} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}.$$

$$\mathcal{L}(\bar{l}_\text{L}, l_\text{L}, \tilde{\Phi}) = \frac{1}{2\Lambda}[({}^\text{c}\bar{l}_\text{L}\tilde{\Phi}^*)(\tilde{\Phi}^\dagger l_\text{L}) + (\bar{l}_\text{L}\tilde{\Phi})(\tilde{\Phi}^\text{T}{}_\text{L})]$$

$$\mathcal{L}(\bar{l}_\text{L}, l_\text{L}, \text{v}) = \frac{\text{v}^2}{2\Lambda}[{}^\text{c}\bar{\nu}_\text{L}\nu_\text{L} + \bar{\nu}_\text{L}{}^\text{c}\nu_\text{L}]$$

$$m_\nu = \frac{\text{v}^2}{\Lambda}$$

$$m_u = f_u\text{v} \approx 2\text{MeV}, m_d = f_d\text{v} \approx 5\text{MeV}, m_e = f_e\text{v} \approx 0.5\text{MeV}$$

$$\begin{aligned} \mathcal{O}_1 &= \epsilon_{abc}(d_\text{R}^a\,{}^\text{c}\bar{u}_\text{R}^b)(u_\text{R}^c\,{}^\text{c}\bar{e}_\text{R}), \quad & \mathcal{O}_2 &= \epsilon_{abc}(d_\text{R}^a\,{}^\text{c}\bar{u}_\text{R}^b)(q_\text{L}^c\text{i}\tau^2\bar{l}_\text{L}), \\ \mathcal{O}_3 &= \epsilon_{abc}(q_\text{L}^a\text{i}\tau^2\bar{q}_\text{L}^b)(u_\text{R}^c\,{}^\text{c}\bar{e}_\text{R}), \quad & \mathcal{O}_4 &= \epsilon_{abc}(q_\text{L}^a\text{i}\tau^2\bar{q}_\text{L}^b)(q_\text{L}^c\text{i}\tau^2\bar{l}_\text{L}). \end{aligned}$$

$$2Y_{l_\text{L}} - Y_{e_\text{R}} + N_\text{c}(2Y_{q_\text{L}} - Y_{u_\text{R}} - Y_{d_\text{R}}) = 0$$

$$N_\text{c}(2Y_{q_\text{L}} - Y_{u_\text{R}} - Y_{d_\text{R}}) = 0$$

$$2Y_{l_\text{L}} = Y_{e_\text{R}}$$

$$Y_{l_\text{L}} = -\frac{1}{2}, Y_{e_\text{R}} = -1$$

$$m_e \neq 0 \ \Rightarrow \ Y_{l_\text{L}} - \frac{1}{2} = Y_{e_\text{R}}$$

$$Y_{q\,\mathrm{L}}=\frac{1}{2N_{\mathrm{c}}},Y_{u\mathrm{R}}=Q_u=\frac{1}{2}\left(\frac{1}{N_{\mathrm{c}}}+1\right),Y_{d_{\mathrm{R}}}=Q_d=\frac{1}{2}\left(\frac{1}{N_{\mathrm{c}}}-1\right)$$

$$Y_{q_{\mathrm{L}}}+\frac{1}{2}=Y_{u_{\mathrm{R}}},Y_{q_{\mathrm{L}}}-\frac{1}{2}=Y_{d_{\mathrm{R}}}$$

$$\begin{aligned}\mathcal{L}(\bar{l}_{\mathrm{L}},l_{\mathrm{L}},\bar{\nu}_{\mathrm{R}},\nu_{\mathrm{R}},\widetilde{\Phi})&=f_\nu\bar{l}_{\mathrm{L}}\tilde{\Phi}\nu_{\mathrm{R}}+f_\nu^*\bar{\nu}_{\mathrm{R}}\widetilde{\Phi}l_{\mathrm{L}}+\frac{1}{2}(M^{\mathrm{c}}\bar{\nu}_{\mathrm{R}}\nu_{\mathrm{R}}+M^*\bar{\nu}_{\mathrm{R}}\,{}^{\mathrm{c}}\nu_{\mathrm{R}})\\&=f_\nu(\bar{\nu}_{\mathrm{L}},\bar{e}_{\mathrm{L}})\begin{pmatrix}\widetilde{\Phi}^0\\\widetilde{\Phi}^-\end{pmatrix}\nu_{\mathrm{R}}+f_\nu^*\bar{\nu}_{\mathrm{R}}(\widetilde{\Phi}^{0*},\widetilde{\Phi}^{-*})\begin{pmatrix}\nu_{\mathrm{L}}\\e_{\mathrm{L}}\end{pmatrix}\\&+\frac{1}{2}(M^{\mathrm{c}}\bar{\nu}_{\mathrm{R}}\nu_{\mathrm{R}}+M^*\bar{\nu}_{\mathrm{R}}\,{}^{\mathrm{c}}\nu_{\mathrm{R}})\end{aligned}$$

$$\mathcal{L}(\bar{l}_{\mathrm{L}},l_{\mathrm{L}},\bar{\nu}_{\mathrm{R}},\nu_{\mathrm{R}},\mathrm{v})=\frac{1}{2}(\bar{\nu}_{\mathrm{L}},\,{}^{\mathrm{c}}\bar{\nu}_{\mathrm{R}})\begin{pmatrix}0&f_\nu\mathrm{V}\\f_\nu\mathrm{V}&M\end{pmatrix}\begin{pmatrix}{}^{\mathrm{c}}\nu_{\mathrm{L}}\\\nu_{\mathrm{R}}\end{pmatrix}+\frac{1}{2}({}^{\mathrm{c}}\bar{\nu}_{\mathrm{L}},\bar{\nu}_{\mathrm{R}})\begin{pmatrix}0&f_\nu^*\mathrm{V}\\f_\nu^*\mathrm{V}&M^*\end{pmatrix}\begin{pmatrix}\nu_{\mathrm{L}}\\{}^{\mathrm{c}}\nu_{\mathrm{R}}\end{pmatrix}=\frac{1}{2}(\bar{\nu}_{\mathrm{L}},\,{}^{\mathrm{c}}\bar{\nu}_{\mathrm{R}})\mathcal{M}_\nu\begin{pmatrix}{}^{\mathrm{c}}\nu_{\mathrm{L}}\\\nu_{\mathrm{R}}\end{pmatrix}+\frac{1}{2}({}^{\mathrm{c}}\bar{\nu}_{\mathrm{L}},\bar{\nu}_{\mathrm{R}})\mathcal{M}_\nu^*\begin{pmatrix}\nu_{\mathrm{L}}\\{}^{\mathrm{c}}\nu_{\mathrm{R}}\end{pmatrix}$$

,

$$\mathcal{M}_\nu=\begin{pmatrix}0&f_\nu\mathrm{v}\\f_\nu\mathrm{v}&M\end{pmatrix}$$

$$\begin{aligned}&\begin{pmatrix}\nu'_{\mathrm{L}}\\{}^{\mathrm{c}}\nu'_{\mathrm{R}}\end{pmatrix}=N\begin{pmatrix}\nu_{\mathrm{L}}\\{}^{\mathrm{c}}\nu_{\mathrm{R}}\end{pmatrix},(\bar{\nu}'_{\mathrm{L}},\,{}^{\mathrm{c}}\bar{\nu}'_{\mathrm{R}})=(\bar{\nu}_{\mathrm{L}},\,{}^{\mathrm{c}}\bar{\nu}_{\mathrm{R}})N^\dagger,\\&\begin{pmatrix}{}^{\mathrm{c}}\nu'_{\mathrm{L}}\\\nu'_{\mathrm{R}}\end{pmatrix}=N^*\begin{pmatrix}{}^{\mathrm{c}}\nu_{\mathrm{L}}\\\nu_{\mathrm{R}}\end{pmatrix},({}^{\mathrm{c}}\bar{\nu}'_{\mathrm{L}},\bar{\nu}'_{\mathrm{R}})=({}^{\mathrm{c}}\bar{\nu}_{\mathrm{L}},\bar{\nu}_{\mathrm{R}})N^{\mathrm{T}},\\&N\mathcal{M}_\nu N^{\mathrm{T}}=N^*\mathcal{M}_\nu^*N^\dagger=\begin{pmatrix}m_\nu&0\\0&M_\nu\end{pmatrix}.\end{aligned}$$

$$\begin{aligned}N&=\begin{pmatrix}A\exp(\mathrm{i}\varphi)&B\exp(\mathrm{i}\varphi)\\-B^*&A^*\end{pmatrix}\\&=\begin{pmatrix}\exp(\mathrm{i}(\alpha+\beta+\varphi))&0\\0&1\end{pmatrix}\begin{pmatrix}|A|&|B|\\-|B|&|A|\end{pmatrix}\begin{pmatrix}\exp(-\mathrm{i}\beta)&0\\0&\exp(-\mathrm{i}\alpha)\end{pmatrix}\end{aligned}$$

$$\exp(\mathrm{i}\chi)=\exp(\mathrm{i}(\alpha+\beta)),\exp(\mathrm{i}\theta)=\exp(2\mathrm{i}\alpha).$$

$$\begin{pmatrix}\exp(-\mathrm{i}\beta)&0\\0&\exp(-\mathrm{i}\alpha)\end{pmatrix}\begin{pmatrix}0&f_\nu\mathrm{v}\\f_\nu\mathrm{v}&M\end{pmatrix}\begin{pmatrix}\exp(-\mathrm{i}\beta)&0\\0&\exp(-\mathrm{i}\alpha)\end{pmatrix}^{\mathrm{T}}=\begin{pmatrix}0&|f_\nu|\mathrm{v}\\|f_\nu|\mathrm{v}&|M|\end{pmatrix}$$

$$\begin{pmatrix}|A|&|B|\\-|B|&|A|\end{pmatrix}\begin{pmatrix}0&|f_\nu|\mathrm{v}\\|f_\nu|\mathrm{v}&|M|\end{pmatrix}\begin{pmatrix}|A|&|B|\\-|B|&|A|\end{pmatrix}^{\mathrm{T}}=\begin{pmatrix}-m_\nu&0\\0&M_\nu\end{pmatrix},$$

$$M_\nu=\frac{1}{2}\left(\sqrt{|M|^2+4|f_\nu|^2\mathrm{v}^2}+|M|\right),m_\nu=\frac{1}{2}\left(\sqrt{|M|^2+4|f_\nu|^2\mathrm{v}^2}-|M|\right)$$

$$|A|^2=\frac{1}{2}\left(1+\frac{|M|}{\sqrt{|M|^2+4|f_\nu|^2\mathrm{v}^2}}\right),|B|^2=\frac{1}{2}\left(1-\frac{|M|}{\sqrt{|M|^2+4|f_\nu|^2\mathrm{v}^2}}\right).$$

$$\begin{pmatrix}\exp(\mathrm{i}(\alpha+\beta+\varphi))&0\\0&1\end{pmatrix}\begin{pmatrix}-m_\nu&0\\0&M_\nu\end{pmatrix}\begin{pmatrix}\exp(\mathrm{i}(\alpha+\beta+\varphi))&0\\0&1\end{pmatrix}^{\mathrm{T}}=\begin{pmatrix}m_\nu&0\\0&M_\nu\end{pmatrix}$$

$$M_\nu\simeq M,m_\nu\simeq\frac{|f_\nu|^2\mathrm{v}^2}{|M|}$$

$$|A|^2 \simeq 1 - \frac{|f_v|^2 v^2}{|M|^2}, |B|^2 \simeq \frac{|f_v|^2 v^2}{|M|^2}$$

$$2Y_{l_L}^3 - Y_{\nu_R}^3 - Y_{e_R}^3 + N_c\left(2Y_{q_L}^3 - Y_{u_R}^3 - Y_{d_R}^3\right) = 0$$
$$2Y_{l_L} + 2N_c Y_{q_L} = 0 \Rightarrow Y_{l_L} = -N_c Y_{q_L}$$
$$N_c\left(2Y_{q_L} - Y_{u_R} - Y_{d_R}\right) = 0 \Rightarrow Y_{q_L} = \frac{1}{2}\left(Y_{u_R} + Y_{d_R}\right)$$

$$\left(Y_{u_R} - Y_{d_R}\right)^2 = \frac{2}{3Y_{l_L}}\left(Y_{\nu_R}^3 + Y_{e_R}^3 - 2Y_{l_L}^3\right)$$

$$2Y_{l_L} - Y_{\nu_R} - Y_{e_R} + N_c\left(2Y_{q_L} - Y_{u_R} - Y_{d_R}\right) = 0 \;\Rightarrow\; 2Y_{l_L} - Y_{\nu_R} - Y_{e_R} = 0$$

$$m_e \neq 0 \;\Rightarrow\; Y_{l_L} - \frac{1}{2} = Y_{e_R}$$

$$Y_{l_L} + \frac{1}{2} = Y_{\nu_R}$$

$$\begin{aligned}\left(Y_{u_R} - Y_{d_R}\right)^2 &= \frac{2}{3Y_{l_L}}\left(Y_{\nu_R}^3 + Y_{e_R}^3 - 2Y_{l_L}^3\right) = \frac{2}{3Y_{l_L}}\left(\left(Y_{\nu_R} + Y_{e_R}\right)^3 - 3Y_{\nu_R}Y_{e_R}\left(Y_{\nu_R} + Y_{e_R}\right) - 2Y_{l_L}^3\right)\\ &= \frac{2}{3Y_{l_L}}\left(8Y_{l_L}^3 - 6Y_{\nu_R}Y_{e_R}Y_{l_L} - 2Y_{l_L}^3\right) = 4Y_{l_L}^2 - 4Y_{\nu_R}Y_{e_R} = \left(Y_{\nu_R} + Y_{e_R}\right)^2 - 4Y_{\nu_R}Y_{e_R} = \left(Y_{\nu_R} - Y_{e_R}\right)^2 = 1\end{aligned}$$

PARTE III.

$$Q_v = T_{L\nu_L}^3 + Y_{l_L} = \frac{1}{2} + Y_{l_L} = Y_{\nu_R}$$

$$Q_e = -\frac{1}{2} + Y_{l_L} = -1 + Q_v, Q_u = \frac{1}{2}\left(\frac{1}{N_c} + 1\right) - \frac{Q_v}{N_c}, Q_d = \frac{1}{2}\left(\frac{1}{N_c} - 1\right) - \frac{Q_v}{N_c}.$$

$$Q_p = 1 - Q_v = -Q_e, Q_n = -Q_v$$

$$M \neq 0 \;\Rightarrow\; Y_{\nu_R} = 0$$

$$q_L = \begin{pmatrix} p_L \\ n_L \end{pmatrix}, p_R, n_R, l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, e_R$$

$${}^c\bar{n}_R n_R,\; {}^c\bar{p}_R e_R,\; {}^c\bar{q}_L (i\tau^2) l_L$$

$$\begin{aligned}\mathcal{L}(\bar{e}, e, \bar{\nu}, \nu, \bar{p}, p, \bar{n}, n, \Phi) = {} & f_e \bar{l}_L \Phi e_R + f_e^* \bar{e}_R \Phi^\dagger l_L + f_p \bar{q}_L \tilde{\Phi} p_R + f_p^* \bar{p}_R \tilde{\Phi}^\dagger q_L\\ & + M_R\, {}^c\bar{p}_R e_R + M_R^* \bar{e}_R\, {}^c p_R + M_L \bar{l}_L (i\tau^2)^\dagger\, {}^c q_L + M_L^{*c} \bar{q}_L (i\tau^2) l_L\\ & + f_n \bar{q}_L \Phi n_R + f_n^* \bar{n}_R \Phi^\dagger q_L + \frac{M_n}{2}\left({}^c\bar{n}_R n_R + \bar{n}_R\, {}^c n_R\right)\end{aligned}$$

$$\Phi(x) = \begin{pmatrix} \Phi^+(x) \\ \Phi^0(x) \end{pmatrix}, \tilde{\Phi}(x) = i\tau^2 \Phi(x)^* = \begin{pmatrix} \Phi^0(x)^* \\ -\Phi^+(x)^* \end{pmatrix}$$

$$\mathcal{L}(\bar{e},e,\bar{v},v,\bar{p},p,\bar{n},n,\mathrm{v})$$

$$= (\bar{e}_\mathrm{L},\ {}^c\bar{p}_\mathrm{R})\begin{pmatrix} f_e\,\mathrm{V} & M_\mathrm{L} \\ M_\mathrm{R} & f_p\mathrm{v}\end{pmatrix}\begin{pmatrix} e_\mathrm{R} \\ {}^cp_\mathrm{L}\end{pmatrix} + (\bar{e}_\mathrm{R},\ {}^c\bar{p}_\mathrm{L})\begin{pmatrix} f_e^*\mathrm{v} & M_\mathrm{R}^* \\ M_\mathrm{L}^* & f_p^*\mathrm{v}\end{pmatrix}\begin{pmatrix} e_\mathrm{L} \\ {}^cp_\mathrm{R}\end{pmatrix}$$

$$+\frac{1}{2}(\bar{v}_\mathrm{L},\bar{n}_\mathrm{L},\ {}^c\bar{n}_\mathrm{R})\begin{pmatrix} 0 & -M_\mathrm{L} & 0 \\ -M_\mathrm{L} & 0 & f_n\mathrm{v} \\ 0 & f_n\mathrm{v} & M_n\end{pmatrix}\begin{pmatrix} {}^c\nu_\mathrm{L} \\ {}^cn_\mathrm{L} \\ n_\mathrm{R}\end{pmatrix}$$

$$+\frac{1}{2}(\ {}^c\bar{v}_\mathrm{L},\ {}^c\bar{n}_\mathrm{L},\bar{n}_\mathrm{R})\begin{pmatrix} 0 & -M_\mathrm{L}^* & 0 \\ -M_\mathrm{L}^* & 0 & f_n^*\mathrm{v} \\ 0 & f_n^*\mathrm{v} & M_n\end{pmatrix}\begin{pmatrix} \nu_\mathrm{L} \\ n_\mathrm{L} \\ {}^cn_\mathrm{R}\end{pmatrix}$$

$$= (\bar{e}_\mathrm{L},\ {}^c\bar{p}_\mathrm{R})\mathcal{M}_C\begin{pmatrix} e_\mathrm{R} \\ {}^cp_\mathrm{L}\end{pmatrix} + (\bar{e}_\mathrm{R},\ {}^c\bar{p}_\mathrm{L})\mathcal{M}_C^\dagger\begin{pmatrix} e_\mathrm{L} \\ {}^cp_\mathrm{R}\end{pmatrix} + \frac{1}{2}(\bar{v}_\mathrm{L},\bar{n}_\mathrm{L},\ {}^c\bar{n}_\mathrm{R})\mathcal{M}_N\begin{pmatrix} {}^c\nu_\mathrm{L} \\ {}^cn_\mathrm{L} \\ n_\mathrm{R}\end{pmatrix}$$

$$+\frac{1}{2}(\ {}^c\bar{v}_\mathrm{L},\ {}^c\bar{n}_\mathrm{L},\bar{n}_\mathrm{R})\mathcal{M}_N^*\begin{pmatrix} \nu_\mathrm{L} \\ n_\mathrm{L} \\ {}^cn_\mathrm{R}\end{pmatrix}$$

$$\mathcal{M}_C = \begin{pmatrix} f_e\mathrm{v} & M_\mathrm{L} \\ M_\mathrm{R} & f_p\mathrm{v}\end{pmatrix}, \mathcal{M}_N = \begin{pmatrix} 0 & -M_\mathrm{L} & 0 \\ -M_\mathrm{L} & 0 & f_n\mathrm{v} \\ 0 & f_n\mathrm{v} & M_n\end{pmatrix}.$$

$$\begin{pmatrix} e_\mathrm{R}' \\ {}^cp_\mathrm{L}'\end{pmatrix} = C_\mathrm{R}\begin{pmatrix} e_\mathrm{R} \\ {}^cp_\mathrm{L}\end{pmatrix}, \begin{pmatrix} e_\mathrm{L}' \\ {}^cp_\mathrm{R}'\end{pmatrix} = C_\mathrm{L}\begin{pmatrix} e_\mathrm{L} \\ {}^cp_\mathrm{R}\end{pmatrix}, C_\mathrm{L}\mathcal{M}_C C_\mathrm{R}^\dagger = \begin{pmatrix} m_e & 0 \\ 0 & m_p\end{pmatrix},$$

$$\begin{pmatrix} {}^c\nu_\mathrm{L}' \\ {}^cn_\mathrm{L}' \\ n_\mathrm{R}'\end{pmatrix} = N_\mathrm{R}\begin{pmatrix} {}^cn_\mathrm{L} \\ n_\mathrm{R}\end{pmatrix}, \begin{pmatrix} \nu_\mathrm{L}' \\ n_\mathrm{L}' \\ {}^cn_\mathrm{R}'\end{pmatrix} = N_\mathrm{L}\begin{pmatrix} \nu_\mathrm{L} \\ n_\mathrm{L} \\ {}^cn_\mathrm{R}\end{pmatrix}, N_\mathrm{R}\mathcal{M}_N N_\mathrm{L}^\dagger = \begin{pmatrix} m_\nu & 0 & 0 \\ 0 & m_n & 0 \\ 0 & 0 & M\end{pmatrix}.$$

$$\begin{pmatrix} \nu_{e\,\mathrm{L}}(x) \\ e_\mathrm{L}(x)\end{pmatrix}, e_\mathrm{R}(x), \begin{pmatrix} u_\mathrm{L}(x) \\ d_\mathrm{L}(x)\end{pmatrix}, u_\mathrm{R}(x), d_\mathrm{R}(x)$$

$$\begin{pmatrix} \nu_{\mu\mathrm{L}}(x) \\ \mu_\mathrm{L}(x)\end{pmatrix}, \mu_\mathrm{R}(x), \begin{pmatrix} c_\mathrm{L}(x) \\ s_\mathrm{L}(x)\end{pmatrix}, c_\mathrm{R}(x), s_\mathrm{R}(x)$$

$$\begin{pmatrix} \nu_{\tau\mathrm{L}}(x) \\ \tau_\mathrm{L}(x)\end{pmatrix}, \tau_\mathrm{R}(x), \begin{pmatrix} t_\mathrm{L}(x) \\ b_\mathrm{L}(x)\end{pmatrix}, t_\mathrm{R}(x), b_\mathrm{R}(x)$$

$$N_\mathrm{L}(x) = \begin{pmatrix} \nu_{e\,\mathrm{L}}(x) \\ \nu_{\mu\mathrm{L}}(x) \\ \nu_{\tau\mathrm{L}}(x)\end{pmatrix}, E_{\mathrm{L,R}}(x) = \begin{pmatrix} e_{\mathrm{L,R}}(x) \\ \mu_{\mathrm{L,R}}(x) \\ \tau_{\mathrm{L,R}}(x)\end{pmatrix}$$

$$U_{\mathrm{L,R}}(x) = \begin{pmatrix} u_{\mathrm{L,R}}(x) \\ c_{\mathrm{L,R}}(x) \\ t_{\mathrm{L,R}}(x)\end{pmatrix}, D_{\mathrm{L,R}}(x) = \begin{pmatrix} d_{\mathrm{L,R}}(x) \\ s_{\mathrm{L,R}}(x) \\ b_{\mathrm{L,R}}(x)\end{pmatrix}$$

$$L_\mathrm{L}(x) = \begin{pmatrix} N_\mathrm{L}(x) \\ E_\mathrm{L}(x)\end{pmatrix}, Q_\mathrm{L}(x) = \begin{pmatrix} U_\mathrm{L}(x) \\ D_\mathrm{L}(x)\end{pmatrix}$$

$$N'_{\mathrm{L}}(x) = U^{N_{\mathrm{L}}} N_{\mathrm{L}}(x), E'_{\mathrm{L,R}}(x) = U^{E_{\mathrm{L,R}}} E_{\mathrm{L,R}}(x)$$
$$U'_{\mathrm{L,R}}(x) = U^{U_{\mathrm{L,R}}} U_{\mathrm{L,R}}(x), D'_{\mathrm{L,R}}(x) = U^{D_{\mathrm{L,R}}} D_{\mathrm{L,R}}(x)$$

$$\mathcal{L}(\bar{L}'_{\mathrm{L}}, L'_{\mathrm{L}}, \bar{E}'_{\mathrm{R}}, E'_{\mathrm{R}}, B, W) = \bar{L}'_{\mathrm{L}} \bar{\sigma}_\mu D_\mu L'_{\mathrm{L}} + \bar{E}'_{\mathrm{R}} \sigma_\mu D_\mu E'_{\mathrm{R}}$$

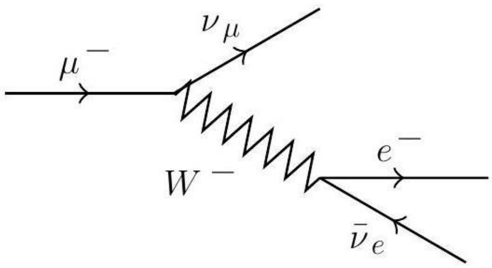
$$D_\mu L'_{\mathrm{L}}(x) = \left(\partial_\mu - \mathrm{i}\frac{g'}{2} B_\mu(x) + W_\mu(x)\right) L'_{\mathrm{L}}(x), D_\mu E'_{\mathrm{R}}(x) = \left(\partial_\mu - \mathrm{i}g' B_\mu(x)\right) E'_{\mathrm{R}}(x)$$

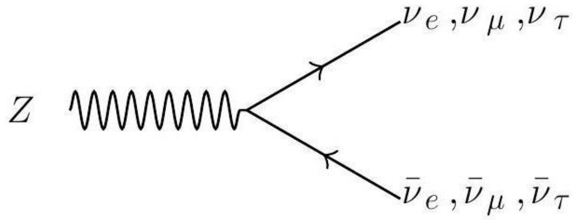
$$\mathcal{L}(\bar{L}'_{\mathrm{L}}, L'_{\mathrm{L}}, \bar{E}'_{\mathrm{R}}, E'_{\mathrm{R}}, \Phi) = \bar{L}'_{\mathrm{L}} \Phi \mathcal{F}_E E'_{\mathrm{R}} + \bar{E}'_{\mathrm{R}} \mathcal{F}_E^\dagger \Phi^\dagger L'_{\mathrm{L}}.$$

$$\mathcal{L}(\bar{L}_{\mathrm{L}}, L_{\mathrm{L}}, \bar{E}_{\mathrm{R}}, E_{\mathrm{R}}, \mathrm{v}) = \bar{E}_{\mathrm{L}} \mathcal{M}_E E_{\mathrm{R}} + \bar{E}_{\mathrm{R}} \mathcal{M}_E E_{\mathrm{L}}$$

$$U^{E_{\mathrm{L}}\dagger} \mathrm{v} \mathcal{F}_E U^{E_{\mathrm{R}}} = \mathcal{M}_E$$

$$\begin{aligned}\mathcal{L}(\bar{L}_{\mathrm{L}}, L_{\mathrm{L}}, \bar{E}_{\mathrm{R}}, E_{\mathrm{R}}, B, W) &= \bar{L}_{\mathrm{L}} U^{E_{\mathrm{L}}\dagger} \bar{\sigma}_\mu D_\mu U^{E_{\mathrm{L}}} L_{\mathrm{L}} + \bar{E}_{\mathrm{R}} U^{E_{\mathrm{R}}\dagger} \sigma_\mu D_\mu U^{E_{\mathrm{R}}} E_{\mathrm{R}} \\ &= \bar{L}_{\mathrm{L}} \bar{\sigma}_\mu D_\mu L_{\mathrm{L}} + \bar{E}_{\mathrm{R}} \sigma_\mu D_\mu E_{\mathrm{R}}\end{aligned}$$





$$\mathcal{L}(\bar{Q}'_{\mathrm{L}}, Q'_{\mathrm{L}}, \bar{U}'_{\mathrm{R}}, U'_{\mathrm{R}}, \bar{D}'_{\mathrm{R}}, D'_{\mathrm{R}}, B, W, G) = \bar{Q}'_{\mathrm{L}} \bar{\sigma}_\mu D_\mu Q'_{\mathrm{L}} + \bar{U}'_{\mathrm{R}} \sigma_\mu D_\mu U'_{\mathrm{R}} + \bar{D}'_{\mathrm{R}} \sigma_\mu D_\mu D'_{\mathrm{R}}$$

$$\begin{aligned}D_\mu Q'_{\mathrm{L}}(x) &= \left(\partial_\mu + \mathrm{i}\frac{g'}{2N_{\mathrm{c}}} B_\mu(x) + W_\mu(x) + G_\mu(x)\right) Q'_{\mathrm{L}}(x), \\ D_\mu U'_{\mathrm{R}}(x) &= \left(\partial_\mu + \mathrm{i}\frac{g'}{2}\left(\frac{1}{N_{\mathrm{c}}} + 1\right) B_\mu(x) + G_\mu(x)\right) U'_{\mathrm{R}}(x), \\ D_\mu D'_{\mathrm{R}}(x) &= \left(\partial_\mu + \mathrm{i}\frac{g'}{2}\left(\frac{1}{N_{\mathrm{c}}} - 1\right) B_\mu(x) + G_\mu(x)\right) D'_{\mathrm{R}}(x).\end{aligned}$$

$$\mathcal{L}(\bar{Q}'_{\mathrm{L}}, Q'_{\mathrm{L}}, \bar{U}'_{\mathrm{R}}, U'_{\mathrm{R}}, \bar{D}'_{\mathrm{R}}, D'_{\mathrm{R}}, \Phi) = \bar{Q}'_{\mathrm{L}} \tilde{\Phi} \mathcal{F}_U U'_{\mathrm{R}} + \bar{U}'_{\mathrm{R}} \mathcal{F}_U^\dagger \tilde{\Phi}^\dagger Q'_{\mathrm{L}} + \bar{Q}'_{\mathrm{L}} \Phi \mathcal{F}_D D'_{\mathrm{R}} + \bar{D}'_{\mathrm{R}} \mathcal{F}_D^\dagger \Phi^\dagger Q'_{\mathrm{L}}$$

$$\mathcal{L}(\bar{Q}_{\mathrm{L}}, Q_{\mathrm{L}}, \bar{U}_{\mathrm{R}}, U_{\mathrm{R}}, \bar{D}_{\mathrm{R}}, D_{\mathrm{R}}, \mathrm{v}) = \bar{U}_{\mathrm{L}} \mathcal{M}_U U_{\mathrm{R}} + \bar{U}_{\mathrm{R}} \mathcal{M}_U U_{\mathrm{L}} + \bar{D}_{\mathrm{L}} \mathcal{M}_D D_{\mathrm{R}} + \bar{D}_{\mathrm{R}} \mathcal{M}_D D_{\mathrm{L}}$$

$$U^{U_{\mathrm{L}}\dagger} \mathrm{v} \mathcal{F}_U U^{U_{\mathrm{R}}} = \mathcal{M}_U, U^{D_{\mathrm{L}}\dagger} \mathrm{v} \mathcal{F}_D U^{D_{\mathrm{R}}} = \mathcal{M}_D$$

$$\begin{aligned}\bar{U}'_{\mathrm{L,R}}(x) \gamma_\mu U'_{\mathrm{L,R}}(x) &= \bar{U}_{\mathrm{L,R}}(x) U^{U^\dagger_{\mathrm{L,R}}\dagger} \gamma_\mu U^{U_{\mathrm{L,R}}} U_{\mathrm{L,R}}(x) = \bar{U}_{\mathrm{L,R}}(x) \gamma_\mu U_{\mathrm{L,R}}(x), \\ \bar{D}'_{\mathrm{L,R}}(x) \gamma_\mu D'_{\mathrm{L,R}}(x) &= \bar{D}_{\mathrm{L,R}}(x) U^{D^\dagger_{\mathrm{L,R}}\dagger} \gamma_\mu U^{D_{\mathrm{L,R}}} D_{\mathrm{L,R}}(x) = \bar{D}_{\mathrm{L,R}}(x) \gamma_\mu D_{\mathrm{L,R}}(x).\end{aligned}$$

$$\begin{aligned}\sqrt{2} j_\mu^+(x)/g &= \bar{U}'_{\mathrm{L}}(x) \bar{\sigma}_\mu D'_{\mathrm{L}}(x) = \bar{U}_{\mathrm{L}}(x) U^{U_{\mathrm{L}}^\dagger} \bar{\sigma}_\mu U^{D_{\mathrm{L}}} D_{\mathrm{L}}(x) = \bar{U}_{\mathrm{L}}(x) \bar{\sigma}_\mu V D_{\mathrm{L}}(x) \\ \sqrt{2} j_\mu^-(x)/g &= \bar{D}'_{\mathrm{L}}(x) \bar{\sigma}_\mu U'_{\mathrm{L}}(x) = \bar{D}_{\mathrm{L}}(x) U^{D_{\mathrm{L}}^\dagger} \bar{\sigma}_\mu U^{U_{\mathrm{L}}} U_{\mathrm{L}}(x) = \bar{D}_{\mathrm{L}}(x) \bar{\sigma}_\mu V^\dagger U_{\mathrm{L}}(x)\end{aligned}$$

$$V = U^{U_L\dagger} U^{D_L} \in \mathrm{U}(N_g)$$

$$U^{U_L'} = U^{U_L} D^{U_L}, U^{D_L'} = U^{D_L} D^{D_L}$$

$$V' = U^{U_L'\dagger} U^{D_L'} = D^{U_L^\dagger} V D^{D_L}$$

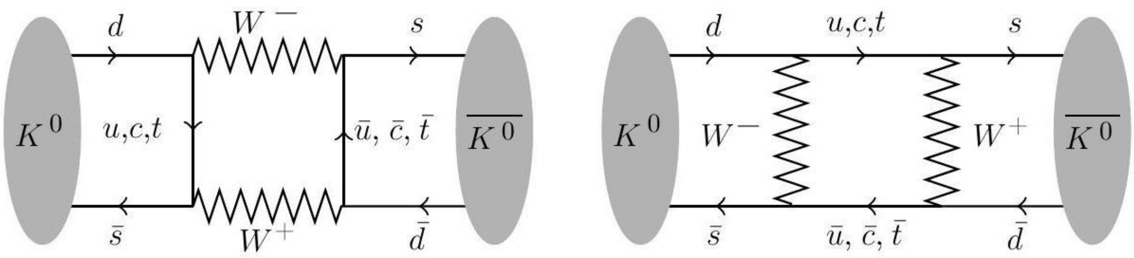
$$N_g^2 - (2N_g - 1) = (N_g - 1)^2$$

$$V = \begin{pmatrix} A\exp(\mathrm{i}\varphi) & B\exp(\mathrm{i}\varphi) \\ -B^* & A^* \end{pmatrix} = \begin{pmatrix} |A|\exp(\mathrm{i}(\alpha+\varphi)) & |B|\exp(\mathrm{i}(\beta+\varphi)) \\ -|B|\exp(-\mathrm{i}\beta) & |A|\exp(-\mathrm{i}\alpha) \end{pmatrix},$$

$$V' = D^{U_L\dagger} V D^{D_L} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}.$$

$$D^{U_L} = \mathrm{diag}(\exp(\mathrm{i}(\varphi+\beta)), \exp(-\mathrm{i}\alpha)), D^{D_L} = \mathrm{diag}(\exp(\mathrm{i}(\beta-\alpha)), 1),$$

$$(N_g - 1)^2 - \frac{N_g(N_g-1)}{2} = \frac{(N_g-1)(N_g-2)}{2} > 0.$$



$$V = \begin{pmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}.$$

$$V_{du}V_{ud}V_{su}V_{us} + V_{du}V_{ud}V_{sc}V_{cs} + V_{dc}V_{cd}V_{su}V_{us} + V_{dc}V_{cd}V_{sc}V_{cs} = 0;$$

$$|K_L\rangle = \frac{1}{\sqrt{2}}\left[|K^0\rangle + \left|\overline{K^0}\right\rangle\right], |K_S\rangle = \frac{1}{\sqrt{2}}\left[|K^0\rangle - \left|\overline{K^0}\right\rangle\right].$$

$$|K_L\rangle = \frac{1}{\sqrt{2(1+|\varepsilon|^2)}}\left[(1+\varepsilon)|K^0\rangle + (1-\varepsilon)\left|\overline{K^0}\right\rangle\right]$$
$$|K_S\rangle = \frac{1}{\sqrt{2(1+|\varepsilon|^2)}}\left[(1-\varepsilon)|K^0\rangle - (1+\varepsilon)\left|\overline{K^0}\right\rangle\right]$$

$$\mathcal{L}(\bar{L}_L', L_L', \tilde{\Phi}) = \frac{1}{2\Lambda}\left[\left({}^{c}\bar{L}_L'\tilde{\Phi}^*\right)\mathcal{G}_N\left(\tilde{\Phi}^\dagger L_L'\right) + \left(\bar{L}_L'\tilde{\Phi}\right)\mathcal{G}_N^*\left(\tilde{\Phi}^\mathrm{T}\,{}^{c}L_L'\right)\right]$$

$$\mathcal{L}(\bar{L}_L', L_L', \mathrm{v}) = \frac{\mathrm{v}^2}{2\Lambda}\left[{}^{c}\bar{N}_L'\mathcal{G}_N N_L' + \bar{N}_L'\mathcal{G}_N^{*c}N_L'\right]$$

$$N_L'(x) = U^{N_L}N_L(x), \bar{N}_L'(x) = \bar{N}_L(x)U^{N_L\dagger}$$
$${}^{c}N_L'(x) = \mathrm{i}\sigma^2\bar{N}_L'(x)^\mathrm{T} = \mathrm{i}\sigma^2\left[\bar{N}_L(x)U^{N_L\dagger}\right]^\mathrm{T} = U^{N_L*}\mathrm{i}\sigma^2\bar{N}_L(x)^\mathrm{T} = U^{N_L*}\,{}^{c}N_L(x)$$
$${}^{c}\bar{N}_L'(x) = -N_L'(x)^\mathrm{T}\mathrm{i}\sigma^2 = -[U^{N_L}N_L(x)]^\mathrm{T}\mathrm{i}\sigma^2 = -N_L(x)^\mathrm{T}\mathrm{i}\sigma^2U^{N_L\,\mathrm{T}} = {}^{c}\bar{N}_L(x)U^{N_L\,\mathrm{T}}$$

$$\mathcal{L}(\bar{L}_\mathrm{L}, L_\mathrm{L}, \mathrm{v}) = \frac{\mathrm{v}^2}{2\Lambda}\left[\,{}^\mathrm{c}\bar{N}_\mathrm{L} U^{N_\mathrm{L}\ \mathrm{T}} \mathcal{G}_N U^{N_\mathrm{L}} N_\mathrm{L} + \bar{N}_\mathrm{L} U^{N_\mathrm{L}^\dagger} \mathcal{G}_N^* U^{N_\mathrm{L}*\mathrm{c}} N_\mathrm{L}\right]$$

$$\mathcal{G}_N' = U^{N_\mathrm{L}\ \mathrm{T}} \mathcal{G}_N U^{N_\mathrm{L}}$$

$$\mathcal{G}_N'^\mathrm{T} = U^{N_\mathrm{L}^\mathrm{T}} \mathcal{G}_N^\mathrm{T} U^{N_\mathrm{L}} = \mathcal{G}_N'$$

$$U^{N_\mathrm{L}\ \mathrm{T}} \frac{\mathrm{v}^2}{\Lambda} \mathcal{G}_N U^{N_\mathrm{L}} = \mathcal{M}_N = \mathrm{diag}\left(m_{v_1}, m_{v_2}, m_{v_3}\right)$$

$$\sqrt{2} j_\mu^+(x)/g = \bar{N}_\mathrm{L}'(x) \bar{\sigma}_\mu E_\mathrm{L}'(x) = \bar{N}_\mathrm{L}(x) U^{N_\mathrm{L}^\dagger} \bar{\sigma}_\mu U^{E_\mathrm{L}} E_\mathrm{L}(x) = \bar{N}_\mathrm{L}(x) \bar{\sigma}_\mu U E_\mathrm{L}(x)$$
$$\sqrt{2} j_\mu^-(x)/g = \bar{E}_\mathrm{L}'(x) \bar{\sigma}_\mu N_\mathrm{L}'(x) = \bar{E}_\mathrm{L}(x) U^{E_\mathrm{L}\dagger} \bar{\sigma}_\mu U^{N_\mathrm{L}} N_\mathrm{L}(x) = \bar{E}_\mathrm{L}(x) \bar{\sigma}_\mu U^\dagger N_\mathrm{L}(x)$$

$$U = U^{N_\mathrm{L}\dagger} U^{E_\mathrm{L}} \in \mathrm{U}\left(N_\mathrm{g}\right)$$

$$\begin{pmatrix} v_{e\,\mathrm{L}}(x) \\ v_{\mu\mathrm{L}}(x) \\ v_{\tau\mathrm{L}}(x) \end{pmatrix} = U^{E_\mathrm{L}^\dagger} N_\mathrm{L}'(x)$$

$$\begin{pmatrix} v_{1\,\mathrm{L}}(x) \\ v_{2\,\mathrm{L}}(x) \\ v_{3\,\mathrm{L}}(x) \end{pmatrix} = N_\mathrm{L}(x) = U^{N_\mathrm{L}^\dagger} N_\mathrm{L}'(x) = U^{N_\mathrm{L}^\dagger} U^{E_\mathrm{L}} \begin{pmatrix} v_{e\,\mathrm{L}}(x) \\ v_{\mu\mathrm{L}}(x) \\ v_{\tau\mathrm{L}}(x) \end{pmatrix} = U \begin{pmatrix} v_{e\,\mathrm{L}}(x) \\ v_{\mu\mathrm{L}}(x) \\ v_{\tau\mathrm{L}}(x) \end{pmatrix}$$

$$U^{E_\mathrm{L}'} = U^{E_\mathrm{L}} D^{E_\mathrm{L}}$$

$$U' = U^{N_\mathrm{L}\dagger} U^{E_\mathrm{L}'} = U D^{E_\mathrm{L}}.$$

$$|v_i(t)\rangle = \exp\left(-\mathrm{i}E_i t\right)|v_i\rangle = \exp\left(-\mathrm{i}m_{v_i} t\right)|v_i\rangle$$

$$\left|v_f\right\rangle = \sum_{i=1}^{N_\mathrm{g}} U_{fi}^\dagger |v_i\rangle = \sum_{i=1}^{N_\mathrm{g}} U_{if}^* |v_i\rangle,$$

$$|\Psi(t)\rangle = \sum_{i=1}^{N_\mathrm{g}} U_{if}^* |v_i(t)\rangle = \sum_{i=1}^{N_\mathrm{g}} U_{if}^* \exp\left(-\mathrm{i}m_{v_i} t\right)|v_i\rangle$$

$$\left\langle v_{f'} \mid \Psi(t)\right\rangle = \sum_{i=1}^{N_\mathrm{g}} U_{if}^* U_{if'} \exp\left(-\mathrm{i}m_{v_i} t\right)$$

$$\left|\left\langle v_\mu \mid \Psi(t)\right\rangle\right|^2 = \left|\sum_{i=1}^{2} U_{ie}^* U_{i\mu} \exp\left(-\mathrm{i}m_{v_i} t\right)\right|^2 = |U_{1e}|^2 \left|U_{1\mu}\right|^2 + |U_{2e}|^2 \left|U_{2\mu}\right|^2 + 2\mathrm{Re}\left[U_{1e} U_{1\mu}^* U_{2e}^* U_{2\mu} \exp\left(\mathrm{i}\left(m_{v_1} - m_{v_2}\right)t\right)\right]$$

$$U = \begin{pmatrix} U_{1e} & U_{1\mu} \\ U_{2e} & U_{2\mu} \end{pmatrix} = \begin{pmatrix} \cos\,\theta & \sin\,\theta \exp\left(\mathrm{i}\phi\right) \\ -\sin\,\theta \exp\left(-\mathrm{i}\phi\right) & \cos\,\theta \end{pmatrix} \in \mathrm{SU}(2).$$

$$\left|\left\langle v_\mu \mid \Psi(t)\right\rangle\right|^2 = \sin^2\left(2\theta\right) \sin^2\left(\frac{1}{2}\left(m_{v_1} - m_{v_2}\right)t\right)$$

$$2 + N_g + 2N_g + (N_g - 1)^2 = N_g(N_g + 1) + 3$$

$$3 + 2 + N_g(N_g + 1) + 3 = N_g(N_g + 1) + 8$$

$$G_\mu(x) = \mathrm{i}g_s G_\mu^a(x)\frac{\lambda^a}{2}, a \in \{1,2,\dots,8\}$$

$$G'_\mu(x) = \Omega(x)(G_\mu(x) + \partial_\mu)\Omega(x)^\dagger,$$

$$G_{\mu\nu}(x) = \partial_\mu G_\nu(x) - \partial_\nu G_\mu(x) + [G_\mu(x), G_\nu(x)]$$

$$G'_{\mu\nu}(x) = \Omega(x)G_{\mu\nu}(x)\Omega(x)^\dagger$$

$$\mathcal{L}_{\mathrm{QCD}}(\bar{q}, q, G) = \sum_{\mathrm{f}} \bar{q}^{\mathrm{f}}[\gamma_\mu(\partial_\mu + G_\mu) + m_{\mathrm{f}}]q^{\mathrm{f}} - \frac{1}{2g_{\mathrm{s}}^2}\mathrm{Tr}[G_{\mu\nu}G_{\mu\nu}]$$

$$Z = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}G\exp\left(-\int d^4x\mathcal{L}_{\mathrm{QCD}}(\bar{q}, q, G)\right)$$

$$q(x) = Z_q(\varepsilon)^{1/2}q^{\mathrm{r}}(x), G_\mu(x) = Z_G(\varepsilon)^{1/2}G_\mu^{\mathrm{r}}(x),$$

$$g_{\mathrm{s}} = \frac{Z(\varepsilon)}{Z_q(\varepsilon)Z_G(\varepsilon)^{1/2}}g_{\mathrm{s}}^{\mathrm{r}}$$

$$\Gamma_{n_q,n_G}^{\mathrm{r}}(k_i, p_j) = \lim_{\varepsilon\to 0} Z_q(\varepsilon)^{n_q/2}Z_G(\varepsilon)^{n_G/2}\Gamma_{n_q,n_G}(k_i, p_j, \varepsilon)$$

$$\Gamma_{0,2}^{\mathrm{r}}(p, -p)_{\nu\rho}^{ab}\big|_{p^2=\mu^2} = (p_\nu p_\rho - \delta_{\nu\rho}p^2)\delta_{ab},$$
$$\Gamma_{2,0}^{\mathrm{r}}(k, k)\big|_{k^2=\mu^2} = \gamma_\nu k_\nu$$
$$\Gamma_{2,1}^{\mathrm{r}}(k, k, k)_\nu^a\big|_{k^2=\mu^2} = \mathrm{i}g_{\mathrm{s}}^{\mathrm{r}}\gamma_\nu\frac{\lambda^a}{2}.$$

$$g_{\mathrm{s}}^{\mathrm{r}} = g_{\mathrm{s}}^{\mathrm{r}}(g_{\mathrm{s}}, \varepsilon, \mu),$$

$$\beta(g_{\mathrm{s}}^{\mathrm{r}}) = \lim_{\varepsilon\to 0}\mu\frac{\partial}{\partial\mu}g_{\mathrm{s}}^{\mathrm{r}}(g_{\mathrm{s}}, \varepsilon, \mu)$$

$$\beta(g_{\mathrm{s}}^{\mathrm{r}}) = -\frac{(g_{\mathrm{s}}^{\mathrm{r}})^3}{16\pi^2}\left(11 - \frac{2}{3}N_{\mathrm{f}}\right)$$

$$11 - \frac{2}{3}N_{\mathrm{f}} > 0 \;\Rightarrow\; N_{\mathrm{f}} \leq 16$$

$$\beta(g_{\mathrm{s}}^{\mathrm{r}}) = \mu\frac{\partial}{\partial\mu}g_{\mathrm{s}}^{\mathrm{r}} = -\frac{(g_{\mathrm{s}}^{\mathrm{r}})^3}{16\pi^2}\left(11 - \frac{2}{3}N_{\mathrm{f}}\right) \Rightarrow$$
$$\frac{\partial g_{\mathrm{s}}^{\mathrm{r}}}{\partial\mu}\frac{1}{(g_{\mathrm{s}}^{\mathrm{r}})^3} = \frac{1}{2}\frac{\partial(g_{\mathrm{s}}^{\mathrm{r}})^2}{\partial\mu}\frac{1}{(g_{\mathrm{s}}^{\mathrm{r}})^4} = -\frac{11 - \frac{2}{3}N_{\mathrm{f}}}{16\pi^2}\frac{1}{\mu} \Rightarrow$$
$$\frac{\partial(g_{\mathrm{s}}^{\mathrm{r}})^2}{(g_{\mathrm{s}}^{\mathrm{r}})^4} = -\frac{33 - 2N_{\mathrm{f}}}{24\pi^2}\frac{\partial\mu}{\mu} \Rightarrow \frac{1}{(g_{\mathrm{s}}^{\mathrm{r}})^2} = \frac{33 - 2N_{\mathrm{f}}}{24\pi^2}\log\frac{\mu}{\Lambda_{\mathrm{QCD}}}$$

$$\alpha_s^r(\mu) = \frac{6\pi}{33 - 2N_f}\frac{1}{\log\left(\mu/\Lambda_{QCD}\right)}.$$

$$\mathcal{L}(\bar{q}, q, G) = \bar{q}\left[\gamma_\mu\left(\partial_\mu + G_\mu\right) + \mathcal{M}\right]q$$

$$q_L(x) = \frac{1-\gamma_5}{2}q(x), \quad q_R(x) = \frac{1+\gamma_5}{2}q(x), \quad q(x) = q_L(x) + q_R(x)$$
$$\bar{q}_L(x) = \bar{q}(x)\frac{1+\gamma_5}{2}, \quad \bar{q}_R(x) = \bar{q}(x)\frac{1-\gamma_5}{2}, \quad \bar{q}(x) = \bar{q}_L(x) + \bar{q}_R(x)$$

$$\mathcal{L}(\bar{q}_L, q_L, \bar{q}_R, q_R, G) = \bar{q}_L\bar{\sigma}_\mu\left(\partial_\mu + G_\mu\right)q_L + \bar{q}_R\sigma_\mu\left(\partial_\mu + G_\mu\right)q_R + \bar{q}_L\mathcal{M}q_R + \bar{q}_R\mathcal{M}q_L$$

$$q'_L(x) = Lq_L(x), \quad \bar{q}'_L(x) = \bar{q}_L(x)L^\dagger, \quad L \in U(N_f)_L$$
$$q'_R(x) = Rq_R(x), \quad \bar{q}'_R(x) = \bar{q}_R(x)R^\dagger, \quad R \in U(N_f)_R$$

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_{L=R} = SU(N_f)_L \times SU(N_f)_R \times U(1)_B.$$

$$\bar{q}'_L(x)\mathcal{M}q'_R(x) = \bar{q}_L(x)R^\dagger m\mathbb{1}Lq_R(x) = \bar{q}_L(x)R^\dagger L\mathcal{M}q_R(x)$$

$$SU(N_f)_{L=R} \times U(1)_{L=R} = SU(N_f)_f \times U(1)_B$$

$$\prod_f U(1)_f = U(1)_u \times U(1)_d \times U(1)_s \times \cdots \times U(1)_{N_f}.$$

$$j_\mu^{La}(x) = \bar{q}_L(x)\bar{\sigma}_\mu\frac{\tau^a}{2}q_L(x), j_\mu^{Ra}(x) = \bar{q}_R(x)\sigma_\mu\frac{\tau^a}{2}q_R(x),$$

$$j_\mu^a(x) = j_\mu^{Ra}(x) + j_\mu^{La}(x) = \bar{q}(x)\gamma_\mu\frac{\tau^a}{2}q(x)$$

$$j_\mu^{5a}(x) = j_\mu^{Ra}(x) - j_\mu^{La}(x) = \bar{q}(x)\gamma_\mu\gamma_5\frac{\tau^a}{2}q(x)$$

$$\langle\Phi|j_\mu^{La}(x)j_\nu^{Rb}(y)|\Phi\rangle = \langle\Phi|j_\mu^{Ra}(x)j_\nu^{Lb}(y)|\Phi\rangle = 0 \Rightarrow$$
$$\langle\Phi|j_\mu^a(x)j_\nu^b(y)|\Phi\rangle = \langle\Phi|j_\mu^{5a}(x)j_\nu^{5b}(y)|\Phi\rangle$$

$$\hat{Q}^a = \int d^3x\hat{q}(\vec{x})^\dagger\frac{\tau^a}{2}\hat{q}(\vec{x}), \hat{Q}^{5a} = \int d^3x\hat{q}(\vec{x})^\dagger\gamma^5\frac{\tau^a}{2}\hat{q}(\vec{x})$$

$$\hat{Q}^a|\Phi\rangle = \hat{Q}^{5a}|\Phi\rangle = 0$$

$$\hat{Q}^{5a}|0\rangle \neq 0$$

$$\left[\hat{H}_{QCD}, \hat{Q}^{5a}\right] = 0 \;\Rightarrow\; \hat{H}_{QCD}\hat{Q}^{5a}|0\rangle = \hat{Q}^{5a}\hat{H}_{QCD}|0\rangle = 0$$

$$\langle\bar{q}q\rangle = \langle 0|\bar{q}(x)q(x)|0\rangle = \langle 0|(\bar{q}_L(x)q_R(x) + \bar{q}_R(x)q_L(x))|0\rangle.$$

$$S_{\mathrm{QCD}}[\bar{Q},Q,U] = a^4 \sum_{x,\mu} \frac{1}{2a}\left(\bar{Q}_x\gamma_\mu U_{x,\mu}Q_{x+\hat{\mu}} - \bar{Q}_{x+\hat{\mu}}\gamma_\mu U^\dagger_{x,\mu}Q_x\right) + a^4 \sum_x \bar{Q}_x \mathcal{M} Q_x$$

$$+ a^4 \sum_{x,\mu} \frac{1}{2a}\left(2\bar{Q}_x Q_x - \bar{Q}_x U_{x,\mu}Q_{x+\hat{\mu}} - \bar{Q}_{x+\hat{\mu}} U^\dagger_{x,\mu}Q_x\right)$$

$$- a^4 \sum_{x,\mu,\nu} \frac{1}{g_s^2 a^4} \mathrm{Tr}\left[U_{x,\mu}U_{x+\hat{\mu},\nu}U^\dagger_{x+\hat{\nu},\mu}U^\dagger_{x,\nu} + U_{x,\nu}U_{x+\hat{\nu},\mu}U^\dagger_{x+\hat{\mu},\nu}U^\dagger_{x,\mu}\right]$$

$$Z = \prod_x \int d\bar{Q}_x dQ_x \prod_{y,\mu} \int_{\mathrm{SU}(3)} dU_{y,\mu} \exp\left(-S_{\mathrm{QCD}}[\bar{Q},Q,U]\right)$$

$$S[\bar{Q},Q,U] = a^4 \bar{Q} D[U] Q = a^4 \sum_{x,y} \bar{Q}_x D[U]_{xy} Q_y$$

$$\{D[U]^{-1},\gamma_5\} = D[U]^{-1}\gamma_5 + \gamma_5 D[U]^{-1} = a\gamma_5.$$

$$\gamma_5 D[U] + D[U]\gamma_5 = aD[U]\gamma_5 D[U].$$

$$A_{\mathrm{overlap}}[U] = A_{\mathrm{W}}[U] / \sqrt{A_{\mathrm{W}}[U]^\dagger A_{\mathrm{W}}[U]}$$

$$D_{\mathrm{overlap}}[U] = \frac{1}{a}\left(A_{\mathrm{overlap}}[U] + \mathbb{1}\right)$$

$$= (D_{\mathrm{W}}[U] - \mathbb{1}/a) / \sqrt{(aD_{\mathrm{W}}[U]^\dagger - \mathbb{1})(aD_{\mathrm{W}}[U] - \mathbb{1})} + \frac{1}{a}\mathbb{1}.$$

$$Q' = Q + \delta Q = (\mathbb{1} + \mathrm{i}\varepsilon^a T^a \gamma_5) Q$$
$$\bar{Q}' = \bar{Q} + \delta\bar{Q} = \bar{Q}(\mathbb{1} + \mathrm{i}\varepsilon^a T^a \gamma_5)$$

$$Q' = Q + \delta Q = \left(\mathbb{1} + \mathrm{i}\varepsilon^a T^a \gamma_5 \left(1 - \frac{a}{2}D[U]\right)\right) Q$$
$$\bar{Q}' = \bar{Q} + \delta\bar{Q} = \bar{Q}\left(\mathbb{1} + \mathrm{i}\varepsilon^a T^a \left(1 - \frac{a}{2}D[U]\right)\gamma_5\right)$$

$$\bar{Q}' D[U] Q' = \bar{Q}\left(\mathbb{1} + \mathrm{i}\varepsilon^a T^a \left(\mathbb{1} - \frac{a}{2}D[U]\right)\gamma_5\right) D[U] \left(\mathbb{1} + \mathrm{i}\varepsilon^a T^a \gamma_5 \left(\mathbb{1} - \frac{a}{2}D[U]\right)\right) Q$$

$$= \bar{Q}D[U]Q + \bar{Q}(\mathrm{i}\varepsilon^a T^a[\gamma_5 D[U] + D[U]\gamma_5 - aD[U]\gamma_5 D[U]])Q + \mathcal{O}(\varepsilon^2)$$

$$= \bar{Q}D[U]Q + \mathcal{O}(\varepsilon^2)$$

$$\mathcal{D}\bar{Q}'\mathcal{D}Q' = \mathcal{D}\bar{Q}\left(\mathbb{1} + \mathrm{i}\varepsilon^a T^a \left(\mathbb{1} - \frac{a}{2}D[U]\right)\gamma_5\right)\left(\mathbb{1} + \mathrm{i}\varepsilon^a T^a \gamma_5 \left(\mathbb{1} - \frac{a}{2}D[U]\right)\mathcal{D}Q\right.$$
$$= \mathcal{D}\bar{Q}\mathcal{D}Q\left(\mathbb{1} + \mathrm{i}\varepsilon^a \mathrm{Tr}[T^a\gamma_5(2\mathbb{1} - aD[U])] + \mathcal{O}(\varepsilon^2)\right)$$
$$= \mathcal{D}\bar{Q}\mathcal{D}Q\left(\mathbb{1} - \mathrm{i}\varepsilon^0 a \mathrm{Tr}[\gamma_5 D[U]] + \mathcal{O}(\varepsilon^2)\right)$$

$$j^5_\mu(x) = \sum_{\mathrm{f}} \bar{q}^{\mathrm{f}}(x)\gamma_\mu\gamma_5 q^{\mathrm{f}}(x) = \sum_{\mathrm{f}} \left(\bar{q}^{\mathrm{f}}_{\mathrm{R}}(x)\sigma_\mu q^{\mathrm{f}}_{\mathrm{R}}(x) - \bar{q}^{\mathrm{f}}_{\mathrm{L}}(x)\bar{\sigma}_\mu q^{\mathrm{f}}_{\mathrm{L}}(x)\right)$$

$$\langle \partial_\mu j^5_\mu(x) \rangle = 2N_\mathrm{f} P(x)$$

$$P(x) = -\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[G_{\mu\nu}(x)G_{\rho\sigma}(x)\right]$$

$$j^B_\mu(x) = \sum_\mathrm{f} \bar{q}^\mathrm{f}(x)\gamma_\mu q^\mathrm{f}(x) = \sum_\mathrm{f} \left(\bar{q}^\mathrm{f}_\mathrm{R}(x)\sigma_\mu q^\mathrm{f}_\mathrm{R}(x) + \bar{q}^\mathrm{f}_\mathrm{L}(x)\bar{\sigma}_\mu q^\mathrm{f}_\mathrm{L}(x)\right)$$

$$\langle \partial_\mu j^B_\mu(x) \rangle = N_\mathrm{g} P(x).$$

$$P(x) = -\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[W_{\mu\nu}(x)W_{\rho\sigma}(x)\right]$$

$$P(x) = \partial_\mu \Omega^{(0)}_\mu(x),$$

$$\Omega^{(0)}_\mu(x) = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[G_\nu(x)\left(\partial_\rho G_\sigma(x) + \frac{2}{3}G_\rho(x)G_\sigma(x)\right)\right].$$

$$\tilde{j}^5_\mu(x) = j^5_\mu(x) - 2N_\mathrm{f}\Omega^{(0)}_\mu(x),$$

$$\partial_\mu \tilde{j}^5_\mu(x) = \partial_\mu j^5_\mu(x) - 2N_\mathrm{f}P(x) = 0.$$

$$Q = -\frac{1}{32\pi^2}\int d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[G_{\mu\nu}G_{\rho\sigma}\right] = \int d^4xP = \int d^4x\partial_\mu\Omega^{(0)}_\mu = \int_{S^3} d^3\sigma_\mu\Omega^{(0)}_\mu$$

$$G_\mu(x) = \Omega(x)\partial_\mu\Omega(x)^\dagger, \Omega(x) \in \mathrm{SU}(N)$$

$$\begin{aligned}
Q &= -\frac{1}{8\pi^2}\int_{S^3} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\Omega\partial_\nu\Omega^\dagger)\left\{\partial_\rho(\Omega\partial_\sigma\Omega^\dagger) + \frac{2}{3}(\Omega\partial_\rho\Omega^\dagger)(\Omega\partial_\sigma\Omega^\dagger)\right\}\right] \\
&= -\frac{1}{8\pi^2}\int_{S^3} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[-(\Omega\partial_\nu\Omega^\dagger)(\Omega\partial_\rho\Omega^\dagger)(\Omega\partial_\sigma\Omega^\dagger) + \frac{2}{3}(\Omega\partial_\nu\Omega^\dagger)(\Omega\partial_\rho\Omega^\dagger)(\Omega\partial_\sigma\Omega^\dagger)\right] \\
&= \frac{1}{24\pi^2}\int_{S^3} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\Omega\partial_\nu\Omega^\dagger)(\Omega\partial_\rho\Omega^\dagger)(\Omega\partial_\sigma\Omega^\dagger)\right]
\end{aligned}$$

$$\Omega: S^3 \to \mathrm{SU}(N).$$

$$\Pi_3[\mathrm{SU}(N)] = \mathbb{Z}$$

$$\Omega = VW, W = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & \dots & \tilde{\Omega}_{1,N-1} \\ 0 & \tilde{\Omega}_{21} & \tilde{\Omega}_{22} & \dots & \tilde{\Omega}_{1,N-1} \\ . & . & . & & . \\ . & . & \tilde{\Omega}_{N-1,1} & \tilde{\Omega}_{N-1,2} & \dots \\ 0 & & & & \tilde{\Omega}_{N-1,N-1} \end{pmatrix},$$

$$V=\begin{pmatrix} \Omega_{11} & -\Omega_{21}^* & -\dfrac{\Omega_{31}^*(1+\Omega_{11})}{1+\Omega_{11}^*} & \cdots & -\dfrac{\Omega_{N1}^*(1+\Omega_{11})}{1+\Omega_{11}^*} \\ \Omega_{21} & \dfrac{1+\Omega_{11}^*-|\Omega_{21}|^2}{1+\Omega_{11}} & -\dfrac{\Omega_{31}^*\Omega_{21}}{1+\Omega_{11}^*} & \cdots & -\dfrac{\Omega_{N1}^*\Omega_{21}}{1+\Omega_{11}^*} \\ \Omega_{31} & -\dfrac{\Omega_{21}^*\Omega_{31}}{1+\Omega_{11}} & \dfrac{1+\Omega_{11}^*-|\Omega_{31}|^2}{1+\Omega_{11}^*} & \cdots & -\dfrac{\Omega_{N1}^*\Omega_{31}}{1+\Omega_{11}^*} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \Omega_{N1} & -\dfrac{\Omega_{21}^*\Omega_{N1}}{1+\Omega_{11}} & -\dfrac{\Omega_{31}^*\Omega_{N1}}{1+\Omega_{11}^*} & \cdots & \dfrac{1+\Omega_{11}^*-|\Omega_{N1}|^2}{1+\Omega_{11}^*} \end{pmatrix} \in \mathrm{SU}(N)$$

$$\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(VW)\partial_\nu(VW)^\dagger(VW)\partial_\rho(VW)^\dagger(VW)\partial_\sigma(VW)^\dagger\right]=\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(V\partial_\nu V^\dagger)(V\partial_\rho V^\dagger)(V\partial_\sigma V^\dagger)\right]$$
$$+\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(W\partial_\nu W^\dagger)(W\partial_\rho W^\dagger)(W\partial_\sigma W^\dagger)\right]+3\epsilon_{\mu\nu\rho\sigma}\partial_\nu\mathrm{Tr}\left[(V\partial_\rho V^\dagger)(W\partial_\sigma W^\dagger)\right]$$

$$\begin{aligned} Q =& \frac{1}{24\pi^2}\int_{S^3} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\Omega\partial_\nu\Omega^\dagger)(\Omega\partial_\rho\Omega^\dagger)(\Omega\partial_\sigma\Omega^\dagger)\right] \\ =& \frac{1}{24\pi^2}\int_{S^3} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(V\partial_\nu V^\dagger)(V\partial_\rho V^\dagger)(V\partial_\sigma V^\dagger)\right. \\ & \left.+(W\partial_\nu W^\dagger)(W\partial_\rho W^\dagger)(W\partial_\sigma W^\dagger)\right]. \end{aligned}$$

$$V: S^3 \to S^{2N-1}$$

$$\Pi_3[S^{2N-1}]=\{0\}$$

$$\Pi_1[S^2]=\{0\}$$

$$Q=\frac{1}{24\pi^2}\int_{S^3} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\tilde\Omega\partial_\nu\tilde\Omega^\dagger)(\tilde\Omega\partial_\rho\tilde\Omega^\dagger)(\tilde\Omega\partial_\sigma\tilde\Omega^\dagger)\right]$$

$$(N^2-1)-[(N-1)^2-1]=2N-1>3,$$

$$\tilde\Omega: S^3 \to \mathrm{SU}(2)=S^3$$

$$\Pi_3[\mathrm{SU}(2)]=\Pi_3[S^3]=\mathbb{Z}$$

$$\Omega=\exp(\mathrm{i}\alpha): S^1 \to \mathrm{U}(1)=S^1.$$

$$\Pi_1[\mathrm{U}(1)]=\Pi_1[S^1]=\mathbb{Z}$$

$$Q=-\frac{1}{2\pi\mathrm{i}}\int_{S^1} d\sigma_\mu\epsilon_{\mu\nu}\Omega\partial_\nu\Omega^\dagger=\frac{1}{2\pi}\int_{S^1} d\sigma_\mu\epsilon_{\mu\nu}\partial_\nu\alpha=\frac{1}{2\pi}(\alpha(2\pi)-\alpha(0))$$

$$\tilde\Omega(x)=\exp(\mathrm{i}\vec\alpha(x)\cdot\vec\tau)=\cos\alpha(x)+\mathrm{i}\sin\alpha(x)\vec e_\alpha(x)\cdot\vec\tau$$
$$\vec\alpha(x)=\alpha(x)\vec e_\alpha(x),\ \vec e_\alpha(x)=(\sin\theta(x)\sin\varphi(x),\sin\theta(x)\cos\varphi(x),\cos\theta(x))$$

$$\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\tilde\Omega\partial_\nu\tilde\Omega^\dagger)(\tilde\Omega\partial_\rho\tilde\Omega^\dagger)(\tilde\Omega\partial_\sigma\tilde\Omega^\dagger)\right]=12\sin^2\alpha\sin\theta\,\epsilon_{\mu\nu\rho\sigma}\partial_\nu\alpha\partial_\rho\theta\partial_\sigma\varphi$$

$$Q = \frac{1}{2\pi^2}\int_{S^3} d^3\sigma_\mu \sin^2\,\alpha \sin\,\theta \epsilon_{\mu\nu\rho\sigma}\partial_\nu\alpha\partial_\rho\theta\partial_\sigma\varphi = \frac{1}{2\pi^2}\int_{S^3} d\tilde{\Omega}$$

$$Q = \int_M d^4xP$$

$$P(x) = \partial_\mu\Omega_\mu^{(0)}(x),$$

$$Q = \int_M d^4x\partial_\mu\Omega_\mu^{(0)} = \int_{\partial M} d^3\sigma_\mu\Omega_\mu^{(0)} = 0$$

$$G_\mu^i(x) = \Omega_i(x)[G_\mu(x) + \partial_\mu]\Omega_i(x)^\dagger$$

$$Q = \sum_i \int_{c_i} d^4xP = \sum_i \int_{\partial c_i} d^3\sigma_\mu\Omega_{\mu,i}^{(0)} = \frac{1}{2}\sum_{ij} \int_{c_i\cap c_j} d^3\sigma_\mu \left[\Omega_{\mu,i}^{(0)} - \Omega_{\mu,j}^{(0)}\right]$$

$$G_\mu^i(x) = \mathrm{v}_{ij}(x)[G_\mu^j(x) + \partial_\mu]\mathrm{v}_{ij}(x)^\dagger$$

$$\mathrm{v}_{ij}(x) = \Omega_i(x)\Omega_j(x)^\dagger.$$

$$\mathrm{v}_{ik}(x) = \mathrm{v}_{ij}(x)\mathrm{v}_{jk}(x).$$

$$\Delta\Omega_{\mu,ij}^{(0)} = \Omega_{\mu,i}^{(0)} - \Omega_{\mu,j}^{(0)}.$$

$$\Delta\Omega_{\mu,ij}^{(0)} = -\frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[(\mathrm{v}_{ij}\partial_\nu\mathrm{v}_{ij}^\dagger)(\mathrm{v}_{ij}\partial_\rho\mathrm{v}_{ij}^\dagger)(\mathrm{v}_{ij}\partial_\sigma\mathrm{v}_{ij}^\dagger)] - \frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\mathrm{Tr}[\partial_\rho\mathrm{v}_{ij}^\dagger\mathrm{v}_{ij}G_\sigma^i]$$

$$Q = -\frac{1}{48\pi^2}\sum_{ij} \int_{c_i\cap c_j} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[(\mathrm{v}_{ij}\partial_\nu\mathrm{v}_{ij}^\dagger)(\mathrm{v}_{ij}\partial_\rho\mathrm{v}_{ij}^\dagger)(\mathrm{v}_{ij}\partial_\sigma\mathrm{v}_{ij}^\dagger)]$$
$$-\frac{1}{16\pi^2}\sum_{ij} \int_{\partial(c_i\cap c_j)} d^2\sigma_{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[\partial_\rho\mathrm{v}_{ij}^\dagger\mathrm{v}_{ij}G_\sigma^i].$$

$$Q = -\frac{1}{48\pi^2}\sum_{ij} \int_{c_i\cap c_j} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[(\mathrm{v}_{ij}\partial_\nu\mathrm{v}_{ij}^\dagger)(\mathrm{v}_{ij}\partial_\rho\mathrm{v}_{ij}^\dagger)(\mathrm{v}_{ij}\partial_\sigma\mathrm{v}_{ij}^\dagger)]$$
$$-\frac{1}{48\pi^2}\sum_{ijk} \int_{c_i\cap c_j\cap c_k} d^2\sigma_{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[(\mathrm{v}_{ij}\partial_\rho\mathrm{v}_{ij}^\dagger)(\mathrm{v}_{jk}\partial_\sigma\mathrm{v}_{jk}^\dagger)].$$

$$Q = \sum_i Q_i = \frac{1}{24\pi^2}\sum_i \int_{\partial c_i} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[(\Omega_i\partial_\nu\Omega_i^\dagger)(\Omega_i\partial_\rho\Omega_i^\dagger)(\Omega_i\partial_\sigma\Omega_i^\dagger)]$$

$$\Omega_i: \partial c_i \to \mathrm{SU}(N)$$

$$Q_i \in \Pi_3[\mathrm{SU}(N)] = \mathbb{Z}$$

$$G_\mu(x)\Big|_{|x|\to\infty} = \Omega(x)\partial_\mu\Omega(x)^\dagger, |x| = \sqrt{\vec{x}^2 + x_4^2}$$

$$\Omega(x) = \frac{x_4 + \mathrm{i}\vec{x}\cdot\vec{\tau}}{|x|}.$$

$$G_\mu(x) = f(|x|)\Omega(x)\partial_\mu\Omega(x)^\dagger, f(\infty) = 1, f(0) = 0.$$

$$S[G] = -\frac{1}{2g_s^2}\int d^4x\mathrm{Tr}[G_{\mu\nu}G_{\mu\nu}]$$

$$-\int d^4x\mathrm{Tr}\left[\left(G_{\mu\nu} \pm \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}\right)\left(G_{\mu\nu} \pm \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}G_{\kappa\lambda}\right)\right] =$$
$$-\int d^4x\mathrm{Tr}[G_{\mu\nu}G_{\mu\nu} \pm \epsilon_{\mu\nu\rho\sigma}G_{\mu\nu}G_{\rho\sigma} + G_{\mu\nu}G_{\mu\nu}] = 4g_s^2S[G] \pm 32\pi^2Q[G]$$

$$S[G] \pm \frac{8\pi^2}{g_s^2}Q[G] \geq 0 \;\Rightarrow\; S[G] \geq \frac{8\pi^2}{g_s^2}|Q[G]|$$

$$S[G] = \frac{8\pi^2}{g_s^2}|Q[G]|$$

$$G_{\mu\nu}(x) = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}(x)$$

$$G_\mu(x) = \frac{|x|^2}{|x|^2 + \rho^2}\Omega(x)\partial_\mu\Omega(x)^\dagger$$

$$G_i(\vec{x}) = \Omega(\vec{x})\partial_i\Omega(\vec{x})^\dagger$$

$$n[\Omega] \in \Pi_3[\mathrm{SU}(N)] = \mathbb{Z}$$

$$n[\Omega] = \frac{1}{24\pi^2}\int_{S^3} d^3x\epsilon_{ijk}\mathrm{Tr}[(\Omega\partial_i\Omega^\dagger)(\Omega\partial_j\Omega^\dagger)(\Omega\partial_k\Omega^\dagger)]$$

$$Q = n - m.$$

$$\langle n|\hat{U}(\infty,-\infty)|m\rangle = \int \mathcal{D}G_\mu^{(n-m)}\exp\left(-S[G_\mu]\right)$$

$$\hat{T}\Psi[G] = \Psi[\,^\Omega G],\ ^\Omega G_i(\vec{x}) = \Omega(G_i(\vec{x}) + \partial_i)\Omega(\vec{x})^\dagger$$

$$\hat{T}|n\rangle = |n+1\rangle;$$

$$\hat{T}|\theta\rangle = \exp\,(\mathrm{i}\theta)|\theta\rangle$$

$$|\theta\rangle = \sum_{n\in\mathbb{Z}} c_n|n\rangle$$

$$\hat{T}|\theta\rangle = \sum_n c_n\hat{T}|n\rangle = \sum_n c_n|n+1\rangle = \sum_n c_{n-1}|n\rangle = \exp\,(\mathrm{i}\theta)\sum_n c_n|n\rangle,$$

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}}\sum_n \exp\,(-\mathrm{i}n\theta)|n\rangle$$

$$\langle\theta \mid \theta'\rangle = \frac{1}{2\pi}\sum_{n,m} \exp\left(\mathrm{i}(n\theta - m\theta')\right)\langle n \mid m\rangle = \frac{1}{2\pi}\sum_{m} \exp\left(\mathrm{i}m(\theta-\theta')\right) = \delta_{2\pi}(\theta-\theta')$$

$$\begin{aligned}\langle\theta|\hat{U}(\infty,-\infty)|\theta'\rangle &= \sum_{n,m} \exp(\mathrm{i}n\theta)\langle n|\hat{U}(\infty,-\infty)|m\rangle \exp(-\mathrm{i}m\theta') \\ &= \sum_{m,Q=n-m} \exp(\mathrm{i}m(\theta-\theta')+\mathrm{i}\theta Q)\int \mathcal{D}G_\mu^{(Q)}\exp(-S[G]) \\ &= \delta_{2\pi}(\theta-\theta')\sum_{Q}\int \mathcal{D}G_\mu^{(Q)}\exp(-S[G]+\mathrm{i}\theta Q[G]) \\ &= \delta_{2\pi}(\theta-\theta')\int \mathcal{D}G_\mu \exp(-S_\theta[G])\end{aligned}$$

$$S_\theta[G] = S[G] - \mathrm{i}\theta Q[G]$$

$$\hat{H} = \frac{\hat{p}^2}{2M} + V(x), V(x+a) = V(x)$$

$$\hat{T}\Psi(x) = \Psi(x+a)$$

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \exp(-\mathrm{i}n\theta)|n\rangle,$$

$$\Psi_\theta(x+a) = \hat{T}\Psi_\theta(x) = \exp(\mathrm{i}\theta)\Psi_\theta(x)$$

$$\widetilde{\Psi}(x) = U_\theta\Psi_\theta(x) = \exp(-\mathrm{i}\theta x/a)\Psi_\theta(x),$$

$$\begin{aligned}\widetilde{\Psi}(x+a) &= \exp(-\mathrm{i}\theta(x+a)/a)\Psi_\theta(x+a) = \exp(-\mathrm{i}\theta(x+a)/a)\exp(\mathrm{i}\theta)\Psi_\theta(x) \\ &= \exp(-\mathrm{i}\theta x/a)\Psi_\theta(x) = \widetilde{\Psi}(x)\end{aligned}$$

$$\hat{p}_\theta = U_\theta\hat{p}U_\theta^\dagger = \exp(-\mathrm{i}\theta x/a)(-\mathrm{i}\hbar\partial_x)\exp(\mathrm{i}\theta x/a) = -\mathrm{i}\hbar\partial_x + \hbar\theta/a = \hat{p} + \hbar\theta/a$$

$$\hat{H}_\theta = U_\theta\hat{H}U_\theta^\dagger = \frac{(\hat{p}+\hbar\theta/a)^2}{2M} + V(x)$$

$$\hat{T}_\theta = U_\theta\hat{T}U_\theta^\dagger = \exp(\mathrm{i}\hat{p}_\theta a/\hbar) = \exp(\mathrm{i}\hat{p}a/\hbar)\exp(\mathrm{i}\theta) = \exp(\mathrm{i}\theta)\hat{T}$$

$$\hat{T}_\theta\widetilde{\Psi}(x) = \exp(\mathrm{i}\theta)\hat{T}\widetilde{\Psi}(x) = \exp(\mathrm{i}\theta)\widetilde{\Psi}(x+a) = \exp(\mathrm{i}\theta)\widetilde{\Psi}(x)$$

$$\hat{T}\widetilde{\Psi}[G] = \widetilde{\Psi}[\,^{\Omega}G] = \widetilde{\Psi}[G].$$

$$\hat{T}\Psi_\theta[G] = \exp(\mathrm{i}\theta)\Psi_\theta[G])$$

$$D[G] = \gamma_\mu[\partial_\mu + G_\mu(x)].$$

$$D[G]\psi_\lambda(x) = \lambda\psi_\lambda(x)$$

$$\{D[G],\gamma_5\} = D[G]\gamma_5 + \gamma_5 D[G] = 0.$$

$$D[G]\gamma_5\psi_\lambda(x) = -\gamma_5 D[G]\psi_\lambda(x) = -\lambda\gamma_5\psi_\lambda(x).$$

$$\psi_{-\lambda}(x) = \mathcal{N}_\lambda \gamma_5 \psi_\lambda(x)$$

$$D[G]\psi_0(x) = 0$$

$$\gamma_5\psi_0(x) = \pm\psi_0(x) \;\Rightarrow\; (1 \mp \gamma_5)\psi_0(x) = 0.$$

$$n_{\mathrm{R}} - n_{\mathrm{L}} = Q[G]$$

$$\begin{aligned}
Z &= \int \mathcal{D}G\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-S_\theta[G]\right)\exp\left(-\bar{\psi}D[G]\psi\right) \\
&= \int \mathcal{D}G\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-S[G] + \mathrm{i}\theta Q[G]\right)\exp\left(-\bar{\psi}D[G]\psi\right) \\
&= \sum_{Q\in\mathbb{Z}} \int \mathcal{D}G^{(Q)} \exp\left(-S[G] + \mathrm{i}\theta Q\right)\det D[G] \\
&= \int \mathcal{D}G^{(0)} \exp\left(-S[G]\right)\det D[G]
\end{aligned}$$

$$\det D[G] = \delta_{Q[G],0}\det D[G]$$

$$\tau_\mu = (-\mathrm{i}\vec{\tau}, \mathbb{1}), \bar{\tau}_\mu = (\mathrm{i}\vec{\tau}, \mathbb{1}), \bar{\tau}_\mu\tau_\nu = \delta_{\mu\nu}\mathbb{1} + \eta_{\mu\nu}, \tau_\mu\bar{\tau}_\nu = \delta_{\mu\nu}\mathbb{1} + \bar{\eta}_{\mu\nu}.$$

$$\begin{aligned}
&\tilde{\eta}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta_{\rho\sigma} = \eta_{\mu\nu}, \tilde{\bar{\eta}}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\eta}_{\rho\sigma} = -\bar{\eta}_{\mu\nu}, \\
&[\eta_{\mu\nu}, \eta_{\rho\sigma}] = 2\left(\delta_{\mu\sigma}\eta_{\nu\rho} - \delta_{\mu\rho}\eta_{\nu\sigma} - \delta_{\nu\sigma}\eta_{\mu\rho} + \delta_{\nu\rho}\eta_{\mu\sigma}\right), \mathrm{Tr}\left(\eta_{\mu\nu}\eta_{\mu\nu}\right) = -24.
\end{aligned}$$

$$G_\mu(x) = \frac{x_\nu}{x^2 + \rho^2}\eta_{\nu\mu}, G_{\mu\nu}(x) = \frac{2\rho^2}{(x^2 + \rho^2)^2}\eta_{\mu\nu}$$

$$\sigma_\mu = (-\mathrm{i}\vec{\sigma}, \mathbb{1}), \bar{\sigma}_\mu = (\mathrm{i}\vec{\sigma}, \mathbb{1}), \bar{\sigma}_\mu\sigma_\nu = \delta_{\mu\nu}\mathbb{1} + \zeta_{\mu\nu}, \sigma_\mu\bar{\sigma}_\nu = \delta_{\mu\nu}\mathbb{1} + \bar{\zeta}_{\mu\nu}$$

$$\frac{1}{2}[\gamma_\mu, \gamma_\nu] = \frac{1}{2}\left[\begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_\nu \\ \bar{\sigma}_\nu & 0 \end{pmatrix}\right] = \begin{pmatrix} \bar{\zeta}_{\mu\nu} & 0 \\ 0 & \zeta_{\mu\nu} \end{pmatrix}$$

$$\begin{aligned}
D[G] &= \gamma_\mu\left(\partial_\mu + G_\mu(x)\right) = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}\left(\partial_\mu + G_\mu(x)\right) \Rightarrow \\
D[G]^2 &= \begin{pmatrix} \sigma_\mu\bar{\sigma}_\nu & 0 \\ 0 & \bar{\sigma}_\mu\sigma_\nu \end{pmatrix}\left(\partial_\mu + G_\mu(x)\right)\left(\partial_\nu + G_\nu(x)\right) \\
&= \left(\partial_\mu + G_\mu(x)\right)\left(\partial_\mu + G_\mu(x)\right) + \frac{1}{2}\begin{pmatrix} \bar{\zeta}_{\mu\nu} & 0 \\ 0 & \zeta_{\mu\nu} \end{pmatrix}\left[\partial_\mu + G_\mu(x), \partial_\nu + G_\nu(x)\right] \\
&= \left(\partial_\mu + G_\mu(x)\right)\left(\partial_\mu + G_\mu(x)\right) + \frac{1}{2}\begin{pmatrix} \bar{\zeta}_{\mu\nu} & 0 \\ 0 & \zeta_{\mu\nu} \end{pmatrix}G_{\mu\nu}(x)
\end{aligned}$$

$$\begin{aligned}
&D[G]^2 = \left(\partial_\mu + G_\mu(x)\right)\left(\partial_\mu + G_\mu(x)\right) + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \zeta_{\mu\nu} \end{pmatrix}G_{\mu\nu}(x) \Rightarrow \\
&D[G]^2\psi_{0\mathrm{R}}(x) = \left\{\left(\partial_\mu + G_\mu(x)\right)\left(\partial_\mu + G_\mu(x)\right) + \frac{1}{2}\zeta_{\mu\nu}G_{\mu\nu}(x)\right\}\psi_{0\mathrm{R}}(x) = 0 \\
&D[G]^2\psi_{0\,\mathrm{L}}(x) = \left(\partial_\mu + G_\mu(x)\right)\left(\partial_\mu + G_\mu(x)\right)\psi_{0\,\mathrm{L}}(x) = 0
\end{aligned}$$

$$\sigma_\mu\left(\partial_\mu + G_\mu(x)\right)\psi_{0\mathrm{R}}(x) = 0 \;\Rightarrow\; (x^2 + \rho^2)\sigma_\nu\partial_\nu\psi_{0\mathrm{R}}(x) = x_\nu\sigma_\mu\eta_{\mu\nu}\psi_{0\mathrm{R}}(x)$$

$$\sigma_\mu \eta_{\mu\nu} \psi_{0\mathrm{R}}^{sc}(x) = -3\sigma_\nu \psi_{0\mathrm{R}}^{sc}(x) \Rightarrow (x^2+\rho^2)\partial_\nu \psi(x) = -3x_\nu \psi(x) \Rightarrow$$
$$\psi(x) = \frac{A}{(x^2+\rho^2)^{3/2}}$$

$$\psi_{0\mathrm{R}}^{sc}(x) = \frac{\rho}{\pi} \frac{1}{(x^2+\rho^2)^{3/2}} \epsilon_{sc}$$

$$\{D[U], \gamma_5\} = D[U]\gamma_5 + \gamma_5 D[U] = aD[U]\gamma_5 D[U].$$

$$D[U]^\dagger = \gamma_5 D[U] \gamma_5.$$

$$D[U] + D[U]^\dagger = aD[U]^\dagger D[U] = aD[U]D[U]^\dagger.$$

$$A[U]^\dagger A[U] = a^2 D[U]^\dagger D[U] - aD[U]^\dagger - aD[U] + \mathbb{1} = \mathbb{1}$$
$$A[U]A[U]^\dagger = a^2 D[U]D[U]^\dagger - aD[U] - aD[U]^\dagger + \mathbb{1} = \mathbb{1}$$

$$D[U]\Psi_{\omega,x} = \frac{1}{a}[1+\exp(\mathrm{i}\omega)]\Psi_{\omega,x}$$

$$D[U]^\dagger \gamma_5 \Psi_{\omega,x} = \frac{1}{a}[1+\exp(\mathrm{i}\omega)]\gamma_5 \Psi_{\omega,x}$$

$$\Psi_{-\omega,x} = \gamma_5 \Psi_{\omega,x}$$

$$\gamma_5 \Psi_{0,x} = \pm\Psi_{0,x}, \gamma_5 \Psi_{\pi,x} = \pm\Psi_{\pi,x}$$

$$\mathrm{Tr}(\gamma_5) = n_\mathrm{R} - n_\mathrm{L} + n_\mathrm{R}' - n_\mathrm{L}' = 0 \;\Rightarrow\; n_\mathrm{R} - n_\mathrm{L} = n_\mathrm{L}' - n_\mathrm{R}'$$

$$\mathrm{Tr}(\gamma_5 D[U]) = \frac{2}{a}(n_\mathrm{R}' - n_\mathrm{L}') \;\Rightarrow\; n_\mathrm{R} - n_\mathrm{L} = -\frac{a}{2}\mathrm{Tr}(\gamma_5 D[U])$$

$$q_x[U] = -\frac{a}{2}\mathrm{tr}(\gamma_5 D[U]_{xx}) \;\Rightarrow\; Q[U] = \sum_x q_x[U]$$

$$n_\mathrm{R} - n_\mathrm{L} = Q[U].$$

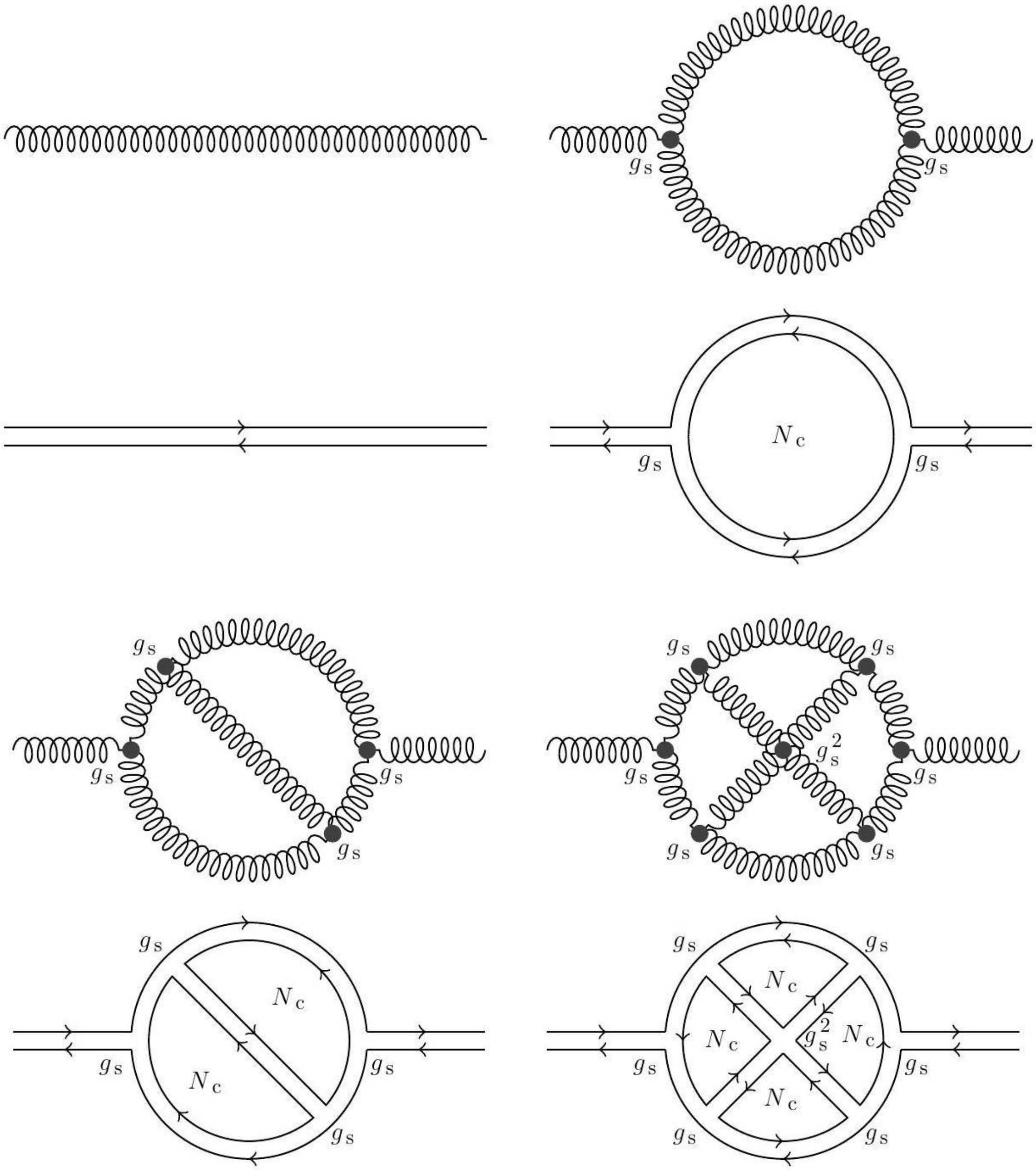
$$G_\mu'(x) = \Omega(x)\big(G_\mu(x) + \partial_\mu\big)\Omega(x)^\dagger.$$

$$I[U] = \frac{1}{24\pi^2}\int_{H^3} d^3x \epsilon_{ijk} \mathrm{Tr}\big[(U\partial_i U^\dagger)(U\partial_j U^\dagger)(U\partial_k U^\dagger)\big]$$

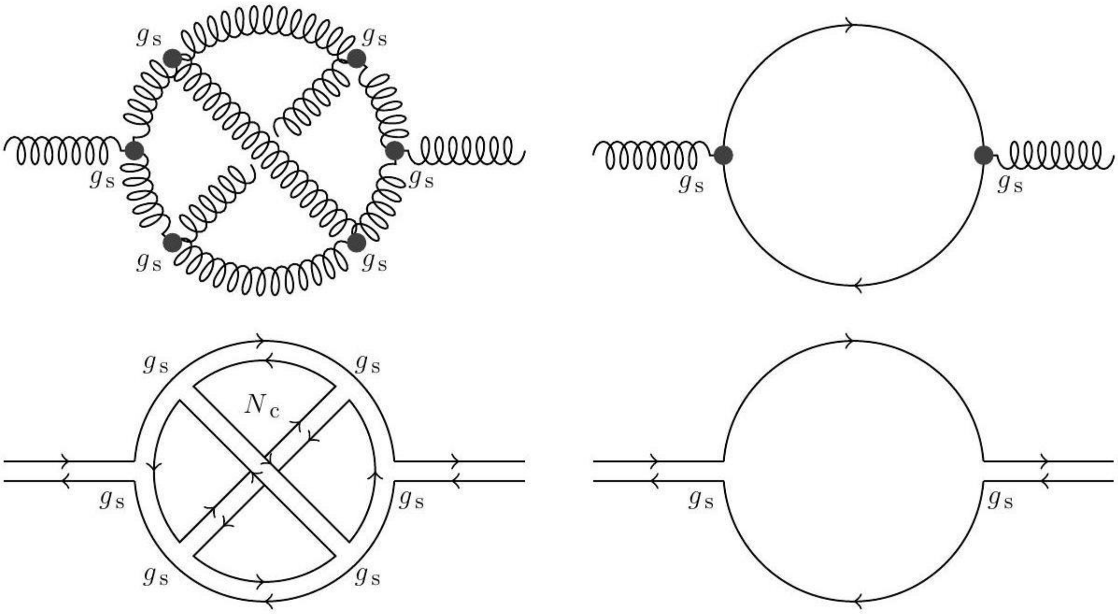
$$\frac{1}{24\pi^2}\epsilon_{ijk}\mathrm{Tr}\big[(U\partial_i U^\dagger)(U\partial_j U^\dagger)(U\partial_k U^\dagger)\big] = \partial_i \Omega_i^{(1)},$$

$$\Omega_i^{(1)} = -\frac{1}{8\pi^2}(\alpha - \sin\alpha\cos\alpha)\epsilon_{ijk}\vec{e}_\alpha \cdot \big(\partial_j \vec{e}_\alpha \times \partial_k \vec{e}_\alpha\big).$$

$$f(|x|) = \frac{|x|^2}{|x|^2+\rho^2}$$



$$g_{\mathrm{tH}}^2 = g_{\mathrm{s}}^2 N_{\mathrm{c}}$$



$$S[G] \geq \frac{8\pi^2}{g_s^2}|Q[G]| = \frac{8\pi^2 N_c}{g_{tH}^2}|Q[G]|$$

$$S[G] = -\frac{1}{2g_s^2}\int d^4x \mathrm{Tr}\left[G_{\mu\nu}G_{\mu\nu}\right] = -\frac{N_c}{2g_{tH}^2}\int d^4x \mathrm{Tr}\left[G_{\mu\nu}G_{\mu\nu}\right],$$

$$\chi_t = \int_V d^4x_{\mathrm{YM}}\langle 0|P(0)P(x)|0\rangle_{\mathrm{YM}} = \frac{\langle Q^2\rangle}{V},$$

$$\int d^4x\langle 0|P(0)P(x)|0\rangle = 0$$

$$\chi_t - \sum_m \frac{\langle 0|P|m\rangle\langle m|P|0\rangle}{M_m^2} = 0.$$

$$\chi_t = \frac{|\langle 0|P|\eta'\rangle \quad|^2}{M_{\eta'}^2}$$

$$\langle 0|P|\eta'\rangle = \frac{1}{2N_f}\langle 0|\partial_\mu j_\mu^5|\eta'\rangle = \frac{1}{\sqrt{2N_f}}M_{\eta'}^2 F_{\eta'}$$

$$\chi_t = \frac{F_\pi^2 M_{\eta'}^2}{2N_f}$$

$$\chi_t = \frac{F_\pi^2}{6}\left(M_{\eta'}^2 + M_\eta^2 - 2M_K^2\right) \simeq (0.180\mathrm{GeV})^4$$

$$\chi_t = (0.483(2)/R_0)^4 = (0.203(2)\mathrm{GeV})^4.$$

$$M_p \simeq 0.9383\mathrm{GeV}, M_n \simeq 0.9396\mathrm{GeV}$$

$$M_{\pi^+} = M_{\pi^-} \simeq 0.140\mathrm{GeV}, M_{\pi^0} \simeq 0.135\mathrm{GeV}$$

$$M_{\Delta^{++}} \simeq M_{\Delta^{+}} \simeq M_{\Delta^{0}} \simeq M_{\Delta^{-}} \simeq 1.230\text{GeV}$$

$$M_{\rho^{+}} \simeq M_{\rho^{0}} \simeq M_{\rho^{-}} \simeq 0.770\text{GeV}$$

$$S_z = -S, -S+1, \dots, S-1, S$$

$$I_3 = -I, -I+1, \dots, I-1, I$$

$$\{2\} \times \{3\} = \{2\} + \{4\}$$

$$\{2\} \times \{2\} \times \{2\} = (\{1\} + \{3\}) \times \{2\} = \{2\} + \{2\} + \{4\},$$

$$Q = I_3 + \frac{1}{2} = \sum_{q=1}^{3} \left(I_{3q} + \frac{1}{6} \right) = \sum_{q=1}^{3} Q_q$$

$$Q_q = I_{3q} + \frac{1}{6} \Rightarrow Q_u = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, Q_d = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,3/2} = uuu \equiv \Delta^{++},$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,1/2} = \frac{1}{\sqrt{3}}(uud + udu + duu) \equiv \Delta^{+},$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,-1/2} = \frac{1}{\sqrt{3}}(udd + dud + ddu) \equiv \Delta^{0},$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,-3/2} = ddd \equiv \Delta^{-}.$$

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \times \begin{array}{|c|}\hline 2 \\ \hline \end{array} \times \begin{array}{|c|}\hline 3 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} + \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{2} \\ \hline 3 \\ \cline{1-1} \end{array} + \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{3} \\ \hline 2 \\ \cline{1-1} \end{array} + \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

$$\{2\} \times \{2\} \times \{2\} = \{4\} + \{2\} + \{2\} + \{0\}.$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,3/2} = \uparrow\uparrow\uparrow,$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,1/2} = \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow),$$

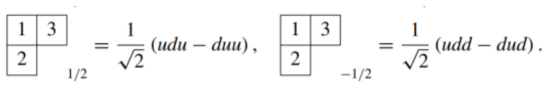
$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,-1/2} = \frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow),$$

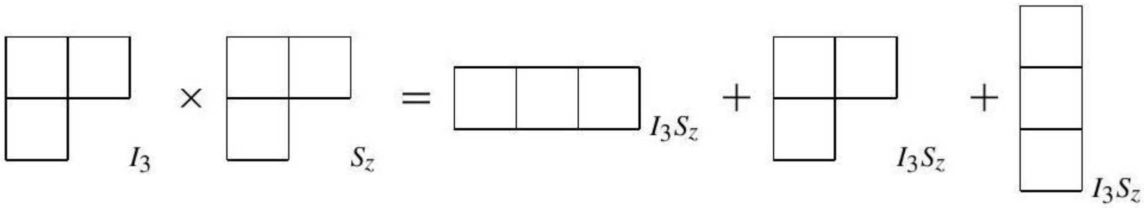
$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,-3/2} = \downarrow\downarrow\downarrow.$$

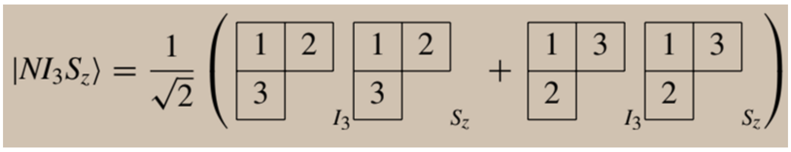
$$|\Delta I_3 S_z\rangle = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,I_3} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}_{\,S_z}$$

$$\left|\Delta \frac{1}{2}\frac{1}{2}\right\rangle = \frac{1}{3}(u\uparrow u\uparrow d\downarrow + u\uparrow u\downarrow d\uparrow + u\downarrow u\uparrow d\uparrow$$
$$+ u\uparrow d\uparrow u\downarrow + u\uparrow d\downarrow u\uparrow + u\downarrow d\uparrow u\uparrow$$
$$+ d\uparrow u\uparrow u\downarrow + d\uparrow u\downarrow u\uparrow + d\downarrow u\uparrow u\uparrow).$$

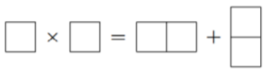
$$\begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{2} \\ \hline 3 \\ \cline{1-1} \end{array}_{\,1/2} = \frac{1}{\sqrt{6}}(2uud - udu - duu), \quad \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{2} \\ \hline 3 \\ \cline{1-1} \end{array}_{\,-1/2} = \frac{1}{\sqrt{6}}(udd + dud - 2ddu).$$



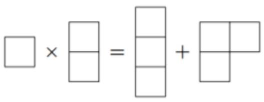




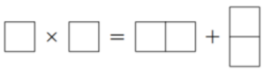
$$\square = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr).$$



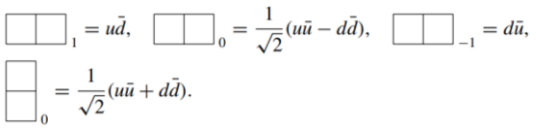
$\{3\} \times \{3\} = \{6\} + \{\bar{3}\},$

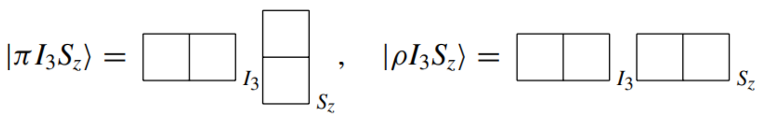


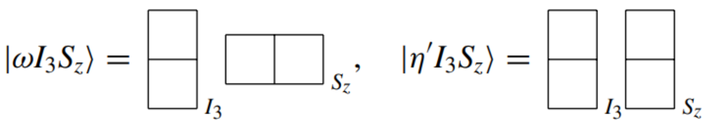
$\{3\} \times \{\bar{3}\} = \{1\} + \{8\},$



$\{2\} \times \{2\} = \{3\} + \{1\},$







$$M_\pi \simeq 0.138\text{GeV}, M_K \simeq 0.496\text{GeV}, M_\eta \simeq 0.549\text{GeV}, M_{\eta'} \simeq 0.958\text{GeV},$$

$$M_\rho \simeq 0.770\text{GeV}, M_\omega \simeq 0.783\text{GeV}, M_{K^*} \simeq 0.892\text{GeV}, M_\varphi \simeq 1.020\text{GeV}$$

$$\{2\} \times \{2\} = \{1\} + \{3\}.$$

$$\{3\} \times \{\bar{3}\} = \{1\} + \{8\}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$I_1 = \frac{1}{2}\lambda^1, I_2 = \frac{1}{2}\lambda^2, I_3 = \frac{1}{2}\lambda^3$$

$$Y = \frac{1}{\sqrt{3}}\lambda^8$$

$$I^2 u = \frac{1}{2}\left(\frac{1}{2} + 1\right) u = \frac{3}{4} u, I_3 u = \frac{1}{2} u, Y u = \frac{1}{3} u,$$
$$I^2 d = \frac{1}{2}\left(\frac{1}{2} + 1\right) d = \frac{3}{4} d, I_3 d = -\frac{1}{2} d, Y d = \frac{1}{3} d,$$
$$I^2 s = 0, I_3 s = 0, Y s = -\frac{2}{3} s.$$

$$Q = I_3 + \frac{1}{2} Y$$

$$Q_u = \frac{2}{3}, Q_d = -\frac{1}{3}, Q_s = -\frac{1}{3}$$

$$\{3\} \times \{3\} \times \{3\} = \{10\} + 2\{8\} + \{1\}$$

$$\{2\} \times \{2\} \times \{2\} = \{4\} + 2\{2\} + \{0\}$$

$$M_N \simeq 0.939\mathrm{GeV}, M_\Lambda \simeq 1.116\mathrm{GeV}, M_\Sigma \simeq 1.193\mathrm{GeV}, M_\Xi \simeq 1.318\mathrm{GeV},$$

$$M_\Delta \simeq 1.232\mathrm{GeV}, M_{\Sigma^*} \simeq 1.385\mathrm{GeV}, M_{\Xi^*} \simeq 1.530\mathrm{GeV}, M_\Omega \simeq 1.672\mathrm{GeV}.$$

$$M_{\Sigma^*} - M_\Delta \simeq M_{\Xi^*} - M_{\Sigma^*} \simeq M_\Omega - M_{\Xi^*}$$

$$M_{\Sigma^*} - M_\Delta \simeq 0.153\mathrm{GeV}, M_{\Xi^*} - M_{\Sigma^*} \simeq 0.145\mathrm{GeV}, M_\Omega - M_{\Xi^*} \simeq 0.142\mathrm{GeV}$$

$$\mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$$\mathcal{M} = \frac{2m_q + m_s}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_q - m_s}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$
$$= \frac{2m_q + m_s}{3} \mathbb{1} + \frac{m_q - m_s}{\sqrt{3}} \lambda^8$$

$$\hat{H}_{\text{QCD}} = \hat{H}_1 + \hat{H}_8.$$

$$\hat{H}_1|B_1YII_3\rangle = M_{B_1}|B_1YII_3\rangle.$$

$$M_B = M_{B_1} + \langle B_1YII_3|\hat{H}_8|B_1YII_3\rangle.$$

$$\langle B_1YI_3|\hat{H}_8|B_1YII_3\rangle = \langle B_1\|\hat{H}_8\|B_1\rangle\langle\{10\}YII_3 \mid \{8\}000\{10\}YII_3\rangle,$$

$$\langle\{10\}YII_3 \mid \{8\}000\{10\}YII_3\rangle = Y/\sqrt{8}$$

$$M_B = M_{B_1} + \langle B_1\|\hat{H}_8\|B_1\rangle Y/\sqrt{8}$$

$$M_{\Sigma^*} - M_\Delta = M_{\Xi^*} - M_{\Sigma^*} = M_\Omega - M_{\Xi^*} = -\langle B_1\|\hat{H}_8\|B_1\rangle/\sqrt{8}$$

$$\{8\} \times \{8\} = \{27\} + \{10\} + \{\overline{10}\} + 2\{8\} + \{1\}.$$

$$\langle B_1YI_3|\hat{H}_8|B_1YI_3\rangle = \langle B_1\|\hat{H}_8\|B_1\rangle_{\text{s}}\langle\{8\}YI_3 \mid \{8\}000\{8\}YI_3\rangle_{\text{s}} + \langle B_1\|\hat{H}_8\|B_1\rangle_{\text{a}}\langle\{8\}YI_3 \mid \{8\}000\{8\}YI_3\rangle_{\text{a}}$$

$$\langle\{8\}YI_3 \mid \{8\}000\{8\}YI_3\rangle_{\text{s}} = (I(I+1) - Y^2/4 - 1)/\sqrt{5},$$
$$\langle\{8\}YII_3 \mid \{8\}000\{8\}YII_3\rangle_{\text{a}} = \sqrt{3/4}Y,$$

$$M_B = M_{B_1} + \langle B_1\|\hat{H}_8\|B_1\rangle_{\text{s}}(I(I+1) - Y^2/4 - 1)/\sqrt{5} + \langle B_1\|\hat{H}_8\|B_1\rangle_{\text{a}}\sqrt{3/4}Y$$

$$2M_N + 2M_\Xi = 4M_{B_1} + \langle B_1\|\hat{H}_8\|B_1\rangle_{\text{s}}4(3/4 - 1/4 - 1)/\sqrt{5},$$
$$M_\Sigma + 3M_\Lambda = 4M_{B_1} + \langle B_1\|\hat{H}_8\|B_1\rangle_{\text{s}}[(2-1) + 3(-1)]/\sqrt{5},$$

$$2M_N + 2M_\Xi = M_\Sigma + 3M_\Lambda$$

$$\hat{H}_1|M_1YII_3\rangle = M_{M_1}|M_1YII_3\rangle.$$

$$M_M = M_{M_1} + \langle M_1YII_3|\hat{H}_8|M_1YII_3\rangle.$$

$$M_M = M_{M_1} + \langle M_1\|\hat{H}_8\|M_1\rangle_{\text{s}}(I(I+1) - Y^2/4 - 1)/\sqrt{5} + \langle M_1\|\hat{H}_8\|M_1\rangle_{\text{a}}\sqrt{3/4}Y.$$

$$M_{K^+} = M_{M_1} + \langle M_1\|\hat{H}_8\|M_1\rangle_{\text{s}}(3/4 - 1/4 - 1)/\sqrt{5} + \langle M_1\|\hat{H}_8\|M_1\rangle_{\text{a}}\sqrt{3/4}$$
$$M_{K^-} = M_{M_1} + \langle M_1\|\hat{H}_8\|M_1\rangle_{\text{s}}(3/4 - 1/4 - 1)/\sqrt{5} - \langle M_1\|\hat{H}_8\|M_1\rangle_{\text{a}}\sqrt{3/4}$$

$$\langle M_1\|\hat{H}_8\|M_1\rangle_{\text{a}} = 0$$

$$\langle\omega_1|\hat{H}_8|\omega_1\rangle = 0$$
$$\langle\omega_8|\hat{H}_8|\omega_8\rangle = \langle M_1\|\hat{H}_8\|M_1\rangle_{\text{s}}\langle\{8\}000 \mid \{8\}000\{8\}000\rangle_{\text{s}}$$
$$= \langle M_1\|\hat{H}_8\|M_1\rangle_{\text{s}}(-1/\sqrt{5})$$

$$\mathcal{M}_M = \begin{pmatrix} M_{\omega_1} & \langle\omega_1|\hat{H}_8|\omega_8\rangle \\ \langle\omega_8|\hat{H}_8|\omega_1\rangle & M_{\omega_8} - \langle M_1\|\hat{H}_8\|M_1\rangle_{\text{s}}/\sqrt{5} \end{pmatrix}.$$

$$\begin{pmatrix} |\varphi\rangle \\ |\omega\rangle \end{pmatrix} = \begin{pmatrix} \cos\,\theta & -\sin\,\theta \\ \sin\,\theta & \cos\,\theta \end{pmatrix} \begin{pmatrix} |\omega_1\rangle \\ |\omega_8\rangle \end{pmatrix}.$$

$$M_\varphi + M_\omega = M_{\omega_1} + M_{\omega_8} - \langle M_1 \| \hat{H}_8 \| M_1 \rangle_s / \sqrt{5}$$
$$M_\varphi M_\omega = M_{\omega_1}\left(M_{\omega_8} - \langle M_1 \| \hat{H}_8 \| M_1 \rangle_s \sqrt{5}\right) - \left|\langle \omega_1 | \hat{H}_8 | \omega_8 \rangle \quad \right|^2$$

$$M_\rho = M_{\omega_8} + \langle M_1 \| \hat{H}_8 \| M_1 \rangle_s (2-1)/\sqrt{5}$$
$$M_{K^*} = M_{\omega_8} + \langle M_1 \| \hat{H}_8 \| M_1 \rangle_s (3/4 - 1/4 - 1)/\sqrt{5}$$

$$4M_{K^*}/3 - M_\rho/3 = M_{\omega_8} + \langle M_1 \| \hat{H}_8 \| M_1 \rangle_s (4/3(-1/2) - 1/3)/\sqrt{5}$$
$$= M_{\omega_8} - \langle M_1 \| \hat{H}_8 \| M_1 \rangle_s / \sqrt{5}$$

$$M_{\omega_1} = M_\varphi + M_\omega - 4M_{K^*}/3 + M_\rho/3 = 0.870\text{GeV}$$
$$\left|\langle \omega_1 | \hat{H}_8 | \omega_8 \rangle \quad \right|^2 = M_{\omega_1}\left(4M_{K^*}/3 - M_\rho/3\right) - M_\varphi M_\omega \simeq (0.113\text{GeV})^2$$

$$M_{\omega_1} \cos\,\theta - \langle \omega_1 | \hat{H}_8 | \omega_8 \rangle \sin\,\theta = M_\varphi \cos\,\theta$$
$$\langle \omega_8 | \hat{H}_8 | \omega_1 \rangle \cos\,\theta - \left(M_{\omega_8} - \langle M_1 \| \hat{H}_8 \| M_1 \rangle_s / \sqrt{5}\right) \sin\,\theta = -M_\varphi \sin\,\theta$$

$$\left(M_{\omega_1} + M_{\omega_8} - \langle M_1 \| \hat{H}_8 \| M_1 \rangle_s / \sqrt{5}\right) \sin\,\theta \cos\,\theta - \langle \omega_1 | \hat{H}_8 | \omega_8 \rangle = 2M_\varphi \sin\,\theta \cos\,\theta \Rightarrow$$
$$\frac{1}{2}\sin(2\theta) = \pm \frac{\sqrt{\left(M_\varphi + M_\omega - 4M_{K^*}/3 + M_\rho/3\right)\left(4M_{K^*}/3 - M_\rho/3\right) - M_\varphi M_\omega}}{M_\varphi - M_\omega}$$

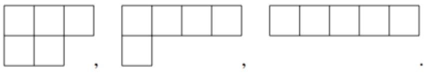
$$|\varphi\rangle \approx s\bar{s} \text{ or } \frac{1}{3}(2u\bar{u} + 2d\bar{d} - s\bar{s}),$$
$$|\omega\rangle \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \text{ or } -\frac{1}{\sqrt{18}}(u\bar{u} + d\bar{d} + 4s\bar{s}).$$

$$|\varphi\rangle \approx s\bar{s}, |\omega\rangle \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$

$$\eta_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}), \eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\,\bar{\theta} & -\sin\,\bar{\theta} \\ \sin\,\bar{\theta} & \cos\,\bar{\theta} \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_1 \end{pmatrix}.$$

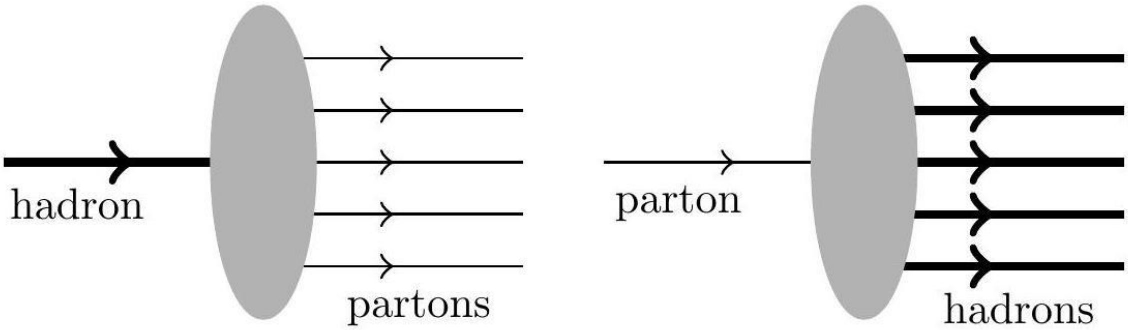
$$Q_u = \frac{1}{2}\left(\frac{1}{N_c} + 1\right), Q_d = Q_s = \frac{1}{2}\left(\frac{1}{N_c} - 1\right)$$

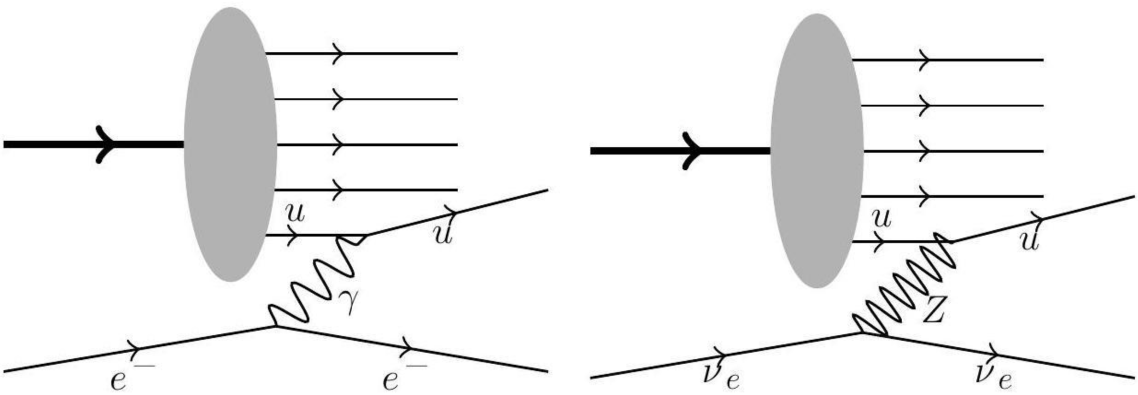
$$Q_{\Delta^{++}} = \frac{N_c + 3}{2} Q_u + \frac{N_c - 3}{2} Q_d = 2$$
$$Q_{\Delta^{+}} = \frac{N_c + 1}{2} Q_u + \frac{N_c - 1}{2} Q_d = 1$$
$$Q_{\Delta^{0}} = \frac{N_c - 1}{2} Q_u + \frac{N_c + 1}{2} Q_d = 0$$
$$Q_{\Delta^{-}} = \frac{N_c - 3}{2} Q_u + \frac{N_c + 3}{2} Q_d = -1$$

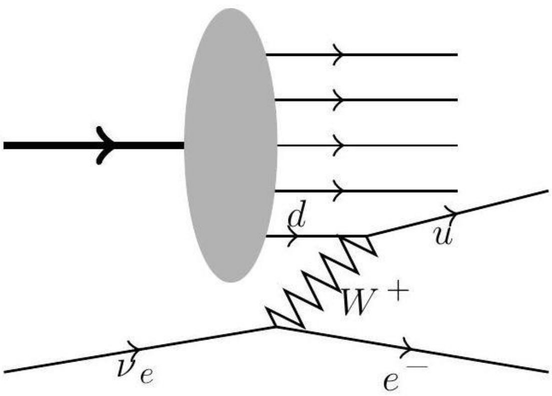


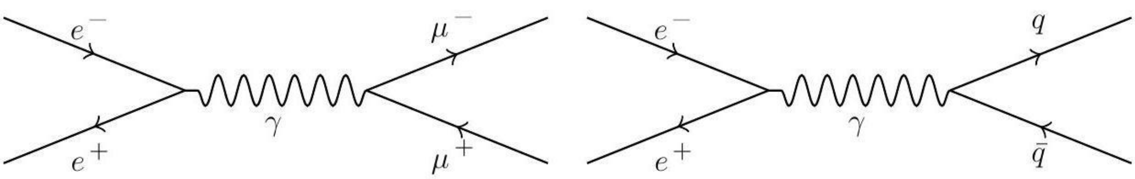
$$\mu_{B_{I_3}} = \langle BI_3S|\mu_{B_z}|BI_3S\rangle.$$

$$\vec{\mu}_B = \mu_0 \sum_{q=1}^{3} Q_q\vec{\sigma}_q$$









$$R=\frac{\sigma(e^+e^-\to \text{ hadrons })}{\sigma(e^+e^-\to\mu^+\mu^-)}=\sum_{\text{partons } q}\frac{\sigma(e^+e^-\to q\bar{q})}{\sigma(e^+e^-\to\mu^+\mu^-)}$$

$$p_{e^-}=\left(\sqrt{m_e^2+|\vec{p}|^2},\vec{p}\right),p_{e^+}=\left(\sqrt{m_e^2+|\vec{p}|^2},-\vec{p}\right)$$
$$p_{\mu^-}=\left(\sqrt{m_\mu^2+|\vec{p}\,'|^2},\vec{p}\,'\right),p_{\mu^+}=\left(\sqrt{m_\mu^2+|\vec{p}\,'|^2},-\vec{p}\,'\right)$$

$$\sqrt{m_e^2+|\vec{p}|^2}=\sqrt{m_\mu^2+|\vec{p}\,'|^2}$$

$$s=(p_{e^-}+p_{e^+})^2=\left(p_{\mu^-}+p_{\mu^+}\right)^2=4(m_e^2+|\vec{p}|^2)=4\left(m_\mu^2+|\vec{p}\,'|^2\right)$$

$$\sigma(e^+e^-\to\mu^+\mu^-)=\frac{4\pi\alpha^2}{3s},$$

$$\sigma(e^+e^-\to q\bar{q})=\frac{4\pi\alpha^2}{3s}Q_q^2$$

$$R=\sum_{\text{partons } q}\frac{\sigma(e^+e^-\to q\bar{q})}{\sigma(e^+e^-\to\mu^+\mu^-)}=\sum_{\text{partons } q}Q_q^2$$

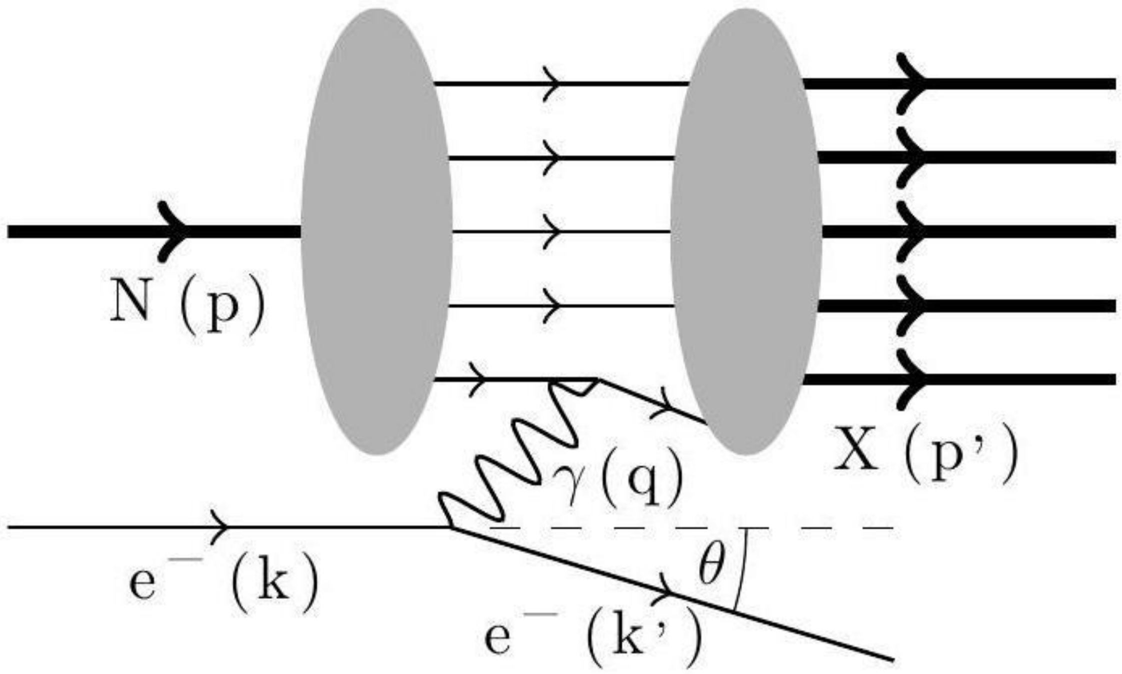
$$Q_u=\frac{1}{2}\left(\frac{1}{N_c}+1\right),Q_d=Q_s=\frac{1}{2}\left(\frac{1}{N_c}-1\right)$$

$$\sum_{\text{partons } q}Q_q^2=N_c\sum_{f=u,d,s}Q_f^2=N_c\left[\frac{1}{4}\left(\frac{1}{N_c}+1\right)^2+\frac{1}{2}\left(\frac{1}{N_c}-1\right)^2\right]$$
$$=\frac{3N_c}{4}-\frac{1}{2}+\frac{3}{4N_c}\overset{N_c=3}{\to}2$$

$$\sum_{\text{partons } q}Q_q^2=N_c\sum_{f=u,d,s,c}Q_f^2=N_c\left[\frac{1}{2}\left(\frac{1}{N_c}+1\right)^2+\frac{1}{2}\left(\frac{1}{N_c}-1\right)^2\right]$$
$$=N_c+\frac{1}{N_c}\overset{N_c=3}{\to}\frac{10}{3}$$

$$\sum_{\text{partons } q}Q_q^2=N_c\sum_{f=u,d,s,c,b}Q_f^2=N_c\left[\frac{1}{2}\left(\frac{1}{N_c}+1\right)^2+\frac{3}{4}\left(\frac{1}{N_c}-1\right)^2\right]$$
$$=\frac{5N_c}{4}-\frac{1}{2}+\frac{5}{4N_c}\overset{N_c=3}{\to}\frac{11}{3}$$

$$\sum_{\text{partons } q}Q_q^2=N_c\sum_{f=u,d,s,c,b,t}Q_f^2=N_c\left[\frac{3}{4}\left(\frac{1}{N_c}+1\right)^2+\frac{3}{4}\left(\frac{1}{N_c}-1\right)^2\right]$$
$$=\frac{3N_c}{2}+\frac{3}{2N_c}\overset{N_c=3}{\to}5$$



$$e + N \rightarrow e + X$$

$$W^2 = (p+q)^2 = p^2 + 2pq + q^2 = M_N^2 + 2M_N v - Q^2.$$

$$v = \frac{pq}{M_N}, Q = \sqrt{-q^2}$$

$$p = (M_N, \vec{0}), k = (E, \vec{k}), k' = (E', \vec{k}'),$$

$$v = E - E' = \frac{pq}{M_N} = \frac{p}{M_N}(k - k').$$

$$\cos\ \theta = \frac{\vec{k} \cdot \vec{k}'}{|\vec{k}||\vec{k}'|}$$

$$E \simeq |\vec{k}|, E' \simeq |\vec{k}'| \Rightarrow$$
$$\cos\ \theta = 1 - 2\sin^2\frac{\theta}{2} \simeq \frac{EE' - kk'}{EE'}, kk' \simeq 2EE'\sin^2\frac{\theta}{2}$$

$$Q^2 = -q^2 = -(k - k')^2 = 2kk' = 4EE'\sin^2\frac{\theta}{2}$$

$$W^2 = M_N^2 + 2M_N v - Q^2 \geq M_N^2$$

$$x = \frac{Q^2}{2M_N v} \leq 1$$

$$\langle e(k')X(p')|S|e(k)N(p)\rangle =$$
$$-\mathrm{i}(2\pi)^4\delta(p' + k' - p - k)4\pi\alpha\bar{e}(k')\gamma^\mu e(k)\frac{1}{q^2}\langle X(p')|j_\mu|N(p)\rangle.$$

$$j^{\mu}(x) = \sum_{\mathrm{f}=u,d,s,c,b,t} Q_{\mathrm{f}} e \bar{q}^{\mathrm{f}}(x)\gamma^{\mu} q^{\mathrm{f}}(x)$$

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{\alpha^2}{Q^4}\frac{E'}{E}\frac{1}{2}\sum_{\text{spins}} [\bar{e}(k')\gamma^{\mu}e(k)][\bar{e}(k')\gamma^{\nu}e(k)]^{*}$$
$$\times \sum_{X} \frac{1}{M_N}(2\pi)^3\delta(p'-p-q)\frac{1}{2}\sum_{\text{spins}} \langle N(p)|j_{\mu}|X(p')\rangle\langle X(p')|j_{\nu}|N(p)\rangle$$
$$= \frac{\alpha^2}{Q^4}\frac{E'}{E}L^{\mu\nu}W_{\mu\nu}$$

$$L^{\mu\nu} = \frac{1}{2}\sum_{\text{spins}} [\bar{e}(k')\gamma^{\mu}e(k)][\bar{e}(k')\gamma^{\nu}e(k)]^{*} = 2(k^{\mu}k'^{\nu} + k^{\nu}k'^{\mu} - kk'g^{\mu\nu})$$

$$W_{\mu\nu} = \sum_{X} \frac{1}{M_N}(2\pi)^3\delta(p'-p-q)\frac{1}{2}\sum_{\text{spins}} \langle N(p)|j_{\mu}|X(p')\rangle\langle X(p')|j_{\nu}|N(p)\rangle.$$

$$p^2 = p^{\mu}p_{\mu} = M_N^2, pq = p^{\mu}q_{\mu} = \nu M_N, q^2 = q^{\mu}q_{\mu} = -Q^2$$

$$W_{\mu\nu} = Ap_{\mu}p_{\nu} + Bp_{\mu}q_{\nu} + Cq_{\mu}p_{\nu} + Dq_{\mu}q_{\nu} + Eg_{\mu\nu} + F\epsilon_{\mu\nu\rho\sigma}p^{\rho}q^{\sigma}$$

$$\partial^{\mu}j_{\mu} = 0 \Rightarrow q^{\mu}W_{\mu\nu} = q^{\nu}W_{\mu\nu} = 0$$

$$q^{\mu}W_{\mu\nu} = A\nu M_N p_{\nu} + B\nu M_N q_{\nu} - CQ^2p_{\nu} - DQ^2q_{\nu} + Eq_{\nu} = 0 \quad ,$$
$$q^{\nu}W_{\mu\nu} = A\nu M_N p_{\mu} - BQ^2p_{\mu} + C\nu M_N q_{\mu} - DQ^2q_{\mu} + Eq_{\mu} = 0 \Rightarrow$$
$$A\nu M_N - CQ^2 = 0, B\nu M_N - DQ^2 + E = 0,$$
$$A\nu M_N - BQ^2 = 0, C\nu M_N - DQ^2 + E = 0 \Rightarrow B = C,$$

$$B = \frac{\nu M_N}{Q^2}A, E = DQ^2 - B\nu M_N = DQ^2 - \frac{M_N^2\nu^2}{Q^2}A,$$
$$W_{\mu\nu} = A\left(p_{\mu}p_{\nu} + \frac{M_N\nu}{Q^2}(p_{\mu}q_{\nu} + p_{\nu}q_{\mu} - M_N\nu g_{\mu\nu})\right) + D(q_{\mu}q_{\nu} + Q^2g_{\mu\nu})$$
$$= A\left(p_{\mu} + \frac{M_N\nu}{Q^2}q_{\mu}\right)\left(p_{\nu} + \frac{M_N\nu}{Q^2}q_{\nu}\right) + \left(D - A\frac{\nu^2M_N^2}{Q^4}\right)(q_{\mu}q_{\nu} + Q^2g_{\mu\nu})$$

$$W_1^{eN}(\nu,Q^2) = \frac{\nu^2M_N^2}{Q^2}A(\nu,Q^2) - Q^2D(\nu,Q^2)$$
$$W_2^{eN}(\nu,Q^2) = M_N^2A(\nu,Q^2)$$

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{\alpha^2}{Q^4}\frac{E'}{E}L^{\mu\nu}W_{\mu\nu} = \frac{\alpha^2}{Q^4}\frac{E'}{E}2(k^{\mu}k'^{\nu} + k^{\nu}k'^{\mu} - kk'g^{\mu\nu})$$

$$\times\left[-W_1^{eN}(\nu,Q^2)\frac{1}{Q^2}(q_{\mu}q_{\nu} + Q^2g_{\mu\nu}) + W_2^{eN}(\nu,Q^2)\frac{1}{M_N^2}\left(p_{\mu} + \frac{M_N\nu}{Q^2}q_{\mu}\right)\left(p_{\nu} + \frac{M_N\nu}{Q^2}q_{\nu}\right)\right]$$
$$= 4\frac{\alpha^2E'^2}{Q^4}\left[2W_1^{eN}(\nu,Q^2)\sin^2\frac{\theta}{2} + W_2^{eN}(\nu,Q^2)\cos^2\frac{\theta}{2}\right]$$

$$q=(0,\vec{q})=(0,0,0,-Q), p=\left(\sqrt{P^2+M_N^2},0,0,P\right)$$

$$q^2=-Q^2, M_N\nu=pq=PQ \;\Rightarrow\; P=\frac{M_N\nu}{Q}=\frac{Q}{2x}$$

$$p_{\mathrm{t}}=(\xi P,0,0,\xi P), \xi\in[0,1].$$

$$\sum_{\mathrm{t}\in\{\mathrm{f},\bar{\mathrm{f}},\,\mathrm{g}\}}\int_0^1 d\xi\xi N_{\mathrm{t}}(\xi)=1$$

$$P_{\mu\nu}^{\mathrm{f}}=\frac{Q_{\mathrm{f}}^2}{p_{\mathrm{f}}^0}\delta(2p_{\mathrm{f}}q+q^2)\frac{1}{4}\sum_{\text{spins}}\left[\bar{q}^{\mathrm{f}}(p_{\mathrm{f}}')g_{\mu\rho}\gamma^{\rho}q^{\mathrm{f}}(p_{\mathrm{f}})\right]\left[\bar{q}^{\mathrm{f}}(p_{\mathrm{f}}')g_{\nu\sigma}\gamma^{\sigma}q^{\mathrm{f}}(p_{\mathrm{f}})\right]^*$$

$$=\frac{Q_{\mathrm{f}}^2}{p_{\mathrm{f}}^0}\delta(2p_{\mathrm{f}}q+q^2)\left(p_{\mathrm{f}\mu}p_{\mathrm{f}\nu}'+p_{\mathrm{f}\nu}p_{\mathrm{f}\mu}'-p_{\mathrm{f}}p_{\mathrm{f}}'g_{\mu\nu}\right)$$

$$p_{\mathrm{f}\mu}p_{\mathrm{f}\nu}'+p_{\mathrm{f}\nu}p_{\mathrm{f}\mu}'-p_{\mathrm{f}}p_{\mathrm{f}}'g_{\mu\nu}=\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{q^2}{2}+2\left(p_{\mathrm{f}\mu}-\frac{p_{\mathrm{f}}q}{q^2}q_\mu\right)\left(p_{\mathrm{f}\nu}-\frac{p_{\mathrm{f}}q}{q^2}q_\nu\right)$$

$$P_{\mu\nu}^{\mathrm{f}}=\frac{Q_{\mathrm{f}}^2}{p_{\mathrm{f}}^0}\delta(2p_{\mathrm{f}}q+q^2)\left[\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\left(-\frac{q^2}{2}\right)+2\left(p_{\mathrm{f}\mu}-\frac{p_{\mathrm{f}}q}{q^2}q_\mu\right)\left(p_{\mathrm{f}\nu}-\frac{p_{\mathrm{f}}q}{q^2}q_\nu\right)\right]$$

$$2p_{\mathrm{f}}q+q^2=0 \;\Rightarrow\; p_{\mathrm{f}}^2=p_{\mathrm{f}}'^2=(p_{\mathrm{f}}+q)^2=0$$

$$p_{\mathrm{f}}=\xi p \;\Rightarrow\; \xi=\frac{p_{\mathrm{f}}q}{pq}=\frac{-q^2}{2M_N\nu}=x$$

$$\sum_{\mathrm{t}\in\{\mathrm{f},\bar{\mathrm{f}}\}}\int_0^1 d\xi N_{\mathrm{t}}(\xi)P_{\mu\nu}^{\mathrm{t}}=\sum_{\mathrm{t}\in\{\mathrm{f},\bar{\mathrm{f}}\}}\int_0^1 d\xi N_{\mathrm{t}}(\xi)\frac{Q_{\mathrm{t}}^2}{\xi P}\delta(2p_{\mathrm{f}}q+q^2)$$
$$\times\left[\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\left(-\frac{q^2}{2}\right)+2\xi^2\left(p_\mu-\frac{pq}{q^2}q_\mu\right)\left(p_\nu-\frac{pq}{q^2}q_\nu\right)\right]=$$
$$\sum_{\mathrm{t}\in\{\mathrm{f},\bar{\mathrm{f}}\}}\int_0^1 d\xi N_{\mathrm{t}}(\xi)\frac{Q_{\mathrm{t}}^2}{\xi P}\frac{1}{2pq}\delta(\xi-x)\times\left[\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\left(-\frac{q^2}{2}\right)+2\xi^2\left(p_\mu-\frac{pq}{q^2}q_\mu\right)\left(p_\nu-\frac{pq}{q^2}q_\nu\right)\right]=$$

$$\sum_{\mathrm{t}\in\{\mathrm{f},\bar{\mathrm{f}}\}}N_{\mathrm{t}}(x)Q_{\mathrm{t}}^2\left[\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{Q^2}{4xPpq}+\left(p_\mu-\frac{pq}{q^2}q_\mu\right)\left(p_\nu-\frac{pq}{q^2}q_\nu\right)\frac{2x^2}{2xPpq}\right]=$$
$$\sum_{\mathrm{t}\in\{\mathrm{f},\bar{\mathrm{f}}\}}N_{\mathrm{t}}(x)Q_{\mathrm{t}}^2\left[\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{Q}{2M_N\nu}+\left(p_\mu-\frac{pq}{q^2}q_\mu\right)\left(p_\nu-\frac{pq}{q^2}q_\nu\right)\frac{xQ}{(M_N\nu)^2}\right]$$

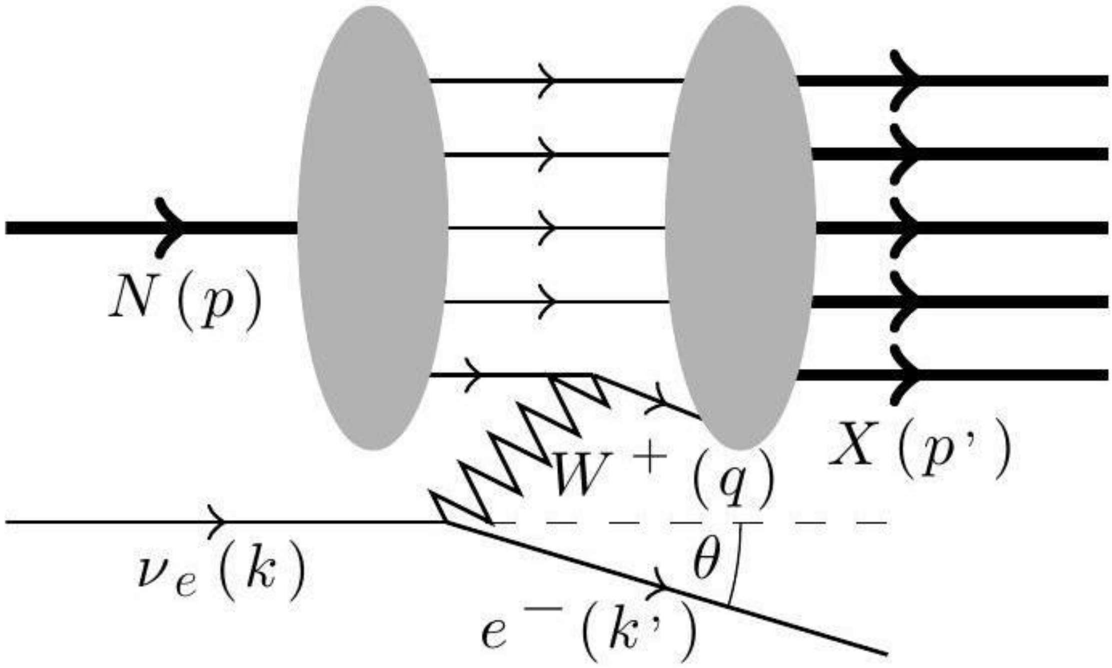
$$\frac{M_N}{p^0}W_{\mu\nu}=\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{Q}{\nu}W_1^{eN}(\nu,Q^2)+\left(p_\mu-\frac{pq}{q^2}q_\mu\right)\left(p_\nu-\frac{pq}{q^2}q_\nu\right)\frac{Q}{\nu M_N^2}W_2^{eN}(\nu,Q^2)$$

$$W_1^{eN}(v,Q^2)=\frac{1}{2M_N}\sum_{t\in\{f,\bar{f}\}} N_t(x)Q_t^2\,,$$
$$W_2^{eN}(v,Q^2)=\frac{x}{v}\sum_{t\in\{f,\bar{f}\}} N_t(x)Q_t^2.$$

$$F_1^{eN}(x)=\lim_{v,Q^2\to\infty} 2M_N W_1^{eN}(v,Q^2)=\sum_{t\in\{f,\bar{f}\}} N_t(x)Q_t^2$$
$$F_2^{eN}(x)=\lim_{v,Q^2\to\infty} vW_2^{eN}(v,Q^2)=x\sum_{t\in\{f,\bar{f}\}} N_t(x)Q_t^2$$

$$F_2^{eN}(x)=xF_1^{eN}(x)$$

$$\left\langle\frac{F_2^{eN}(x)-xF_1^{eN}(x)}{F_2^{eN}(x)}\right\rangle_x=-0.01\pm0.11$$



$$j^{+\mu}=\frac{g}{\sqrt{2}}(\bar{u}_L\bar{\sigma}^\mu\cos\ \theta_C d_L+\bar{u}_L\bar{\sigma}^\mu\sin\ \theta_C s_L)$$

$$\frac{d^2\sigma}{dE'd\Omega}=\frac{G_F^2E'^2}{2\pi^2}\left[2W_1^{vN}(v,Q^2)\sin^2\frac{\theta}{2}+W_2^{vN}(v,Q^2)\cos^2\frac{\theta}{2}-W_3^{vN}(v,Q^2)\frac{E+E'}{M_N}\sin^2\frac{\theta}{2}\right]$$

$$F_1^{vN}(x)=\lim_{v,Q^2\to\infty} 2M_N W_1^{vN}(v,Q^2)=2\cos^2\ \theta_C N_d(x)+2\sin^2\ \theta_C N_s(x)+2N_{\bar{u}}(x),$$
$$F_2^{vN}(x)=\lim_{v,Q^2\to\infty} vW_2^{vN}(v,Q^2)=xF_1^{vN}(x),$$
$$F_3^{vN}(x)=\lim_{v,Q^2\to\infty} vW_3^{vN}(v,Q^2)=-2\cos^2\ \theta_C N_d(x)-2\sin^2\ \theta_C N_s(x)+2N_{\bar{u}}(x).$$

$$F_1^{ep}(x)=\frac{4}{9}p_u(x)+\frac{1}{9}p_d(x)+\frac{1}{9}p_s(x)+\frac{4}{9}p_{\bar{u}}(x)+\frac{1}{9}p_{\bar{d}}(x)+\frac{1}{9}p_{\bar{s}}(x)$$
$$F_1^{en}(x)=\frac{4}{9}n_u(x)+\frac{1}{9}n_d(x)+\frac{1}{9}n_s(x)+\frac{4}{9}n_{\bar{u}}(x)+\frac{1}{9}n_{\bar{d}}\ (x)+\frac{1}{9}n_{\bar{s}}(x)$$
$$F_1^{vp}(x)=2\cos^2\ \theta_C p_d(x)+2\sin^2\ \theta_C p_s(x)+2p_{\bar{u}}(x)$$
$$F_1^{vn}(x)=2\cos^2\ \theta_C n_d(x)+2\sin^2\ \theta_C n_s(x)+2n_{\bar{u}}(x).$$

$$p_u(x)=n_d(x), p_d(x)=n_u(x), p_{\bar{u}}(x)=n_{\bar{d}}(x), p_{\bar{d}}(x)=n_{\bar{u}}(x).$$

$$F_1^{ep}(x)+F_1^{en}(x)=\frac{5}{9}(p_u(x)+p_d(x)+p_{\bar{u}}(x)+p_{\bar{d}}(x))$$
$$F_1^{vp}(x)+F_1^{vn}(x)=2\cos^2\ \theta_C(p_d(x)+p_u(x))+2(p_{\bar{u}}(x)+p_{\bar{d}}(x))$$

$$\frac{F_2^{vp}(x)+F_2^{vn}(x)}{F_2^{ep}(x)+F_2^{en}(x)}=\frac{F_1^{vp}(x)+F_1^{vn}(x)}{F_1^{ep}(x)+F_1^{en}(x)}\approx\frac{18}{5}.$$

$$Q_N=\int_0^1 dx\sum_{\mathrm{f}=u,d,s}\left(Q_\mathrm{f}N_\mathrm{f}(x)-Q_\mathrm{f}N_{\bar{\mathrm{f}}}(x)\right)$$

$$1=\int_0^1 dx\sum_{\mathrm{f}=u,d,s}\left(\frac{1}{3}N_\mathrm{f}(x)-\frac{1}{3}N_{\bar{\mathrm{f}}}(x)\right)$$

$$1=\int_0^1 dxx\sum_i N_i(x)$$

$$I_q=\int_0^1 dxx\sum_{\mathrm{f}=u,d,s}\left(N_\mathrm{f}(x)+N_{\bar{\mathrm{f}}}(x)\right)$$

$$F_1^{eN}(x)=\frac{4}{9}N_u(x)+\frac{1}{9}N_d(x)+\frac{1}{9}N_s(x)+\frac{4}{9}N_{\bar{u}}(x)+\frac{1}{9}N_{\bar{d}}(x)+\frac{1}{9}N_{\bar{s}}(x)$$
$$F_1^{vN}(x)=2N_d(x)+2N_{\bar{u}}(x)$$

$$F_1^{ep}(x)=\frac{4}{9}p_u(x)+\frac{1}{9}p_d(x)+\frac{1}{9}p_s(x)+\frac{4}{9}p_{\bar{u}}(x)+\frac{1}{9}p_{\bar{d}}(x)+\frac{1}{9}p_{\bar{s}}(x)$$
$$F_1^{en}(x)=\frac{4}{9}n_u(x)+\frac{1}{9}n_d(x)+\frac{1}{9}n_s(x)+\frac{4}{9}n_{\bar{u}}(x)+\frac{1}{9}n_{\bar{d}}(x)+\frac{1}{9}n_{\bar{s}}(x)$$
$$=\frac{4}{9}p_d(x)+\frac{1}{9}p_u(x)+\frac{1}{9}p_s(x)+\frac{4}{9}p_{\bar{d}}(x)+\frac{1}{9}p_{\bar{u}}(x)+\frac{1}{9}p_{\bar{s}}(x)$$
$$F_1^{vp}(x)=2p_d(x)+2p_{\bar{u}}(x)$$
$$F_1^{vn}(x)=2n_d(x)+2n_{\bar{u}}(x)=2p_u(x)+2p_{\bar{d}}(x)$$

$$\frac{9}{2}\left(F_1^{ep}(x)+F_1^{en}(x)\right)-\frac{3}{4}\left(F_1^{vp}(x)+F_1^{vn}(x)\right)=$$
$$\frac{5}{2}p_u(x)+\frac{5}{2}p_d(x)+p_s(x)+\frac{5}{2}p_{\bar{u}}(x)+\frac{5}{2}p_{\bar{d}}(x)+p_{\bar{s}}(x)$$
$$-\frac{3}{2}p_u(x)-\frac{3}{2}p_d(x)-\frac{3}{2}p_{\bar{u}}(x)-\frac{3}{2}p_{\bar{d}}(x)=$$
$$p_u(x)+p_d(x)+p_s(x)+p_{\bar{u}}(x)+p_{\bar{d}}(x)+p_{\bar{s}}(x)$$

$$I_q = \int_0^1 dx\left[\frac{9}{2}\left(F_1^{ep}(x) + F_1^{en}(x)\right) - \frac{3}{4}\left(F_1^{vp}(x) + F_1^{vn}(x)\right)\right]$$

$$\begin{aligned} G &= \mathrm{SU}(N_\mathrm{f})_L \times \mathrm{SU}(N_\mathrm{f})_R \times \mathrm{U}(1)_B \\ H &= \mathrm{SU}(N_\mathrm{f})_{L=R} \times \mathrm{U}(1)_B \end{aligned}$$

$$G/H = \mathrm{SU}(N_\mathrm{f})$$

$$U'(x) = LU(x)R^\dagger, L, R \in \mathrm{SU}(N_\mathrm{f})$$

$$\partial_\mu U'(x) = L\partial_\mu U(x) R^\dagger$$

$$\mathcal{L}(U) = \frac{F_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right]$$

$$\mathcal{L}(U') = \frac{F_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu U'^\dagger \partial_\mu U'\right] = \frac{F_\pi^2}{4}\mathrm{Tr}\left[R\partial_\mu U^\dagger L^\dagger L \partial_\mu U R^\dagger\right] = \mathcal{L}(U)$$

$$\begin{aligned} \mathcal{L}(\bar{q}, q, G) = &\ \bar{q}_\mathrm{L}\bar{\sigma}_\mu\left(\partial_\mu + G_\mu\right)q_\mathrm{L} + \bar{q}_\mathrm{R}\sigma_\mu\left(\partial_\mu + G_\mu\right)q_\mathrm{R} \\ &+ \bar{q}_\mathrm{L}\mathcal{M}q_\mathrm{R} + \bar{q}_\mathrm{R}\mathcal{M}^\dagger q_\mathrm{L} - \frac{1}{2g_s^2}\mathrm{Tr}\left[G_{\mu\nu}G_{\mu\nu}\right] \end{aligned}$$

$$\begin{aligned} q'_\mathrm{L}(x) &= \begin{pmatrix} u'_\mathrm{L}(x) \\ d'_\mathrm{L}(x) \\ s'_\mathrm{L}(x) \end{pmatrix} = L\begin{pmatrix} u_\mathrm{L}(x) \\ d_\mathrm{L}(x) \\ s_\mathrm{L}(x) \end{pmatrix} = Lq_\mathrm{L}(x), \bar{q}'_\mathrm{L}(x) = \bar{q}_\mathrm{L}(x)L^\dagger, L \in \mathrm{SU}(3)_\mathrm{L}, \\ q'_\mathrm{R}(x) &= \begin{pmatrix} u'_\mathrm{R}(x) \\ d'_\mathrm{R}(x) \\ s'_\mathrm{R}(x) \end{pmatrix} = R\begin{pmatrix} u_\mathrm{R}(x) \\ d_\mathrm{R}(x) \\ s_\mathrm{R}(x) \end{pmatrix} = Rq_\mathrm{R}(x), \bar{q}'_\mathrm{R}(x) = \bar{q}_\mathrm{R}(x)R^\dagger, R \in \mathrm{SU}(3)_\mathrm{R}, \end{aligned}$$

$$\mathcal{M}' = L\mathcal{M}R^\dagger$$

$$\mathcal{L}(U) = \frac{F_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right] - \frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}^\dagger U + U^\dagger \mathcal{M}\right]$$

$$\mathrm{Tr}\left[\mathcal{M}^\dagger U' + U'^\dagger \mathcal{M}\right] = \mathrm{Tr}\left[\mathcal{M}\left(LUR^\dagger + RU^\dagger L^\dagger\right)\right]$$

$$\langle 0|\partial_{m_\mathrm{f}}\mathcal{L}\big|_{\mathcal{M}=0}|0\rangle = \langle 0|\left(\bar{q}_\mathrm{L}^\mathrm{f} q_\mathrm{R}^\mathrm{f} + \bar{q}_\mathrm{R}^\mathrm{f} q_\mathrm{L}^\mathrm{f}\right)|0\rangle = \frac{1}{N_\mathrm{f}}\langle\bar{q}q\rangle$$

$$\langle 0|\partial_{m_\mathrm{f}}\mathcal{L}\big|_{\mathcal{M}=0}|0\rangle = -\frac{\Sigma}{2}\mathrm{Tr}[\mathrm{diag}(1,0,\ldots,0)(\mathbb{1}+\mathbb{1})] = -\Sigma \ \Rightarrow\ \Sigma = -\frac{\langle\bar{q}q\rangle}{N_\mathrm{f}}$$

$$\mathcal{L}(\bar{q}_\mathrm{L}, q_\mathrm{L}, \bar{q}_\mathrm{R}, q_\mathrm{R}, \mathbf{\Phi}) = \bar{q}_\mathrm{L}\mathbf{\Phi}\mathcal{F}q_\mathrm{R} + \bar{q}_\mathrm{R}\mathcal{F}^\dagger\mathbf{\Phi}^\dagger q_\mathrm{L}$$

$$U(x) = \exp\left(\mathrm{i}\pi^a(x)\lambda^a/F_\pi\right), a \in \left\{1,2,\ldots,N_\mathrm{f}^2 - 1\right\}$$

$$U(x) \simeq 1 + \mathrm{i}\pi^a(x)\lambda^a/F_\pi, \partial_\mu U(x) \simeq \mathrm{i}\partial_\mu\pi^a(x)\lambda^a/F_\pi,$$

$$\mathcal{L}(\pi) \simeq \frac{1}{4}\mathrm{Tr}\left[\partial_\mu\pi^a\lambda^a\partial_\mu\pi^b\lambda^b\right] - \frac{\Sigma}{2}\mathrm{Tr}[\mathcal{M}(\mathbb{1}+\mathbb{1})] = \frac{1}{2}\partial_\mu\pi^a\partial_\mu\pi^a - \Sigma\mathrm{Tr}\mathcal{M}$$

$$U(x) \simeq 1 + \mathrm{i}\pi^a(x)\lambda^a/F_\pi + \frac{1}{2}(\mathrm{i}\pi^a(x)\lambda^a/F_\pi)^2$$

$$\begin{aligned}\mathrm{Tr}\left[\mathcal{M}\left(U+U^\dagger\right)\right] &= m\mathrm{Tr}\left[U+U^\dagger\right] \\ &\simeq 2mN_\mathrm{f} - m\frac{1}{F_\pi^2}\pi^a\pi^b\mathrm{Tr}[\lambda^a\lambda^b] = 2mN_\mathrm{f} - 2m\frac{1}{F_\pi^2}\pi^a\pi^a\end{aligned}$$

$$\mathcal{L}(\pi) \simeq \frac{1}{2}\partial_\mu\pi^a\partial_\mu\pi^a - \Sigma\left(mN_\mathrm{f} - m\frac{1}{F_\pi^2}\pi^a\pi^a\right)$$

$$M_\pi^2 = \frac{2m\Sigma}{F_\pi^2}$$

$$M_{\pi^+}^2 = M_{\pi^0}^2 = M_{\pi^-}^2 = \frac{(m_u+m_d)\Sigma}{F_\pi^2}$$

$$\pi^a\lambda^a = \sqrt{2}\begin{pmatrix} \pi^3(x)/\sqrt{2} + \pi^8(x)/\sqrt{6} & \pi^+(x) & K^+(x) \\ \pi^-(x) & -\pi^3(x)/\sqrt{2} + \pi^8(x)/\sqrt{6} & K^0(x) \\ K^-(x) & \overline{K^0}(x) & -2\pi^8(x)/\sqrt{6} \end{pmatrix},$$

$$\pi^\pm(x) = \frac{1}{\sqrt{2}}(\pi^1(x) \mp \mathrm{i}\pi^2(x)),$$

$$K^0(x) = \frac{1}{\sqrt{2}}(\pi^6(x) - \mathrm{i}\pi^7(x)), \overline{K^0}(x) = \frac{1}{\sqrt{2}}(\pi^4(x) \mp \mathrm{i}\pi^5(x)),$$

$$\begin{aligned}\mathrm{Tr}\left[\mathcal{M}U^\dagger + U\mathcal{M}^\dagger\right] &= \mathrm{Tr}\left[\mathcal{M}\left(U+U^\dagger\right)\right] \\ &= 2\mathrm{Tr}[\mathcal{M}] - \frac{1}{F_\pi^2}\mathrm{Tr}[\mathcal{M}(\pi^a\lambda^a)^2)\Big] + \mathcal{O}\left(\left(\frac{\pi^a\lambda^a}{F_\pi}\right)^4\right)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}[\mathcal{M}(\pi^a\lambda^a)^2] &= (m_u+m_d)\pi^+\pi^- + (m_u+m_s)K^+K^- + (m_d+m_s)K^0\overline{K^0} \\ &+(m_u+m_d)\frac{(\pi^3)^2}{2} + \frac{1}{\sqrt{3}}(m_u-m_d)\pi^3\pi^8 + \frac{1}{3}(m_u+m_d+4m_s)\frac{(\pi^8)^2}{2}\end{aligned}$$

$$M_{\pi^+}^2 = M_{\pi^-}^2 = \frac{(m_u+m_d)\Sigma}{F_\pi^2}$$

$$M_{K^+}^2 = M_{K^-}^2 = \frac{(m_u+m_s)\Sigma}{F_\pi^2}, M_{K^0}^2 = M_{\bar{K}^0}^2 = \frac{(m_d+m_s)\Sigma}{F_\pi^2}$$

$$M_{\pi^0}^2 = \frac{\Sigma}{F_\pi^2}\left[\frac{2}{3}(m_u+m_d+m_s) - \frac{1}{3}\sqrt{(2m_s-m_u-m_d)^2 + 3(m_u-m_d)^2}\right],$$

$$M_\eta^2 = \frac{\Sigma}{F_\pi^2}\left[\frac{2}{3}(m_u+m_d+m_s) + \frac{1}{3}\sqrt{(2m_s-m_u-m_d)^2 + 3(m_u-m_d)^2}\right].$$

$$M_{\pi^0}^2 = \frac{(m_u+m_d)\Sigma}{F_\pi^2}, M_\eta^2 = \frac{(m_u+m_d+4m_s)\Sigma}{3F_\pi^2}.$$

$$M_\pi^2 + 3M_\eta^2 = 2M_{K^\pm}^2 + 2M_{K^0}^2$$

$$\mathcal{L}(\bar{q},q,G_\mu,A_\mu)=\bar{q}_{\rm L}\bar{\sigma}_\mu(\partial_\mu+G_\mu+{\rm i}eQ_{\rm L}A_\mu)q_{\rm L}+\bar{q}_{\rm R}\sigma_\mu(\partial_\mu+G_\mu+{\rm i}eQ_{\rm R}A_\mu)q_{\rm R}$$
$$+\bar{q}_{\rm L}\mathcal{M}q_{\rm R}+\bar{q}_{\rm R}\mathcal{M}^\dagger q_{\rm L}-\frac{1}{2g_s^2}{\rm Tr}[G_{\mu\nu}G_{\mu\nu}]+\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$
$$Q_{\rm L}=Q_{\rm R}=Q={\rm diag}(Q_u,Q_d,Q_s)={\rm diag}\left(\frac{2}{3},-\frac{1}{3},-\frac{1}{3}\right)$$

$$Q'_{\rm L}=LQ_{\rm L}L^\dagger,Q'_{\rm R}=RQ_{\rm L}R^\dagger$$

$$\mathcal{L}(U)=\frac{F_\pi^2}{4}{\rm Tr}[D_\mu U^\dagger D_\mu U]-\frac{\Sigma}{2}{\rm Tr}[\mathcal{M}^\dagger U+U^\dagger\mathcal{M}]$$
$$-Ce^2{\rm Tr}[Q_{\rm L}UQ_{\rm R}U^\dagger]+\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

$$D_\mu U(x)=\partial_\mu U(x)+{\rm i}eA_\mu(x)Q_{\rm L}U(x)-{\rm i}eA_\mu(x)U(x)Q_{\rm R}.$$

$$W'_\mu(x)=L(x)(W_\mu(x)+\partial_\mu)L(x)^\dagger,X'_\mu(x)=R(x)(X_\mu(x)+\partial_\mu)R(x)^\dagger,$$
$$U'(x)=L(x)U(x)R(x)^\dagger\Rightarrow$$
$$D'_\mu U'(x)=\partial_\mu U'(x)+W'_\mu(x)U'(x)-U'(x)X'_\mu(x)$$
$$=L(x)[\partial_\mu U(x)+W_\mu(x)U(x)-U(x)X_\mu(x)]R(x)^\dagger$$
$$=L(x)D_\mu U(x)R(x)^\dagger.$$

$$L(x)=R(x)={\rm diag}(\exp({\rm i}g'\varphi(x)/2),\exp(-{\rm i}g'\varphi(x)/2)),$$
$$B'_\mu(x)=B_\mu-\partial_\mu\varphi(x).$$

$$W_\mu^3(x)=\frac{g'A_\mu(x)+gZ_\mu(x)}{\sqrt{g^2+g'^2}},B_\mu(x)=\frac{gA_\mu(x)-g'Z_\mu(x)}{\sqrt{g^2+g'^2}}\Rightarrow$$
$$A_\mu(x)=\frac{g'W_\mu^3(x)+gB_\mu(x)}{\sqrt{g^2+g'^2}},A'_\mu(x)=A_\mu(x)-\partial_\mu\alpha(x)$$
$$e\alpha(x)=g'\varphi(x),e=\frac{gg'}{\sqrt{g^2+g'^2}}$$

$$D_\mu U(x)=\partial_\mu U(x)+{\rm i}\frac{gg'A_\mu(x)}{\sqrt{g^2+g'^2}}\frac{\tau^3}{2}U(x)-U(x){\rm i}\frac{g'gA_\mu(x)}{\sqrt{g^2+g'^2}}\frac{\tau^3}{2}$$
$$=\partial_\mu U(x)+[{\rm i}eA_\mu(x)Q',U(x)]=\partial_\mu U(x)+[{\rm i}eA_\mu(x)Q,U(x)].$$

$$Q=\begin{pmatrix}2/3 & 0\\ 0 & -1/3\end{pmatrix}=\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}+\frac{1}{6}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=Q'+\frac{B}{2}\mathbb{1}.$$

$$L(x)=R(x)={\rm diag}(\exp({\rm i}2e\alpha(x)/3),\exp(-{\rm i}e\alpha(x)/3),\exp(-{\rm i}e\alpha(x)/3))$$

$$\pi^{\pm'}(x)=\exp(\pm{\rm i}e\alpha(x))\pi^\pm(x),\quad \pi^{0'}(x)=\pi^0(x),\quad \eta'(x)=\eta(x)$$
$$K^{\pm'}(x)=\exp(\pm{\rm i}e\alpha(x))K^\pm(x),\quad K^{0'}(x)=K^0(x),\quad \overline{K^0}'(x)=\overline{K^0}(x)$$

$${\rm Tr}[Q_{\rm L}UQ_{\rm R}U^\dagger]={\rm Tr}[QUQU^\dagger]$$
$$={\rm Tr}[Q^2]+\frac{1}{F_\pi^2}{\rm Tr}\left[Q\pi^a\lambda^a[Q,\pi^b\lambda^b]\right]+\mathcal{O}\left(\left(\frac{\pi^a\lambda^a}{F_\pi}\right)^3\right).$$

$$\mathrm{Tr}\left[Q\pi^a\lambda^a[Q,\pi^b\lambda^b]\right] = -2(\pi^+\pi^- + K^+K^-)$$

$$M_{\pi^\pm}^2 = \frac{(m_u+m_d)\Sigma}{F_\pi^2} + \frac{2Ce^2}{F_\pi^2}, M_{K^\pm}^2 = \frac{(m_u+m_s)\Sigma}{F_\pi^2} + \frac{2Ce^2}{F_\pi^2}.$$

$$M_{\pi^+}^2 + M_{\pi^-}^2 - M_{\pi^0}^2 + 3M_\eta^2 = 2(m_u+m_d+2m_s)\frac{\Sigma}{F_\pi^2} + \frac{4Ce^2}{F_\pi^2} = M_{K^+}^2 + M_{K^-}^2 + M_{K^0}^2 + M_{\overline{K^0}}^2$$

$$M_{\pi^\pm} = 139.6\mathrm{MeV}, \quad M_{\pi^0} = 135.0\mathrm{MeV}, M_\eta = 547.9\mathrm{MeV}$$
$$M_{K^\pm} = 493.7\mathrm{MeV}, \quad M_{K^0} = M_{\overline{K^0}} = 497.6\mathrm{MeV}$$

$$\Delta M = M_n - M_p = 939.56\mathrm{MeV} - 938.27\mathrm{MeV} = 1.29\mathrm{MeV} = \Delta M_{\mathrm{QCD}} + \Delta M_{\mathrm{QED}}$$
$$\Delta M_{\mathrm{QCD}} = 1.87(16)\mathrm{MeV}, \Delta M_{\mathrm{QED}} = -0.58(16)\mathrm{MeV}$$

$$\frac{2m_s}{m_u+m_d} = \frac{M_{K^0}^2 + M_{K^\pm}^2 - M_{\pi^\pm}^2}{M_{\pi^0}^2} = 25.9, \frac{m_s - m_d}{m_u+m_d} = \frac{M_{K^\pm}^2 - M_{\pi^\pm}^2}{M_{\pi^0}^2} = 12.3,$$
$$\frac{m_s - m_u}{m_u+m_d} = \frac{M_{K^0}^2 - M_{\pi^0}^2}{M_{\pi^0}^2} = 12.6, \frac{m_d - m_u}{m_u+m_d} = \frac{\left(M_{\pi^\pm}^2 - M_{\pi^0}^2\right) - \left(M_{K^\pm}^2 - M_{K^0}^2\right)}{M_{\pi^0}^2} = 0.28$$
$$\frac{m_u}{m_d} = \frac{2M_{\pi^0}^2 - M_{\pi^\pm}^2 + M_{K^\pm}^2 - M_{K^0}^2}{M_{\pi^\pm}^2 - M_{K^\pm}^2 + M_{K^0}^2} = 0.56, \frac{m_s}{m_d} = \frac{M_{K^\pm}^2 - M_{\pi^\pm}^2 + M_{K^0}^2}{M_{\pi^\pm}^2 - M_{K^\pm}^2 + M_{K^0}^2} = 19.2$$

$$m_u = 2.16_{-0.26}^{+0.49}\mathrm{MeV}, m_d = 4.67_{-0.17}^{+0.48}\mathrm{MeV}, m_s = 93.4_{-3.4}^{+8.6}\mathrm{MeV}.$$

$$\frac{m_u}{m_d} = 0.474_{-0.074}^{+0.056}, \frac{m_s}{m_d} = 19.5_{-2.5}^{+2.5}, \frac{2m_s}{m_u+m_d} = 27.33_{-0.77}^{+0.67},$$

$$\psi'(x) = V(x)\psi(x), \bar{\psi}'(x) = \bar{\psi}(x)V(x)^\dagger.$$

$$V(x) = R\left[R^\dagger L U(x)\right]^{1/2}\left[U(x)^{1/2}\right]^\dagger = L\left[L^\dagger R U(x)^\dagger\right]^{1/2} U(x)^{1/2}$$

$$u(x) = U(x)^{1/2}$$

$$u(x)' = Lu(x)V(x)^\dagger = V(x)u(x)R^\dagger$$

$$\mathrm{v}_\mu(x) = \frac{1}{2}\left[u(x)^\dagger\partial_\mu u(x) + u(x)\partial_\mu u(x)^\dagger\right]$$

$$\begin{aligned}\mathrm{v}'_\mu(x) &= \frac{1}{2}\left[V(x)u(x)^\dagger L^\dagger\partial_\mu\left(Lu(x)V(x)^\dagger\right) + V(x)u(x)R^\dagger\partial_\mu\left(Ru(x)^\dagger V(x)^\dagger\right)\right]\\ &= V(x)\left(\mathrm{v}_\mu(x) + \partial_\mu\right)V(x)^\dagger\end{aligned}$$

$$a_\mu(x) = \frac{\mathrm{i}}{2}\left[u(x)^\dagger\partial_\mu u(x) - u(x)\partial_\mu u(x)^\dagger\right]$$

$$\begin{aligned}a'_\mu(x) &= \frac{1}{2}\left[V(x)u(x)^\dagger L^\dagger\partial_\mu\left(Lu(x)V(x)^\dagger\right) - V(x)u(x)R^\dagger\partial_\mu\left(Ru(x)^\dagger V(x)^\dagger\right)\right]\\ &= V(x)a_\mu(x)V(x)^\dagger\end{aligned}$$

$$\begin{aligned}\mathcal{L}(U,\bar{\psi},\psi) = {} & M\bar{\psi}\psi + \bar{\psi}\gamma_\mu(\partial_\mu + \mathrm{v}_\mu)\psi + \mathrm{i}g_A\bar{\psi}\gamma_\mu\gamma_5 a_\mu\psi \\ & + \frac{F_\pi^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U] - \frac{\Sigma}{2}\mathrm{Tr}[\mathcal{M}(U^\dagger + U^\dagger)]\end{aligned}$$

$$g_{\pi NN}F_\pi = g_A M$$

$$D_\mu U(x) = (\partial_\mu + W_\mu(x))U(x)$$

$$U'(x) = L(x)U(x), W'_\mu(x) = L(x)(W_\mu(x) + \partial_\mu)L(x)^\dagger$$

$$\mathcal{L}(U,W) = \frac{F_\pi^2}{4}\mathrm{Tr}[D_\mu U^\dagger D_\mu U] - \frac{1}{2g^2}\mathrm{Tr}[W_{\mu\nu}W_{\mu\nu}]$$

$$\mathcal{L}(\mathbb{1},W) = \frac{F_\pi^2}{4}\mathrm{Tr}[W_\mu^\dagger W_\mu] - \frac{1}{2g^2}\mathrm{Tr}[W_{\mu\nu}W_{\mu\nu}]$$

$$M_W = \frac{1}{2}gF_\pi$$

$$D_\mu U(x) = \partial_\mu U(x) + W_\mu(x)U(x) - U(x)X_\mu(x)$$

$$\mathcal{L}(U,W,X) = \frac{F_\pi^2}{4}\mathrm{Tr}[D_\mu U^\dagger D_\mu U] - \frac{1}{2g^2}\mathrm{Tr}[W_{\mu\nu}W_{\mu\nu}] - \frac{1}{2g'^2}\mathrm{Tr}[X_{\mu\nu}X_{\mu\nu}]$$

$$\mathcal{L}(\mathbb{1},W,B) = \frac{F_\pi^2}{4}\mathrm{Tr}[(W_\mu - X_\mu)(W_\mu - X_\mu)] - \frac{1}{2g^2}\mathrm{Tr}[W_{\mu\nu}W_{\mu\nu}] - \frac{1}{2g'^2}\mathrm{Tr}[X_{\mu\nu}X_{\mu\nu}]$$

$$M_Z = \frac{1}{2}\sqrt{g^2 + g'^2}F_\pi$$

$$\frac{M_W}{M_Z} = \frac{g}{\sqrt{g^2 + g'^2}} = \cos\,\theta_\mathrm{W}$$

$$\begin{aligned} Q_\mathrm{L}(x) = \begin{pmatrix} U_\mathrm{L}(x) \\ D_\mathrm{L}(x)\end{pmatrix}, \quad & U_\mathrm{L}(x) = \begin{pmatrix} u_\mathrm{L}(x) \\ c_\mathrm{L}(x) \\ t_\mathrm{L}(x)\end{pmatrix}, D_\mathrm{L}(x) = \begin{pmatrix} d_\mathrm{L}(x) \\ s_\mathrm{L}(x) \\ b_\mathrm{L}(x)\end{pmatrix} \\ & U_\mathrm{R}(x) = \begin{pmatrix} u_\mathrm{R}(x) \\ c_\mathrm{R}(x) \\ t_\mathrm{R}(x)\end{pmatrix}, D_\mathrm{R}(x) = \begin{pmatrix} d_\mathrm{R}(x) \\ s_\mathrm{R}(x) \\ b_\mathrm{R}(x)\end{pmatrix}, \\ & L_\mathrm{L}(x) = \begin{pmatrix} N_\mathrm{L}(x) \\ E_\mathrm{L}(x)\end{pmatrix}, N_\mathrm{L}(x) = \begin{pmatrix} \nu_{e\,\mathrm{L}}(x) \\ \nu_{\mu\mathrm{L}}(x) \\ \nu_{\tau_\mathrm{L}}(x)\end{pmatrix}, E_\mathrm{L}(x) = \begin{pmatrix} e_\mathrm{L}(x) \\ \mu_\mathrm{L}(x) \\ \tau_\mathrm{L}(x)\end{pmatrix} \\ & E_\mathrm{R}(x) = \begin{pmatrix} e_\mathrm{R}(x) \\ \mu_\mathrm{R}(x) \\ \tau_\mathrm{R}(x)\end{pmatrix}\end{aligned}$$

$$\begin{aligned} & Q'_\mathrm{L}(x) = V^{Q_\mathrm{L}}Q_\mathrm{L}(x), U'_\mathrm{R}(x) = V^{U_\mathrm{R}}U_\mathrm{R}(x), D'_\mathrm{R}(x) = V^{D_\mathrm{R}}D_\mathrm{R}(x) \\ & L'_\mathrm{L}(x) = V^{L_\mathrm{L}}L_\mathrm{L}(x), E'_\mathrm{R}(x) = V^{E_\mathrm{R}}E_\mathrm{R}(x)\end{aligned}$$

$$\mathcal{L}(\bar{L}'_{\mathrm{L}}, L'_{\mathrm{L}}, \bar{E}'_{\mathrm{R}}, E'_{\mathrm{R}}, \bar{Q}'_{\mathrm{L}}, Q'_{\mathrm{L}}, \bar{U}'_{\mathrm{R}}, U'_{\mathrm{R}}, \bar{D}'_{\mathrm{R}}, D'_{\mathrm{R}}, \Phi) = \bar{L}'_{\mathrm{L}}\Phi\mathcal{F}_E E'_{\mathrm{R}} + \bar{E}'_{\mathrm{R}}\mathcal{F}_E^\dagger\Phi^\dagger L'_{\mathrm{L}}$$
$$+\bar{Q}'_{\mathrm{L}}\Phi\mathcal{F}_D D'_{\mathrm{R}} + \bar{D}'_{\mathrm{R}}\mathcal{F}_D^\dagger\Phi^\dagger Q'_{\mathrm{L}} + \bar{Q}'_{\mathrm{L}}\widetilde{\Phi}\mathcal{F}_U U'_{\mathrm{R}} + \bar{U}'_{\mathrm{R}}\mathcal{F}_U^\dagger\widetilde{\Phi}^\dagger Q'_{\mathrm{L}}$$

$$\mathcal{F}'_E = V^{L_{\mathrm{L}}}\mathcal{F}_E V^{E_{\mathrm{R}}^\dagger}, \mathcal{F}'_D = V^{Q_{\mathrm{L}}}\mathcal{F}_D V^{D_{\mathrm{R}}^\dagger}, \mathcal{F}'_U = V^{Q_{\mathrm{L}}}\mathcal{F}_U V^{U_{\mathrm{R}}^\dagger}$$

$$U(x) \in \mathrm{SU}(N_{\mathrm{f}})_L \times \mathrm{SU}(N_{\mathrm{f}})_R / \mathrm{SU}(N_{\mathrm{f}})_{L=R} = \mathrm{SU}(N_{\mathrm{f}})$$

$$B = \frac{1}{24\pi^2}\int d^3x\epsilon_{ijk}\mathrm{Tr}\big[(U^\dagger\partial_i U)(U^\dagger\partial_j U)(U^\dagger\partial_k U)\big] \in \Pi_3[\mathrm{SU}(N_{\mathrm{f}})] = \mathbb{Z}$$

$$j_\mu^{\mathrm{S}}(x) = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[\big(U(x)^\dagger\partial_\nu U(x)\big)\big(U(x)^\dagger\partial_\rho U(x)\big)\big(U(x)^\dagger\partial_\sigma U(x)\big)\big]$$

$$\partial_\mu j_\mu^{\mathrm{S}}(x) = 0$$

$$B = \int d^3x j_4^{\mathrm{S}}$$

$$\partial_4 B = \int d^3x\partial_4 j_4^{\mathrm{S}} = -\int d^3x\partial_i j_i^{\mathrm{S}} = -\int d^2\sigma_i j_i^{\mathrm{S}} = 0$$

$$U(\vec{x}) = \exp\left(\mathrm{i}f(|\vec{x}|)\frac{\vec{x}}{|\vec{x}|}\cdot\vec{\tau}\right)$$

$$\mathrm{Sign}[U] = \pm 1$$

$$Z = \int \mathcal{D}U\exp\left(-S[U]\right)\mathrm{Sign}[U]^{N_{\mathrm{c}}}$$

$$\mathrm{Sign}[U']^{N_{\mathrm{c}}} = \mathrm{Sign}[LU]^{N_{\mathrm{c}}} = \mathrm{Sign}[L]^{N_{\mathrm{c}}}\mathrm{Sign}[U]^{N_{\mathrm{c}}}$$

$$\partial_\mu j_\mu^B(x) = -\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[W_{\mu\nu}(x)W_{\rho\sigma}(x)\big].$$

$$D_\mu U(x) = \big(\partial_\mu + W_\mu(x)\big)U(x), \big(D_\mu U(x)\big)^\dagger = \partial_\mu U(x)^\dagger - U(x)^\dagger W_\mu(x).$$

$$j_\mu^{\mathrm{GW}}(x) = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[\big(U(x)^\dagger D_\nu U(x)\big)\big(U(x)^\dagger D_\rho U(x)\big)\big(U(x)^\dagger D_\sigma U(x)\big)\big]$$
$$-\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[W_{\nu\rho}(x)\big(D_\sigma U(x)U(x)^\dagger\big)\big]$$

$$\partial_\mu j_\mu^{\mathrm{GW}}(x) = -\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[W_{\mu\nu}(x)W_{\rho\sigma}(x)\big].$$

$$B(x_4 = \infty) - B(x_4 = -\infty) = \int d^3x j_4^{\mathrm{GW}}(x_4 = \infty) - \int d^3x j_4^{\mathrm{GW}}(x_4 = -\infty)$$
$$= \int d^4x\partial_\mu j_\mu^{\mathrm{GW}} = -\frac{1}{32\pi^2}\int d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[W_{\mu\nu}W_{\rho\sigma}\big]$$
$$= Q[W] = 1$$

$$\tilde{L}(x) = \begin{pmatrix} \exp\left(\mathrm{i}g'Y_{q_L}\varphi(x)\right) & 0 \\ 0 & \exp\left(\mathrm{i}g'Y_{q_L}\varphi(x)\right) \end{pmatrix}$$

$$\tilde{R}(x) = \begin{pmatrix} \exp\left(\mathrm{i}g'Y_{u_R}\varphi(x)\right) & 0 \\ 0 & \exp\left(\mathrm{i}g'Y_{d_R}\varphi(x)\right) \end{pmatrix}$$

$$U'(x) = U(x)R(x)^{\dagger}, R(x) = \begin{pmatrix} \exp\left(\mathrm{i}g'\varphi(x)/2\right) & 0 \\ 0 & \exp\left(-\mathrm{i}g'\varphi(x)/2\right) \end{pmatrix}$$

$$\begin{aligned} D_\mu U(x) &= \partial_\mu U(x) + W_\mu(x)U(x) - U(x)X_\mu(x) \\ &= \partial_\mu U(x) + \mathrm{i}gW_\mu^a(x)\frac{\tau^a}{2}U(x) - \mathrm{i}g'U(x)B_\mu(x)\frac{\tau^3}{2} \end{aligned}$$

$$\begin{aligned} j_\mu^{\mathrm{GW}}(x) &= \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\left(U(x)^\dagger D_\nu U(x)\right)\left(U(x)^\dagger D_\rho U(x)\right)\left(U(x)^\dagger D_\sigma U(x)\right)\right] \\ &- \frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[W_{\nu\rho}(x)\left(D_\sigma U(x)U(x)^\dagger\right)\right] \\ &- \frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[X_{\nu\rho}(x)\left(U(x)^\dagger D_\sigma U(x)\right)\right] \end{aligned}$$

$$\begin{aligned} \partial_\mu j_\mu^{\mathrm{GW}}(x) &= -\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[W_{\mu\nu}(x)W_{\rho\sigma}(x)\right] + \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[X_{\mu\nu}(x)X_{\rho\sigma}(x)\right] \\ &= -\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[W_{\mu\nu}(x)W_{\rho\sigma}(x)\right] - \frac{g'^2}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}B_{\mu\nu}(x)B_{\rho\sigma}(x) \end{aligned}$$

$$S_{\mathrm{GW}}[U,W,B] = \frac{g'}{2}\int d^4x B_\mu j_\mu^{\mathrm{GW}}$$

$$\begin{aligned} &S_{\mathrm{GW}}[U',W,B'] - S_{\mathrm{GW}}[U,W,B] = -\frac{g'}{2}\int d^4x\partial_\mu\varphi j_\mu^{\mathrm{GW}} = \frac{g'}{2}\int d^4x\varphi\partial_\mu j_\mu^{\mathrm{GW}} \\ &= -\frac{g'}{2}\int d^4x\varphi\left\{\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[W_{\mu\nu}W_{\rho\sigma}\right] + \frac{g'^2}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}B_{\mu\nu}B_{\rho\sigma}\right\} \end{aligned}$$

$$S[U,W,B] = \int d^4x\frac{F_\pi^2}{4}\mathrm{Tr}\left[D_\mu U^\dagger D_\mu U\right]$$

$$Z[W,B] = \int \mathcal{D}U\exp\left(-S[U,W,B]\right)\mathrm{Sign}[U]\exp\left(\mathrm{i}S_{\mathrm{GW}}[U,W,B]\right)$$

$$\bar{L}(x) = \bar{R}(x) = \begin{pmatrix} \exp\left(\mathrm{i}eQ_u\alpha(x)\right) & 0 \\ 0 & \exp\left(\mathrm{i}eQ_d\alpha(x)\right) \end{pmatrix}$$

$$V(x) = \begin{pmatrix} \exp\left(\mathrm{i}e\alpha(x)/2\right) & 0 \\ 0 & \exp\left(-\mathrm{i}e\alpha(x)/2\right) \end{pmatrix}$$

$$D_\mu U = \partial_\mu U(x) + \mathrm{i}eA_\mu(x)\left[\frac{\tau^3}{2}, U(x)\right]$$

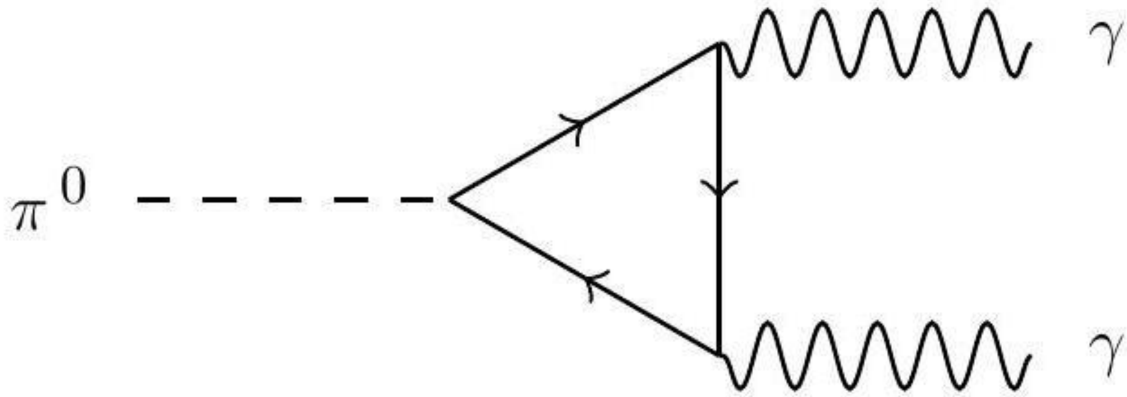
$$\begin{aligned} j_\mu^{\mathrm{GW}}(x) &= \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\left(U(x)^\dagger D_\nu U(x)\right)\left(U(x)^\dagger D_\rho U(x)\right)\left(U(x)^\dagger D_\sigma U(x)\right)\right] \\ &- \frac{\mathrm{i}e}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F_{\nu\rho}\mathrm{Tr}\left[\frac{\tau^3}{2}\left(D_\sigma U(x)U(x)^\dagger\right)\right] \end{aligned}$$

$$S_{\mathrm{GW}}[U,A] = \frac{e}{2}\int d^4x A_\mu j_\mu^{\mathrm{GW}}$$

$${}^{\mathrm{C}}U(x) = U(x)^{\mathrm{T}}, U(x) = \exp\left(\mathrm{i}\pi^a(x)\tau^a/F_\pi\right), \pi^a(x)\tau^a = \begin{pmatrix} \pi^0(x) & \sqrt{2}\pi^+(x) \\ \sqrt{2}\pi^-(x) & -\pi^0(x) \end{pmatrix} \Rightarrow$$

$${}^{\mathrm{C}}\pi^\pm(x) = \pi^\mp(x),\ {}^{\mathrm{C}}\pi^0(x) = \pi^0(x)$$

$${}^{\mathrm{G}}U(x) = \mathrm{i}\tau^{2\mathrm{C}}U(x)(\mathrm{i}\tau^2)^\dagger = U(x)^{*\,\mathrm{T}} = U(x)^\dagger \Rightarrow\ {}^{\mathrm{G}}\pi^a(x) = -\pi^a(x)$$



$$U(x) = \exp\left(\mathrm{i}\pi^0(x)\tau^3/F_\pi\right) \approx 1 + \mathrm{i}\pi^0(x)\tau^3/F_\pi$$

$$D_\mu U(x) = \partial_\mu U(x) + \mathrm{i}eA_\mu(x)\left[\frac{\tau^3}{2}, U(x)\right] \approx \mathrm{i}\partial_\mu\pi^0(x)\tau^3/F_\pi$$

$$\mathcal{L}_{\pi^0\gamma\gamma} = -\mathrm{i}\frac{e^2}{32\pi^2F_\pi}\pi^0\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},$$

$$\mathcal{L}_{\pi^0\gamma\gamma} = -\mathrm{i}N_{\mathrm{c}}\left(Q_u^2 - Q_d^2\right)\frac{e^2}{32\pi^2F_\pi}\pi^0\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$\Gamma(\pi^0 \to \gamma\gamma) = \left[N_{\mathrm{c}}\left(Q_u^2 - Q_d^2\right)\right]^2\frac{e^4M_\pi^3}{1024\pi^5F_\pi^2}$$

$$Q_p = \frac{N_{\mathrm{c}}+1}{2}Q_u + \frac{N_{\mathrm{c}}-1}{2}Q_d = 1,$$
$$Q_n = \frac{N_{\mathrm{c}}-1}{2}Q_u + \frac{N_{\mathrm{c}}+1}{2}Q_d = 0,$$

$$Q_u = \frac{1}{2}\left(\frac{1}{N_{\mathrm{c}}}+1\right), Q_d = \frac{1}{2}\left(\frac{1}{N_{\mathrm{c}}}-1\right)$$

$$N_{\mathrm{c}}\left(Q_u^2 - Q_d^2\right) = \frac{N_{\mathrm{c}}}{4}\left[\left(\frac{1}{N_{\mathrm{c}}}+1\right)^2 - \left(\frac{1}{N_{\mathrm{c}}}-1\right)^2\right] = 1,$$

$$S_{\mathrm{WZNW}}[U] = \frac{1}{240\pi^2\mathrm{i}}\int_{H^5} d^5x\epsilon_{\mu\nu\rho\sigma\lambda}\mathrm{Tr}\left[(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\sigma U)(U^\dagger\partial_\lambda U)\right]$$

$$w[U]=\frac{1}{480\pi^3\mathrm{i}}\int_{S^5} d^5x\epsilon_{\mu\nu\rho\sigma\lambda}\mathrm{Tr}\left[(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\sigma U)(U^\dagger\partial_\lambda U)\right]$$
$$\in \Pi_5[\mathrm{SU}(N_\mathrm{f})]=\mathbb{Z}$$

$$S_{\mathrm{WZNW}}\left[U^{(1)}\right]-S_{\mathrm{WZNW}}\left[U^{(2)}\right]=2\pi w[U]$$

$$Z=\int \mathcal{D}U\exp\left(-S[U]\right)\exp\left(\mathrm{i}nS_{\mathrm{WZNW}}[U]\right)$$

$$\exp\left(\mathrm{i}nS_{\mathrm{WZNW}}\left[U^{(1)}\right]\right)=\exp\left(\mathrm{i}nS_{\mathrm{WZNW}}\left[U^{(2)}\right]\right)\exp\left(2\pi\mathrm{i}nw[U]\right)$$

$$\exp\left(2\pi\mathrm{i}nw[U]\right)=1 \;\Rightarrow\; n\in\mathbb{Z}$$

$$U(x)=\begin{pmatrix}\widetilde{U}(x) & 0\\ 0 & \mathbb{1}\end{pmatrix},$$

$$\exp\left(\mathrm{i}nS_{\mathrm{WZNW}}[U]\right)=\mathrm{Sign}[\widetilde{U}]^n$$

$$\mathrm{P}^{\mathrm{P}}\pi^a(\vec{x},x_4)=-\pi^a(-\vec{x},x_4)$$

$$^{\mathrm{P}_0}\pi^a(\vec{x},x_4)=-\pi^a(\vec{x},x_4)$$

$$^{\mathrm{P}}U(\vec{x},x_4)=U(-\vec{x},x_4)^\dagger,\mathrm{P}_0U(\vec{x},x_4)=U(\vec{x},x_4)^\dagger.$$

$$S_{\mathrm{WZNW}}\left[\,^{\mathrm{P}_0}U\right]=S_{\mathrm{WZNW}}\left[U^\dagger\right]=-S_{\mathrm{WZNW}}[U],$$

$$\mathrm{Sign}\left[\,^{\mathrm{P}_0}\widetilde{U}\right]=\mathrm{Sign}\left[\widetilde{U}^\dagger\right]=\mathrm{Sign}[\widetilde{U}].$$

$$Q'=\mathrm{diag}(Q'_u,Q'_d,Q'_s)=\mathrm{diag}\left(\frac{2}{3},-\frac{1}{3},-\frac{1}{3}\right).$$

$$S_{\mathrm{WZNW}}[U,A]=S_{\mathrm{WZNW}}[U]+\frac{e}{48\pi^2}\int d^4x\epsilon_{\mu\nu\rho\sigma}A_\mu$$
$$\times\mathrm{Tr}\left[Q'(\partial_\nu UU^\dagger)(\partial_\rho UU^\dagger)(\partial_\sigma UU^\dagger)+Q'(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\sigma U)\right]$$
$$-\frac{\mathrm{i}e^2}{48\pi^2}\int d^4x\epsilon_{\mu\nu\rho\sigma}A_\mu F_{\nu\rho}\mathrm{Tr}\left[Q'(\partial_\sigma UU^\dagger)\left(Q'+\frac{1}{2}UQ'U^\dagger\right)\right.$$
$$\left.+Q'(U^\dagger\partial_\sigma U)\left(Q'+\frac{1}{2}U^\dagger Q'U\right)\right].$$

$$n=N_\mathrm{c}$$

$$Q=\mathrm{diag}(Q_u,Q_d,Q_s)=\mathrm{diag}\left(\frac{1}{2}\left(\frac{1}{N_\mathrm{c}}+1\right),\frac{1}{2}\left(\frac{1}{N_\mathrm{c}}-1\right),\frac{1}{2}\left(\frac{1}{N_\mathrm{c}}-1\right)\right)$$
$$=Q'+\left(1-\frac{N_\mathrm{c}}{3}\right)\frac{1}{2}B$$

$$\mathrm{diag}(Q_u,Q_d)=\mathrm{diag}\left(\frac{1}{2}\left(\frac{1}{N_\mathrm{c}}+1\right),\frac{1}{2}\left(\frac{1}{N_\mathrm{c}}-1\right)\right)=\mathrm{diag}\left(\frac{1}{2},-\frac{1}{2}\right)+\frac{1}{2}B.$$

$$S_{\mathrm{GW}}[U,A]=\frac{e}{48\pi^2}\int d^4x\epsilon_{\mu\nu\rho\sigma}A_\mu \mathrm{Tr}\left[(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\sigma U)\right] \\ -\frac{\mathrm{i}e^2}{32\pi^2}\int d^4x\epsilon_{\mu\nu\rho\sigma}A_\mu F_{\nu\rho}\mathrm{Tr}\left[Q'(\partial_\sigma U U^\dagger + U^\dagger\partial_\sigma U)\right]$$

$$Z[A]=\int \mathcal{D}U\exp\left(-S[U,A]\right)\exp\left(\mathrm{i}N_\mathrm{c}S_{\mathrm{WZNW}}[U,A]\right)\exp\left(\mathrm{i}(1-N_\mathrm{c}/3)S_{\mathrm{GW}}[U,A]\right)$$

$$N_\mathrm{c}\left({Q'_u}^2-{Q'_d}^2\right)=N_\mathrm{c}\left[\left(\frac{2}{3}\right)^2-\left(-\frac{1}{3}\right)^2\right]=\frac{N_\mathrm{c}}{3}.$$

$$Q=Q'+\left(1-N_\mathrm{c}\frac{N_d-N_u}{N_\mathrm{f}}\right)\frac{1}{2}B$$

$$Z[A]=\int \mathcal{D}U\exp\left(-S[U,A]\right)\exp\left(\mathrm{i}N_\mathrm{c}S_{\mathrm{WZNW}}[U,A]\right) \\ \times\exp\left(\mathrm{i}\left(1-N_\mathrm{c}\frac{N_d-N_u}{N_\mathrm{f}}\right)S_{\mathrm{GW}}[U,A]\right)$$

$$Q'_u=Q_u-\frac{1}{2}\left(\frac{1}{N_\mathrm{c}}-\frac{N_d-N_u}{N_\mathrm{f}}\right)=\frac{N_d}{N_\mathrm{f}}, \\ Q'_d=Q_d-\frac{1}{2}\left(\frac{1}{N_\mathrm{c}}-\frac{N_d-N_u}{N_\mathrm{f}}\right)=-\frac{N_u}{N_\mathrm{f}}.$$

$$N_\mathrm{c}\left({Q'_u}^2-{Q'_d}^2\right)=N_\mathrm{c}\frac{N_d^2-N_u^2}{N_\mathrm{f}^2}=N_\mathrm{c}\frac{N_d-N_u}{N_\mathrm{f}}$$

$$U(x)=\begin{pmatrix}\widetilde{U}(x) & 0\\ 0 & \mathbb{1}\end{pmatrix}$$

$$\tilde{Q}'=\begin{pmatrix}Q'_u & 0\\ 0 & Q'_d\end{pmatrix}=\frac{N_d-N_u}{2N_\mathrm{f}}+\frac{1}{2}\tau^3 \\ \tilde{Q}'^2=\begin{pmatrix}Q'^2_u & 0\\ 0 & Q'^2_d\end{pmatrix}=\frac{N_d^2+N_u^2}{2N_\mathrm{f}^2}+\frac{N_d-N_u}{2N_\mathrm{f}}\tau^3$$

$$N_\mathrm{c}\left(S_{\mathrm{WZNW}}[U,A]-S_{\mathrm{WZNW}}[U]\right)+\left(1-N_\mathrm{c}\frac{N_d-N_u}{N_\mathrm{f}}\right)S_{\mathrm{GW}}[U,A]=S_{\mathrm{GW}}[\widetilde{U},A]$$

$$U(x)=\exp\left(\mathrm{i}\eta^8(x)\lambda^8/F_\pi\right),$$

$$N_\mathrm{c}\left(S_{\mathrm{WZNW}}[U,A]-S_{\mathrm{WZNW}}[U]\right)+(1-N_\mathrm{c}/3)S_{\mathrm{GW}}[U,A] \\ =\frac{e^2}{32\sqrt{3}\pi^2F_\pi}\int d^4x\eta^8\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$\mathcal{L}_{\eta^8\pi^+\pi^-\gamma}=\frac{eN_\mathrm{c}}{4\sqrt{3}\pi^2F_\pi^3}\epsilon_{\mu\nu\rho\sigma}A_\mu\partial_\nu\eta^8\partial_\rho\pi^+\partial_\sigma\pi^-$$

$$\mathrm{Tr}\left(\frac{1}{\sqrt{6}}Q^2\right)=\frac{N_\mathrm{c}}{\sqrt{6}}\left(Q_u^2+2Q_d^2\right)=\frac{3}{4\sqrt{6}}\left(N_\mathrm{c}+\frac{1}{N_\mathrm{c}}-\frac{2}{3}\right),$$

$$j_\mu^{\mathrm{S}}(x)=\frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[\big(U(x)^\dagger\partial_\nu U(x)\big)\big(U(x)^\dagger\partial_\rho U(x)\big)\big(U(x)^\dagger\partial_\sigma U(x)\big)\big]$$

$$^{\mathrm{C}}\pi^{\pm}(\vec{x},x_4)=\pi^{\mp}(\vec{x},x_4),\ ^{\mathrm{C}}\pi^{0}(\vec{x},x_4)=\pi^{0}(\vec{x},x_4),$$
$$^{\mathrm{P}}\pi^{a}(\vec{x},x_4)=-\pi^{a}(-\vec{x},x_4),\ ^{\mathrm{P_0}}\pi^{a}(\vec{x},x_4)=-\pi^{a}(\vec{x},x_4).$$

$$j_\mu^{\mathrm{GW}}(x)=\frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[\big(U(x)^\dagger D_\nu U(x)\big)\big(U(x)^\dagger D_\rho U(x)\big)\big(U(x)^\dagger D_\sigma U(x)\big)\big]$$
$$+C\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[W_{\nu\rho}(x)\big(D_\sigma U(x)U(x)^\dagger\big)\big].$$

$$\partial_\mu j_\mu^{\mathrm{GW}}(x)=-\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[W_{\mu\nu}(x)W_{\rho\sigma}(x)\big].$$

$$\epsilon_{\mu\nu\rho\sigma\lambda}\mathrm{Tr}\big[(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\sigma U)(U^\dagger\partial_\lambda U)\big]=$$
$$\epsilon_{\mu\nu\rho\sigma\lambda}\mathrm{Tr}\big[(V^\dagger\partial_\mu V)(V(x)^\dagger\partial_\nu V)(V^\dagger\partial_\rho V)(V^\dagger\partial_\sigma V)(V^\dagger\partial_\lambda V)\big]+$$
$$\epsilon_{\mu\nu\rho\sigma\lambda}\mathrm{Tr}\big[(W^\dagger\partial_\mu W)(W^\dagger\partial_\nu W)(W^\dagger\partial_\rho W)(W^\dagger\partial_\sigma W)(W^\dagger\partial_\lambda W)\big]+\partial_\mu K_\mu,$$

$$Q[G]=-\frac{1}{32\pi^2}\int d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big[G_{\mu\nu}G_{\rho\sigma}\big]$$

$$q_{\mathrm{L}}^{\mathrm{f}'}(x)=\exp(\mathrm{i}\theta/2N_{\mathrm{f}})q_{\mathrm{L}}^{\mathrm{f}}(x)$$
$$q_{\mathrm{R}}^{\mathrm{f}'}(x)=\exp(-\mathrm{i}\theta/2N_{\mathrm{f}})q_{\mathrm{R}}^{\mathrm{f}}(x)$$

$$\delta_{\mathrm{A}}\theta=-N_{\mathrm{f}}\frac{\theta}{2N_{\mathrm{f}}}-N_{\mathrm{f}}\frac{\theta}{2N_{\mathrm{f}}}=-\theta\ \Rightarrow\ \theta'=\theta+\delta_{\mathrm{A}}\theta=0$$

$$\mathcal{M}'=\mathrm{diag}\Big(m_u\exp(\mathrm{i}\theta/N_{\mathrm{f}}),m_d\exp(\mathrm{i}\theta/N_{\mathrm{f}}),\dots,m_{N_{\mathrm{f}}}\exp(\mathrm{i}\theta/N_{\mathrm{f}})\Big);$$

$$S[U]=\int d^4x\left\{\frac{F_\pi^2}{4}\mathrm{Tr}\big[\partial_\mu U^\dagger\partial_\mu U\big]-\frac{\Sigma}{2}\mathrm{Tr}\big[\mathcal{M}'^\dagger U+U^\dagger\mathcal{M}'\big]\right\}$$

$$Z(\theta)=\int \mathcal{D}U\exp(-S[U])$$

$$z=\exp(2\pi\mathrm{i}/N_{\mathrm{f}})\mathbb{1}\in\mathbb{Z}(N_{\mathrm{f}})$$

$$\mathcal{M}'=\mathrm{diag}(m_u\exp(\mathrm{i}\theta/2),m_d\exp(\mathrm{i}\theta/2)).$$

$$\mathrm{Tr}\big[\mathcal{M}'^\dagger U+U^\dagger\mathcal{M}'\big]=m_u\cos\left(\frac{\theta}{2}+\varphi\right)+m_d\cos\left(\frac{\theta}{2}-\varphi\right)$$

$$\tan\ \varphi=\frac{m_d-m_u}{m_u+m_d}\tan\ \frac{\theta}{2}.$$

$$M_{\eta'}^2=\frac{2N_{\mathrm{f}}\chi_{\mathrm{t}}}{F_\pi^2}$$

$$\mathrm{U}(N_{\mathrm{f}})_L\times\mathrm{U}(N_{\mathrm{f}})_R/\mathrm{U}(N_{\mathrm{f}})_{L=R}=\mathrm{U}(N_{\mathrm{f}})$$

$$\det\widetilde{U}(x)=\exp\left(\mathrm{i}\sqrt{2N_{\mathrm{f}}}\eta'(x)/F_\pi\right)$$

$$S[\tilde{U}]=\int d^4x\left\{\frac{F_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu\tilde{U}^\dagger\partial_\mu\tilde{U}\right]-\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}'^\dagger\tilde{U}+\tilde{U}^\dagger\mathcal{M}'\right]+\frac{\chi_t}{2}(\mathrm{i}\log\,\det\tilde{U})^2\right\}_\zeta$$

$$S[\tilde{U}]=\int d^4x\left\{\frac{F_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu\tilde{U}^\dagger\partial_\mu\tilde{U}\right]-\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}^\dagger\tilde{U}+\tilde{U}^\dagger\mathcal{M}\right]+\frac{\chi_t}{2}(\mathrm{i}\log\,\det\tilde{U}-\theta)^2\right\}$$

$$\mathcal{L}(\bar{q}_\mathrm{L},q_\mathrm{L},\bar{d}_\mathrm{R},d_\mathrm{R},\Phi)=f_d\left[(\bar{u}_\mathrm{L},\bar{d}_\mathrm{L})\begin{pmatrix}\Phi^+\\ \Phi^0\end{pmatrix}d_\mathrm{R}+\bar{d}_\mathrm{R}(\Phi^{+*},\Phi^{0*})\begin{pmatrix}u_\mathrm{L}\\ d_\mathrm{L}\end{pmatrix}\right]$$

$$\mathcal{L}(\bar{q}_\mathrm{L},q_\mathrm{L},\bar{u}_\mathrm{R},u_\mathrm{R},\tilde{\Phi})=f_u\left[(\bar{u}_\mathrm{L},\bar{d}_\mathrm{L})\begin{pmatrix}\tilde{\Phi}^0\\ \tilde{\Phi}^-\end{pmatrix}u_\mathrm{R}+\bar{u}_\mathrm{R}(\tilde{\Phi}^{0*},\tilde{\Phi}^{-*})\begin{pmatrix}u_\mathrm{L}\\ d_\mathrm{L}\end{pmatrix}\right].$$

$$\tilde{\Phi}(x)=\begin{pmatrix}\tilde{\Phi}^0(x)\\ \tilde{\Phi}^-(x)\end{pmatrix}=\begin{pmatrix}\Phi_0(x)^*\\ -\Phi^+(x)^*\end{pmatrix}$$

$$\mathbf{\Phi}(x)=\begin{pmatrix}\Phi^0(x)^* & \Phi^+(x)\\ -\Phi^+(x)^* & \Phi^0(x)\end{pmatrix}=(\tilde{\Phi}(x),\Phi(x))$$

$$\mathcal{L}(\bar{q}_\mathrm{L},q_\mathrm{L},\bar{u}_\mathrm{R},u_\mathrm{R},\bar{d}_\mathrm{R},d_\mathrm{R},\mathbf{\Phi})=(\bar{u}_\mathrm{L},\bar{d}_\mathrm{L})\mathbf{\Phi}\mathcal{F}\begin{pmatrix}u_\mathrm{R}\\ d_\mathrm{R}\end{pmatrix}+(\bar{u}_\mathrm{R},\bar{d}_\mathrm{R})\mathcal{F}^\dagger\mathbf{\Phi}^\dagger\begin{pmatrix}u_\mathrm{L}\\ d_\mathrm{L}\end{pmatrix},$$

$$\begin{pmatrix}u'_\mathrm{L}(x)\\ d'_\mathrm{L}(x)\end{pmatrix}=\exp\,(\mathrm{i}\theta/4)\begin{pmatrix}u_\mathrm{L}(x)\\ d_\mathrm{L}(x)\end{pmatrix},\begin{pmatrix}u'_\mathrm{R}(x)\\ d'_\mathrm{R}(x)\end{pmatrix}=\exp\,(-\mathrm{i}\theta/4)\begin{pmatrix}u_\mathrm{R}(x)\\ d_\mathrm{R}(x)\end{pmatrix}.$$

$$\mathcal{F}'=\mathrm{diag}(f_u\exp\,(\mathrm{i}\theta/2),f_d\exp\,(\mathrm{i}\theta/2))$$

$$S[U,\mathbf{\Phi}]=\int d^4x\left\{\frac{F_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger\partial_\mu U\right]-\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{F}'^\dagger\mathbf{\Phi}^\dagger U+U^\dagger\mathbf{\Phi}\mathcal{F}'\right]\right\}$$

$$S[U,\mathbf{\Phi}']=\int d^4x\left\{\frac{F_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger\partial_\mu U\right]-\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{F}^\dagger\mathbf{\Phi}'^\dagger U+U^\dagger\mathbf{\Phi}'\mathcal{F}\right]\right\}$$

$$\mathcal{L}(a,G)=\frac{1}{2}\partial_\mu a\partial_\mu a-\mathrm{i}\frac{a}{\Lambda_\mathrm{PQ}}\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[G_{\mu\nu}G_{\rho\sigma}\right]$$

$$\mathbf{\Phi}'(x)=\mathrm{vdiag}\left(\exp\,(\mathrm{i}a(x)/\Lambda_\mathrm{PQ}),\exp\,(\mathrm{i}a(x)/\Lambda_\mathrm{PQ})\right),$$

$$U(x)=\mathrm{diag}\left(\exp\,(\mathrm{i}\pi^0(x)/F_\pi),\exp\,(-\mathrm{i}\pi^0(x)/F_\pi)\right)$$

$$\mathrm{Tr}\left[\mathcal{F}^\dagger\mathbf{\Phi}'^\dagger U+U^\dagger\mathbf{\Phi}'\mathcal{F}\right]=m_u\cos\,(a/\Lambda_\mathrm{PQ}+\pi^0/F_\pi)+m_d\cos\,(a/\Lambda_\mathrm{PQ}-\pi^0/F_\pi)$$

$$M^2=\Sigma\begin{pmatrix}(m_u+m_d)/F_\pi^2 & (m_u-m_d)/(F_\pi\Lambda_\mathrm{PQ})\\ (m_u-m_d)/(F_\pi\Lambda_\mathrm{PQ}) & (m_u+m_d)/\Lambda_\mathrm{PQ}^2\end{pmatrix}.$$

$$M^2=\Sigma\begin{pmatrix}(m_u+m_d)/F_\pi^2 & 0\\ 0 & 0\end{pmatrix}$$

$$M_\pi^2=\frac{(m_u+m_d)\Sigma}{F_\pi^2}.$$

$$\Sigma^2\left[\frac{(m_u+m_d)^2}{F_\pi^2\Lambda_{\text{PQ}}^2}-\frac{(m_u-m_d)^2}{F_\pi^2\Lambda_{\text{PQ}}^2}\right]=\Sigma^2\frac{4m_um_d}{F_\pi^2\Lambda_{\text{PQ}}^2}=M_\pi^2M_a^2$$

$$M_a^2=\frac{4m_um_d\Sigma}{(m_u+m_d)\Lambda_{\text{PQ}}^2}$$

$$\frac{M_a^2}{M_\pi^2}=\frac{4m_um_dF_\pi^2}{(m_u+m_d)^2\Lambda_{\text{PQ}}^2}$$

$$\frac{M_a}{M_\pi}=\frac{F_\pi}{\text{V}}\approx\frac{0.1\text{GeV}}{250\text{GeV}}=\frac{1}{2500}\ \Rightarrow\ M_a\approx\frac{0.14\text{GeV}}{2500}\approx 50\text{keV}$$

$$Q[W]=-\frac{1}{32\pi^2}\int\ d^4x\epsilon_{\mu\nu\rho\sigma}\text{Tr}\left[W_{\mu\nu}W_{\rho\sigma}\right]\in\Pi_3[\text{SU}(2)]=\mathbb{Z}$$

$$\begin{aligned}Q'_{\text{L}}(x)&=\exp\,(\text{i}\rho/N_{\text{c}})Q_{\text{L}}(x),\\ U'_{\text{R}}(x)&=\exp\,(\text{i}\rho/N_{\text{c}})U_{\text{R}}(x),\\ D'_{\text{R}}(x)&=\exp\,(\text{i}\rho/N_{\text{c}})D_{\text{R}}(x),\end{aligned}$$

$$\delta_B\theta_2=-N_{\text{g}}N_{\text{c}}\rho/N_{\text{c}}=-N_{\text{g}}\rho,$$

$$\theta'_2=\theta_2+\delta_B\theta_2=\theta_2-N_{\text{g}}\rho=0$$

$$Q[B]=-\frac{1}{32\pi^2}\int\ d^4x\epsilon_{\mu\nu\rho\sigma}\text{Tr}\left[X_{\mu\nu}X_{\rho\sigma}\right]=\frac{g'^2}{64\pi^2}\int\ d^4x\epsilon_{\mu\nu\rho\sigma}B_{\mu\nu}B_{\rho\sigma}$$

$$\begin{aligned}\delta_B\theta_1&=-N_{\text{g}}N_{\text{c}}2\left[2Y_{Q_L}^2-Y_{U_R}^2-Y_{D_R}^2\right]\rho/N_{\text{c}}\\ &=-N_{\text{g}}2\left[2\frac{1}{4N_{\text{c}}^2}-\frac{1}{4}\left(\frac{1}{N_{\text{c}}}+1\right)^2-\frac{1}{4}\left(\frac{1}{N_{\text{c}}}-1\right)^2\right]\rho=N_{\text{g}}\rho\end{aligned}$$

$$\theta'_1=\theta_1+\delta_B\theta_1=\theta_1+N_{\text{g}}\rho=\theta_1+\theta_2$$

$$\theta'_1+\theta'_2=\theta_1+\theta_2$$

$$\begin{aligned}Q'_{\text{L}}(x)&=\exp\,(\text{i}\rho/N_{\text{c}})Q_{\text{L}}(x),\\ U'_{\text{R}}(x)&=\exp\,(\text{i}\rho/N_{\text{c}})U_{\text{R}}(x),\\ D'_{\text{R}}(x)&=\exp\,(\text{i}\rho/N_{\text{c}})D_{\text{R}}(x),\\ L'_{\text{L}}(x)&=\exp\,(\text{i}\rho)L_{\text{L}}(x),\\ E'_{\text{R}}(x)&=\exp\,(\text{i}\rho)E_{\text{R}}(x),\end{aligned}$$

$$\begin{aligned}&\delta_L\theta_1=-N_{\text{g}}2\left[2Y_{L_L}^2-Y_{E_R}^2\right]\rho=-N_{\text{g}}2\left[2\left(-\frac{1}{2}\right)^2-(-1)^2\right]\rho=N_{\text{g}}\rho\\ &\delta_L\theta_2=-N_{\text{g}}\rho\end{aligned}$$

$$\begin{aligned}&\theta'_1=\theta_1+\delta_B\theta_1+\delta_L\theta_1=\theta_1+2N_{\text{g}}\rho\\ &\theta'_2=\theta_2+\delta_B\theta_2+\delta_L\theta_2=\theta_2-2N_{\text{g}}\rho\ \Rightarrow\\ &\theta'_1+\theta'_2=\theta_1+\theta_2\end{aligned}$$

$$W_\mu^3(x) = \frac{g' A_\mu(x) + g Z_\mu(x)}{\sqrt{g^2 + g'^2}} \to \frac{g' A_\mu(x)}{\sqrt{g^2 + g'^2}},$$
$$B_\mu(x) = \frac{g A_\mu(x) - g' Z_\mu(x)}{\sqrt{g^2 + g'^2}} \to \frac{g A_\mu(x)}{\sqrt{g^2 + g'^2}}.$$

$$\mathrm{i}\theta_1 \frac{g'^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma} - \mathrm{i}\theta_2 \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[W_{\mu\nu} W_{\rho\sigma}\right] \to$$
$$\mathrm{i}(\theta_1 + \theta_2) \frac{g^2 g'^2}{g^2 + g'^2} \frac{1}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \mathrm{i}\theta_{\mathrm{QED}} \frac{e^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$\theta_{\mathrm{QED}} = \theta_1 + \theta_2, e = \frac{g g'}{\sqrt{g^2 + g'^2}}$$

$$\Xi(x) = \Xi_a(x) \eta^a, a \in \{1,2, \dots ,24\}, \mathrm{Tr}\left(\eta^a \eta^b\right) = 2\delta_{ab}$$

$$\Xi'(x) = \Upsilon(x) \Xi(x) \Upsilon(x)^\dagger$$

$$V(\Xi) = \frac{M^2}{2} \mathrm{Tr}(\Xi^2) + \frac{\kappa}{3!} \mathrm{Tr}(\Xi^3) + \frac{\Lambda_1}{4!} \mathrm{Tr}(\Xi^4) + \frac{\Lambda_2}{4!} \left(\mathrm{Tr}(\Xi^2)\right)^2.$$

$$\Xi(x) = \begin{pmatrix} \xi_1(x) & 0 & 0 & 0 & 0 \\ 0 & \xi_2(x) & 0 & 0 & 0 \\ 0 & 0 & \xi_3(x) & 0 & 0 \\ 0 & 0 & 0 & \xi_4(x) & 0 \\ 0 & 0 & 0 & 0 & \xi_5(x) \end{pmatrix}, \xi_i(x) \in \mathbb{R}, \sum_i \xi_i = 0$$

$$V(\Xi) = \frac{M^2}{2} \sum_i \xi_i^2 + \frac{\kappa}{3!} \sum_i \xi_i^3 + \frac{\Lambda_1}{4!} \sum_i \xi_i^4 + \frac{\Lambda_2}{4!} \left(\sum_i \xi_i^2\right)^2.$$

$$\frac{\partial V(\Xi)}{\partial \xi_i} = M^2 \xi_i + \frac{\kappa}{2} \xi_i^2 + \frac{\Lambda_1}{3!} \xi_i^3 + \frac{\Lambda_2}{3!} \sum_j \xi_j^2 \xi_i = C$$

$$\Xi(x) = \Xi_0 = \mathcal{V} \eta^{24} = \mathcal{V} \sqrt{\frac{3}{5}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2/3 & 0 & 0 \\ 0 & 0 & 0 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & -2/3 \end{pmatrix}$$

$$V_\mu(x) = \mathrm{i} g_5 V_\mu^a(x) \frac{\eta^a}{2}$$

$$V_\mu'(x) = \Upsilon(x) \left(V_\mu(x) + \partial_\mu\right) \Upsilon(x)^\dagger$$

$$V_\mu(x) = \mathrm{i} \frac{g_5}{2} \begin{pmatrix} W_\mu^a(x) \tau^a + \sqrt{3/5} B_\mu(x) \mathbb{1}_{2\times2} & \bar{X}_\mu^1(x) & \bar{X}_\mu^2(x) & \bar{X}_\mu^3(x) \\ X_\mu^1(x) & Y_\mu^1(x) & \bar{Y}_\mu^2(x) & \bar{Y}_\mu^3(x) \\ X_\mu^2(x) & Y_\mu^2(x) & & \\ X_\mu^3(x) & Y_\mu^3(x) & & G_\mu^a(x) \lambda^a - 2/\sqrt{15} B_\mu(x) \mathbb{1}_{3\times3} \end{pmatrix}$$

$$\Upsilon(x)=\begin{pmatrix}\exp\left(\mathrm{i}g'/2\varphi(x)\right)L(x) & 0_{2\times 3}\\ 0_{3\times 2} & \exp\left(-\mathrm{i}g'/3\varphi(x)\right)\Omega(x)\end{pmatrix},$$
$$L(x)\in \mathrm{SU}(2)_L,$$
$$\Omega(x)\in \mathrm{SU}(3)_c,$$

$$G'_\mu(x)=\Omega(x)\left(G_\mu(x)+\partial_\mu\right)\Omega(x)^\dagger$$
$$W'_\mu(x)=L(x)\left(W_\mu(x)+\partial_\mu\right)L(x)^\dagger$$
$$B'_\mu(x)=B_\mu(x)-\partial_\mu\varphi(x)$$
$$\begin{pmatrix}X^{1\prime}_\mu(x) & Y^{1\prime}_\mu(x)\\ X^{2\prime}_\mu(x) & Y^{2\prime}_\mu(x)\\ X^{3\prime}_\mu(x) & Y^{3\prime}_\mu(x)\end{pmatrix}=\exp\left(-\mathrm{i}g'\frac{5}{6}\varphi(x)\right)\Omega(x)\begin{pmatrix}X^{1}_\mu(x) & Y^{1}_\mu(x)\\ X^{2}_\mu(x) & Y^{2}_\mu(x)\\ X^{3}_\mu(x) & Y^{3}_\mu(x)\end{pmatrix}L(x)^\dagger,$$
$$\begin{pmatrix}\bar X^{1\prime}_\mu(x) & \bar X^{2\prime}_\mu(x) & \bar X^{3\prime}_\mu(x)\\ \bar Y^{1\prime}_\mu(x) & \bar Y^{2\prime}_\mu(x) & \bar Y^{3\prime}_\mu(x)\end{pmatrix}=\exp\left(\mathrm{i}g'\frac{5}{6}\varphi(x)\right)L(x)\begin{pmatrix}\bar X^{1}_\mu(x) & \bar X^{2}_\mu(x) & \bar X^{3}_\mu(x)\\ \bar Y^{1}_\mu(x) & \bar Y^{2}_\mu(x) & \bar Y^{3}_\mu(x)\end{pmatrix}\Omega(x)^\dagger$$

$$Q_X=-\frac{1}{2}-\frac{5}{6}=-\frac{4}{3},Q_Y=\frac{1}{2}-\frac{5}{6}=-\frac{1}{3},Q_{\bar X}=\frac{4}{3},Q_{\bar Y}=\frac{1}{3}$$

$$\{24\}=\{8,1\}_0+\{1,3\}_0+\{1,1\}_0+\{\bar 3,2\}_{5/6}+\{3,\bar 2\}_{-5/6}$$

$$\{2\}\times\{\bar 2\}=\{2\}\times\{2\}=\{3\}+\{1\}$$
$$\{3\}\times\{\bar 3\}=\{8\}+\{1\}$$
$$\{5\}\times\{\bar 5\}=\{24\}+\{1\}$$

$$\{5\}=\{3,1\}_{-1/3}+\{1,2\}_{1/2}$$

$$\{5\}\times\{\bar 5\}=\left(\{3,1\}_{-1/3}+\{1,2\}_{1/2}\right)\times\left(\{\bar 3,1\}_{1/3}+\{1,\bar 2\}_{-1/2}\right)$$
$$=\{3,1\}_{-1/3}\times\{\bar 3,1\}_{1/3}+\{1,2\}_{1/2}\times\{\bar 3,1\}_{1/3}+\{3,1\}_{-1/3}\times\{1,\bar 2\}_{-1/2}+\{1,2\}_{1/2}\times\{1,\bar 2\}_{-1/2}$$

$$=\{8,1\}_0+\{1,1\}_0+\{\bar 3,2\}_{5/6}+\{3,\bar 2\}_{-5/6}+\{1,3\}_0+\{1,1\}_0$$

$$g_s=g=g_5,g'=\sqrt{\frac{3}{5}}g_5\ \Rightarrow\ e=\frac{gg'}{\sqrt{g^2+g'^2}}=\sqrt{\frac{3}{8}}g_5,\sin^2\ \theta_\mathrm{W}=\frac{3}{8}$$

$$D_\mu\Xi(x)=\partial_\mu\Xi(x)+\left[V_\mu(x),\Xi(x)\right].$$

$$V_{\mu\nu}(x)=\partial_\mu V_\nu(x)-\partial_\nu V_\mu(x)+\left[V_\mu(x),V_\nu(x)\right]$$

$$\mathcal{L}\left(\Xi,V_\mu\right)=\frac{1}{4}\mathrm{Tr}\left(D_\mu\Xi D_\mu\Xi\right)+V(\Xi)-\frac{1}{2g_5^2}\mathrm{Tr}\left(V_{\mu\nu}V_{\mu\nu}\right)$$

$$\frac{1}{4}\mathrm{Tr}\left(D_\mu\Xi D_\mu\Xi\right)=\frac{1}{4}\mathrm{Tr}\left(\left[V_\mu,\Xi_0\right]\left[V_\mu,\Xi_0\right]\right)=\frac{5}{24}g_5^2\mathcal{V}^2\left(\bar X_\mu X_\mu+\bar Y_\mu Y_\mu\right)$$

$$[V_\mu,\Xi_0]=\mathrm{i}\sqrt{\frac{5}{3}}\frac{g_5}{2}\mathcal{V}\begin{pmatrix}0_{2\times2} & -\bar{X}_\mu^1 & -\bar{X}_\mu^2 & & -\bar{X}_\mu^3\\ X_\mu^1 & Y_\mu^1 & -\bar{Y}_\mu^1 & -\bar{Y}_\mu^2 & -\bar{Y}_\mu^3\\ X_\mu^2 & Y_\mu^2 & & & \\ X_\mu^3 & Y_\mu^3 & & 0_{3\times3} & \end{pmatrix}$$

$$M_X=M_Y=\sqrt{\frac{5}{12}}g_5\mathcal{V}$$

$$\begin{pmatrix}u_L^1\\ d_L^1\end{pmatrix},u_R^1,d_R^1,\begin{pmatrix}u_L^2\\ d_L^2\end{pmatrix},u_R^2,d_R^2,\begin{pmatrix}u_L^3\\ d_L^3\end{pmatrix},u_R^3,d_R^3,\begin{pmatrix}\nu_L\\ e_L\end{pmatrix},e_R.$$

$$\{2\}\times\{2\}=\{3\}+\{1\}$$
$$\{3\}\times\{3\}=\{6\}+\{\bar{3}\}$$
$$\{5\}\times\{5\}=\{15\}+\{10\}$$

$$\begin{aligned}\{5\}\times\{5\}&=\left(\{3,1\}_{-1/3}+\{1,2\}_{1/2}\right)\times\left(\{3,1\}_{-1/3}+\{1,2\}_{1/2}\right)\\ &=\{3,1\}_{-1/3}\times\{3,1\}_{-1/3}+\{1,2\}_{1/2}\times\{3,1\}_{-1/3}\\ &+\{3,1\}_{-1/3}\times\{1,2\}_{1/2}+\{1,2\}_{1/2}\times\{1,2\}_{1/2}\\ &=\{6,1\}_{-2/3}+\{\bar{3},1\}_{-2/3}+\{3,2\}_{1/6}+\{3,2\}_{1/6}\\ &+\{1,3\}_1+\{1,1\}_1=\{15\}+\{10\}\end{aligned}$$

$$\{15\}=\{6,1\}_{-2/3}+\{3,2\}_{1/6}+\{1,3\}_1$$
$$\{10\}=\{\bar{3},1\}_{-2/3}+\{3,2\}_{1/6}+\{1,1\}_1$$

$$\left\{\begin{pmatrix}u_L^1\\ d_L^1\end{pmatrix},\begin{pmatrix}u_L^2\\ d_L^2\end{pmatrix},\begin{pmatrix}u_L^3\\ d_L^3\end{pmatrix}\right\}=\{3,2\}_{1/6}$$
$$\{u_R^1,u_R^2,u_R^3\}=\{3,1\}_{2/3},\{d_R^1,d_R^2,d_R^3\}=\{3,1\}_{-1/3}$$
$$\left\{\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\right\}=\{1,2\}_{-1/2},\{e_R\}=\{1,1\}_{-1}$$

$$\left\{\begin{pmatrix}u_L^1\\ d_L^1\end{pmatrix},\begin{pmatrix}u_L^2\\ d_L^2\end{pmatrix},\begin{pmatrix}u_L^3\\ d_L^3\end{pmatrix}\right\}=\{3,2\}_{1/6,}$$
$$\{{}^{c}u_R^1,{}^{c}u_R^2,{}^{c}u_R^3\}=\{\bar{3},1\}_{-2/3},\{{}^{c}d_R^1,{}^{c}d_R^2,{}^{c}d_R^3\}=\{\bar{3},1\}_{1/3},$$
$$\left\{\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\right\}=\{1,2\}_{-1/2},\{{}^{c}e_R\}=\{1,1\}_1.$$

$$\left\{\begin{pmatrix}u_L^1\\ d_L^1\end{pmatrix},\begin{pmatrix}u_L^2\\ d_L^2\end{pmatrix},\begin{pmatrix}u_L^3\\ d_L^3\end{pmatrix},{}^{c}u_R^1,{}^{c}u_R^2,{}^{c}u_R^3,{}^{c}e_R\right\}=\{3,2\}_{1/6}+\{\bar{3},1\}_{-2/3}+\{1,1\}_1=\{10\}$$

$$\chi(x)=\begin{pmatrix}0 & -{}^{c}e_R(x) & u_L^1(x) & u_L^2(x) & u_L^3(x)\\ {}^{c}e_R(x) & 0 & d_L^1(x) & d_L^2(x) & d_L^3(x)\\ -u_L^1(x) & -d_L^1(x) & 0 & {}^{c}u_R^3(x) & -{}^{c}u_R^2(x)\\ -u_L^2(x) & -d_L^2(x) & -{}^{c}u_R^3(x) & 0 & {}^{c}u_R^1(x)\\ -u_L^3(x) & -d_L^3(x) & {}^{c}u_R^2(x) & -{}^{c}u_R^1(x) & 0\end{pmatrix}=-\chi(x)^{\mathrm{T}},$$

$$\chi'(x) = \Upsilon(x)\chi(x)\Upsilon(x)^{\top}, \Upsilon(x) \in \mathrm{SU}(5)$$

$$\chi'(x)^{\top} = (\Upsilon(x)\chi(x)\Upsilon(x)^{\top})^{\top} = \Upsilon(x)\chi(x)^{\top}\Upsilon(x)^{\top} = -\Upsilon(x)\chi(x)\Upsilon(x)^{\top} = -\chi'(x)$$

$$D_{\mu}\chi(x) = \partial_{\mu}\chi(x) + V_{\mu}(x)\chi(x) + \chi(x)V_{\mu}(x)^{\top}$$

$$\left(D_{\mu}\chi(x)\right)^{\top} = -D_{\mu}\chi(x), D_{\mu}\chi'(x) = \Upsilon(x)D_{\mu}\chi(x)\Upsilon(x)^{\top}.$$

$$\begin{aligned} -{}^{\mathrm{c}}e_{\mathrm{R}}'(x)\mathrm{i}\tau^2 &= \begin{pmatrix} 0 & -{}^{\mathrm{c}}e_{\mathrm{R}}'(x) \\ {}^{\mathrm{c}}e_{\mathrm{R}}'(x) & 0 \end{pmatrix} \\ &= \exp\left(\mathrm{i}\frac{g'}{2}\varphi(x)\right)L(x)\begin{pmatrix} 0 & -{}^{\mathrm{c}}e_{\mathrm{R}}(x) \\ {}^{\mathrm{c}}e_{\mathrm{R}}(x) & 0 \end{pmatrix}\exp\left(\mathrm{i}\frac{g'}{2}\varphi(x)\right)L(x)^{\mathrm{T}} \\ &= -\exp\left(\mathrm{i}g'\varphi(x)\right){}^{\mathrm{c}}e_{\mathrm{R}}(x)L(x)\mathrm{i}\tau^2L(x)^{\mathrm{T}} = -\exp\left(\mathrm{i}g'\varphi(x)\right){}^{\mathrm{c}}e_{\mathrm{R}}(x)\mathrm{i}\tau^2 \end{aligned}$$

$$\begin{aligned} &\begin{pmatrix} u_{\mathrm{L}}^{1'}(x) & u_{\mathrm{L}}^{2'}(x) & u_{\mathrm{L}}^{3'}(x) \\ d_{\mathrm{L}}^{1'}(x) & d_{\mathrm{L}}^{2'}(x) & d_{\mathrm{L}}^{3'}(x) \end{pmatrix} = \\ &\exp\left(\mathrm{i}\frac{g'}{2}\varphi(x)\right)L(x)\begin{pmatrix} u_{\mathrm{L}}^{1}(x) & u_{\mathrm{L}}^{2}(x) & u_{\mathrm{L}}^{3}(x) \\ d_{\mathrm{L}}^{1}(x) & d_{\mathrm{L}}^{2}(x) & d_{\mathrm{L}}^{3}(x) \end{pmatrix}\exp\left(-\mathrm{i}\frac{g'}{3}\varphi(x)\right)\Omega(x)^{\mathrm{T}} = \\ &\exp\left(\mathrm{i}\frac{g'}{6}\varphi(x)\right)L(x)\begin{pmatrix} u_{\mathrm{L}}^{1}(x) & u_{\mathrm{L}}^{2}(x) & u_{\mathrm{L}}^{3}(x) \\ d_{\mathrm{L}}^{1}(x) & d_{\mathrm{L}}^{2}(x) & d_{\mathrm{L}}^{3}(x) \end{pmatrix}\Omega(x)^{\mathrm{T}} \end{aligned}$$

$$\begin{aligned} &\begin{pmatrix} 0 & {}^{\mathrm{c}}u_{\mathrm{R}}^{3'}(x) & -{}^{\mathrm{c}}u_{\mathrm{R}}^{2'}(x) \\ -{}^{\mathrm{c}}u_{\mathrm{R}}^{3'}(x) & 0 & {}^{\mathrm{c}}u_{\mathrm{R}}^{1'}(x) \\ {}^{\mathrm{c}}u_{\mathrm{R}}^{2'}(x) & -{}^{\mathrm{c}}u_{\mathrm{R}}^{1'}(x) & 0 \end{pmatrix} = \\ &\exp\left(-\mathrm{i}g'\frac{2}{3}\varphi(x)\right)\Omega(x)\begin{pmatrix} 0 & {}^{\mathrm{c}}u_{\mathrm{R}}^{3}(x) & -{}^{\mathrm{c}}u_{\mathrm{R}}^{2}(x) \\ -{}^{\mathrm{c}}u_{\mathrm{R}}^{3}(x) & 0 & {}^{\mathrm{c}}u_{\mathrm{R}}^{1}(x) \\ {}^{\mathrm{c}}u_{\mathrm{R}}^{2}(x) & -{}^{\mathrm{c}}u_{\mathrm{R}}^{1}(x) & 0 \end{pmatrix}\Omega(x)^{\mathrm{T}} \end{aligned}$$

$$\Omega_{ad}(x)\epsilon_{def}\,{}^{\mathrm{c}}u_{\mathrm{R}}^{f}(x)\Omega_{be}(x) = \epsilon_{abc}\Omega_{cf}(x)^{*\mathrm{c}}u_{\mathrm{R}}^{f}(x) = \epsilon_{abc}\,{}^{\mathrm{c}}u_{\mathrm{R}}^{\prime c}(x).$$

$$\epsilon_{def}\Omega_{ad}(x)\Omega_{be}(x) = \epsilon_{abc}\Omega_{cf}(x)^{*}$$

$$\left\{{}^{\mathrm{c}}d_{\mathrm{R}}^{1}, {}^{\mathrm{c}}d_{\mathrm{R}}^{2}, {}^{\mathrm{c}}d_{\mathrm{R}}^{3}\right\} = \{\overline{3}, 1\}_{1/3}, \left\{\begin{pmatrix} \nu_{\mathrm{L}} \\ e_{\mathrm{L}} \end{pmatrix}\right\} = \{1,2\}_{-1/2}$$

$$\{\overline{5}\} = \{\overline{3}, 1\}_{1/3} + \{1, \overline{2}\}_{-1/2}$$

$$\left\{{}^{\mathrm{c}}d_{\mathrm{R}}^{1}, {}^{\mathrm{c}}d_{\mathrm{R}}^{2}, {}^{\mathrm{c}}d_{\mathrm{R}}^{3}, \mathrm{i}\tau^2\begin{pmatrix} \nu_{\mathrm{L}} \\ e_{\mathrm{L}} \end{pmatrix}\right\} = \{\overline{3}, 1\}_{1/3} + \{1, \overline{2}\}_{-1/2} = \{\overline{5}\}$$

$$\psi(x) = \begin{pmatrix} e_{\mathrm{L}}(x) \\ -\nu_{\mathrm{L}}(x) \\ {}^{\mathrm{c}}d_{\mathrm{R}}^{1}(x) \\ {}^{\mathrm{c}}d_{\mathrm{R}}^{2}(x) \\ {}^{\mathrm{c}}d_{\mathrm{R}}^{3}(x) \end{pmatrix}, \psi'(x) = \Upsilon(x)^{*}\psi(x)$$

$$D_{\mu}\psi(x) = \partial_{\mu}\psi(x) + V_{\mu}(x)^{*}\psi(x), D_{\mu}\psi'(x) = \Upsilon(x)^{*}D_{\mu}\psi(x)$$

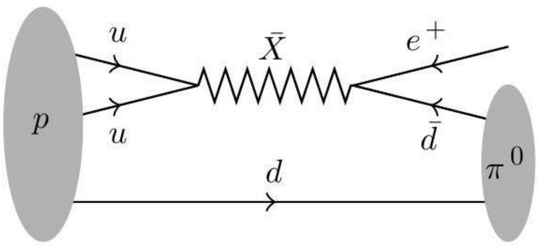
$$\begin{pmatrix} e_L'(x) \\ -\nu_L'(x) \end{pmatrix} = \exp\left(-\mathrm{i}\frac{g'}{2}\varphi(x)\right) L(x)^* \begin{pmatrix} e_L(x) \\ -\nu_L(x) \end{pmatrix} = \exp\left(-\mathrm{i}\frac{g'}{2}\varphi(x)\right) L(x)^* \mathrm{i}\tau^2 \begin{pmatrix} \nu_L(x) \\ e_L(x) \end{pmatrix}$$
$$= \exp\left(-\mathrm{i}\frac{g'}{2}\varphi(x)\right) \mathrm{i}\tau^2 L(x) \begin{pmatrix} \nu_L(x) \\ e_L(x) \end{pmatrix}$$

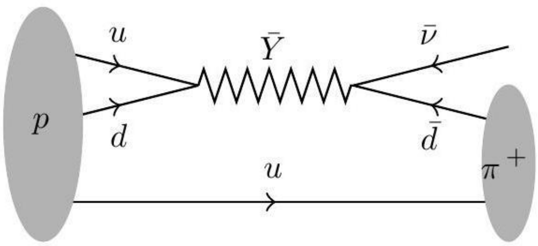
$$\begin{pmatrix} {}^c d_R^{1'}(x) \\ {}^c d_R^{2'}(x) \\ {}^c d_R^{3'}(x) \end{pmatrix} = \exp\left(\mathrm{i}\frac{g'}{3}\varphi(x)\right) \Omega(x)^* \begin{pmatrix} {}^c d_R^{1}(x) \\ {}^c d_R^{2}(x) \\ {}^c d_R^{3}(x) \end{pmatrix}$$

$$F = B - L - \frac{4}{5}Y, F_\chi = \frac{1}{5}, F_\psi = -\frac{3}{5}.$$

$$\mathcal{L}(\bar{\chi}, \chi, \bar{\psi}, \psi, V_\mu) = \mathrm{Tr}(\bar{\chi}\bar{\sigma}_\mu D_\mu \chi) + \bar{\psi}\bar{\sigma}_\mu D_\mu \psi$$

$$\mathcal{L}(\bar{\chi}, \chi, \bar{\psi}, \psi, V_\mu) - \mathcal{L}(\bar{\chi}, \chi, \bar{\psi}, \psi, V_\mu)\big|_{X_\mu = Y_\mu = 0} =$$
$$\mathrm{i}g_5 \left\{ (\bar{u}_L, \bar{d}_L)(\mathrm{i}\tau^2)\bar{\sigma}_\mu \begin{pmatrix} X_\mu \\ Y_\mu \end{pmatrix} {}^c e_R + {}^c\bar{e}_R \bar{\sigma}_\mu (\bar{X}_\mu, \bar{Y}_\mu)(-\mathrm{i}\tau^2) \begin{pmatrix} u_L \\ d_L \end{pmatrix} \right\}$$
$$-\mathrm{i}g_5 \epsilon_{abc} \left\{ (\bar{u}_L^a, \bar{d}_L^a)\bar{\sigma}_\mu \begin{pmatrix} \bar{X}_\mu^b \\ \bar{Y}_\mu^b \end{pmatrix} {}^c u_R^c + {}^c\bar{u}_R^a \bar{\sigma}_\mu (X_\mu^b, Y_\mu^b) \begin{pmatrix} u_L^c \\ d_L^c \end{pmatrix} \right\}$$
$$+\mathrm{i}\frac{g_5}{2} \left\{ (\bar{\nu}_L, \bar{e}_L)(\mathrm{i}\tau^2)\bar{\sigma}_\mu \begin{pmatrix} X_\mu \\ Y_\mu \end{pmatrix} {}^c d_R + {}^c\bar{d}_R \bar{\sigma}_\mu (\bar{X}_\mu, \bar{Y}_\mu)(-\mathrm{i}\tau^2) \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \right\}$$





$$d + e \to X, u + e \to Y, d + v \to Y, \bar{u} + \bar{u} \to X, \bar{u} + \bar{d} \to Y.$$

$$p \sim uud \to \bar{X}d \to \bar{d}d + \bar{e} \to \pi^0 + \bar{e}$$
$$p \sim uud \to u\bar{Y} \to u\bar{d} + \bar{v} \sim \pi^+ + \bar{v}$$

$$\tau_p \propto \frac{M_X^4}{M_p^5} \approx 10^{31} \text{ years.}$$

$$u + u \to \bar{X} \to e^+ + \bar{d}, u + d \to \bar{Y} \to \bar{\nu}_e + \bar{d}, u + d \to \bar{Y} \to \bar{u} + e^+.$$

$$\Phi(\vec{x}) = \begin{pmatrix} \Phi^+(\vec{x}) \\ \Phi^0(\vec{x}) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{v} \end{pmatrix}, W_i(\vec{x}) = 0$$

$$L(\vec{x}): \mathbb{R}^3 \to \mathrm{SU}(2)_L$$

$$L(\vec{x}): S^3 \to S^3$$

$$n[L] \in \Pi_3[\mathrm{SU}(2)_L] = \mathbb{Z}$$

$$\Phi^{(n)}(\vec{x}) = L_n(\vec{x})\begin{pmatrix}0\\ \mathrm{v}\end{pmatrix}, W_i^{(n)}(\vec{x}) = L_n(\vec{x})\partial_i L_n(\vec{x})^\dagger$$

$$n_{\mathrm{CS}}[W] = \int d^3x\Omega_4^{(0)} = -\frac{1}{8\pi^2}\int d^3x\epsilon_{ijk}\mathrm{Tr}\left[W_i\left(\partial_j W_k + \frac{2}{3}W_j W_k\right)\right]$$

$$n[L_n] = \frac{1}{24\pi^2}\int d^3x\epsilon_{ijk}\mathrm{Tr}\left[\left(L_n\partial_i L_n^\dagger\right)\left(L_n\partial_j L_n^\dagger\right)\left(L_n\partial_k L_n^\dagger\right)\right] = n \in \Pi_3[\mathrm{SU}(2)_L] = \mathbb{Z}$$

$$\Delta(B_{\mathrm{f}} + L_{\mathrm{f}}) = 0$$

$$\Delta(B_{\mathrm{f}} - L_{\mathrm{f}}) = \Delta(B_{\mathrm{i}} - L_{\mathrm{i}})$$

$$\Delta B_{\mathrm{f}} = -\Delta L_{\mathrm{f}} = \frac{1}{2}\Delta(B_{\mathrm{i}} - L_{\mathrm{i}})$$

$$\mathcal{M} = \{\Phi \mid \Phi \text{ is a global minimum of } V(\Phi)\}.$$

$$\Phi(z=-\infty), \Phi(z=\infty) \in \mathcal{M}$$

$$\Phi: S^0 \to \mathcal{M}$$

$$\Pi_0[\mathcal{M}] = \{0\}$$

$$\Pi_1[\mathcal{M}] = \Pi_1[\mathrm{U}(1)/\{\mathbb{1}\}] = \Pi_1[\mathrm{U}(1)] = \Pi_1[S^1] = \mathbb{Z}.$$

$$\Pi_1[\mathcal{M}] = \Pi_1[\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)/\mathrm{SU}(3)\times\mathrm{U}(1)] = \Pi_1[\mathrm{SU}(2)] = \Pi_1[S^3] = \{0\}$$

$$\Pi_2[\mathcal{M}] = \Pi_2[G/H] = \Pi_2[\mathrm{SU}(2)/\mathrm{U}(1)] = \Pi_2[S^2] = \mathbb{Z}.$$

$$\Pi_2[\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)/\mathrm{SU}(3)\times\mathrm{U}(1)] = \Pi_2[\mathrm{SU}(2)] = \Pi_2[S^3] = \{0\}.$$

$$\begin{aligned}\Pi_2[\mathrm{SU}(5)/\mathrm{SU}(3)\times\mathrm{SU}(2)\times U(1)] &= \Pi_1[\mathrm{SU}(3)\times\mathrm{SU}(2)\times U(1)]\\ &= \Pi_1[U(1)] = \Pi_1[S^1] = \mathbb{Z}\end{aligned}$$

$$\Xi(x) = \Xi_a(x)\tau^a.$$

$$\mathcal{L}(\Xi, V) = \frac{1}{4}\mathrm{Tr}\left[D_\mu\Xi D^\mu\Xi\right] - V(\Xi) + \frac{1}{2e^2}\mathrm{Tr}\left[V_{\mu\nu}V^{\mu\nu}\right]$$

$$\begin{aligned}&V_{\mu\nu}(x) = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) + \left[V_\mu(x), V_\nu(x)\right], V_\mu(x) = \mathrm{i}eV_\mu^a(x)\frac{\tau^a}{2}\\ &D_\mu\Xi(x) = \partial_\mu\Xi(x) + \left[V_\mu(x), \Xi(x)\right].\end{aligned}$$

$$\Xi_a(r,\theta,\varphi) = \xi(r)\frac{x^a}{r}, V_i^a(r,\theta,\varphi) = \mathrm{v}(r)\epsilon_{iab}\frac{x^b}{r^2}$$

$$\Upsilon(x)(\Xi_3(x)\tau^3)\Upsilon(x)^\dagger = \Xi_3(x)\tau^3, \Upsilon(x) = \begin{pmatrix}\exp\left(\mathrm{i}e\alpha(x)/2\right) & 0\\ 0 & \exp\left(-\mathrm{i}e\alpha(x)/2\right)\end{pmatrix}$$

$$A_\mu(x) = V_\mu^3(x), X^\pm(x) = \frac{1}{\sqrt{2}}\left(V_\mu^1(x) \mp \mathrm{i}V_\mu^2(x)\right)$$

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x), X_\mu^{\pm'}(x) = \exp\left(\pm \mathrm{i}e\alpha(x)\right) X_\mu^{\pm}(x)$$

$$B_i(\vec{x}) = \frac{1}{2}\epsilon_{ijk}F^{jk}(\vec{x}) = \frac{g}{4\pi}\frac{x_i}{r^3}, g = \frac{2\pi}{e}$$

$$\Pi_2[\mathrm{SU}(2)/\mathrm{U}(1)] = \Pi_2[S^3/S^1] = \Pi_1[S^1] = \mathbb{Z}$$

$$m_\mu(x) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma}(x)$$

$$\partial^\mu j_\mu^{\mathrm{GW}}(x) = -\frac{\mathrm{ie}}{8\pi^2} m_\mu(x) \mathrm{Tr}\left[T^3\left(D^\mu U(x) U(x)^\dagger + U(x)^\dagger D^\mu U(x)\right)\right]$$

$$m_0(\vec{x}, t) = g\delta(\vec{x}), m_i(\vec{x}, t) = 0$$

$$\vec{A}(\vec{x}) = g\frac{1 - \cos\,\theta}{r\sin\,\theta}\vec{e}_\varphi$$

$$\partial_t B(t) = \frac{eg}{\pi F_\pi}\partial_t \pi^0(\vec{0}, t)$$

$$B(\infty) - B(-\infty) = \frac{1}{2\pi F_\pi}\left[\pi^0(\vec{0}, \infty) - \pi^0(\vec{0}, -\infty)\right]$$

$$eg = 2\pi n\hbar c, n \in \mathbb{Z}$$

$$\partial_\mu F^{\mu\nu}(x) = \frac{1}{c}j^\nu(x), \partial_\mu \tilde{F}^{\mu\nu}(x) = \frac{1}{c}m^\nu(x), \tilde{F}_{\mu\nu}(x) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}(x)$$

$$\begin{pmatrix} F^{\mu\nu\prime}(x) \\ \tilde{F}^{\mu\nu}(x) \end{pmatrix} = \begin{pmatrix} \cos\,\gamma & \sin\,\gamma \\ -\sin\,\gamma & \cos\,\gamma \end{pmatrix} \begin{pmatrix} F^{\mu\nu}(x) \\ \tilde{F}^{\mu\nu}(x) \end{pmatrix},$$
$$\begin{pmatrix} j^{\mu\prime}(x) \\ m^{\mu\prime}(x) \end{pmatrix} = \begin{pmatrix} \cos\,\gamma & \sin\,\gamma \\ -\sin\,\gamma & \cos\,\gamma \end{pmatrix} \begin{pmatrix} j^\mu(x) \\ m^\mu(x) \end{pmatrix}.$$

$$\begin{pmatrix} \cos\,\gamma & \sin\,\gamma \\ -\sin\,\gamma & \cos\,\gamma \end{pmatrix} \begin{pmatrix} e_1 \\ g_1 \end{pmatrix} = \begin{pmatrix} e'_1 \\ 0 \end{pmatrix} \Rightarrow e_1 \sin\,\gamma = g_1 \cos\,\gamma$$

$$\begin{pmatrix} \cos\,\gamma & \sin\,\gamma \\ -\sin\,\gamma & \cos\,\gamma \end{pmatrix} \begin{pmatrix} e_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} e_2 \cos\,\gamma + g_2 \sin\,\gamma \\ -e_2 \sin\,\gamma + g_2 \cos\,\gamma \end{pmatrix} = \begin{pmatrix} e'_2 \\ g'_2 \end{pmatrix}.$$

$$\begin{aligned} e'_1 g'_2 &= (e_1 \cos\,\gamma + g_1 \sin\,\gamma)(-e_2 \sin\,\gamma + g_2 \cos\,\gamma) \\ &= -e_1 e_2 \cos\,\gamma \sin\,\gamma + e_1 g_2 \cos^2\,\gamma - g_1 e_2 \sin^2\,\gamma + g_1 g_2 \sin\gamma\cos\gamma \\ &= -g_1 e_2 \cos^2\,\gamma + e_1 g_2 \cos^2\,\gamma - g_1 e_2 \sin^2\,\gamma + e_1 g_2 \sin^2\,\gamma \\ &= e_1 g_2 - e_2 g_1 = 2\pi n\hbar c, n \in \mathbb{Z} \end{aligned}$$

$$\vec{E}(\vec{x}) = \frac{e_1}{4\pi}\frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3} + \frac{e_2}{4\pi}\frac{\vec{x} - \vec{x}_2}{|\vec{x} - \vec{x}_2|^3}, \vec{B}(\vec{x}) = \frac{g_1}{4\pi}\frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3} + \frac{g_2}{4\pi}\frac{\vec{x} - \vec{x}_2}{|\vec{x} - \vec{x}_2|^3}$$

$$\vec{J} = \frac{1}{c}\int d^3x\vec{x} \times (\vec{E}(\vec{x}) \times \vec{B}(\vec{x}))$$
$$= \frac{e_1g_2 - e_2g_1}{16\pi^2 c}\int d^3x\vec{x} \times \left(\frac{\vec{x}-\vec{x}_1}{|\vec{x}-\vec{x}_1|^3} \times \frac{\vec{x}-\vec{x}_2}{|\vec{x}-\vec{x}_2|^3}\right)$$
$$= \frac{e_1g_2 - e_2g_1}{16\pi^2 c}\int d^3x \frac{\vec{x} \times (\vec{x} \times \vec{d})}{|\vec{x}-\vec{x}_1|^3|\vec{x}-\vec{x}_2|^3}$$

$$\vec{J} \cdot \frac{\vec{d}}{d} = -\frac{e_1g_2 - e_2g_1}{16\pi^2 c}\int d^3x \frac{|\vec{x} \times \vec{d}|^2}{d|\vec{x}-\vec{x}_1|^3|\vec{x}-\vec{x}_2|^3} = -\frac{e_1g_2 - e_2g_1}{4\pi c}$$

$$\mathcal{L}_\theta(V) = -\hbar\frac{\theta}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[V^{\mu\nu}V^{\rho\sigma}].$$

$$X_\mu^{\pm'}(x) = \exp(\pm \mathrm{i}e\alpha)X_\mu^{\pm}(x).$$

$$L(\alpha, \dot{\alpha}) = \int d^3x[\mathcal{L}(\Xi, V) + \mathcal{L}_\theta(V)] = \frac{I}{2}\dot{\alpha}^2 + \hbar\frac{\theta}{2\pi}e\dot{\alpha}$$

$$p_\alpha = \frac{\delta L(\alpha, \dot{\alpha})}{\delta\dot{\alpha}} = I\dot{\alpha} + \hbar\frac{\theta}{2\pi}e.$$

$$H = p_\alpha\dot{\alpha} - L = \frac{I}{2}\dot{\alpha}^2 = \frac{1}{2I}\left(p_\alpha - \hbar\frac{\theta}{2\pi}e\right)^2$$

$$\hat{p}_\alpha = -\mathrm{i}\hbar\partial_\alpha$$

$$\hat{H} = \frac{\hbar^2}{2I}\left(-\mathrm{i}\partial_\alpha - \frac{\theta}{2\pi}e\right)^2 = \frac{\hbar^2\hat{q}^2}{2I}.$$

$$\hat{q} = -\mathrm{i}\partial_\alpha - \frac{\theta}{2\pi}e$$

$$\hat{q}\Psi_n(\alpha) = \left(n - \frac{\theta}{2\pi}\right)e\Psi_n(\alpha)$$

$$\Phi(x) = \begin{pmatrix} \Phi^+(x) \\ \Phi^0(x) \\ \phi^1(x) \\ \phi^2(x) \\ \phi^3(x) \end{pmatrix} \in \mathbb{C}^5, \Phi'(x) = \Upsilon(x)\Phi(x)$$

$$F = B - L - \frac{4}{5}Y, F_\Phi = -\frac{2}{5}$$

$$F_\Xi = 0$$

$$\{5\} = \{3\}_{-1} + \{1\}_1 + \{1\}_0.$$

$$\{\bar{5}\} \times \{\bar{5}\} = \{\overline{15}\} + \{\overline{10}\}$$
$$\{\bar{5}\} \times \{10\} = \{5\} + \{45\}$$
$$\{10\} \times \{10\} = \{\bar{5}\} + \{\overline{45}\} + \{50\}$$

$$\mathcal{L}(\bar{\chi},\chi,\bar{\psi},\psi,\Phi) = -f_d({}^{c}\bar{\psi}_a\chi_{ab}\Phi_b^* + \Phi_a^{*c}\bar{\chi}_{ab}\psi_b) + f_u\frac{1}{4}\epsilon_{abcde}\,{}^{c}\bar{\chi}_{ab}\chi_{cd}\Phi_e$$

$$\epsilon_{abcde}\Upsilon_{af}\Upsilon_{bg}\Upsilon_{ch}\Upsilon_{di} = \epsilon_{fghij}\Upsilon_{ej}^*$$

$$F_\psi + F_\chi - F_\Phi = 0, 2F_\chi + F_\Phi = 0$$

$$\mathcal{L}(\bar{\chi},\chi,\bar{\psi},\psi,\mathrm{v}) = f_d\mathrm{v}\left(\bar{e}_\mathrm{L}e_\mathrm{R} + \bar{e}_\mathrm{R}e_\mathrm{L} + \bar{d}_\mathrm{L}d_\mathrm{R} + \bar{d}_\mathrm{R}d_\mathrm{L}\right) + f_u\mathrm{v}(\bar{u}_\mathrm{L}u_\mathrm{R} + \bar{u}_\mathrm{R}u_\mathrm{L})$$

$$m_e = m_d = f_d\mathrm{v}, m_u = f_u\mathrm{v}, m_\nu = 0$$

$$\{54\} = \{24\}_0 + \{15\}_{-4/5} + \{\overline{15}\}_{4/5} \;\Rightarrow\; \mathrm{Tr}(F) = 15\left(-\frac{4}{5}+\frac{4}{5}\right) = 0$$

$$\{45\} = \{24\}_0 + \{10\}_{-4/5} + \{\overline{10}\}_{4/5} + \{1\}_0 \;\Rightarrow\; \mathrm{Tr}(F) = 10\left(-\frac{4}{5}+\frac{4}{5}\right) = 0$$

$$\{10\} = \{5\}_{-2/5} + \{\bar{5}\}_{2/5} \Rightarrow \mathrm{Tr}(F) = 5\left(-\frac{2}{5}+\frac{2}{5}\right) = 0$$

$$\{16\} = \{10\}_{1/5} + \{\bar{5}\}_{-3/5} + \{1\}_1 \Rightarrow \mathrm{Tr}(F) = 10\cdot\frac{1}{5} - 5\cdot\frac{3}{5} + 1 = 0$$

$$\mathrm{Spin}(10) \underset{\{16\}}{\longrightarrow} \mathrm{SU}(5) \underset{\{45\}}{\longrightarrow} \mathrm{SU}(3)_\mathrm{c}\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y \underset{\{10\}}{\longrightarrow} \mathrm{SU}(3)_\mathrm{c}\times\mathrm{U}(1)_\mathrm{em}.$$

$$\mathrm{Spin}(10) \supset \mathrm{Spin}(6)\times\mathrm{Spin}(4) = \mathrm{SU}(4)\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R.$$

$$\begin{aligned}\{54\} &= \{20,1,1\} + \{6,2,2\} + \{1,3,3\} + \{1,1,1\}\\ \{45\} &= \{15,1,1\} + \{6,2,2\} + \{1,3,1\} + \{1,1,3\}\\ \{16\} &= \{4,2,1\} + \{\bar{4},1,2\}\\ \{10\} &= \{6,1,1\} + \{1,2,2\}\end{aligned}$$

$$\begin{aligned}\mathrm{Spin}(10) &\underset{\{54\}}{\to} \mathrm{Spin}(6)\times\mathrm{Spin}(4) = \mathrm{SU}(4)\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\\ &\underset{\{45\}}{\to} \mathrm{SU}(3)_\mathrm{c}\times\mathrm{U}(1)_{B-L}\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\\ &\underset{\{16\}}{\to} \mathrm{SU}(3)_\mathrm{c}\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y\\ &\underset{\{10\}}{\to} \mathrm{SU}(3)_\mathrm{c}\times\mathrm{U}(1)_\mathrm{em}\end{aligned}$$

$$\mathrm{SU}(5) = \mathrm{E}(4) \subset \mathrm{Spin}(10) = \mathrm{E}(5) \subset \mathrm{E}(6) \subset \mathrm{E}(7) \subset \mathrm{E}(8)$$

$$\{27\} = \{1\} + \{10\} + \{16\}$$

$$\{16\}\times\{16\} = \{10\} + \{120\} + \{126\}.$$

PARTE IV.

$$q_\mathrm{L} = \begin{pmatrix}p_\mathrm{L}\\ n_\mathrm{L}\end{pmatrix}, p_\mathrm{R}, n_\mathrm{R}, l_\mathrm{L} = \begin{pmatrix}\nu_\mathrm{L}\\ e_\mathrm{L}\end{pmatrix}, e_\mathrm{R}$$

$$\left\{\begin{pmatrix}p_\mathrm{L}\\ n_\mathrm{L}\end{pmatrix}, {}^{\mathrm{c}}p_\mathrm{R}\right\} = \{2\}_{1/2} + \{1\}_{-1} = \{3\}, \{n_\mathrm{R}\} = \{1\}_0 = \{1\},$$

$$\left\{\mathrm{i}\tau^2\begin{pmatrix}\nu_\mathrm{L}\\ e_\mathrm{L}\end{pmatrix}, {}^{\mathrm{c}}e_\mathrm{R}\right\} = \{\bar{2}\}_{-1/2} + \{1\}_1 = \{\bar{3}\}.$$

$$\chi(x)=\begin{pmatrix} p_{\mathrm{L}}(x)\\ n_{\mathrm{L}}(x)\\ -{}^{\mathrm{c}}p_{\mathrm{R}}(x)\end{pmatrix}, \chi'(x)=\Upsilon(x)\chi(x),$$

$$\psi(x)=\begin{pmatrix} e_{\mathrm{L}}(x)\\ -\nu_{\mathrm{L}}(x)\\ -{}^{\mathrm{c}}e_{\mathrm{R}}(x)\end{pmatrix}, \psi'(x)=\Upsilon(x)^{*}\psi(x).$$

$$F=B-L-\frac{4}{3}Y$$

$$F_{\chi}=\frac{1}{3}, F_{n_{\mathrm{R}}}=1, F_{\psi}=-\frac{1}{3}, F_{\Phi}=-\frac{2}{3}, F_{\Xi}=0$$

$$\mathrm{SU}(3)\underset{\{8\}}{\longrightarrow}\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y\underset{\{3\}}{\longrightarrow}\mathrm{U}(1)_{\mathrm{em}}$$

$$\{8\}=\{3\}_0+\{2\}_1+\{2\}_{-1}+\{1\}_0$$

$$\{3\}=\{1\}_1+\{1\}_0+\{1\}_{-1},$$

$$\{3\}\times\{3\}=\{\bar{3}\}+\{6\},\{\bar{3}\}\times\{\bar{3}\}=\{3\}+\{\bar{6}\},$$

$$\{3\}\times\{\bar{3}\}=\{1\}+\{8\}$$

$$\begin{aligned}\{14\}&=\{8\}+\{3\}+\{\bar{3}\}\\ &=\{3\}_0+\{2\}_1+\{2\}_{-1}+\{1\}_0+\{2\}_{1/3}+\{1\}_{-2/3}+\{2\}_{-1/3}+\{1\}_{2/3}\end{aligned}$$

$$\mathrm{G}(2)\underset{\{14\}}{\longrightarrow}\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y\underset{\{7\}}{\longrightarrow}\mathrm{U}(1)_{\mathrm{em}}$$

$$\{7\}=\{3\}+\{\bar{3}\}+\{1\}$$

$$\{7\}\times\{7\}=\{1\}+\{7\}+\{14\}+\{27\}$$

$$\left\{\begin{pmatrix}p_{\mathrm{L}}\\ n_{\mathrm{L}}\end{pmatrix}, {}^{\mathrm{c}}p_{\mathrm{R}}, {}^{\mathrm{c}}n_{\mathrm{R}}\right\}=\{3\}_{1/3}+\{1\}_{-1}=\{4\},$$

$$\left\{\mathrm{i}\tau^2\begin{pmatrix}\nu_{\mathrm{L}}\\ e_{\mathrm{L}}\end{pmatrix}, {}^{\mathrm{c}}e_{\mathrm{R}}, {}^{\mathrm{c}}\nu_{\mathrm{R}}\right\}=\{\bar{3}\}_{-1/3}+\{1\}_1=\{\bar{4}\}.$$

$$\{20\}=\{8\}_0+\{6\}_{-4/3}+\{\bar{6}\}_{4/3}\Rightarrow \mathrm{Tr}(F)=6\left(-\frac{4}{3}+\frac{4}{3}\right)=0$$

$$\{15\}=\{8\}_0+\{\bar{3}\}_{-4/3}+\{3\}_{4/3}+\{1\}_0\Rightarrow \mathrm{Tr}(F)=3\left(-\frac{4}{3}+\frac{4}{3}\right)=0$$

$$\{6\}=\{3\}_{-2/3}+\{\bar{3}\}_{2/3}\Rightarrow \mathrm{Tr}(F)=3\left(-\frac{2}{3}+\frac{2}{3}\right)=0$$

$$\{4\}=\{3\}_{1/3}+\{1\}_{-1}\Rightarrow \mathrm{Tr}(F)=3\cdot\frac{1}{3}-1=0$$

$$\mathrm{Spin}(6)\underset{\{4\}}{\longrightarrow}\mathrm{SU}(3)\underset{\{15\}}{\longrightarrow}\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y\underset{\{6\}}{\longrightarrow}\mathrm{U}(1)_{\mathrm{em}}.$$

$$\mathrm{Spin}(6)\supset\mathrm{Spin}(4)\times\mathrm{Spin}(2)=\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_{B-L}.$$

$$\{20\} = \{3,3\}_0 + \{2,2\}_2 + \{2,2\}_{-2} + \{1,1\}_0 + \{1,1\}_4 + \{1,1\}_{-4}$$
$$\{15\} = \{2,2\}_2 + \{2,2\}_{-2} + \{3,1\}_0 + \{1,3\}_0 + \{1,1\}_0$$
$$\{6\} = \{2,2\}_0 + \{1,1\}_2 + \{1,1\}_{-2}$$
$$\{4\} = \{2,1\}_1 + \{1,2\}_{-1}$$

$$\mathrm{Spin}(6) \underset{\{20\}}{\longrightarrow} \mathrm{Spin}(4) \times \mathrm{Spin}(2) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$$
$$\underset{\{15\}}{\longrightarrow} \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \underset{\{6\}}{\longrightarrow} \mathrm{U}(1)_{\mathrm{em}}.$$

$$\{4\} \times \{\bar{4}\} = \{1\} + \{15\},$$

$$\{4\} \times \{4\} = \{6\} + \{10\},$$

$$\{\bar{4}\} \times \{\bar{4}\} = \{6\} + \{\overline{10}\}.$$

$$\{8\} = \{4\} + \{\bar{4}\}.$$

$$\{8\} \times \{8\} = \{1\} + \{7\} + \{21\} + \{35\}.$$

$$1\ \mathrm{m} = 3.33564095 \times 10^{-9} c\ \mathrm{sec}.$$

$$c = 2.99792458 \times 10^{8}\ \mathrm{msec}^{-1},$$

$$\hbar = \frac{h}{2\pi} = 1.054571817 \times 10^{-34}\ \mathrm{kg\ m^2 sec^{-1}},$$

$$G = 6.67430(15) \times 10^{-11}\ \mathrm{m^3\ kg^{-1} sec^{-2}}$$

$$l_{\mathrm{Planck}} = \sqrt{\frac{G\hbar}{c^3}} = 1.61626(2) \times 10^{-35}\ \mathrm{m},$$

$$t_{\mathrm{Planck}} = \sqrt{\frac{G\hbar}{c^5}} = 5.39125(6) \times 10^{-44} \mathrm{sec}$$

$$M_{\mathrm{Planck}} = \sqrt{\frac{\hbar c}{G}} = 2.17643(2) \times 10^{-8}\ \mathrm{kg}.$$

$$M_p = 1.67262 \times 10^{-27}\ \mathrm{kg} = 7.6852 \times 10^{-20} M_{\mathrm{Planck}},$$

$$1\mathrm{eV} \simeq 1.602176634 \times 10^{-19}\ \mathrm{kg\ m^2 sec^{-2}}$$

$$M_p c^2 = 0.9382720882(3)\mathrm{GeV}$$

$$\hbar c \simeq 3.1615 \times 10^{-26}\ \mathrm{kg\ m^3 sec^{-2}},$$

$$1\mathrm{fm} = 10^{-15}\ \mathrm{m} \simeq (0.1973\mathrm{GeV})^{-1}$$

$$\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137.03599908(2)}.$$

$$F_g = G\frac{M_p^2}{R^2}, F_e = \frac{1}{4\pi}\frac{e^2}{R^2} \Rightarrow \frac{F_g}{F_e} = \frac{4\pi G M_p^2}{e^2} = \frac{1}{\alpha}\left(\frac{M_p}{M_{\text{Planck}}}\right)^2 \approx 10^{-36}$$

$$\text{v} \simeq 246\text{GeV} \simeq 2.02 \times 10^{-17} M_{\text{Planck}}$$

Particle type	Particle	Electric charge	Mass [GeV]	Life-time
scalar boson	Higgs particle	0	125.25(17)	$\simeq 1.9(7) \times 10^{-22}$sec
gauge bosons	photon γ	0	$< 10^{-27}$	stable
	$W^{\pm}$-bosons	± 1	80.377(12)	$3.16(6) \times 10^{-25}$sec
	Z-boson	0	91.1876(21)	$2.638(2) \times 10^{-25}$sec
leptons	neutrino ν_e	0	$< 0.8 \times 10^{-9}$	expected to be stable
	electron e	-1	$0.51099895000(15) \times 10^{-3}$	$> 6.6 \times 10^{28}$ years
baryons	proton p	1	0.93827208816(29)	$> 10^{31} \ldots 10^{34}$ years
	neutron n	0	0.9395654205(5)	878.4(5) sec
mesons	pion π^0	0	0.1349768(5)	$8.43(13) \times 10^{-17}$sec
	pions $\pi^{\pm}$	± 1	0.13957039(18)	$2.6033(5) \times 10^{-8}$sec

Lepton	Electric charge	Mass [GeV]	Life-time
electron-neutrino ν_e electron e	0 -1	$< 0.8 \times 10^{-9}$ $0.51099895000(15) \times 10^{-3}$	$> 6.6 \times 10^{28}$ years
muon-neutrino ν_μ muon μ	0 -1	$< 0.19 \times 10^{-3}$ $0.1056583755(23)$	$2.1969811(22) \times 10^{-6}$sec
tau-neutrino ν_τ tau-lepton τ	0 -1	$< 18.2 \times 10^{-3}$ $1.77686(12)$	$2.903(5) \times 10^{-13}$sec

Generation	Quark flavor	Electric charge	Mass [GeV]
1.	up$u$	2/3	$0.00216^{+0.00049}_{-0.00026}$
	down d	-1/3	$0.00467^{+0.00048}_{-0.00017}$
2.	charm c	2/3	1.27(2)
	strange s	-1/3	$0.0934^{+0.0086}_{-0.0034}$
3.	top$t$	2/3	172.7(3)
	bottom b	-1/3	4.18(3)

$$\Lambda_{\text{QCD}} = 0.332(14)\text{GeV}$$

$$\alpha_s(M_Z) = 0.1179(9)$$

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}$$

$$\frac{M_W}{M_Z} = \frac{g}{\sqrt{g^2 + g'^2}} = \cos\ \theta_\mathrm{W}$$

$$\sin^2\ \theta_\mathrm{W} = \frac{g'^2}{g^2 + g'^2} = 0.23121(4)$$

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, \vec{x}), x^0 = ct$$

$$s^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -\vec{x})$$

$$x_0 = x^0, x_1 = -x^1, x_2 = -x^2, x_3 = -x^3$$

$$\sum_{\mu=0}^{3} x_\mu x^\mu = x_\mu x^\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = s^2$$

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = g_{\mu\nu} x^\mu x^\nu$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x_\mu = g_{\mu\nu} x^\nu$$

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$g g^{-1} = \mathbb{1}$$

$$g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$$

$$x^\mu = g^{\mu\nu} x_\nu.$$

$$x'^\mu = \Lambda_\nu^\mu x^\nu$$

$$x'_\mu = g_{\mu\nu} x'^\nu = g_{\mu\nu} \Lambda_\rho^\nu x^\rho.$$

$$s'^2 = x'_\mu x'^\mu = g_{\mu\nu} \Lambda_\rho^\nu x^\rho \Lambda_\sigma^\mu x^\sigma$$

$$g_{\mu\nu} \Lambda_\rho^\nu \Lambda_\sigma^\mu = g_{\rho\sigma}$$

$$(\Lambda^\top)_\sigma^\mu g_{\mu\nu} \Lambda_\rho^\nu = g_{\sigma\rho}^\top,$$

$$\Lambda^\top g \Lambda = g^\top = g.$$

$$x'_\mu = g_{\mu\nu} \Lambda_\rho^\nu x^\rho = g_{\mu\nu} \Lambda_\rho^\nu g^{\rho\sigma} x_\sigma = [g \Lambda g^{-1}]_\mu^\sigma x_\sigma = x_\sigma([g \Lambda g^{-1}]^\top)_\mu^\sigma = x_\sigma (\Lambda^{-1})_\mu^\sigma$$

$$[g\Lambda g^{-1}]^{\top} = g^{-1}\Lambda^{\top} g = \Lambda^{-1}$$

$$x_{\nu} = x'_{\mu}\Lambda^{\mu}_{\nu}$$

$$(\Delta s)^2 = (x^0 - y^0)^2 - (x^1 - y^1)^2 - (x^2 - y^2)^2 - (x^3 - y^3)^2,$$

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + d^{\mu}, y'^{\mu} = \Lambda^{\mu}_{\nu} y^{\nu} + d^{\mu}.$$

$$\partial^{\mu} = (\partial^0, \partial^1, \partial^2, \partial^3) = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{1}{c}\partial_t, -\vec{\nabla}\right).$$

$$\partial^{\mu\prime} = \frac{\partial}{\partial x'_{\mu}} = \frac{\partial x_{\nu}}{\partial x'_{\mu}}\frac{\partial}{\partial x_{\nu}} = \Lambda^{\mu}_{\nu}\partial^{\nu}$$

$$\partial_{\mu} = (\partial_0, \partial_1, \partial_2, \partial_3) = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\partial_t, \vec{\nabla}\right).$$

$$\Box = \partial_{\mu}\partial^{\mu} = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$$

$$\partial_t \rho(\vec{x}, t) + \vec{\nabla} \cdot \vec{j}(\vec{x}, t) = 0.$$

$$j^{\mu}(x) = \big(c\rho(\vec{x}, t), j_x(\vec{x}, t), j_y(\vec{x}, t), j_z(\vec{x}, t)\big).$$

$$\partial_{\mu} j^{\mu}(x) = \frac{1}{c}\partial_t c\rho(\vec{x}, t) + \partial_x j_x(\vec{x}, t) + \partial_y j_y(\vec{x}, t) + \partial_z j_z(\vec{x}, t) = 0.$$

$$j'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} j^{\nu}(x) = \Lambda^{\mu}{}_{\nu} j^{\nu}(\Lambda^{-1}x').$$

$$A^{\mu}(x) = \big(\phi(\vec{x}, t), A_x(\vec{x}, t), A_y(\vec{x}, t), A_z(\vec{x}, t)\big)$$

$$A'^{\mu}(x') = \Lambda^{\mu}_{\nu} A^{\nu}(\Lambda^{-1}x')$$

$${}^{\alpha}\phi(\vec{x}, t) = \phi(\vec{x}, t) - \frac{1}{c}\partial_t \alpha(\vec{x}, t),\ {}^{\alpha}\vec{A}(\vec{x}, t) = \vec{A}(\vec{x}, t) + \vec{\nabla}\alpha(\vec{x}, t)$$

$${}^{\alpha}A^{\mu}(x) = A^{\mu}(x) - \partial^{\mu}\alpha(x)$$

$$\alpha'(x') = \alpha(\Lambda^{-1}x')$$

$$\frac{1}{c}\partial_t \phi(\vec{x}, t) + \vec{\nabla} \cdot \vec{A}(\vec{x}, t) = 0$$

$$\partial_{\mu} A^{\mu}(x) = 0$$

$$\frac{1}{c^2}\partial_t^2 \phi(\vec{x}, t) - \Delta\phi(\vec{x}, t) = \rho(\vec{x}, t), \frac{1}{c^2}\partial_t^2 \vec{A}(\vec{x}, t) - \Delta\vec{A}(\vec{x}, t) = \frac{1}{c}\vec{j}(\vec{x}, t),$$

$$\Box\, A^{\mu}(x) = \frac{1}{c} j^{\mu}(x)$$

$$\vec{E}(\vec{x}, t) = -\vec{\nabla}\phi(\vec{x}, t) - \frac{1}{c}\partial_t \vec{A}(\vec{x}, t), \vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t).$$

$$\partial_\mu A^\mu(x) = \frac{1}{c}\partial_t\phi(\vec{x},t) + \vec{\nabla}\cdot\vec{A}(\vec{x},t)$$

$$D^{\mu\nu}(x) = \partial^\mu A^\nu(x) + \partial^\nu A^\mu(x)$$

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$$

$$\begin{aligned}
{}^\alpha D^{\mu\nu}(x) &= \partial^\mu\, {}^\alpha A^\nu(x) + \partial^\nu\, {}^\alpha A^\mu(x)\\
&= \partial^\mu A^\nu(x) - \partial^\mu\partial^\nu\alpha(x) + \partial^\nu A^\mu(x) - \partial^\nu\partial^\mu\alpha(x) = D^{\mu\nu}(x) - 2\partial^\mu\partial^\nu\alpha(x),\\
{}^\alpha F^{\mu\nu}(x) &= \partial^\mu \alpha A^\nu(x) - \partial^\nu A^\mu(x)\\
&= \partial^\mu A^\nu(x) - \partial^\mu\partial^\nu\alpha(x) - \partial^\nu A^\mu(x) + \partial^\nu\partial^\mu\alpha(x) = F^{\mu\nu}(x),
\end{aligned}$$

$$\begin{aligned}
F^{01}(x) &= \partial^0 A^1(x) - \partial^1 A^0(x) = \frac{1}{c}\partial_t A_x(\vec{x},t) + \partial_x\phi(\vec{x},t) = -E_x(\vec{x},t),\\
F^{02}(x) &= \partial^0 A^2(x) - \partial^2 A^0(x) = \frac{1}{c}\partial_t A_y(\vec{x},t) + \partial_y\phi(\vec{x},t) = -E_y(\vec{x},t),\\
F^{03}(x) &= \partial^0 A^3(x) - \partial^3 A^0(x) = \frac{1}{c}\partial_t A_z(\vec{x},t) + \partial_z\phi(\vec{x},t) = -E_z(\vec{x},t),\\
F^{12}(x) &= \partial^1 A^2(x) - \partial^2 A^1(x) = -\partial_x A_y(\vec{x},t) + \partial_y A_x(\vec{x},t) = -B_z(\vec{x},t),\\
F^{23}(x) &= \partial^2 A^3(x) - \partial^3 A^2(x) = -\partial_y A_z(\vec{x},t) + \partial_z A_y(\vec{x},t) = -B_x(\vec{x},t),\\
F^{31}(x) &= \partial^3 A^1(x) - \partial^1 A^3(x) = -\partial_z A_x(\vec{x},t) + \partial_x A_z(\vec{x},t) = -B_y(\vec{x},t).
\end{aligned}$$

$$F^{\mu\nu}(x) = \begin{pmatrix} 0 & -E_x(\vec{x},t) & -E_y(\vec{x},t) & -E_z(\vec{x},t)\\ E_x(\vec{x},t) & 0 & -B_z(\vec{x},t) & B_y(\vec{x},t)\\ E_y(\vec{x},t) & B_z(\vec{x},t) & 0 & -B_x(\vec{x},t)\\ E_z(\vec{x},t) & -B_y(\vec{x},t) & B_x(\vec{x},t) & 0 \end{pmatrix}$$

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & E_x(\vec{x},t) & E_y(\vec{x},t) & E_z(\vec{x},t)\\ -E_x(\vec{x},t) & 0 & -B_z(\vec{x},t) & B_y(\vec{x},t)\\ -E_y(\vec{x},t) & B_z(\vec{x},t) & 0 & -B_x(\vec{x},t)\\ -E_z(\vec{x},t) & -B_y(\vec{x},t) & B_x(\vec{x},t) & 0 \end{pmatrix}$$

$$\vec{\nabla}\cdot\vec{E}(\vec{x},t) = \rho(\vec{x},t), \vec{\nabla}\times\vec{B}(\vec{x},t) - \frac{1}{c}\partial_t\vec{E}(\vec{x},t) = \frac{1}{c}\vec{j}(\vec{x},t)$$

$$\begin{aligned}
\partial_\mu F^{\mu 0}(x) &= \partial_x E_x(\vec{x},t) + \partial_y E_y(\vec{x},t) + \partial_z E_z(\vec{x},t)\\
&= \vec{\nabla}\cdot\vec{E}(\vec{x},t) = \rho(\vec{x},t)\\
\partial_\mu F^{\mu 1}(x) &= -\frac{1}{c}\partial_t E_x(\vec{x},t) + \partial_y B_z(\vec{x},t) - \partial_z B_y(\vec{x},t)\\
&= [\vec{\nabla}\times\vec{B}]_x(\vec{x},t) - \frac{1}{c}\partial_t E_x(\vec{x},t) = \frac{1}{c}j_x(\vec{x},t)\\
\partial_\mu F^{\mu 2}(x) &= -\frac{1}{c}\partial_t E_y(\vec{x},t) - \partial_x B_z(\vec{x},t) + \partial_z B_x(\vec{x},t)\\
&= [\vec{\nabla}\times\vec{B}]_y(\vec{x},t) - \frac{1}{c}\partial_t E_y(\vec{x},t) = \frac{1}{c}j_y(\vec{x},t)\\
\partial_\mu F^{\mu 3}(x) &= -\frac{1}{c}\partial_t E_z(\vec{x},t) + \partial_x B_y(\vec{x},t) - \partial_y B_x(\vec{x},t)\\
&= [\vec{\nabla}\times\vec{B}]_z(\vec{x},t) - \frac{1}{c}\partial_t E_z(\vec{x},t) = \frac{1}{c}j_z(\vec{x},t)
\end{aligned}$$

$$\partial_\mu F^{\mu\nu}(x) = \frac{1}{c} j^\nu(x)$$

$$\partial_\mu F^{\mu\nu}(x) = \partial_\mu(\partial^\mu A^\mu(x) - \partial^\nu A^\mu(x)) = \Box\, A^\nu(x) - \partial^\nu \partial_\mu A^\mu(x) = \frac{1}{c} j^\nu(x)$$

$$\vec{\nabla} \cdot \vec{B}(\vec{x},t) = 0, \vec{\nabla} \times \vec{E}(\vec{x},t) + \frac{1}{c}\partial_t \vec{B}(\vec{x},t) = 0$$

$$\vec{E}(\vec{x},t) \longrightarrow -\vec{B}(\vec{x},t), \vec{B}(\vec{x},t) \longrightarrow \vec{E}(\vec{x},t).$$

$$\tilde{F}^{\mu\nu}(x) = \begin{pmatrix} 0 & B_x(\vec{x},t) & B_y(\vec{x},t) & B_z(\vec{x},t) \\ -B_x(\vec{x},t) & 0 & -E_z(\vec{x},t) & E_y(\vec{x},t) \\ -B_y(\vec{x},t) & E_z(\vec{x},t) & 0 & -E_x(\vec{x},t) \\ -B_z(\vec{x},t) & -E_y(\vec{x},t) & E_x(\vec{x},t) & 0 \end{pmatrix}$$

$$\partial_\mu \tilde{F}^{\mu\nu}(x) = 0$$

$$\tilde{F}_{\mu\nu}(x) = \begin{pmatrix} 0 & -B_x(\vec{x},t) & -B_y(\vec{x},t) & -B_z(\vec{x},t) \\ B_x(\vec{x},t) & 0 & -E_z(\vec{x},t) & E_y(\vec{x},t) \\ B_y(\vec{x},t) & E_z(\vec{x},t) & 0 & -E_x(\vec{x},t) \\ B_z(\vec{x},t) & -E_y(\vec{x},t) & E_x(\vec{x},t) & 0 \end{pmatrix}.$$

$$\tilde{F}_{\mu\nu}(x) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}(x)$$

$$\tilde{F}_{01}(x) = \frac{1}{2}\epsilon_{01\rho\sigma} F^{\rho\sigma}(x) = \frac{1}{2}(\epsilon_{0123} F^{23}(x) - \epsilon_{0132} F^{32}(x)) = F^{23}(x) = -B_x(\vec{x},t)$$

$$\tilde{F}_{12}(x) = \frac{1}{2}\epsilon_{12\rho\sigma} F^{\rho\sigma}(x) = \frac{1}{2}(\epsilon_{1203} F^{03}(x) - \epsilon_{1230} F^{30}(x)) = F^{03}(x) = -E_z(\vec{x},t)$$

$$\partial^\mu \tilde{F}_{\mu\nu}(x) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma}(x) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\mu(\partial^\rho A^\sigma(x) - \partial^\sigma A^\rho(x)) = 0$$

$$\partial^\mu F^{\rho\sigma}(x) + \partial^\rho F^{\sigma\mu}(x) + \partial^\sigma F^{\mu\rho}(x) = 0$$

$$\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) = \frac{1}{2}\left(\vec{B}(\vec{x},t)^2 - \vec{E}(\vec{x},t)^2\right)$$

$$\frac{1}{4}\tilde{F}_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x) = \frac{1}{2}\left(\vec{E}(\vec{x},t)^2 - \vec{B}(\vec{x},t)^2\right)$$

$$\frac{1}{4}F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x) = \vec{E}(\vec{x},t)\cdot\vec{B}(\vec{x},t) = \frac{1}{4}\tilde{F}_{\mu\nu}(x)F^{\mu\nu}(x)$$

$$F^{\mu\nu'}(x') = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}(x) = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}(\Lambda^{-1}x').$$

$$F^{\mu\nu'}(x') = \Lambda^\mu_\rho F^{\rho\sigma}(\Lambda^{-1}x')\Lambda^{\top\nu}_\sigma \Rightarrow F'(x') = \Lambda F(\Lambda^{-1}x')\Lambda^\top.$$

$$\mathcal{L}(A) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_\nu j^\nu$$

$$S[A] = \int dtd^3x\mathcal{L}(A) = \int d^4x\frac{1}{c}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_\nu j^\nu\right).$$

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu(x)} - \frac{\delta\mathcal{L}}{\delta A_\nu(x)} = \partial_\mu F^{\mu\nu}(x) - \frac{1}{c}j^\nu(x) = 0.$$

$$\partial_\mu F^{\mu\nu}(x) = \frac{1}{c}j^\nu(x).$$

$$\mathcal{T}^{\mu\nu}(x) = -F^{\mu\rho}(x)F^\nu{}_\rho(x) - \mathcal{L}g^{\mu\nu},$$

$$\partial_\mu\mathcal{T}^{\mu\nu}(x) = 0.$$

$$\begin{aligned}\mathcal{T}^{00}(x) &= -F^{0\rho}(x)F^0_\rho(x) + \frac{1}{4}F_{\rho\sigma}(x)F^{\rho\sigma}(x)g^{00}\\ &= \vec{E}(\vec{x},t)^2 + \frac{1}{2}\left(\vec{B}(\vec{x},t)^2 - \vec{E}(\vec{x},t)^2\right) = \frac{1}{2}\left(\vec{E}(\vec{x},t)^2 + \vec{B}(\vec{x},t)^2\right)\end{aligned}$$

$$\begin{aligned}\mathcal{T}^{i0}(x) &= -F^{i\rho}(x)F^0_\rho(x) + \frac{1}{4}F_{\rho\sigma}(x)F^{\rho\sigma}(x)g^{i0}\\ &= \epsilon_{ijk}E_j(\vec{x},t)B_k(\vec{x},t) = [\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)]_i\end{aligned}$$

$$\partial_\mu\mathcal{T}^{\mu 0}(x) = \partial_0\mathcal{T}^{00}(x) + \partial_i\mathcal{T}^{i0}(x) = 0$$

$$\begin{aligned}&[\hat{P}_i,\hat{H}] = 0, [\hat{J}_i,\hat{H}] = 0, [\hat{G}_i,\hat{H}] = \mathrm{i}\hat{P}_i, [\hat{P}_i,\hat{P}_j] = 0, [\hat{J}_i,\hat{P}_j] = \mathrm{i}\epsilon_{ijk}\hat{P}_k\\ &[\hat{G}_i,\hat{P}_j] = \mathrm{i}\delta_{ij}\mathcal{M}, [\hat{J}_i,\hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{J}_k, [\hat{J}_i,\hat{G}_j] = \mathrm{i}\epsilon_{ijk}\hat{G}_k, [\hat{G}_i,\hat{G}_j] = 0\end{aligned}$$

$$[\hat{J}_i,\hat{V}_j] = \mathrm{i}\epsilon_{ijk}\hat{V}_k$$

$$\begin{aligned}&[\hat{P}_i,\hat{H}] = 0, [\hat{J}_i,\hat{H}] = 0, [\hat{K}_i,\hat{H}] = \mathrm{i}\hat{P}_i, [\hat{P}_i,\hat{P}_j] = 0, [\hat{J}_i,\hat{P}_j] = \mathrm{i}\epsilon_{ijk}\hat{P}_k\\ &[\hat{K}_i,\hat{P}_j] = \mathrm{i}\delta_{ij}\hat{H}, [\hat{J}_i,\hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{J}_k, [\hat{J}_i,\hat{K}_j] = \mathrm{i}\epsilon_{ijk}\hat{K}_k, [\hat{K}_i,\hat{K}_j] = -\mathrm{i}\epsilon_{ijk}\hat{J}_k\end{aligned}$$

$$\hat{P}_\mu = (\hat{H},\hat{P}_1,\hat{P}_2,\hat{P}_3)$$

$$\hat{M}_{\mu\nu} = \begin{pmatrix} 0 & \hat{K}_1 & \hat{K}_2 & \hat{K}_3\\ -\hat{K}_1 & 0 & \hat{J}_3 & -\hat{J}_2\\ -\hat{K}_2 & -\hat{J}_3 & 0 & \hat{J}_1\\ -\hat{K}_3 & \hat{J}_2 & -\hat{J}_1 & 0\end{pmatrix}$$

$$\begin{aligned}&[\hat{P}_\mu,\hat{P}_\nu] = 0, [\hat{M}_{\mu\nu},\hat{P}_\rho] = \mathrm{i}(g_{\nu\rho}\hat{P}_\mu - g_{\mu\rho}\hat{P}_\nu),\\ &[\hat{M}_{\mu\nu},\hat{M}_{\rho\sigma}] = \mathrm{i}(g_{\mu\sigma}\hat{M}_{\nu\rho} - g_{\mu\rho}\hat{M}_{\nu\sigma} - g_{\nu\sigma}\hat{M}_{\mu\rho} + g_{\nu\rho}\hat{M}_{\mu\sigma}),\end{aligned}$$

$$\hat{\mathcal{M}}^2 = \hat{P}_\mu\hat{P}^\mu = \hat{H}^2 - \hat{\vec{P}}^2$$

$$\hat{M}_{\mu\nu}\hat{M}^{\mu\nu} = 2\left(\hat{\vec{J}}^2 - \hat{\vec{K}}^2\right), \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{M}^{\mu\nu}\hat{M}^{\rho\sigma} = 4\hat{\vec{J}}\cdot\hat{\vec{K}}$$

$$\hat{I}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{M}^{\nu\rho}\hat{P}^\sigma$$

$$\hat{I}_0 = \hat{P}_i\hat{J}_i, \hat{I}_i = \epsilon_{ijk}\hat{K}_j\hat{P}_k - \hat{H}\hat{J}_i$$

$$[\hat{P}_\mu, \hat{I}_\nu] = 0, [\hat{M}_{\mu\nu}, \hat{I}_\rho] = \mathrm{i}(g_{\nu\rho}\hat{I}_\mu - g_{\mu\rho}\hat{I}_\nu).$$

$$[\hat{I}_\mu, \hat{I}_\nu] = \mathrm{i}\epsilon_{\mu\nu\rho\sigma}\hat{I}^\rho\hat{P}^\sigma$$

$$\hat{I}_\mu\hat{I}^\mu = \hat{I}_0^2 - \hat{I}_i^2 = -\hat{\mathcal{M}}^2\hat{\vec{J}}^2$$

$$H = \sum_{a=1}^{n_G} \omega^a T^a = \omega^a T^a$$

$$H_1 + H_2 = H, H_1 = \omega_1^a T^a, H_2 = \omega_2^a T^a, \omega_1^a + \omega_2^a = \omega^a$$

$$[H_1, H_2] = H_1H_2 - H_2H_1 = \omega_1^a\omega_2^b[T^a, T^b].$$

$$[T^a, T^b] = \mathrm{i}f_{abc}T^c$$

$$T^{a\dagger} = T^a$$

$$[T^a, T^b]^\dagger = (T^aT^b - T^bT^a)^\dagger = T^{b\dagger}T^{a\dagger} - T^{a\dagger}T^{b\dagger} = T^bT^a - T^aT^b = -[T^a, T^b],$$

$$\left[[T^a, T^b], T^c\right] + \left[[T^b, T^c], T^a\right] + \left[[T^c, T^a], T^b\right] = 0$$

$$[S^a, S^b] = \mathrm{i}f_{abc}S^c$$

$$[T^a, S^b] = \mathrm{i}f_{abc}S^c$$

$$T^a_{bc} = -\mathrm{i}f_{abc}$$

$$[T^a, T^b]_{de} = T^a_{df}T^b_{fe} - T^b_{df}T^a_{fe} = -f_{adf}f_{bfe} + f_{bdf}f_{afe}$$

$$-f_{adf}f_{bfe} + f_{bdf}f_{afe} = f_{abc}f_{cde}$$

$$\mathrm{i}\bar{T}^a = (\mathrm{i}T^a)^* = -\mathrm{i}(T^a)^\top \Rightarrow \bar{T}^a = -(T^a)^\top$$

$$[\bar{T}^a, \bar{T}^b] = \left[(T^a)^\top, (T^b)^\top\right] = (T^a)^\top(T^b)^\top - (T^b)^\top(T^a)^\top = (T^bT^a - T^aT^b)^\top$$
$$= -[T^a, T^b]^\top = -\mathrm{i}f_{abc}(T^c)^\top = \mathrm{i}f_{abc}\bar{T}^c$$

$$[T^a, T^b] = \mathrm{i}\epsilon_{abc}T^c$$

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} = -\mathrm{i}\hbar\vec{r} \times \vec{\nabla}$$

$$\hat{\vec{L}}^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle, \hat{L}^3|lm\rangle = \hbar m|lm\rangle, l \in \mathbb{N}, m \in \{-l, -l+1, \ldots, l\}.$$

$$\Psi(\theta,\varphi) = \langle \theta,\varphi \mid \Psi \rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} \langle \theta,\varphi \mid lm \rangle,$$

$$T^1 = T^2 = T^3 = 0$$

$$T^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, T^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$T^1 = \frac{\sigma^1}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T^2 = \frac{\sigma^2}{2} = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, T^3 = \frac{\sigma^3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, T^2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i}\sqrt{3} & 0 & 0 \\ \mathrm{i}\sqrt{3} & 0 & -2\mathrm{i} & 0 \\ 0 & 2\mathrm{i} & 0 & -\mathrm{i}\sqrt{3} \\ 0 & 0 & \mathrm{i}\sqrt{3} & 0 \end{pmatrix}$$

$$T^3 = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$UU^\dagger = U^\dagger U = \mathbb{1}$$

$$U = \exp\left(\mathrm{i}\omega^a T^a\right) = \exp\left(\frac{\mathrm{i}}{2}\vec{\omega}\cdot\vec{\sigma}\right) = \cos\left(\frac{\omega}{2}\right)\mathbb{1} + \mathrm{i}\sin\left(\frac{\omega}{2}\right)\frac{\vec{\omega}}{\omega}\cdot\vec{\sigma}$$

$$U = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}$$

$$OO^\top = O^\top O = \mathbb{1}, \det O = 1$$

$$O = \exp\left(\mathrm{i}\omega^a T^a\right) = \exp\begin{pmatrix} 0 & \omega^3 & -\omega^2 \\ -\omega^3 & 0 & \omega^1 \\ \omega^2 & -\omega^1 & 0 \end{pmatrix}$$

$$O_{ab} = \frac{1}{2}\mathrm{Tr}\left[U\sigma^a U^\dagger \sigma^b\right]$$

$$O_{ab} = \frac{1}{2}\mathrm{Tr}\left[\sigma^a\sigma^b\right] = \delta_{ab}$$

$$\frac{1}{2}\mathrm{Tr}\left[U^\dagger\sigma^a U\sigma^b\right] = \frac{1}{2}\mathrm{Tr}\left[U\sigma^b U^\dagger\sigma^a\right] = O_{ba} = O^\top_{ab}$$

$$UU^\dagger = U^\dagger U = \mathbb{1}, \det U = 1, VV^\dagger = V^\dagger V = \mathbb{1}, \det V = 1 \Rightarrow$$
$$UV(UV)^\dagger = UVV^\dagger U^\dagger = \mathbb{1}, \det(UV) = \det U \det V = 1.$$

$$U = \exp\left(\mathrm{i}H\right)$$

$$r = n - 1$$

$$\mathrm{Tr}\left[T^a T^b\right] = \frac{1}{2}\delta_{ab}$$

$$\mathrm{SU}(n) = S^3 \times S^5 \times \cdots \times S^{2n-1}$$

$$U = VW, W = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \tilde{U}_{11} & \tilde{U}_{12} & \dots & \tilde{U}_{1n-1} \\ 0 & \tilde{U}_{21} & \tilde{U}_{22} & \dots & \tilde{U}_{2n-1} \\ . & . & . & & . \\ . & . & . & & . \\ 0 & \tilde{U}_{n-11} & \tilde{U}_{n-12} & \dots & \tilde{U}_{n-1n-1} \end{pmatrix},$$

$$V = \begin{pmatrix} U_{11} & -U_{21}^* & -\frac{U_{31}^*(1+U_{11})}{1+U_{11}^*} & \dots & -\frac{U_{n1}^*(1+U_{11})}{1+U_{11}^*} \\ U_{21} & \frac{1+U_{11}^*-|U_{21}|^2}{1+U_{11}} & -\frac{U_{31}^*U_{21}}{1+U_{11}^*} & \dots & -\frac{U_{n1}^*U_{21}}{1+U_{11}^*} \\ U_{31} & -\frac{U_{21}^*U_{31}}{1+U_{11}} & \frac{1+U_{11}^*-|U_{31}|^2}{1+U_{11}^*} & \dots & -\frac{U_{n1}^*U_{31}}{1+U_{11}^*} \\ . & . & . & & . \\ . & . & . & & . \\ U_{n1} & -\frac{U_{21}^*U_{n1}}{1+U_{11}} & -\frac{U_{31}^*U_{n1}}{1+U_{11}^*} & \dots & \frac{1+U_{11}^*-|U_{n1}|^2}{1+U_{11}^*} \end{pmatrix} \in \mathrm{SU}(n)$$

$$\mathrm{SU}(n)/\mathrm{SU}(n-1) = S^{2n-1}$$

$$\mathbb{Z}(n) = \{\exp(2\pi \mathrm{i} m/n)\mathbb{1}, m = 1, \dots, n\}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix},$$

$$f_{abc} = \frac{1}{4\mathrm{i}}\mathrm{Tr}\left[[\lambda^a, \lambda^b]\lambda^c\right], d_{abc} = \frac{1}{4}\mathrm{Tr}[\{\lambda^a, \lambda^b\}\lambda^c]$$

abc	123	147	156	246	257	345	367	458	678
f_{abc}	1	1/2	−1/2	1/2	1/2	1/2	−1/2	$\sqrt{3}/2$	$\sqrt{3}/2$

abc	118	146	157	228	247	256	338	344

d_{abc}	$1/\sqrt{3}$	$1/2$	$1/2$	$1/\sqrt{3}$	$-1/2$	$1/2$	$1/\sqrt{3}$	$1/2$
abc	355	366	377	448	558	668	778	888
d_{abc}	$1/2$	$-1/2$	$-1/2$	$-1/(2\sqrt{3})$	$-1/(2\sqrt{3})$	$-1/(2\sqrt{3})$	$-1/(2\sqrt{3})$	$-1/\sqrt{3}$

$$\{\lambda^a,\lambda^b\}=\frac{4}{3}\delta_{ab}\mathbb{1}+2d_{abc}\lambda^c$$

$$T_\pm=T^1\pm \mathrm{i}T^2, V_\pm=T^4\pm \mathrm{i}T^5, U_\pm=T^6\pm \mathrm{i}T^7$$

$$T^3=\frac{\lambda^3}{2}, Y=\frac{\lambda^8}{\sqrt{3}}=\frac{2}{\sqrt{3}}T^8$$

$$\begin{aligned}
&[T^3,T_\pm]=\pm T_\pm, [T_+,T_-]=2T^3\\
&[T^3,V_\pm]=\pm\frac{1}{2}V_\pm, [V_+,V_-]=\frac{3}{2}Y+T^3\\
&[T^3,U_\pm]=\mp\frac{1}{2}U_\pm, [U_+,U_-]=\frac{3}{2}Y-T^3\\
&[Y,T^3]=[Y,T_\pm]=0, [Y,V_\pm]=\pm V_\pm, [Y,U_\pm]=\pm U_\pm\\
&[T_+,V_+]=[T_+,U_-]=[U_+,V_+]=0,\\
&[T_+,V_-]=-U_-, [T_+,U_+]=V_+, [U_+,V_-]=T_-
\end{aligned}$$

$$u=\left|\{3\}\frac{1}{3}\frac{1}{2}\frac{1}{2}\right\rangle, d=\left|\{3\}\frac{1}{3}\frac{1}{2}-\frac{1}{2}\right\rangle, s=\left|\{3\}-\frac{2}{3}00\right\rangle$$

$$\bar{u}=\left|\{\bar{3}\}-\frac{1}{3}\frac{1}{2}-\frac{1}{2}\right\rangle, \bar{d}=\left|\{\bar{3}\}-\frac{1}{3}\frac{1}{2}\frac{1}{2}\right\rangle, \bar{s}=\left|\{\bar{3}\}\frac{2}{3}00\right\rangle.$$

$$T_\pm|\{\Gamma\}YTT^3\rangle=\sqrt{T(T+1)-T^3(T^3\pm 1)}|\{\Gamma\}YTT^3\pm 1\rangle.$$

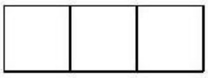
$$YV_\pm|\{\Gamma\}YTT^3\rangle=([Y,V_\pm]+V_\pm Y)|\{\Gamma\}YTT^3\rangle$$

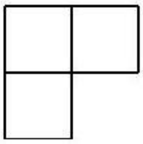
$$\begin{aligned}
T^3V_\pm|\{\Gamma\}YTT^3\rangle&=(\pm V_\pm+V_\pm Y)|\{\Gamma\}YTT^3\rangle=(Y\pm 1)V_\pm|\{\Gamma\}YTT^3\rangle=([T^3,V_\pm]+V_\pm T^3)|\{\Gamma\}YTT^3\rangle\\
&=\left(\pm\frac{1}{2}V_\pm+V_\pm T^3\right)|\{\Gamma\}YTT^3\rangle=\left(T^3\pm\frac{1}{2}\right)V_\pm|\{\Gamma\}YTT^3\rangle\\
YU_\pm|\{\Gamma\}YTT^3\rangle&=([Y,U_\pm]+U_\pm Y)|\{\Gamma\}YTT^3\rangle=(\pm U_\pm+U_\pm Y)|\{\Gamma\}YTT^3\rangle=(Y\pm 1)U_\pm|\{\Gamma\}YTT^3\rangle
\end{aligned}$$

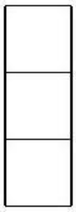
$$\begin{aligned}
T^3U_\pm|\{\Gamma\}YTT^3\rangle&=([T^3,U_\pm]+U_\pm T^3)|\{\Gamma\}YTT^3\rangle\\
&=\left(\mp\frac{1}{2}U_\pm+U_\pm T^3\right)|\{\Gamma\}YTT^3\rangle=\left(T^3\mp\frac{1}{2}\right)U_\pm|\{\Gamma\}YTT^3\rangle
\end{aligned}$$

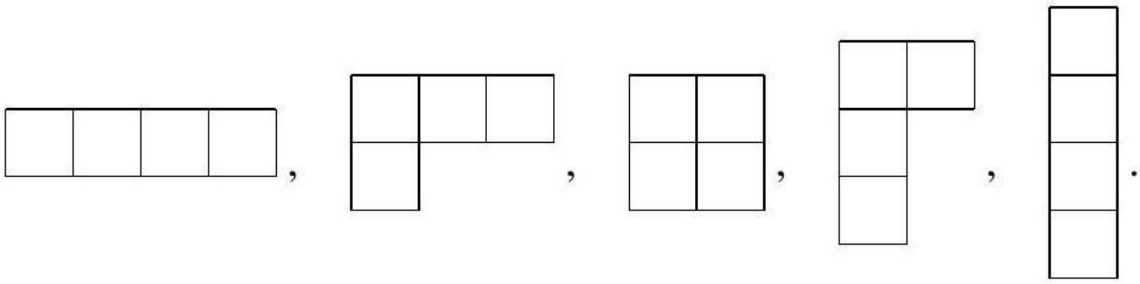
$$V_{\pm}|\{\Gamma\}YTT^3\rangle = \sum_{T'} C_{T'YTT^3}\left|\{\Gamma\}Y \pm 1, T', T^3 \pm \frac{1}{2}\right\rangle$$

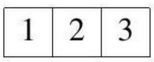
$$U_{\pm}|\{\Gamma\}YTT^3\rangle = \sum_{T'} C'_{T'YTT^3}\left|\{\Gamma\}Y \pm 1, T', T^3 \mp \frac{1}{2}\right\rangle$$

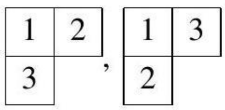

1-dimensional symmetric representation,


2-dimensional representation of mixed symmetry,


1-dimensional anti-symmetric representation.



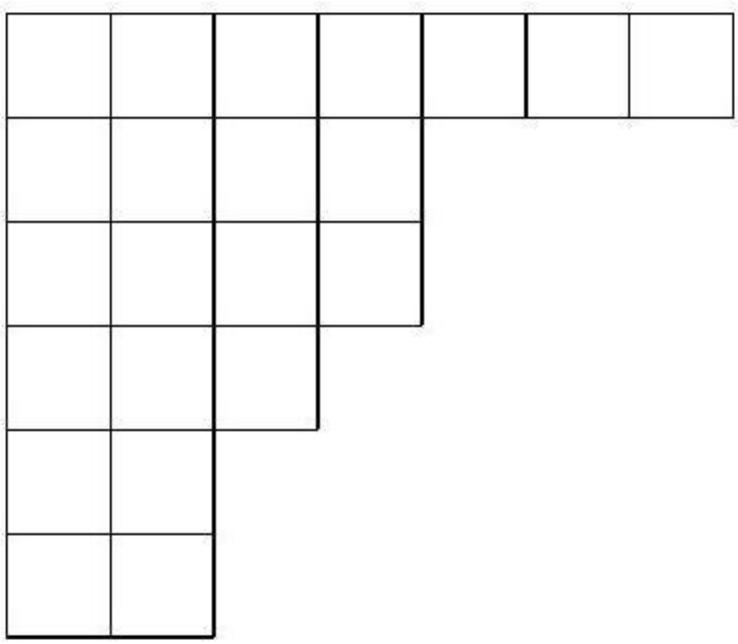


1-dimensional,


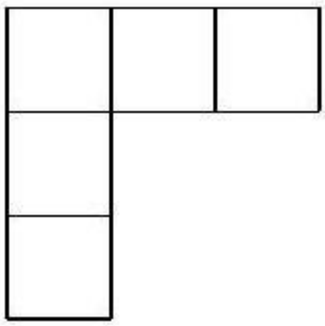
2-dimensional,


1-dimensional.

$$\sum_{\Gamma} d_{\Gamma}^{2} = N!$$

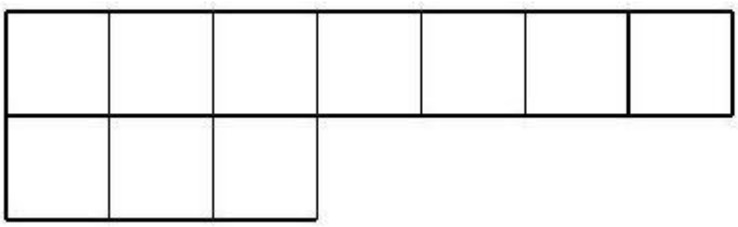


$$d_{m_1,m_2,\ldots,m_n} = N!\frac{\prod_{i<k}(l_i - l_k)}{l_1!\,l_2!\ldots l_n!}, l_i = m_i + n - i,$$



$$d_{3,1,1} = 5!\frac{(l_1 - l_2)(l_1 - l_3)(l_2 - l_3)}{l_1!\,l_2!\,l_3!} = 5!\frac{3 \cdot 4 \cdot 1}{5!\,2!\,1!} = 6.$$

 $= \{2\}.$



n	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$
$n-1$	n	$n+1$	$n+2$			
$n-2$	$n-1$	n	$n+1$			
$n-3$	$n-2$	$n-1$				
$n-4$	$n-3$					
$n-5$	$n-4$					

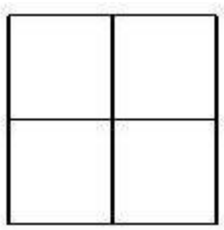
.

$$S = \frac{1}{2}(m_1 - m_2)$$

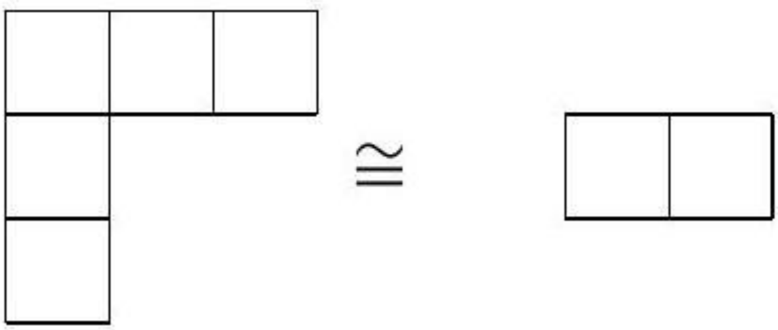
$$D^n_{m_1,m_2,\dots,m_n} = \frac{(n+m_1-1)!}{(n-1)!}\frac{(n+m_2-2)!}{(n-2)!}\dots\frac{m_n!}{0!}\frac{1}{N!}N!\frac{\prod_{i<k}(l_i-l_k)}{l_1!\,l_2!\dots l_n!}$$

$$= \frac{\prod_{i<k}(m_i-m_k-i+k)}{(n-1)!\,(n-2)!\dots 0!}$$

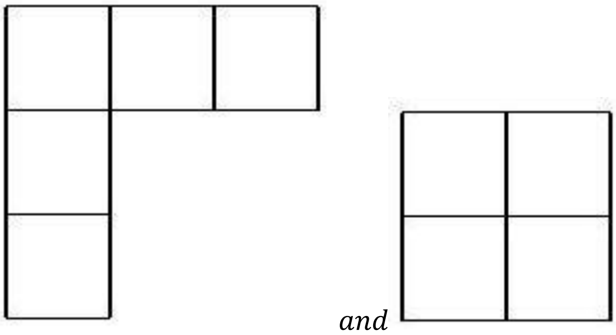
$$D^2_{m_1,m_2} = \frac{m_1-m_2-1+2}{1!\,0!} = m_1 - m_2 + 1 = q_1 + 1 = 2S + 1$$

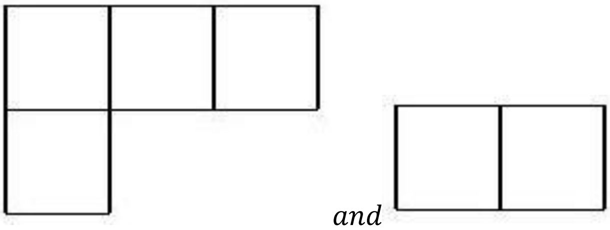


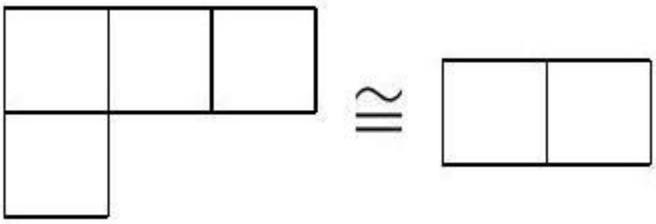
$$D^n_{m,m,\dots,m} = \frac{\prod_{i<k}(m_i-m_k-i+k)}{(n-1)!\,(n-2)!\dots 0!} = \frac{(n-1)!\,(n-2)!\dots 0!}{(n-1)!\,(n-2)!\dots 0!} = 1$$



$$\bar{m}_i = m_1 - m_{n-i+1}, \bar{q}_i = \bar{m}_i - \bar{m}_{i+1} = m_{n-i} - m_{n-i+1} = q_{n-i}$$







$$D^n_{\bar{m}_1,\bar{m}_2,\dots,\bar{m}_n} = D^n_{m_1,m_2,\dots,m_n}.$$

$$D^n_{2,1,1,\dots,1,0} = n^2 - 1.$$

$$\{n\} \times \{n\} \times \cdots \times \{n\} = \square \times \square \times \cdots \times \square.$$

$$\{n\} \times \{n\} \times \cdots \times \{n\} = \sum_{\Gamma} d_{m_1,m_2,\dots,m_n}\{D^n_{m_1,m_2,\dots,m_n}\}.$$

$$\square \times \square \times \square = \square + 2\square + \square$$

$$\{n\} \times \{n\} \times \{n\} = \left\{\frac{n(n+1)(n+2)}{6}\right\} + 2\left\{\frac{(n-1)n(n+1)}{3}\right\} + \left\{\frac{(n-2)(n-1)n}{6}\right\}$$

$$\frac{n(n+1)(n+2)}{6} + 2\frac{(n-1)n(n+1)}{3} + \frac{(n-2)(n-1)n}{6} = n^3$$

$$\{2\} \times \{2\} \times \{2\} = \{4\} + 2\{2\} + \{0\},$$

$$\{3\} \times \{3\} \times \{3\} = \{10\} + 2\{8\} + \{1\}$$

$$\{3\} \times \{\bar{3}\} = \{1\} + \{8\}$$

$$T_- u = d, T_+ d = u, U_- d = s, U_+ s = d, V_- u = s, V_+ s = u.$$

$$\begin{aligned} \bar{T}_- \bar{d} &= -\bar{u}, \quad \bar{T}_+ \bar{u} = -\bar{d}, \bar{U}_- \bar{s} = -\bar{d}, \bar{U}_+ \bar{s} = -\bar{u} \\ \bar{V}_- \bar{s} &= -\bar{u}, \quad \bar{V}_+ \bar{s} = -\bar{d} \end{aligned}$$

$$|\{8\}011\rangle = \left|\{3\}\frac{1}{3}\frac{1}{2}\frac{1}{2}\right\rangle \left|\{\bar{3}\} - \frac{1}{3}\frac{1}{2}\frac{1}{2}\right\rangle = u\bar{d}$$

$$-\sqrt{2}|\{8\}010\rangle = (T_- + \bar{T}_-)u\bar{d} = -u\bar{u} + d\bar{d} \;\Rightarrow\; |\{8\}010\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

$$\sqrt{2}|\{8\}01-1\rangle = (T_- + \bar{T}_-)\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) = \sqrt{2}d\bar{u} \;\Rightarrow\; |\{8\}010\rangle = d\bar{u}$$

$$-\left|\{8\}1\frac{1}{2}\frac{1}{2}\right\rangle = (U_+ + \bar{U}_+)u\bar{d} = -u\bar{s} \;\Rightarrow\; \left|\{8\}1\frac{1}{2}\frac{1}{2}\right\rangle = u\bar{s},$$

$$\left|\{8\}1\frac{1}{2} - \frac{1}{2}\right\rangle = d\bar{s}, \left|\{8\} - 1\frac{1}{2}\frac{1}{2}\right\rangle = s\bar{d}, \left|\{8\} - 1\frac{1}{2} - \frac{1}{2}\right\rangle = s\bar{u}.$$

$$(V_- + \bar{V}_-)\left|\{8\}1\frac{1}{2}\frac{1}{2}\right\rangle = (V_- + \bar{V}_-)u\bar{s} = s\bar{s} - u\bar{u} = \alpha|\{8\}000\rangle + \beta|\{8\}010\rangle.$$

$$|\{8\}000\rangle = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

$$|\{1\}000\rangle = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

$$\mathrm{Spin}(n) = S^1 \times S^2 \times \cdots \times S^{n-1}$$

$$S^3/S^1 = S^2, S^7/S^3 = S^4, S^{15}/S^7 = S^8$$

$$S^3 = S^1 \times S^2, S^7 = S^3 \times S^4, S^{15} = S^7 \times S^8$$

$$\begin{aligned} &\mathrm{Spin}(3) = S^1 \times S^2 = S^3 = \mathrm{SU}(2) \\ &\mathrm{Spin}(6) = S^1 \times S^2 \times S^3 \times S^4 \times S^5 = S^3 \times S^5 \times S^7 = \mathrm{SU}(4) \end{aligned}$$

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma^2 \otimes \mathbb{1}$$

$$U^* = JUJ^\dagger$$

$$U = \begin{pmatrix} W & X \\ -X^* & W^* \end{pmatrix}$$

$$H^* = -JHJ^\dagger = JHJ.$$

$$H = \begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix},$$

$$n_G = n^2 + (n+1)n = (2n+1)n.$$

$$\mathrm{Sp}(n) = S^3 \times S^7 \times \cdots \times S^{4n-1}$$

$$\mathrm{Sp}(1) = S^3 = \mathrm{SU}(2), \mathrm{Sp}(2) = S^3 \times S^7 = S^1 \times S^2 \times S^3 \times S^4 = \mathrm{Spin}(5)$$

$$\begin{aligned}\mathrm{Sp}(3) &= S^3 \times S^7 \times S^{11} = S^1 \times S^2 \times S^3 \times S^4 \times S^{11} \\ &\neq S^1 \times S^2 \times S^3 \times S^4 \times S^5 \times S^6 = \mathrm{Spin}(7)\end{aligned}$$

$$O_{ab}O_{ac} = \delta_{bc}$$

$$T_{abc} = T_{def}O_{da}O_{eb}O_{fc}.$$

$$T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1$$

$$\Lambda^a = \frac{1}{\sqrt{2}}\begin{pmatrix} \lambda^a & 0 & 0 \\ 0 & -\lambda^{a*} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\{7\} = \{3\} + \{\overline{3}\} + \{1\}$$

$$\begin{aligned}
T^+ &= \frac{1}{\sqrt{2}}(\Lambda^1 + \mathrm{i}\Lambda^2) = |1\rangle\langle 2| - |5\rangle\langle 4|, T^- = \frac{1}{\sqrt{2}}(\Lambda^1 - \mathrm{i}\Lambda^2) = |2\rangle\langle 1| - |4\rangle\langle 5|, \\
V^+ &= \frac{1}{\sqrt{2}}(\Lambda^4 + \mathrm{i}\Lambda^5) = |2\rangle\langle 3| - |6\rangle\langle 5|, V^- = \frac{1}{\sqrt{2}}(\Lambda^4 - \mathrm{i}\Lambda^5) = |3\rangle\langle 2| - |5\rangle\langle 6|, \\
U^+ &= \frac{1}{\sqrt{2}}(\Lambda^6 + \mathrm{i}\Lambda^7) = |1\rangle\langle 3| - |6\rangle\langle 4|, U^- = \frac{1}{\sqrt{2}}(\Lambda^6 - \mathrm{i}\Lambda^7) = |3\rangle\langle 1| - |4\rangle\langle 6|,
\end{aligned}$$

$$\begin{aligned}
X^+ &= \frac{1}{\sqrt{2}}(\Lambda^9 + \mathrm{i}\Lambda^{10}) = |2\rangle\langle 4| - |1\rangle\langle 5| - \sqrt{2}|7\rangle\langle 3| - \sqrt{2}|6\rangle\langle 7|, \\
X^- &= \frac{1}{\sqrt{2}}(\Lambda^9 - \mathrm{i}\Lambda^{10}) = |4\rangle\langle 2| - |5\rangle\langle 1| - \sqrt{2}|3\rangle\langle 7| - \sqrt{2}|7\rangle\langle 6|, \\
Y^+ &= \frac{1}{\sqrt{2}}(\Lambda^{11} + \mathrm{i}\Lambda^{12}) = |6\rangle\langle 1| - |4\rangle\langle 3| - \sqrt{2}|2\rangle\langle 7| - \sqrt{2}|7\rangle\langle 5|, \\
Y^- &= \frac{1}{\sqrt{2}}(\Lambda^{11} - \mathrm{i}\Lambda^{12}) = |1\rangle\langle 6| - |3\rangle\langle 4| - \sqrt{2}|7\rangle\langle 2| - \sqrt{2}|5\rangle\langle 7|, \\
Z^+ &= \frac{1}{\sqrt{2}}(\Lambda^{13} + \mathrm{i}\Lambda^{14}) = |3\rangle\langle 5| - |2\rangle\langle 6| - \sqrt{2}|7\rangle\langle 1| - \sqrt{2}|4\rangle\langle 7|, \\
Z^- &= \frac{1}{\sqrt{2}}(\Lambda^{13} - \mathrm{i}\Lambda^{14}) = |5\rangle\langle 3| - |6\rangle\langle 2| - \sqrt{2}|1\rangle\langle 7| - \sqrt{2}|7\rangle\langle 4|.
\end{aligned}$$

$$\{14\} = \{8\} + \{3\} + \{\overline{3}\}$$

$$Z = \begin{pmatrix} z\mathbb{1} & 0 & 0 \\ 0 & z^*\mathbb{1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\{7\} \times \{7\} = \{1\} + \{7\} + \{14\} + \{27\},$$

$$\begin{aligned}\{7\} \times \{14\} &= \{7\} + \{27\} + \{64\} \\ \{14\} \times \{14\} &= \{1\} + \{14\} + \{27\} + \{77\} + \{77'\}\end{aligned}$$

$$\mathrm{G}(2) = \mathrm{SU}(3) \times S^6 = S^3 \times S^5 \times S^6$$

$$\mathrm{SO}(7) = S^1 \times S^2 \times S^3 \times S^4 \times S^5 \times S^6 = \mathrm{G}(2) \times S^1 \times S^2 \times S^4 = \mathrm{G}(2) \times S^3 \times S^4 \\ = \mathrm{G}(2) \times S^7$$

$$\{40\} \times \{16\} = \{55\} + 2\{30\} + \{154\} + 3\{14\} + 2\{81\} + 3\{5\} + 3\{35\} + \{105\} + 2\{1\} + 3\{10\} + 2\{35'\} \\ -[\{35\} + \{30\} + \{10\} + 2\{14\} + 2\{5\} + \{1\}] - [\{1\} + \{10\} + \{35'\}] \\ = \{154\} + \{105\} + 2\{81\} + \{55\} + 2\{35\} + \{35'\} + \{30\} + \{14\} + \{10\} + \{5\}$$

$$n[U] = \frac{1}{2\pi \mathrm{i}} \int_{S^1} dx U(x)^* \partial_x U(x) = \frac{1}{2\pi} \int_0^L dx \partial_x \alpha(x) = \frac{1}{2\pi} [\alpha(L) - \alpha(0)] \\ \in \Pi_1[\mathrm{U}(1)] = \mathbb{Z}$$

$$n[U] = n[U_1] + n[U_2].$$

$$n[\vec{e}] = \frac{1}{8\pi} \int d^2x \epsilon_{\mu\nu} \vec{e} \cdot (\partial_\mu \vec{e} \times \partial_\nu \vec{e}) \in \Pi_2[S^2] = \mathbb{Z}$$

$$S[\vec{e}] = \frac{1}{2g^2} \int d^2x \partial_\mu \vec{e} \cdot \partial_\mu \vec{e}$$

$$0 \leq \int d^2x (\partial_\mu \vec{e} \pm \epsilon_{\mu\nu} \partial_\nu \vec{e} \times \vec{e})^2 \\ = \int d^2x [2\partial_\mu \vec{e} \cdot \partial_\mu \vec{e} \pm 2\epsilon_{\mu\nu} \vec{e} \cdot (\partial_\mu \vec{e} \times \partial_\nu \vec{e})] = 4g^2 S[\vec{e}] \pm 16\pi |n[\vec{e}]|$$

$$S[\vec{e}] \geq \frac{4\pi}{g^2} |n[\vec{e}]|$$

$$\partial_\mu \vec{e}(x) + \sigma \epsilon_{\mu\nu} \partial_\nu \vec{e}(x) \times \vec{e}(x) = 0$$

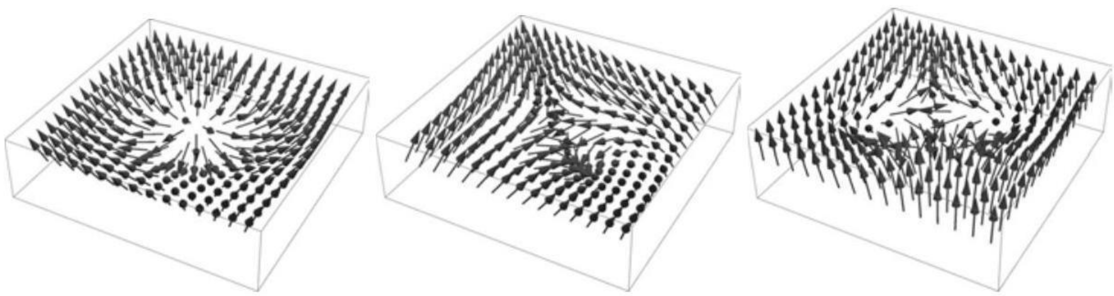


Figura 3. Ondas gravitacionales a escala cuántica.

$$\vec{e}_n(r, \chi) = \left(\frac{2r^n \rho^n}{r^{2n} + \rho^{2n}} \cos(n\chi), \frac{2r^n \rho^n}{r^{2n} + \rho^{2n}} \sigma \sin(n\chi), \frac{r^{2n} - \rho^{2n}}{r^{2n} + \rho^{2n}} \right),$$

$$H[\vec{e}] \in \Pi_3[S^2] = \mathbb{Z}.$$

$$n[U]=\frac{1}{24\pi^2}\int_{S^3} d^3x\epsilon_{ijk}\mathrm{Tr}\big[(U^\dagger\partial_i U)(U^\dagger\partial_j U)(U^\dagger\partial_k U)\big]\in\Pi_3[\mathrm{SU}(2)]=\mathbb{Z}$$

$$S_{\mathrm{WZNW}}[U]=\frac{1}{240\pi^2\mathrm{i}}\int_{H^5} d^5x\epsilon_{\mu\nu\rho\sigma\lambda}\mathrm{Tr}\big[(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\sigma U)(U^\dagger\partial_\lambda U)\big]$$

	S^1	S^2	S^3	S^4	S^5	S^6
Π_1	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
Π_2	$\{0\}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
Π_3	$\{0\}$	$\mathbb{Z}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$
Π_4	$\{0\}$	$\mathbb{Z}(2)$	$\mathbb{Z}(2)$	$\mathbb{Z}$	$\{0\}$	$\{0\}$
Π_5	$\{0\}$	$\mathbb{Z}(2)$	$\mathbb{Z}(2)$	$\mathbb{Z}(2)$	$\mathbb{Z}$	$\{0\}$
Π_6	$\{0\}$	$\mathbb{Z}(12)$	$\mathbb{Z}(12)$	$\mathbb{Z}(2)$	$\mathbb{Z}(2)$	$\mathbb{Z}$

$$w[U]=\frac{1}{480\pi^3\mathrm{i}}\int_{S^5} d^5x\epsilon_{\mu\nu\rho\sigma\lambda}\mathrm{Tr}\big[(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\sigma U)(U^\dagger\partial_\lambda U)\big]$$
$$\in\Pi_5[\mathrm{SU}(N_\mathrm{f}\geq 3)]=\mathbb{Z}$$

$$S[A]=\frac{1}{2e^2}\int d^2xF_{\mu\nu}F_{\mu\nu}$$

$$q(x)=\frac{1}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x)=\frac{1}{2\pi}E(x),$$

$$Q=\int d^2xq=\frac{1}{4\pi}\int d^2x\epsilon_{\mu\nu}F_{\mu\nu}$$

$$q(x)=\frac{1}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x)=\frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu(x)=\partial_\mu\Omega_\mu^{(0)}(x),$$

$$\Omega_\mu^{(0)}(x)=\frac{1}{2\pi}\epsilon_{\mu\nu}A_\nu(x)$$

$$Q = \int d^2x \partial_\mu \Omega_\mu^{(0)} = \int_{S^1} d\sigma_\mu \Omega_\mu^{(0)} = \frac{1}{2\pi} \int_{S^1} d\sigma_\mu \epsilon_{\mu\nu} A_\nu$$

$$0 = A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x) \Rightarrow A_\mu(x) = \partial_\mu \alpha(x)$$

$$Q = \frac{1}{2\pi} \int_{S^1} d\sigma_\mu \epsilon_{\mu\nu} A_\nu = \frac{1}{2\pi} \int_{S^1} d\sigma_\mu \epsilon_{\mu\nu} \partial_\nu \alpha \in \Pi_1[S^1] = \mathbb{Z}$$

$$\mathrm{SU}(n) \quad = S^3 \times S^5 \times \cdots \times S^{2n-1} \qquad (n \geq 2)$$

$$\mathrm{Spin}(n) \quad = S^1 \times S^2 \times \cdots \times S^{n-1} \qquad (n \geq 3)$$

$$\mathrm{Sp}(n) \quad = S^3 \times S^7 \times \cdots \times S^{4n-1} \qquad (n \geq 1)$$

$$\mathrm{G}(2) \quad = \mathrm{SU}(3) \times S^6 = S^3 \times S^5 \times S^6$$

$$\Pi_3[\mathrm{SU}(n)] = \Pi_3[\mathrm{Spin}(n)] = \Pi_3[\mathrm{Sp}(n)] = \Pi_3[\mathrm{G}(2)] = \mathbb{Z}$$

$$Z = \int \mathcal{D}\Phi \exp(-S[\Phi]) = \prod_x \int_{S^{N-1}} d\Phi_x \exp(-S[\Phi])$$

$$S[\Phi] = a^d \sum_{x,\mu} \frac{1}{2a^2} \left|\Phi_{x+\hat{\mu}} - \Phi_x\right|^2 = \sum_{x,\mu} a^{d-2} \left(\mathrm{v}^2 - \Phi_x \cdot \Phi_{x+\hat{\mu}}\right)$$

$$\left[\Phi^{(0)}\right] \to \left[\Phi^{(1)}\right] \to \left[\Phi^{(2)}\right] \to \cdots \to \left[\Phi^{(M)}\right]$$

$$p^{(0)}[\Phi] = \frac{1}{Z^{(0)}}, Z^{(0)} = \int \mathcal{D}\Phi 1 = \prod_x \int_{S^{N-1}} d\Phi 1$$

$$\int \mathcal{D}\Phi p^{(0)}[\Phi] = 1$$

$$p^{(1)}[\Phi'] = \int \mathcal{D}\Phi p^{(0)}[\Phi] w[\Phi, \Phi']$$

$$\int \mathcal{D}\Phi' w[\Phi, \Phi'] = 1$$

$$\int \mathcal{D}\Phi' p^{(1)}[\Phi'] = \int \mathcal{D}\Phi p^{(0)}[\Phi] \int \mathcal{D}\Phi' w[\Phi, \Phi'] = \int \mathcal{D}\Phi p^{(0)}[\Phi] = 1$$

$$p^{(2)}[\Phi''] = \int \mathcal{D}\Phi' p^{(1)}[\Phi'] w[\Phi', \Phi''] = \int \mathcal{D}\Phi' \int \mathcal{D}\Phi p^{(0)}[\Phi] w[\Phi, \Phi'] w[\Phi', \Phi'']$$
$$= \int \mathcal{D}\Phi p^{(0)}[\Phi] \int \mathcal{D}\Phi' w[\Phi, \Phi'] w[\Phi', \Phi''] = \int \mathcal{D}\Phi p^{(0)}[\Phi] w^2[\Phi, \Phi'']$$

$$w^2[\Phi,\Phi''] = \int \mathcal{D}\Phi' w[\Phi,\Phi'] w[\Phi',\Phi'']$$

$$p^{(i)}[\Phi'] = \int \mathcal{D}\Phi p^{(0)}[\Phi] w^i[\Phi,\Phi']$$

$$p[\Phi] = \frac{1}{Z}\exp\left(-S[\Phi]\right)$$

$$\langle \mathcal{O} \rangle = \lim_{M\to\infty} \frac{1}{M - M_0 + 1} \sum_{i=M_0}^{M} \mathcal{O}\left[\Phi^{(i)}\right]$$

$$\exp\left(-S[\Phi]\right) w[\Phi,\Phi'] = \exp\left(-S[\Phi']\right) w[\Phi',\Phi]$$

$$\int \mathcal{D}\Phi \mathrm{v}_1[\Phi] w[\Phi,\Phi'] = \mathrm{v}_1[\Phi']$$

$$\begin{aligned} \int \mathcal{D}\Phi p[\Phi] w[\Phi,\Phi'] &= \int \mathcal{D}\Phi \frac{1}{Z}\exp\left(-S[\Phi]\right) w[\Phi,\Phi'] \\ &= \int \mathcal{D}\Phi \frac{1}{Z}\exp\left(-S[\Phi']\right) w[\Phi',\Phi] \\ &= \frac{1}{Z}\exp\left(-S[\Phi']\right) \int \mathcal{D}\Phi w[\Phi',\Phi] \\ &= \frac{1}{Z}\exp\left(-S[\Phi']\right) = p[\Phi']. \end{aligned}$$

$$\int \mathcal{D}\Phi \mathrm{v}_\lambda[\Phi] w[\Phi,\Phi'] = \lambda \mathrm{v}_\lambda[\Phi']$$

$$\widetilde{w}[\Phi,\Phi'] = \sqrt{\frac{p[\Phi]}{p[\Phi']}} w[\Phi,\Phi'] = \sqrt{\frac{p[\Phi']}{p[\Phi]}} w[\Phi',\Phi] = \widetilde{w}[\Phi',\Phi],$$

$$\tilde{\mathrm{v}}_\lambda[\Phi] = \frac{\mathrm{v}_\lambda[\Phi]}{\sqrt{p[\Phi]}},$$

$$\int \mathcal{D}\Phi \tilde{\mathrm{v}}_\lambda[\Phi] \widetilde{w}[\Phi,\Phi'] = \int \mathcal{D}\Phi \frac{\mathrm{v}_\lambda[\Phi]}{\sqrt{p[\Phi']}} w[\Phi,\Phi'] = \lambda \tilde{\mathrm{v}}_\lambda[\Phi']$$

$$\int \mathcal{D}\Phi \tilde{\mathrm{v}}_\lambda[\Phi] \tilde{\mathrm{v}}_{\lambda'}[\Phi] = \int \mathcal{D}\Phi \frac{1}{p[\Phi]} \mathrm{v}_\lambda[\Phi] \mathrm{v}_{\lambda'}[\Phi] = \delta_{\lambda,\lambda'}$$

$$\int \mathcal{D}\Phi \frac{1}{p[\Phi]} \mathrm{v}_1[\Phi]^2 = \int \mathcal{D}\Phi p[\Phi] = 1.$$

$$\lambda \int \mathcal{D}\Phi' \mathrm{v}_\lambda[\Phi'] = \int \mathcal{D}\Phi \mathrm{v}_\lambda[\Phi] \int \mathcal{D}\Phi' w[\Phi,\Phi'] = \int \mathcal{D}\Phi \mathrm{v}_\lambda[\Phi] \Rightarrow$$
$$(\lambda - 1) \int \mathcal{D}\Phi \mathrm{v}_\lambda[\Phi] = 0$$

$$|\lambda|\int \mathcal{D}\Phi'|\mathrm{v}_\lambda[\Phi']| = \int \mathcal{D}\Phi'|\lambda \mathrm{v}_\lambda[\Phi']| = \int \mathcal{D}\Phi'\left|\int \mathcal{D}\Phi \mathrm{v}_\lambda[\Phi]w[\Phi,\Phi']\right|$$
$$< \int \mathcal{D}\Phi' \int \mathcal{D}\Phi|\mathrm{v}_\lambda[\Phi]||w[\Phi,\Phi']$$
$$= \int \mathcal{D}\Phi|\mathrm{v}_\lambda[\Phi]|\int \mathcal{D}\Phi' w[\Phi,\Phi'] = \int \mathcal{D}\Phi|\mathrm{v}_\lambda[\Phi]|.$$

$$\left|\int \mathcal{D}\Phi \mathrm{v}_\lambda[\Phi]w[\Phi,\Phi']\right| < \int \mathcal{D}\Phi|\mathrm{v}_\lambda[\Phi]||w[\Phi,\Phi'].$$

$$\left|\int \mathcal{D}\Phi \mathrm{v}_1[\Phi]w[\Phi,\Phi']\right| = \int \mathcal{D}\Phi|\mathrm{v}_1[\Phi]||w[\Phi,\Phi']$$

$$\int \mathcal{D}\Phi(\mathrm{v}_1[\Phi] - \mathrm{v}_1'[\Phi])w[\Phi,\Phi'] = \mathrm{v}_1[\Phi'] - \mathrm{v}_1'[\Phi']$$

$$\int \mathcal{D}\Phi \mathrm{v}_1[\Phi] = \int \mathcal{D}\Phi \mathrm{v}_1'[\Phi] = 1 \Rightarrow \int \mathcal{D}\Phi(\mathrm{v}_1[\Phi] - \mathrm{v}_1'[\Phi]) = 0$$

$$\mathrm{v}_1[\Phi] = p[\Phi] = \frac{1}{Z}\exp(-S[\Phi])$$

$$p^{(0)}[\Phi] = \sum_{\lambda} c_\lambda \mathrm{V}_\lambda[\Phi] = c_1 \mathrm{V}_1[\Phi] + \sum_{\lambda\neq 1} c_\lambda \mathrm{V}_\lambda[\Phi] = p[\Phi] + \sum_{\lambda\neq 1} c_\lambda \mathrm{V}_\lambda[\Phi]$$

$$1 = \int \mathcal{D}\Phi p^{(0)}[\Phi] = c_1 \int \mathcal{D}\Phi \mathrm{v}_1[\Phi] + \sum_{\lambda\neq 1} c_\lambda \int \mathcal{D}\Phi \mathrm{v}_\lambda[\Phi] = c_1$$

$$p^{(i)}[\Phi] = \int \mathcal{D}\Phi' p^{(0)}[\Phi']w^i[\Phi',\Phi] = p[\Phi] + \sum_{\lambda\neq 1} \lambda^i c_\lambda \mathrm{V}_\lambda[\Phi] \underset{i\to\infty}{\to} p[\Phi]$$

$$1 > |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$$

$$\int_{C_\delta(\Phi_x)} d\Phi_x' q = 1$$

$$S[\Phi'] \leq S[\Phi] \Rightarrow w[\Phi,\Phi'] = q p_{\mathrm{acc}} = q.$$

$$S[\Phi'] > S[\Phi] \Rightarrow w[\Phi,\Phi'] = q p_{\mathrm{acc}} = q\exp(-S[\Phi'] + S[\Phi]).$$

$$p_{\mathrm{acc}} = \min\{1, \exp(-S[\Phi'] + S[\Phi])\} \in [0,1].$$

$$\exp(-S[\Phi])w[\Phi,\Phi'] = \exp(-S[\Phi])q$$
$$= \exp(-S[\Phi'])q\exp(-S[\Phi] + S[\Phi'])$$
$$= \exp(-S[\Phi'])w[\Phi',\Phi]$$

$$\Delta\mathcal{O} = \frac{1}{\sqrt{M - M_0}}\sqrt{\langle(\mathcal{O} - \langle\mathcal{O}\rangle)^2\rangle} = \frac{1}{\sqrt{M - M_0}}\sqrt{\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2}.$$

$$\overline{\mathcal{O}}_j = \frac{1}{M_{\mathrm{b}}}\sum_{i=M_0+(j-1)M_{\mathrm{b}}}^{M_0+jM_{\mathrm{b}}-1} \mathcal{O}[\Phi^{(i)}], j \in \{1,2,\dots,m\}$$

$$\Delta\bar{\mathcal{O}} = \frac{1}{\sqrt{m-1}}\sqrt{\frac{1}{m}\sum_{j=1}^{m}\left(\bar{\mathcal{O}}_j - \langle\mathcal{O}\rangle\right)^2}$$

$$\left\langle\mathcal{O}^{(i)}\mathcal{O}^{(i+t)}\right\rangle = \lim_{M\to\infty}\frac{1}{M-M_0+1-t}\sum_{i=M_0}^{M-t}\mathcal{O}\left[\Phi^{(i)}\right]\mathcal{O}\left[\Phi^{(i+t)}\right] \sim c_0 + c_1\exp\left(-t/\tau\right)$$

$$\tau \propto \xi^z$$

$$p(\Phi_x') \propto \exp\left(a^{d-2}\Phi_x'\cdot\bar{\Phi}_x\right), \bar{\Phi}_x = \sum_{\mu=1}^{d}\left(\Phi_{x+\hat{\mu}} + \Phi_{x-\hat{\mu}}\right), e_x = \frac{\bar{\Phi}_x}{|\bar{\Phi}_x|},$$

$$\Phi_x' = 2(\Phi_x\cdot e_x)e_x - \Phi_x$$

$$\Phi_x'' = 2(\Phi_x'\cdot e_x)e_x - \Phi_x' = 2(\Phi_x\cdot e_x)e_x - 2(\Phi_x\cdot e_x)e_x + \Phi_x = \Phi_x,$$

$$Z = \int \mathcal{D}\Phi\,\mathrm{Sign}[\Phi]\exp\left(-S[\Phi]\right)$$

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\,\mathcal{O}[\Phi]\mathrm{Sign}[\Phi]\exp\left(-S[\Phi]\right) = \frac{\langle\mathcal{O}\mathrm{Sign}\rangle'}{\langle\mathrm{Sign}\rangle'}$$

$$\langle\mathrm{Sign}\rangle' = \frac{1}{Z'}\int \mathcal{D}\Phi\,\mathrm{Sign}[\Phi]\exp\left(-S[\Phi]\right) = \frac{Z}{Z'}, Z' = \int \mathcal{D}\Phi\exp\left(-S[\Phi]\right),$$

$$\langle\mathcal{O}\mathrm{Sign}\rangle' = \frac{1}{Z'}\int \mathcal{D}\Phi\,\mathcal{O}[\Phi]\mathrm{Sign}[\Phi]\exp\left(-S[\Phi]\right).$$

$$\langle\mathrm{Sign}\rangle' = \frac{Z}{Z'} = \exp\left(-(f-f')\beta V\right)$$

$$\Delta\mathrm{Sign} = \frac{1}{\sqrt{M-M_0}}\sqrt{\langle\mathrm{Sign}^2\rangle' - \langle\mathrm{Sign}\rangle'^2} = \frac{1}{\sqrt{M-M_0}}\sqrt{1-\langle\mathrm{Sign}\rangle'^2} \approx \frac{1}{\sqrt{M-M_0}}$$

$$\frac{\langle\mathrm{Sign}\rangle'}{\Delta\mathrm{Sign}} \approx \sqrt{M-M_0}\exp\left(-(f-f')\beta V\right)$$

$$M - M_0 + 1 \propto \exp\left(2(f-f')\beta V\right)$$

$$\rho - \rho_c, M = \begin{cases}\neq 0 & T < T_c \\ = 0 & T \geq T_c\end{cases}.$$

$$F = -T\log Z$$

$$T \lesssim T_c: M = -\left.\frac{\partial F}{\partial B}\right|_{T,B=0} \propto (T_c - T)^\beta$$

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{T,B=0} = -\left.\frac{\partial^2 F}{\partial B^2}\right|_{T,B=0} \propto |T - T_c|^{-\gamma}$$

$$B \gtrsim 0\colon\ M|_{T=T_c} \propto B^{1/\delta}$$

$$C = -T\frac{\partial^2 f}{\partial T^2} \propto |T_c - T|^{-\alpha}$$

$$\xi \propto |T - T_c|^{-\nu}$$

$$C = -T\frac{\partial^2 f}{\partial T^2} \propto \frac{\xi^{-d}}{|T - T_c|^2} \propto |T - T_c|^{\nu d - 2}$$

$$\alpha = 2 - \nu d.$$

$$\tau = 1 - T/T_c.$$

$$F(\tau, B) = \frac{1}{\lambda}F(\lambda^u \tau, \lambda^{\mathrm{v}} B),$$

$$\lambda = \tau^{-1/u}\colon F(\tau, B) = \tau^{1/u}F\left(1, B/\tau^{\mathrm{v}/u}\right) = \tau^{1/u}\phi_1\left(B/\tau^{\mathrm{v}/u}\right),$$

$$\lambda = B^{-1/\mathrm{v}}\colon F(\tau, B) = B^{1/\mathrm{v}}F\left(\tau/B^{u/\mathrm{v}}, 1\right) = B^{1/\mathrm{v}}\phi_2\left(\tau/B^{u/\mathrm{v}}\right).$$

$$M|_{B=0} = -\left.\frac{\partial F}{\partial B}\right|_{B=0} = -\tau^{(1-\mathrm{v})/u}\phi_1'(0) \propto \tau^{\beta} \Rightarrow \beta = \frac{1-\mathrm{v}}{u}$$

$$\chi|_{B=0} = -\left.\frac{\partial^2 F}{\partial B^2}\right|_{B=0} = -\tau^{(1-2\mathrm{v})/u}\phi_1''(0) \propto \tau^{-\gamma} \Rightarrow \gamma = \frac{2\mathrm{v}-1}{u}$$

$$C|_{B=0} = -T\left.\frac{\partial^2 f}{\partial T^2}\right|_{B=0} \propto -\frac{T}{T_c^2}\frac{1}{u}\left(\frac{1}{u} - 1\right)\tau^{1/u-2}\phi_1(0) \propto \tau^{-\alpha} \Rightarrow \alpha = 2 - \frac{1}{u}$$

$$\alpha + 2\beta + \gamma = 2$$

$$M = -\left.\frac{\partial F}{\partial B}\right|_{\tau=0} = -\frac{1}{\mathrm{v}}B^{1/\mathrm{v}-1}\phi_2(0) \propto B^{1/\delta} \ \Rightarrow\ \delta = \frac{\mathrm{v}}{1-\mathrm{v}}$$

$$\alpha + \beta(\delta + 1) = 2, \gamma - \beta(\delta - 1) = 0$$

$$M = -\left.\frac{\partial F}{\partial B}\right|_{T,B=0} \propto (T_c - T)^{\beta}$$

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{T,B=0} \propto |T_c - T|^{-\gamma}$$

$$M|_{T=T_c} \propto B^{1/\delta}$$

$$C = -T\left.\frac{\partial^2 f}{\partial T^2}\right|_{B=0} \propto |T - T_c|^{-\alpha}$$

$$\xi \propto |T - T_c|^{-\nu}$$

		β	γ	α	δ	v
Fe(T_c = 1044 K)		0.385(5)	1.333	-0.103(11)	4.350(5)	
Ni(T_c = 632 K)		0.395(10)	1.345(10)	-0.091(2)	4.58(5)	
Liquid-gas		0.35	1.37(20)	≈ 0	4.4(4)	0.64
Ising model	$d = 2$	1/8	7/4	0	15	1
	$d = 3$	0.326419(3)	1.237075(10)	0.11008(1)	4.78984(1)	0.629971(4)
	$d = 4$	1/2	1	0	3	1/2
	MFA	1/2	1	0	3	1/2
O(2) model	$d = 3$	0.3486(1)	1.3178(2)	-0.0151(3)	4.780(1)	0.6717(1)
O(3) model	$d = 3$	0.3689(3)	1.3960(9)	-0.1336(15)	4.783(3)	0.7112(5)

$$\begin{aligned} vd + \alpha &= 2 \\ \alpha + 2\beta + \gamma &= 2 \\ \alpha + \beta(1+\delta) &= 2 \\ \gamma &= \beta(\delta - 1) \end{aligned}$$

$$\gamma(\delta + 1) \qquad = \qquad (2 - \alpha)(\delta - 1)$$

$$\left\langle \vec{s}_x \cdot \vec{s}_y \right\rangle_{\mathrm{c}} \Big|_{T_{\mathrm{c}}} \propto |x - y|^{2+\eta-d}$$

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