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TEORÍA CUÁNTICA DE CAMPOS RELATIVISTAS. CUANTIZACIÓN. VOLUMEN II

QUANTUM THEORY OF RELATIVISTIC FIELDS. QUANTIZATION. VOLUME II

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Teoría Cuántica de Campos Relativistas. Cuantización. Volumen II

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RESUMEN

En este trabajo, me propondré formalizar matemáticamente la Teoría Cuántica de Campos Relativistas (TCCR), en escenarios de gravedad y supergravedad cuánticas. Se profundizará principalmente en la necesidad ineludible de encontrar y determinar simetrías y supersimetrías isométricas y homeomorfas, esto a propósito de campos de gauge asimétricos y entrópicos. La finalidad de este artículo, es dotar a la TCCR, de un modelo formal que reconcilie la relatividad especial y general con la mecánica cuántica, desde dos conceptos esenciales, siendo éstos, gravedad y supergravedad cuánticas, y la relación con sus consecuencias lógicas. Intentaremos aquí, compactar un modelo que permita predecir y detectar el comportamiento de una partícula blanca y oscura respectivamente, así como sus interacciones con el supergravitón y el gravitón, según corresponda, esto, cuando existe interferencia gravitónica, pero quiero en lo principal, describir un modelo de simulación por deformación del espacio – tiempo cuántico, cuando interactúa, exclusivamente una partícula supermasiva o una partícula estrella, debido a la inmensa densidad de su masa y/o energía y entender por tanto, a la gravedad, como una cualidad propia de las partículas subatómicas cuyo centro de masa – energía es extremo, lo que permite la distorsión del plano cuántico, a tal punto de hacerlo, en dimensiones altas o en dimensiones sin desprendimiento y la formación de materia y energía oscuras, integradas éstas últimas, con la existencia de partículas oscuras, en extremo densas, en la que la gravedad, es el único elemento transdimensional y multisectorial.

Palabras Clave: teoría cuántica de campos Relativistas, supergravedad cuántica, gravedad cuántica, dimensiones D, supersimetría

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Quantum Theory of Relativistic Fields. Quantization. Volume II

ABSTRACT

In this work, I will propose to mathematically formalize the Quantum Theory of Relativistic Fields (TCCR), in quantum gravity and supergravity scenarios. The unavoidable need to find and determine isometric and homeomorphic symmetries and supersymmetries will be deepened, this with regard to asymmetric and entropic gauge fields. The purpose of this article is to provide the TCCR with a formal model that reconciles special and general relativity with quantum mechanics, from two essential concepts, these being, quantum gravity and supergravity, and the relationship with their logical consequences. We will try here to compact a model that allows predicting and detecting the behavior of a white and dark particle respectively, as well as their interactions with the supergraviton and the graviton, as appropriate, this, when there is gravitonic interference, but I want to describe a simulation model by deformation of quantum space-time, when it interacts, exclusively, a supermassive particle or a star particle, due to the immense density of its mass and/or energy and therefore understand gravity as a quality of subatomic particles whose center of mass-energy is extreme, which allows the distortion of the quantum plane, to the point of doing so, in high dimensions or in dimensions without detachment and the formation of dark matter and energy, The latter are integrated with the existence of dark, extremely dense particles, in which gravity is the only transdimensional and multisectoral element.

Keywords: quantum theory of relativistic fields, quantum supergravity, quantum gravity, D dimensions, supersymmetry

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INTRODUCCIÓN

La Teoría Cuántica de Campos Relativistas, plantea algunas hipótesis teóricas, las mismas que serán abordadas en este trabajo, entre ellas, la existencia de partículas supermasivas y partículas estrella respectivamente, la existencia de agujeros negros a escala cuántica, la existencia de ondas cuánticas gravitacionales por colapso de una partícula estrella u oscura o por colisión de dos partículas con centros de masa y/o energía extremadamente densos, pero sobre todo, la supergravedad y gravedad cuánticas, entendidas como un fenómeno de distorsión y deformación del espacio – tiempo cuántico, por interacción de las antes referidas partículas, con o sin intervención gravitónica y en un marco de dualidad holográfica y simetrías de gauge compactas e isométricas. El objetivo de este trabajo es cuantizar los fenómenos antes referidos, robusteciendo la física de partículas que le es inherente a este sistema y por ende, la aplicación de la mecánica relativista a la física cuántica y viceversa.

RESULTADOS Y DISCUSIÓN.

Retomo el desarrollo matemático desplegado en el volumen I de este manuscrito.

Ecuaciones de campo: Cálculos complementarios.

$$\frac{\delta(D_\delta)}{D_8 D_{11}},\frac{\delta(D_\delta)}{D_1 D_8 D_{11}},\frac{\delta(D_\delta)}{D_2 D_9 D_{10}},\frac{\delta(D_\delta)}{D_5 D_8 D_{10}},\frac{\delta(D_\delta)}{D_8 D_9},\frac{\delta(D_\delta)}{D_1 D_8 D_9}.$$

$$\sum_S S_1(x_k;\zeta_{ij}) \otimes S_2(x_k;\zeta_{ij})$$

$$\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{1}{x-b} - \frac{1}{x-a}\right)$$

$$F_{6\mu}\big|_\partial = 0,\ D_\mu \vec{X}\big|_\partial = 0,\ D_6 \vec{Y}\big|_\partial = 0$$
$$F_{\mu\nu}\big|_\partial = 0, D_6\vec{X} + \vec{X}\times\vec{X}\big|_\partial = 0,\ D_\mu\vec{Y}\big|_\partial = 0,$$

$$\vec{X}(x^6) = \frac{\vec{\mathfrak{t}}}{x^6},$$

$$\rho: \mathfrak{su}(2) \to \mathfrak{g}.$$

$$V = \bigoplus_{i=1}^{s} \mathcal{R}_{j_i}.$$

$$N = n_1 + \cdots + n_s$$

$$\mathcal{X} = X_1 + iX_2, \mathcal{A} = A_6 + iX_3$$

$$\frac{\mathcal{D}\mathcal{X}}{\mathcal{D}x^6} = 0,$$

$$\frac{\mathcal{D}\mathcal{X}}{dx^6} = \frac{d}{dx^6} + [\mathcal{A},\mathcal{X}]$$

$$\mathcal{X} = \frac{t_1 + it_2}{x^6}, \mathcal{A} = \frac{it_3}{x^6}$$

$$\mathcal{X} \sim \begin{pmatrix} 0 & * & & & \\ & 0 & * & & \\ & & 0 & * & \\ & & & 0 & * \\ & & & & 0 \end{pmatrix}$$

$$W_{\mathcal{R}} = \mathrm{Tr}_{\mathcal{R}} P\exp\left[\oint \; i\left(A_\mu \dot{x}^\mu + \sqrt{-\dot{x}^2}Y\right) ds\right]$$

$$\mathcal{X}(z) \sim \begin{pmatrix} 0 & z^m \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{X}(z) \sim \begin{pmatrix} 0 z^{m_1} & & & \\ & 0 & z^{m_2} & \\ & & 0 & z^{m_{N-2}} \\ & & & \\ & & & 0 \end{pmatrix}$$

$$\tau^\vee = -\frac{1}{n_g \tau},$$

G	$\mathcal{R}$	$G^\vee$	B
$U(N)$	(1^k)	$U(N)$	$(1^k, 0^{N-k})$
$Spin(2N+1)$	spinor	$USp(2N)/\mathbb{Z}_2$	$(\frac{1}{2}^N)$
$USp(2N)$	(1)	$SO(2N+1)$	$(1, 0^{N-1})$
$Spin(2N)$	(anti)chiral spinor	$SO(2N)/\mathbb{Z}_2$	$(\frac{1}{2}^N)$
$SO(2N)$	(1)	$SO(2N)$	$(1, 0^{N-1})$
$O(2N)$	(1)	$O(2N)$	$(1, 0^{N-1})$

	$SO(2N+1)$	$USp(2N)$	$O(2N)$	$USp(2N)'$
θ_{RR}	$1/2$	0	0	$1/2$
θ_{NS}	0	$1/2$	0	$1/2$
D3-brane charge	$1/4$	$1/4$	$-1/4$	$1/4$
orientifold	$\widetilde{\mathrm{O3}}^-$	$\mathrm{O3}^+$	$\mathrm{O3}^-$	$\widetilde{\mathrm{O3}}^+$
S operation	$\mathrm{O3}^+$	$\widetilde{\mathrm{O3}}^-$	$\mathrm{O3}^-$	$\widetilde{\mathrm{O3}}^+$

.

	0	1	2	3	4	5	6	7	8	9
N D3	○	○	○				○			
O3	○	○	○				○			
NS5	○	○	○	○	○	○				
D5	○	○	○					○	○	○
F1	○					○				
$\widetilde{\mathrm{D5}}$	○			○	○			○	○	○

	0	1	2	3	4	5	6	7	8	9
N D3	○	○	○				○			
O3	○	○	○				○			
NS5	○	○	○	○	○	○				
D5	○	○	○					○	○	○
D1	○									○
$\widetilde{\text{NS5}}$	○			○	○	○		○	○	

$$\mathbb{II}_{\mathcal{B}}^{4\,\mathrm{d}G}(t,z;q)=\mathrm{Tr}_{\mathcal{H}}(-1)^F q^{J+\frac{H+C}{4}}t^{H-C}z^f,$$

$$(a;q)_0:=1,(a;q)_n:=\prod_{k=0}^{n-1}(1-aq^k),(q)_n:=\prod_{k=1}^{n}(1-q^k),$$

$$(a;q)_\infty:=\prod_{k=0}^{\infty}(1-aq^k),(q)_\infty:=\prod_{k=1}^{\infty}(1-q^k),$$

$$(x^{\pm};q)_n:=(x;q)_n(x^{-1};q)_n,$$
$$(x_1,\cdots,x_k;q)_n:=(x_1;q)_n\cdots(x_k;q)_n.$$

	$\partial^n X$	$\partial^n Y$	$\partial^n\lambda$	$\partial^n\overline{\lambda}$
G	adj	adj	adj	adj
$U(1)_J$	n	n	$n+\frac{1}{2}$	$n+\frac{1}{2}$
$U(1)_C$	0	2	$+$	$+$
$U(1)_H$	2	0	$+$	$+$
fugacity	$q^{n+\frac{1}{2}}t^2e^\alpha$	$q^{n+\frac{1}{2}}t^{-2}e^\alpha$	$-q^{n+1}e^\alpha$	$-q^{n+1}e^\alpha$

.

$$\mathcal{I}^{3\,\mathrm{dHM}}(t,x;q)=\frac{\left(q^{\frac{3}{4}}t^{-1}x^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}}tx^{\mp};q\right)_\infty}$$

$$\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle_{\mathcal{N}}^{G}=\frac{1}{|W_R|}\oiiint ds\prod_{\alpha\in R}\frac{(e^\alpha;q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}e^\alpha;q\right)_\infty}\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathfrak{g}}(s)$$

$$\mathbf{q}=q,\mathbf{t}=q^{\frac{1}{2}}t^{-2}$$

$$\langle f,g\rangle=\frac{1}{|W_R|}\oiiint ds\,w_R(s;\mathbf{q},\mathrm{t})f(s)\overline{g(s)}$$
$$w_R(s;\mathbf{q},\mathbf{t})=\prod_{\alpha\in R}\frac{(e^\alpha;\mathbf{q})_\infty}{(\mathbf{t}e^\alpha;\mathbf{q})_\infty}$$

$$\langle P_\lambda,P_\lambda\rangle=\prod_{\alpha\in R^+}\frac{\left(\mathrm{t}^{(\rho,\alpha^\vee)}\mathrm{q}^{(\lambda,\alpha^\vee)};\mathrm{q}\right)_\infty\left(\mathrm{t}^{(\rho,\alpha^\vee)}\mathrm{q}^{(\lambda,\alpha^\vee)+1};\mathrm{q}\right)_\infty}{\mathrm{t}^{(\rho,\alpha^\vee)+1}\mathrm{q}^{(\lambda,\alpha^\vee)};\mathrm{q})_\infty(\mathrm{t}^{(\rho,\alpha^\vee)-1}\mathrm{q}^{(\lambda,\alpha^\vee)+1};\mathrm{q})_\infty}$$

$$\alpha^\vee=\frac{2\alpha}{(\alpha,\alpha)},\rho=\frac{1}{2}\sum_{\alpha\in R^+}\alpha$$

$$\frac{\langle P_\lambda,P_\lambda\rangle}{\langle 1,1\rangle}=\prod_{\alpha\in R^+}\frac{\left(\mathrm{t}^{(\rho,\alpha^\vee)+1};\mathrm{q}\right)_{(\lambda,\alpha^\vee)}\left(\mathrm{t}^{(\rho,\alpha^\vee)-1}\mathrm{q};\mathrm{q}\right)_{(\lambda,\alpha^\vee)}}{(\mathrm{t}^{(\rho,\alpha^\vee)};\mathrm{q})_{(\lambda,\alpha^\vee)}(\mathrm{t}^{(\rho,\alpha^\vee)}\mathrm{q};\mathrm{q})_{(\lambda,\alpha^\vee)}},$$

$$\langle 1,1\rangle = \prod_{\alpha\in R^+} \frac{(t^{\langle\rho,\alpha^\vee\rangle};q)_\infty (t^{\langle\rho,\alpha^\vee\rangle}q;q)_\infty}{(t^{\langle\rho,\alpha^\vee\rangle+1};q)_\infty (t^{\langle\rho,\alpha^\vee\rangle-1}q;q)_\infty}$$

$$\mathbb{II}_{\mathcal{N}}^{4\,\mathrm{d}\mathcal{N}=4U(2)}(t;q) = \frac{1}{2}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^2} \oint \prod_{i=1}^{2} \frac{ds_i}{2\pi i s_i} \prod_{i\neq j} \frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}$$

$$\mathbb{II}_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ U(2)}(t;q) = \frac{(q)_\infty (q^{\frac{3}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty (qt^4;q)_\infty}.$$

$$\mathbb{III}_{\mathcal{D}}^{4\,\mathrm{d}\mathcal{N}=4U(2)}(t,x_1,x_2;q) = \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty^2} \frac{\left(q\frac{x_1}{x_2};q\right)_\infty \left(q\frac{x_2}{x_1};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2\frac{x_1}{x_2};q\right)_\infty \left(q^{\frac{1}{2}}t^2\frac{x_2}{x_1};q\right)_\infty}$$

$$x_1 = q^{\frac{1}{4}}t, x_2 = q^{\frac{3}{4}}t^3$$

$$\mathbb{II}_{\mathcal{D}}^{4\,\mathrm{d}\mathcal{N}=4U(2)}\left(t, x_1 = q^{\frac{1}{4}}t, x_2 = q^{\frac{3}{4}}t^3; q\right) = \mathbb{II}_{\mathrm{Nahm}}^{4\,\mathrm{N}\mathcal{N}=4U(2)}(t;q)\mathcal{I}^{3\,\mathrm{dHM}}\left(t, x = q^{\frac{1}{4}}t; q\right)$$

$$x_1 = q^{\frac{5}{4}}t, x_2 = q^{\frac{7}{4}}t^3$$

$$\langle W_{\square} W_{\overline{\square}}\rangle_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ U(2)}(t;q) = \frac{1}{2}\frac{(q)_\infty^2}{(q^{\frac{1}{2}}t^{-2};q)_\infty^2} \oint \prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i} \prod_{i\neq j} \frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty} (s_1+s_2)(s_1^{-1}+s_2^{-1}).$$

$$x_1 = q^{\frac{1}{4}}t, x_2 = q^{\frac{7}{4}}t^3$$

$$\mathbb{II}_{\mathcal{D}}^{4\mathrm{d}\ \mathcal{N}=4\ U(2)}\left(t, x_1 = q^{\frac{1}{4}}t, x_2 = q^{\frac{7}{4}}t^3; q\right) = \langle T_{(1,0)} T_{(1,0)}\rangle_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ U(2)}(t;q)\mathcal{I}^{3\mathrm{d\ HM}}(t, x = q^{\frac{5}{4}}t; q),$$

$$\langle W_{\square} W_{\overline{\square}}\rangle_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ U(2)}(t;q) = \langle T_{(1,0)} T_{(1,0)}\rangle_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ U(2)}(t^{-1};q).$$

$$\langle W_{\square\square} W_{\overline{\square\square}}\rangle_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ U(2)}(t;q) = 3\frac{(q)_\infty (q^{\frac{3}{2}}t^{-2};q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty (qt^{-4};q)_\infty} - 2\frac{(q^2;q)_\infty^2 (q^{\frac{5}{2}}t^{-2};q)_\infty}{(qt^{-4};q)_\infty (q^{\frac{3}{2}}t^{-2};q)_\infty (q^3;q)_\infty} - q^{\frac{1}{2}}t^{-2}\frac{(q^{\frac{1}{2}}t^2;q)_\infty (q)_\infty (q^3;q)_\infty (q^{\frac{7}{2}}t^{-2};q)_\infty (q^4;q)_\infty}{(qt^{-4};q)_\infty (q^{\frac{3}{2}}t^{-2};q)_\infty (q^{\frac{3}{2}}t^2;q)_\infty (q^2;q)_\infty (q^5;q)_\infty}.$$

$$
\begin{aligned}
&\langle W_{\square\square\square} W_{\overline{\square\square\square}} \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ U(2)}(t;q)\\
&= 4\frac{(q)_\infty (q^{\frac{3}{2}}t^{-2};q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty (qt^{-4};q)_\infty} - 3\frac{(q^2;q)_\infty^2 (q^{\frac{5}{2}}t^{-2};q)_\infty}{(qt^{-4};q)_\infty (q^{\frac{3}{2}}t^{-2};q)_\infty (q^3;q)_\infty}\\
&- 2q^{\frac{1}{2}}t^{-2}\frac{(q^{\frac{1}{2}}t^{2};q)_\infty (q)_\infty (q^3;q)_\infty (q^{\frac{7}{2}}t^{-2};q)_\infty (q^4;q)_\infty}{(qt^{-4};q)_\infty (q^{\frac{3}{2}}t^{-2};q)_\infty (q^{\frac{3}{2}}t^{2};q)_\infty (q^2;q)_\infty (q^5;q)_\infty}\\
&- qt^{-4}\frac{(q^{\frac{1}{2}}t^{2};q)_\infty (q)_\infty (q^4;q)_\infty (q^{\frac{9}{2}}t^{-2};q)_\infty (q^6;q)_\infty}{(qt^{-4};q)_\infty (q^{\frac{3}{2}}t^{-2};q)_\infty (q^{\frac{5}{2}}t^{2};q)_\infty (q^3;q)_\infty (q^7;q)_\infty}.
\end{aligned}
$$

$$
\begin{aligned}
&\langle W_{(k)} W_{\overline{(k)}} \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ U(2)}(t;q) = (k+1)\frac{(q)_\infty (q^{\frac{3}{2}}t^{-2};q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty (qt^{-4};q)_\infty}\\
&- \frac{(q^{\frac{1}{2}}t^{2};q)_\infty (q)_\infty}{(qt^{-4};q)_\infty (q^{\frac{3}{2}}t^{-2};q)_\infty} \sum_{i=1}^{k} (k-i+1) q^{\frac{i-1}{2}} t^{-2(2i-1)} (1+q^i) \frac{(q^{\frac{2i+3}{2}}t^{-2};q)_\infty}{(q^{\frac{2i-1}{2}}t^{2};q)_\infty}.
\end{aligned}
$$

$$
\begin{aligned}
&\langle W_{(k)} W_{\overline{(k)}} \rangle^{4\text{d}\ \mathcal{N}=4\ U(2)}(q)\\
&= (k+1)\mathcal{I}^{U(2)}(t;q) - q^{-1}t^{4} \sum_{i=1}^{k} (k-i+1) \frac{t^{2i}-t^{-2i}}{q^{\frac{i}{2}}-q^{-\frac{i}{2}}} P_1 \begin{bmatrix} q^{-1}t^{4} \\ 1 \end{bmatrix} (\zeta,\tau),
\end{aligned}
$$

$$
\begin{aligned}
&\langle W_{(k)} W_{\overline{(k)}} \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ U(2)}(q)\\
&= \frac{k+1}{1-q^{\frac{1}{2}}} + (1-q^{\frac{1}{2}}) \sum_{i=1}^{k} (k-i+1) q^{\frac{i-1}{2}} \frac{(1+q^i)}{(1-q^{\frac{2i-1}{2}})(1-q^{\frac{2i+1}{2}})}.
\end{aligned}
$$

$$
\begin{aligned}
\langle W_{(k=\infty)} W_{\overline{(k=\infty)}} \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ U(2)}(q) &= \sum_{n=1}^{\infty} \frac{q^{\frac{n-1}{2}}}{1-q^{n-\frac{1}{2}}}\\
&= q^{-\frac{1}{2}} \sum_{n=1}^{\infty} \sigma_0(2n-1) q^{\frac{n}{2}},
\end{aligned}
$$

$$
\sigma_k(n) = \sum_{d|n} d^k
$$

$$
\begin{aligned}
\langle W_{(k=\infty)} W_{\overline{(k=\infty)}} \rangle^{4\text{d}\ \mathcal{N}=4\ U(2)}(q) &= \sum_{n=1}^{\infty} \frac{n^2 q^{\frac{n-1}{2}}}{1-q^{n}}\\
&= q^{-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\sigma_2(2n)-\sigma_2(n)}{4} q^{\frac{n}{2}}.
\end{aligned}
$$

$$
\mathbb{II}_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ U(3)}(t;q) = \frac{1}{3!} \frac{(q)_\infty^3}{(q^{\frac{1}{2}}t^{-2};q)_\infty^3} \oint \prod_{i=1}^{3} \frac{ds_i}{2\pi i s_i} \prod_{i\neq j} \frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}.
$$

$$
\mathbb{II}_{\text{Nahm}}^{4\text{d}\ \mathcal{N}=4\ U(3)}(t;q) = \frac{(q)_\infty (q^{\frac{3}{2}}t^{2};q)_\infty (q^2 t^4;q)_\infty}{(q^{\frac{1}{2}}t^{2};q)_\infty (qt^{4};q)_\infty (q^{\frac{3}{2}}t^{6};q)_\infty}.
$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ U(3)}(t,x_1,x_2,x_3;q)$$
$$=\frac{(q)_\infty^3}{(q^{\frac{1}{2}}t^2;q)_\infty^3}\frac{(q\frac{x_1}{x_2};q)_\infty(q\frac{x_1}{x_3};q)_\infty(q\frac{x_2}{x_1};q)_\infty(q\frac{x_2}{x_3};q)_\infty(q\frac{x_3}{x_1};q)_\infty(q\frac{x_3}{x_2};q)_\infty}{(q^{\frac{1}{2}}t^2\frac{x_1}{x_2};q)_\infty(q^{\frac{1}{2}}t^2\frac{x_1}{x_3};q)_\infty(q^{\frac{1}{2}}t^2\frac{x_2}{x_1};q)_\infty(q^{\frac{1}{2}}t^2\frac{x_2}{x_3};q)_\infty(q^{\frac{1}{2}}t^2\frac{x_3}{x_1};q)_\infty(q^{\frac{1}{2}}t^2\frac{x_3}{x_2};q)_\infty}$$

$$x_1=q^{\frac{1}{4}}t,x_2=q^{\frac{3}{4}}t^3,x_3=q^{\frac{5}{4}}t^5$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ U(3)}\left(t,x_1=q^{\frac{1}{4}}t,x_2=q^{\frac{3}{4}}t^3,x_3=q^{\frac{5}{4}}t^5;q\right)$$
$$=\mathbb{II}_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ U(3)}(t;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{1}{4}}t;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{3}{4}}t^3;q).$$

$$x_1=q^{\frac{5}{4}}t,x_2=q^{\frac{7}{4}}t^3,x_3=q^{\frac{9}{4}}t^5$$

$$\langle W_{\Box}W_{\overline{\Box}}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ U(3)}(t;q)$$
$$=\frac{1}{3!}\frac{(q)_\infty^3}{(q^{\frac{1}{2}}t^{-2};q)_\infty^3}\oint\prod_{i=1}^3\frac{ds_i}{2\pi is_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}(s_1+s_2+s_3)(s_1^{-1}+s_2^{-1}+s_3^{-1}).$$

$$x_1=q^{\frac{1}{4}}t,x_2=q^{\frac{3}{4}}t^3,x_3=q^{\frac{9}{4}}t^5$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ U(3)}\left(t,x_1=q^{\frac{1}{4}}t,x_2=q^{\frac{3}{4}}t^3,x_3=q^{\frac{9}{4}}t^5;q\right)$$
$$=\langle T_{(1,0,0)}T_{(1,0,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ U(3)}(t;q)$$
$$\times\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{1}{4}}t;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{5}{4}}t;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{7}{4}}t^3;q).$$

$$\langle T_{(1,0,0)}T_{(1,0,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ U(3)}(t;q)=\frac{(q)_\infty^2(q^{\frac{3}{2}}t^2;q)_\infty^2(q^3t^4;q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty^2(qt^4;q)_\infty(q^2;q)_\infty(q^{\frac{5}{2}}t^6;q)_\infty}.$$

$$\langle W_{\Box}W_{\overline{\Box}}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ U(3)}(t;q)=\langle T_{(1,0,0)}T_{(1,0,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ U(3)}(t^{-1};q).$$

$$x_1=q^{\frac{1}{4}}t,x_2=q^{\frac{7}{4}}t^3,x_3=q^{\frac{9}{4}}t^5$$

$$\langle W_{\Box\Box}W_{\overline{\Box\Box}}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ U(3)}(t;q)$$
$$=\frac{(q)_\infty^2(q^{\frac{3}{2}}t^{-2};q)_\infty^2(q^3t^{-4};q)_\infty}{(q^{\frac{5}{2}}t^{-2};q)_\infty^2(qt^{-4};q)_\infty(q^2;q)_\infty(q^{\frac{5}{2}}t^{-6};q)_\infty}$$
$$+qt^{-4}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q)_\infty^2(q^{\frac{3}{2}}t^{-2};q)_\infty(q^{\frac{5}{2}}t^{-2};q)_\infty(q^4;q)_\infty(q^4t^{-4};q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty^2(qt^{-4};q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty(q^{\frac{5}{2}}t^{-6};q)_\infty(q^3;q)_\infty(q^4;q)_\infty}.$$

$$\langle W_1W_1W_{-2}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ U(3)}(t;q)$$
$$=qt^{-4}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q)_\infty^2(q^{\frac{3}{2}}t^{-2};q)_\infty(q^{\frac{5}{2}}t^{-2};q)_\infty(q^4;q)_\infty(q^4t^{-4};q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty^2(qt^{-4};q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty(q^{\frac{5}{2}}t^{-6};q)_\infty(q^3;q)_\infty(q^4;q)_\infty}.$$

$$\mathbb{II}_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ U(N)}(t;q)=\frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi is_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}.$$

$$\mathbb{III}^{\text{4d } \mathcal{N}=4 \ U(N)}_{\text{Nahm}}(t;q) = \prod_{k=1}^{N} \frac{(q^{\frac{1+k}{2}} t^{2(k-1)};q)_\infty}{(q^{\frac{k}{2}} t^{2k};q)_\infty}.$$

$$\mathbb{III}^{\text{4d } \mathcal{N}=4 \ U(N)}_{\mathcal{D}}(t, x_i; q) = \frac{(q)_\infty^N}{(q^{\frac{1}{2}} t^2;q)_\infty^N} \prod_{i\neq j} \frac{(q\frac{x_i}{x_j};q)_\infty}{(q^{\frac{1}{2}} t^2 \frac{x_i}{x_j};q)_\infty}$$

$$x_i = q^{\frac{2i-1}{4}} t^{2i-1}$$

$$\mathbb{III}^{\text{4d } \mathcal{N}=4 \ U(N)}_{\mathcal{D}}\left(t, x_i = q^{\frac{2i-1}{4}} t^{2i-1}; q\right) = \mathbb{III}^{\text{4d } \mathcal{N}=4 \ U(N)}_{\text{Nahm}}(t;q) \prod_{i=1}^{N-1} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{2i-1}{4}} t^{2i-1}; q)^{N-i}.$$

$$\langle W_{\square} W_{\overline{\square}} \rangle_{\mathcal{N}}^{\text{4d } \mathcal{N}=4U(N)}(t;q) = \frac{1}{N!} \frac{(q)_\infty^N}{\left(q^{\frac{1}{2}} t^{-2};q\right)_\infty^N} \oiiint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \prod_{i\neq j} \frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}} t^{-2} \frac{s_i}{s_j};q\right)_\infty} \left(\sum_i s_i\right)\left(\sum_i s_i^{-1}\right)$$

$$x_i = q^{\frac{2i-1}{4}} t^{2i-1}, i = 1, \cdots, N-1$$
$$x_N = q^{\frac{2N+3}{4}} t^{2N-1}$$

$$\begin{aligned}
&\mathbb{III}^{\text{4d } \mathcal{N}=4 \ U(N)}_{\mathcal{D}}\left(t, \left\{x_i = q^{\frac{2i-1}{4}} t^{2i-1}\right\}_{i=1}^{N-1}, x_N = q^{\frac{2N+3}{4}} t^{2N-1}; q\right) \\
&= \langle T_{(1,0,\cdots,0)} T_{(1,0,\cdots,0)} \rangle^{\text{4d } \mathcal{N}=4 \ U(N)}_{\text{Nahm}}(t;q) \\
&\times \prod_{i=1}^{N-2} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{2i-1}{4}} t^{2i-1}; q)^{N-1-i} \prod_{i=1}^{N-1} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{2i+1}{4}} t^{2i-1}; q).
\end{aligned}$$

$$\langle T_{(1,0,\cdots,0)} T_{(1,0,\cdots,0)} \rangle^{\text{4d } \mathcal{N}=4 \ U(N)}_{\text{Nahm}}(t;q) = \frac{(q)_\infty (q^{\frac{3}{2}} t^2;q)_\infty (q^{\frac{3+N}{2}} t^{2(N-1)};q)_\infty}{(q^{\frac{1}{2}} t^2;q)_\infty (q^2;q)_\infty (q^{\frac{2+N}{2}} t^{2N};q)_\infty} \prod_{k=1}^{N-1} \frac{(q^{\frac{k+1}{2}} t^{2(k-1)};q)_\infty}{(q^{\frac{k}{2}} t^{2k};q)_\infty}.$$

$$\begin{aligned}
&\langle T_{(1,0,0,0)} T_{(1,0,0,0)} \rangle^{\text{4d } \mathcal{N}=4 \ U(4)}_{\text{Nahm}}(t;q) \\
&= \frac{(q)_\infty^2 (q^{\frac{3}{2}} t^2;q)_\infty^2 (q^2 t^4;q)_\infty (q^{\frac{7}{2}} t^6;q)_\infty}{(q^{\frac{1}{2}} t^2;q)_\infty (q t^4;q)_\infty (q^{\frac{3}{2}} t^6;q)_\infty (q^2;q)_\infty (q^3 t^6;q)_\infty}, \\
&\langle T_{(1,0,0,0,0)} T_{(1,0,0,0,0)} \rangle^{\text{4d } \mathcal{N}=4 \ U(5)}_{\text{Nahm}}(t;q) \\
&= \frac{(q)_\infty^2 (q^{\frac{3}{2}} t^2;q)_\infty^2 (q^2 t^4;q)_\infty (q^{\frac{5}{2}} t^6;q)_\infty (q^4 t^8;q)_\infty}{(q^{\frac{1}{2}} t^2;q)_\infty (q t^4;q)_\infty (q^{\frac{3}{2}} t^6;q)_\infty (q^2;q)_\infty (q^2 t^8;q)_\infty (q^{\frac{7}{2}} t^{10};q)_\infty}.
\end{aligned}$$

$$\langle W_{\Box} W_{\overline{\Box}} \rangle_{\mathcal{N}}^{4d\ \mathcal{N}=4\ U(N)}(t;q) = \langle T_{(1,0,\cdots,0)} T_{(1,0,\cdots,0)} \rangle_{\text{Nahm}}^{4d\ \mathcal{N}=4\ U(N)}(t^{-1};q).$$

$$\langle W_{(1^k)} W_{\overline{(1^k)}} \rangle_{\mathcal{N}}^{4d\ \mathcal{N}=4\ U(N)}(t;q)$$
$$= \frac{1}{N!} \frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \prod_{i\neq j} \frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty} e_k(s) e_k(s^{-1}),$$

$$e_k(s) = \sum_{1\le i_1<\cdots<i_k\le N} s_{i_1}\cdots s_{i_k}$$

$$x_i = q^{\frac{2i-1}{4}} t^{2i-1}, \quad i = 1,\cdots,N-k$$
$$x_j = q^{\frac{2j+3}{4}} t^{2j-1}, \quad j = N-k+1,\cdots,N$$

$$\mathbb{III}_{\mathcal{D}}^{4d\ \mathcal{N}=4\ U(N)}\left(t, \left\{x_i = q^{\frac{2i-1}{4}} t^{2i-1}\right\}_{i=1}^{N-k}, \left\{x_j = q^{\frac{2j+3}{4}} t^{2j-1}\right\}_{j=N-k+1}^{N}; q\right)$$
$$= \langle T_{(1^k,0^{N-k})} T_{(1^k,0^{N-k})} \rangle_{\text{Nahm}}^{4d\ \mathcal{N}=4\ U(N)}(t;q)$$
$$\times \prod_{i=1}^{k-1} \mathcal{I}^{3d\ \text{HM}}\left(t, x = q^{\frac{2i-1}{4}} t^{2i-1}; q\right)^{k-i} \prod_{i=1}^{N-k-1} \mathcal{I}^{3d\ \text{HM}}\left(t, x = q^{\frac{2i-1}{4}} t^{2i-1}; q\right)^{N-k-i}$$
$$\times \prod_{i=1}^{k-1} \mathcal{I}^{3d\ \text{HM}}\left(t, x = q^{\frac{2i+k+1}{4}} t^{2i+k-3}; q\right)^{i} \prod_{i=k}^{N-k} \mathcal{I}^{3d\ \text{HM}}\left(t, x = q^{\frac{2i+k+1}{4}} t^{2i+k-3}; q\right)^{k}$$
$$\times \prod_{i=N-k+1}^{N-1} \mathcal{I}^{3d\ \text{HM}}\left(t, x = q^{\frac{2i+k+1}{4}} t^{2i+k-3}; q\right)^{N-i}.$$

$$\langle T_{(1^k,0^{N-k})} T_{(1^k,0^{N-k})} \rangle_{\text{Nahm}}^{4d\ \mathcal{N}=4\ U(N)}(t;q)$$
$$= \prod_{l=1}^{N-k} \frac{(q^{\frac{l+1}{2}} t^{2(l-1)};q)_\infty}{(q^{\frac{l}{2}} t^{2l};q)_\infty} \prod_{m=1}^{k} \frac{(q^{\frac{m+1}{2}} t^{2(m-1)};q)_\infty}{(q^{\frac{m}{2}} t^{2m};q)_\infty} \frac{(q^{\frac{m+2}{2}} t^{2m};q)_\infty}{(q^{\frac{m+3}{2}} t^{2(m-1)};q)_\infty} \frac{(q^{\frac{m+N-k+3}{2}} t^{2m+2N-2k-2};q)_\infty}{(q^{\frac{m+N-k+2}{2}} t^{2m+2N-2k};q)_\infty}.$$

$$\langle T_{(1,1,0,0)} T_{(1,1,0,0)} \rangle_{\text{Nahm}}^{4d\ \mathcal{N}=4\ U(4)}(t;q)$$
$$= \frac{(q)_\infty^2 (q^{\frac{3}{2}}t^2;q)_\infty^3 (q^2t^4;q)_\infty (q^3t^4;q)_\infty (q^{\frac{7}{2}}t^6;q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty^2 (qt^4;q)_\infty^2 (q^2;q)_\infty (q^{\frac{5}{2}}t^2;q)_\infty (q^{\frac{5}{2}}t^6;q)_\infty (q^3t^8;q)_\infty},$$
$$\langle T_{(1,1,0,0,0)} T_{(1,1,0,0,0)} \rangle_{\text{Nahm}}^{4d\ \mathcal{N}=4\ U(5)}(t;q)$$
$$= \frac{(q)_\infty^2 (q^{\frac{3}{2}}t^2;q)_\infty^3 (q^2t^4;q)_\infty^2 (q^{\frac{7}{2}}t^6;q)_\infty (q^4t^8;q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty^2 (qt^4;q)_\infty^2 (q^{\frac{3}{2}}t^6;q)_\infty (q^2;q)_\infty (q^{\frac{5}{2}}t^2;q)_\infty (q^3t^8;q)_\infty (q^{\frac{7}{2}}t^{10};q)_\infty},$$
$$\langle T_{(1,1,0,0,0,0)} T_{(1,1,0,0,0,0)} \rangle_{\text{Nahm}}^{4d\ \mathcal{N}=4\ U(6)}(t;q)$$
$$= \frac{(q)_\infty^2 (q^{\frac{3}{2}}t^2;q)_\infty^3 (q^2t^4;q)_\infty^2 (q^{\frac{5}{2}}t^6;q)_\infty (q^4t^8;q)_\infty (q^{\frac{9}{2}}t^{10};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty^2 (qt^4;q)_\infty^2 (q^{\frac{3}{2}}t^6;q)_\infty (q^2;q)_\infty (q^2t^8;q)_\infty (q^{\frac{5}{2}}t^2;q)_\infty (q^{\frac{7}{2}}t^{10};q)_\infty (q^4t^{12};q)_\infty},$$

$$\langle T_{(1,1,1,0,0,0)}T_{(1,1,1,0,0,0)}\rangle^{\text{4d }\mathcal{N}=4\ U(6)}_{\text{Nahm}}(t;q)$$
$$=\frac{(q)_\infty^2(q^{\frac{3}{2}}t^2;q)_\infty^3(q^2t^4;q)_\infty^3(q^{\frac{5}{2}}t^6;q)_\infty(q^{\frac{7}{2}}t^6;q)_\infty(q^4t^8;q)_\infty(q^{\frac{9}{2}}t^{10};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty^2(qt^4;q)_\infty^2(q^{\frac{3}{2}}t^6;q)_\infty^2(q^2;q)_\infty(q^{\frac{5}{2}}t^2;q)_\infty(q^3t^4;q)_\infty(q^3t^8;q)_\infty(q^{\frac{7}{2}}t^{10};q)_\infty(q^5t^{12};q)_\infty}.$$

$$\langle W_{(1^k)}W_{\overline{(1^k)}}\rangle^{\text{4d }\mathcal{N}=4\ U(N)}_{\mathcal{N}}(t;q)=\langle T_{(1^k,0^{N-k})}T_{(1^k,0^{N-k})}\rangle^{\text{4d }\mathcal{N}=4\ U(N)}_{\text{Nahm}}(t^{-1};q).$$

$$R\ =\{\varepsilon_i-\varepsilon_j \mid 1\le i\ne j\le N\}$$
$$R^+\ =\{\varepsilon_i-\varepsilon_j \mid 1\le i<j\le N\}$$

$$\rho=\frac{1}{2}\sum_{i=1}^{N}(N-2i+1)\varepsilon_i,\lambda=\sum_{i=1}^{N}\lambda_i\varepsilon_i$$

$$w_{A_{N-1}}(s;\mathrm{q},\mathrm{t})=\prod_{1\le i\ne j\le N}\frac{(s_i/s_j;\mathrm{q})_\infty}{(\mathrm{t}s_i/s_j;\mathrm{q})_\infty},$$

$$\frac{\langle P_\lambda,P_\lambda\rangle}{\langle 1,1\rangle}=\prod_{1\le i<j\le N}\frac{(\mathrm{t}^{j-i+1};\mathrm{q})_{\lambda_i-\lambda_j}(\mathrm{t}^{j-i-1}\mathrm{q};\mathrm{q})_{\lambda_i-\lambda_j}}{(\mathrm{t}^{j-i};\mathrm{q})_{\lambda_i-\lambda_j}(\mathrm{t}^{j-i}\mathrm{q};\mathrm{q})_{\lambda_i-\lambda_j}},$$

$$\langle 1,1\rangle=\prod_{k=1}^{N}\frac{(\mathrm{t}^{k-1}\mathrm{q};\mathrm{q})_\infty}{(\mathrm{t}^k;\mathrm{q})_\infty}$$

$$\frac{\left\langle W_{(1^k)}W_{\overline{(1^k)}}\right\rangle^{U(N)}_{\mathcal{N}}}{\mathbb{II}^{U(N)}_{\mathcal{N}}}=\frac{\left\langle P_{(1^k)},P_{(1^k)}\right\rangle}{\langle 1,1\rangle}=\frac{(\mathrm{q};\mathrm{t})_k(\mathrm{t}^{N-k+1};\mathrm{t})_k}{(\mathrm{t};\mathrm{t})_k(\mathrm{t}^{N-k}\mathrm{q};\mathrm{q})_k}$$

$$\chi_{(2)}=h_2=P_{(2)}+\frac{\mathrm{t}-\mathrm{q}}{1-\mathrm{qt}}P_{(1^2)},$$

$$\frac{\left\langle W_{(2)}W_{\overline{(2)}}\right\rangle^{U(N)}_{\mathcal{N}}}{\mathbb{II}I^{U(N)}_{\mathcal{N}}}=\frac{(1-\mathrm{q})(1-\mathrm{t}^N)}{(1-\mathrm{t})(1-\mathrm{qt})(1-\mathrm{qt}^{N-1})}\left[\frac{(1-\mathrm{q}^2)(1-\mathrm{qt}^N)}{1-\mathrm{q}^2\mathrm{t}^{N-1}}+\frac{(\mathrm{t}-\mathrm{q})^2(1-\mathrm{t}^{N-1})}{(1-\mathrm{t}^2)(1-\mathrm{qt}^{N-2})}\right]$$

$$\mathbb{II}^{\text{4d }\mathcal{N}=4\ SO(3)}_{\mathcal{N}}(t;q)=\frac{1}{2}\frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds}{2\pi is}\frac{(s^\pm;q)_\infty}{(q^{\frac{1}{2}}t^{-2}s^\pm;q)_\infty}.$$

$$\mathbb{II}^{\text{4d }\mathcal{N}=4\ USp(2)}_{\text{Nahm}}(t;q)=\frac{(q^{\frac{3}{2}}t^2;q)_\infty}{(qt^4;q)_\infty}.$$

$$\mathbb{II}^{\text{4d }\mathcal{N}=4\ USp(2)}_{\mathcal{D}}(t,x;q)=\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty}\frac{(qx^2;q)_\infty(qx^{-2};q)_\infty}{(q^{\frac{1}{2}}t^2x^2;q)_\infty(q^{\frac{1}{2}}t^2x^{-2};q)_\infty}.$$

$$\mathbb{II}^{\text{4d }\mathcal{N}=4\ USp(2)}_{\mathcal{D}}\left(t,x=q^{\frac{1}{4}}t;q\right)$$
$$=\mathbb{II}^{\text{4d }\mathcal{N}=4\ USp(2)}_{\text{Nahm}}(t;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{1}{4}}t;q).$$

$$\langle W_{\rm sp}W_{\rm sp}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ Spin(3)}(t;q) = \frac{1}{2}\frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds}{2\pi is}\frac{(s^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s^{\pm};q)_\infty}(2+s+s^{-1}).$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ USp(2)}\left(t,x=q^{\frac{3}{4}}t;q\right) = \langle T_{(\frac{1}{2})}T_{(\frac{1}{2})}\rangle_{\rm Nahm}^{\text{4d }\mathcal{N}=4\ USp(2)/\mathbb{Z}_2}(t;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{5}{4}}t;q).$$

$$\langle T_{(\frac{1}{2})}T_{(\frac{1}{2})}\rangle_{\rm Nahm}^{\text{4d }\mathcal{N}=4\ USp(2)/\mathbb{Z}_2}(t;q) = \frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(q^2;q)_\infty}\frac{(q^{\frac{5}{2}}t^2;q)_\infty}{(q^2t^4;q)_\infty}.$$

$$\langle W_{\Box}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(3)}(t;q) = q^{\frac{1}{2}}t^{-2}\frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{3}{2}}t^2;q)_\infty}\frac{(q^{\frac{5}{2}}t^{-2};q)_\infty}{(qt^{-4};q)_\infty}.$$

$$\langle W_{\Box}W_{\Box}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(3)}(t;q) = \frac{(q^{\frac{3}{2}}t^{-2};q)_\infty}{(qt^{-4};q)_\infty} + q^{\frac{1}{2}}t^{-2}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q^{\frac{5}{2}}t^{-2};q)_\infty}{(qt^{-4};q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty} + qt^{-4}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q^{\frac{7}{2}}t^{-2};q)_\infty}{(q^{\frac{5}{2}}t^2;q)_\infty(qt^{-4};q)_\infty}.$$

$$\langle \underbrace{W_{\Box}\cdots W_{\Box}}_{k}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(3)}(t;q) = \sum_{i=0}^{k} C_{k,i}q^{\frac{i}{2}}t^{-2i}\frac{\left(q^{\frac{i}{2}}t^2;q\right)_\infty\left(q^{\frac{2i+3}{2}}t^{-2};q\right)_\infty}{\left(q^{\frac{2i+1}{2}}t^2;q\right)_\infty(qt^{-4};q)_\infty},$$

$$C_{k,i} = T(k,i) - T(k,i-1) = \sum_{j=0}^{k}\binom{k}{j}\left[\binom{j}{i-j} - \binom{j}{i-1-j}\right]$$

$$(1+x+x^2)^n = \sum_{k=0}^{2n} T(n,k)x^k$$

$$\langle W_{(k)}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(3)}(t;q) = q^{\frac{k}{2}}t^{-2k}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q^{\frac{2k+3}{2}}t^{-2};q)_\infty}{(q^{\frac{2k+1}{2}}t^2;q)_\infty(qt^{-4};q)_\infty}.$$

$$\mathbb{II}_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(5)}(t;q) = \frac{1}{8}\frac{(q)_\infty^2}{(q^{\frac{1}{2}}t^{-2})^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi is_i}\frac{(s_i^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q)_\infty}\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q)_\infty}.$$

$$\mathbb{II}_{\rm Nahm}^{\text{4d }\mathcal{N}=4\ USp(4)}(t;q) = \frac{(q^{\frac{3}{2}}t^2;q)_\infty(q^{\frac{5}{2}}t^6;q)_\infty}{(qt^4;q)_\infty(q^2t^8;q)_\infty}.$$

$$\mathbb{III}_{\mathcal{D}}^{\text{4d } \mathcal{N}=4 \ USp(4)}(t,x_1,x_2;q) = \frac{(q)_\infty^2}{(q^{\frac{1}{2}}t^2;q)_\infty^2}\prod_{i=1}^{2}\frac{(qx_i^{\pm 2};q)_\infty}{(q^{\frac{1}{2}}t^2x_i^{\pm 2};q)_\infty}\prod_{i<j}\frac{(qx_i^{\pm}x_j^{\mp};q)_\infty(qx_i^{\pm}x_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\mp};q)_\infty(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\pm};q)_\infty}.$$

$$x_1 = q^{\frac{1}{4}}t, x_2 = q^{\frac{3}{4}}t^3$$

$$\mathbb{III}_{\mathcal{D}}^{\text{4d } \mathcal{N}=4 \ USp(4)}\left(t, x_1 = q^{\frac{1}{4}}t, x_2 = q^{\frac{3}{4}}t^3; q\right) = \mathbb{III}_{\text{Nahm}}^{\text{4d } \mathcal{N}=4 \ USp(4)}(t;q)\mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{1}{4}}t;q)^2 \times \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{3}{4}}t^3;q)\mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{5}{4}}t^5;q).$$

$$\langle W_{\text{sp}}W_{\text{sp}}\rangle_{\mathcal{N}}^{\text{4d } \mathcal{N}=4 \ Spin(5)}(t;q) = \frac{1}{8}\frac{(q)_\infty^2}{(q^{\frac{1}{2}}t^{-2})^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q)_\infty}\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q)_\infty} \times (4+2(s_1+s_2+s_1^{-1}+s_2^{-1})+s_1s_2+s_1^{-1}s_2^{-1}+s_1s_2^{-1}+s_1^{-1}s_2).$$

$$x_1 = q^{\frac{3}{4}}t, x_2 = q^{\frac{5}{4}}t^3$$

$$\mathbb{III}_{\mathcal{D}}^{\text{4d } \mathcal{N}=4 \ USp(4)}\left(t, x_1 = q^{\frac{3}{4}}t, x_2 = q^{\frac{5}{4}}t^3; q\right) = \langle T_{(\frac{1}{2},\frac{1}{2})}T_{(\frac{1}{2},\frac{1}{2})}\rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4 \ USp(4)/\mathbb{Z}_2}(t;q)\mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{1}{4}}t;q)\mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{5}{4}}t;q) \times \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{7}{4}}t^3;q)\mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{9}{4}}t^5;q).$$

$$\langle T_{(\frac{1}{2},\frac{1}{2})}T_{(\frac{1}{2},\frac{1}{2})}\rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4 \ USp(4)/\mathbb{Z}_2}(t;q) = \frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty^2(q^{\frac{7}{2}}t^6;q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(qt^4;q)_\infty(q^2;q)_\infty(q^3t^8;q)_\infty}.$$

$$\langle W_{\square}\rangle_{\mathcal{N}}^{\text{4d } \mathcal{N}=4 \ SO(5)}(t;q) = qt^{-4}\frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{3}{2}}t^2;q)_\infty}\frac{(q^{\frac{3}{2}}t^{-2};q)_\infty(q^{\frac{7}{2}}t^{-6};q)_\infty}{(qt^{-4};q)_\infty(q^2t^{-8};q)_\infty}.$$

$$\langle W_{\square}W_{\square}\rangle_{\mathcal{N}}^{\text{4d } \mathcal{N}=4 \ SO(5)}(t;q) = \frac{(q^{\frac{3}{2}}t^{-2};q)_\infty(q^{\frac{5}{2}}t^{-6};q)_\infty}{(qt^{-4};q)_\infty(q^2t^{-8};q)_\infty} + q^{\frac{1}{2}}t^{-2}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q^{\frac{3}{2}}t^{-2};q)_\infty(q^2t^{-4};q)_\infty(q^{\frac{7}{2}}t^{-6};q)_\infty}{(qt^{-4};q)_\infty^2(q^{\frac{3}{2}}t^2;q)_\infty(q^3t^{-8};q)_\infty} + qt^{-4}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q^{\frac{3}{2}}t^{-2};q)_\infty(q^2t^{-4};q)_\infty(q^{\frac{5}{2}}t^{-2};q)_\infty(q^{\frac{9}{2}}t^{-6};q)_\infty}{(qt^{-4};q)_\infty^2(q^{\frac{3}{2}}t^2;q)_\infty(q^3t^{-8};q)_\infty(q^{\frac{7}{2}}t^{-2};q)_\infty}.$$

$$\langle W_{\substack{\square\\\square}}\rangle_{\mathcal{N}}^{\text{4d } \mathcal{N}=4 \ SO(5)}(t;q) = \langle W_{\text{sp}}W_{\text{sp}}\rangle_{\mathcal{N}}^{\text{4d } \mathcal{N}=4 \ Spin(5)}(t;q) - \langle W_{\square}\rangle_{\mathcal{N}}^{\text{4d } \mathcal{N}=4 \ SO(5)}(t;q) - \mathbb{III}_{\mathcal{N}}^{\text{4d } \mathcal{N}=4 \ SO(5)}(t;q)$$

$$= q^{\frac{1}{2}}t^{-2}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q^{\frac{3}{2}}t^{-2};q)_\infty(q^2t^{-4};q)_\infty(q^{\frac{7}{2}}t^{-6};q)_\infty}{(qt^{-4};q)_\infty^2(q^{\frac{3}{2}}t^2;q)_\infty(q^3t^{-8};q)_\infty}.$$

$$\langle W_{\square\square}\rangle_{\mathcal{N}}^{4d\ \mathcal{N}=4\ SO(5)}(t;q)$$
$$= qt^{-4}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q^{\frac{3}{2}}t^{-2};q)_\infty(q^2t^{-4};q)_\infty(q^{\frac{5}{2}}t^{-2};q)_\infty(q^{\frac{9}{2}}t^{-6};q)_\infty}{(qt^{-4};q)_\infty^2(q^{\frac{3}{2}}t^2;q)_\infty(q^3t^{-8};q)_\infty(q^{\frac{7}{2}}t^{-2};q)_\infty}.$$

$$\mathbb{II}_{\mathcal{N}}^{4d\ \mathcal{N}=4\ SO(2N+1)}(t;q)$$
$$= \frac{1}{2^N N!}\frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2})^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}\frac{(s_i^\pm;q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^\pm;q)_\infty}\prod_{i<j}\frac{(s_i^\pm s_j^\mp;q)_\infty(s_i^\pm s_j^\pm;q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\mp;q)_\infty(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\pm;q)_\infty}.$$

$$\mathbb{II}_{\text{Nahm}}^{4d\ \mathcal{N}=4\ USp(2N)}(t;q) = \prod_{k=1}^N\frac{(q^{k+\frac{1}{2}}t^{4k-2};q)_\infty}{(q^kt^{4k};q)_\infty}.$$

$$\mathbb{II}_{\mathcal{D}}^{4d\ \mathcal{N}=4\ USp(2N)}(t,x_i;q)$$
$$= \frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^2;q)_\infty^N}\prod_{i=1}^N\frac{(qx_i^{\pm 2};q)_\infty}{(q^{\frac{1}{2}}t^2x_i^{\pm 2};q)_\infty}\prod_{i<j}\frac{(qx_i^\pm x_j^\mp;q)_\infty(qx_i^\pm x_j^\pm;q)_\infty}{(q^{\frac{1}{2}}t^2x_i^\pm x_j^\mp;q)_\infty(q^{\frac{1}{2}}t^2x_i^\pm x_j^\pm;q)_\infty}.$$

$$x_i = q^{\frac{2i-1}{4}}t^{2i-1}$$

$$\mathbb{II}_{\mathcal{D}}^{4d\ \mathcal{N}=4\ USp(2N)}\left(t,x_i=q^{\frac{2i-1}{4}}t^{2i-1};q\right)$$
$$= \mathbb{II}_{\text{Nahm}}^{4d\ \mathcal{N}=4\ USp(2N)}(t;q)\prod_{i=1}^N\mathcal{I}^{3d\ \text{HM}}(t,x=q^{\frac{2i-1}{2}}t^{4i-2};q)$$
$$\times\prod_{i=1}^{N-1}\mathcal{I}^{3d\ \text{HM}}(t,x=q^{\frac{2i-1}{4}}t^{2i-1};q)^{N-i}\times\prod_{i=1}^{2N-3}\mathcal{I}^{3d\ \text{HM}}(t,x=q^{\frac{2i+1}{4}}t^{2i+1};q)^{a_N(i)}.$$

$$\binom{N}{2}_q = \sum_{i=1}^{2N-3}a_N(i)q^{i-1}$$

$$\binom{n}{k}_q := \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[1]_q[2]_q\cdots[k]_q}$$
$$[n]_q := \frac{1-q^n}{1-q}$$

$$\langle W_{\text{sp}}W_{\text{sp}}\rangle_{\mathcal{N}}^{4d\ \mathcal{N}=4\ Spin(2N+1)}(t;q)$$
$$= \frac{1}{2^N N!}\frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2})^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}\frac{(s_i^\pm;q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^\pm;q)_\infty}\prod_{i<j}\frac{(s_i^\pm s_j^\mp;q)_\infty(s_i^\pm s_j^\pm;q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\mp;q)_\infty(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\pm;q)_\infty}$$

$$\times \prod_{i=1}^{N} (s_i^{\frac{1}{2}} + s_i^{-\frac{1}{2}})^2.$$

$$x_i = q^{\frac{2i+1}{4}} t^{2i-1}, i = 1, \cdots, N$$

$$\mathbb{III}_{\mathcal{D}}^{\text{4d } \mathcal{N}=4\ USp(2N)} \left(t, x_i = q^{\frac{2i+1}{4}} t^{2i-1}; q\right)$$
$$= \langle T_{(\frac{1}{2},\cdots,\frac{1}{2})} T_{(\frac{1}{2},\cdots,\frac{1}{2})} \rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4\ USp(2N)/\mathbb{Z}_2}(t;q) \prod_{i=1}^{N-1} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{2i-1}{4}} t^{2i-1}; q)^{N-i}$$

$$\times \prod_{i=1}^{2N-1} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{2i+3}{4}} t^{2i-1}; q)^{a_{N+1}(i)},$$

$$\langle T_{(\frac{1}{2},\cdots,\frac{1}{2})} T_{(\frac{1}{2},\cdots,\frac{1}{2})} \rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4\ USp(2N)/\mathbb{Z}_2}(t;q)$$
$$= \prod_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} \frac{(q^{\frac{2k+1}{2}} t^{4k-2}; q)_\infty}{(q^{k+1} t^{4k-4}; q)_\infty} \prod_{k=1}^{N} \frac{(q^{\frac{k+1}{2}} t^{2(k-1)}; q)_\infty}{(q^{\frac{k}{2}} t^{2k}; q)_\infty} \prod_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} \frac{(q^{\frac{2(k+\lfloor \frac{N}{2} \rfloor)+3}{2}} t^{4(k+\lfloor \frac{N}{2} \rfloor)-2}; q)_\infty}{(q^{k+\lfloor \frac{N}{2} \rfloor+1} t^{4(k+\lfloor \frac{N}{2} \rfloor)}; q)_\infty},$$

$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\}$$

$$\langle T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} \rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4\ USp(6)/\mathbb{Z}_2}(t;q)$$
$$= \frac{(q)_\infty (q^{\frac{3}{2}} t^2; q)_\infty^2 (q^2 t^4; q)_\infty (q^{\frac{5}{2}} t^6; q)_\infty (q^{\frac{7}{2}} t^6; q)_\infty (q^{\frac{9}{2}} t^{10}; q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty (q t^4; q)_\infty (q^{\frac{3}{2}} t^6; q)_\infty (q^2; q)_\infty (q^3 t^4; q)_\infty (q^3 t^8; q)_\infty (q^4 t^{12}; q)_\infty},$$

$$\langle T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} \rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4\ USp(8)/\mathbb{Z}_2}(t;q)$$
$$= \frac{(q)_\infty (q^{\frac{3}{2}} t^2; q)_\infty^2 (q^2 t^4; q)_\infty (q^{\frac{5}{2}} t^6; q)_\infty^2 (q^{\frac{9}{2}} t^{10}; q)_\infty (q^{\frac{11}{2}} t^{14}; q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty (q t^4; q)_\infty (q^{\frac{3}{2}} t^6; q)_\infty (q^2; q)_\infty (q^2 t^8; q)_\infty (q^3 t^4; q)_\infty (q^4 t^{12}; q)_\infty (q^5 t^{16}; q)_\infty},$$

$$\langle T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} \rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4\ USp(10)/\mathbb{Z}_2}(t;q)$$
$$= \frac{(q)_\infty (q^{\frac{3}{2}} t^2; q)_\infty^2 (q^2 t^4; q)_\infty (q^{\frac{5}{2}} t^6; q)_\infty^2}{(q^{\frac{1}{2}} t^2; q)_\infty (q t^4; q)_\infty (q^{\frac{3}{2}} t^6; q)_\infty (q^2; q)_\infty (q^2 t^8; q)_\infty}$$
$$\times \frac{(q^3 t^8; q)_\infty (q^{\frac{7}{2}} t^{10}; q)_\infty (q^{\frac{9}{2}} t^{10}; q)_\infty (q^{\frac{11}{2}} t^{14}; q)_\infty (q^{\frac{13}{2}} t^{18}; q)_\infty}{(q^{\frac{5}{2}} t^{10}; q)_\infty (q^3 t^4; q)_\infty (q^4 t^8; q)_\infty (q^4 t^{12}; q)_\infty (q^5 t^{16}; q)_\infty (q^6 t^{20}; q)_\infty}.$$

$$\langle W_{\text{sp}} W_{\text{sp}} \rangle_{\mathcal{N}}^{\text{4d } \mathcal{N}=4\ Spin(2N+1)}(t;q) = \langle T_{(\frac{1}{2},\cdots,\frac{1}{2})} T_{(\frac{1}{2},\cdots,\frac{1}{2})} \rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4\ USp(2N)/\mathbb{Z}_2}(t^{-1};q).$$

$$R = \{\pm\varepsilon_i \mid 1 \leq i \leq N\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq N\},$$
$$R^+ = \{\varepsilon_i \mid 1 \leq i \leq N\} \cup \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq N\},$$

$$\rho = \sum_{i=1}^{N} \left(N - i + \frac{1}{2}\right) \varepsilon_i, \lambda = \sum_{i=1}^{N} \lambda_i \varepsilon_i$$

$$w_{B_N}(s;\mathbf{q},\mathbf{t})=\prod_{i=1}^{N}\frac{\left(s_i^{\pm1};\mathbf{q}\right)_\infty}{\left(\mathbf{t}s_i^{\pm1};\mathbf{q}\right)_\infty}\prod_{1\le i<j\le N}\frac{\left(s_i^{\pm1}s_j^{\pm1};\mathbf{q}\right)_\infty}{\left(\mathbf{t}s_i^{\pm1}s_j^{\pm1};\mathbf{q}\right)_\infty}$$

$$\frac{\langle P_\lambda,P_\lambda\rangle}{\langle 1,1\rangle}=\prod_{i=1}^{N}\frac{\left(\mathrm{t}^{2N-2i+2};\mathrm{q}\right)_{2\lambda_i}\left(\mathrm{t}^{2N-2i}\mathrm{q};\mathrm{q}\right)_{2\lambda_i}}{\left(\mathrm{t}^{2N-2i+1};\mathrm{q}\right)_{2\lambda_i}\left(\mathrm{t}^{2N-2i+1}\mathrm{q};\mathrm{q}\right)_{2\lambda_i}}$$
$$\times\prod_{1\le i<j\le N}\frac{\left(\mathrm{t}^{j-i+1};\mathrm{q}\right)_{\lambda_i-\lambda_j}\left(\mathrm{t}^{j-i-1}\mathrm{q};\mathrm{q}\right)_{\lambda_i-\lambda_j}}{\left(\mathrm{t}^{j-i};\mathrm{q}\right)_{\lambda_i-\lambda_j}\left(\mathrm{t}^{j-i}\mathrm{q};\mathrm{q}\right)_{\lambda_i-\lambda_j}}\frac{\left(\mathrm{t}^{2N-i-j+2};\mathrm{q}\right)_{\lambda_i+\lambda_j}\left(\mathrm{t}^{2N-i-j}\mathrm{q};\mathrm{q}\right)_{\lambda_i+\lambda_j}}{\left(\mathrm{t}^{2N-i-j+1};\mathrm{q}\right)_{\lambda_i+\lambda_j}\left(\mathrm{t}^{2N-i-j+1}\mathrm{q};\mathrm{q}\right)_{\lambda_i+\lambda_j}}$$

$$\langle 1,1\rangle=\frac{(\mathrm{t};\mathrm{q})_\infty^N}{(\mathrm{q};\mathrm{q})_\infty^N}\prod_{k=1}^{N}\frac{\left(\mathrm{t}^{2k-1}\mathrm{q};\mathrm{q}\right)_\infty}{\left(\mathrm{t}^{2k};\mathrm{q}\right)_\infty}$$

$$\chi_{(1^r)}(s)=E_r(s)+E_{r-1}(s)$$

$$E_r(s)=\sum_{k=0}^{\lfloor\frac{r}{2}\rfloor}\frac{\left(\mathrm{t}^{N-r+1};\mathrm{t}\right)_{2k}}{\left(\mathrm{t}^{N-r};\mathrm{t}\right)_{2k}}\frac{\left(\mathrm{t}^{2N-2r};\mathrm{t}^2\right)_{2k}}{\left(\mathrm{t}^{2N-2r-1}\mathrm{q};\mathrm{t}^2\right)_{2k}}\frac{\left(\mathrm{q}/\mathrm{t},\mathrm{t}^{2N-2r-1};\mathrm{t}^2\right)_k}{\left(\mathrm{t}^2,\mathrm{t}^{2N-2r+2};\mathrm{t}^2\right)_k}\mathrm{t}^k$$
$$\times\sum_{j=0}^{r-2k}(-1)^j\frac{\left(\mathrm{t}^{N-r+2k+1},-\mathrm{t}^{N-r+2k}\mathrm{q},-\mathrm{t}^{N-r+2k}\mathrm{q}^{1/2},\mathrm{t}^{N-r+2k}\mathrm{q}^{1/2};\mathrm{t}\right)_j}{\left(\mathrm{t}^{2N-2r+4k}\mathrm{q};\mathrm{t}\right)_{2j}}P_{(1^{r-2k-j})}(s;\mathrm{q},\mathrm{t})$$

$$\bar{E}(s\mid z)=\prod_{i=1}^{N}\left(1+zs_i\right)\left(1+zs_i^{-1}\right)=\sum_{r=0}^{2N}z^rE_r(s)$$

$$\chi_{(1)}(s)=E_1(s)+E_0(s)=P_{(1)}(s;\mathbf{q},\mathbf{t})+\frac{\mathrm{t}^{N-1}(\mathrm{t}-\mathbf{q})}{1-\mathrm{q}\mathrm{t}^{2N-1}}$$

$$\left\langle\chi_{(1)},1\right\rangle=\frac{\mathrm{t}^{N-1}(\mathrm{t}-\mathrm{q})}{1-\mathrm{q}\mathrm{t}^{2N-1}}\langle 1,1\rangle$$
$$\left\langle\chi_{(1)},\chi_{(1)}\right\rangle=\left\langle P_{(1)},P_{(1)}\right\rangle+\frac{\mathrm{t}^{2N-2}(\mathrm{t}-\mathrm{q})^2}{\left(1-\mathrm{q}\mathrm{t}^{2N-1}\right)^2}\langle 1,1\rangle$$

$$\frac{\left\langle P_{(1)},P_{(1)}\right\rangle}{\langle 1,1\rangle}=\frac{(1-\mathrm{q})\left(1-\mathrm{q}^2\mathrm{t}^{2N-2}\right)\left(1-\mathrm{t}^{2N}\right)\left(1-\mathrm{q}\mathrm{t}^{2N}\right)}{(1-\mathrm{t})\left(1-\mathrm{q}\mathrm{t}^{2N-1}\right)^2\left(1-\mathrm{q}^2\mathrm{t}^{2N-1}\right)}$$

$$\frac{\langle W_\square\rangle_{\mathcal{N}}^{SO(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}}=\frac{\mathrm{t}^{N-1}(\mathrm{t}-\mathrm{q})}{1-\mathrm{q}\mathrm{t}^{2N-1}},$$
$$\frac{\langle W_\square W_\square\rangle_{\mathcal{N}}^{SO(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}}=\frac{(1-\mathrm{q})\left(1-\mathrm{q}^2\mathrm{t}^{2N-2}\right)\left(1-\mathrm{t}^{2N}\right)\left(1-\mathrm{q}\mathrm{t}^{2N}\right)}{(1-\mathrm{t})\left(1-\mathrm{q}\mathrm{t}^{2N-1}\right)^2\left(1-\mathrm{q}^2\mathrm{t}^{2N-1}\right)}+\frac{\mathrm{t}^{2N-2}(\mathrm{t}-\mathrm{q})^2}{\left(1-\mathrm{q}\mathrm{t}^{2N-1}\right)^2}.$$

$$\langle W_\square W_\square\rangle_{\mathcal{N},c}^{SO(2N+1)}:=\frac{\langle W_\square W_\square\rangle_{\mathcal{N}}^{SO(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}}-\left(\frac{\langle W_\square\rangle_{\mathcal{N}}^{SO(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}}\right)^2$$
$$=\frac{(1-\mathrm{q})\left(1-\mathrm{q}^2\mathrm{t}^{2N-2}\right)\left(1-\mathrm{t}^{2N}\right)\left(1-\mathrm{q}\mathrm{t}^{2N}\right)}{(1-\mathrm{t})\left(1-\mathrm{q}\mathrm{t}^{2N-1}\right)^2\left(1-\mathrm{q}^2\mathrm{t}^{2N-1}\right)}$$

$$\frac{\left\langle W_{(1^r)}\right\rangle_{\mathcal{N}}^{SO(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}}=\frac{\left\langle\chi_{(1^r)},1\right\rangle}{\langle 1,1\rangle}=\mathcal{E}_{N,r}+\mathcal{E}_{N,r-1}$$

$$\mathcal{E}_{N,r}:=\frac{\langle E_r,1\rangle}{\langle 1,1\rangle}=\sum_{k=0}^{\lfloor\frac{r}{2}\rfloor}(-1)^{r-2k}\frac{\left(\mathrm{t}^{N-r+1};\mathrm{t}\right)_{2k}}{\left(\mathrm{t}^{N-r};\mathrm{t}\right)_{2k}}\frac{\left(\mathrm{t}^{2N-2r};\mathrm{t}^2\right)_{2k}}{\left(\mathrm{t}^{2N-2r-1}\mathrm{q};\mathrm{t}^2\right)_{2k}}\frac{\left(\mathrm{q}/\mathrm{t},\mathrm{t}^{2N-2r-1};\mathrm{t}^2\right)_k}{\left(\mathrm{t}^2,\mathrm{t}^{2N-2r+2};\mathrm{t}^2\right)_k}$$
$$\times\frac{\left(\mathrm{t}^{N-r+2k+1},-\mathrm{t}^{N-r+2k}\mathrm{q},-\mathrm{t}^{N-r+2k}\mathrm{q}^{1/2},\mathrm{t}^{N-r+2k}\mathrm{q}^{1/2};\mathrm{t}\right)_{r-2k}}{\left(\mathrm{t}^{2N-2r+4k}\mathrm{q};\mathrm{t}\right)_{2r-4k}}\mathrm{t}^k$$

$$\frac{\langle W_{(1^{2m+1})}\rangle_{\mathcal{N}}^{SO(2N+1)}}{\langle W_{(1^{2m})}\rangle_{\mathcal{N}}^{SO(2N+1)}} = \frac{\mathrm{t}^{N-2m}(1-\mathrm{qt}^{2m-1})}{1-\mathrm{qt}^{2N-2m-1}}$$

$$\frac{\langle W_{(1^{2m})}\rangle_{\mathcal{N}}^{SO(2N+1)}}{\langle W_{(1^{2m-1})}\rangle_{\mathcal{N}}^{SO(2N+1)}} = \frac{1-\mathrm{t}^{2N-2m+2}}{\mathrm{t}^{N-2m+1}(1-\mathrm{t}^{2m})}$$

$$\frac{\langle W_{(1^2)}\rangle_{\mathcal{N}}^{SO(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}} = \frac{(\mathrm{t}-\mathrm{q})(1-\mathrm{t}^{2N})}{(1-\mathrm{t}^2)(1-\mathrm{qt}^{2N-1})}$$

$$\frac{\langle W_{(1^3)}\rangle_{\mathcal{N}}^{SO(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}} = \frac{\mathrm{t}^{N-2}(\mathrm{t}-\mathrm{q})(1-\mathrm{qt})(1-\mathrm{t}^{2N})}{(1-\mathrm{t}^2)(1-\mathrm{qt}^{2N-1})(1-\mathrm{qt}^{2N-3})}$$

$$\frac{\langle W_{(1^4)}\rangle_{\mathcal{N}}^{SO(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}} = \frac{\mathrm{t}(\mathrm{t}-\mathrm{q})(1-\mathrm{qt})(1-\mathrm{t}^{2N})(1-\mathrm{t}^{2N-2})}{(1-\mathrm{t}^2)(1-\mathrm{t}^4)(1-\mathrm{qt}^{2N-1})(1-\mathrm{qt}^{2N-3})}$$

$$\chi_{\mathrm{sp}}^2(s) = \prod_{i=1}^{N}(1+s_i)(1+s_i^{-1}) = \bar{E}(s\mid 1) = 2\sum_{r=0}^{N-1} E_r(s) + E_N(s)$$

$$\frac{\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}} = 2\sum_{r=0}^{N-1}\mathcal{E}_{N,r} + \mathcal{E}_{N,N}$$

$$\frac{\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(2N+1)}/\mathbb{II}_{\mathcal{N}}^{SO(2N+1)}}{\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(2N-1)}/\mathbb{II}_{\mathcal{N}}^{SO(2N-1)}} = \frac{(1+\mathrm{t}^N)(1-\mathrm{qt}^{N-1})}{1-\mathrm{qt}^{2N-1}}.$$

$$\frac{\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(2N+1)}}{\mathbb{II}_{\mathcal{N}}^{\mathrm{SO}(2N+1)}} = \prod_{k=0}^{N-1}\frac{(1+\mathrm{t}^{k+1})(1-\mathrm{qt}^k)}{1-\mathrm{qt}^{2k+1}} = \frac{(-\mathrm{t};\mathrm{t})_N(\mathrm{q};\mathrm{t})_N}{(\mathrm{qt};\mathrm{t}^2)_N}.$$

$$\mathbb{II}_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ USp(2)}(t;q) = \frac{1}{2}\frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds}{2\pi is}\frac{(s^{\pm2};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s^{\pm2};q)_\infty}.$$

$$\mathbb{II}_{\mathrm{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(3)}(t;q) = \frac{(q^{\frac{3}{2}}t^2;q)_\infty}{(qt^4;q)_\infty}.$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ SO(3)}(t,x;q) = \frac{(q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty}\frac{(qx;q)_\infty(qx^{-1};q)_\infty}{(q^{\frac{1}{2}}t^2x;q)_\infty(q^{\frac{1}{2}}t^2x^{-1};q)_\infty}$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ SO(3)}\left(t,x=q^{\frac{1}{2}}t^2;q\right) = \mathbb{II}_{\mathrm{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(3)}(t;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{1}{4}}t;q).$$

$$\mathbb{II}_{\mathcal{N}+\mathrm{hyp}}^{\text{4d }\mathcal{N}=4\ USp(2)'}(t;q) = \frac{1}{2}\frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds}{2\pi is}\frac{(s^{\pm2};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s^{\pm2};q)_\infty}\frac{(q^{\frac{3}{4}}t^{-1}s^{\mp};q)_\infty}{(q^{\frac{1}{4}}ts^{\pm};q)_\infty}.$$

$$\langle W_{\square} W_{\square} \rangle_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ USp(2)}(t;q) = \frac{1}{2}\frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty} \oint \frac{ds}{2\pi i s} \frac{(s^{\pm 2};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s^{\pm 2};q)_\infty}(s+s^{-1})^2.$$

$$\begin{aligned}&\mathbb{II}_{\mathcal{D}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(3)}\left(t, x=q^{\frac{3}{2}}t^2;q\right)\\ &= \langle T_{(1)}T_{(1)}\rangle_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(3)}(t;q)\mathcal{I}^{3\mathrm{d\ HM}}(t,x=q^{\frac{5}{4}}t;q).\end{aligned}$$

$$\langle T_{(1)}T_{(1)}\rangle_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(3)}(t;q) = \frac{(q)_\infty (q^{\frac{3}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty (q^2;q)_\infty}\frac{(q^{\frac{5}{2}}t^2;q)_\infty}{(q^2t^4;q)_\infty}.$$

$$\langle W_{\square} W_{\square} \rangle_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ USp(2)}(t;q) = \langle T_{(1)}T_{(1)}\rangle_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(3)}(t^{-1};q).$$

$$\langle W_{\square}\rangle_{\mathcal{N}+\mathrm{hyp}}^{4\mathrm{d}\ \mathcal{N}=4\ USp(2)'}(t;q) = q^{\frac{1}{4}}t\frac{(q^{\frac{5}{2}}t^{-2};q)_\infty}{(q^2t^{-4};q)_\infty}.$$

$$\begin{aligned}\langle W_{(2k)}\rangle_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ USp(2)}(t;q) &= \langle W_{(k)}\rangle_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(3)}(t;q)\\ &= q^{\frac{k}{2}}t^{-2k}\frac{(q^{\frac{1}{2}}t^2;q)_\infty (q^{\frac{2k+3}{2}}t^{-2};q)_\infty}{(q^{\frac{2k+1}{2}}t^2;q)_\infty (qt^{-4};q)_\infty}.\end{aligned}$$

$$\begin{aligned}&\mathbb{II}_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ USp(4)}(t;q)\\ &= \frac{1}{8}\frac{(q)_\infty^2}{(q^{\frac{1}{2}}t^{-2};q)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm 2};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm 2};q)_\infty}\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty (s_i^{\pm}s_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty (q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q)_\infty}.\end{aligned}$$

$$\mathbb{II}_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(5)}(t;q) = \frac{(q^{\frac{3}{2}}t^2;q)_\infty (q^{\frac{5}{2}}t^6;q)_\infty}{(qt^4;q)_\infty (q^2t^8;q)_\infty}.$$

$$\begin{aligned}&\mathbb{II}_{\mathcal{D}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(5)}(t,x_1,x_2;q)\\ &= \frac{(q)_\infty^2}{(q^{\frac{1}{2}}t^2;q)_\infty^2}\prod_{i=1}^{2}\frac{(qx_i^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^2x_i^{\pm};q)_\infty}\prod_{i<j}\frac{(qx_i^{\pm}x_j^{\mp};q)_\infty (qx_i^{\pm}x_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\mp};q)_\infty (q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\pm};q)_\infty}.\end{aligned}$$

$$x_1 = q^{\frac{1}{2}}t^2, x_2 = qt^4,$$

$$\begin{aligned}&\mathbb{II}_{\mathcal{D}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(5)}\left(t, x_1=q^{\frac{1}{2}}t^2, x_2=qt^4;q\right)\\ &= \mathbb{II}_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(5)}(t;q)\mathcal{I}^{3\mathrm{d\ HM}}(t,x=q^{\frac{1}{4}}t;q)^2\\ &\times \mathcal{I}^{3\mathrm{d\ HM}}(t,x=q^{\frac{3}{4}}t^3;q)\mathcal{I}^{3\mathrm{d\ HM}}(t,x=q^{\frac{5}{4}}t^5;q).\end{aligned}$$

$$\langle W_{\square}W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ USp(4)}(t;q)$$
$$=\frac{1}{8}\frac{(q)_\infty^2}{(q^{\frac{1}{2}}t^{-2};q)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm 2};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm 2};q)_\infty}\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q)_\infty}$$
$$\times(s_1+s_1^{-1}+s_2+s_2^{-1}).$$

$$x_1=q^{\frac{1}{2}}t^2,x_2=q^2t^4$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ SO(5)}\left(t,x_1=q^{\frac{1}{2}}t^2,x_2=q^2t^4;q\right)$$
$$=\langle T_{(1,1)}T_{(1,1)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(5)}(t;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{1}{4}}t;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{5}{4}}t;q)$$
$$\times\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{7}{4}}t^3;q)\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{9}{4}}t^5;q).$$

$$\langle T_{(1,0)}T_{(1,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(5)}(t;q)=\frac{(q)_\infty\left(q^{\frac{3}{2}}t^2;q\right)_\infty^2\left(q^{\frac{7}{2}}t^6;q\right)_\infty}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty(qt^4;q)_\infty(q^2;q)_\infty(q^3t^8;q)_\infty}$$

$$\langle W_{\square}W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ USp(4)}(t;q)=\langle T_{(1,0)}T_{(1,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(5)}(t^{-1};q).$$

$$\langle W_{\square}\rangle_{\mathcal{N}+\text{hyp}}^{\text{4d }\mathcal{N}=4\ USp(4)'}(t;q)=q^{\frac{1}{4}}t\frac{\left(q^{\frac{3}{2}}t^{-2};q\right)_\infty\left(q^{\frac{7}{2}}t^{-6};q\right)_\infty}{(qt^{-4};q)_\infty(q^3t^{-8};q)_\infty}$$

$$\langle W_{\tiny\begin{array}{|c|}\hline\\\hline\\\hline\end{array}}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ USp(4)}(t;q)=\langle W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(5)}(t;q).$$

$$\langle W_{\tiny\begin{array}{|c|}\hline\\\hline\\\hline\end{array}}\rangle_{\mathcal{N}+\text{hyp}}^{\text{4d }\mathcal{N}=4\ USp(4)'}(t;q)=\langle W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(5)}(t;q).$$

$$\langle W_{\square\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ USp(4)}(t;q)=\langle W_{\tiny\begin{array}{|c|}\hline\\\hline\\\hline\end{array}}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(5)}(t;q).$$

$$\mathbb{II}_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ USp(2N)}(t;q)$$
$$=\frac{1}{2^NN!}\frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm 2};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm 2};q)_\infty}\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q)_\infty}.$$

$$\mathbb{II}_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(2N+1)}(t;q)=\prod_{k=1}^{N}\frac{(q^{k+\frac{1}{2}}t^{4k-2};q)_\infty}{(q^kt^{4k};q)_\infty}.$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d } \mathcal{N}=4\ SO(2N+1)}(t,x_i;q) = \frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^2;q)_\infty^N} \prod_{i=1}^{N} \frac{(qx_i^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^2x_i^{\pm};q)_\infty} \prod_{i<j} \frac{(qx_i^{\pm}x_j^{\mp};q)_\infty (qx_i^{\pm}x_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\mp};q)_\infty (q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\pm};q)_\infty}.$$

$$x_i = q^{\frac{i}{2}}t^{2i}$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d } \mathcal{N}=4\ SO(2N+1)}\left(t, x_i = q^{\frac{i}{2}}t^{2i}; q\right) = \mathbb{II}_{\text{Nahm}}^{\text{4d } \mathcal{N}=4\ SO(2N+1)}(t;q) \prod_{i=1}^{N} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{2i-1}{4}}t^{2i-1};q) \times \prod_{i=1}^{N-1} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{2i-1}{4}}t^{2i-1};q)^{N-i} \prod_{i=1}^{2N-3} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{2i+3}{4}}t^{2i+3};q)^{a_N(i)},$$

$$\langle W_\square W_\square \rangle_{\mathcal{N}}^{\text{4d } \mathcal{N}=4\ USp(2N)}(t;q) = \frac{1}{2^N N!} \frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \frac{(s_i^{\pm 2};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm 2};q)_\infty} \prod_{i<j} \frac{(s_i^{\pm}s_j^{\mp};q)_\infty (s_i^{\pm}s_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty (q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q)_\infty} \times \left[\sum_{i=1}^{N}(s_i + s_i^{-1})\right]^2$$

$$x_i = q^{\frac{i}{2}}t^{2i}, i = 1, \cdots, N-1$$
$$x_N = q^{\frac{N}{2}+1}t^{2N}$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d } \mathcal{N}=4\ SO(2N+1)}\left(t, \left\{x_i = q^{\frac{i}{2}}t^{2i}\right\}_{i=1}^{N-1}, x_N = q^{\frac{N}{2}+1}t^{2N}; q\right) = \langle T_{(1,0,\cdots,0)} T_{(1,0,\cdots,0)} \rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4\ SO(2N+1)}(t;q) \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{1}{4}}t;q)^{N-1} \times \prod_{i=1}^{N-2} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{4i-1}{4}}t^{4i-1};q)^{N-i-1} \prod_{i=1}^{N-2} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{4i+1}{4}}t^{4i+1};q)^{N-i-1} \times \prod_{i=1}^{2N-1} \mathcal{I}^{\text{3d HM}}(t, x = q^{\frac{2i+3}{4}}t^{2i-1};q).$$

$$\langle T_{(1,0,\cdots,0)} T_{(1,0,\cdots,0)} \rangle_{\text{Nahm}}^{\text{4d } \mathcal{N}=4\ SO(2N+1)}(t;q) = \frac{(q)_\infty (q^{\frac{3}{2}}t^2;q)_\infty^2 (q^{\frac{2N+3}{2}}t^{4N-2};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty (q^2;q)_\infty (qt^4;q)_\infty (q^{N+1}t^{4N};q)_\infty} \prod_{k=1}^{N-2} \frac{(q^{\frac{2k+3}{2}}t^{4k+2};q)_\infty}{(q^{k+1}t^{4k+4};q)_\infty}.$$

$$\langle T_{(1,0,0)} T_{(1,0,0)} \rangle^{\text{4d } \mathcal{N}=4\ SO(7)}_{\text{Nahm}}(t;q)$$
$$= \frac{(q)_\infty (q^{\frac{3}{2}}t^2;q)^2_\infty (q^{\frac{5}{2}}t^6;q)_\infty (q^{\frac{9}{2}}t^{10};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty (qt^4;q)_\infty (q^2;q)_\infty (q^2t^8;q)_\infty (q^4t^{12};q)_\infty},$$
$$\langle T_{(1,0,0,0)} T_{(1,0,0,0)} \rangle^{\text{4d } \mathcal{N}=4\ SO(9)}_{\text{Nahm}}(t;q)$$
$$= \frac{(q)_\infty (q^{\frac{3}{2}}t^2;q)^2_\infty (q^{\frac{5}{2}}t^6;q)_\infty (q^{\frac{7}{2}}t^{10};q)_\infty (q^{\frac{11}{2}}t^{14};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty (qt^4;q)_\infty (q^2;q)_\infty (q^2t^8;q)_\infty (q^3t^{12};q)_\infty (q^5t^{16};q)_\infty},$$
$$\langle T_{(1,0,0,0,0)} T_{(1,0,0,0,0)} \rangle^{\text{4d } \mathcal{N}=4\ SO(11)}_{\text{Nahm}}(t;q)$$
$$= \frac{(q)_\infty (q^{\frac{3}{2}}t^2;q)^2_\infty (q^{\frac{5}{2}}t^6;q)_\infty (q^{\frac{7}{2}}t^{10};q)_\infty (q^{\frac{9}{2}}t^{14};q)_\infty (q^{\frac{13}{2}}t^{18};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty (qt^4;q)_\infty (q^2;q)_\infty (q^2t^8;q)_\infty (q^3t^{12};q)_\infty (q^4t^{16};q)_\infty (q^6t^{20};q)_\infty}.$$

$$\langle W_\square W_\square \rangle^{\text{4d } \mathcal{N}=4\ USp(2N)}_{\mathcal{N}}(t;q) = \langle T_{(1,0,\cdots,0)} T_{(1,0,\cdots,0)} \rangle^{\text{4d } \mathcal{N}=4\ SO(2N+1)}_{\text{Nahm}}(t^{-1};q).$$

$$R = \{\pm 2\varepsilon_i \mid 1 \le i \le N\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le N\}$$
$$R^+ = \{2\varepsilon_i \mid 1 \le i \le N\} \cup \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le N\}$$

$$\rho = \sum_{i=1}^{N} (N-i+1)\varepsilon_i, \lambda = \sum_{i=1}^{N} \lambda_i \varepsilon_i$$

$$w_{C_N}(s;\mathbf{q},\mathbf{t}) = \prod_{i=1}^{N} \frac{(s_i^{\pm 2};\mathbf{q})_\infty}{(\mathbf{t}s_i^{\pm 2};\mathbf{q})_\infty} \prod_{1\le i<j\le N} \frac{(s_i^{\pm 1}s_j^{\pm 1};\mathbf{q})_\infty}{(\mathbf{t}s_i^{\pm 1}s_j^{\pm 1};\mathbf{q})_\infty}$$

$$\frac{\langle P_\lambda, P_\lambda \rangle}{\langle 1,1 \rangle} = \prod_{i=1}^{N} \frac{(\mathrm{t}^{N-i+2};\mathrm{q})_{\lambda_i} (\mathrm{t}^{N-i}\mathrm{q};\mathrm{q})_{\lambda_i}}{(\mathrm{t}^{N-i+1};\mathrm{q})_{\lambda_i} (\mathrm{t}^{N-i+1}\mathrm{q};\mathrm{q})_{\lambda_i}}$$
$$\times \prod_{1\le i<j\le N} \frac{(\mathrm{t}^{j-i+1};\mathrm{q})_{\lambda_i-\lambda_j} (\mathrm{t}^{j-i-1}\mathrm{q};\mathrm{q})_{\lambda_i-\lambda_j}}{(\mathrm{t}^{j-i};\mathrm{q})_{\lambda_i-\lambda_j} (\mathrm{t}^{j-i}\mathrm{q};\mathrm{q})_{\lambda_i-\lambda_j}} \frac{(\mathrm{t}^{2N-i-j+3};\mathrm{q})_{\lambda_i+\lambda_j} (\mathrm{t}^{2N-i-j+1}\mathrm{q};\mathrm{q})_{\lambda_i+\lambda_j}}{(\mathrm{t}^{2N-i-j+2};\mathrm{q})_{\lambda_i+\lambda_j} (\mathrm{t}^{2N-i-j+2}\mathrm{q};\mathrm{q})_{\lambda_i+\lambda_j}},$$

$$\langle 1,1 \rangle = \frac{(\mathrm{t};\mathrm{q})^N_\infty}{(\mathrm{q};\mathrm{q})^N_\infty} \prod_{k=1}^{N} \frac{(\mathrm{t}^{2k-1}\mathrm{q};\mathrm{q})_\infty}{(\mathrm{t}^{2k};\mathrm{q})_\infty}$$

$$\chi_{(1^r)}(s) = E_r(s) - E_{r-2}(s)$$
$$E_r(s) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \frac{(\mathrm{qt};\mathrm{t}^2)_j (\mathrm{t}^{2N-2r+2j+2};\mathrm{t}^2)_j}{(\mathrm{t}^2;\mathrm{t}^2)_j (\mathrm{qt}^{2N-2r+2j+1};\mathrm{t}^2)_j} P_{(1^{r-2j})}(s;\mathrm{q},\mathrm{t})$$

$$\chi_{(1)}(s) = E_1(s) = P_{(1)}(s;\mathbf{q},\mathbf{t})$$

$$\langle \chi_{(1)}, \chi_{(1)} \rangle = \langle P_{(1)}, P_{(1)} \rangle = \frac{(1-\mathrm{q})(1-\mathrm{t}^{2N})}{(1-\mathrm{t})(1-\mathrm{qt}^{2N-1})} \langle 1,1 \rangle,$$

$$\frac{\langle W_\square W_\square \rangle^{USp(2N)}_{\mathcal{N}}}{\mathbb{II}^{USp(2N)}_{\mathcal{N}}} = \frac{(1-\mathrm{q})(1-\mathrm{t}^{2N})}{(1-\mathrm{t})(1-\mathrm{qt}^{2N-1})}.$$

$$\langle \chi_{(1^{2m})}, 1 \rangle = \langle E_{2m}, 1 \rangle - \langle E_{2m-2}, 1 \rangle$$
$$= \frac{(\mathrm{qt};\mathrm{t}^2)_m (\mathrm{t}^{2N-2m+2};\mathrm{t}^2)_m}{(\mathrm{t}^2;\mathrm{t}^2)_m (\mathrm{qt}^{2N-2m+1};\mathrm{t}^2)_m} \langle 1,1 \rangle - \frac{(\mathrm{qt};\mathrm{t}^2)_{m-1} (\mathrm{t}^{2N-2m};\mathrm{t}^2)_{m-1}}{(\mathrm{t}^2;\mathrm{t}^2)_{m-1} (\mathrm{qt}^{2N-2m-1};\mathrm{t}^2)_{m-1}} \langle 1,1 \rangle$$

$$\frac{\langle W_{(1^{2m})}\rangle_{\mathcal{N}}^{USp(2N)}}{\mathbb{III}_{\mathcal{N}}^{USp(2N)}} = \frac{(\mathrm{qt};\mathrm{t}^2)_m(\mathrm{t}^{2N-2m+2};\mathrm{t}^2)_m}{(\mathrm{t}^2;\mathrm{t}^2)_m(\mathrm{qt}^{2N-2m+1};\mathrm{t}^2)_m} - \frac{(\mathrm{qt};\mathrm{t}^2)_{m-1}(\mathrm{t}^{2N-2m};\mathrm{t}^2)_{m-1}}{(\mathrm{t}^2;\mathrm{t}^2)_{m-1}(\mathrm{qt}^{2N-2m-1};\mathrm{t}^2)_{m-1}}.$$

$$\frac{\langle W_{(1^2)}\rangle_{\mathcal{N}}^{USp(2N)}}{\mathbb{II}_{\mathcal{N}}^{USp(2N)}} = \frac{(1-\mathrm{qt})(1-\mathrm{t}^{2N})}{(1-\mathrm{t}^2)(1-\mathrm{qt}^{2N-1})} - 1 = \frac{\mathrm{t}(\mathrm{t}-\mathrm{q})(1-\mathrm{t}^{2N-2})}{(1-\mathrm{t}^2)(1-\mathrm{qt}^{2N-1})}.$$

$$\mathbb{II}_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(4)}(t;q) = \frac{1}{4}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^2}\oiiint \prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\mathbb{II}_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(4)}(t;q) = \frac{\left(q^{\frac{3}{2}}t^2;q\right)_\infty^2}{(qt^4;q)_\infty^2}$$

$$\mathbb{II}_{\mathcal{D}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(4)}(t;q) = \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty}\prod_{i<j}\frac{\left(qx_i^{\pm}x_j^{\mp};q\right)_\infty\left(qx_i^{\pm}x_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\pm};q\right)_\infty}$$

$$x_1 = 1, x_2 = q^{\frac{1}{2}}t^2$$

$$\begin{aligned}&\mathbb{II}_{\mathcal{D}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(4)}\left(t, x_1 = 1, x_2 = q^{\frac{1}{2}}t^2;q\right)\\ &= \mathbb{II}_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(4)}(t;q)\mathcal{I}^{3\mathrm{d\ HM}}\left(t, x = q^{\frac{1}{4}}t;q\right)^2\end{aligned}$$

$$\mathbb{II}_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4SO(4)^-}(t;q) = \frac{1}{2}\frac{(\pm q;q)_\infty}{\left(\pm q^{\frac{1}{2}}t^{-2};q\right)_\infty}\oiiint\frac{ds}{2\pi i s}\frac{(s^{\pm};q)_\infty(-s^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{-2}s^{\pm};q\right)_\infty}$$

$$\mathbb{II}_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(4)^-}(t;q) = \frac{\left(\pm q^{\frac{3}{2}}t^2;q\right)_\infty}{(\pm qt^4;q)_\infty}$$

$$\mathbb{II}_{\mathcal{D}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(4)^-}(t,x;q) = \frac{(\pm q;q)_\infty}{\left(\pm q^{\frac{1}{2}}t^2;q\right)_\infty}\frac{(qx^{\pm};q)_\infty(-qx^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2x^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^2x^{\pm};q\right)_\infty}$$

$$\mathbb{III}_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ O(4)^{\pm}}(t;q) = \frac{1}{2}\frac{\left(q^{\frac{3}{2}}t^2;q\right)_\infty}{(qt^4;q)_\infty}\left[\frac{\left(q^{\frac{3}{2}}t^2;q\right)_\infty}{(qt^4;q)_\infty} \pm \frac{\left(-q^{\frac{3}{2}}t^2;q\right)_\infty}{(-qt^4;q)_\infty}\right]$$

$$\begin{aligned}&\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle_{\mathcal{N}}^{4\mathrm{d}\ \mathcal{N}=4\ \mathrm{Spin}(4)}(t;q)\\ &= \frac{1}{4}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^2}\oiiint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}\\ &\times\left(2 + s_1s_2 + s_1^{-1}s_2^{-1}\right)\end{aligned}$$

$$x_1 = q^{\frac{1}{2}}, x_2 = qt^2$$

$$\begin{aligned}&\mathbb{III}_{\mathcal{D}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(4)}\left(t, x_1 = q^{\frac{1}{2}}, x_2 = qt^2;q\right)\\ &= \left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle_{\mathrm{Nahm}}^{4\mathrm{d}\ \mathcal{N}=4\ SO(4)/\mathbb{Z}_2}(t;q)\mathcal{I}^{3\mathrm{d\ HM}}\left(t, x = q^{\frac{1}{4}}t;q\right)\mathcal{I}^{3\mathrm{d\ HM}}\left(t, x = q^{\frac{5}{4}}t;q\right),\end{aligned}$$

$$\left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle_{\text{Nahm}}^{4\text{d }\mathcal{N}=4\ SO(4)/\mathbb{Z}_2}(t;q)=\frac{(q)_\infty\left(q^{\frac{3}{2}}t^2;q\right)_\infty^2\left(q^{\frac{5}{2}}t^2;q\right)_\infty}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty(qt^4;q)_\infty(q^2;q)_\infty(q^2t^4;q)_\infty}$$

$$\langle W_{\text{sp}}W_{\text{sp}}\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ \text{Spin}(4)}(t;q)=\left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle_{\text{Nahm}}^{4\text{ d}\mathcal{N}=4SO(4)/\mathbb{Z}_2}(t^{-1};q)$$

$$\langle W_{\text{sp}}W_{\text{sp}}\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ \text{Spin}(4)}(t;q)=\mathbb{II}_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ SU(2)}(t;q)\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ SU(2)}(t;q)$$

$$\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ SO(4)}(t;q)$$
$$=\frac{1}{4}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi is_i}\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}(s_1+s_2+s_1^{-1}+s_2^{-1})^2$$

$$x_1=1,x_2=q^{\frac{3}{2}}t^2$$

$$\mathbb{II}_{\mathcal{D}}^{4\text{d }\mathcal{N}=4\ SO(4)}\left(t,x_1=1,x_2=q^{\frac{3}{2}}t^2;q\right)$$
$$=\langle T_{(1,0)}T_{(1,0)}\rangle_{\text{Nahm}}^{4\text{d }\mathcal{N}=4\ SO(4)}(t;q)\mathcal{I}^{3\text{d HM}}\left(t,x=q^{\frac{5}{4}}t;q\right)^2$$

$$\langle T_{(1,0)}T_{(1,0)}\rangle_{\text{Nahm}}^{4\text{d }\mathcal{N}=4\ SO(4)}(t;q)=\frac{(q)_\infty^2\left(q^{\frac{3}{2}}t^2;q\right)_\infty^2\left(q^{\frac{5}{2}}t^2;q\right)_\infty^2}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty^2(q^2;q)_\infty^2(q^2t^4;q)_\infty^2}$$

$$\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d}\mathcal{N}=4SO(4)}(t;q)=\langle T_{(1,0)}T_{(1,0)}\rangle_{\text{Nahm}}^{4\text{d }\mathcal{N}=4\ SO(4)}(t^{-1};q)$$

$$\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ SO(4)}(t;q)=\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ SU(2)}(t;q)^2$$

$$\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ SO(4)^-}(t;q)$$
$$=\frac{1}{2}\frac{(\pm q;q)_\infty}{\left(\pm q^{\frac{1}{2}}t^{-2};q\right)_\infty}\oint\frac{ds}{2\pi is}\frac{(s^{\pm};q)_\infty(-s^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{-2}s^{\pm};q\right)_\infty}$$

$$\mathbb{II}_{\mathcal{D}}^{4\text{d }\mathcal{N}=4\ SO(4)^-}\left(t,x=q^{\frac{3}{2}}t^2;q\right)$$
$$=\langle T_{(1)}T_{(1)}\rangle_{\text{Nahm}}^{4\text{d }\mathcal{N}=4\ SO(4)^-}(t;q)\mathcal{I}^{3\text{d HM}}\left(t,x=q^{\frac{5}{4}}t;q\right)\mathcal{I}^{3\text{d HM}}\left(t,x=-q^{\frac{5}{4}}t;q\right)$$

$$\langle T_{(1)}T_{(1)}\rangle_{\text{Nahm}}^{4\text{d }\mathcal{N}=4\ SO(4)^-}(t;q)=\frac{(\pm q;q)_\infty\left(\pm q^{\frac{3}{2}}t^2;q\right)_\infty\left(\pm q^{\frac{5}{2}}t^2;q\right)_\infty}{\left(\pm q^{\frac{1}{2}}t^2;q\right)_\infty(\pm q^2;q)_\infty(\pm q^2t^4;q)_\infty}$$

$$\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ O(4)^{\pm}}(t;q)$$
$$=\frac{1}{2}\Big[\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ SO(4)}$$
$$(t;q)\pm\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ SO(4)^-}(t;q)\Big]$$

$$\langle W_\square W_\square\rangle_{\mathcal{N}}^{4\text{d }\mathcal{N}=4\ O(4)^{\pm}}(t;q)=\langle T_{(1,0)}T_{(1,0)}\rangle_{\text{Nahm}}^{4\text{d }\mathcal{N}=4\ O(4)}(t^{-1};q)$$

$$\langle T_{(1,0)}T_{(1,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ O(4)}(t;q) = \frac{1}{2}\frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty(q^{\frac{5}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(q^2;q)_\infty(q^2t^4;q)_\infty}$$

$$\times\left[\frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty(q^{\frac{5}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(q^2;q)_\infty(q^2t^4;q)_\infty}\pm\frac{(-q;q)_\infty(-q^{\frac{3}{2}}t^2;q)_\infty(-q^{\frac{5}{2}}t^2;q)_\infty}{(-q^{\frac{1}{2}}t^2;q)_\infty(-q^2;q)_\infty(-q^2t^4;q)_\infty}\right].$$

$$\langle W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(4)}(t;q) = \langle W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(4)}(t;q)$$

$$= q^{\frac{1}{2}}t^{-2}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q^{\frac{3}{2}}t^{-2};q)_\infty(q^{\frac{5}{2}}t^{-2};q)_\infty}{(qt^{-4};q)^2(q^{\frac{3}{2}}t^2;q)_\infty}.$$

$$\langle (W_{\square})^k(W_{\square})^m\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(4)}(t;q)$$

$$=\frac{(qt^{-4};q)_\infty^2}{(q^{\frac{3}{2}}t^{-2};q)_\infty^2}\langle (W_{\square})^k\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(4)}(t;q)\langle (W_{\square})^m\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(4)}(t;q).$$

$$\langle W_{\square\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(4)}(t;q) = qt^{-4}\frac{(q^{\frac{1}{2}}t^2;q)_\infty^2(q^{\frac{5}{2}}t^{-2};q)_\infty^2}{(qt^{-4};q)_\infty^2(q^{\frac{3}{2}}t^2;q)_\infty^2}.$$

$$\langle\underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(4)}(t;q) = \langle\underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SU(2)}(t;q)^2,$$

$$\langle\underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(4)}(t;q) = \langle\underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SU(2)}(t^2;q^2),$$

$$\mathbb{II}_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(6)}(t;q) = \frac{1}{24}\frac{(q)_\infty^3}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^3}\oiiint\prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\mathbb{II}_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(6)}(t;q) = \frac{\left(q^{\frac{3}{2}}t^2;q\right)_\infty(q^2t^4;q)_\infty\left(q^{\frac{5}{2}}t^6;q\right)_\infty}{(qt^4;q)_\infty\left(q^{\frac{3}{2}}t^6;q\right)_\infty(q^2t^8;q)_\infty}$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ SO(6)}(t,x_1,x_2,x_3;q) = \frac{(q)_\infty^3}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty^3}\prod_{i<j}\frac{\left(qx_i^{\pm}x_j^{\mp};q\right)_\infty\left(qx_i^{\pm}x_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\pm};q\right)_\infty}.$$

$$x_1=1, x_2=q^{\frac{1}{2}}t^2, x_3=qt^4,$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ SO(6)}\left(t,x_1=1,x_2=q^{\frac{1}{2}}t^2,x_3=qt^4;q\right)$$

$$=\mathbb{II}_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(6)}(t;q)\mathcal{I}^{\text{3d HM}}\left(t,x=q^{\frac{1}{4}}t;q\right)^3$$

$$\times\mathcal{I}^{\text{3d HM}}\left(t,x=q^{\frac{3}{4}}t^3;q\right)^2\mathcal{I}^{\text{3d HM}}\left(t,x=q^{\frac{5}{4}}t^5;q\right)$$

$$\mathbb{III}_{\mathcal{N}}^{4d\ \mathcal{N}=4\ SO(6)^-}(t;q)$$
$$= \frac{1}{8}\frac{(q)_\infty^2(-q;q)_\infty}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty^2\left(-q^{\frac{1}{2}}t^2;q\right)_\infty}\oiiint \prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{\left(s_i^{\pm};q\right)_\infty\left(-s_i^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\mathbb{II}_{\text{Nahm}}^{4d\ \mathcal{N}=4\ SO(6)^-}(t;q) = \frac{\left(q^{\frac{3}{2}}t^2;q\right)_\infty\left(q^{\frac{5}{2}}t^6;q\right)_\infty(-q^2t^4;q)_\infty}{(qt^4;q)_\infty(q^2t^8;q)_\infty\left(-q^{\frac{3}{2}}t^6;q\right)_\infty}$$

$$\mathbb{II}_{\mathcal{D}}^{4d\ \mathcal{N}=4\ SO(6)^-}(t;q)$$
$$= \frac{(q)_\infty^2(-q;q)_\infty}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty^2\left(-q^{\frac{1}{2}}t^2;q\right)_\infty}\oiiint \prod_{i=1}^{3}\frac{\left(qx_i^{\pm};q\right)_\infty\left(-qx_i^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2x_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^2x_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{\left(qx_i^{\pm}x_j^{\mp};q\right)_\infty\left(qx_i^{\pm}x_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}x_i^{\pm}x_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}x_i^{\pm}x_j^{\pm};q\right)_\infty}$$

$$x_1 = q^{\frac{1}{2}}t^2, x_2 = qt^4$$

$$\langle W_{\text{sp}}W_{\overline{\text{sp}}}\rangle_{\mathcal{N}}^{4d\ \mathcal{N}=4\ \text{Spin}(6)}(t;q)$$
$$= \frac{1}{24}\frac{(q)_\infty^3}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^3}\oiiint \prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$
$$\times\left(4+\sum_{i<j}s_is_j + s_i^{-1}sj^{-1} + s_is_j^{-1} + s_i^{-1}s_j\right)$$

$$x_1 = q^{\frac{1}{2}}, x_2 = qt^2, x_3 = q^{\frac{3}{2}}t^4$$

$$\mathbb{II}_{\mathcal{D}}^{4d\ \mathcal{N}=4\ SO(6)}\left(t, x_1 = q^{\frac{1}{2}}, x_2 = qt^2, x_3 = q^{\frac{3}{2}}t^4; q\right)$$
$$= \left\langle T_{\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)}T_{\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)}\right\rangle_{\text{Nahm}}^{4d\ \mathcal{N}=4\ SO(6)/\mathbb{Z}_2}(t;q)\mathcal{I}^{3d\ \text{HM}}\left(t,x=q^{\frac{1}{4}}t;q\right)^2\mathcal{I}^{3d\ \text{HM}}\left(t,x=q^{\frac{3}{4}}t^3;q\right)$$
$$\times\mathcal{I}^{3d\ \text{HM}}\left(t,x=q^{\frac{5}{4}}t;q\right)\mathcal{I}^{3d\ \text{HM}}\left(t,x=q^{\frac{7}{4}}t^3;q\right)\mathcal{I}^{3d\ \text{HM}}\left(t,x=q^{\frac{9}{4}}t^5;q\right)$$

$$\left\langle T_{\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)}T_{\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)}\right\rangle_{\text{Nahm}}^{4d\ \mathcal{N}=4\ SO(6)/\mathbb{Z}_2}(t;q)$$
$$= \frac{(q)_\infty\left(q^{\frac{3}{2}}t^2;q\right)_\infty^2(q^2t^4;q)_\infty\left(q^{\frac{7}{2}}t^6;q\right)_\infty}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty(qt^4;q)_\infty\left(q^{\frac{3}{2}}t^6;q\right)_\infty(q^2;q)_\infty(q^3t^8;q)_\infty}$$

$$\langle W_{\text{sp}}W_{\overline{\text{sp}}}\rangle_{\mathcal{N}}^{4d\ \mathcal{N}=4\ \text{Spin}(6)}(t;q) = \left\langle T_{\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)}T_{\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)}\right\rangle_{\text{Nahm}}^{4d\ \mathcal{N}=4\ SO(6)/\mathbb{Z}_2}(t^{-1};q)$$

$$\langle W_\square W_\square \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ SO(6)}(t;q)$$
$$= \frac{1}{24} \frac{(q)_\infty^3}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^3} \oint\oint\oint \prod_{i=1}^{3} \frac{ds_i}{2\pi i s_i} \prod_{i<j} \frac{(s_i^\pm s_j^\mp;q)_\infty (s_i^\pm s_j^\pm;q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\mp;q\right)_\infty \left(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\pm;q\right)_\infty}$$
$$\times (s_1+s_2+s_3+s_1^{-1}+s_2^{-1}+s_3^{-1})^2$$

$$x_1 = 1, x_2 = q^{\frac{1}{2}}t^2, x_3 = q^2t^4$$

$$\mathbb{II}_{\mathcal{D}}^{4\text{d}\ \mathcal{N}=4\ SO(6)}\left(t, x_1=1, x_2=q^{\frac{1}{2}}t^2, x_3=q^2t^4;q\right)$$
$$= \langle T_{(1,0,0)}T_{(1,0,0)}\rangle_{\text{Nahm}}^{4\text{d}\ \mathcal{N}=4\ SO(6)}(t;q)\mathcal{I}^{3\text{d HM}}(t,x=q^{\frac{1}{4}}t;q)^2$$
$$\times \mathcal{I}^{3\text{d HM}}(t,x=q^{\frac{5}{4}}t;q)\mathcal{I}^{3\text{d HM}}(t,x=q^{\frac{7}{4}}t^3;q)^2\mathcal{I}^{3\text{d HM}}(t,x=q^{\frac{9}{4}}t^5;q).$$

$$\langle T_{(1,0,0)}T_{(1,0,0)}\rangle_{\text{Nahm}}^{4\text{d}\ \mathcal{N}=4\ SO(6)}(t;q)$$
$$= \frac{(q)_\infty \left(q^{\frac{3}{2}}t^2;q\right)_\infty^3 (q^2t^4;q)_\infty (q^3t^4;q)_\infty \left(q^{\frac{7}{2}}t^6;q\right)_\infty}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty (qt^4;q)_\infty (q^2;q)_\infty \left(q^{\frac{5}{2}}t^2;q\right)_\infty \left(q^{\frac{5}{2}}t^6;q\right)_\infty (q^3t^8;q)_\infty}$$

$$\langle W_\square W_\square \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ SO(6)}(t;q) = \langle T_{(1,0,0)}T_{(1,0,0)}\rangle_{\text{Nahm}}^{4\text{d}\ \mathcal{N}=4\ SO(6)}(t^{-1};q).$$

$$\langle W_\square W_\square \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ SO(6)^-}(t;q)$$
$$= \frac{1}{8} \frac{(q)_\infty^2(-q;q)_\infty}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty^2 \left(-q^{\frac{1}{2}}t^2;q\right)_\infty} \oint\oint \prod_{i=1}^{2} \frac{ds_i}{2\pi i s_i} \frac{(s_i^\pm;q)_\infty (-s_i^\pm;q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^\pm;q\right)_\infty \left(-q^{\frac{1}{2}}t^{-2}s_i^\pm;q\right)_\infty}$$
$$\times \prod_{i\neq j} \frac{(s_i^\pm s_j^\mp;q)_\infty (s_i^\pm s_j^\pm;q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\mp;q\right)_\infty \left(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\pm;q\right)_\infty}$$
$$\times (s_1+s_2+s_1^{-1}+s_2^{-1})^2$$

$$x_1 = q^{\frac{1}{2}}t^2, x_2 = q^2t^4$$

$$\mathbb{II}_{\mathcal{D}}^{4\text{d}\ \mathcal{N}=4\ SO(6)^-}\left(t, x_1=q^{\frac{1}{2}}t^2, x_2=q^2t^4;q\right)$$
$$= \langle T_{(1,0)}T_{(1,0)}\rangle_{\text{Nahm}}^{4\text{d}\ \mathcal{N}=4\ SO(6)^-}(t;q)\mathcal{I}^{3\text{d HM}}(t,x=q^{\frac{1}{4}}t;q)\mathcal{I}^{3\text{d HM}}(t,x=-q^{\frac{1}{4}}t;q)$$
$$\times \mathcal{I}^{3\text{d HM}}(t,x=q^{\frac{5}{4}}t;q)\mathcal{I}^{3\text{d HM}}(t,x=-q^{\frac{5}{4}}t;q)$$
$$\times \mathcal{I}^{3\text{d HM}}(t,x=q^{\frac{7}{4}}t^3;q)\mathcal{I}^{3\text{d HM}}(t,x=q^{\frac{9}{4}}t^5;q),$$

$$\langle T_{(1,0)}T_{(1,0)}\rangle_{\text{Nahm}}^{4\text{d}\ \mathcal{N}=4\ SO(6)^-}(t;q)$$
$$= \frac{(q)_\infty \left(q^{\frac{3}{2}}t^2;q\right)_\infty^2 \left(q^{\frac{7}{2}}t^6;q\right)_\infty \left(-q^{\frac{3}{2}}t^2;q\right)_\infty (-q^2t^4;q)_\infty(-q^3t^4;q)_\infty}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty (qt^4;q)_\infty(q^2;q)_\infty(q^3t^8;q)_\infty(-qt^4;q)_\infty \left(-q^{\frac{5}{2}}t^2;q\right)_\infty \left(-q^{\frac{5}{2}}t^6;q\right)_\infty}$$

$$\langle W_\square W_\square \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ O(6)^\pm}(t;q)$$
$$= \frac{1}{2}\left[\langle W_\square W_\square \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ SO(6)}(t;q) \pm \langle W_\square W_\square \rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ SO(6)^-}(t;q)\right]$$

$$\langle T_{(1,0,0)}T_{(1,0,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ O(6)^{\pm}}(t;q)=\frac{1}{2}\frac{(q)_\infty\left(q^{\frac{3}{2}}t^2;q\right)_\infty^2\left(q^{\frac{7}{2}}t^6;q\right)_\infty}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty(qt^4;q)_\infty(q^2;q)_\infty(q^3t^8;q)_\infty}$$

$$\times\left[\frac{\left(q^{\frac{3}{2}}t^2;q\right)_\infty(q^2t^4;q)_\infty(q^3t^4;q)_\infty}{(qt^4;q)_\infty\left(q^{\frac{5}{2}}t^2;q\right)_\infty\left(q^{\frac{5}{2}}t^6;q\right)_\infty}\pm\frac{\left(-q^{\frac{3}{2}}t^2;q\right)_\infty(-q^2t^4;q)_\infty(-q^3t^4;q)_\infty}{(-qt^4;q)_\infty\left(-q^{\frac{5}{2}}t^2;q\right)_\infty\left(-q^{\frac{5}{2}}t^6;q\right)_\infty}\right]$$

$$\langle W_{\square}W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ O(6)^{\pm}}(t;q)=\langle T_{(1,0,0)}T_{(1,0,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ O(6)^{\pm}}(t^{-1};q)$$

$$\langle W_{\substack{\square\\\square}}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(6)}(t;q)=\langle W_{\overline{\substack{\square\\\square}}}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(6)}(t;q)$$

$$=q^{\frac{1}{2}}t^{-2}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q^{\frac{3}{2}}t^{-2};q)_\infty^2(q^2t^{-4};q)_\infty(q^{\frac{7}{2}}t^{-6};q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty(qt^{-4};q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty(q^2t^{-8};q)_\infty(q^{\frac{5}{2}}t^{-6};q)_\infty}.$$

$$\langle W_{\square\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(6)}(t;q)$$

$$=qt^{-4}\frac{(q^{\frac{1}{2}}t^2;q)_\infty(q)_\infty(q^{\frac{3}{2}}t^{-2};q)_\infty(q^2t^{-4};q)_\infty(q^3t^{-4};q)_\infty(q^{\frac{7}{2}}t^{-6};q)_\infty}{(qt^{-4};q)_\infty^2(q^{\frac{3}{2}}t^2;q)_\infty(q^{\frac{3}{2}}t^{-6};q)_\infty(q^2;q)_\infty(q^3t^{-8};q)_\infty}.$$

$$\mathbb{II}_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t;q)$$

$$=\frac{1}{2^{N-1}N!}\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi is_i}\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\mathbb{II}_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t;q)=\frac{\left(q^{\frac{1}{2}+\frac{N}{2}}t^{2N-2};q\right)_\infty}{\left(q^{\frac{N}{2}}t^{2N};q\right)_\infty}\prod_{k=1}^{N-1}\frac{\left(q^{\frac{1}{2}+k}t^{4k-2};q\right)_\infty}{(q^kt^{4k};q)_\infty}$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t;q)$$

$$=\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^2;q\right)_\infty^N}\prod_{i<j}\frac{\left(qx_i^{\pm}x_j^{\mp};q\right)_\infty\left(qx_i^{\pm}x_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2x_i^{\pm}x_j^{\pm};q\right)_\infty}$$

$$x_i=q^{\frac{i-1}{2}}t^{2(i-1)}$$

$$\mathbb{II}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ SO(2N)}\left(t,x_i=q^{\frac{i-1}{2}}t^{2(i-1)};q\right)$$

$$=\mathbb{II}_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t;q)\prod_{i=1}^{N-1}\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{2i-1}{4}}t^{2i-1};q)^{N-i}$$

$$\times\prod_{i=1}^{2N-3}\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{2i-1}{4}}t^{2i-1};q)^{a_N(i)}$$

$$\mathbb{II}_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ SO(2N)^-}(t;q)$$
$$=\frac{1}{2^{N-1}(N-1)!}\frac{(q)_\infty^{N-1}(-q;q)_\infty}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^{N-1}\left(-q^{\frac{1}{2}}t^{-2};q\right)_\infty}\oint\prod_{i=1}^{N-1}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm};q)_\infty(-s_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\mp};q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty}$$

$$\mathbb{II}_{\text{Nahm}}^{4\text{d}\ \mathcal{N}=4\ SO(2N)^-}(t;q)=\frac{\left(-q^{\frac{N+1}{2}}t^{2N-2};q\right)_\infty}{\left(-q^{\frac{N}{2}}t^{2N};q\right)_\infty}\prod_{k=1}^{N-1}\frac{\left(q^{\frac{2k+1}{2}}t^{4k-2};q\right)_\infty}{(q^k t^{4k};q)_\infty}$$

$$\mathbb{II}_{\mathcal{D}}^{4\text{d}\ \mathcal{N}=4\ SO(2N)^-}(t;q)$$
$$=\frac{(q)_\infty^{N-1}(-q;q)_\infty}{\left(q^{\frac{1}{2}}t^{2};q\right)_\infty^{N-1}\left(-q^{\frac{1}{2}}t^{2};q\right)_\infty}\prod_{i=1}^{N}\frac{(qx_i^{\pm};q)_\infty(-qx_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^{2}x_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{2}x_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{(qx_i^{\pm}x_j^{\mp};q)_\infty(qx_i^{\pm}x_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}x_i^{\pm}x_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}x_i^{\pm}x_j^{\pm};q\right)_\infty}$$

$$x_i=q^{\frac{i}{2}}t^{2i}$$

$$\mathbb{II}_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ O(2N)^{\pm}}(t;q)=\frac{1}{2}\left[\mathbb{II}_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ SO(2N)}(t;q)\pm\mathbb{II}_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ SO(2N)^-}(t;q)\right]$$

$$\mathbb{II}_{\text{Nahm}}^{4\text{d}\ \mathcal{N}=4\ O(2N)^{\pm}}(t;q)=\frac{1}{2}\left[\frac{\left(q^{\frac{N+1}{2}}t^{2N-2};q\right)_\infty}{\left(q^{\frac{N}{2}}t^{2N};q\right)_\infty}\pm\frac{\left(-q^{\frac{N+1}{2}}t^{2N-2};q\right)_\infty}{\left(-q^{\frac{N}{2}}t^{2N};q\right)_\infty}\right]\prod_{k=1}^{N-1}\frac{\left(q^{\frac{2k+1}{2}}t^{4k-2};q\right)_\infty}{(q^k t^{4k};q)_\infty}$$

$$\langle W_{\text{sp}}W_{\text{sp}}\rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ \text{Spin}(2N)}(t;q)$$
$$=\frac{1}{2^{N-1}N!}\frac{(q)_\infty^{N}}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^{N}}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$
$$\times\frac{1}{4}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)+\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right]^2.$$

$$\langle W_{\text{sp}}W_{\overline{\text{sp}}}\rangle_{\mathcal{N}}^{4\text{d}\ \mathcal{N}=4\ \text{Spin}(2N)}(t;q)$$
$$=\frac{1}{2^{N-1}N!}\frac{(q)_\infty^{N}}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^{N}}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$
$$\times\frac{1}{4}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)+\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right]\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)-\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right].$$

$$x_i=q^{\frac{i}{2}}t^{2(i-1)},\ i=1,\cdots,N.$$

$$\mathbb{II}_{\mathcal{D}}^{4\text{d}\ \mathcal{N}=4\ SO(2N)}\left(t,x_i=q^{\frac{i}{2}}t^{2(i-1)};q\right)$$
$$=\left\langle T_{\left(\frac{1}{2},\cdots,\frac{1}{2}\right)}T_{\left(\frac{1}{2},\cdots,\frac{1}{2}\right)}\right\rangle_{\text{Nahm}}^{4\text{d}\ \mathcal{N}=4\ SO(2N)/\mathbb{Z}_2}(t;q)\prod_{i=1}^{N-1}\mathcal{I}^{\text{3d HM}}\left(t,x=q^{\frac{2i-1}{4}}t^{2i-1};q\right)^{N-i}$$
$$\times\prod_{i=1}^{2N-3}\mathcal{I}^{\text{3d HM}}\left(t,x=q^{\frac{2i+3}{4}}t^{2i-1};q\right)^{a_N(i)}$$

$$\left\langle T_{\left(\frac{1}{2},\cdots,\frac{1}{2}\right)}T_{\left(\frac{1}{2},\cdots,\frac{1}{2}\right)}\right\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(2N)/\mathbb{Z}_2}(t;q)$$

$$=\prod_{k=1}^{\left\lfloor\frac{N}{2}\right\rfloor}\frac{\left(q^{\frac{2k+1}{2}}t^{4k-2};q\right)_\infty}{(q^{k+1}t^{4k-4};q)_\infty}\prod_{k=1}^{N}\frac{\left(q^{\frac{k+1}{2}}t^{2(k-1)};q\right)_\infty}{\left(q^{\frac{k}{2}}t^{2k};q\right)_\infty}\prod_{k=1}^{\left\lfloor\frac{N}{2}\right\rfloor}\frac{\left(q^{\frac{2\left(k+\left\lfloor\frac{N+1}{2}\right\rfloor\right)+3}{2}}t^{4\left(k+\left\lfloor\frac{N-1}{2}\right\rfloor\right)-2};q\right)_\infty}{\left(q^{k+\left\lfloor\frac{N+1}{2}\right\rfloor+1}t^{4\left(k+\left\lfloor\frac{N-1}{2}\right\rfloor\right)};q\right)_\infty}$$

$$\langle T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(8)/\mathbb{Z}_2}(t;q)$$
$$=\frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty^2(q^2t^4;q)_\infty(q^{\frac{5}{2}}t^6;q)_\infty^2(q^{\frac{7}{2}}t^6;q)_\infty(q^{\frac{9}{2}}t^{10};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(qt^4;q)_\infty(q^{\frac{3}{2}}t^6;q)_\infty(q^2;q)_\infty(q^3t^4;q)_\infty(q^3t^8;q)_\infty(q^4t^{12};q)_\infty},$$

$$\langle T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(10)/\mathbb{Z}_2}(t;q)$$
$$=\frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty^2(q^2t^4;q)_\infty(q^{\frac{5}{2}}t^6;q)_\infty^2(q^3t^8;q)_\infty(q^{\frac{9}{2}}t^{10};q)_\infty(q^{\frac{11}{2}}t^{14};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(qt^4;q)_\infty(q^{\frac{3}{2}}t^6;q)_\infty(q^2;q)_\infty(q^2t^8;q)_\infty(q^{\frac{5}{2}}t^{10};q)_\infty(q^3t^4;q)_\infty(q^4t^{12};q)_\infty(q^5t^{16};q)_\infty}.$$

$$\langle W_{\text{sp}}W_{\text{sp}}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ \text{Spin}(4k)}(t;q)=\left\langle T_{\left(\frac{1}{2},\cdots,\frac{1}{2}\right)}T_{\left(\frac{1}{2},\cdots,\frac{1}{2}\right)}\right\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(4k)/\mathbb{Z}_2}(t^{-1};q)$$
$$\langle W_{\text{sp}}W_{\overline{\text{sp}}}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ \text{Spin}(4k+2)}(t;q)=\left\langle T_{\left(\frac{1}{2},\cdots,\frac{1}{2}\right)}T_{\left(\frac{1}{2},\cdots,\frac{1}{2}\right)}\right\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(4k+2)/\mathbb{Z}_2}(t^{-1};q)$$

$$\langle W_{\square}W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t;q)$$
$$=\frac{1}{2^{N-1}N!}\frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi is_i}\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q)_\infty}$$
$$\times\left[\sum_{i=1}^{N}(s_i+s_i^{-1})\right]^2.$$

$$x_i=q^{\frac{i-1}{2}}t^{2(i-1)},i=1,\cdots,N-1$$
$$x_N=q^{\frac{N+1}{2}}t^{2(N-1)}$$

$$\mathbb{III}_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ SO(2N)}\left(t,\left\{x_i=q^{\frac{i-1}{2}}t^{2(i-1)}\right\}_{i=1}^{N-1},x_N=q^{\frac{N+1}{2}}t^{2(N-1)};q\right)$$
$$=\langle T_{(1,0,\cdots,0)}T_{(1,0,\cdots,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t;q)$$
$$\times\prod_{i=1}^{N-2}\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{2i-1}{4}}t^{2i-1};q)^{N-1-\lfloor\frac{i}{2}\rfloor}\prod_{i=1}^{N-3}\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{4N-2i-9}{4}}t^{4N-2i-9};q)^{\lfloor\frac{i}{2}\rfloor}$$
$$\times\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{2N+1}{4}}t^{2N-3};q)\prod_{i=1}^{2N-3}\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{2i+3}{4}}t^{2i-1};q).$$

$$\langle T_{(1,0,\cdots,0)}T_{(1,0,\cdots,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t;q)$$
$$=\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty}\frac{(q^{\frac{3}{2}}t^2;q)_\infty}{(q^2;q)_\infty}\prod_{k=1}^{N-2}\frac{(q^{\frac{2k+1}{2}}t^{4k-2};q)_\infty}{(q^k t^{4k};q)_\infty}$$
$$\times\frac{(q^{\frac{N}{2}}t^{2N-4};q)_\infty}{(q^{\frac{N-1}{2}}t^{2N-2};q)_\infty}\frac{(q^{\frac{N+1}{2}}t^{2N-2};q)_\infty}{(q^{\frac{N+2}{2}}t^{2N-4};q)_\infty}\frac{(q^{\frac{N+3}{2}}t^{2N-2};q)_\infty}{(q^{\frac{N+2}{2}}t^{2N};q)_\infty}\frac{(q^{\frac{2N+1}{2}}t^{4N-6};q)_\infty}{(q^N t^{4N-4};q)_\infty}.$$

$$\langle T_{(1,0,0,0)}T_{(1,0,0,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(8)}(t;q)$$
$$=\frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty^2(q^2t^4;q)_\infty(q^{\frac{5}{2}}t^6;q)_\infty^2(q^{\frac{7}{2}}t^6;q)_\infty(q^{\frac{9}{2}}t^{10};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(qt^4;q)_\infty(q^{\frac{3}{2}}t^6;q)_\infty(q^2;q)_\infty(q^3t^4;q)_\infty(q^3t^8;q)_\infty(q^4t^{12};q)_\infty},$$
$$\langle T_{(1,0,0,0,0)}T_{(1,0,0,0,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(10)}(t;q)$$
$$=\frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty^2(q^{\frac{5}{2}}t^6;q)_\infty^2(q^3t^8;q)_\infty(q^{\frac{7}{2}}t^{10};q)_\infty(q^4t^8;q)_\infty(q^{\frac{11}{2}}t^{14};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(qt^4;q)_\infty(q^2;q)_\infty(q^2t^8;q)_\infty^2(q^{\frac{5}{2}}t^6;q)_\infty(q^{\frac{7}{2}}t^{10};q)_\infty(q^5t^{16};q)_\infty}.$$

$$\langle W_\square W_\square\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t;q)=\langle T_{(1,0,\cdots,0)}T_{(1,0,\cdots,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t^{-1};q)$$

$$\langle W_\square W_\square\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(2N)^-}(t;q)$$
$$=\frac{1}{2^{N-1}(N-1)!}\frac{(q)_\infty^{N-1}(-q;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty^{N-1}(-q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\prod_{i=1}^{N-1}\frac{ds_i}{2\pi is_i}\frac{(s_i^{\pm};q)_\infty(-s_i^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q)_\infty(-q^{\frac{1}{2}}t^{-2}s_i^{\pm};q)_\infty}$$
$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty}\left[\sum_{i=1}^{N-1}(s_i+s_i^{-1})\right]^2$$

$$x_{N-1}=q^{\frac{N+1}{2}}t^{2N-2},x_i=q^{\frac{i}{2}}t^{2i},i=1,\cdots,N-2$$

$$Ⅲ_{\mathcal{D}}^{\text{4d }\mathcal{N}=4\ SO(2N)^-}\left(t,\left\{x_i=q^{\frac{i}{2}}t^{2i}\right\}_{i=1}^{N-2},x_{N-1}=q^{\frac{N+1}{2}}t^{2N-2};q\right)$$
$$=\langle T_{(1,0,\cdots,0)}T_{(1,0,\cdots,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(2N)^-}(t;q)$$
$$\times\prod_{i=1}^{2N-5}\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{2i-1}{4}}t^{2i-1};q)^{N-2-\lfloor\frac{i}{2}\rfloor}\prod_{i=1}^{N-2}\mathcal{I}^{\text{3d HM}}(t,x=-q^{\frac{2i-1}{4}}t^{2i-1};q)$$
$$\times\mathcal{I}^{\text{3d HM}}(t,x=-q^{\frac{2N+1}{4}}t^{2N-3};q)\prod_{i=1}^{2N-3}\mathcal{I}^{\text{3d HM}}(t,x=q^{\frac{2i+3}{4}}t^{2i-1};q),$$

$$\langle T_{(1,0,\cdots,0)}T_{(1,0,\cdots,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ SO(2N)^-}(t;q)$$
$$=\frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty(q^{\frac{2N+1}{2}}t^{4N-6};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(q^2;q)_\infty(q^Nt^{4N-4};q)_\infty}\prod_{k=1}^{N-2}\frac{(q^{\frac{2k+1}{2}}t^{4k-2};q)_\infty}{(q^kt^{4k};q)_\infty}$$
$$\times\frac{(-q^{\frac{N}{2}}t^{2N-4})_\infty(-q^{\frac{N+1}{2}}t^{2N-2};q)_\infty(-q^{\frac{N+3}{2}}t^{2N-2};q)_\infty}{(-q^{\frac{N-1}{2}}t^{2N-2};q)_\infty(-q^{\frac{N+2}{2}}t^{2N-4};q)_\infty(-q^{\frac{N+2}{2}}t^{2N};q)_\infty}.$$

$$\langle W_{\square}W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ O(2N)^{\pm}}(t;q)$$
$$=\frac{1}{2}\left[\langle W_{\square}W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(2N)}(t;q)\pm\langle W_{\square}W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ SO(2N)^{-}}(t;q)\right].$$

$$\langle T_{(1,0,\cdots,0)}T_{(1,0,\cdots,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ O(2N)^{\pm}}(t;q)$$
$$=\frac{1}{2}\frac{(q)_\infty(q^{\frac{3}{2}}t^2;q)_\infty(q^{\frac{2N+1}{2}}t^{4N-6};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty(q^2;q)_\infty(q^Nt^{4N-4};q)_\infty}\prod_{k=1}^{N-2}\frac{(q^{\frac{2k+1}{2}}t^{4k-2};q)_\infty}{(q^kt^{4k};q)_\infty}$$
$$\times\left[\frac{(q^{\frac{N}{2}}t^{2N-4})_\infty(q^{\frac{N+1}{2}}t^{2N-2};q)_\infty(q^{\frac{N+3}{2}}t^{2N-2};q)_\infty}{(q^{\frac{N-1}{2}}t^{2N-2};q)_\infty(q^{\frac{N+2}{2}}t^{2N-4};q)_\infty(q^{\frac{N+2}{2}}t^{2N};q)_\infty}\right.$$
$$\left.\pm\frac{(-q^{\frac{N}{2}}t^{2N-4})_\infty(-q^{\frac{N+1}{2}}t^{2N-2};q)_\infty(-q^{\frac{N+3}{2}}t^{2N-2};q)_\infty}{(-q^{\frac{N-1}{2}}t^{2N-2};q)_\infty(-q^{\frac{N+2}{2}}t^{2N-4};q)_\infty(-q^{\frac{N+2}{2}}t^{2N};q)_\infty}\right].$$

$$\langle W_{\square}W_{\square}\rangle_{\mathcal{N}}^{\text{4d }\mathcal{N}=4\ O(2N)^{\pm}}(t;q)=\langle T_{(1,0,\cdots,0)}T_{(1,0,\cdots,0)}\rangle_{\text{Nahm}}^{\text{4d }\mathcal{N}=4\ O(2N)^{\pm}}(t^{-1};q).$$

$$R=\{\pm\varepsilon_i\pm\varepsilon_j\mid 1\le i<j\le N\},$$
$$R^+=\{\varepsilon_i\pm\varepsilon_j\mid 1\le i<j\le N\}.$$

$$\rho=\sum_{i=1}^{N}(N-i)\varepsilon_i,\lambda=\sum_{i=1}^{N}\lambda_i\varepsilon_i$$

$$w_{D_N}(s;\mathrm{q},\mathrm{t})=\prod_{1\le i<j\le N}\frac{(s_i^{\pm1}s_j^{\pm1};\mathrm{q})_\infty}{(\mathrm{t}s_i^{\pm1}s_j^{\pm1};\mathrm{q})_\infty}.$$

$$\frac{\langle P_\lambda,P_\lambda\rangle}{\langle 1,1\rangle}=\prod_{1\le i<j\le N}\frac{(\mathrm{t}^{j-i+1};\mathrm{q})_{\lambda_i-\lambda_j}(\mathrm{t}^{j-i-1}\mathrm{q};\mathrm{q})_{\lambda_i-\lambda_j}}{(\mathrm{t}^{j-i};\mathrm{q})_{\lambda_i-\lambda_j}(\mathrm{t}^{j-i}\mathrm{q};\mathrm{q})_{\lambda_i-\lambda_j}}$$
$$\times\frac{(\mathrm{t}^{2N-i-j+1};\mathrm{q})_{\lambda_i+\lambda_j}(\mathrm{t}^{2N-i-j-1}\mathrm{q};\mathrm{q})_{\lambda_i+\lambda_j}}{(\mathrm{t}^{2N-i-j};\mathrm{q})_{\lambda_i+\lambda_j}(\mathrm{t}^{2N-i-j}\mathrm{q};\mathrm{q})_{\lambda_i+\lambda_j}}$$

$$\langle 1,1\rangle=\frac{(\mathrm{t};\mathrm{q})_\infty^N}{(\mathrm{q};\mathrm{q})_\infty^N}\prod_{k=1}^{N}\frac{(\mathrm{t}^{2k-1}\mathrm{q};\mathrm{q})_\infty}{(\mathrm{t}^{2k};\mathrm{q})_\infty}$$

$$\chi_{(1^r)}(s)=E_r(s)\ (1\le r\le N-1),\chi_{(1^N)}(s)+\chi_{(1^{N-1},-1)}(s)=E_N(s)$$
$$E_r(s)=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\frac{(\mathbf{q}/\mathrm{t};\mathrm{t}^2)_j(\mathrm{t}^{2N-2r+2j+2};\mathrm{t}^2)_j}{(\mathrm{t}^2;\mathrm{t}^2)_j(\mathrm{q}\mathrm{t}^{2N-2r+2j-1};\mathrm{t}^2)_j}\cdot\frac{1+\mathrm{t}^{N-r}}{1+\mathrm{t}^{N-r+2j}}\mathrm{t}^jP_{(1^{r-2j})}(s;\mathbf{q},\mathrm{t})$$

$$\chi_{(1)}(s)=E_1(s)=P_{(1)}(s;\mathbf{q},\mathbf{t})$$

$$\langle\chi_{(1)},\chi_{(1)}\rangle=\langle P_{(1)},P_{(1)}\rangle=\frac{(1-\mathrm{q})(1-\mathrm{q}\mathrm{t}^{N-2})(1-\mathrm{t}^N)(1-\mathrm{t}^{2N-2})}{(1-\mathrm{t})(1-\mathrm{t}^{N-1})(1-\mathrm{q}\mathrm{t}^{N-1})(1-\mathrm{q}\mathrm{t}^{2N-3})}\langle 1,1\rangle,$$

$$\frac{\langle W_\square W_\square\rangle_{\mathcal{N}}^{SO(2N)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N)}}=\frac{(1-\mathrm{q})(1-\mathrm{q}\mathrm{t}^{N-2})(1-\mathrm{t}^N)(1-\mathrm{t}^{2N-2})}{(1-\mathrm{t})(1-\mathrm{t}^{N-1})(1-\mathrm{q}\mathrm{t}^{N-1})(1-\mathrm{q}\mathrm{t}^{2N-3})}.$$

$$\langle\chi_{(1^{2m})},1\rangle=\langle E_{2m},1\rangle$$
$$=\frac{(\mathrm{q}/\mathrm{t};\mathrm{t}^2)_m(\mathrm{t}^{2N-2m+2};\mathrm{t}^2)_m}{(\mathrm{t}^2;\mathrm{t}^2)_m(\mathrm{q}\mathrm{t}^{2N-2m-1};\mathrm{t}^2)_m}\frac{1+\mathrm{t}^{N-2m}}{1+\mathrm{t}^N}\mathrm{t}^m\langle 1,1\rangle$$

$$\frac{\langle W_{(1^{2m})}\rangle_{\mathcal{N}}^{SO(2N)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N)}} = \frac{(\mathrm{q/t;t^2})_m(\mathrm{t}^{2N-2m+2};\mathrm{t}^2)_m}{(\mathrm{t^2;t^2})_m(\mathrm{qt}^{2N-2m-1};\mathrm{t}^2)_m}\frac{1+\mathrm{t}^{N-2m}}{1+\mathrm{t}^N}\mathrm{t}^m.$$

$$\frac{\langle W_{(1^2)}\rangle_{\mathcal{N}}^{SO(2N)}}{\mathbb{II}_{\mathcal{N}}^{SO(2N)}} = \frac{(\mathrm{t-q})(1-\mathrm{t}^N)(1+\mathrm{t}^{N-2})}{(1-\mathrm{t}^2)(1-\mathrm{qt}^{2N-3})}$$

$$\chi_{\mathrm{sp}}^2(s) = \frac{1}{4}\bar{E}(s\mid 1)+\frac{(-1)^N}{4}\bar{E}(s\mid -1)+\frac{1}{2}\prod_{i=1}^{N}\left(s_i-s_i^{-1}\right)$$

$$\chi_{\overline{\mathrm{sp}}}^2(s) = \frac{1}{4}\bar{E}(s\mid 1)+\frac{(-1)^N}{4}\bar{E}(s\mid -1)-\frac{1}{2}\prod_{i=1}^{N}\left(s_i-s_i^{-1}\right)$$

$$\chi_{\mathrm{sp}}(s)\chi_{\overline{\mathrm{sp}}}(s) = \frac{1}{4}\bar{E}(s\mid 1)-\frac{(-1)^N}{4}\bar{E}(s\mid -1)$$

$$\frac{1}{4}\bar{E}(s\mid 1)+\frac{(-1)^N}{4}\bar{E}(s\mid -1)=\sum_{m=0}^{M-1}E_{2m}(s)+\frac{1}{2}E_{2M}(s)$$

$$\frac{1}{4}\bar{E}(s\mid 1)-\frac{(-1)^N}{4}\bar{E}(s\mid -1)=\sum_{m=0}^{M-1}E_{2m+1}(s)$$

$$\begin{aligned}\frac{\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(4M)}}{\mathbb{III}_{\mathcal{N}}^{\mathrm{SO}(4M)}} &= \frac{\langle W_{\overline{\mathrm{sp}}}W_{\overline{\mathrm{sp}}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(4M)}}{\mathbb{II}_{\mathcal{N}}^{SO(4M)}} = \sum_{m=0}^{M-1}\frac{\langle E_{2m},1\rangle}{\langle 1,1\rangle}+\frac{1}{2}\frac{\langle E_{2M},1\rangle}{\langle 1,1\rangle}\\ &=\sum_{m=0}^{M-1}\frac{(\mathrm{q/t;t^2})_m(\mathrm{t}^{4M-2m+2};\mathrm{t}^2)_m}{(\mathrm{t^2;t^2})_m(\mathrm{qt}^{4M-2m-1};\mathrm{t}^2)_m}\frac{1+\mathrm{t}^{2M-2m}}{1+\mathrm{t}^{2M}}\mathrm{t}^m\\ &\quad+\frac{1}{2}\frac{(\mathrm{q/t;t^2})_M(\mathrm{t}^{2M+1};\mathrm{t}^2)_M}{(\mathrm{t^2;t^2})_M(\mathrm{qt}^{2M-1};\mathrm{t}^2)_M}\frac{1}{1+\mathrm{t}^{2M}}\mathrm{t}^M\\ \langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(4M)} &=0,\end{aligned}$$

$$\frac{1}{4}\bar{E}(s\mid 1)+\frac{(-1)^N}{4}\bar{E}(s\mid -1)=\sum_{m=0}^{M-1}E_{2m+1}(s)+\frac{1}{2}E_{2M+1}(s)$$

$$\frac{1}{4}\bar{E}(s\mid 1)-\frac{(-1)^N}{4}\bar{E}(s\mid -1)=\sum_{m=0}^{M}E_{2m}(s)$$

$$\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(4M+2)} = \langle W_{\overline{\mathrm{sp}}}W_{\overline{\mathrm{sp}}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(4M+2)} = 0$$

$$\frac{\langle W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}\rangle_{\mathcal{N}}^{\mathrm{Spin}(4M+2)}}{\mathbb{III}_{\mathcal{N}}^{SO(4M+2)}} = \sum_{m=0}^{M}\frac{\langle E_{2m},1\rangle}{\langle 1,1\rangle}$$

$$=\sum_{m=0}^{M}\frac{(\mathrm{q/t;t^2})_m(\mathrm{t}^{4M-2m+4};\mathrm{t}^2)_m}{(\mathrm{t^2;t^2})_m(\mathrm{qt}^{4M-2m+1};\mathrm{t}^2)_m}\frac{1+\mathrm{t}^{2M-2m+1}}{1+\mathrm{t}^{2M+1}}\mathrm{t}^m$$

$$\chi_\lambda^{\mathfrak{u}(N)} = s_\lambda(s) = \frac{\det s_j^{\lambda_i+N-i}}{\det s_j^{N-j}}$$

$$\chi_\lambda^{\mathfrak{so}(2N+1)} = \frac{\det\left(s_j^{\lambda_i+N-i+1/2}-s_j^{-(\lambda_i+N-i+1/2)}\right)}{\det\left(s_j^{N-i+1/2}-s_j^{-(N-i+1/2)}\right)}.$$

$$\chi_{\mathrm{sp}}^{\mathfrak{so}(2N+1)} = \prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)$$

$$\chi_{\square}^{\mathfrak{usp}(2N)} = \frac{\det\left(s_j^{\lambda_i+N-i+1}-s_j^{-\lambda_i-N+i-1}\right)}{\det\left(s_j^{N-i+1}-s_j^{-N+i-1}\right)}.$$

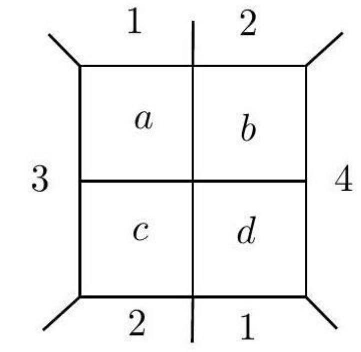
$$\chi_\lambda^{\mathfrak{so}(2N)} = \frac{\det\left(s_j^{\lambda_i+N-i} + s_j^{-\lambda_i-N+i}\right) + \det\left(s_j^{\lambda_i+N-i} - s_j^{-\lambda_i-N+i}\right)}{\det\left(s_j^{N-i} + s_j^{-N+i}\right)}.$$

$$\chi_{\mathrm{sp}}^{\mathfrak{so}(2N)} = \frac{1}{2}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}} + s_i^{-\frac{1}{2}}\right) + \prod_{i=1}^{N}\left(s_i^{\frac{1}{2}} - s_i^{-\frac{1}{2}}\right)\right],$$

$$\chi_{\overline{\mathrm{sp}}}^{\mathfrak{so}(2N)} = \frac{1}{2}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}} + s_i^{-\frac{1}{2}}\right) - \prod_{i=1}^{N}\left(s_i^{\frac{1}{2}} - s_i^{-\frac{1}{2}}\right)\right].$$

$$(AB) = (Z_1 + xZ_2 - wZ_4, yZ_2 + Z_3 + zZ_4)$$

$I_G^{12;34}$:



$$\Delta^2 I_G^{12;34}$$

$$\left(x^{\alpha\dot{\alpha}}, \theta_\alpha^{+a}, \bar{\theta}_{-a'}^{\dot{\alpha}}, u_A^{+a}\right), \theta_\alpha^{+a} = \theta_\alpha^A u_A^{+a}, \bar{\theta}_{-a'}^{\dot{\alpha}} = \bar{\theta}_A^{\dot{\alpha}} u_{-a'}^A$$

$$\mathcal{T}(x, \theta^+, 0, u) = \mathcal{O}_{20'}^{+4}(x) + \cdots + (\theta^+)^4 \mathcal{L}(x)$$

$$G_4 = \langle \mathcal{O}(x_1, y_1) \cdots \mathcal{O}(x_4, y_4) \rangle = \sum_{\ell=0}^{\infty} a^\ell G_4^{(\ell)}$$

$$G_4^{(0)} = \frac{(N_c^2 - 1)^2}{(4\pi^2)^4}(d_{12}^2 d_{34}^2 + 2\,\text{perms.}) + \frac{N_c^2 - 1}{(4\pi^2)^2}(d_{12}d_{23}d_{34}d_{41} + 2\,\text{perms.})$$

$$G_4^{(\ell)} = \frac{2(N_c^2 - 1)}{(4\pi^2)^4} R(1,2,3,4) \times F^{(L)}(x_1, x_2, x_3, x_4) \text{ for } \ell \geq 1$$

$$R(1,2,3,4) := \left(d_{1,2}d_{2,3}d_{3,4}d_{4,1}(1 - U - V) + d_{1,3}^2 d_{2,4}^2\right)x_{1,3}^2 x_{2,4}^2 + (1 \leftrightarrow 2) + (1 \leftrightarrow 4)$$

$$F^{(L)} := \frac{\prod_{1 \leq i < j \leq 4} x_{i,j}^2}{L!} \int \prod_{a=1}^{L} d^4 x_{4+a} \mathcal{F}^{(L)}(x_1, \cdots, x_{4+L})$$

$$\mathcal{F}^{(L)} = \sum_i c_i f_i^{(L)}$$

$$\mathcal{T}(x, \theta^+, 0, u) = \int d^4\theta^- L_{\text{int}}(x, \theta)$$

$$\langle \mathcal{T}(x_1, \theta_1^+, 0, u_1) \cdots \mathcal{T}(x_n, \theta_n^+, 0, u_n) \rangle = \int \prod_{i=1}^{n} d\theta_i^- \langle L_{\text{int}}(x_1, \theta_1) \cdots L_{\text{int}}(x_n, \theta_n) \rangle$$

$$\langle \mathcal{T}(x_1, u_1) \cdots \mathcal{T}(x_4, u_4) \rangle = \left[\prod_{i=1}^{4} D_i^4\right] \langle L_{\text{int}}(x_1, \theta_1) \cdots L_{\text{int}}(x_4, \theta_4) \rangle$$
$$\equiv \left[\prod_{i=1}^{4} D_i^4\right] \mathcal{G}(x, \theta)$$

$$D_i^4 = y_i^{AB} y_i^{CD} \frac{\partial}{\partial\theta_i^{A\alpha}} \frac{\partial}{\partial\theta_i^{B\beta}} \frac{\partial}{\partial\theta_{i\alpha}^{C}} \frac{\partial}{\partial\theta_{i\beta}^{D}}$$

$$\langle i,i\rangle \equiv \epsilon_{IJKL} X_i^{IJ} X_i^{KL} = 0$$

$$x_{i,j}^2 \equiv (x_i - x_j)^2 = \frac{\langle i,j\rangle}{\langle i,I\rangle\langle j,I\rangle}$$

$$\mathbf{X}_i = \begin{pmatrix} Z_{i,\alpha} \\ \theta_i^\alpha \cdot \phi_1 \\ \vdots \\ \theta_i^\alpha \cdot \phi_4 \end{pmatrix}$$

$$\langle\langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \rangle\rangle \equiv \mathrm{Det}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4).$$

$$\mathcal{G}(\mathbf{X}, Y) \prod_{\alpha=1}^{4} \langle\langle Y d^4 Y_\alpha \rangle\rangle$$

$$\mathcal{G}(x,\theta) = \int d^{16}\phi \left[\mathcal{G}(\mathbf{X}, Y)|_{\langle\langle Y, \mathbf{X}_i, \mathbf{X}_j\rangle\rangle \to x_{i,j}^2} \right]$$

$$\left[\prod_{i=1}^{4} D_i^4\right] \int d^{16}\phi \langle\langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \rangle\rangle^4 = R(1,2,3,4) \prod_{i<j} x_{i,j}^2$$

$$\langle i,j\rangle \equiv \frac{1}{4!} \epsilon_{IJKL} X_i^{IJ} X_j^{KL} > 0$$

$$\{X_1, X_2, X_3, X_4\} = \begin{pmatrix} \mathbb{I}_{2\times2} & 0 & \mathbb{I}_{2\times2} & \mathbb{I}_{2\times2} \\ 0 & \mathbb{I}_{2\times2} & \mathbb{I}_{2\times2} & c_1 \\ & 0 & & c_2 \end{pmatrix}$$

$$\Delta^2 = \epsilon(X_1, X_2, X_3, X_4, *, *)\epsilon(X_1, X_2, X_3, X_4, *, *) = s^2 + t^2 + u^2 - 2(st + tu + us) > 0$$

$$\mathbb{T}_4 \colon \langle i,j\rangle > 0 \cup \Delta^2 > 0$$

$$\{X\} = \begin{pmatrix} \frac{218}{23} & \frac{86}{9} & \frac{138}{7} & \frac{137}{12} & \frac{4700}{161} & \frac{755}{36} & \frac{36440}{161} & \frac{583}{18} \\ \frac{15}{4} & \frac{155}{11} & \frac{209}{12} & \frac{84}{11} & \frac{127}{6} & \frac{239}{11} & \frac{586}{3} & \frac{323}{11} \\ \frac{37}{8} & \frac{130}{11} & \frac{190}{11} & \frac{166}{9} & \frac{1927}{88} & \frac{2996}{99} & \frac{1557}{8} & \frac{4822}{99} \\ \frac{239}{12} & \frac{4}{9} & \frac{221}{12} & \frac{127}{11} & \frac{115}{3} & \frac{1187}{99} & \frac{445}{2} & \frac{2330}{99} \end{pmatrix}$$

$$f(c_1, c_2) dc_1 dc_2$$

$$f^+(c_1, c_2) = \frac{(1 - c_2 + c_1 c_2) dc_1 dc_2}{(c_1 - 1)(c_2 - 1) c_1 c_2}$$

$$f^\pm(c_1, c_2) dc_1 dc_2 = \left(\frac{s + t + u \pm \Delta}{stu}\right) \frac{\langle\langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \rangle\rangle^4}{\Delta^2} \prod_{\alpha=1}^{4} \langle\langle Y d^4 Y_\alpha \rangle\rangle \equiv \omega^\pm(Y, \mathbf{X}) d\mu_Y$$

$$d\mu_Y = \prod_{\alpha=1}^{4} \langle\langle Y d^4 Y_\alpha \rangle\rangle$$

$$\langle i,j\rangle = \langle\langle Y, \mathbf{X}_i, \mathbf{X}_j \rangle\rangle.$$

$$\langle (A_{\ell_i}, B_{\ell_i}), X_j \rangle > 0, \left\langle (A_{\ell_i}, B_{\ell_i}), \left(A_{\ell_j}, B_{\ell_j}\right) \right\rangle > 0.$$

$$\frac{\langle (A_{\ell_i}, B_{\ell_i}), X_j \rangle}{\langle (A_{\ell_i}, B_{\ell_i}), X_i \rangle} > 0, \frac{\left\langle (A_{\ell_i}, B_{\ell_i}), \left(A_{\ell_j}, B_{\ell_j}\right) \right\rangle}{\langle (A_{\ell_i}, B_{\ell_i}), X_\ell) \rangle \left\langle \left(A_{\ell_j}, B_{\ell_j}\right), X_k \right\rangle} > 0.$$

$$\mathbb{T}_4 = \sum_{\alpha} \mathbb{T}_{4,\alpha}$$

$$\begin{array}{ll} r_1 : s < t < u, & r_2 : s < u < t, r_3 : t < s < u, \\ r_4 : t < u < s, & r_5 : u < s < t, r_6 : u < t < s. \end{array}$$

$$\begin{aligned}
&r_1 : \left(0 < c_2 \leq \frac{1}{2} \wedge 1 - c_2 < c_1 < 1\right) \vee \left(\frac{1}{2} < c_2 < 1 \wedge c_2 < c_1 < 1\right), \\
&r_2 : \left(1 < c_2 < 2 \wedge c_2 < c_1 < \frac{c_2}{c_2 - 1}\right), \\
&r_3 : \left(0 < c_2 < \frac{1}{2} \wedge c_2 < c_1 < 1 - c_2\right), \\
&r_4 : \left(c_2 \leq -1 \wedge \frac{1}{c_2} < c_1 < 0\right) \vee (-1 < c_2 < 0 \wedge c_2 < c_1 < 0), \\
&r_5 : \left(1 < c_2 \leq 2 \wedge c_1 > \frac{c_2}{c_2 - 1}\right) \vee (c_2 > 2 \wedge c_1 > c_2), \\
&r_6 : \left(c_2 < -1 \wedge c_2 < c_1 < \frac{1}{c_2}\right).
\end{aligned}$$

$$\omega_{r_1}^{\pm} = \frac{\langle \langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \rangle \rangle^4}{\Delta^2} \times \left(\frac{1}{s(t-s)} \pm \frac{\Delta}{s(t-s)(u-t)} \right).$$

$$\sum_{\sigma} \omega_{\sigma}^{\pm}(Y, \mathbf{X}) = \omega^{\pm}(Y, \mathbf{X}) = \left(\frac{s + t + u \pm \Delta}{stu} \right) \frac{\langle \langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \rangle \rangle^4}{\Delta^2} d\mu_Y.$$

$$\mathcal{G}^{(L)}(\mathbf{X}, Y) = \frac{1}{2} \sum_{\sigma, \pm} \omega_{\sigma}^{\pm}(Y, \mathbf{X}) \Omega_{\sigma}^{(L)\pm}.$$

$$\begin{pmatrix} Z_{A_\ell} \\ Z_{B_\ell} \end{pmatrix} = \begin{pmatrix} 1 & x_\ell & 0 & -w_\ell \\ 0 & y_\ell & 1 & z_\ell \end{pmatrix} \begin{pmatrix} Z_{1,i} \\ Z_{2,i} \\ Z_{1,j} \\ Z_{2,j} \end{pmatrix}$$

$$\begin{pmatrix} Z_{A_\ell} \\ Z_{B_\ell} \end{pmatrix} = \begin{pmatrix} 1 & x_\ell & 0 & -w_\ell \\ 0 & y_\ell & -1 & z_\ell \end{pmatrix} \begin{pmatrix} Z_{1,i} \\ Z_{2,i} \\ Z_{1,j} \\ Z_{2,j} \end{pmatrix}$$

$$\Omega_{r_i}^{(L)\pm} = 2\Delta^2 A_{\sigma_3} \pm 2\Delta\left(B + (\sigma_2 - \sigma_1) C_{\sigma_2, \sigma_1} + (\sigma_3 - \sigma_1) C_{\sigma_3, \sigma_1} + (\sigma_3 - \sigma_2) C_{\sigma_3, \sigma_2}\right),$$

$$\langle (A, B), X_j \rangle > 0 \text{ or } \langle (A, B), X_j \rangle < 0, j = 1,2,3,4.$$

$$x_{a,*}^2 = \langle (A, B), Z_{2,2}, Z_{2,1} \rangle$$

$$\begin{aligned}
\Omega_{\pm}^{(1)+} = \frac{\Delta}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,4}^2} &\pm \frac{x_{3,4}^2 x_{2,*}^2 + x_{2,4}^2 x_{3,*}^2 - x_{2,3}^2 x_{4,*}^2}{x_{a,2}^2 x_{a,3}^2 x_{a,4}^2 x_{a,*}^2} \pm \frac{x_{3,4}^2 x_{1,*}^2 + x_{1,3}^2 x_{4,*}^2 - x_{1,4}^2 x_{3,*}^2}{x_{a,1}^2 x_{a,3}^2 x_{a,4}^2 x_{a,*}^2} \\
&\pm \frac{x_{1,2}^2 x_{4,*}^2 + x_{2,4}^2 x_{1,*}^2 - x_{1,4}^2 x_{2,*}^2}{x_{a,1}^2 x_{a,2}^2 x_{a,4}^2 x_{a,*}^2} \pm \frac{x_{2,3}^2 x_{1,*}^2 + x_{1,3}^2 x_{2,*}^2 - x_{1,2}^2 x_{3,*}^2}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,*}^2}
\end{aligned}$$

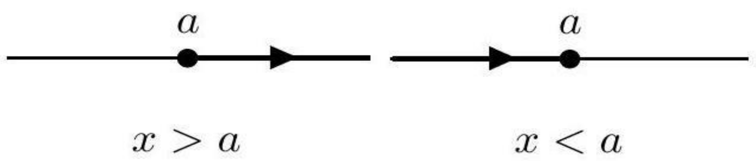
$$\Omega^{(1)\pm} = \frac{\pm 2\Delta}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,4}^2} d^4 x_a = \pm 2\Delta g(1,2,3,4)$$

$$\mathcal{G}^{(1)}(\mathbf{X},Y) = \frac{1}{2}\sum_{\pm} \omega^{\pm}\Omega^{(1)\pm} = \frac{\langle\langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \rangle\rangle^4}{stu\Delta}[\tilde{\Delta} g(1,2,3,4)]$$

$$G_4^{(1)} = R(1,2,3,4)\left(\prod_{i<j} x_{ij}^2\right)\frac{g(1,2,3,4)}{\prod_{i<j} x_{ij}^2}$$

$$\begin{array}{lll} \langle (AB)X_i\rangle > 0, & \langle (CD)X_i\rangle > 0, & \langle (AB)(CD)\rangle > 0, \\ \langle (AB)X_i\rangle < 0, & \langle (CD)X_i\rangle > 0, & \langle (AB)(CD)\rangle < 0, \\ \langle (AB)X_i\rangle > 0, & \langle (CD)X_i\rangle < 0, & \langle (AB)(CD)\rangle < 0, \\ \langle (AB)X_i\rangle < 0, & \langle (CD)X_i\rangle < 0, & \langle (AB)(CD)\rangle > 0. \end{array}$$

$$x \geq a: \frac{dx}{x-a}, x \leq a: -\frac{dx}{x-a}$$



$$\langle ABX_1\rangle = w_1, \langle CDX_3\rangle = (1-c_1)(1-c_2)w_2.$$

$$\Omega^{(2)+} = \Omega^{(2)+}_{+,+} - \Omega^{(2)+}_{+,-} - \Omega^{(2)+}_{-,+} + \Omega^{(2)+}_{-,-}.$$

$$\Omega^{(2)\pm} = 2\Delta^2 g(1,2,3,4)^2 \pm 2\Delta(h(1,2;3,4) + 5\,\text{perm.}),$$

$$g(1,2,3,4)^2 := \frac{d^4x_a d^4x_b}{2x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,4}^2 x_{b,1}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2} + (a \leftrightarrow b),$$
$$h(1,2;3,4) := \frac{d^4x_a d^4x_b x_{3,4}^2}{x_{a,1}^2 x_{a,3}^2 x_{a,4}^2 x_{a,b}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2} + (a \leftrightarrow b).$$

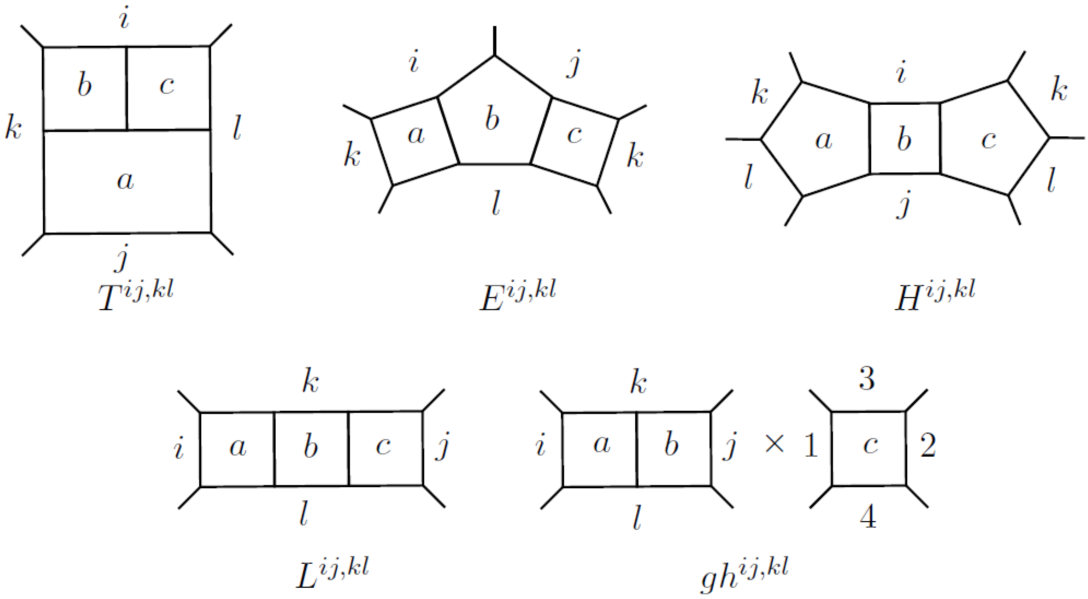
$$\mathcal{G}^{(2)}(\mathbf{X},Y) = \frac{\langle\langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \rangle\rangle^4}{stu}((s+t+u)g(1,2,3,4)^2 + (h(1,2;3,4) + 5\,\text{perm.}))$$

$$\Omega_{r_i}^{(3)\pm} = 2\Delta^2 A_{\sigma_3} \pm 2\Delta\big(B + (\sigma_2-\sigma_1)C_{\sigma_2,\sigma_1} + (\sigma_3-\sigma_1)C_{\sigma_3,\sigma_1} + (\sigma_3-\sigma_2)C_{\sigma_3,\sigma_2}\big),$$

$$\Omega_{r_1}^{(3)\pm} = 2\Delta^2 A_u \pm 2\Delta\big(B + (t-s)C_{s,t} + (u-s)C_{s,u} + (u-t)C_{t,u}\big).$$

$$\text{LS}[A_{\sigma_i}] \propto \frac{1}{\Delta^2}, \text{LS}[B] \propto \frac{1}{\Delta}$$
$$\text{LS}[C_{s,t}] \propto \frac{1}{\Delta(t-s)}, \text{LS}[C_{s,u}] \propto \frac{1}{\Delta(u-s)}, \text{LS}[C_{t,u}] \propto \frac{1}{\Delta(u-t)}$$

$$B := T^{12;34} + E^{12;34} + 11\,\text{perms.} + L^{12;34} + 5\,\text{perms.}$$
$$A_s := [H^{14;23} + (1,4) \leftrightarrow (2,3)] + (3 \leftrightarrow 4) + gh^{12;34} + gh^{34;12},$$
$$C_{t,u} := 2(E'^{12;34} + E'^{34;12}).$$

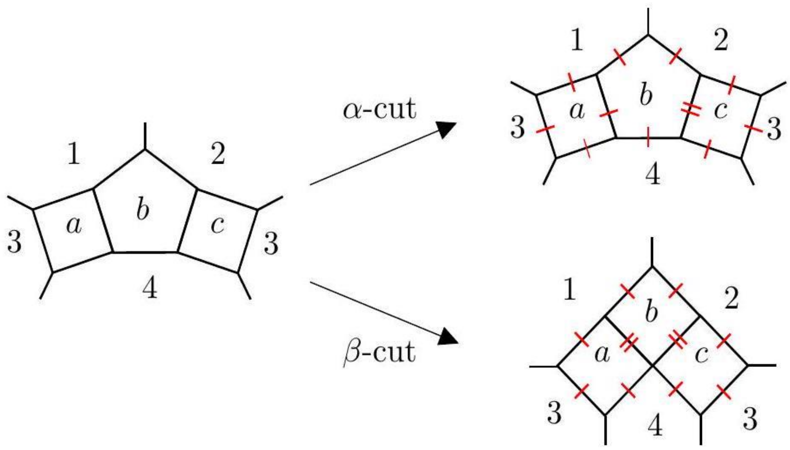


$$n_T = x_{a,i}^2 x_{k,l}^2, \quad n_E = x_{b,k}^2 x_{i,l}^2 x_{j,l}^2 - x_{b,l}^2 \left(x_{i,l}^2 x_{j,k}^2 + x_{i,k}^2 x_{j,l}^2\right)/2, \quad n_E' = x_{b,l}^2,$$

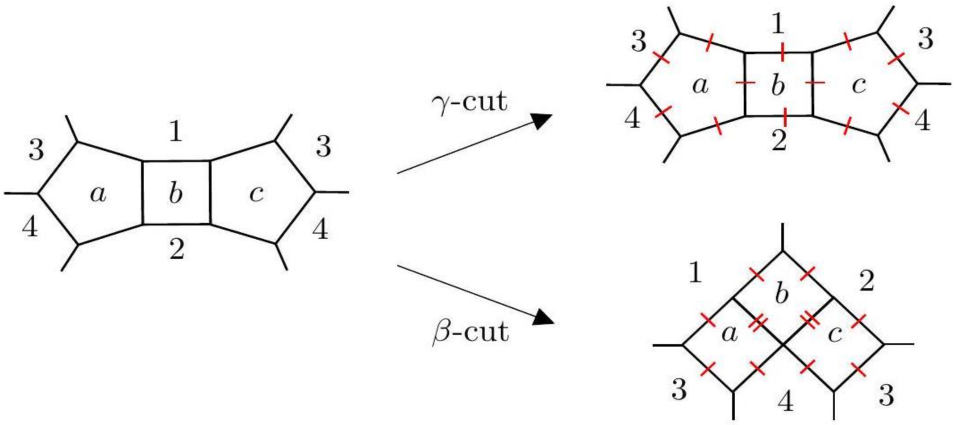
$$n_L = x_{k,l}^4, \quad n_H = \frac{1}{2} x_{a,c}^2 x_{i,j}^2 - x_{a,j}^2 x_{c,i}^2, \quad n_{gh} = x_{k,l}^2.$$

$$E_c^{12;34} = E^{12;34} + \frac{1}{2}\left(x_{1,4}^2 x_{2,3}^2 + x_{1,3}^2 x_{2,4}^2\right) E'^{12;34}$$

$$H_c^{12;34} = H^{12;34} + E^{12;34}$$

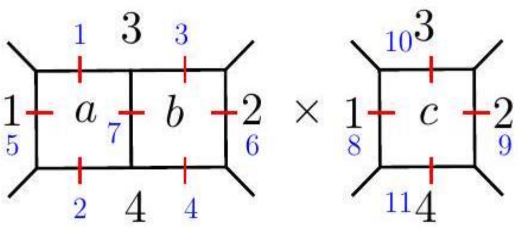


$$\Delta E^{12;34}|_\beta + \Delta E^{12;43}|_\beta = \frac{1}{2} - \frac{1}{2} = 0.$$



$$\frac{1}{\Delta}\ \text{on the}\ \alpha\text{-cut},\ \frac{t}{(t-u)\Delta},\frac{u}{(t-u)\Delta}\ \ \text{on the}\ \beta\text{-cut}\,,$$

$$\frac{1}{\Delta^2}\ \text{on the}\ \gamma\text{-cut},\ \frac{1}{(t-u)\Delta}\ \text{on the}\ \beta\text{-cut}.$$

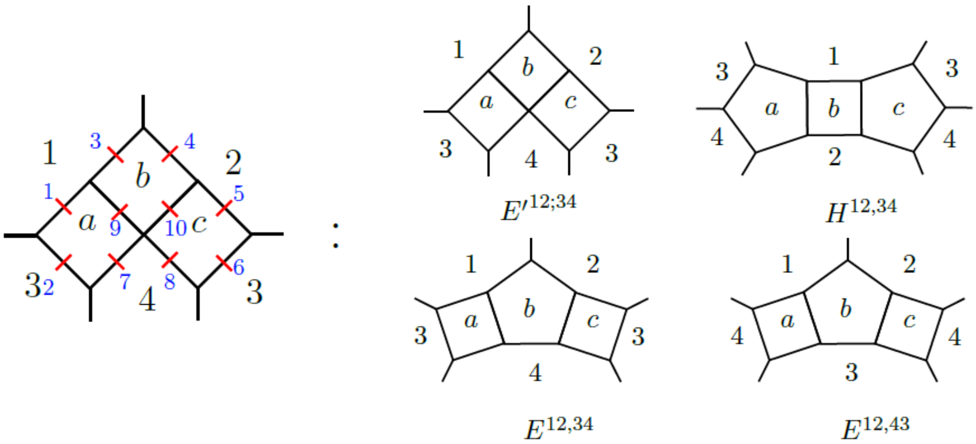


$$(A,B)=(Z_{1,3}+x_1Z_{2,3}-w_1Z_{2,4},\quad y_1Z_{2,3}+Z_{1,4}+z_1Z_{2,4})\,,$$
$$(C,D)=(Z_{1,3}+x_2Z_{2,3}-w_2Z_{2,4},\quad y_2Z_{2,3}+Z_{1,4}+z_2Z_{2,4})\,,$$
$$(E,F)=(Z_{1,1}+x_3Z_{2,1}-w_3Z_{2,2},\quad y_3Z_{2,1}+Z_{1,2}+z_3Z_{2,2})\,.$$

$$(w_1,y_1,w_2,y_2,x_1,x_2,z_2,w_3,y_3,x_3,z_3).$$

$$z_1<0,\ \text{for}\ \max(s,t,u)=s,$$

$$z_1>0,\ \text{for}\ \max(s,t,u)=s.$$



$$(A,B)=(Z_{1,1}+x_1Z_{2,1}-w_1Z_{2,3},\quad y_1Z_{2,1}+Z_{1,3}+z_1Z_{2,3}),$$
$$(C,D)=(Z_{1,1}+x_2Z_{2,1}-w_2Z_{2,2},\quad y_2Z_{2,1}+Z_{1,2}+z_2Z_{2,2}),$$
$$(E,F)=(Z_{1,2}+x_3Z_{2,2}-w_3Z_{2,3},\quad y_3Z_{2,2}+Z_{1,3}+z_3Z_{2,3}).$$

$$(w_1,y_1,w_2,y_2,w_3,y_3,x_1,x_3,x_2,z_2).$$

$$t<u:\quad (z_3-z_1)\left(z_3-\frac{(1-c_1)z_1}{1-c_2}\right)<0\vee\left(z_3-\frac{c_1z_1}{c_2}\right)\left(z_3-\frac{c_1(1-c_2)z_1}{c_2(1-c_1)}\right)<0$$
$$t>u:\quad \left(z_3-\frac{c_1z_1}{c_2}\right)\left(z_3-\frac{(1-c_1)z_1}{1-c_2}\right)<0\vee(z_3-z_1)\left(z_3-\frac{c_1(1-c_2)z_1}{c_2(1-c_1)}\right)<0$$

$$\max(s,t,u)\neq s, t<u: z_3\left(z_3-\frac{(1-c_1)z_1}{1-c_2}\right)<0\vee z_1\left(z_3-\frac{c_1(1-c_2)z_1}{c_2(1-c_1)}\right)>0$$
$$\max(s,t,u)\neq s, t>u: z_3\left(z_3-\frac{c_1(1-c_2)z_1}{c_2(1-c_1)}\right)<0\vee z_1\left(z_3-\frac{(1-c_1)z_1}{1-c_2}\right)>0$$
$$\max(s,t,u)=s: \left(z_3-\frac{c_1(1-c_2)z_1}{c_2(1-c_1)}\right)\left(z_3-\frac{(1-c_1)z_1}{1-c_2}\right)<0$$

$$t<u: z_3\to z_1^{\pm}, z_1\frac{c_1-c_2}{1-c_2}\to 0^{\mp}$$
$$t>u: z_3\to z_1^{\pm}, z_1\frac{c_2-c_1}{(1-c_1)c_2}\to 0^{\mp}$$

$$t<u:\quad z_3\to z_1^{\pm},\quad z_1\to 0^{\mp}$$
$$t>u:\quad z_3\to z_1^{\pm},\quad z_1\to 0^{\mp}$$

$$t<u:\quad z_3\to 0^{\pm},\quad z_1\to 0^{\pm}$$
$$u<t:\quad z_3\to 0^{\pm},\quad z_1\to 0^{\pm}$$

$$z_3\to\left(\frac{1-c_1}{1-c_2}z_1\right)^{\pm},\frac{1-c_1}{1-c_2}z_1\to 0^{\mp}$$

$$z_3\to\left(\frac{1-c_1}{1-c_2}z_1\right)^{\pm},z_1\to 0^{\pm}$$

$$z_3\to\left(\frac{1-c_1}{1-c_2}z_1\right)^{\pm},z_1\to 0^{\mp}$$

$L_1: z_3=\frac{c_1(1-c_2)}{c_2(1-c_1)}z_1, L_2: z_3=\frac{1-c_2}{1-c_1}z_1, L_3: z_3=\frac{c_1}{c_2}z_1, L_4: z_3=z_1, L_5: z_1=0$, and $L_6: z_3=0$

$$G_4^{(L)}=R(1,2,3,4)stuF^{(L)}$$

$$\frac{1}{stu\Delta},\frac{1}{st\Delta^2},\frac{1}{ut\Delta^2},\frac{1}{su\Delta^2}$$
$$\frac{1}{ts\Delta(t-u)},\frac{1}{us\Delta(t-u)},\frac{1}{ts\Delta(u-s)},\frac{1}{tu\Delta(u-s)},\frac{1}{tu\Delta(s-t)},\frac{1}{su\Delta(s-t)}$$

$$\frac{1}{tu\Delta(s-t)}=\frac{1}{2}\sum_{\pm}\pm\left(\sum_{i=3,4,6}\omega_{r_i}^{\pm}\right)\frac{1}{\langle\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4\rangle^4}$$

$$
\begin{aligned}
E'^{12;34} &:= \frac{1}{\Delta(u-t)}\left(\mathcal{E}(z,\bar{z}) + \mathcal{E}\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right)\right)\\
E_c^{12;34} &:= \frac{u}{\Delta(u-t)}\left(\mathcal{E}(z,\bar{z}) + \frac{t}{u}\mathcal{E}\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right)\right)\\
&= \frac{1}{2\Delta}\left(\mathcal{E}(z,\bar{z}) - \mathcal{E}\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right)\right) - \frac{t+u}{2}E'^{12;34}\\
H_c^{12;34} &+ E'^{12;34} := -\frac{1}{\Delta^2}\mathcal{H}_a(z,\bar{z})
\end{aligned}
$$

$$
\mathcal{L}_{a_n,a_{n-1},\cdots,a_1} = \int_{sv}\frac{dz}{z-a_n}\mathcal{L}_{a_{n-1},\cdots,a_1}, a_n = 0,1;
$$
$$
\mathcal{L}_{\emptyset} = 1; \mathcal{L}_0 = \log z\bar{z}, \mathcal{L}_1 = \log(1-z)(1-\bar{z})
$$

$$
G_{a_n,a_{n-1},\dots,a_1}(z) = \int_0^z \frac{\mathrm{d}z'}{z'-a_n}G_{a_{n-1},\dots,a_1}(z), a_n = 0,\pm 1;\ G_{\emptyset}(z) := 1
$$

$$
\frac{(-1)^L}{\Delta}\times(\mathcal{L}^{0,\dots,}_{\underbrace{0,\dots,0}_{L}}1,\underbrace{0,\dots,0}_{L-1} - \mathcal{L}^{0,\dots,}_{\underbrace{0,\dots,0}_{L-1}}1,\underbrace{0,\dots,0}_{L})
$$

$$
\begin{aligned}
\mathcal{E}(z,\bar{z}) =& 2\mathcal{L}_{0,0,0,1,0,1} - 2\mathcal{L}_{0,0,1,0,1,0} - \mathcal{L}_{0,0,1,1,0,1} + \mathcal{L}_{0,0,1,1,1,0} - 2\mathcal{L}_{0,1,0,0,0,1} + 2\mathcal{L}_{0,1,0,1,0,0} - \mathcal{L}_{0,1,0,1,1,0}\\
&+\mathcal{L}_{0,1,1,0,0,1} + \mathcal{L}_{0,1,1,0,1,0} - \mathcal{L}_{0,1,1,1,0,0} + 2\mathcal{L}_{1,0,0,0,1,0} - \mathcal{L}_{1,0,0,1,0,1} - \mathcal{L}_{1,0,0,1,1,0} - 2\mathcal{L}_{1,0,1,0,0,0}\\
&+\mathcal{L}_{1,0,1,0,0,1} + \mathcal{L}_{1,0,1,1,0,0} - 20\zeta_5\mathcal{L}_1 + 8\zeta_3\mathcal{L}_{0,0,1} - 4\zeta_3\mathcal{L}_{0,1,1} - 2\zeta_3\mathcal{L}_{1,0,1}
\end{aligned}
$$

$$
(z-\bar{z})(z+\bar{z}+z\bar{z})E' = \mathcal{E}(z,\bar{z}) + \mathcal{E}\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right)
$$

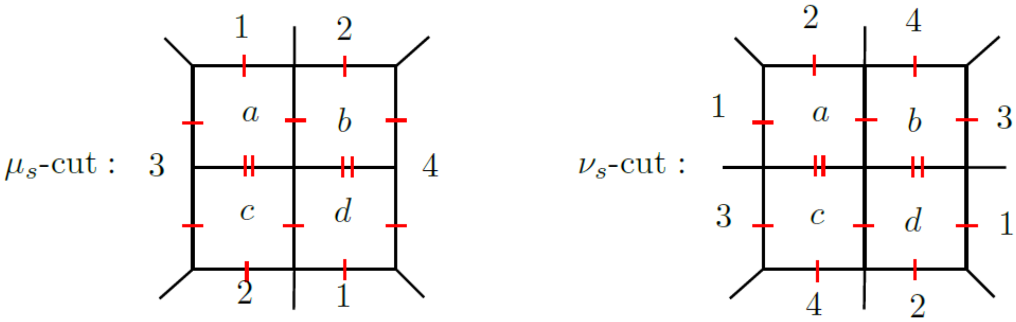
$$
\begin{aligned}
&4\mathcal{L}_{0,0,0,1,0,1} - 2\mathcal{L}_{0,0,0,1,1,1} - 4\mathcal{L}_{0,0,1,0,1,0} + 2\mathcal{L}_{0,0,1,0,1,1} - 2\mathcal{L}_{0,0,1,1,0,1} + 2\mathcal{L}_{0,0,1,1,1,0} - 4\mathcal{L}_{0,1,0,0,0,1}\\
&+2\mathcal{L}_{0,1,0,0,1,1} + 4\mathcal{L}_{0,1,0,1,0,0} - 2\mathcal{L}_{0,1,0,1,0,1} - 2\mathcal{L}_{0,1,0,1,1,0} + \mathcal{L}_{0,1,0,1,1,1} + 2\mathcal{L}_{0,1,1,0,0,1} + 2\mathcal{L}_{0,1,1,0,1,0}\\
&-2\mathcal{L}_{0,1,1,0,1,1} - 2\mathcal{L}_{0,1,1,1,0,0} + \mathcal{L}_{0,1,1,1,0,1} + 4\mathcal{L}_{1,0,0,0,1,0} - 2\mathcal{L}_{1,0,0,0,1,1} - 2\mathcal{L}_{1,0,0,1,0,1} - 2\mathcal{L}_{1,0,0,1,1,0}\\
&+2\mathcal{L}_{1,0,0,1,1,1} - 4\mathcal{L}_{1,0,1,0,0,0} + 2\mathcal{L}_{1,0,1,0,0,1} + 2\mathcal{L}_{1,0,1,0,1,0} - \mathcal{L}_{1,0,1,0,1,1} + 2\mathcal{L}_{1,0,1,1,0,0} - \mathcal{L}_{1,0,1,1,1,0}\\
&+2\mathcal{L}_{1,1,0,0,0,1} - 2\mathcal{L}_{1,1,0,0,1,0} - 2\mathcal{L}_{1,1,0,1,0,0} + \mathcal{L}_{1,1,0,1,0,1} + 2\mathcal{L}_{1,1,0,1,1,0} - \mathcal{L}_{1,1,0,1,1,1} + 2\mathcal{L}_{1,1,1,0,0,0}\\
&-2\mathcal{L}_{1,1,1,0,0,1} - \mathcal{L}_{1,1,1,0,1,0} + \mathcal{L}_{1,1,1,0,1,1} + 4\zeta_3\mathcal{L}_{1,0,1} - 2\zeta_3\mathcal{L}_{1,1,1}
\end{aligned}
$$

$$
\begin{aligned}
&-4\mathcal{L}_{0,0,0,1,0,1} + 2\mathcal{L}_{0,0,0,1,1,1} + 4\mathcal{L}_{0,0,1,0,1,0} - 2\mathcal{L}_{0,0,1,0,1,1} + 2\mathcal{L}_{0,0,1,1,0,1} - 2\mathcal{L}_{0,0,1,1,1,0}\\
&+4\mathcal{L}_{0,1,0,0,0,1} + 8\mathcal{L}_{0,1,0,0,1,0} - 10\mathcal{L}_{0,1,0,0,1,1} - 4\mathcal{L}_{0,1,0,1,0,0} + 2\mathcal{L}_{0,1,0,1,0,1} + 2\mathcal{L}_{0,1,0,1,1,0}\\
&+\mathcal{L}_{0,1,0,1,1,1} - 8\mathcal{L}_{0,1,1,0,0,0} + 6\mathcal{L}_{0,1,1,0,0,1} - 2\mathcal{L}_{0,1,1,0,1,0} + 2\mathcal{L}_{0,1,1,1,0,0} - \mathcal{L}_{0,1,1,1,0,1}\\
&+4\mathcal{L}_{1,0,0,0,1,0} - 6\mathcal{L}_{1,0,0,0,1,1} - 2\mathcal{L}_{1,0,0,1,0,1} + 2\mathcal{L}_{1,0,0,1,1,0} + 2\mathcal{L}_{1,0,0,1,1,1} - 4\mathcal{L}_{1,0,1,0,0,0}\\
&+6\mathcal{L}_{1,0,1,0,0,1} + 2\mathcal{L}_{1,0,1,0,1,0} - 3\mathcal{L}_{1,0,1,0,1,1} - 2\mathcal{L}_{1,0,1,1,0,0} + 2\mathcal{L}_{1,0,1,1,0,1} - \mathcal{L}_{1,0,1,1,1,0}\\
&+2\mathcal{L}_{1,1,0,0,0,1} - 2\mathcal{L}_{1,1,0,0,1,0} - 2\mathcal{L}_{1,1,0,1,0,0} + \mathcal{L}_{1,1,0,1,0,1} + 2\mathcal{L}_{1,1,1,0,0,0} - 2\mathcal{L}_{1,1,1,0,0,1}\\
&+\mathcal{L}_{1,1,1,0,1,0} + 20\zeta_3\mathcal{L}_{0,1,1} + 16\zeta_3\mathcal{L}_{1,0,1} + 12\zeta_3\mathcal{L}_{1,1,0} - 8\zeta_3\mathcal{L}_{1,1,1}\\
&+8\mathbf{L}_{0,\bar{z},0,0,1,1} - 8\mathbf{L}_{0,\bar{z},0,1,1,0} - 8\mathbf{L}_{0,\bar{z},1,0,0,1} + 8\mathbf{L}_{0,\bar{z},1,1,0,0} + 4\mathbf{L}_{1,\bar{z},0,0,1,1} - 4\mathbf{L}_{1,\bar{z},0,1,1,0} - 4\mathbf{L}_{1,\bar{z},1,0,0,1}\\
&+4\mathbf{L}_{1,\bar{z},1,1,0,0} + 8\mathbf{L}_{\bar{z},0,0,0,1,1} + 8\mathbf{L}_{\bar{z},0,0,1,0,1} - 8\mathbf{L}_{\bar{z},0,0,1,1,0} - 4\mathbf{L}_{\bar{z},0,0,1,1,1} - 8\mathbf{L}_{\bar{z},0,1,0,0,1} - 8\mathbf{L}_{\bar{z},0,1,0,1,0}\\
&+4\mathbf{L}_{\bar{z},0,1,0,1,1} + 8\mathbf{L}_{\bar{z},0,1,1,0,0} - 4\mathbf{L}_{\bar{z},0,1,1,0,1} + 4\mathbf{L}_{\bar{z},0,1,1,1,0} - 8\mathbf{L}_{\bar{z},1,0,0,0,1} - 8\mathbf{L}_{\bar{z},1,0,0,1,0} + 12\mathbf{L}_{\bar{z},1,0,0,1,1}\\
&+8\mathbf{L}_{\bar{z},1,0,1,0,0} - 4\mathbf{L}_{\bar{z},1,0,1,0,1} - 4\mathbf{L}_{\bar{z},1,0,1,1,0} + 8\mathbf{L}_{\bar{z},1,1,0,0,0} - 4\mathbf{L}_{\bar{z},1,1,0,0,1} + 4\mathbf{L}_{\bar{z},1,1,0,1,0} - 4\mathbf{L}_{\bar{z},1,1,1,0,0}\\
&-16\mathbf{L}_{\bar{z},\bar{z},0,0,1,1} + 16\mathbf{L}_{\bar{z},\bar{z},0,1,1,0} + 16\mathbf{L}_{\bar{z},\bar{z},1,0,0,1} - 16\mathbf{L}_{\bar{z},\bar{z},1,1,0,0} - 32\zeta_3\mathbf{L}_{\bar{z},1,1}
\end{aligned}
$$

$$
\Omega_{r_i}^{(4)\pm} = 2\Delta^2 A_{\sigma_3} \pm 2\Delta\big(B + (\sigma_2-\sigma_1)C_{\sigma_2,\sigma_1} + (\sigma_3-\sigma_1)C_{\sigma_3,\sigma_1} + (\sigma_3-\sigma_2)C_{\sigma_3,\sigma_2}\big),
$$

$$
\begin{aligned}
B :=& -\frac{1}{2}I_{\Theta'}^{12;34} + 2\text{ perms.} + I_S^{12;34} + 3\text{ perms.} + \big[I_{X''}^{12;34} + I_Q^{12;34} + I_{K'}^{12;34} + 5\text{ perms.}\big]\\
&+\big[I_P^{12;34} - I_{M''}^{12;34} + I_J^{12;34} + I_{Y''}^{12;34} + I_{N'}^{12;34} + I_V^{12;34} + I_{R'}^{12;34} + 11\text{ perms.}\big]\\
&+\big[I_{P''}^{12;34} + I_Y^{12;34} + I_U^{12;34} + 23\text{ perms.}\big]\\
A_s :=& 8I_F^{12;34} + I_{hh'}^{12;34} + \big[4I_G^{12;34} + I_{gL}^{12;34} + I_{hh}^{12;34} + (1,2)\leftrightarrow(3,4)\big] + \big[I_Z^{12;34}\\
&+I_N^{12;34} + 7\text{ perms.}|_s\big] + \big[I_{S''}^{12;34} + I_{gT}^{12;34} + I_W^{12;34} + 3\text{ perms.}|_s\big] + \big[I_K^{14;23}\\
&+I_X^{14;23} + (1,4)\leftrightarrow(2,3) + I_M^{14;23} + (3\leftrightarrow 4)\big]
\end{aligned}
$$

$$C_{t,u}^{(1)} := \frac{1}{2} I_{\Theta}^{12;34} + \left[I_{X'}^{12;34} + I_{\Xi}^{12;34} + 2 I_{M'}^{12;34} + I_{K''}^{12;34} + (1,2) \leftrightarrow (3,4) \right]$$
$$+ \left[\frac{1}{2} I_{P'}^{12;34} + I_{Y'}^{12;34} + \frac{1}{2} I_{S'}^{12;34} + 3 \text{ perms.}|_s \right] + \left[I_{O}^{12;34} + 7 \text{ perms.}|_s \right]$$
$$C_{t,u}^{(2)} := \left[2 I_{R}^{12;43} + I_{\text{I}}^{12;43} + 2 I_{U'}^{12;43} + (1,3) \leftrightarrow (2,4) \right] + (1,2) \leftrightarrow (3,4)$$
$$C_{u,t}^{(2)} := \left[2 I_{R}^{12;34} + I_{\text{I}}^{12;34} + 2 I_{U'}^{12;34} + (1,3) \leftrightarrow (2,4) \right] + (1,2) \leftrightarrow (3,4)$$



$$\begin{aligned}
(A,B) &= (Z_{1,1} + x_1 Z_{2,1} - w_1 Z_{2,3}, \quad y_1 Z_{2,1} + Z_{1,3} + z_1 Z_{2,3}),\\
(C,D) &= (Z_{1,2} + x_2 Z_{2,2} - w_2 Z_{2,4}, \quad y_2 Z_{2,2} + Z_{1,4} + z_2 Z_{2,4}),\\
(E,F) &= (Z_{1,2} + x_3 Z_{2,2} - w_3 Z_{2,3}, \quad y_3 Z_{2,2} + Z_{1,3} + z_3 Z_{2,3}),\\
(G,H) &= (Z_{1,1} + x_4 Z_{2,1} - w_4 Z_{2,4}, \quad y_4 Z_{2,1} + Z_{1,4} + z_4 Z_{2,4}),
\end{aligned}$$

$$z_1^{-2}\Big((-c_1(c_1+1)(c_2-1) + (c_1-c_2)(c_1-1)(c_2-1)X + (c_1-1)c_2(c_2+1)X^2)^2$$
$$-4X(c_1 + c_1^2 c_2 (X-1) - c_2 X)(c_1 + c_1 c_2^2 (X-1) - c_2 X)\Big)^{-\frac{1}{2}}$$

$$Y^2 = \frac{(sX-t)(4uX^2 - sX + t)}{su} = \frac{(UX-V)(4X^2 - UX + V)}{U}.$$

$$\frac{(U^4 - 16U^2V + 16V^2)^3}{U^2V^4(U^2 - 16V)}$$

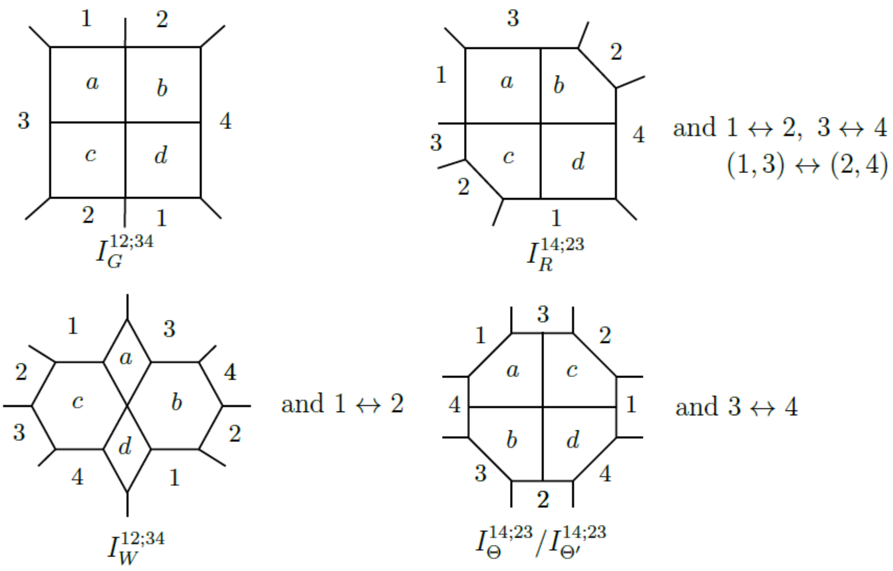
$$\begin{aligned}
(A,B) &= (Z_{1,1} + x_1 Z_{2,1} - w_1 Z_{2,2},\\
(C,D) &= (Z_{1,3} Z_{2,1} + Z_2 Z_{2,3} - w_2 Z_{2,4}, y_2 Z_{2,3} + Z_{1,4} + z_2 Z_{2,4})\\
(E,F) &= (Z_{1,3} + x_3 Z_{2,3} - w_3 Z_{2,4}, y_3 Z_{2,3} + Z_{1,4} + z_3 Z_{2,4})\\
(G,H) &= (Z_{1,1} + x_4 Z_{2,1} - w_4 Z_{2,2}, y_4 Z_{2,1} + Z_{1,2} + z_4 Z_{2,2}),
\end{aligned}$$

$$\big(4(c_1-1)(c_2-1)(X-1)(c_1-c_2X)(c_1-1+X-c_2X)(c_1-c_1c_2-c_2X+c_1c_2X)\big)^{-\frac{1}{2}}$$

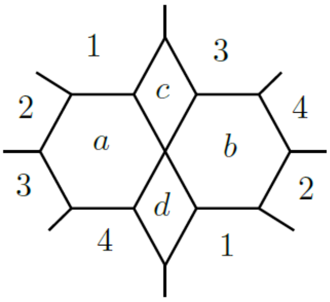
$$Y^2 = \frac{4X(sX+u)(sX+t)}{s^2} = \frac{4X(UX+1)(UX+V)}{U^2}$$

$$\frac{256(1-V-V^2)^3}{(-1+V)^2V^2}.$$

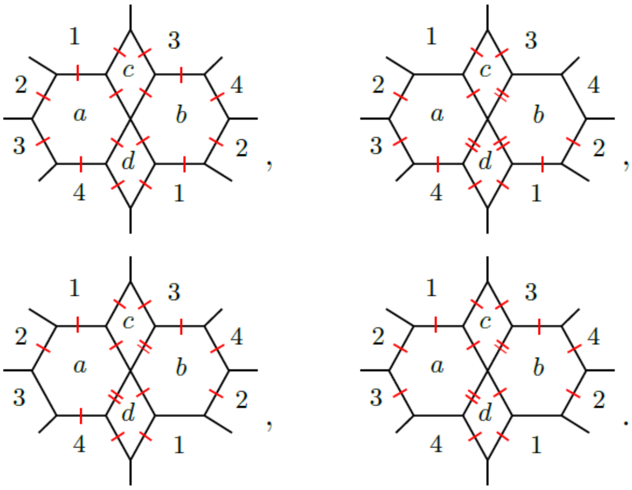
μ_s-cut :



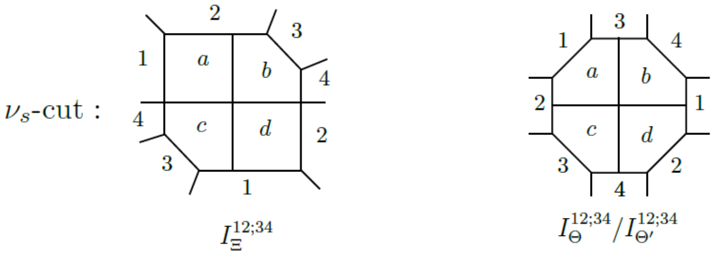
$W^{12;34}$:

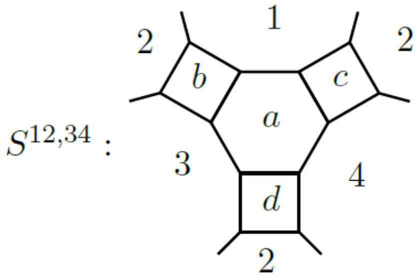


$$x_{1,3}^2 x_{1,4}^2 x_{a,b}^4 \to (x_{a,b}^2 x_{1,3}^2 - x_{a,1}^2 x_{b,3}^2 - x_{a,3}^2 x_{b,1}^2) \times (x_{a,b}^2 x_{1,4}^2 - x_{a,1}^2 x_{b,4}^2 - x_{a,4}^2 x_{b,1}^2)$$



$$\frac{1}{\Delta^2}, \frac{t}{\Delta s(s-t)}, \frac{1}{\Delta(s-t)}, \text{ and } \frac{1}{\Delta(s-u)}.$$

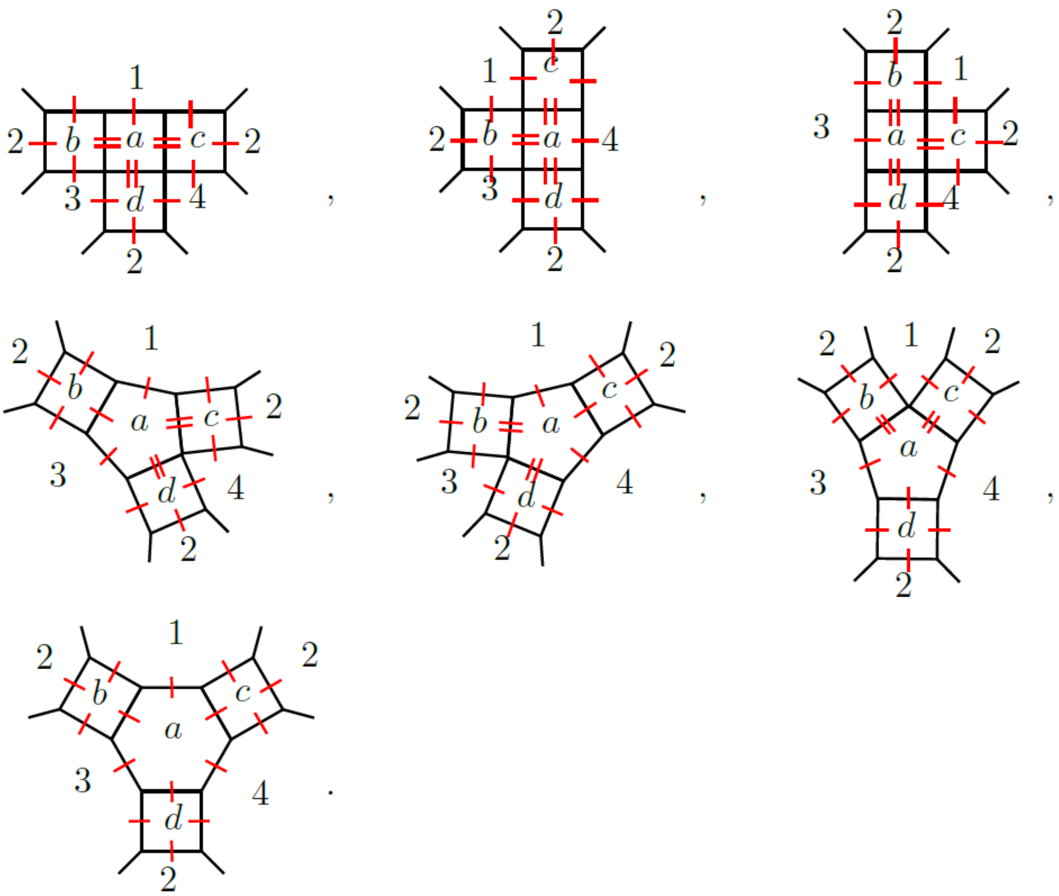




$$I_S^{12;34}: n_S^{12;34} = \left(x_{1,3}^2 x_{1,4}^2 x_{3,4}^2 x_{a,2}^4 + 3x_{1,4}^2 x_{2,3}^2 x_{a,1}^2\left(x_{2,4}^2 x_{a,3}^2 - x_{3,4}^2 x_{a,2}^2\right)\right)/6 + \text{perms}(1,3,4)$$

$$I_{S'}^{12;34}, I_{S'}^{12;43}, I_{S'}^{32;41}: n_S'^{ij;kl} = x_{a,l}^2\left(x_{i,j}^2 x_{a,k}^2 - x_{j,k}^2 x_{a,i}^2 - x_{i,k}^2 x_{a,j}^2\right)$$

$$I_{S''}^{12;34}, I_{S''}^{23;14}, I_{S''}^{42;13}: n_S''^{ij;kl} = x_{i,j}^2 x_{a,k}^2 x_{a,l}^2$$

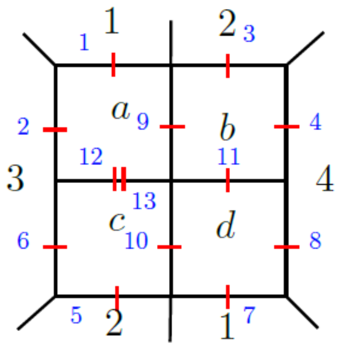


$$\frac{s}{\Delta^2}, \frac{t}{\Delta^2}, \frac{u}{\Delta^2}, \frac{s}{(s-t)\Delta}/\frac{t}{(s-t)\Delta}, \frac{s}{(s-u)\Delta}/\frac{u}{(s-u)\Delta}, \frac{t}{(t-u)\Delta}/\frac{u}{(t-u)\Delta}, \frac{1}{\Delta}$$

$\Delta: I_S^{12;34}$

$(s-t)\Delta: I_{S'}^{12,34}, \Delta(s-u): I_{S'}^{12,43}, \Delta(s-u): I_{S'}^{32,41}$

$\Delta^2: I_{S''}^{12;34}, I_{S''}^{32;14}, I_{S''}^{42;13}$



$$(w_1, y_1, w_2, y_2, w_3, y_3, w_4, y_4, x_1, z_4, x_2, z_3, x_3)$$

$$r_5(u<s<t): z_1<0 \wedge \frac{c_1^2(1-c_2)z_1}{c_2^2(1-c_1)} < x_4 < z_1 \wedge \frac{c_1 z_1}{c_1} < z_2 < \frac{c_2(1-c_1)x_4}{c_1(1-c_2)}$$

$$r_3(t<s<u): z_1>0 \wedge \frac{c_1 z_1}{c_2} < x_4 < \frac{(1-c_2)z_1}{1-c_1} \wedge x_4 < z_2$$

$$\wedge\, z_2 < \frac{z_1\big((-1+c_1)(1+c_1-c_2)c_2x_4 - c_1(-1+c_2)z_1\big)}{(-1+c_2)c_2x_4 + c_1(c_1-c_2-1)(c_2-1)z_1}.$$

$$\mathrm{p} := \left(z_2 - \frac{z_1\big((c_1-1)c_2(c_2+1)X^2 + (c_1-1)(c_2-1)(c_1-c_2)X - c_1(c_1+1)(c_2-1)\big)}{2c_1(c_2^2(X-1)+1) - 2c_2X}\right)^2$$

$$-\frac{z_1^2}{4(c_1c_2^2(X-1)+c_1-c_2X)^2}(-4X(c_1+c_1^2c_2(X-1)-c_2X)(c_1+c_1c_2^2(X-1)-c_2X)$$

$$+(-c_1(c_1+1)(c_2-1)+(c_1-c_2)(c_1-1)(c_2-1)X+(c_1-1)c_2(c_2+1)X^2)^2)$$

$$z_1>0 \wedge \left(x_4 - \frac{c_1z_1}{c_2}\right)^2 + \frac{1}{c_2}\left(\frac{c_1z_1}{c_2}\right)^2 > 0 \wedge \left[\left(\frac{c_2(c_1-1)x_4}{c_1(c_2-1)} < z_2 < \frac{c_1z_1}{c_2}\right)\right.$$

$$\left.\vee \left(z_1 < x_4 < \frac{c_1(c_2^2-1)z_1}{c_2(c_1c_2-1)} \wedge \frac{c_2(c_1-1)x_4}{c_1(c_2-1)} < z_2\right)\right]$$

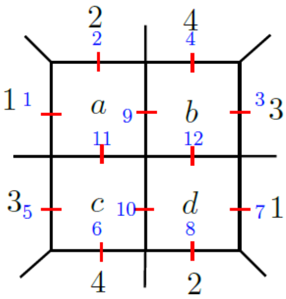
$$\vee\, z_1>0 \wedge x_4<0 \wedge z_2<x_4$$

$$z_1>0 \wedge x_4 > \frac{c_1(c_2^2-1)z_1}{c_2(c_1c_2-1)} \wedge \left[\left(\frac{c_2(c_1-1)x_4}{c_1(c_2-1)} < z_2\right) \vee \left(z_2 < \frac{c_1z_1}{c_2}\right)\right]$$

$$z_2 = \frac{z_1\big((c_1-1)c_2(c_2+1)X^2 + (c_1-1)(c_2-1)(c_1-c_2)X - c_1(c_1+1)(c_2-1)\big)}{2c_1(c_2^2(X-1)+1) - 2c_2X}$$

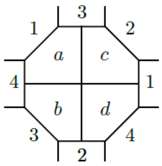
$$+\frac{z_1}{2(c_1c_2^2(X-1)+c_1-c_2X)}(-4X(c_1+c_1^2c_2(X-1)-c_2X)(c_1+c_1c_2^2(X-1)-c_2X)$$

$$+(-c_1(c_1+1)(c_2-1)+(c_1-c_2)(c_1-1)(c_2-1)X+(c_1-1)c_2(c_2+1)X^2)^2)^{\frac{1}{2}}$$



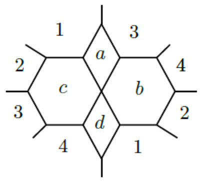
$$(w_1, y_1, w_2, y_2, w_3, y_3, w_4, y_4, x_1, x_4, x_2, z_3).$$

$I_{14} =$



$\times\ x_{1,2}^2 x_{1,3}^2 x_{2,4}^2 x_{3,4}^2 x_{a,d}^2 x_{b,c}^2\ ,$

$I_{30} =$



$\times\ x_{1,2}^2 x_{3,4}^2 x_{1,3}^2 x_{1,4}^2 x_{b,c}^4\ ,$

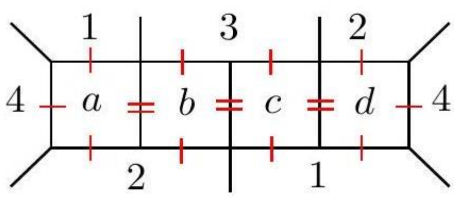
$$I'_{14} = I_{14}|_{3\leftrightarrow 4}\,, \quad I'_{30} = I_{30}|_{1\leftrightarrow 2}\,.$$

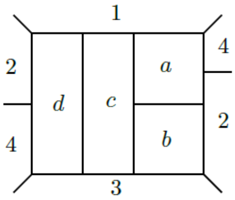
$$4 \times \frac{1}{stu} \frac{s}{\Delta\sqrt{(t - sX)(4uX^2 - sX + t)}} dX.$$

$$\frac{1}{2}\left(\sum_{\pm} \sum_{i=4,6} \omega_{r_i}^{\pm} \Delta^2\right) \frac{1}{\Delta} \frac{4}{\sqrt{(sX - t)(4uX^2 - sX + t)}} dX\bigg|_{\langle\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle\rangle^4 \to 1}$$

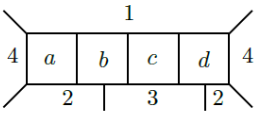
$$= \frac{1}{tu\Delta} \frac{4}{\sqrt{(sX - t)(4uX^2 - sX + t)}} dX.$$

$$I''_{14} = I_{14}|_{2\leftrightarrow 4}.$$



$I_7 =$ 

$\times\ x_{1,2}^2 x_{1,3}^2 x_{3,4}^2 x_{c,2}^2 x_{a,d}^2$

$I_{13} =$ 

$\times\ x_{1,2}^4 x_{3,4}^2 x_{1,3}^2 x_{b,d}^2$

$$I'_7 = I_7|_{1\leftrightarrow 2}\,, \quad I'_{13} = I_{13}|_{1\leftrightarrow 2}\,.$$

$$\frac{x_{1,2}^2 x_{1,3}^2 x_{3,4}^2 x_{c,2}^2 x_{a,d}^2}{x_{a,1}^2 x_{a,2}^2 x_{a,4}^2 x_{c,1}^2 x_{c,3}^2 x_{d,1}^2 x_{d,2}^2 x_{3,d}^2 x_{d,4}^2 x_{a,c}^2 x_{c,d}^2 \lambda_{2,3,a,c}},$$

$$\lambda_{i,j,k,l}^2 := \det|x_{a,b}^2|_{a,b=i,j,k,l}$$

$$\frac{x_{1,2}^2 x_{1,3}^2 x_{2,3}^2 x_{3,4}^2 x_{a,d}^2}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,4}^2 x_{d,1}^2 x_{d,2}^2 x_{d,3}^2 x_{d,4}^2 \lambda_{1,3,d,\{a2\}}}.$$

$$\frac{x_{1,2}^2 x_{3,4}^2 \left(x_{2,3}^2 x_{a,1}^2 - x_{1,2}^2 x_{a,3}^2\right)}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,4}^2 \lambda_{1,2,4,\{a3\}}},$$

$$I_7^{final} = -\frac{s}{t+u-s}\frac{1}{\Delta}.$$

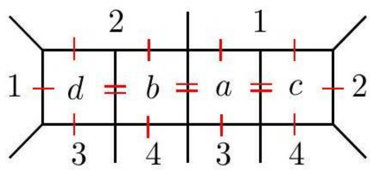
cut: $x_{a,1}^2, x_{a,2}^2, x_{a,4}^2, x_{a,b}^2, x_{c,1}^2, x_{c,3}^2, x_{c,d}^2, x_{b,c}^2,$

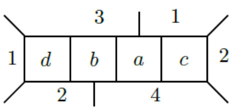
$$\frac{x_{1,2}^4 x_{3,4}^2}{x_{d,2}^2 x_{d,4}^2 x_{d,1}^2 \lambda_{2,3\{14\}\{d1\}}}.$$

$$I_{13}^{\text{final}} = \frac{s}{t+u-s}\frac{1}{\Delta}.$$

$$-I_7|_{(5.36)} = -I'_7|_{(5.36)} = I_{13}|_{(5.36)} = I'_{13}|_{(5.36)} = \frac{s}{t+u-s}\frac{1}{\Delta}.$$

$$I_7 + I'_7 + I_{13} + I'_{13} = 0.$$



$I_{10} =$ 

$\times\ x_{1,4}^2 x_{2,3}^2 x_{3,4}^2 x_{a,2}^2 x_{b,1}^2\,.$

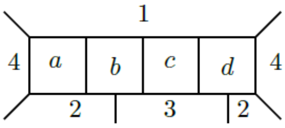
$$\pm\frac{s}{\Delta\sqrt{(s-t)^2 - 4tu}}$$

$$I_{10} + I'_{14} = 0$$

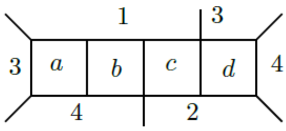
$$\Delta T^{12;34} + \Delta E_c^{12;34} + 11\text{ perms.} + \Delta L^{12;34} + 5\text{ perms.} + \Delta^2(gh^{12;34} + gh^{34;12})$$
$$+\Delta^2[H_c^{14;23} + H_c^{23;14} + H_c^{13;24} + H_c^{24;13}] + (t-3u)\Delta[E'^{12;34} + E'^{34;12}]$$
$$+ (s-3u-\Delta)\Delta[E'^{14;23} + E'^{23;14}] + (s-3t-\Delta)\Delta[E'^{13;24} + E'^{24;13}].$$

$$I_G^{12;34} := \int_{\ell_1,\ell_2} \frac{n_G^{12;34}}{x_{a,1}^2 x_{a,3}^2 x_{b,2}^2 x_{b,4}^2 x_{c,2}^2 x_{c,3}^2 x_{d,1}^2 x_{d,4}^2 x_{a,b}^2 x_{a,c}^2 x_{b,d}^2 x_{c,d}^2} + \text{perms}(a,b,c,d)$$

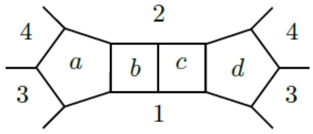
$$I_P^{12;34} = \int_{a,b,c,d} \frac{n_P^{12;34}}{x_{a,2}^2 x_{a,4}^2 x_{b,1}^2 x_{c,1}^2 x_{c,2}^2 x_{c,3}^2 x_{d,1}^2 x_{d,3}^2 x_{d,4}^2 x_{a,b}^2 x_{a,c}^2 x_{a,d}^2 x_{b,c}^2 x_{b,d}^2} + 3 \leftrightarrow 4$$
$$+\text{perms}(a,b,c,d)$$



$N^{12;34}$



$O^{12;34}$

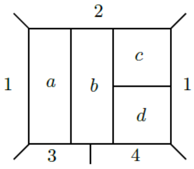


$K^{12;34}$

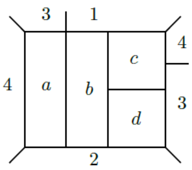
$$n_N^{12;34} = x_{1,2}^2(x_{b,d}^2 x_{1,3}^2 - x_{b,3}^2 x_{d,1}^2 - x_{b,1}^2 x_{d,3}^2),$$
$$n_N'^{12;34} = x_{1,2}^2 x_{1,3}^2 x_{1,4}^2 x_{b,3}^2 x_{d,2}^2.$$

$$n_O^{12;34} = x_{1,4}^2.$$

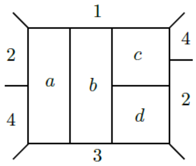
$$n_K^{12;34} = x_{1,2}^2(x_{1,2}^2 x_{a,d}^2 - x_{a,1}^2 x_{d,2}^2 - x_{a,2}^2 x_{d,1}^2),$$
$$n_K'^{12;34} = x_{1,2}^6 x_{a,3}^2 x_{d,4}^2,$$
$$n_K''^{12;34} = x_{1,2}^2 x_{a,1}^2 x_{d,2}^2.$$



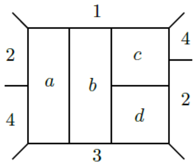
$U^{12;34}$



$\text{I}^{12;34}$



$Z^{12;34}$



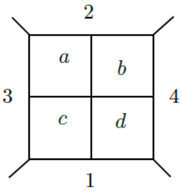
$Y^{12;34}$

$$n_U^{12;34} = x_{2,4}^2 x_{b,1}^2(x_{2,3}^2 x_{b,1}^2 - x_{1,3}^2 x_{b,2}^2)$$
$$n_U'^{12;34} = x_{b,1}^2 x_{b,2}^2$$

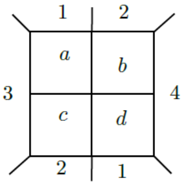
$$n_{\text{I}}^{12;34} = x_{2,3}^2 x_{b,c}^2 + x_{b,2}^2 x_{c,3}^2 - x_{b,3}^2 x_{c,2}^2.$$

$$n_Z^{12;34} = x_{1,3}^2 x_{b,2}^2 x_{a,c}^2 + x_{2,3}^2 x_{a,1}^2 x_{b,c}^2 - x_{1,3}^2 x_{a,2}^2 x_{b,c}^2 - x_{a,1}^2 x_{b,2}^2 x_{c,3}^2$$
$$+x_{a,1}^2 x_{b,3}^2 x_{c,2}^2 - x_{a,3}^2 x_{b,2}^2 x_{c,1}^2 + x_{1,2}^2 x_{a,3}^2 x_{b,c}^2$$

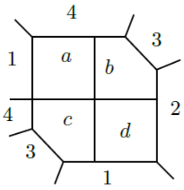
$$n_Y^{12;34} = x_{2,4}^2 x_{d,1}^2(x_{1,4}^2 x_{b,3}^2 - x_{1,3}^2 x_{b,4}^2)$$
$$n_Y'^{12;34} = x_{d,1}^2 x_{b,4}^2$$
$$n_Y''^{12;34} = x_{1,4}^4 x_{b,2}^2 x_{d,3}^2.$$



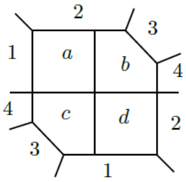
$F^{12;34}$



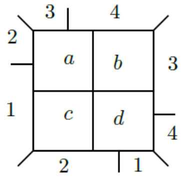
$G^{12;34}$



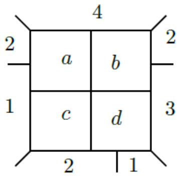
$R^{12;34}$



$\Xi^{12;34}$



$M^{12;34}$



$J^{12;34}$

$$n_F^{12;34} = 1.$$

$$n_G^{12;34} = 1.$$

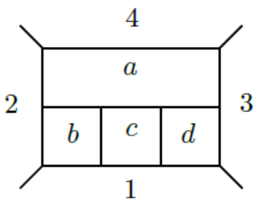
$$n_R^{12;34} = x_{b,c}^2 x_{1,4}^2 - x_{b,1}^2 x_{c,4}^2 - x_{b,4}^2 x_{c,1}^2$$
$$n_R'^{12;34} = x_{b,1}^2 x_{c,4}^2 x_{1,3}^2 x_{2,4}^2$$

$$n_\Xi^{12;34} = x_{1,2}^2 x_{a,b}^2 + x_{a,2}^2 x_{b,1}^2 - x_{a,1}^2 x_{b,2}^2$$

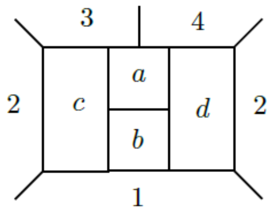
$$n_M^{12;34} = \left(x_{a,d}^2 x_{1,2}^2 - x_{a,1}^2 x_{d,2}^2 - x_{a,2}^2 x_{d,1}^2\right) \times \left(x_{a,d}^2 x_{3,4}^2 - x_{a,3}^2 x_{d,4}^2 - x_{a,4}^2 x_{d,3}^2\right)$$
$$n_M'^{12;34} = x_{a,3}^2 x_{d,4}^2 \left(x_{a,d}^2 x_{1,2}^2 - x_{a,1}^2 x_{d,2}^2 - x_{a,2}^2 x_{d,1}^2\right)$$
$$n_M''^{12;34} = x_{a,2}^2 x_{a,3}^2 x_{d,1}^2 x_{d,4}^2 \left(x_{1,2}^2 x_{3,4}^2 + x_{1,3}^2 x_{2,4}^2 - x_{1,4}^2 x_{2,3}^2\right)/2 - x_{1,2}^2 x_{1,3}^2 x_{3,4}^2 x_{a,2}^2 x_{d,4}^2 x_{a,d}^2.$$

$$n_J^{12;34} = x_{1,2}^2 x_{2,3}^2 x_{a,d}^2 \left(x_{b,1}^2 x_{2,4}^2 - x_{b,2}^2 x_{1,4}^2\right)$$
$$+ x_{1,2}^2 x_{2,3}^2 x_{a,b}^2 \left(x_{1,2}^2 x_{d,4}^2 - x_{1,4}^2 x_{d,2}^2 - x_{2,4}^2 x_{d,1}^2\right)$$

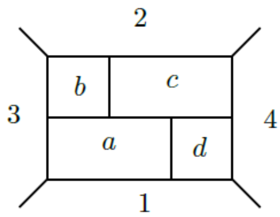
$$n_Q^{12;34} = \left(-x_{1,2}^2 x_{3,4}^2 - x_{1,4}^2 x_{2,3}^2 + x_{1,3}^2 x_{2,4}^2\right) x_{b,c}^2$$
$$+ x_{1,2}^2 x_{b,3}^2 x_{c,4}^2 + x_{3,4}^2 x_{b,1}^2 x_{c,2}^2$$



$V^{12;34}$



$P^{12;34}$

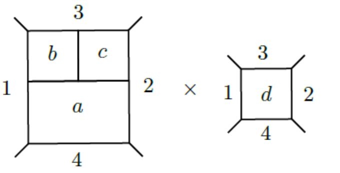


$Q^{12;34}$

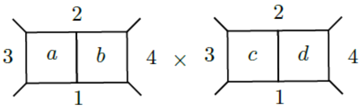
$$n_V^{12;34} = x_{2,3}^2 x_{a,1}^4.$$

$$n_P^{12;34} = x_{a,b}^2 x_{1,4}^2 \left(x_{1,3}^2 x_{c,2}^2 - x_{2,3}^2 x_{c,1}^2\right)/2 + x_{a,c}^2 x_{b,1}^2 \left(x_{1,4}^2 x_{2,3}^2 + x_{1,3}^2 x_{2,4}^2\right)/2$$
$$- x_{1,3}^2 x_{1,4}^2 x_{b,2}^2 x_{a,c}^2$$
$$n_P'^{12;34} = x_{a,c}^2 x_{d,1}^2 + x_{a,d}^2 x_{c,1}^2 - x_{c,d}^2 x_{a,1}^2$$
$$n_P''^{12;34} = x_{1,3}^2 x_{a,1}^2 x_{c,d}^2 x_{d,2}^2$$

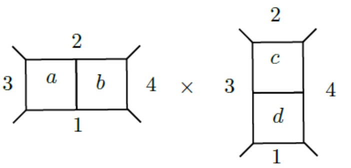
$$n_Q^{12;34} = \left(-x_{1,2}^2 x_{3,4}^2 - x_{1,4}^2 x_{2,3}^2 + x_{1,3}^2 x_{2,4}^2\right) x_{b,c}^2 + x_{1,2}^2 x_{b,3}^2 x_{c,4}^2 + x_{3,4}^2 x_{b,1}^2 x_{c,2}^2$$



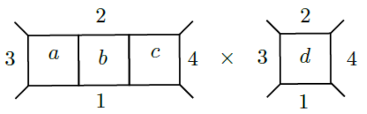
$gT^{12;34}$



$hh^{12;34}$



$hh'^{12;34}$



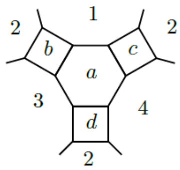
$gL^{12;34}$

$$n_{gT}^{12;34} = x_{1,2}^2 x_{a,3}^2 - x_{1,3}^2$$

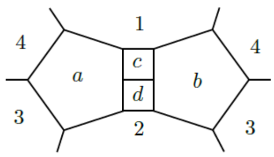
$$n_{hh}^{12;34} = x_{1,2}^4.$$

$$n_{hh'}^{12;34} = x_{1,2}^2 x_{3,4}^2.$$

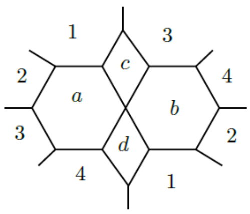
$$n_{L\times g}^{12;34} = x_{1,2}^4.$$



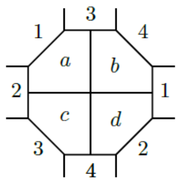
$S^{12;34}$



$X^{12;34}$



$W^{12;34}$



$\Theta^{12;34}$

$$n_S^{12;34} = \left(x_{1,3}^2 x_{1,4}^2 x_{3,4}^2 x_{a,2}^4 + 3 x_{14}^2 x_{2,3}^2 x_{a,1}^2 \left(x_{2,4}^2 x_{a,3}^2 - x_{3,4}^2 x_{a,2}^2\right)\right)/6$$
$$n_{S_2}'^{12;34} = x_{a,4}^2 \left(x_{1,2}^2 x_{a,3}^2 - x_{2,3}^2 x_{a,1}^2 - x_{1,3}^2 x_{a,2}^2\right)$$
$$n_S''^{12;34} = x_{1,2}^2 x_{a,3}^2 x_{a,4}^2$$

$$n_X^{12;34} = x_{a,b}^2 \left(x_{a,b}^2 x_{1,2}^2 - x_{a,1}^2 x_{b,2}^2 - x_{a,2}^2 x_{b,1}^2\right)$$
$$n_X'^{12;34} = x_{a,1}^2 x_{b,2}^2 x_{a,b}^2$$
$$n_X''^{12;34} = x_{1,2}^2 x_{a,3}^2 x_{b,4}^2 \left(x_{a,b}^2 x_{1,2}^2 - x_{a,1}^2 x_{b,2}^2 - x_{a,2}^2 x_{b,1}^2\right).$$

$$n_W^{12;34} = \left(x_{a,b}^2 x_{1,3}^2 - x_{a,1}^2 x_{b,3}^2 - x_{a,3}^2 x_{b,1}^2\right) \times \left(x_{a,b}^2 x_{1,4}^2 - x_{a,1}^2 x_{b,4}^2 - x_{a,4}^2 x_{b,1}^2\right).$$

$$
\begin{aligned}
n_{\Theta}^{12;34} =& \frac{1}{16}\left(-x_{a,4}^2x_{b,2}^2x_{c,1}^2x_{d,3}^2 + x_{a,3}^2x_{b,1}^2x_{c,2}^2x_{d,4}^2 - x_{1,2}^2x_{3,4}^2x_{a,d}^2x_{b,c}^2\right) \\
&+\frac{1}{8}\left(-x_{a,2}^2x_{b,4}^2x_{c,3}^2x_{d,1}^2 - x_{a,1}^2x_{b,4}^2x_{c,3}^2x_{d,2}^2 + x_{1,2}^2x_{a,4}^2x_{d,3}^2x_{b,c}^2 + x_{1,4}^2x_{2,3}^2x_{a,d}^2x_{b,c}^2\right. \\
&\quad \left.-x_{1,4}^2x_{2,3}^2x_{a,b}^2x_{c,d}^2 - x_{1,3}^2x_{2,4}^2x_{a,b}^2x_{c,d}^2 + x_{1,2}^2x_{3,4}^2x_{a,b}^2x_{c,d}^2\right) \\
&+\frac{1}{4}\left(x_{a,2}^2x_{b,4}^2x_{c,1}^2x_{d,3}^2 + x_{a,1}^2x_{b,3}^2x_{c,2}^2x_{d,4}^2 - x_{2,4}^2x_{c,3}^2x_{d,1}^2x_{a,b}^2 + x_{2,4}^2x_{c,1}^2x_{d,3}^2x_{a,b}^2\right. \\
&\quad \left.+x_{1,3}^2x_{c,4}^2x_{d,2}^2x_{a,b}^2 - x_{1,3}^2x_{c,2}^2x_{d,4}^2x_{a,b}^2 + x_{3,4}^2x_{a,2}^2x_{d,1}^2x_{b,c}^2\right) \\
&+\frac{1}{2}\left(x_{a,2}^2x_{b,3}^2x_{c,1}^2x_{d,4}^2 + x_{2,3}^2x_{c,4}^2x_{d,1}^2x_{a,b}^2 - x_{1,2}^2x_{c,4}^2x_{d,3}^2x_{a,b}^2 + x_{2,3}^2x_{c,1}^2x_{d,4}^2x_{a,b}^2\right. \\
&\quad \left.-x_{1,2}^2x_{c,3}^2x_{d,4}^2x_{a,b}^2 - x_{1,4}^2x_{a,2}^2x_{d,3}^2x_{b,c}^2 - x_{1,3}^2x_{a,2}^2x_{d,4}^2x_{b,c}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&n_{\Theta}'^{12;34} = \frac{1}{8}\Big(-x_{1,4}^2x_{2,3}^2x_{a,4}^2x_{b,2}^2x_{c,1}^2x_{d,3}^2 - x_{1,4}^2x_{2,3}^2x_{a,2}^2x_{b,4}^2x_{c,3}^2x_{d,1}^2 - x_{1,3}^2x_{2,4}^2x_{a,2}^2x_{b,4}^2x_{c,3}^2x_{d,1}^2 \\
&-x_{1,4}^2x_{2,3}^2x_{a,3}^2x_{b,1}^2x_{c,2}^2x_{d,4}^2 + x_{1,4}^4x_{2,3}^4x_{a,d}^2x_{b,c}^2 - x_{1,4}^4x_{2,3}^4x_{a,c}^2x_{b,d}^2 - x_{1,3}^4x_{2,4}^4x_{a,c}^2x_{b,d}^2 \\
&-x_{1,4}^2x_{2,3}^2x_{1,3}^2x_{2,4}^2x_{a,d}^2x_{b,c}^2 - x_{1,2}^2x_{3,4}^2x_{1,4}^2x_{2,3}^2x_{a,d}^2x_{b,c}^2 + x_{1,2}^2x_{3,4}^2x_{1,4}^2x_{2,3}^2x_{a,c}^2x_{b,d}^2 \\
&+x_{1,2}^2x_{3,4}^2x_{1,3}^2x_{2,4}^2x_{a,c}^2x_{b,d}^2\Big) + \frac{5}{4}x_{1,2}^2x_{3,4}^2x_{1,4}^2x_{2,3}^2x_{b,4}^2x_{c,3}^2x_{a,d}^2 + \frac{3}{4}\Big(-x_{1,4}^2x_{2,3}^2x_{a,3}^2 \\
&x_{b,4}^2x_{c,2}^2x_{d,1}^2 - x_{1,4}^2x_{2,3}^2x_{a,1}^2x_{b,4}^2x_{c,3}^2x_{d,2}^2 + x_{1,4}^2x_{2,3}^2x_{2,4}^2x_{c,3}^2x_{d,1}^2x_{a,b}^2 - x_{1,4}^2x_{2,3}^2x_{2,4}^2x_{c,1}^2 \\
&x_{d,3}^2x_{a,b}^2\Big) + \frac{1}{4}\Big(x_{1,4}^2x_{2,3}^2x_{a,4}^2x_{b,3}^2x_{c,1}^2x_{d,2}^2 + x_{1,3}^2x_{2,4}^2x_{a,4}^2x_{b,3}^2x_{c,1}^2x_{d,2}^2 + x_{1,4}^2x_{2,3}^2x_{a,4}^2x_{b,1}^2 \\
&x_{c,2}^2x_{d,3}^2 + x_{1,3}^2x_{2,4}^2x_{a,3}^2x_{b,4}^2x_{c,2}^2x_{d,1}^2 - x_{1,2}^2x_{3,4}^2x_{a,1}^2x_{b,4}^2x_{c,3}^2x_{d,2}^2 - x_{1,3}^2x_{2,4}^4x_{c,3}^2x_{d,1}^2x_{a,b}^2 \\
&+x_{1,3}^2x_{2,4}^4x_{c,1}^2x_{d,3}^2x_{a,b}^2 + x_{1,3}^2x_{1,4}^2x_{2,3}^2x_{c,4}^2x_{d,2}^2x_{a,b}^2 + x_{1,3}^4x_{2,4}^2x_{c,4}^2x_{d,2}^2x_{a,b}^2 - x_{1,3}^2x_{1,4}^2x_{2,3}^2 \\
&x_{c,2}^2x_{d,4}^2x_{a,b}^2 - x_{1,3}^4x_{2,4}^2x_{c,2}^2x_{d,4}^2x_{a,b}^2 + x_{1,4}^2x_{2,3}^2x_{3,4}^2x_{b,2}^2x_{c,1}^2x_{a,d}^2 - x_{1,4}^2x_{2,3}^2x_{3,4}^2x_{b,1}^2x_{c,2}^2 \\
&x_{a,d}^2 + x_{1,2}^2x_{1,4}^2x_{2,3}^2x_{b,4}^2x_{c,3}^2x_{a,d}^2 + x_{1,4}^2x_{2,3}^2x_{1,3}^2x_{2,4}^2x_{a,c}^2x_{b,d}^2\Big) + \frac{3}{2}\Big(-x_{1,3}^2x_{2,4}^2x_{a,4}^2x_{b,3}^2x_{c,2}^2 \\
&x_{d,1}^2 + x_{1,3}^2x_{1,4}^2x_{2,3}^2x_{b,3}^2x_{c,2}^2x_{a,d}^2\Big) + \frac{1}{2}\Big(x_{1,4}^2x_{2,3}^2x_{a,4}^2x_{b,3}^2x_{c,2}^2x_{d,1}^2 - x_{1,4}^2x_{2,3}^2x_{3,4}^2x_{c,2}^2x_{d,1}^2 \\
&x_{a,b}^2 - x_{1,3}^2x_{2,4}^2x_{3,4}^2x_{c,2}^2x_{d,1}^2x_{a,b}^2 - x_{1,3}^2x_{2,4}^2x_{3,4}^2x_{c,2}^2x_{d,1}^2x_{a,b}^2 - x_{1,4}^2x_{2,3}^2x_{3,4}^2x_{c,1}^2x_{d,2}^2x_{a,b}^2 \\
&-x_{1,3}^2x_{2,4}^2x_{3,4}^2x_{c,1}^2x_{d,2}^2x_{a,b}^2 + x_{1,4}^4x_{2,3}^2x_{c,3}^2x_{d,2}^2x_{a,b}^2 + x_{1,3}^2x_{2,4}^2x_{1,4}^2x_{c,3}^2x_{d,2}^2x_{a,b}^2 + x_{1,4}^4x_{2,3}^2 \\
&x_{c,2}^2x_{d,3}^2x_{a,b}^2 - x_{1,3}^2x_{2,4}^2x_{1,4}^2x_{c,2}^2x_{d,3}^2x_{a,b}^2 - x_{1,4}^2x_{2,3}^2x_{2,4}^2x_{b,3}^2x_{c,1}^2x_{a,d}^2 - x_{1,3}^2x_{2,4}^4x_{b,3}^2x_{c,1}^2x_{a,d}^2 \\
&-x_{1,3}^2x_{2,4}^4x_{b,3}^2x_{c,1}^2x_{a,d}^2 - x_{1,4}^4x_{2,3}^2x_{b,3}^2x_{c,2}^2x_{a,d}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
n_{\Theta}^{12;34} =& \frac{1}{16}(-x_{a,4}^2x_{b,2}^2x_{c,1}^2x_{d,3}^2+x_{a,3}^2x_{b,1}^2x_{c,2}^2x_{d,4}^2-x_{1,2}^2x_{3,4}^2x_{a,d}^2x_{b,c}^2) \\
&+\frac{1}{8}(-x_{a,2}^2x_{b,4}^2x_{c,3}^2x_{d,1}^2-x_{a,1}^2x_{b,4}^2x_{c,3}^2x_{d,2}^2+x_{1,2}^2x_{a,4}^2x_{d,3}^2x_{b,c}^2+x_{1,4}^2x_{2,3}^2x_{a,d}^2x_{b,c}^2 \\
&\qquad -x_{1,4}^2x_{2,3}^2x_{a,b}^2x_{c,d}^2-x_{1,3}^2x_{2,4}^2x_{a,b}^2x_{c,d}^2+x_{1,2}^2x_{3,4}^2x_{a,b}^2x_{c,d}^2) \\
&+\frac{1}{4}(x_{a,2}^2x_{b,4}^2x_{c,1}^2x_{d,3}^2+x_{a,1}^2x_{b,3}^2x_{c,2}^2x_{d,4}^2-x_{2,4}^2x_{c,3}^2x_{d,1}^2x_{a,b}^2+x_{2,4}^2x_{c,1}^2x_{d,3}^2x_{a,b}^2 \\
&\qquad +x_{1,3}^2x_{c,4}^2x_{d,2}^2x_{a,b}^2-x_{1,3}^2x_{c,2}^2x_{d,4}^2x_{a,b}^2+x_{3,4}^2x_{a,2}^2x_{d,1}^2x_{b,c}^2) \\
&+\frac{1}{2}(x_{a,2}^2x_{b,3}^2x_{c,1}^2x_{d,4}^2+x_{2,3}^2x_{c,4}^2x_{d,1}^2x_{a,b}^2-x_{1,2}^2x_{c,4}^2x_{d,3}^2x_{a,b}^2+x_{2,3}^2x_{c,1}^2x_{d,4}^2x_{a,b}^2 \\
&\qquad -x_{1,2}^2x_{c,3}^2x_{d,4}^2x_{a,b}^2-x_{1,4}^2x_{a,2}^2x_{d,3}^2x_{b,c}^2-x_{1,3}^2x_{a,2}^2x_{d,4}^2x_{b,c}^2),
\end{aligned}
$$

$$
\begin{aligned}
n_{\Theta}^{\prime\,12;34} = \frac{1}{8}(&-x_{1,4}^2x_{2,3}^2x_{a,4}^2x_{b,2}^2x_{c,1}^2x_{d,3}^2-x_{1,4}^2x_{2,3}^2x_{a,2}^2x_{b,4}^2x_{c,3}^2x_{d,1}^2-x_{1,3}^2x_{2,4}^2x_{a,2}^2x_{b,4}^2x_{c,3}^2x_{d,1}^2\\
&-x_{1,4}^2x_{2,3}^2x_{a,3}^2x_{b,1}^2x_{c,2}^2x_{d,4}^2+x_{1,4}^4x_{2,3}^4x_{a,d}^2x_{b,c}^2-x_{1,4}^4x_{2,3}^4x_{a,c}^2x_{b,d}^2-x_{1,3}^4x_{2,4}^4x_{a,c}^2x_{b,d}^2\\
&-x_{1,4}^2x_{2,3}^2x_{1,3}^2x_{2,4}^2x_{a,d}^2x_{b,c}^2-x_{1,2}^2x_{3,4}^2x_{1,4}^2x_{2,3}^2x_{a,d}^2x_{b,c}^2+x_{1,2}^2x_{3,4}^2x_{1,4}^2x_{2,3}^2x_{a,c}^2x_{b,d}^2\\
&+x_{1,2}^2x_{3,4}^2x_{1,3}^2x_{2,4}^2x_{a,c}^2x_{b,d}^2)+\frac{5}{4}x_{1,2}^2x_{3,4}^2x_{1,4}^2x_{2,3}^2x_{b,4}^2x_{c,3}^2x_{a,d}^2+\frac{3}{4}(-x_{1,4}^2x_{2,3}^2x_{a,3}^2\\
&x_{b,4}^2x_{c,2}^2x_{d,1}^2-x_{1,4}^2x_{2,3}^2x_{a,1}^2x_{b,4}^2x_{c,3}^2x_{d,2}^2+x_{1,4}^2x_{2,3}^2x_{2,4}^2x_{c,3}^2x_{d,1}^2x_{a,b}^2-x_{1,4}^2x_{2,3}^2x_{2,4}^2x_{c,1}^2\\
&x_{d,3}^2x_{a,b}^2)+\frac{1}{4}(x_{1,4}^2x_{2,3}^2x_{a,4}^2x_{b,3}^2x_{c,1}^2x_{d,2}^2+x_{1,3}^2x_{2,4}^2x_{a,4}^2x_{b,3}^2x_{c,1}^2x_{d,2}^2+x_{1,4}^2x_{2,3}^2x_{a,4}^2x_{b,1}^2\\
&x_{c,2}^2x_{d,3}^2+x_{1,3}^2x_{2,4}^2x_{a,3}^2x_{b,4}^2x_{c,2}^2x_{d,1}^2-x_{1,2}^2x_{3,4}^2x_{a,1}^2x_{b,4}^2x_{c,3}^2x_{d,2}^2-x_{1,3}^2x_{2,4}^4x_{c,3}^2x_{d,1}^2x_{a,b}^2\\
&+x_{1,3}^2x_{2,4}^4x_{c,1}^2x_{d,3}^2x_{a,b}^2+x_{1,3}^2x_{1,4}^2x_{2,3}^2x_{c,4}^2x_{d,2}^2x_{a,b}^2+x_{1,3}^4x_{2,4}^2x_{c,4}^2x_{d,2}^2x_{a,b}^2-x_{1,3}^2x_{1,4}^2x_{2,3}^2\\
&x_{c,2}^2x_{d,4}^2x_{a,b}^2-x_{1,3}^4x_{2,4}^2x_{c,2}^2x_{d,4}^2x_{a,b}^2+x_{1,4}^2x_{2,3}^2x_{3,4}^2x_{b,2}^2x_{c,1}^2x_{a,d}^2-x_{1,4}^2x_{2,3}^2x_{3,4}^2x_{b,1}^2x_{c,2}^2\\
&x_{a,d}^2+x_{1,2}^2x_{1,4}^2x_{2,3}^2x_{b,4}^2x_{c,3}^2x_{a,d}^2+x_{1,4}^2x_{2,3}^2x_{1,3}^2x_{2,4}^2x_{a,c}^2x_{b,d}^2)+\frac{3}{2}(-x_{1,3}^2x_{2,4}^2x_{a,4}^2x_{b,3}^2x_{c,2}^2\\
&x_{d,1}^2+x_{1,3}^2x_{1,4}^2x_{2,3}^2x_{b,3}^2x_{c,2}^2x_{a,d}^2)+\frac{1}{2}(x_{1,4}^2x_{2,3}^2x_{a,4}^2x_{b,3}^2x_{c,2}^2x_{d,1}^2-x_{1,4}^2x_{2,3}^2x_{3,4}^2x_{c,2}^2x_{d,1}^2\\
&x_{a,b}^2-x_{1,3}^2x_{2,4}^2x_{3,4}^2x_{c,2}^2x_{d,1}^2x_{a,b}^2-x_{1,3}^2x_{2,4}^2x_{3,4}^2x_{c,2}^2x_{d,1}^2x_{a,b}^2-x_{1,4}^2x_{2,3}^2x_{3,4}^2x_{c,1}^2x_{d,2}^2x_{a,b}^2\\
&-x_{1,3}^2x_{2,4}^2x_{3,4}^2x_{c,1}^2x_{d,2}^2x_{a,b}^2+x_{1,4}^4x_{2,3}^2x_{c,3}^2x_{d,2}^2x_{a,b}^2+x_{1,3}^2x_{2,4}^2x_{1,4}^2x_{c,3}^2x_{d,2}^2x_{a,b}^2+x_{1,4}^4x_{2,3}^2\\
&x_{c,2}^2x_{d,3}^2x_{a,b}^2-x_{1,3}^2x_{2,4}^2x_{1,4}^2x_{c,2}^2x_{d,3}^2x_{a,b}^2-x_{1,4}^2x_{2,3}^2x_{2,4}^2x_{b,3}^2x_{c,1}^2x_{a,d}^2-x_{1,3}^2x_{2,4}^4x_{b,3}^2x_{c,1}^2x_{a,d}^2\\
&-x_{1,3}^2x_{2,4}^4x_{b,3}^2x_{c,1}^2x_{a,d}^2-x_{1,4}^4x_{2,3}^2x_{b,3}^2x_{c,2}^2x_{a,d}^2)
\end{aligned}
$$

$$z\bar{z} \equiv U, (1-z)(1-\bar{z}) \equiv V$$

$$E_{\mathrm{c}}^{12;34} := \frac{x_{b,3}^2 x_{1,4}^2 x_{2,4}^2}{x_{a,1}^2 x_{a,3}^2 x_{a,4}^2 x_{a,b}^2 x_{b,1}^2 x_{b,2}^2 x_{b,4}^2 x_{b,c}^2 x_{c,2}^2 x_{c,3}^2 x_{c,4}^2}$$

$$H_{\mathrm{c}}^{12;34} := \frac{x_{a,c}^2 x_{1,2}^2}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,4}^2 x_{a,b}^2 x_{b,1}^2 x_{b,2}^2 x_{b,c}^2 x_{c,1}^2 x_{c,2}^2 x_{c,3}^2 x_{c,4}^2}$$

$$(X,Y) \mapsto \left(1 + \frac{1}{X}, \frac{Y}{X^2}\right)$$

$$X \to X + \frac{c_1^2(c_2-1)^2 - 2c_1(c_2+1)^2 + c_2(c_2+10) + 1}{12(c_1-c_2)}$$

$$(X,Y) \to (a^2X, a^3Y)$$

$$a = \frac{\sqrt{-(c_1-1)(c_2-1)}}{(c_1-c_2)^2}$$

$$X \to X - \frac{(c_1-1)(c_2-1)^2 + 4c_1c_2}{12(c_1-1)(c_2-1)}$$

$$\langle f|a_n|g\rangle = \int f(\Phi(1))g(\Phi(0))\exp(-S[\Phi])d\Phi$$

$$\sum_{p_k,p_{k+n}} \frac{f(p_{k+n})g(p_k)}{\Pi TM_{k,k+n}(p_k,p_{k+n})}$$

$$M_{k,k+n} = \{I_{k+n} \subset I_k \ \mathrm{supp}(I_k/I_{k+n}) \subset (0,0) \in \mathbb{C}^2 \ I_{k+n} \in M_{k+n} \ I_k \in M_k\}$$

$$[a_n, a_m] = (-1)^n n \delta_{n+m} \epsilon_1 \epsilon_2$$

$$[a_{-1}^{6d}, a_1^{6d}] = \sinh(\epsilon_1)\sinh(\epsilon_2)$$

$$Z_{\text{inst}}^{U(1)}(\epsilon_1,\epsilon_2,\Lambda)=\langle\Lambda\mid\Lambda\rangle$$

$$a_{-1}|\Lambda\rangle=\Lambda^{1/2}|\Lambda\rangle\ a_n|\Lambda\rangle=0,\forall n\leq -2$$

$$\begin{aligned}a_{-1}|\Lambda,m_1\rangle &= m_1\Lambda^{1/2}|\Lambda,m_1\rangle\\ a_{-1}|\Lambda,m_1,m_2\rangle &= \Lambda^{1/2}(m_1m_2-\epsilon_1\epsilon_2K)|\Lambda,m_1,m_2\rangle\end{aligned}$$

$$\langle f\mid g\rangle=\sum_{p\in\text{ fixed points}}\frac{f(p)g(p)}{\Pi TM(p)}\propto\int_M f\wedge g$$

$$\langle f|M_{12}|g\rangle=\sum_{(p_1,p_2)\in\text{ fixed points of }M_{12}}\frac{f(p_2)g(p_1)}{\prod\ TM_{12}(p_1,p_2)}$$

$$\begin{aligned}\pi_1&:M_{12}\to M_1\quad (p_1,p_2)\mapsto p_1\in M_1\\ \pi_2&:M_{12}\to M_2\quad (p_1,p_2)\mapsto p_2\in M_2\end{aligned}$$

$$\langle f|M_{23}\circ M_{12}|g\rangle=\sum_{(p_1,p_2,p_3)\in\text{ fixed points of }M_{123}}\frac{f(p_3)g(p_1)}{\prod\ TM_{123}(p_1,p_2,p_3)_{\text{vir}}}$$

$$\frac{TM_1\oplus TM_2\oplus TM_3}{T(M_{12}\times M_3)+T(M_1\times M_{23})}$$

$$M_{12}=\sqrt{KM_2^{-1}}\otimes(\pi_2)_*\left(\sqrt{KM_{12}}\otimes\pi_1^*\right)$$

$$\frac{1}{2}\partial_0\phi\partial_0\phi+\psi\partial_0\bar{\psi}-\frac{1}{4}R(\bar{\psi},\bar{\psi},\psi,\psi)-\frac{1}{2}FF$$

$$\begin{aligned}S[\phi,\psi,\bar{\psi},F]=&\frac{1}{2}\partial_0\phi\partial_0\phi+\psi\partial_0\bar{\psi}-\frac{1}{4}R(\bar{\psi},\bar{\psi},\psi,\psi)-\frac{1}{2}FF\\ &+\frac{1}{2}|V|^2-\frac{1}{2}\nabla_iV_j\psi^i\psi^j+\frac{1}{2}\nabla_iV_j\bar{\psi}^i\bar{\psi}^j\end{aligned}$$

$$\begin{aligned}\delta\phi=&\bar{\epsilon}\psi+\epsilon\bar{\psi}\\ \delta F=&\frac{1}{2}\epsilon R(\bar{\psi},\bar{\psi})\psi-\frac{1}{2}\bar{\epsilon}R(\psi,\psi)\bar{\psi}+\epsilon\partial_0\bar{\psi}-\bar{\epsilon}\partial_0\psi\\ &+\epsilon\psi^i\nabla_iV^j\partial_j-\bar{\epsilon}\bar{\psi}^i\nabla_iV^j\partial_j\\ \delta\psi=&-\epsilon\partial_0\phi+\bar{\epsilon}V+\epsilon F\\ \delta\bar{\psi}=&-\bar{\epsilon}\partial_0\phi+\epsilon V-\bar{\epsilon}F\end{aligned}$$

$$\delta_\eta\delta_\epsilon=\epsilon\eta\mathcal{L}_V\ \delta_{\bar{\eta}}\delta_{\bar{\epsilon}}=\bar{\epsilon}\bar{\eta}\mathcal{L}_V\ [\delta_\epsilon,\delta_{\bar{\epsilon}}]=2\epsilon\bar{\epsilon}\partial_0$$

$$\mathcal{L}_V\phi=V(\phi)\ \mathcal{L}_VF=F^i\nabla_iV\ \mathcal{L}_V\psi=\psi^i\nabla_iV\ \mathcal{L}_V\bar{\psi}=\bar{\psi}^i\nabla_iV$$

$$\frac{1}{2}\bar{\psi}V-\frac{1}{2}\psi(\partial_0\phi+F)$$

$$\langle f\mid g\rangle:=\int\ d\phi dFd\psi d\bar{\psi}\exp\left(-S[\phi,\psi,\bar{\psi},F]\right)f(\phi(1),\bar{\psi}(1))g(\phi(0),\bar{\psi}(0))$$

$$\phi:[0,1]\to M\ \psi,\bar{\psi},F\in\phi^*TM$$

$$\partial_0\phi=\partial_0\bar{\psi}=F=\psi=0\text{ at }x^0=0,1$$

$$\phi\equiv p\ \psi,\bar{\psi},F=0\ p\in\text{ fixed points of }V$$

$$\frac{1}{2}(\bar{\psi},\psi)\begin{pmatrix}\nabla V & \partial_0\\ \partial_0 & -\nabla V\end{pmatrix}\begin{pmatrix}\bar{\psi}\\ \psi\end{pmatrix}$$

$$-\frac{1}{2}\delta\phi^i\left(g_{ik}\partial_0\partial_0+\nabla_iV_j\nabla^jV_k\right)\delta\phi^k$$

$$\nabla_i V_j \delta\phi^j = \lambda_V \delta\phi_i \; \partial_0 \delta\phi^i = \lambda_0 \delta\phi^i$$

$$\begin{pmatrix} \bar{\psi} \\ \psi \end{pmatrix} \in \left\langle \begin{pmatrix} \delta\phi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_0 \delta\phi \end{pmatrix} \right\rangle$$

$$\frac{1}{2}\begin{pmatrix} \lambda_V & \lambda_0^2 \\ 1 & -\lambda_V \end{pmatrix}$$

$$\text{one-loop determinant } = \frac{\mathrm{pf}(\nabla V)}{\det(\nabla V)} = \frac{1}{\mathrm{pf}(\nabla V)}$$

$$\frac{1}{\mathrm{pf}(\nabla V)} = \frac{1}{\prod \; TM(p)}$$

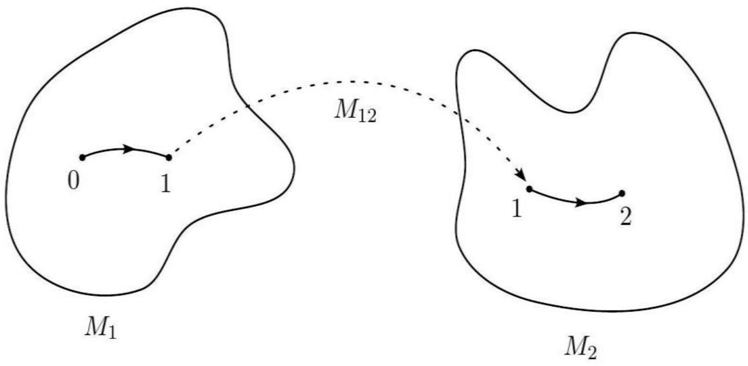
$$\langle f \mid g \rangle = \sum_{p \in \text{ fixed points}} \frac{f(p)g(p)}{\Pi TM(p)} \propto \int_M f \wedge g$$

$$\langle f \mid g \rangle = \frac{f(0{,}0)g(0{,}0)}{\epsilon_1 \epsilon_2} = \frac{1}{\epsilon_1 \epsilon_2}$$

$$\phi_1 : [0{,}1] \to M_1 \; \psi_1, \bar{\psi}_1, F_1 \in \phi_1^* TM_1 i \phi_2 : [1{,}2] \to M_2 \; \psi_2, \bar{\psi}_2, F_2 \in \phi_2^* TM_2$$

$$(\phi_1(1), \phi_2(1)) \in M_{12}$$
$$(\bar{\psi}_1(1), \bar{\psi}_2(1)) \in TM_{12}$$
$$\partial_0 \phi_1(0) = \partial_0 \bar{\psi}_1(0) = F_1(0) = \psi_1(0) = 0$$
$$\partial_0 \phi_2(2) = \partial_0 \bar{\psi}_2(2) = F_2(2) = \psi_2(2) = 0$$

$$\langle f | M_{12} | g \rangle := \int \; d\phi dF d\psi d\bar{\psi} \exp\left(-S[\phi, \psi, \bar{\psi}, F]\right) f(\phi(2), \bar{\psi}(2)) g(\phi(0), \bar{\psi}(0))$$



$$\phi = (\phi_1, \phi_2) \; \psi = (\psi_1, \psi_2) \; \bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) \; F = (F_1, F_2)$$

$$\phi_1 \equiv p_1 \; \phi_2 \equiv p_2 \; (p_1, p_2) \in M_{12} \text{ is a fixed point}$$

$$\sum_{(p_1, p_2) \in \text{ fixed points of } M_{12}} \frac{1}{\prod \; TM_{12}(p_1, p_2)}$$

$$\langle f | M_{12} | g \rangle = \sum_{(p_1, p_2) \in \text{ fixed points of } M_{12}} \frac{f(p_2) g(p_1)}{\prod \; TM_{12}(p_1, p_2)}$$

$$\pi_1 : M_{12} \to M_1 \; (p_1, p_2) \mapsto p_1$$
$$\pi_2 : M_{12} \to M_2 \; (p_1, p_2) \mapsto p_2$$

$$\pi_1^* | p_1 \rangle = \sum_{(p_1, q) \in \text{ fixed points of } M_{12}} | p_1, q \rangle$$

$$(\pi_2)_* | p_1, q \rangle = \frac{\prod \; TM_2(q)}{\prod \; TM_{12}(p_1, q)} | q \rangle$$

$$M_{12} = \{I_2 \subset I_1, \mathrm{supp}(I_1/I_2) \subset (0{,}0) \in \mathbb{C}^2 \; I_2 \in M_2, I_1 \in M_1\}$$

$$\left(|\square\rangle\,,|\square\square\rangle\right), \quad \left(|\square\rangle\,,|\begin{smallmatrix}\square\\\square\end{smallmatrix}\rangle\right)$$

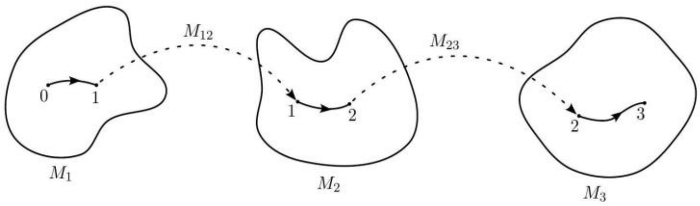
$$\square = \langle z_1, z_2\rangle \quad \square\square = \langle z_1^2, z_2\rangle \quad \begin{smallmatrix}\square\\\square\end{smallmatrix} = \langle z_1, z_2^2\rangle$$

$$T(|\square\rangle\,,|\square\square\rangle) = q_2/q_1 + q_1 \quad T(|\square\rangle\,,|\begin{smallmatrix}\square\\\square\end{smallmatrix}\rangle) = q_1/q_2 + q_2.$$

$$\langle\square\square|\, M_{12}\, |\square\rangle = \frac{1}{(\epsilon_2 - \epsilon_1)\epsilon_1} \quad \langle\begin{smallmatrix}\square\\\square\end{smallmatrix}|\, M_{12}\, |\square\rangle = \frac{1}{(\epsilon_1 - \epsilon_2)\epsilon_2}.$$

$$\phi_1\colon [0{,}1] \to M_1 \; \psi_1, \bar{\psi}_1, F_1 \in \phi_1^* TM_1$$
$$\phi_2\colon [1{,}2] \to M_2 \; \psi_2, \bar{\psi}_2, F_2 \in \phi_2^* TM_2$$
$$\phi_3\colon [2{,}3] \to M_3 \; \psi_3, \bar{\psi}_3, F_3 \in \phi_3^* TM_3$$

$$(\phi_1(1), \phi_2(1)) \in M_{12} \quad (\bar{\psi}_1(1), \bar{\psi}_2(1)) \in TM_{12}$$
$$(\phi_2(2), \phi_3(2)) \in M_{23} \quad (\bar{\psi}_2(2), \bar{\psi}_3(2)) \in TM_{23}$$
$$\partial_0\phi_1(0) = \partial_0\bar{\psi}_1(0) = F_1(0) = \psi_1(0) = 0$$
$$\partial_0\phi_3(3) = \partial_0\bar{\psi}_3(3) = F_3(3) = \psi_3(3) = 0$$



$$\langle f|M_{23} \circ M_{12}|g\rangle := \int d\phi dF d\psi d\bar{\psi} \exp\left(-S[\phi, \psi, \bar{\psi}, F]\right) f(\phi(3), \bar{\psi}(3)) g(\phi(0), \bar{\psi}(0))$$

$$\phi = (\phi_1, \phi_2, \phi_3) \; \psi = (\psi_1, \psi_2, \phi_3) \; \bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3) \; F = (F_1, F_2, F_3)$$

$$\phi_1 \equiv p_1 \; \phi_2 \equiv p_2 \; \phi_3 \equiv p_3 \; (p_1, p_2, p_3) \in M_{123} \text{ is a fixed point}$$

$$M_{123} = (M_{12} \times M_3) \cap (M_1 \times M_{23}) \subset M_1 \times M_2 \times M_3$$

$$\frac{1}{TM_{123}(p_1, p_2, p_3)}$$

$$\int \partial_0 \delta\phi\psi = -\delta\phi(1)\psi(1) = -\delta\phi_2(1)\psi(1)$$

$$\prod \frac{TM_1 \oplus TM_2 \oplus TM_3}{T(M_{12} \times M_3) + T(M_1 \times M_{23})}$$

$$T(M_{12} \times M_3) + T(M_1 \times M_{23}) = TM_1 \oplus TM_2 \oplus TM_3$$

$$\langle f|M_{23} \circ M_{12}|g\rangle = \sum_{(p_1,p_2,p_3) \in \text{ fixed points of } M_{123}} f(p_3) g(p_1) \frac{\prod TM_{123}(p_1, p_2, p_3)_{\mathrm{obs}}}{\prod TM_{123}(p_1, p_2, p_3)_{\mathrm{tan}}}$$

$$\langle f|M_{23} \circ M_{12}|g\rangle = \sum_{(p_1,p_2,p_3) \in \text{ fixed points of } M_{123}} \frac{f(p_3) g(p_1)}{\prod TM_{123}(p_1, p_2, p_3)_{\mathrm{vir}}}$$

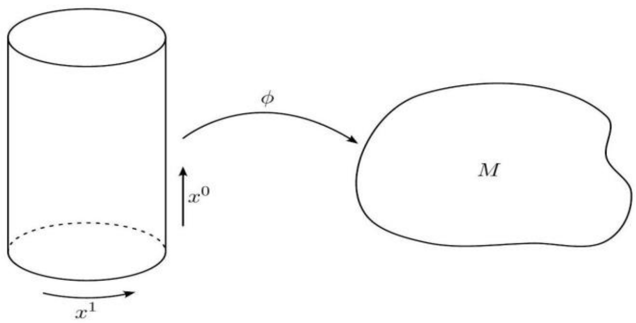
$$\langle p_3|(\pi_3)_*\pi_2^*(\pi_2)_*\pi_1^*|p_1\rangle = \langle p_3|(\pi_3)_*\pi_2^*\sum_q \frac{\prod\ TM_2(q)}{\prod\ TM_{12}(p_1,q)}|q\rangle$$
$$= \sum_q \frac{\prod\ TM_2(q)}{\prod\ M_{12}(p_1,q)\prod\ M_{23}(q,p_3)} = \sum_q \frac{1}{\prod\ TM_{123}(p_1,q,p_3)}$$

$$L = \partial_+\phi\partial_-\phi - \psi_+\partial_-\psi_+ - \psi_-\partial_+\psi_- + \frac{1}{2}R(\psi_+,\psi_+,\psi_-,\psi_-) + FF$$

$$\partial_\pm = \frac{1}{2}(\partial_1 \pm i\partial_0)$$

$$\begin{aligned}
&\phi(x^0,x^1=2\pi) = \exp(\epsilon)\phi(x^0,x^1=0)\\
&\psi_+(x^0,x^1=2\pi) = \exp(\epsilon)_*\psi_+(x^0,x^1=0)\\
&\psi_-(x^0,x^1=2\pi) = \exp(\epsilon)_*\psi_-(x^0,x^1=0)\\
&F(x^0,x^1=2\pi) = \exp(\epsilon)_*F(x^0,x^1=0)
\end{aligned}$$

$$\begin{aligned}
\delta\phi &= \epsilon_-\psi_+ + \epsilon_+\psi_-\\
\delta F &= -\frac{1}{2}\epsilon_-R(\psi_+,\psi_+)\psi_- + \frac{1}{2}\epsilon_+R(\psi_-,\psi_-)\psi_+\\
&\quad -\epsilon_-\partial_+\psi_- + \epsilon_+\partial_-\psi_+\\
\delta\psi_+ &= \epsilon_-\partial_+\phi + \epsilon_+F\\
\delta\psi_- &= \epsilon_+\partial_-\phi - \epsilon_-F
\end{aligned}$$



$$\delta_{\eta_+}\delta_{\epsilon_+} = \epsilon_+\eta_+\partial_-\ \delta_{\eta_-}\delta_{\epsilon_-} = \epsilon_-\eta_-\partial_+\ [\delta_{\epsilon_+},\delta_{\epsilon_-}] = 0$$

$$\partial_0\phi = F = 0 \text{ and } \psi_+ - \psi_- = \partial_0\psi_+ + \partial_0\psi_- = 0.$$

$$\partial_-\phi\psi_+ - F\psi_-.$$

$$\langle f \mid g\rangle := \int\ d\phi dF d\psi_+ d\psi_- \exp(-S[\phi,\psi_\pm,F]) f(\phi(1),\psi_\pm(1)) g(\phi(0),\psi_\pm(0))$$

$$\bar\psi = \frac{1}{2}(\psi_+ + \psi_-)\ \psi = \frac{1}{2}(\psi_+ - \psi_-)$$

$$\psi_+\partial_-\psi_+ + \psi_-\partial_+\psi_- = \bar\psi\partial_1\bar\psi + \psi\partial_1\psi - i\bar\psi\partial_0\psi - i\psi\partial_0\bar\psi,$$

$$(\bar\psi\psi)\begin{pmatrix}\partial_1 & -i\partial_0\\ -i\partial_0 & \partial_1\end{pmatrix}\begin{pmatrix}\bar\psi\\ \psi\end{pmatrix}$$

$$\partial_1\delta\phi = \lambda_1\delta\phi, \partial_0\delta\phi = \lambda_0\delta\phi,$$

$$\begin{pmatrix}\bar\psi\\ \psi\end{pmatrix} \in \left\langle\begin{pmatrix}\delta\phi\\ 0\end{pmatrix},\begin{pmatrix}0\\ \partial_0\delta\phi\end{pmatrix}\right\rangle.$$

$$\begin{pmatrix}\lambda_1 & -i\lambda_0^2\\ -i & \lambda_1\end{pmatrix}$$

$$\delta\phi, \bar{\psi} \sim \exp\left(\left(\frac{2\pi i n}{\beta} + j\epsilon\right)x^1\right), n \in \mathbb{Z}$$

$$\sqrt{\frac{\det(\partial_1)_{\bar{\psi}}}{\det(\partial_1\partial_1)_{\delta\phi}}} = \prod_j \frac{\beta}{2\pi i n + \beta j\epsilon} = \prod_j \frac{1}{2\sinh(\beta j\epsilon/2)}$$

$$\langle f \mid g\rangle = \sum_{p\in \text{ fixed points}} \frac{f(p)g(p)}{\prod\ 2\sinh(TM(p)/2)} = \chi(f \otimes g \otimes \sqrt{KM}),$$

$$(\phi_1, \phi_2) \in M_{12}\ (\psi_+^1 + \psi_-^1, \psi_+^2 + \psi_-^2) \in TM_{12}$$

$$\phi_1: [0,1] \times [0,\beta] \to M_1$$
$$\phi_2: [1,2] \times [0,\beta] \to M_2$$
$$\psi_\pm^1 \in \phi_1^* TM_1, \psi_\pm^2 \in \phi_2^* TM_2$$

$$M_{12}(g) = \sqrt{KM_2^{-1}} \otimes (\pi_2)_*\left(\pi_1^* g \otimes \sqrt{KM_{12}}\right)$$

$$\delta A_+ = -\epsilon_+\lambda_+, \delta A_- = -\epsilon_-\lambda_-,$$
$$\delta\phi = -i\epsilon_+\lambda_- - i\epsilon_-\lambda_+,$$
$$\delta\lambda_+ = i\epsilon_-(\partial_+\phi + [A_+, \phi]) + \epsilon_+ F_{+-},$$
$$\delta\lambda_- = i\epsilon_+(\partial_-\phi + [A_-, \phi]) + \epsilon_- F_{-+}.$$

$$A_0 = \partial_0 A_1 = \partial_0\phi = \partial_0(\lambda_- + \lambda_+) = \lambda_- - \lambda_+ = 0.$$

$$F_{+-}F_{-+} + D_+\phi D_-\phi + \cdots = \delta_{\epsilon_+=\epsilon_-}(-F_{+-}\lambda_+ - i(\partial_+\phi + [A_+, \phi])\lambda_-)$$

$$\bigoplus_{k\geq 0} H_{\epsilon_1,\epsilon_2}(M_k)$$

$$\langle Y_{k+n}|a_n|Y_k\rangle = \frac{1}{\prod\ TM_{k,k+n}(Y_k, Y_{k+n})}$$

$$M_{k,k+n} = \{I_{k+n} \subset I_k, \operatorname{supp}(I_k/I_{k+n}) \subset (0,0) \in \mathbb{C}^2\ I_{k+n} \in M_{k+n}, I_k \in M_k\}$$

$$a_n = (\pi_{k+n})_* \pi_k^*$$

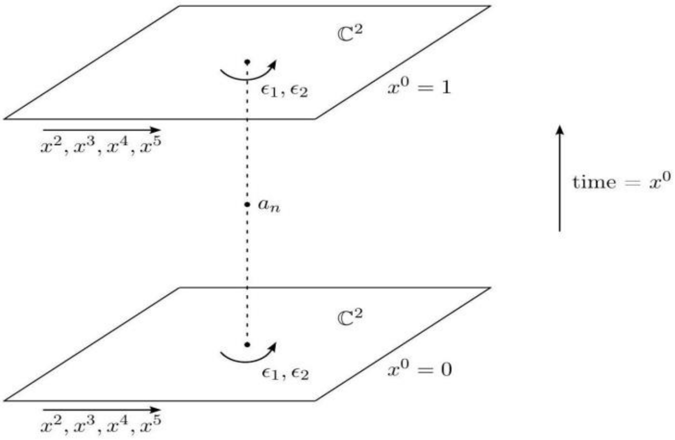
$$\pi_k: M_{k,k+n} \to M_k\ (p_k, p_{k+n}) \mapsto p_k \in M_k$$
$$\pi_{k+n}: M_{k,k+n} \to M_{k+n}\ (p_k, p_{k+n}) \mapsto p_{k+n} \in M_{k+n}$$

$$[a_n, a_m] = (-1)^n n\delta_{n+m}\epsilon_1\epsilon_2$$

$$z_1 = x^2 + ix^3\ z_2 = x^4 + ix^5$$

$$(z_1, z_2) \mapsto (q_1 z_1, q_2 z_2)\ (z_1, z_2) \in \mathbb{C}^2\ q_1 = \exp(\epsilon_1)\ q_2 = \exp(\epsilon_2)$$

$$V = \epsilon_1\left(z_1\frac{\partial}{\partial z_1} - \bar{z}_1\frac{\partial}{\partial \bar{z}_1}\right) + \epsilon_2\left(z_2\frac{\partial}{\partial z_2} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2}\right)$$



$$Q^2 = \tilde{Q}^2 = i\mathcal{L}_V \ \{Q, \tilde{Q}\} = \partial_0$$

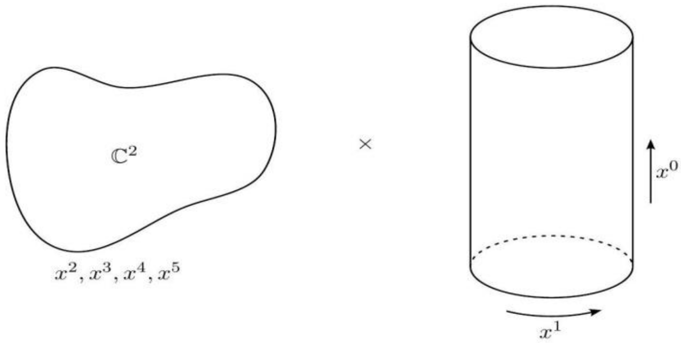
$$\partial_1 \phi = \mathcal{L}_V \phi$$

$$Q_+^2 = \partial_+ \ Q_-^2 = \partial_- \ \{Q_+, Q_-\} = 0$$

$$\partial_\pm = \frac{1}{2}(\partial_1 \pm i\partial_0)$$
$$\partial_\pm = \frac{1}{2}(\partial_1 \pm \partial_0)$$

$$Q_+^2 = \frac{1}{2}(\mathcal{L}_V + i\partial_0) \ Q_-^2 = \frac{1}{2}(\mathcal{L}_V - i\partial_0) \ \{Q_+, Q_-\} = 0$$

$$Q = Q_+ + Q_- \ \tilde{Q} = Q_+ - Q_- \Rightarrow Q^2 = \tilde{Q}^2 = \mathcal{L}_V \ \{Q, \tilde{Q}\} = i\partial_0$$



$$SO(4)_{\mathbb{C}^2} \cong SU(2)^+_{\mathbb{C}^2} \times SU(2)^-_{\mathbb{C}^2} \ SO(4)_R \cong SU(2)^+_R \times SU(2)^-_R$$

$$\delta A_+ = i\epsilon\lambda \ \delta A_- = i\bar{\epsilon}\bar{\lambda}$$
$$\delta\begin{pmatrix} A_\mu \\ \phi \\ \phi^+_{\mu\nu} \end{pmatrix} = \epsilon\begin{pmatrix} \bar{\lambda}_\mu \\ \bar{\lambda} \\ \bar{\lambda}^+_{\mu\nu} \end{pmatrix} + \bar{\epsilon}\begin{pmatrix} \lambda_\mu \\ \lambda \\ \lambda^+_{\mu\nu} \end{pmatrix}$$
$$\delta\begin{pmatrix} \lambda_\mu \\ \lambda \\ \lambda^+_{\mu\nu} \end{pmatrix} = i\bar{\epsilon}(\partial_+ + A_+)\begin{pmatrix} A_\mu \\ \phi \\ \phi^+_{\mu\nu} \end{pmatrix} + \epsilon\begin{pmatrix} -\frac{1}{2}\partial_\mu\phi - \frac{1}{2}H_\mu \\ -F_{+-} \\ -\frac{1}{2}H^+_{\mu\nu} \end{pmatrix}$$
$$\delta\begin{pmatrix} \bar{\lambda}_\mu \\ \bar{\lambda} \\ \bar{\lambda}^+_{\mu\nu} \end{pmatrix} = i\epsilon(\partial_- + A_-)\begin{pmatrix} A_\mu \\ \phi \\ \phi^+_{\mu\nu} \end{pmatrix} + \bar{\epsilon}\begin{pmatrix} -\frac{1}{2}\partial_\mu\phi + \frac{1}{2}H_\mu \\ -F_{-+} \\ +\frac{1}{2}H^+_{\mu\nu} \end{pmatrix}$$
$$\delta H_\mu = -2i\epsilon\partial_-\lambda_\mu - \epsilon\partial_\mu\bar{\lambda} + 2i\bar{\epsilon}\partial_+\bar{\lambda}_\mu + \bar{\epsilon}\partial_\mu\lambda$$
$$\delta H^+_{\mu\nu} = -2i\epsilon\partial_-\lambda^+_{\mu\nu} + 2i\bar{\epsilon}\partial_+\bar{\lambda}^+_{\mu\nu}$$

$$\delta_\eta \delta_\epsilon = i\epsilon\eta(\partial_- + A_-)\ \delta_{\bar\eta}\delta_{\bar\epsilon} = i\bar\epsilon\bar\eta(\partial_+ + A_+)\ [\delta_{\bar\epsilon}, \delta_\epsilon] = \epsilon\bar\epsilon\phi$$

$$\delta A_\mu = -\partial_\mu C\ \delta A_\pm = -\partial_\pm C$$

$$(\partial_+ + A_+)A_\mu = \partial_+ A_\mu - \partial_\mu A_+ = F_{+\mu}$$

$$-\epsilon\bar\epsilon L = \delta_\epsilon \delta_{\bar\epsilon}\left[(\bar\lambda_\mu, \bar\lambda, \bar\lambda^+_{\mu\nu})\begin{pmatrix}\lambda_\mu \\ \lambda \\ \lambda^+_{\mu\nu}\end{pmatrix} + \phi^+_{\mu\nu}F_{\mu\nu}\right]$$

$$F_{+\mu}F_{-\mu} + \partial_+\phi\partial_-\phi + \partial_+\phi^+_{\mu\nu}\partial_-\phi^+_{\mu\nu} - \frac{1}{4}H^+_{\mu\nu}H^+_{\mu\nu} - \frac{1}{4}H_\mu H^\mu$$
$$+F_{+-}F_{-+} + \frac{1}{4}D_\rho\phi D^\rho\phi - \frac{1}{2}H^+_{\mu\nu}F^+_{\mu\nu} + \partial^\mu\phi^+_{\mu\nu}H^\nu$$

$$i(\bar\lambda_\mu, \bar\lambda, \bar\lambda^+_{\mu\nu})\partial_+\begin{pmatrix}\bar\lambda_\mu \\ \bar\lambda \\ \bar\lambda^+_{\mu\nu}\end{pmatrix} + i(\lambda_\mu, \lambda, \lambda^+_{\mu\nu})\partial_-\begin{pmatrix}\lambda_\mu \\ \lambda \\ \lambda^+_{\mu\nu}\end{pmatrix}$$
$$+\lambda_\mu\partial^\mu\bar\lambda + \bar\lambda_\mu\partial_\mu\lambda + 2\bar\lambda^+_{\mu\nu}\partial_\mu\lambda_\nu - 2\lambda^+_{\mu\nu}\partial_\mu\bar\lambda_\nu$$

$$A_0 = \partial_0 A_1 = \partial_0 A_\mu = \partial_0\phi = \phi^+_{\mu\nu} = \partial_0\partial_0\phi^+_{\mu\nu} - \partial_0 H^+_{\mu\nu} = H_\mu = 0$$
$$\bar\lambda_\mu - \lambda_\mu = \lambda - \bar\lambda = \partial_0(\bar\lambda^+_{\mu\nu} - \lambda^+_{\mu\nu}) = 0$$
$$\partial_0(\bar\lambda_\mu + \lambda_\mu) = \partial_0(\lambda + \bar\lambda) = \bar\lambda^+_{\mu\nu} + \lambda^+_{\mu\nu} = 0$$

$$F_{\mu\nu}\lambda^+_{\mu\nu} + 2\phi^+_{\mu\nu}\partial_\mu\lambda_\nu - iF_{+\mu}\bar\lambda_\mu - \frac{1}{2}\lambda_\mu(\partial_\mu\phi - H_\mu)$$
$$+\frac{1}{2}H^+_{\mu\nu}\lambda^+_{\mu\nu} - i\bar\lambda^+_{\mu\nu}\partial_+\phi^+_{\mu\nu} - F_{-+}\lambda - i\bar\lambda\partial_+\phi$$

$$iF_{+\mu} - \frac{1}{2}\partial_\mu\phi - \partial^\nu\phi^+_{\nu\mu} = 0$$
$$i\partial_+\phi - F_{+-} = 0$$
$$i\partial_+\phi^+_{\mu\nu} + \frac{1}{2}F^+_{\mu\nu} = 0$$
$$iF_{-\mu} - \frac{1}{2}\partial_\mu\phi + \partial^\nu\phi^+_{\nu\mu} = 0$$
$$i\partial_-\phi - F_{-+} = 0$$
$$i\partial_-\phi^+_{\mu\nu} - \frac{1}{2}F^+_{\mu\nu} = 0$$

$$H_\mu = 2\partial^\nu\phi^+_{\nu\mu}\ H^+_{\mu\nu} = -F^+_{\mu\nu}$$

$$\phi = \phi^+_{\mu\nu} = A_0 = A_1 = 0$$
$$F^+_{\mu\nu} = 0$$
$$\partial_0 A_\mu = \partial_1 A_\mu = 0$$

$$M_{k,k+n} = \{I_{k+n} \subset I_k, \mathrm{supp}(I_k/I_{k+n}) \subset (0,0) \in \mathbb{C}^2\ I_{k+n} \in M_{k+n}, I_k \in M_k\}$$

$$-\frac{1}{2}F_{0\mu} - \partial^\nu\phi^+_{\nu\mu} = 0$$
$$-\partial_0\phi^+_{\mu\nu} + F^+_{\mu\nu} = 0$$

$$\langle f|a^{6d}_n|g\rangle := \int f(\Phi(x^0 = 1))g(\Phi(x^0 = 0))\exp(-S[\Phi])$$

$$\Phi(x^0 = 1) = (A_\mu(x^0 = 1), \dots)$$
$$\Phi(x^0 = 0) = (A_\mu(x^0 = 0), \dots)$$

$$\Phi(x^1 = \beta) = q_1^{J_1}q_2^{J_2}\Phi(x^1 = 0)$$

$$\phi = \phi^+_{\mu\nu} = F^+_{\mu\nu} = F_{\mu\pm} = F_{+-} = 0$$

$$\frac{1}{4}\left(\sum_{i=0}^{5} \partial_i \delta A^i\right)^2 - \sum_{i=0}^{5} \bar{C}\partial_i \partial^i C$$

$$(\delta A_i, \delta\phi, \phi_{\mu\nu}^+)(\partial_+\partial_- + \partial_\mu \partial^\mu)\begin{pmatrix} \delta A_i \\ \delta\phi \\ \delta\phi_{\mu\nu}^+ \end{pmatrix} - \sum_{i=0}^{5} \bar{C}\partial_i \partial^i C$$

$$\bar{\Psi} = \begin{pmatrix} \bar{\psi}_\mu \\ \bar{\psi} \\ \bar{\psi}_{\mu\nu}^+ \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\lambda_\mu \\ \lambda \\ \lambda_{\mu\nu}^+ \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\bar{\lambda}_\mu \\ -\bar{\lambda} \\ \bar{\lambda}_{\mu\nu}^+ \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \psi_\mu \\ \psi \\ \psi_{\mu\nu}^+ \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \lambda_\mu \\ \lambda \\ \lambda_{\mu\nu}^+ \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \bar{\lambda}_\mu \\ -\bar{\lambda} \\ \bar{\lambda}_{\mu\nu}^+ \end{pmatrix}$$

$$\partial_0 \psi_\mu = \psi = \psi_{\mu\nu}^+ = 0$$
$$\bar{\psi}_\mu = \partial_0 \bar{\psi} = \partial_0 \bar{\psi}_{\mu\nu}^+ = 0$$

$$(\psi_\mu, \psi, \psi_{\mu\nu}^+) i\partial_1 \begin{pmatrix} \psi_\mu \\ \psi \\ \psi_{\mu\nu}^+ \end{pmatrix} + (\bar{\psi}_\mu, \bar{\psi}, \bar{\psi}_{\mu\nu}^+) i\partial_1 \begin{pmatrix} \bar{\psi}_\mu \\ \bar{\psi} \\ \bar{\psi}_{\mu\nu}^+ \end{pmatrix}$$

$$+2(\psi_\mu, \psi, \psi_{\mu\nu}^+)\begin{pmatrix} -\partial_0 & -d & -2d^\dagger \\ d^\dagger & \partial_0 & 0 \\ \frac{1}{2}(d + * d) & 0 & \partial_0 \end{pmatrix}\begin{pmatrix} \bar{\psi}_\mu \\ \bar{\psi} \\ \bar{\psi}_{\mu\nu}^+ \end{pmatrix}$$

$$d^\dagger(\lambda_\mu) = -\nabla^\mu \lambda_\mu \; d^\dagger(\lambda_{\mu\nu}^+) = -\nabla^\mu \lambda_{\mu\nu}^+$$

$$(\bar{\Psi}, \Psi)\begin{pmatrix} i\partial_1 & \not{D}_5 \\ \not{D}_5 & i\partial_1 \end{pmatrix}\begin{pmatrix} \bar{\Psi} \\ \Psi \end{pmatrix}$$

$$\not{D}_5 = \begin{pmatrix} -\partial_0 & -d & -2d^\dagger \\ d^\dagger & \partial_0 & 0 \\ \frac{1}{2}(d + * d) & 0 & \partial_0 \end{pmatrix}$$

$$\begin{pmatrix} \bar{\Psi} \\ \Psi \end{pmatrix} \in \left\langle \begin{pmatrix} \delta A_\mu, \delta A_0, \delta\phi_{\mu\nu}^+ \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \not{D}_5(\delta A_\mu, \delta A_0, \delta\phi_{\mu\nu}^+) \end{pmatrix} \right\rangle$$

$$\frac{1}{\prod \; 2\sinh\left(TM_{k,k+n}(Y_k, Y_{k+n})/2\right)}$$

$$\langle f | a_n^{6d} | g \rangle = \sum_{Y_k \subset Y_{k+n}} \frac{f(Y_{k+n}) g(Y_k)}{\prod \; 2\sinh\left(TM_{k,k+n}(Y_k, Y_{k+n})/2\right)}$$

$$\langle f | a_n | g \rangle = \sum_{Y_k \subset Y_{k+n}} \frac{f(Y_{k+n}) g(Y_k)}{\prod \; TM_{k,k+n}(Y_k, Y_{k+n})}$$

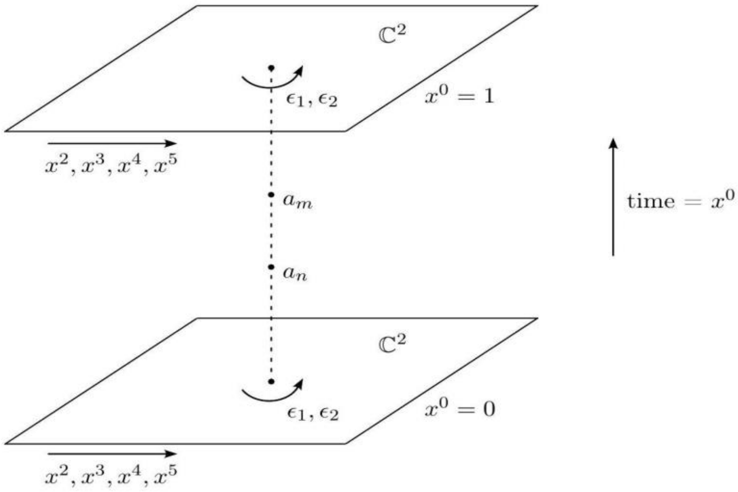
$$\pi_k : M_{k,k+n} \to M_k \; (p_k, p_{k+n}) \mapsto p_k \in M_k$$
$$\pi_{k+n} : M_{k,k+n} \to M_{k+n} \; (p_k, p_{k+n}) \mapsto p_{k+n} \in M_{k+n}$$

$$a_n = (\pi_{n+k})_* \pi_k^*$$
$$a_n^{6d} = \sqrt{KM_{k+n}^{-1}} \otimes (\pi_{n+k})_* \left(\sqrt{KM_{k,k+n}} \otimes \pi_k^*\right)$$

$$[a_n, a_m] = (-1)^n n \delta_{n+m} \epsilon_1 \epsilon_2$$
$$\left[a_{-1}^{6d}, a_1^{6d}\right] = \sinh(\epsilon_1) \sinh(\epsilon_2)$$

$$[a_n, a_m] = (-1)^n n \delta_{n+m} \epsilon_1 \epsilon_2$$

$$M_{k,k+n,k+n+m} = \{I_{k+n+m} \subset I_{k+n} \subset I_k \ \mathrm{supp}(I_k/I_{k+n}), \mathrm{supp}(I_{k+n}/I_{k+n+m}) \subset (0{,}0)\}$$



$$\langle Y_{k+n} | a_n | Y_k \rangle = \frac{1}{\prod TM_{k,k+n}(Y_k, Y_{k+n})}$$

$$M_{k,k+n} = \{I_{k+n} \subset I_k \ \mathrm{supp}(I_k/I_{k+n}) \subset (0{,}0) \ I_k \in M_k, I_{k+n} \in M_{k+n}\}$$

$$\langle Y_k \mid Y_k \rangle = \frac{1}{\prod TM_k(Y_k)} = \prod_{s \in Y_k} \frac{1}{(-\epsilon_2 L(s) + \epsilon_1(A(s)+1))(\epsilon_2(L(s)+1) - \epsilon_1 A(s))}$$

$$\begin{aligned}
\langle \yng(1) | \, a_1 \, | 0 \rangle &= 1 \\
\langle \yng(1,1) | \, a_1 \, | \yng(1) \rangle &= \frac{1}{(\epsilon_1 - \epsilon_2)\epsilon_2} \\
\langle \yng(2) | \, a_1 \, | \yng(1) \rangle &= \frac{1}{(\epsilon_2 - \epsilon_1)\epsilon_1} \\
\langle \yng(3) | \, a_1 \, | \yng(2) \rangle &= \frac{1}{2\epsilon_1} \frac{1}{\epsilon_1} \frac{1}{\epsilon_2 - 2\epsilon_1} \frac{1}{\epsilon_2 - \epsilon_1} \\
\langle \yng(2,1) | \, a_1 \, | \yng(2) \rangle &= \frac{1}{2\epsilon_1 - \epsilon_2} \frac{1}{\epsilon_1} \frac{1}{\epsilon_2 - \epsilon_1} \frac{1}{\epsilon_2} \\
\langle \yng(2,1) | \, a_1 \, | \yng(1,1) \rangle &= \frac{1}{\epsilon_1(\epsilon_1 - \epsilon_2)\epsilon_2(2\epsilon_2 - \epsilon_1)}
\end{aligned}$$

$$\begin{aligned}
a_1 \, | \yng(1) \rangle &= \frac{| \yng(2) \rangle \langle \yng(2) |}{\langle \yng(2) | \yng(2) \rangle} a_1 \, | \yng(1) \rangle + \frac{| \yng(1,1) \rangle \langle \yng(1,1) |}{\langle \yng(1,1) | \yng(1,1) \rangle} a_1 \, | \yng(1) \rangle \\
&= \frac{\epsilon_1 \epsilon_2 (\epsilon_2 - \epsilon_1) 2\epsilon_1}{(\epsilon_2 - \epsilon_1)\epsilon_1} + \frac{\epsilon_1 \epsilon_2 (\epsilon_1 - \epsilon_2) 2\epsilon_2}{(\epsilon_1 - \epsilon_2)\epsilon_2} \\
&= 2\epsilon_1 \epsilon_2 (| \yng(2) \rangle + | \yng(1,1) \rangle)
\end{aligned}$$

$$M_{k,k+1} = \{I_{k+1} \subset I_k \ \mathrm{supp}(I_k/I_{k+1}) \subset (0{,}0) \ I_k \in M_k, I_{k+1} \in M_{k+1}\}$$

$$T\left(\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)=\epsilon_1-\epsilon_2,\epsilon_2\quad T\left(\square,\square\square\right)=\epsilon_2-\epsilon_1,\epsilon_1,$$

$$\left\langle\begin{smallmatrix}\square\\\square\end{smallmatrix}\right|a_1\left|\square\right\rangle=\frac{1}{(\epsilon_1-\epsilon_2)\epsilon_2}\quad\left\langle\square\square\right|a_1\left|\square\right\rangle=\frac{1}{(\epsilon_2-\epsilon_1)\epsilon_1}.$$

$$T\langle Y_{k+1}|a_1|Y_k\rangle=\prod_{s\in Y_k}\frac{1}{\big(\epsilon_1(A(s,Y_k)+1)-\epsilon_2L(s,Y_{k+1})\big)\big(\epsilon_2\big(1+L(s,Y_k)\big)-\epsilon_1A(s,Y_{k+1})\big)},$$

$$\begin{aligned}\left\langle\square\square\right|a_2\left|0\right\rangle&=\frac{1}{\epsilon_2-\epsilon_1}\\\left\langle\begin{smallmatrix}\square\\\square\end{smallmatrix}\right|a_2\left|0\right\rangle&=\frac{1}{\epsilon_1-\epsilon_2}\\\left\langle\square\square\square\right|a_2\left|\square\right\rangle&=\frac{1}{\epsilon_1(\epsilon_2-2\epsilon_1)(\epsilon_2-\epsilon_1)}\\\left\langle\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right|a_2\left|\square\right\rangle&=\frac{2\epsilon_1+2\epsilon_2}{\epsilon_1\epsilon_2(2\epsilon_2-\epsilon_1)(2\epsilon_1-\epsilon_2)}\end{aligned}$$

$$M_{0,2}=\{I_2\subset I_0\ \mathrm{supp}(I_0/I_2)\subset(0{,}0)\ I_0\in M_0,I_2\in M_2\}$$

$$M_{1,3}=\{I_3\subset I_1\ \mathrm{supp}(I_1/I_3)\subset(0{,}0)\ I_1\in M_1,I_3\in M_3\}$$

$$\left(\left|\square\right\rangle,\left|\square\square\square\right\rangle\right)=\left(\langle z_1,z_2\rangle,\langle z_1^3,z_2\rangle\right),$$

$$\begin{aligned}\langle z_1^3+\epsilon z_1^2,z_2\rangle&=q_1\\\langle z_1^3,z_2+\epsilon z_1\rangle&=q_2/q_1\\\langle z_1^3,z_2+\epsilon z_1^2\rangle&=q_2/q_1^2\end{aligned}$$

$$\frac{d}{d\epsilon}\langle z_1^3,z_2+\epsilon z_1\rangle\Big|_{\epsilon=0}=\begin{pmatrix}z_1^3\mapsto 0\\z_2\mapsto z_1\end{pmatrix}$$

$$z_1\mapsto q_1^{-1}z_1\ z_1^3\mapsto q_1^{-3}z_1^3\ z_2\mapsto q_2^{-1}z_2$$

$$\langle q_1^{-3}z_1^3,q_2^{-1}z_2+\epsilon q_1^{-1}z_1\rangle=\langle z_1^3,z_2+(q_2/q_1)\epsilon z_1\rangle$$

$$\left(\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right)=\left(\langle z_1,z_2\rangle,\langle z_1^2,z_1z_2,z_2^2\rangle\right)$$

$$\begin{aligned}\langle z_1^2+\epsilon z_1,z_1z_2,z_2^2\rangle&=q_1\\\langle z_1^2+\epsilon z_2,z_1z_2,z_2^2\rangle&=q_1^2/q_2\\\langle z_1^2,z_1z_2,z_2^2+\epsilon z_1\rangle&=q_2^2/q_1\\\langle z_1^2,z_1z_2,z_2^2+\epsilon z_2\rangle&=q_2\end{aligned}$$

$$M_{0,1,2}=\{I_2\subset I_1\subset I_0\ I_i\in M_i,\mathrm{supp}(I_i/I_{i+1})\subset(0{,}0)\}$$

$$\begin{aligned}I_0&=\mathbb{C}[z_1,z_2],I_1=\langle z_1,z_2\rangle,I_2=\langle z_1,z_2^2\rangle=\epsilon_1-\epsilon_2\\I_0&=\mathbb{C}[z_1,z_2],I_1=\langle z_1,z_2\rangle,I_2=\langle z_1^2,z_2\rangle=\epsilon_2-\epsilon_1\end{aligned}$$

$$\left\langle\begin{smallmatrix}\square\\\square\end{smallmatrix}\right|a_1a_1\left|0\right\rangle=\prod\frac{1}{(TM_{0,1,2})_{\mathrm{vir}}}=\frac{\epsilon_1}{\epsilon_1-\epsilon_2}$$

$$a_1\left|0\right\rangle = \epsilon_1\epsilon_2\left|\square\right\rangle \Rightarrow \left\langle \begin{smallmatrix}\square\\\square\end{smallmatrix}\right| a_1 a_1 \left|0\right\rangle = \epsilon_1\epsilon_2 \left\langle \begin{smallmatrix}\square\\\square\end{smallmatrix}\right| a_1 \left|\square\right\rangle = \frac{\epsilon_1}{\epsilon_1-\epsilon_2}.$$

$$\begin{aligned}\left\langle \begin{smallmatrix}\square\\\square\end{smallmatrix}\right| a_1^{6d} \left|\square\right\rangle &= \frac{1}{2\sinh((\epsilon_1-\epsilon_2)/2)2\sinh(\epsilon_2/2)},\\ \left\langle \square\square\right| a_1^{6d} \left|\square\right\rangle &= \frac{1}{2\sinh((\epsilon_2-\epsilon_1)/2)2\sinh(\epsilon_1/2)}.\end{aligned}$$

$$Z_{\text{inst}}^{U(1)}(\epsilon_1,\epsilon_2,\Lambda) = \langle \Lambda \mid \Lambda\rangle$$

$$Z_{\text{inst}}^{U(1)} = \sum_Y \Lambda^{|Y|}\prod_{s\in Y}\frac{1}{(-\epsilon_2 L(s)+\epsilon_1(A(s)+1))(\epsilon_2(L(s)+1)-\epsilon_1 A(s))}$$

$$[a_n, a_m] = (-1)^n n\delta_{n+m}\epsilon_1\epsilon_2$$

$$a_{-1}|\Lambda\rangle = \Lambda^{1/2}|\Lambda\rangle \; a_n|\Lambda\rangle = 0, \forall n \leq -2$$

$$|\Lambda\rangle = \sum_{k\geq 0}\frac{1}{k!\,(\epsilon_1\epsilon_2)^k}\Lambda^{k/2}a_1^k|0\rangle = \exp\left(\frac{\Lambda^{1/2}a_1}{\epsilon_1\epsilon_2}\right)|0\rangle$$

$$\begin{aligned}\langle \Lambda \mid \Lambda\rangle &= \langle 0|\exp\left(\frac{\Lambda^{1/2}a_{-1}}{\epsilon_1\epsilon_2}\right)\exp\left(\frac{\Lambda^{1/2}a_1}{\epsilon_1\epsilon_2}\right)|0\rangle\\ &= \langle 0|\exp\left(\frac{\Lambda[a_{-1},a_1]}{\epsilon_1^2\epsilon_2^2}\right)|0\rangle = \exp\left(\frac{\Lambda}{\epsilon_1\epsilon_2}\right)\end{aligned}$$

$$|B\rangle = \sum_Y \Lambda^{|Y|/2}|Y\rangle$$

$$|B\rangle = \exp\left(\frac{\Lambda^{1/2}a_1}{\epsilon_1\epsilon_2}\right)|0\rangle$$

$$a_1|\Lambda\rangle = a_1\exp\left(\frac{\Lambda^{1/2}a_1}{\epsilon_1\epsilon_2}\right)|0\rangle = K\frac{\epsilon_1\epsilon_2}{\Lambda^{1/2}}|\Lambda\rangle$$

$$\begin{aligned}\left|\Lambda, m_1\right\rangle &= \sum_Y \Lambda^{|Y|/2}\left(\prod_{w\in Y}(m_1+w(\epsilon_1,\epsilon_2))\right)\left|Y\right\rangle\\ &= \left|0\right\rangle + \Lambda^{1/2}m_1\left|\square\right\rangle + \Lambda m_1(m_1+\epsilon_1)\left|\square\square\right\rangle + \Lambda m_1(m_1+\epsilon_2)\left|\begin{smallmatrix}\square\\\square\end{smallmatrix}\right\rangle + O(\Lambda^{3/2})\end{aligned}$$

$$Y = \begin{array}{|c|c|c|}\hline 0 & \epsilon_1 & 2\epsilon_1\\ \hline \epsilon_2 & \epsilon_1+\epsilon_2 & \\ \hline\end{array}.$$

$$\prod_{w\in\begin{smallmatrix}\square\square\square\\\square\square\end{smallmatrix}}(m_1+w(\epsilon_1,\epsilon_2)) = m_1(m_1+\epsilon_1)(m_1+2\epsilon_1)(m_1+\epsilon_2)(m_1+\epsilon_1+\epsilon_2).$$

$$\langle \Lambda \mid \Lambda, m_1 \rangle = \sum_Y \Lambda^{|Y|} \prod_{s \in Y} \frac{\prod_{w \in Y} \left(m_1 + w(\epsilon_1, \epsilon_2)\right)}{(-\epsilon_2 L(s) + \epsilon_1(A(s)+1))(\epsilon_2(L(s)+1) - \epsilon_1 A(s))}$$

$$a_{-1}|\Lambda, m_1\rangle = m_1 \Lambda^{1/2} |\Lambda, m_1\rangle.$$

$$|\Lambda, m_1, \dots, m_N\rangle = \sum_Y \Lambda^{|Y|/2} \left(\prod_{w \in Y} \prod_{i=1}^{N} \left(m_i + w(\epsilon_1, \epsilon_2)\right) \right) |Y\rangle.$$

$$|\Lambda, m_1, m_2\rangle = |0\rangle + \Lambda^{1/2} m_1 m_2 \,|\square\rangle + \Lambda m_1 m_2 (m_1 + \epsilon_1)(m_2 + \epsilon_1)\,|\square\square\rangle + \Lambda m_1 m_2 (m_1 + \epsilon_2)(m_2 + \epsilon_2)\,|\boxminus\rangle + O(\Lambda^{3/2}).$$

$$a_{-1}|\Lambda, m_1, m_2\rangle = \Lambda^{1/2}(m_1 m_2 - \epsilon_1 \epsilon_2 K)|\Lambda, m_1, m_2\rangle.$$

$$z \mapsto qz, z \in \mathbb{C}, q = \exp(\epsilon) \in U(1)\ V = \epsilon z \frac{\partial}{\partial z} - \epsilon \bar{z} \frac{\partial}{\partial \bar{z}}$$

$$\int_{\mathbb{C}} \omega = \frac{f(0)}{\epsilon} \Rightarrow f(0) = \epsilon$$

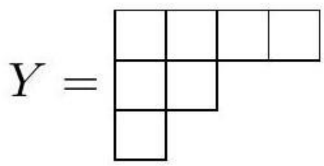
$$|0\rangle = \frac{f}{\epsilon} + \frac{\omega}{\epsilon}$$

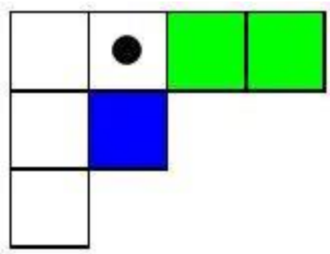
$$f^*|p_2\rangle = \sum_{q \text{ is a fixed point in } M_1, f(q) = p_2,} |q\rangle$$

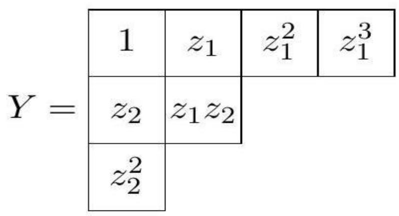
$$f_*\left(\prod TM_1(p_1)|p_1\rangle\right) = \prod TM_2\big(f(p_1)\big)|f(p_1)\rangle$$

$$f_*|p_1\rangle = \frac{\prod TM_2\big(f(p_1)\big)}{\prod TM_1(p_1)} |f(p_1)\rangle.$$

$$f_*|p_1\rangle = \frac{\prod_i \left(1 - \chi_i^{-1}(f(p_1)\right)}{\prod_i \left(1 - \chi_i^{-1}(p_1)\right)} |f(p_1)\rangle$$







$$\langle z_1^4, z_1^3 z_2, z_1^2 z_2, z_1 z_2^2, z_2^3\rangle \subset \mathbb{C}[z_1, z_2].$$

$$\delta_\eta \delta_\epsilon \psi - \delta_\epsilon \delta_\eta \psi - 2\epsilon\eta \mathcal{L}_V \psi = R(\eta\bar{\psi}, \epsilon\bar{\psi})\psi$$

$$H^4\left(\overline{\mathbb{C}^2} \times [0,1] - p\right) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$0 \to H^4\left(\overline{\mathbb{C}^2} \times [0,1]\right) \to H^4\left(\overline{\mathbb{C}^2} \times [0,1] - p\right) \oplus H^4(B) \to H^4\left(B \cap \left(\overline{\mathbb{C}^2} \times [0,1] - p\right)\right) \to 0$$

$$H^3\left(B \cap \left(\overline{\mathbb{C}^2} \times [0,1] - p\right)\right) = H^5\left(\overline{\mathbb{C}^2} \times [0,1]\right) = 0$$

$$0 \to \Omega_0(\mathbb{C}^2) \to \Omega_1(\mathbb{C}^2) \to \Omega_2^+(\mathbb{C}^2) \to 0$$

$$\mathcal{I}_{\mathcal{T}}(b,q) = \mathrm{tr}_{\mathcal{H}} q^{E-\mathcal{R}+\frac{c_{4\,\mathrm{d}}}{2}} b^f = \mathrm{tr}_{\mathbb{V}[\mathcal{T}]} q^{L_0 - \frac{c_{2\,\mathrm{d}}}{24}} b^f = \mathrm{ch}_0[\mathbb{V}[\mathcal{T}]].$$

$$1 + 3\ell(2+\ell)$$

$$\mathcal{T}_{3,2} = (A_2, D_4), \mathcal{T}_{4,3} = (A_3, E_6), \mathcal{T}_{6,5} = (A_5, E_8)$$

$$\Delta_{(N)}\mathrm{ch}(b,q) := b^{-(N^2-1)} q^{-\frac{N^2-1}{2}} \mathrm{ch}(bq,q) - \mathrm{ch}(b,q)$$

$$\mathrm{ch}^{SU(N)}(b,q) \to q^{-\frac{c_{2\,\mathrm{d}}[\mathcal{T}_{p,N}] - p c_{2\,\mathrm{d}}[\mathcal{T}^{SU(N)}]}{24}} \mathrm{ch}^{SU(N)}\left(q^{\frac{p}{2}-1}, q^p\right),$$

$$\mathcal{I}(b,q) = q^{\frac{c_{4\,\mathrm{d}}}{2}} \mathrm{tr}_{\mathcal{H}}(-1)^F q^{E-R} b^f.$$

$$\mathcal{I}(b,q) = \mathrm{ch}_{\mathbb{V}[\mathcal{T}]}(-1)^F q^{L_- - \frac{c_{2\,\mathrm{d}}}{24}} b^f = \mathrm{ch}_0(b,q)$$

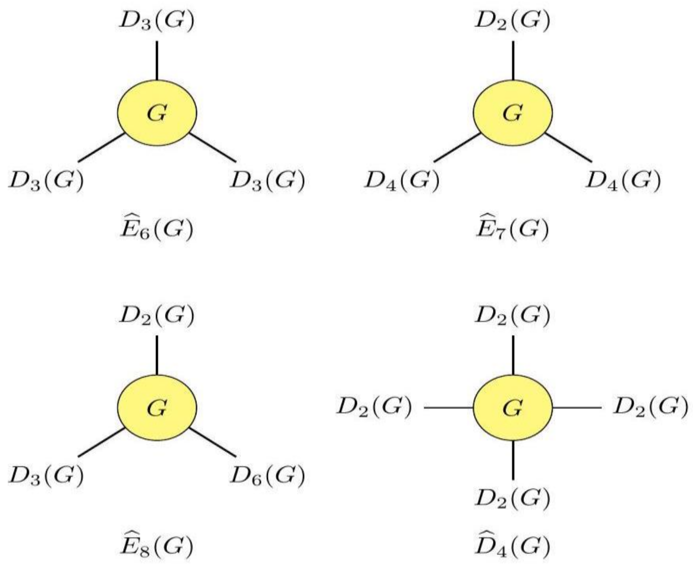
$$\mathrm{ch}_i\left(-\frac{1}{\tau}\right) = \sum_j S_{ij}\mathrm{ch}_j(\tau).$$

$$N_{ijk} = \sum_\ell \frac{S_{i\ell} S_{j\ell} \overline{S_{\ell k}}}{S_{0\ell}}$$

$SU(N)$	$(p,N)=1$
$SO(2N)$	$p \notin 2\mathbb{Z}_{>0}$
E_6	$p \notin 3\mathbb{Z}_{>0}$
E_7	$p \notin 2\mathbb{Z}_{>0}$
E_8	$p \notin 30\mathbb{Z}_{>0}$

$$\sum_{i=1}^n k_i = 4h_G^\vee, k_i = \frac{2(p_i - 1)}{p_i} h^\vee.$$

$$(p_1, p_2, p_3, p_4) = (2,2,2,2), (1,3,3,3), (1,2,4,4), (1,2,3,6).$$



$$\widehat{D}_4(SU(2\ell+1)), \widehat{E}_6(SU(3\ell\pm1)), \widehat{E}_7(SU(4\ell\pm1)), \widehat{E}_8(SU(6\ell\pm1))$$
$$\widehat{E}_6(SO(6\ell)), \widehat{E}_6(SO(6\ell+4)),$$

$$a_{4\,\mathrm{d}} = c_{4\,\mathrm{d}} = \left(1-\frac{1}{p}\right)(N^2-1)$$

$$\mathcal{C}(p,G) = \left\{ j - \frac{h_G^\vee}{p}s \,\middle|\, j - \frac{h_G^\vee}{p}s > 1\,, j \in \mathrm{Cas}(G), s = 1,\cdots,p-1 \right\}$$

$$\mathrm{rank}(\widehat{\Gamma}(G)) = \mathrm{rank}G\left(1+\frac{\mathrm{rank}\Gamma}{2}\right)$$

$$\begin{aligned}
&\mathrm{rank}\mathcal{T}_{2,2\ell+1} = 6\ell, \\
&\mathrm{rank}\mathcal{T}_{3,3\ell+1} = 12\ell, \quad \mathrm{rank}\mathcal{T}_{3,3\ell-1} = 12\ell-8, \\
&\mathrm{rank}\mathcal{T}_{4,4\ell+1} = 18\ell, \quad \mathrm{rank}\mathcal{T}_{4,4\ell-1} = 18\ell-9 \\
&\mathrm{rank}\mathcal{T}_{6,6\ell+1} = 30\ell, \quad \mathrm{rank}\mathcal{T}_{6,6\ell-1} = 30\ell-10.
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{(3,2)} &= (A_2, D_4) = D_4^6[3] = D_4^4[2] \\
\mathcal{T}_{(4,3)} &= (A_3, E_6) = E_6^{12}[4] \\
\mathcal{T}_{(6,5)} &= (A_5, E_8) = E_8^{30}[6]
\end{aligned}$$

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q) = \oiiint [d\boldsymbol{x}] \prod_{j=1}^{\ell} \mathcal{I}_{D_{p_j}(SU(N))}(\boldsymbol{x},q) \times \mathcal{I}_{\mathrm{vec}}(\boldsymbol{x},q)$$

$$\mathcal{I}_{D_p(SU(N))}(\boldsymbol{x},q) = \mathrm{PE}\left[\frac{q-q^p}{(1-q)(1-q^p)}\chi_{\mathrm{adj}}^{SU(N)}(\boldsymbol{x})\right],$$

$$\mathcal{I}_{\mathrm{vec}}(\boldsymbol{x},q) = \mathrm{PE}\left[-\frac{2q}{1-q}\chi_{\mathrm{adj}}^{SU(N)}(\boldsymbol{x})\right]$$

$$[d\boldsymbol{x}] = \prod_{i=1}^{N-1}\frac{dx_i}{2\pi i x_i}\prod_{\substack{j,k=1\\ j\neq k}}^{N}\left(1-\frac{x_j}{x_k}\right)$$

$$I_G^{\mathcal{N}=4}(b,q) = \int [d\vec{z}]\mathrm{PE}\left[\left(-\frac{2q}{1-q}+\frac{q^{\frac{1}{2}}}{1-q}(b+b^{-1})\right)\chi_{\mathrm{adj}}^{G}(\vec{z})\right]$$

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q) = q^{-\frac{c_{2\,\mathrm{d}}[\mathcal{T}_{p,N}]-pc_{2\,\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}} I_{SU(N)}^{\mathcal{N}=4}\left(b=q^{p/2-1},q^p\right)$$

$$\mathcal{I}_{SU(N)}(b,q) = \frac{1}{N!}\vartheta_4(\mathfrak{b})\eta(\tau)^{3(N-1)} \oint\!\!\!\oint\!\!\!\oint \frac{d\mathbf{a}}{2\pi i\mathbf{a}} \frac{\prod_{i\neq j}\vartheta_1(\mathfrak{a}_i-\mathfrak{a}_j)}{\prod_{i,j}\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_j+\mathfrak{b})}$$

$$\frac{\prod_{i\neq j}\vartheta_1(\mathfrak{a}_i-\mathfrak{a}_j)}{\prod_{i,j}\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_j+\mathfrak{b})} = i^{N^2}\frac{u^{-N/2}q^{-\frac{N^2}{8}}b^{-\frac{N^2}{2}}}{\vartheta_1\left(\mathfrak{u}+N\left(\mathfrak{b}+\frac{\tau}{2}\right)\right)\vartheta_1(\mathfrak{u})^{N-1}} \det_{ij}\frac{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_j+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_j+\mathfrak{b})}.$$

$$\det_{ij}\frac{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_j+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_j+\mathfrak{b})} = \sum_{\sigma}(-1)^{\sigma}\prod_{i=1}^{N}\frac{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b})}.$$

$$\sum_{\sigma}(-1)^{\sigma}\oint\!\!\!\oint\!\!\!\oint\frac{d\mathbf{a}}{2\pi i\mathbf{a}}\prod_{i=1}^{N}\frac{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b})}$$

$$\oint\!\!\!\oint\!\!\!\oint\frac{d\mathbf{a}}{2\pi i\mathbf{a}}\prod_{i=1}^{N}\frac{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b})} = \oint\!\!\!\oint\!\!\!\oint\frac{d\mathbf{a}}{2\pi i\mathbf{a}}\prod_{i=1}^{N}\frac{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma'(i)}+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma'(i)}+\mathfrak{b})}.$$

$$\sigma = (c_{11}\cdots c_{1\ell_1})(c_{21}\cdots c_{2\ell_2})\cdots$$

$$\prod_{i=1}^{\ell}\frac{\vartheta_4(\mathfrak{a}_{c_i}-\mathfrak{a}_{c_{i+1}}+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_{c_i}-\mathfrak{a}_{c_{i+1}}+\mathfrak{b})} \subset \prod_{i=1}^{N}\frac{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b})}, \mathfrak{a}_{\ell+1}=\mathfrak{a}_1$$

$$\oint\!\!\!\oint\!\!\!\oint\frac{d\mathbf{a}}{2\pi i\mathbf{a}}\prod_{i=1}^{N}\frac{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_i-\mathfrak{a}_{\sigma(i)}+\mathfrak{b})} = \oint\!\!\!\oint\!\!\!\oint\frac{d\mathbf{a}_c}{2\pi i\mathbf{a}_c}\prod_{i=1}^{\ell}\frac{\vartheta_4(\mathfrak{a}_{c_i}-\mathfrak{a}_{c_{i+1}}+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_{c_i}-\mathfrak{a}_{c_{i+1}}+\mathfrak{b})}\times\oint\!\!\!\oint\!\!\!\oint\cdots$$

$$\mathcal{Z}_{\ell} := \oint\!\!\!\oint\!\!\!\oint\frac{d\mathbf{a}_c}{2\pi i\mathbf{a}_c}\prod_{i=1}^{\ell}\frac{\vartheta_4(\mathfrak{a}_{c_i}-\mathfrak{a}_{c_{i+1}}+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_{c_i}-\mathfrak{a}_{c_{i+1}}+\mathfrak{b})}$$

$$\mathcal{I}_{SU(N)} = \frac{\vartheta_4(\mathfrak{b})\eta(\tau)^{3(N-1)}}{N!}\frac{i^{N^2}u^{-N/2}q^{-\frac{N^2}{8}}b^{-\frac{N^2}{2}}}{\vartheta_1\left(\mathfrak{u}+N\left(\mathfrak{b}+\frac{\tau}{2}\right)\right)\vartheta_1(\mathfrak{u})^{N-1}}\sum_{\vec{\ell}}(-1)^{C_{\vec{\ell}}}|C_{\vec{\ell}}|\mathcal{Z}_{\ell_1}\cdots\mathcal{Z}_{\ell_N},$$

$$\vec{\ell} = \{\ell_1,\ell_2,\cdots\}, \sum_i \ell_i = N, \ell_1\geq\ell_2\geq\cdots\geq 0$$

$$|C_{\vec{n}}| = \frac{N!}{\prod_{\ell=1}^{N}\ell^{n_\ell}n_\ell!}, \sum_{i=1}^{N}n_\ell\ell = N., C_{\vec{\ell}} = C_{\vec{n}}$$

$$\mathcal{I}_{SU(N)} = \frac{\vartheta_4(\mathfrak{b})\eta(\tau)^{3(N-1)}}{N!}\frac{i^{N^2}u^{-N/2}q^{-\frac{N^2}{8}}b^{-\frac{N^2}{2}}}{\vartheta_1\left(\mathfrak{u}+N\left(\mathfrak{b}+\frac{\tau}{2}\right)\right)\vartheta_1(\mathfrak{u})^{N-1}}\sum_{\vec{n}}\frac{(-1)^{C_{\vec{n}}}N!}{\prod_{\ell=1}^{N}\ell^{n_\ell}n_\ell!}\prod_{\ell=1}^{N}\mathcal{Z}_{\ell}^{n_\ell}.$$

$$\ell = 1$$

$$\mathcal{Z}_1 = \oint\!\!\!\oint\!\!\!\oint\frac{da}{2\pi ia}\frac{\vartheta_4(\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{b})} = \frac{\vartheta_4(\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{b})}$$

$$\ell = 2$$

$$\mathcal{Z}_2 = \oint\!\!\!\oint\!\!\!\oint\frac{da_1}{2\pi ia_1}\frac{da_2}{2\pi ia_2}\frac{\vartheta_4(\mathfrak{a}_1-\mathfrak{a}_2+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_1-\mathfrak{a}_2+\mathfrak{b})}\frac{\vartheta_4(\mathfrak{a}_2-\mathfrak{a}_1+\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{a}_2-\mathfrak{a}_1+\mathfrak{b})}$$

$$\mathfrak{a}_1 = \mathfrak{a}_2-\mathfrak{b}+\frac{\tau}{2}, \mathfrak{a}_1 = \mathfrak{a}_2+\mathfrak{b}+\frac{\tau}{2},$$

$$\frac{i\vartheta_1(\mathfrak{u})\vartheta_1(2\mathfrak{b}+\mathfrak{u})}{\eta(\tau)^3\vartheta_1(2\mathfrak{b})}, -\frac{i\vartheta_1(\mathfrak{u})\vartheta_1(2\mathfrak{b}+\mathfrak{u})}{\eta(\tau)^3\vartheta_1(2\mathfrak{b})}.$$

$$\mathcal{Z}_2 = \sum_{\mathfrak{a}_i \text{ real }/\text{Im}} R_j E_1 \begin{bmatrix} -1 \\ a_j/a_0 q^{\pm\frac{1}{2}} \end{bmatrix} = \frac{i\vartheta_1(\mathfrak{u})\vartheta_1(2\mathfrak{b}+\mathfrak{u})}{\eta(\tau)^3\vartheta_1(2\mathfrak{b})}\left(E_1\begin{bmatrix} +1 \\ u \end{bmatrix} - E_1\begin{bmatrix} +1 \\ b^2 u \end{bmatrix}\right)$$

$\underline{\ell = 3}$

$$\prod_{i=1}^{3} \oiiint \frac{d\mathbf{a}}{2\pi i\mathbf{a}} \prod_{i=1}^{3} \frac{\vartheta_4(\mathfrak{a}_i - \mathfrak{a}_{i+1} + \mathfrak{b} + \mathfrak{u})}{\vartheta_4(\mathfrak{a}_i - \mathfrak{a}_{i+1} + \mathfrak{b})}$$

$$\mathfrak{a}_1 = \mathfrak{a}_2 - \mathfrak{b} + \frac{\tau}{2}, \mathfrak{a}_1 = \mathfrak{a}_3 + \mathfrak{b} + \frac{\tau}{2}$$

$$\frac{i\vartheta_1(\mathfrak{u})}{\eta(\tau)^3}\frac{\vartheta_4(\mathfrak{a}_2 - \mathfrak{a}_3 + \mathfrak{b} + \mathfrak{u})\vartheta_1(-\mathfrak{a}_2 + \mathfrak{a}_3 + 2\mathfrak{b} + \mathfrak{u})}{\vartheta_4(\mathfrak{a}_2 - \mathfrak{a}_3 + \mathfrak{b})\vartheta_1(\mathfrak{a}_3 - \mathfrak{a}_2 + 2\mathfrak{b})}\left(E_1\begin{bmatrix} 1 \\ u \end{bmatrix} + E_1\begin{bmatrix} 1 \\ \frac{a_2}{a_3 b^2 u} \end{bmatrix}\right)$$
$$= \frac{i\vartheta_1(\mathfrak{u})}{\eta(\tau)^3}\frac{\vartheta_4(\mathfrak{a}_2 - \mathfrak{a}_3 + \mathfrak{b} + \mathfrak{u})\vartheta_1(-\mathfrak{a}_2 + \mathfrak{a}_3 + 2\mathfrak{b} + \mathfrak{u})}{\vartheta_4(\mathfrak{a}_2 - \mathfrak{a}_3 + \mathfrak{b})\vartheta_1(\mathfrak{a}_3 - \mathfrak{a}_2 + 2\mathfrak{b})}\left(E_1\begin{bmatrix} 1 \\ u \end{bmatrix} - E_1\begin{bmatrix} 1 \\ ub^2\frac{a_3}{a_2} \end{bmatrix}\right)$$

$$\mathfrak{a}_2 = \mathfrak{a}_3 - \mathfrak{b} + \frac{\tau}{2}, \mathfrak{a}_2 = \mathfrak{a}_3 + 2\mathfrak{b}$$

$$-\frac{\vartheta_1(\mathfrak{u})^2\vartheta_4(3\mathfrak{b}+\mathfrak{u})}{\eta(\tau)^6\vartheta_4(3\mathfrak{b})}, \frac{\vartheta_1(\mathfrak{u})^2\vartheta_4(3\mathfrak{b}+\mathfrak{u})}{\eta(\tau)^6\vartheta_4(3\mathfrak{b})}$$

$$\mathcal{I} = -\frac{1}{2}\frac{\vartheta_1(\mathfrak{u})^2\vartheta_4(3\mathfrak{b}+\mathfrak{u})}{\eta(\tau)^6\vartheta_4(3\mathfrak{b})}\left(1 + 8E_1\begin{bmatrix} -1 \\ b^3 u \end{bmatrix}E_1\begin{bmatrix} +1 \\ u \end{bmatrix} - 8E_1\begin{bmatrix} +1 \\ u \end{bmatrix}^2 + 8E_2\begin{bmatrix} -1 \\ b^3 u \end{bmatrix} - 8E_2\begin{bmatrix} +1 \\ u \end{bmatrix}\right)$$

$$\mathcal{Z}_\ell \propto \frac{\vartheta_1(\mathfrak{u})^{\ell-1}\vartheta_{4\text{ or }1}(\ell\mathfrak{b}+\mathfrak{u})}{(\ell-1)!\,\vartheta_{4\text{ or }1}(\ell\mathfrak{b})}$$

$$\mathcal{Z}_{\ell+1} = \oiiint \frac{da_\ell}{2\pi i a_\ell}\frac{\vartheta_4(\mathfrak{a}_\ell + \mathfrak{b} + \mathfrak{u})}{\vartheta_4(\mathfrak{a}_\ell + \mathfrak{b})}[\mathcal{Z}_\ell]_{b^\ell \to b^\ell/a_\ell, \ell\mathfrak{b} \to \ell\mathfrak{b} - \mathfrak{a}_\ell}$$

$$\frac{iu^{-1/2}\eta(\tau)^3}{\vartheta_1(\mathfrak{u})}\mathcal{Z}_1 = \frac{iu^{-1/2}\eta(\tau)^3}{\vartheta_1(\mathfrak{u})}\frac{\vartheta_4(\mathfrak{b}+\mathfrak{u})}{\vartheta_4(\mathfrak{b})} = \sum_{p\in\mathbb{Z}}\frac{b^p q^{\frac{p}{2}}}{1 - uq^p}$$

$$\left(\frac{iu^{-1/2}\eta(\tau)^3}{\vartheta_1(\mathfrak{u})}\right)^\ell \mathcal{Z}_\ell(b,q) = \frac{1}{(\ell-1)!}\partial_u^{\ell-1}\left(\frac{iu^{-1/2}\eta(\tau)^3}{\vartheta_1(\mathfrak{u})}\mathcal{Z}_1\left(b \to b^\ell q^{-\frac{\ell}{2}}\right)\right)$$

$$0 = u\partial_u E_k\begin{bmatrix} 1 \\ u \end{bmatrix} + (k+1)E_{k+1}\begin{bmatrix} 1 \\ u \end{bmatrix} + E_{k+1} + E_1\begin{bmatrix} 1 \\ u \end{bmatrix}E_k\begin{bmatrix} 1 \\ u \end{bmatrix} - \sum_{\ell=2}^{k-1} E_\ell E_{k+1-\ell}\begin{bmatrix} 1 \\ u \end{bmatrix}.$$

$$\mathcal{I}_{SU(2)} = -\frac{i\vartheta_4(\mathfrak{b})}{2\vartheta_4(2\mathfrak{b})}\left(E_1\begin{bmatrix} 1 \\ u \end{bmatrix} - E_1\begin{bmatrix} 1 \\ b^2 u \end{bmatrix}\right) + \frac{\eta(\tau)^3\vartheta_4(\mathfrak{b}+\mathfrak{u})^2}{2\vartheta_1(\mathfrak{u})\vartheta_1(2\mathfrak{b}+\mathfrak{u})\vartheta_4(\mathfrak{b})}.$$

$$\mathcal{I}_{SU(2)} = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_4(2\mathfrak{b})}E_1\begin{bmatrix} -1 \\ b \end{bmatrix},$$

$$\mathcal{I}_{SU(3)} = \frac{\vartheta_4(\mathfrak{b})}{24\vartheta_4(3\mathfrak{b})}\left(1 + 8E_1\begin{bmatrix} -1 \\ b^3 u \end{bmatrix}E_1\begin{bmatrix} 1 \\ u \end{bmatrix} - 8E_1\begin{bmatrix} 1 \\ u \end{bmatrix}^2 + 8E_2\begin{bmatrix} -1 \\ b^3 u \end{bmatrix} - 8E_2\begin{bmatrix} 1 \\ u \end{bmatrix}\right)$$
$$+ \frac{\eta(\tau)^3\vartheta_4(\mathfrak{b}+\mathfrak{u})}{6\vartheta_1(\mathfrak{u})^2\vartheta_4(3\mathfrak{b}+\mathfrak{u})}\left(\frac{\eta(\tau)^3\vartheta_4(\mathfrak{b}+\mathfrak{u})^2}{\vartheta_4(\mathfrak{b})^2} - \frac{3i\vartheta_1(\mathfrak{u})\vartheta_1(2\mathfrak{b}+\mathfrak{u})}{\vartheta_1(2\mathfrak{b})}\left(E_1\begin{bmatrix} 1 \\ u \end{bmatrix} - E_1\begin{bmatrix} 1 \\ b^2 u \end{bmatrix}\right)\right)$$

$$\mathcal{I}_{SU(3)} = -\frac{1}{8}\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\left(-\frac{1}{3} + 4E_1\begin{bmatrix} -1 \\ b \end{bmatrix}^2 - 4E_2\begin{bmatrix} 1 \\ b^2 \end{bmatrix}\right).$$

$$\mathcal{I}_{SU(N)}(b,q)=\begin{cases}\dfrac{\vartheta_4(\mathfrak{b}\mid\tau)}{\vartheta_4(N\mathfrak{b}\mid\tau)}\mathbb{E}_N(b,q), & N=2\ell+1\\ \dfrac{i\vartheta_4(\mathfrak{b}\mid\tau)}{\vartheta_1(N\mathfrak{b}\mid\tau)}\mathbb{E}_N(b,q), & N=2\ell\end{cases},$$

$$\mathcal{I}_L^{(A_1,A_2)}=q^{-\frac{1}{2}}\mathrm{ch}_0-q^{-\frac{1}{2}}\mathrm{ch}_1$$

$$\mathcal{I}_{W_j}=\delta_{j\in\mathbb{Z}}\mathcal{I}_{SU(2)}-\frac{i\vartheta_4(\mathfrak{b})}{2\vartheta_4(2\mathfrak{b})}\sum_{\substack{m=-j\\ m\neq 0}}^{+j}\frac{b^m-b^{-m}}{q^{m/2}-q^{-m/2}}$$

$$\mathcal{I}_{W_j}(M,q)=\left(\delta_{j\in\mathbb{Z}}-\frac{1}{2}\mathcal{M}_{1j}-\frac{1}{2}\mathcal{M}_{2j}\right)\mathrm{ch}_0+\frac{1}{2}(\mathcal{M}_{1j}-\mathcal{M}_{2j})\mathrm{ch}_1+\frac{1}{2}(\mathcal{M}_{3j}-\mathcal{M}_{2j})\mathrm{ch}_2$$
$$+\frac{1}{2}(\mathcal{M}_{3j}+\mathcal{M}_{4j})\mathrm{ch}_3+\frac{1}{2}(\mathcal{M}_{3j}-\mathcal{M}_{4j})\mathrm{ch}_4$$

$$\mathcal{M}_{ij}:=\sum_{\substack{m=-j\\ m\neq 0}}^{+j}\frac{M_i^{2m}-M_i^{-2m}}{q^m-q^{-m}}$$

$$\left[D_q^{(N)}+\sum_{r=1}^{N}\phi_{2N-2r}D_q^{(r)}\right]\mathrm{ch}=0$$

$$D_q^{(k)}:=\partial_{(2k-2)}\circ\cdots\circ\partial(2)\circ\partial_{(0)}$$

$$\partial_{(k)}:=q\partial_q+kE_2(\tau).D_q^{(k)}$$

$$L_{-2}^{\mathfrak{n}}|0\rangle=|c_2\rangle+|\mathcal{N}_T\rangle,|c_2\rangle\in C_2(\mathbb{V}[\mathcal{T}])=\mathrm{span}\{a_{-h_a-1}b\},$$

$$\mathrm{eq}\xrightarrow{S}\tau^{\mathrm{wt}}\mathrm{eq}+\tau^{\mathrm{wt}-1}\mathfrak{b}_i\mathrm{eq}_i+\tau^{\mathrm{wt}-2}\mathfrak{b}_i\mathfrak{b}_j\mathrm{eq}_{ij}+\cdots$$

$$|\mathcal{N}\rangle\xrightarrow{S}\sum_{\ell\geq 0}\sum_{i_1,\ldots,i_\ell}\frac{1}{\ell!}\tau^{h[\mathcal{N}]-\ell}\mathfrak{b}_{i_1}\cdots\mathfrak{b}_{i_\ell}h_1^{i_1}\cdots h_1^{i_\ell}|\mathcal{N}\rangle$$

$$\sigma:\mathfrak{b}_i\to\mathfrak{b}_i+n_i\tau$$

$$\mathrm{eq}\xrightarrow{\sigma}\sum_{\ell\geq 0}\sum_{i_1\leq\cdots i_\ell}\frac{(-1)^\ell}{\ell!}n_{i_1}\cdots n_{i_\ell}\mathrm{eq}_{i_1\cdots i_\ell}.$$

$$|\mathcal{N}\rangle\xrightarrow{\sigma}\sum_{\ell\geq 0}\sum_{i_1\leq\cdots\leq i_\ell}\frac{(-1)^\ell}{\ell!}n_{i_1}\ldots n_{i_\ell}h_1^{i_1}\cdots h_1^{i_\ell}|\mathcal{N}\rangle$$

$$\left[D_q^{(N)}+\sum_{r=1}^{N}\phi_{2N-2r}(\tau)D_q^{(r)}\right]\mathrm{ch}=0$$

$$N\geq\mathfrak{n}$$

$$n_0\leq n_{\min}$$

$$n_0,n_{\mathrm{ord}}\leq n_{\min},n_{\min},\mathfrak{n}\leq N$$

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q)=q^{-\frac{c_{2\mathrm{d}}[\mathcal{T}_{p,N}]-pc_{2\,\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}}\mathcal{I}_{SU(N)}\left(b\to q^{\frac{p}{2}-1},q\to q^p\right).$$

$$\mathcal{I}_{SU(N)}(b,q)=\begin{cases}\frac{\vartheta_4(\mathfrak{b}\mid\tau)}{\vartheta_4(N\mathfrak{b}\mid\tau)}\mathbb{E}_N(b,q), N=2\ell+1\\ \frac{\vartheta_4(\mathfrak{b}\mid\tau)}{\vartheta_1(N\mathfrak{b}\mid\tau)}\mathbb{E}_N(b,q), N=2\ell\end{cases},$$

$$f_N(\mathfrak{b},\tau):=\begin{cases}\frac{\vartheta_4(\mathfrak{b}\mid\tau)}{\vartheta_4(N\mathfrak{b}\mid\tau)}, & N=2\ell+1,\\ \frac{\vartheta_4(\mathfrak{b}\mid\tau)}{i^{N^2-1}\vartheta_1(N\mathfrak{b}\mid\tau)}, & N=2\ell\end{cases}.$$

$$f_N(\mathfrak{b},\tau)\overset{\mathfrak{b}\to\left(\frac{p}{2}-1\right)\tau,\tau\to p\tau}{\to}\frac{\vartheta_4(0\mid 2\tau)}{\vartheta_4(0\mid 2\tau)}=1.$$

$$\text{odd } N:\frac{\vartheta_4(\mathfrak{b}\mid\tau)}{\vartheta_4(N\mathfrak{b}\mid\tau)}\overset{\mathfrak{b}\to\left(\frac{p}{2}-1\right)\tau,\tau\to 3\tau}{\to}\frac{\vartheta_4\left(\frac{1}{2}\tau\middle|\,3\tau\right)}{\vartheta_4\left(N\frac{1}{2}\tau\middle|\,3\tau\right)},$$

$$\text{even } N:\frac{\vartheta_4(\mathfrak{b}\mid\tau)}{\vartheta_1(N\mathfrak{b}\mid\tau)}\overset{\mathfrak{b}\to\left(\frac{p}{2}-1\right)\tau,\tau\to 3\tau}{\to}\frac{\vartheta_4\left(\frac{1}{2}\tau\middle|\,3\tau\right)}{\vartheta_1\left(N\frac{1}{2}\tau\middle|\,3\tau\right)}.$$

$$\vartheta_4\left(\mathfrak{z}+N\frac{\tau}{2}\middle|\,\tau\right)=-(-1)^{\frac{N}{2}-1}b^{-\frac{N}{2}}q^{-\frac{N^2}{8}}\vartheta_1(\mathfrak{z}\mid\tau),$$

$$f_N(\mathfrak{b},\tau)\overset{\mathfrak{b}\to\left(\frac{p}{2}-1\right)\tau,\tau\to 3\tau}{\to}-(-1)^{\left\lfloor\frac{n_N-2}{4}\right\rfloor}q^{\frac{N^2-1}{24}}$$

$$f_N(\mathfrak{b},\tau)\overset{\mathfrak{b}\to\left(\frac{p}{2}-1\right)\tau,\tau\to p\tau}{\to}-(-1)^{\left\lfloor\frac{n_N-3}{2}\right\rfloor}q^{\frac{N^2-1}{8}},$$

$$f_N(\mathfrak{b},\tau)\overset{\mathfrak{b}\to\left(\frac{p}{2}-1\right)\tau,\tau\to p\tau}{\to}q^{\frac{N^2-1}{3}}$$

$$f_N\left(\left(\frac{p}{2}-1\right)\tau,p\tau\right)=q^{\frac{c_{2\mathrm{d}}[\mathcal{T}_{p,N}]-pc_{2\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}}.$$

$$\mathcal{I}_{\mathcal{T}_{p,N}}=\mathbb{E}_N\left(b\to q^{\left(\frac{p}{2}-1\right)},q\to q^p\right)$$

$$\mathcal{I}_{3,2}=E_1\begin{bmatrix}-1\\ \sqrt{q}\end{bmatrix}(3\tau)$$

$$\mathcal{I}_{3,4}=-\frac{1}{72}-\frac{1}{6}E_2(3\tau)+\frac{1}{24}E_1\begin{bmatrix}-1\\ \sqrt{q}\end{bmatrix}(3\tau)$$
$$-\frac{1}{6}E_1\begin{bmatrix}-1\\ \sqrt{q}\end{bmatrix}(3\tau)^3-\frac{1}{2}E_1\begin{bmatrix}-1\\ \sqrt{q}\end{bmatrix}(3\tau)E_2\begin{bmatrix}-1\\ q\end{bmatrix}(3\tau)$$

$$\mathcal{I}_{2,3}=\frac{1}{24}+\frac{1}{2}E_2(2\tau),\mathcal{I}_{2,5}=\frac{3}{640}+\frac{1}{16}E_2(2\tau)+\frac{1}{8}E_2(2\tau)^2-\frac{1}{4}E_4(2\tau)$$

$$\mathcal{I}_{2,7}=\frac{5}{7168}+\frac{37}{3840}E_2(2\tau)+\frac{5}{192}E_2(2\tau)^2+\frac{1}{48}E_2(2\tau)^3$$
$$-\frac{5}{96}E_4(2\tau)-\frac{1}{8}E_2(2\tau)E_4(2\tau)+\frac{1}{6}E_6(2\tau)$$

$$
\begin{aligned}
\mathcal{I}_{4,3} =& -\frac{1}{24} + \frac{1}{2} E_1 \begin{bmatrix} -1 \\ q \end{bmatrix} (4\tau)^2 - \frac{1}{2} E_2 \begin{bmatrix} 1 \\ q^2 \end{bmatrix} (4\tau) \\
\mathcal{I}_{6,5} =& \frac{49}{7680} + \frac{1}{72} E_1 \begin{bmatrix} -1 \\ q^2 \end{bmatrix} (6\tau) + \frac{1}{192} E_1 \begin{bmatrix} -1 \\ q \end{bmatrix} (6\tau) \\
& - \frac{1}{96} E_1 \begin{bmatrix} -1 \\ q^2 \end{bmatrix} (6\tau)^2 - \frac{1}{32} E_1 \begin{bmatrix} -1 \\ q \end{bmatrix} (6\tau)^2 + \frac{1}{12} E_2 \begin{bmatrix} 1 \\ q^2 \end{bmatrix} (6\tau) + \frac{1}{96} E_2 \begin{bmatrix} -1 \\ q \end{bmatrix} (6\tau) \\
& - \frac{1}{8} E_1 \begin{bmatrix} -1 \\ q \end{bmatrix} (6\tau) E_2 \begin{bmatrix} -1 \\ q \end{bmatrix} (6\tau) + \frac{1}{8} E_1 \begin{bmatrix} -1 \\ q \end{bmatrix} (6\tau) E_1 \begin{bmatrix} -1 \\ q^2 \end{bmatrix} (6\tau)^2 \\
& - \frac{1}{3} E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} (6\tau) E_1 \begin{bmatrix} -1 \\ q^2 \end{bmatrix} (6\tau) + \frac{1}{4} E_2 \begin{bmatrix} -1 \\ q \end{bmatrix} (6\tau) E_1 \begin{bmatrix} -1 \\ q^2 \end{bmatrix} (6\tau)^2 \\
& + \frac{1}{4} E_3 \begin{bmatrix} 1 \\ q^2 \end{bmatrix} (6\tau) + \frac{1}{4} E_4 \begin{bmatrix} 1 \\ q^2 \end{bmatrix} (6\tau) - \frac{1}{8} E_2 \begin{bmatrix} -1 \\ q \end{bmatrix} (6\tau)^2 - \frac{1}{24} E_1 \begin{bmatrix} -1 \\ q^2 \end{bmatrix} (6\tau)^4
\end{aligned}
$$

$$
\mathcal{I}_{\mathcal{T}_{2,N}}(q) = q^{-\frac{c_{2\,\mathrm{d}}[\mathcal{T}_{2,N}] - 2c_{2\,\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}} I^{\mathcal{N}=4}_{SU(N)}(b = 1, q^2)
$$

$$
c_{2\,\mathrm{d}}[\mathcal{T}_{2,N}] = -12 \times \frac{N^2 - 1}{2}, c_{2\,\mathrm{d}}[\mathcal{T}_{SU(N)}] = -12 \times \frac{N^2 - 1}{4}.
$$

$$
q^{-\frac{c_{2\,\mathrm{d}}[\mathcal{T}_{2,N}] - 2c_{2\,\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}} = 1
$$

$$
\mathcal{I}_{SU(2\ell+1)}(q) = (-1)^\ell \sum_{k=0}^{\ell} \frac{\tilde{\lambda}^{(2\ell+3)}_{2k+2}(2)}{\max(2k, 1)} \widetilde{\mathbb{E}}_{2k}(\tau)
$$

$$
\widetilde{\mathbb{E}}_0(\tau) = 1, \widetilde{\mathbb{E}}_{2k} := 2k \oint \frac{dy}{2\pi i y} y^{-2k} \exp\left[-\sum_{n=1}^{\infty} \frac{1}{2n} E_{2n}(\tau) y^{2n} \right].
$$

$$
\mathcal{I}_{\mathcal{T}_{2,N}}(q) = (-1)^\ell \sum_{k=0}^{\ell} \frac{\tilde{\lambda}^{(2\ell+3)}_{2k+2}(2)}{\max(2k, 1)} \widetilde{\mathbb{E}}_{2k}(2\tau)
$$

$$
E_4(2\tau) \xrightarrow{S} E_4\left(-\frac{2}{\tau}\right) = \left(\frac{\tau}{2}\right)^4 E_4\left(\frac{\tau}{2}\right), E_6(2\tau) \xrightarrow{S} E_6\left(-\frac{2}{\tau}\right) = \left(\frac{\tau}{2}\right)^6 E_6\left(\frac{\tau}{2}\right).
$$

$$
\sum_{\pm} E_k \begin{bmatrix} \phi \\ \pm z \end{bmatrix} (\tau) = 2 E_k \begin{bmatrix} \phi \\ z^2 \end{bmatrix} (2\tau)
$$

$$
\mathcal{I}_{\mathcal{T}_{2,N}}(q) = (-1)^\ell \sum_{k=0}^{\ell} \frac{\tilde{\lambda}^{(2\ell+3)}_{2k+2}(2)}{\max(2k, 1)} \widetilde{\mathbb{E}}_{2k}(\tau) \Bigg|_{E_k(2\tau) \to \frac{1}{2}\left(E_k\left[\begin{smallmatrix} 1 \\ +1 \end{smallmatrix}\right] + E_k\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]\right)}
$$

$$
\mathcal{I}_{\mathcal{T}_{2,3}}(q) = \frac{1}{24} + \frac{1}{4}\left(E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)
$$

$$
\mathcal{I}_{\mathcal{T}_{2,3}}(q) \xrightarrow{S} S\mathcal{I}_{\mathcal{T}_{2,3}} = \frac{1}{24} + \frac{1}{4}\tau^2 \left(E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + \frac{i\tau}{8\pi}.
$$

$$
T^n S\mathcal{I}_{\mathcal{T}_{2,3}}(q) = \frac{1}{24} + \frac{1}{4}(\tau + n)^2 \left(E_2 \begin{bmatrix} -1 \\ (-1)^n \end{bmatrix} + E_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix}\right) + \frac{(\tau + n)i}{8\pi}
$$

$$
\begin{aligned}
\mathrm{ch}_0 &:= \mathcal{I}_{\mathcal{T}_{2,3}} \\
\mathrm{ch}_1, \mathrm{ch}_2, \cdots, \mathrm{ch}_6 &:= S\mathrm{ch}_0, T\mathrm{ch}_1, \mathbf{T}^{(2)}\mathrm{ch}_2, T\mathrm{ch}_3, \mathbf{T}^{(2)}\mathrm{ch}_4, T\mathrm{ch}_5 \\
\mathrm{ch}_7, \mathrm{ch}_8, \mathrm{ch}_9 &:= S\mathrm{ch}_5, \mathbf{T}^{(1)}\mathrm{ch}_7, \mathbf{T}^{(1)}\mathrm{ch}_8
\end{aligned}
$$

$$
\tau^{\ell=2,1,0}\left(E_2 \begin{bmatrix} -1 \\ \pm 1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)
$$

$$
\tau^{\ell=0,1,2}\left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)
$$

$$\left(E_2\begin{bmatrix}1\\1\end{bmatrix}+E_2\begin{bmatrix}1\\-1\end{bmatrix}\right)\overset{SL(2,\mathbb{Z})}{\rightarrow}\tau^{\ell=0,1,2}\left(E_2\begin{bmatrix}-1\\1\end{bmatrix}+E_2\begin{bmatrix}+1\\+1\end{bmatrix}\right)$$

$$\tau^{\ell=0,1,2}\left(E_2\begin{bmatrix}1\\-1\end{bmatrix}+E_2\begin{bmatrix}+1\\+1\end{bmatrix}\right),\tau^{\ell=0,1,2}\left(E_2\begin{bmatrix}-1\\-1\end{bmatrix}+E_2\begin{bmatrix}+1\\+1\end{bmatrix}\right)$$

$$\dim\mathcal{V}_{\mathcal{J}_{2,3}}=1+3\times 3=10$$

$$\begin{aligned}0=[&D_q^{(12)}-1510E_4D_q^{(10)}-55440E_6D_q^{(9)}-233400E_4^2D_q^{(8)}+2364600E_4E_6D_q^{(7)}\\&+2000(31228E_4^3-41013E_6^2)D_q^{(6)}+1422624000E_4^2E_6D_q^{(5)}\\&+(3925360000E_4^4+40438916000E_4E_6^2)D_q^{(4)}\\&+(420470400000E_4^3E_6+344509200000E_6^3)D_q^{(3)}\\&+(1168824000000E_4^5+7510426000000E_4^2E_6^2)D_q^{(2)}\\&+(23905224000000E_4^4E_6+31682361200000E_4E_6^3)D_q^{(1)}]\,\mathrm{ch}_0\end{aligned}$$

$$\begin{aligned}0=[&\left(\frac{55}{196}E_4^4+E_4E_6^2\right)D_q^{(10)}-\frac{405}{14}E_4^3E_6D_q^{(9)}+\left(-\frac{11775}{49}E_4^5-525E_4^2E_6^2\right)D_q^{(8)}\\&\left(\frac{94125}{7}E_4^4E_6-29400E_4E_6^3\right)D_q^{(7)}+\left(-\frac{2364000}{49}E_4^6+126300E_4^3E_6^2\right)D_q^{(6)}\\&+\left(\frac{14796000}{7}E_4^5E_6-4498200E_4^2E_6^3\right)D_q^{(5)}\\&+\left(\frac{19430000}{7}E_4^7+5253500E_4^4E_6^2-5586000E_4E_6^4\right)D_q^{(4)}\\&+\left(-\frac{1033500000}{7}E_4^6E_6+639555000E_4^3E_6^3\right)D_q^{(3)}\\&\left(\frac{10959000000}{49}E_4^8+575450000E_4^5E_6^2+2830730000E_4^2E_6^4\right)D_q^{(2)}\\&\left(\frac{1683000000}{7}E_4^7E_6+14264950000E_4^2E_6^3+14543200000E_4E_6^5\right)D_q^{(1)}]\,\mathrm{ch}_0\end{aligned}$$

$$S=\begin{pmatrix}0&1&0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&0&1&0&-1&0&1&0&0&0\\0&0&0&0&0&0&0&\frac{1}{2}&-\frac{1}{4}&\frac{1}{8}\\-8&0&8&0&-5&0&4&0&0&\frac{1}{2}\\0&0&0&0&0&0&0&1&0&0\\-8&0&8&0&-4&0&3&0&0&\frac{1}{2}\\0&0&0&0&0&1&0&0&0&0\\-8&8&0&-4&0&2&0&0&0&\frac{1}{2}\\-16&16&0&0&0&0&0&0&0&1\end{pmatrix},T=\begin{pmatrix}+1&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&1&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&1&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&+1&1&0\\0&0&0&0&0&0&0&0&+1&1\\0&0&0&0&0&0&0&0&0&+1\end{pmatrix}.$$

$$\begin{pmatrix}-1&1&0&0&0&0&0&0&0&0\\0&-1&1&0&0&0&0&0&0&0\\0&0&-1&0&0&0&0&0&0&0\\0&0&0&+1&1&0&0&0&0&0\\0&0&0&0&+1&1&0&0&0&0\\0&0&0&0&0&+1&0&0&0&0\\0&0&0&0&0&0&+1&1&0&0\\0&0&0&0&0&0&0&+1&1&0\\0&0&0&0&0&0&0&0&+1&0\\0&0&0&0&0&0&0&0&0&+1\end{pmatrix}.$$

$$1,E_2\begin{bmatrix}1\\-1\end{bmatrix}+E_2\begin{bmatrix}1\\+1\end{bmatrix},E_2\begin{bmatrix}-1\\-1\end{bmatrix}+E_2\begin{bmatrix}1\\+1\end{bmatrix},E_2\begin{bmatrix}-1\\1\end{bmatrix}+E_2\begin{bmatrix}1\\+1\end{bmatrix}.$$

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_{2,5}}=\frac{1}{640}\Bigg[&3+20\left(E_2\begin{bmatrix}1\\-1\end{bmatrix}+E_2\begin{bmatrix}1\\1\end{bmatrix}\right)\\&+20\left(E_2\begin{bmatrix}1\\-1\end{bmatrix}+E_2\begin{bmatrix}1\\1\end{bmatrix}\right)^2-80\left(E_4\begin{bmatrix}1\\-1\end{bmatrix}+E_4\begin{bmatrix}1\\1\end{bmatrix}\right)\Bigg]\end{aligned}$$

$$\tau^{\ell=0,1,2}\left(E_2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}+E_2\begin{bmatrix}1\\ 1\end{bmatrix}\right)\Big|_{(\alpha,\beta)=(1,-1),(-1,1),(-1,-1)}$$

$$\tau^{\ell=0,1,2,3,4}\left[\left(E_2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}+E_2\begin{bmatrix}1\\ 1\end{bmatrix}\right)^2-4\left(E_4\begin{bmatrix}\alpha\\ \beta\end{bmatrix}+E_4\begin{bmatrix}1\\ 1\end{bmatrix}\right)\right]\Big|_{(\alpha,\beta)=(1,-1),(-1,1),(-1,-1)}$$

$$\dim\mathcal{V}_{\mathcal{T}_{2,5}}=1+3\times 3+3\times 5=25.$$

$$\left[D_q^{(37)}+\lambda E_6 D_q^{(34)}+\cdots\right]\mathrm{ch}_0=0,$$

$$\text{numerator}=\begin{matrix}4212824029524904864459922873381906854609227513005531 5\\ 14832418220679017\ 4904798856261657911355047603171139 1\\ 00761626495031848078719518242427794\ 2115734662906185 2\\ 98984911599590791101078722178945684374696685810515210\\ 77474520\end{matrix}$$

$$\text{denominator}=\begin{matrix}19874372165347083106555613490441529620484463842934023\\ 93696952040388320\ 815206246541291181713446874805751 65\\ 86548448495321188586890643377882808\ 26936186828337706\\ 08875281132559897910052369384144317652428941586215542\\ 7\end{matrix}.$$

$$\begin{aligned}&\Big[733377661001283792156623707366556995116638016839211654400000000000000000000E_4^{51}\\ &+282690869316323266546364828331938074747341836739646939747840000000000000000000E_6^{2}E_4^{48}\\ &-788488239387148139158480815011365352662900146444998542130714400000000000000000E_6^{4}E_4^{45}\\ &+936477446565668332715258488848552171843527574494029013614332140000000000000000E_6^{6}E_4^{42}\\ &-562499953760026435445686538976968475851541622284038846026006838800000000000000E_6^{8}E_4^{39}\\ &+123776120446998765213233804570322354284093941230721564683716185710000000000000E_6^{10}E_4^{36}\\ &+600009003146445179966887183561518443971316402509840243143940651386850000000000E_6^{12}E_4^{33}\\ &-605014926028901047282071626645242548514545139282742751057617540681399750000000E_6^{14}E_4^{30}\\ &+253574347551634548911690389512972372549748230485621663095757975176425333750000000E_6^{16}E_4^{27}\\ &-674937213267060398825325560565542490748053015409023766724262837052317705125000000E_6^{18}E_4^{24}\\ &+127122234706541961239228622965045432517328562022811222394567120380632122293468750 0E_6^{20}E_4^{21}\\ &-179641068575147139128864962869074888391491458424686327996657270895772140872270312 5E_6^{22}E_4^{18}\\ &+197119705386085132104453742357954144065717113627093009241920240521777636713634062 5E_6^{24}E_4^{15}\\ &-165550111901841151988749880505973626215752161060815758568909480574954223161439375 0E_6^{26}E_4^{12}\\ &+961307884181455524086753029316646694951039128891591369755961770416467062921587500E_6^{28}E_4^{9}\\ &-302702480531330081358260891093741687255683069618196709811824107244611560172418800E_6^{30}E_4^{6}\\ &+230136951225738911505338258390377062289447427219410925395313080519516428568583 20E_6^{32}E_4^{3}\\ &+604669704120972379283681439557999913127508129846543569232279348643581825075328E_6^{34}\Big]D_q^{(25)}+\cdots\end{aligned}$$

$$\begin{aligned}
&\mathrm{ch}_1 = S\mathrm{ch}_1\\
&\mathrm{ch}_{2\ell}, \mathrm{ch}_{2\ell+1} = T\mathrm{ch}_{2\ell-1}, (T^2-1)\mathrm{ch}_{2\ell-1}, \ell = 1,\cdots,4\\
&\mathrm{ch}_{10} = T\mathrm{ch}_9\\
&\mathrm{ch}_{11} = S\mathrm{ch}_9\\
&\mathrm{ch}_\ell = (T-1)\mathrm{ch}_{\ell-1}, \ell = 12,\cdots,15\\
&\mathrm{ch}_{16} = S\mathrm{ch}_{15}\\
&\mathrm{ch}_{2\ell+1}, \mathrm{ch}_{2\ell+2} = T\mathrm{ch}_{2\ell}, (T^2-1)\mathrm{ch}_{2\ell}, \ell = 8,\cdots,10\\
&\mathrm{ch}_{21} = T\mathrm{ch}_{20}\\
&\mathrm{ch}_{22} = S\mathrm{ch}_{20}\\
&\mathrm{ch}_{23}, \mathrm{ch}_{24} = (T-1)\mathrm{ch}_{22}, (T-1)\mathrm{ch}_{23}
\end{aligned}$$

$$1,\ \left(E_2\begin{bmatrix}\alpha\\ \beta\end{bmatrix} + E_2\begin{bmatrix}1\\ 1\end{bmatrix}\right)\Big|_{(\alpha,\beta)=(1,-1),(-1,1),(-1,-1)}$$

$$\left[\left(E_2\begin{bmatrix}\alpha\\ \beta\end{bmatrix} + E_2\begin{bmatrix}1\\ 1\end{bmatrix}\right)^2 - 4\left(E_4\begin{bmatrix}\alpha\\ \beta\end{bmatrix} + E_4\begin{bmatrix}1\\ 1\end{bmatrix}\right)\right]\Big|_{(\alpha,\beta)=(1,-1),(-1,1),(-1,-1)}$$

$$\begin{aligned}
&1,\ \tau^{0,1,\cdots,2k}\widetilde{\mathbb{E}}_{2k}(2\tau)\Big|_{E_k(2\tau)\to\frac{1}{2}\left(E_k\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right]+E_k\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]\right)}\\
&\tau^{0,1,\cdots,2k}\widetilde{\mathbb{E}}_{2k}(2\tau)\Big|_{E_k(2\tau)\to\frac{1}{2}\left(E_k\left[\begin{smallmatrix}-1\\ 1\end{smallmatrix}\right]+E_k\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]\right)}\\
&\tau^{0,1,\cdots,2k}\widetilde{\mathbb{E}}_{2k}(2\tau)\Big|_{E_k(2\tau)\to\frac{1}{2}\left(E_k\left[\begin{smallmatrix}-1\\ -1\end{smallmatrix}\right]+E_k\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]\right)}
\end{aligned}$$

$$1 + 3\times 3 + 3\times 5 + 3\times 7 + \cdots + 3\times(2\ell+1) = 1 + 3\ell(2+\ell)$$

$$L_{-2}^{\mathfrak{n}}|0\rangle = |c_2\rangle + |\mathcal{N}_T\rangle, |c_2\rangle \in C_2(\mathbb{V}[\mathcal{T}]) := \mathrm{span}\{a_{-h_a-1}b\}$$

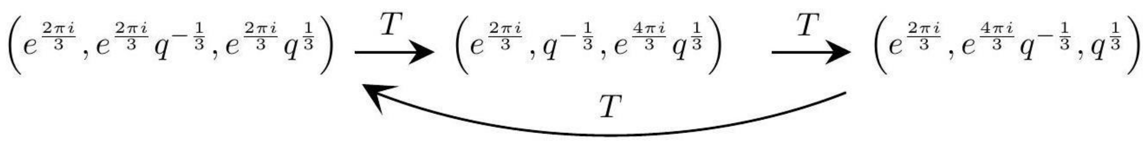
$$\mathfrak{n} - 1 \geq \mathrm{rank}\mathcal{T}$$

$$n_0 - 1 = 6\ell + 3\ell^2 > \mathrm{rank}\mathcal{T}_{2,2\ell+1} = 6\ell$$

$$E_k\begin{bmatrix}-1\\ b^p\end{bmatrix}(p\tau) = \frac{1}{p}\sum_{p=0}^{l-1} E_k\begin{bmatrix}-1\\ e^{2\pi il/p}b\end{bmatrix}, E_k\begin{bmatrix}1\\ b^p\end{bmatrix}(p\tau) = \frac{1}{p}\sum_{p=0}^{l-1} E_k\begin{bmatrix}1\\ e^{2\pi il/p}b\end{bmatrix},$$

$$\mathcal{I}_{3,2} = \frac{1}{3}\left(E_1\begin{bmatrix}-1\\ q^{1/6}\end{bmatrix} + E_1\begin{bmatrix}-1\\ e^{2\pi i/3}q^{1/6}\end{bmatrix} + E_1\begin{bmatrix}-1\\ e^{4\pi i/3}q^{1/6}\end{bmatrix}\right)$$

$$E_1\begin{bmatrix}-1\\ aq^{1/6}\end{bmatrix} \to E_1\begin{bmatrix}-1\\ e^{4\pi i/3}aq^{1/6}\end{bmatrix}$$



$$\begin{aligned}
\mathrm{ch}_1 = S\mathrm{ch}_0 =& -\frac{1}{6} + \frac{\tau}{3}\left(E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}\end{bmatrix} + E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}q^{-1/3}\end{bmatrix} + E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}q^{1/3}\end{bmatrix}\right)\\
\mathrm{ch}_2 = T\mathrm{ch}_1 =& -\frac{1}{6} + \frac{1}{3}\left(E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}\end{bmatrix} + E_1\begin{bmatrix}1\\ q^{-1/3}\end{bmatrix} + E_1\begin{bmatrix}1\\ e^{\frac{4\pi i}{3}}q^{1/3}\end{bmatrix}\right)\\
&+ \frac{\tau}{3}\left(E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}\end{bmatrix} + E_1\begin{bmatrix}1\\ q^{-1/3}\end{bmatrix} + E_1\begin{bmatrix}1\\ e^{\frac{4\pi i}{3}}q^{1/3}\end{bmatrix}\right)\\
\mathrm{ch}_3 = T\mathrm{ch}_2 =& -\frac{1}{6} + \frac{2}{3}\left(E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}\end{bmatrix} + E_1\begin{bmatrix}1\\ e^{\frac{4\pi i}{3}}q^{-1/3}\end{bmatrix} + E_1\begin{bmatrix}1\\ q^{1/3}\end{bmatrix}\right)\\
&+ \frac{\tau}{3}\left(E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}\end{bmatrix} + E_1\begin{bmatrix}1\\ e^{\frac{4\pi i}{3}}q^{-1/3}\end{bmatrix} + E_1\begin{bmatrix}1\\ q^{1/3}\end{bmatrix}\right)\\
\mathrm{ch}_4 = T\mathrm{ch}_3 - \mathrm{ch}_1 =& E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}\end{bmatrix} + E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}q^{-1/3}\end{bmatrix} + E_1\begin{bmatrix}1\\ e^{\frac{2\pi i}{3}}q^{1/3}\end{bmatrix}
\end{aligned}$$

$$T=\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&1&0&0&1\\0&3e^{\frac{\pi i}{3}}&-3&3e^{-\frac{\pi i}{3}}&2e^{\frac{\pi i}{3}}\end{pmatrix}$$

$$S=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&\frac{3}{2}(1-i\sqrt{3})&\frac{3}{2}(1+i\sqrt{3})&-2&\frac{1}{2}(1-i\sqrt{3})\\3&\frac{3}{2}(3+i\sqrt{3})&-\frac{13}{2}+\frac{3i\sqrt{3}}{2}&-3i\sqrt{3}&\frac{1}{2}(5+i\sqrt{3})\\\frac{3(\sqrt{3}+3i)}{\sqrt{3}-i}&-\frac{3(\sqrt{3}-3i)}{\sqrt{3}-i}&-\frac{6(\sqrt{3}+2i)}{\sqrt{3}-i}&6&\frac{\sqrt{3}+5i}{\sqrt{3}-i}\end{pmatrix}.$$

$$S\sim\begin{pmatrix}-1&0&0&0&0\\0&-1&0&0&0\\0&0&+1&0&0\\0&0&0&+1&0\\0&0&0&0&+1\end{pmatrix},T\sim\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&e^{\frac{2i\pi}{3}}&1\\0&0&0&0&e^{\frac{2i\pi}{3}}\end{pmatrix}.$$

$$\mathrm{ch}_0,\mathrm{ch}_1+e^{\frac{2\pi i}{3}}\mathrm{ch}_2-e^{\frac{\pi i}{3}}\mathrm{ch}_3,\mathrm{ch}_4$$

$$1=-6\mathrm{ch}_0-6\left(\mathrm{ch}_1+e^{\frac{2\pi i}{3}}\mathrm{ch}_2-e^{\frac{\pi i}{3}}\mathrm{ch}_3\right)+2\sqrt{3}e^{\frac{5\pi i}{6}}\mathrm{ch}_4$$

$$0=E_1\begin{bmatrix}1\\q^{1/3}\end{bmatrix}-e^{\frac{\pi i}{3}}E_1\begin{bmatrix}1\\e^{-\frac{2\pi i}{3}}q^{1/3}\end{bmatrix}+e^{\frac{2\pi i}{3}}E_1\begin{bmatrix}1\\e^{\frac{2\pi i}{3}}q^{1/3}\end{bmatrix},$$

$$\begin{aligned}1=&-2E_1\begin{bmatrix}-1\\q^{1/6}\end{bmatrix}-2E_1\begin{bmatrix}-1\\e^{-\frac{2\pi i}{3}}q^{1/6}\end{bmatrix}-2E_1\begin{bmatrix}-1\\e^{\frac{2\pi i}{3}}q^{1/6}\end{bmatrix}+2i\sqrt{3}E_1\begin{bmatrix}1\\e^{\frac{2\pi i}{3}}\end{bmatrix}\\&+(1+3\sqrt{3}i)E_1\begin{bmatrix}1\\q^{1/3}\end{bmatrix}+(4-2i\sqrt{3})E_1\begin{bmatrix}1\\e^{-\frac{2\pi i}{3}}q^{1/3}\end{bmatrix}+(-5-i\sqrt{3})E_1\begin{bmatrix}1\\e^{\frac{2\pi i}{3}}q^{1/3}\end{bmatrix}\end{aligned}$$

$$n_0-1=\mathfrak{n}-1=4=\mathrm{rank}\mathcal{T}_{3,2}$$

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_{3,4}}(q)=&-\frac{1}{72}+\frac{1}{72}\left(E_1\begin{bmatrix}-1\\q^{1/6}\end{bmatrix}+E_1\begin{bmatrix}-1\\e^{-\frac{2\pi i}{3}}q^{1/6}\end{bmatrix}+E_1\begin{bmatrix}-1\\e^{\frac{2\pi i}{3}}q^{1/6}\end{bmatrix}\right)\\&+\frac{1}{6}\left(-\frac{1}{3}E_2\begin{bmatrix}1\\1\end{bmatrix}-\frac{1}{3}E_2\begin{bmatrix}1\\e^{-\frac{2\pi i}{3}}\end{bmatrix}-\frac{1}{3}E_2\begin{bmatrix}1\\e^{\frac{2\pi i}{3}}\end{bmatrix}\right)\\&+\frac{1}{18}\left(E_1\begin{bmatrix}-1\\q^{1/6}\end{bmatrix}+E_1\begin{bmatrix}-1\\e^{-\frac{2\pi i}{3}}q^{1/6}\end{bmatrix}+E_1\begin{bmatrix}-1\\e^{\frac{2\pi i}{3}}q^{1/6}\end{bmatrix}\right)\\&\times\left(E_2\begin{bmatrix}1\\q^{1/3}\end{bmatrix}+E_2\begin{bmatrix}1\\e^{-\frac{2\pi i}{3}}q^{1/3}\end{bmatrix}+E_2\begin{bmatrix}1\\e^{\frac{2\pi i}{3}}q^{1/3}\end{bmatrix}\right)\\&-\frac{1}{162}\left(E_1\begin{bmatrix}-1\\q^{1/6}\end{bmatrix}+E_1\begin{bmatrix}-1\\e^{-\frac{2\pi i}{3}}q^{1/6}\end{bmatrix}+E_1\begin{bmatrix}-1\\e^{\frac{2\pi i}{3}}q^{1/6}\end{bmatrix}\right)^3\end{aligned}$$

$$\frac{-5}{324}+\frac{1}{108}\tau(\cdots)+\frac{1}{108}\tau^2(\cdots)+\frac{1}{162}\tau^3(\cdots).$$

$$\begin{aligned}\mathrm{ch}_1,\mathrm{ch}_2,\mathrm{ch}_3&=S\mathrm{ch}_0,T\mathrm{ch}_1,T^2\mathrm{ch}_2\\\mathrm{ch}_4,\mathrm{ch}_5,\mathrm{ch}_6&=T\mathrm{ch}_3-\mathrm{ch}_1,T\mathrm{ch}_4,T^2\mathrm{ch}_5\\\mathrm{ch}_7,\mathrm{ch}_8,\mathrm{ch}_9&=T\mathrm{ch}_6-\mathrm{ch}_4,T\mathrm{ch}_7,T^2\mathrm{ch}_8\\\mathrm{ch}_{10},\mathrm{ch}_{11},\mathrm{ch}_{12}&=T\mathrm{ch}_9-\mathrm{ch}_7,T\mathrm{ch}_{10},T^2\mathrm{ch}_{11}\end{aligned}$$

$$T\mathrm{ch}_{12}=\mathrm{ch}_{10}$$

$$\mathrm{ch}_{13}=S\mathrm{ch}_{10}$$

$$\mathrm{ch}_{14},\mathrm{ch}_{15},\mathrm{ch}_{16}=(T-\mathrm{id})\mathrm{ch}_{13},(T-\mathrm{id})\mathrm{ch}_{14},(T-\mathrm{id})\mathrm{ch}_{15}+972\mathrm{ch}_0$$

$$\mathrm{ch}_{17},\mathrm{ch}_{18},\mathrm{ch}_{19} = S\mathrm{ch}_{16},T\mathrm{ch}_{17},T\mathrm{ch}_{18}$$
$$\mathrm{ch}_{20},\mathrm{ch}_{21},\mathrm{ch}_{22} = T\mathrm{ch}_{19}-\mathrm{ch}_{17},T\mathrm{ch}_{20},T\mathrm{ch}_{21}$$
$$\mathrm{ch}_{23},\mathrm{ch}_{24},\mathrm{ch}_{25} = T\mathrm{ch}_{22}-\mathrm{ch}_{20},T\mathrm{ch}_{23},T\mathrm{ch}_{24}$$
$$\mathrm{ch}_{26},\mathrm{ch}_{27},\mathrm{ch}_{28} = S\mathrm{ch}_{23},(T-\mathrm{id})\mathrm{ch}_{26},(T-\mathrm{id})\mathrm{ch}_{27}-36\mathrm{ch}_{16}$$
$$\mathrm{ch}_{29},\mathrm{ch}_{30},\mathrm{ch}_{31} = S\mathrm{ch}_{28},T\mathrm{ch}_{29},T\mathrm{ch}_{30}$$
$$\mathrm{ch}_{32} = T\mathrm{ch}_{31}-\mathrm{ch}_{29}$$

$$0=-3e^{\frac{i\pi}{3}}\mathrm{ch}_{29}+3\mathrm{ch}_{30}+3e^{\frac{i\pi}{6}}i\mathrm{ch}_{31}-2e^{\frac{i\pi}{3}}\mathrm{ch}_{32}+\mathrm{ch}_{33}$$

$$E_1\begin{bmatrix}1\\q^{1/3}\end{bmatrix}+e^{\frac{4\pi i}{3}}E_1\begin{bmatrix}1\\e^{\frac{4\pi i}{3}}q^{1/3}\end{bmatrix}+e^{\frac{2\pi i}{3}}E_1\begin{bmatrix}1\\e^{-\frac{2\pi i}{3}}q^{1/3}\end{bmatrix}=0$$

$$\mathrm{ch}_0,\mathrm{ch}_{10},\mathrm{ch}_{12},\mathrm{ch}_{13},\mathrm{ch}_{17},\mathrm{ch}_{24},\mathrm{ch}_{25},\mathrm{ch}_{26},\mathrm{ch}_{29}$$
$$\mathrm{ch}_{29}+e^{\frac{2\pi i}{3}}\mathrm{ch}_{30}-e^{\frac{\pi i}{3}}\mathrm{ch}_{31}$$

$$1=\frac{18}{5}\mathrm{ch}_{28}+\frac{18}{5}\mathrm{ch}_9+\frac{18e^{2\pi i/3}}{5}\mathrm{ch}_{30}-\frac{18}{5}e^{\pi i/3}\mathrm{ch}_{31}+\frac{6}{5}\sqrt{3}e^{-\pi i/6}\mathrm{ch}_{32}.$$

$$\dim\mathcal{V}_0-1=\mathfrak{n}-1=32>12=\mathrm{rank}\mathcal{T}_{3,4}$$

$$\mathcal{I}_{\mathcal{T}_{4,3}}=-\frac{1}{48}\left(-1+12E_1\begin{bmatrix}-1\\1\end{bmatrix}(4\tau)-24E_1\begin{bmatrix}-1\\q\end{bmatrix}(4\tau)^2+24E_2\begin{bmatrix}-1\\1\end{bmatrix}(4\tau)\right)$$

$$E_k\begin{bmatrix}-1\\b^p\end{bmatrix}(p\tau)=\frac{1}{p}\sum_{p=0}^{\ell-1}E_k\begin{bmatrix}-1\\e^{2\pi i\ell/p}b\end{bmatrix}$$

$$0=E_1\begin{bmatrix}1\\-q^{1/4}\end{bmatrix}-E_1\begin{bmatrix}1\\q^{1/4}\end{bmatrix}+iE_1\begin{bmatrix}1\\-iq^{1/4}\end{bmatrix}-iE_1\begin{bmatrix}1\\iq^{1/4}\end{bmatrix},$$
$$\begin{aligned}0=&E_2\begin{bmatrix}1\\-q^{1/4}\end{bmatrix}-E_2\begin{bmatrix}1\\q^{1/4}\end{bmatrix}-\frac{1}{4}E_1\begin{bmatrix}1\\-q^{1/4}\end{bmatrix}+\frac{1}{4}E_1\begin{bmatrix}1\\q^{1/4}\end{bmatrix}\\&+\frac{1}{2}E_1\begin{bmatrix}-1\\i\end{bmatrix}E_1\begin{bmatrix}1\\-iq^{1/4}\end{bmatrix}+\frac{1}{2}E_1\begin{bmatrix}1\\i\end{bmatrix}E_1\begin{bmatrix}1\\-iq^{1/4}\end{bmatrix}\\&-\frac{1}{2}E_1\begin{bmatrix}-1\\i\end{bmatrix}E_1\begin{bmatrix}1\\iq^{1/4}\end{bmatrix}-\frac{1}{2}E_1\begin{bmatrix}1\\i\end{bmatrix}E_1\begin{bmatrix}1\\iq^{1/4}\end{bmatrix}\end{aligned}$$

$$\begin{aligned}4(T^2S\mathrm{ch}_0-S\mathrm{ch}_0)|_{\tau^2}=&E_2\begin{bmatrix}1\\-q^{1/4}\end{bmatrix}-E_2\begin{bmatrix}1\\q^{1/4}\end{bmatrix}-\frac{1}{4}E_1\begin{bmatrix}1\\-q^{1/4}\end{bmatrix}+\frac{1}{4}E_1\begin{bmatrix}1\\q^{1/4}\end{bmatrix}\\&+\frac{1}{2}E_1\begin{bmatrix}-1\\i\end{bmatrix}E_1\begin{bmatrix}1\\-iq^{1/4}\end{bmatrix}+\frac{1}{2}E_1\begin{bmatrix}1\\i\end{bmatrix}E_1\begin{bmatrix}1\\-iq^{1/4}\end{bmatrix}\end{aligned}$$

$$-\frac{1}{2}E_1\begin{bmatrix}-1\\i\end{bmatrix}E_1\begin{bmatrix}1\\iq^{1/4}\end{bmatrix}-\frac{1}{2}E_1\begin{bmatrix}1\\i\end{bmatrix}E_1\begin{bmatrix}1\\iq^{1/4}\end{bmatrix}=0$$

$$\mathrm{ch}_1=S\mathrm{ch}_0$$
$$\mathrm{ch}_2,\mathrm{ch}_3,\mathrm{ch}_4,\mathrm{ch}_5=T\mathrm{ch}_1,T\mathrm{ch}_2-\mathrm{ch}_1,T^\ell\mathrm{ch}_3,\ell=1,2$$
$$\mathrm{ch}_6,\mathrm{ch}_7,\mathrm{ch}_8=T\mathrm{ch}_5+i\mathrm{ch}_3-\mathrm{ch}_4-i\mathrm{ch}_5,T^\ell\mathrm{ch}_6,\ell=1,2$$
$$\mathrm{ch}_9=S\mathrm{ch}_8$$
$$\mathrm{ch}_{10},\mathrm{ch}_{11},\mathrm{ch}_{12}=T\mathrm{ch}_9,T\mathrm{ch}_{10}-\mathrm{ch}_9,T\mathrm{ch}_{11}$$

$$
\begin{aligned}
0 = \Big[&(574745600E_4^9 - 1076920320E_6^2E_4^6 + 1096671156E_6^4E_4^3 + 1424635751E_6^6)E_6E_4D_q^{(13)} \\
&-432(124899200E_6^2E_4^9 - 122363780E_6^4E_4^6 + 276220301E_6^6E_4^3)D_q^{(12)} \\
&+\left(\frac{1}{7}(-9326814848000)E_6E_4^{11} + 4224114652800E_6^3E_4^8 - 2783340033180E_6^5E_4^5 - 1838877565469E_6^7E_4^2\right)D_q^{(11)} \\
&+8(7442481904000E_4^9 - 3342340125600E_6^2E_4^6 + 19895117706660E_6^4E_4^3 - 20234101571453E_6^6)E_6^2E_4D_q^{(10)} \\
&-\frac{1}{7}6600(107830112000E_4^9 - 464290614480E_6^2E_4^6 + 3764819611896E_6^4E_4^3 - 5674341339373E_6^6)E_6E_4^3D_q^{(9)} \\
&+\frac{900}{7}(12513100265600 0E_4^9 - 533992662885360E_6^2E_4^6 + 135615545950920E_6^4E_4^3 + 277277681006249E_6^6)E_6^2E_4^2D_q^{(8)} \\
&+\frac{25}{49}(643463494355968000E_4^{12} - 2731343006933587200E_6^2E_4^9 \\
&+3659914196088200400E_6^4E_4^6 - 4407695411247251480E_6^6E_4^3 + 287204123154042009 3E_6^8)E_6E_4D_q^{(7)} \\
&-21000(195522293753600E_4^9 - 227017724188080E_6^2E_4^6 - 287151556728792E_6^4E_4^3 - 652446141959227E_6^6)E_6^2E_4^3D_q^{(6)} \\
&+16000(672100320320000E_4^{12} + 1275146989916800E_6^2E_4^9 - 5549761806160320E_6^4E_4^6 \\
&-5608956608000116E_6^6E_4^3 + 4426262074119193E_6^8)E_6E_4^2D_q^{(5)} \\
&+1000(1862449854833920 00E_4^{12} - 15045107048527696 00E_6^2E_4^9 \\
&+\frac{40000}{7}(45809333636352000E_4^{12} - 106878284832457600E_6^2E_4^9 - 346593764375468400E_6^4E_4^6 \\
&+830194932722473920E_6^6E_4^3 + 903562009907172 77E_6^8)E_6E_4^3D_q^{(3)} \\
&+\frac{60000}{7}(1077666799527296000E_4^{12} - 4957511831704644800E_6^2E_4^9 \\
&\quad +6663480952508528400E_6^4E_4^6 - 4603907186041058980E_6^6E_4^3 + 699409291592300235 1E_6^8)E_6^2E_4^2D_q^{(2)} \\
&+\frac{10000}{49}(434229464543846400 00E_4^{15} - 225296315114048640000E_6^2E_4^{12} \\
&\quad +680500256154933166400E_6^4E_4^9 - 1372309846461167300400E_6^6E_4^6 \\
&\quad +1494610562792323885020E_6^8E_4^3 - 523092474786506400853E_6^{10})E_6E_4D_q^{(1)}\Big]\mathrm{ch}_0^{\mathcal{T}_{4,3}}
\end{aligned}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -2i & 4 & 6i & -6 & -6i & 4 & 2i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -32i & 53 & 96i & -110 & -96i & 61 & 32i & -4 & -1 & 0 & 2 & 0
\end{pmatrix}.
$$

$$
T \to \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1
\end{pmatrix},
$$

$\mathrm{ch}_0, \mathrm{ch}_6, \mathrm{ch}_7, \mathrm{ch}_8$

$\mathrm{ch}_{11} - 4\mathrm{ch}_3 - 8(1+i)\mathrm{ch}_4 + (4-8i)\mathrm{ch}_5, \mathrm{ch}_{12} - 4(2+i)\mathrm{ch}_3 - 8i\mathrm{ch}_4 - 4i\mathrm{ch}_5.$

$$n_0 - 1 = 12 > 9 = \mathrm{rank}\mathcal{T}_{4,3}$$

$$\Phi(z)^{z\to\infty}\left(T_k z^{\frac{k}{b}}+\cdots\right)dz$$

$$\Phi\left(e^{2\pi i}z\right)=g\Phi(z)g^{-1}\Rightarrow gT_kg^{-1}=e^{2\pi ik/b}T_k$$

$$\mathfrak{j}=\bigoplus_{\ell\in\mathbb{Z}/b\mathbb{Z}}\mathfrak{j}_{\ell/b}$$

$$\Phi(z)=\frac{\Phi'(z)}{z}$$

$$\Phi'^{z\to\infty}(T_kz^{\nu}+\cdots)dz.,\nu:=\frac{k}{b}+1>0$$

$$\Phi(z)^{z\to 0}\left(\frac{f^{\vee}}{z}+\cdots\right)dz$$

$$\Phi'\sim(\beta+\cdots)dz,\beta\in\mathfrak{n}^{\vee}$$

$$x\to\lambda^{\alpha}x,z\to\lambda^{\beta}z$$

$$Sp_{\gamma,\mathbf{P}^{\vee}}:=\{g\in\mathbf{P}^{\vee}\setminus\mathbf{J}^{\vee}\mid g\gamma g^{-1}\subset\tilde{\mathfrak{n}}^{\vee}\}$$

$$L_{\nu}=\left\{\alpha+l\delta\in\hat{\Delta}^{\vee}\mid\nu\alpha(\rho)+l=0\right\}$$

$$S_{\nu}=\left\{\alpha+l\delta\in\hat{\Delta}^{\vee}\mid\nu\alpha(\rho))+l=\nu\right\}$$

$$Sp^{T}_{\gamma,\mathbf{P}^{\vee}}=\sqcup H_{\tilde{w}},\left\{\tilde{w}\in W_{\mathbf{P}}\vee W_{aff}/W_{\nu}\mid\mathrm{Ad}(\tilde{w})\gamma\in\tilde{\mathfrak{n}}^{\vee}\right\}$$

$$\dim H_{\tilde{w}}=|\tilde{w}L_{\nu}\setminus\Delta_{\tilde{\mathfrak{n}}}\vee|-|\tilde{w}S_{\nu}\setminus\Delta_{\tilde{\mathfrak{n}}}\vee|.$$

$$\mathcal{T}_{3,2}=(A_2,D_4),\mathcal{T}_{4,3}=(A_3,E_6),\mathcal{T}_{6,5}=(A_5,E_8)$$

$$W_{-h^{\vee}+\frac{b}{b+k}}(\mathfrak{j},f),$$

$$\nu=\frac{k}{b}+1=\frac{3}{2}=\frac{u}{m},u=3,m=2$$

$$\Delta_{\tilde{\mathfrak{n}}^{\vee}}=\hat{\Delta}_{+}\setminus\Delta_{+}=\{\alpha+l\delta\mid l>0,\alpha\in\Delta\},$$

$$\gamma=\sum_{\alpha+l\delta\in S_{\nu},\alpha+l\delta\in\hat{\Delta}_{+}}e_{\alpha}z^{l}$$

$$L_{\nu}=\left\{\alpha+l\delta\in\hat{\Delta}^{\vee}\mid\nu\alpha(\rho)+l=0\right\},$$

$$S_{\nu}=\left\{\alpha+l\delta\in\hat{\Delta}^{\vee}\mid\nu\alpha(\rho)+l=\nu\right\}$$

$$\begin{aligned}L_{\nu}=\{&6\delta-\alpha_1-\alpha_2-\alpha_3-\alpha_4,3\delta-\alpha_2-\alpha_4,3\delta-\alpha_1-\alpha_2,3\delta-\alpha_2-\alpha_3\\&-3\delta+\alpha_1+\alpha_2,-3\delta+\alpha_2+\alpha_3,-6\delta+\alpha_1+\alpha_2+\alpha_3+\alpha_4,-3\delta+\alpha_2+\alpha_4\}\end{aligned}$$

$$=\{\pm(6\delta-e_1-e_3),\pm(3\delta-e_2-e_4),\pm(3\delta-e_1+e_3),\pm(3\delta-e_2+e_4)\}$$

$$S_{\nu}=\begin{Bmatrix}9\delta-\alpha_1-2\alpha_2-\alpha_3-\alpha_4,6\delta-\alpha_1-\alpha_2-\alpha_4,6\delta-\alpha_2-\alpha_3-\alpha_4,3\delta-\alpha_1,\\3\delta-\alpha_2,\alpha_1,3\delta-\alpha_3,-3\delta+\alpha_1+\alpha_2+\alpha_3,\alpha_3,-6\delta+\alpha_1+2\alpha_2+\alpha_3+\alpha_4,\\-3\delta+\alpha_2+\alpha_3+\alpha_4,\alpha_4\end{Bmatrix}$$

$$\left\{\tilde{w}\in W_{\mathfrak{d}_4^{\vee}}\setminus W_{\mathrm{aff}}/W_{\nu}\mid\mathrm{Ad}(\tilde{w})\gamma\in\alpha+l\delta,l>0\right\}$$

$$\begin{aligned}\hat{\beta}_1&=6\delta-\alpha_1-\alpha_2-\alpha_3-\alpha_4\\\hat{\beta}_2&=3\delta-\alpha_2-\alpha_4\\\hat{\beta}_3&=3\delta-\alpha_1-\alpha_2\\\hat{\beta}_4&=3\delta-\alpha_2-\alpha_3\end{aligned}$$

$$s_{\hat{\alpha}}\hat{\lambda} = \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^{\vee})\hat{\alpha}$$

$$W_{\text{aff}} \cong W_{\mathfrak{d}_4^{\vee}} \ltimes Q^{\vee} \;\Rightarrow\; W_{\mathfrak{d}_4^{\vee}} \setminus W_{\text{aff}} \simeq Q^{\vee}$$

$$t_{\sum_{i=1}^{4} m_i\alpha_i}, m_i \in \mathbb{Z}$$

$$t_{\hat{\beta}}(\hat{\alpha}) = \hat{\alpha} - (\alpha, \beta)\delta$$

$$t_{\sum_{i=1}^{4} m_i\alpha_i}(S_{\gamma}) \in \alpha + l\delta, l > 0.$$

$$\begin{gathered} m_1 = -5, m_2 = -8, m_3 = -5, m_4 = -5, \\ m_1 = -4, m_2 = -7, m_3 = -4, m_4 = -4, \\ m_1 = -3, m_2 = -5, m_3 = -3, m_4 = -3, \end{gathered}$$

$$\begin{aligned} L_{\nu} =\pm\{&-4\delta + \alpha_2 + \alpha_4 + \alpha_5, -4\delta + \alpha_2 + \alpha_3 + \alpha_4, -8\delta + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\ &-4\delta + \alpha_3 + \alpha_4 + \alpha_5, -4\delta + \alpha_4 + \alpha_5 + \alpha_6, -8\delta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\ &-8\delta + \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, -12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \\ &-4\delta + \alpha_1 + \alpha_3 + \alpha_4\}, \end{aligned}$$

$$\begin{aligned} S_{\nu} = \{&\alpha_2, -4\delta + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, -4\delta + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \\ &-8\delta + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_3, -4\delta + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\ &\alpha_4, \alpha_5, \alpha_6, -4\delta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\ &-8\delta + \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1, -8\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \\ &-12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, -4\delta + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \\ &4\delta - \alpha_2 - \alpha_4, 8\delta - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, 8\delta - \alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5, \\ &4\delta - \alpha_3 - \alpha_4, 4\delta - \alpha_4 - \alpha_5, 4\delta - \alpha_5 - \alpha_6, \\ &16\delta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6, 8\delta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \\ &12\delta - \alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6, 12\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6, \\ &4\delta - \alpha_1 - \alpha_3, 8\delta - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6\} \end{aligned}$$

$$\begin{aligned} &m_1 = -11 m_2 = -15 m_3 = -21 m_4 = -29 m_5 = -21 m_6 = -11, \\ &m_1 = -11 m_2 = -15 m_3 = -20 m_4 = -28 m_5 = -20 m_6 = -11, \\ &m_1 = -10 m_2 = -14 m_3 = -19 m_4 = -27 m_5 = -19 m_6 = -10, \\ &m_1 = -10 m_2 = -13 m_3 = -18 m_4 = -25 m_5 = -18 m_6 = -10, \\ &m_1 = -9 m_2 = -13 m_3 = -17 m_4 = -24 m_5 = -17 m_6 = -9, \\ &m_1 = -8 m_2 = -11 m_3 = -15 m_4 = -21 m_5 = -15 m_6 = -8. \end{aligned}$$

$$\begin{aligned} L_{\nu} =\pm\{&\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - 6\delta, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 - 6\delta, \\ &2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 - 6\delta, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 - 30\delta, \\ &\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 - 12\delta, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - 6\delta, \\ &\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 - 6\delta, \\ &2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 - 24\delta, \\ &\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 - 18\delta, \\ &\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 - 18\delta, \\ &\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - 6\delta, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 - 18\delta, \\ &\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 - 12\delta, \\ &\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 - 12\delta, \\ &\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 - 12\delta, \\ &\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 - 24\delta, \\ &\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 - 12\delta, \\ &\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 - 12\delta, \\ &\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 - 18\delta, \\ &\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 - 6\delta\}. \end{aligned}$$

$$
\begin{aligned}
&S_\nu = \{\alpha_2, -6\delta + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, -6\delta + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\
&-6\delta + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\
&-30\delta + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \\
&-12\delta + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_3, \\
&-6\delta + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_4, \\
&\alpha_5, -24\delta + 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_6, \alpha_7, \alpha_8, \\
&-12\delta + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\
&-18\delta + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \\
&-18\delta + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \\
&-6\delta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, -6\delta + \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \\
&-12\delta + \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \alpha_1, \\
&-18\delta + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \\
&-12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \\
&-12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \\
&-24\delta + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \\
&-12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\
&-18\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \\
&-6\delta + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, 6\delta - \alpha_2 - \alpha_4 - \alpha_5 - \alpha_6, 6\delta - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \\
&30\delta - 2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 5\alpha_4 - 4\alpha_5 - 3\alpha_6 - 2\alpha_7 - \alpha_8, \\
&12\delta - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7, \\
&12\delta - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7, \\
&36\delta - 2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 6\alpha_4 - 5\alpha_5 - 4\alpha_6 - 3\alpha_7 - 2\alpha_8, \\
&6\delta - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, 6\delta - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7, 6\delta - \alpha_5 - \alpha_6 - \alpha_7 - \alpha_8, \\
&24\delta - 2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - 2\alpha_7 - \alpha_8, \\
&18\delta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7 - \alpha_8, 6\delta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \\
&18\delta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7, \\
&12\delta - \alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6 - \alpha_7 - \alpha_8, \\
&24\delta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 4\alpha_4 - 4\alpha_5 - 3\alpha_6 - 2\alpha_7 - \alpha_8, \\
&12\delta - \alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6 - \alpha_7, \\
&24\delta - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 3\alpha_6 - 2\alpha_7 - \alpha_8, \\
&12\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6 - \alpha_7, \\
&12\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6, 18\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - 2\alpha_6 - 2\alpha_7 - \alpha_8, \\
&18\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 3\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7 - \alpha_8, 6\delta - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5\}.
\end{aligned}
$$

$m_1 = -56,\ m_2 = -83,\ m_3 = -111,\ m_4 = -165,\ m_5 = -134,\ m_6 = -102,\ m_7 = -69,\ m_8 = -35$

$m_1 = -56,\ m_2 = -82,\ m_3 = -110,\ m_4 = -163,\ m_5 = -133,\ m_6 = -102,\ m_7 = -69,\ m_8 = -35$

$m_1 = -55,\ m_2 = -82,\ m_3 = -109,\ m_4 = -162,\ m_5 = -132,\ m_6 = -101,\ m_7 = -69,\ m_8 = -35$

$m_1 = -55,\ m_2 = -81,\ m_3 = -109,\ m_4 = -161,\ m_5 = -131,\ m_6 = -100,\ m_7 = -68,\ m_8 = -35$

$m_1 = -54,\ m_2 = -80,\ m_3 = -107,\ m_4 = -159,\ m_5 = -130,\ m_6 = -99,\ m_7 = -67,\ m_8 = -34$

$m_1 = -54,\ m_2 = -79,\ m_3 = -106,\ m_4 = -157,\ m_5 = -128,\ m_6 = -98,\ m_7 = -67,\ m_8 = -34$

$m_1 = -53,\ m_2 = -79,\ m_3 = -105,\ m_4 = -156,\ m_5 = -127,\ m_6 = -97,\ m_7 = -66,\ m_8 = -34$

$m_1 = -53,\ m_2 = -78,\ m_3 = -105,\ m_4 = -155,\ m_5 = -126,\ m_6 = -96,\ m_7 = -65,\ m_8 = -33$

$m_1 = -52,\ m_2 = -77,\ m_3 = -103,\ m_4 = -153,\ m_5 = -125,\ m_6 = -96,\ m_7 = -65,\ m_8 = -33$

$m_1 = -52,\ m_2 = -76,\ m_3 = -102,\ m_4 = -151,\ m_5 = -123,\ m_6 = -94,\ m_7 = -64,\ m_8 = -33$

$m_1 = -51,\ m_2 = -76,\ m_3 = -101,\ m_4 = -150,\ m_5 = -122,\ m_6 = -93,\ m_7 = -63,\ m_8 = -32$

$m_1 = -50,\ m_2 = -74,\ m_3 = -99,\ m_4 = -147,\ m_5 = -120,\ m_6 = -92,\ m_7 = -63,\ m_8 = -32$

$m_1 = -50,\ m_2 = -73,\ m_3 = -98,\ m_4 = -145,\ m_5 = -118,\ m_6 = -90,\ m_7 = -61,\ m_8 = -31$

$m_1 = -48,\ m_2 = -71,\ m_3 = -95,\ m_4 = -141,\ m_5 = -115,\ m_6 = -88,\ m_7 = -60,\ m_8 = -31$

$m_1 = -46,\ m_2 = -68,\ m_3 = -91,\ m_4 = -135,\ m_5 = -110,\ m_6 = -84,\ m_7 = -57,\ m_8 = -29$

$$\mathrm{ch}^{SU(N)}(b,q) \to q^{-\frac{c_{2\,\mathrm{d}}[\mathcal{T}_{p,N}] - p c_{2\,\mathrm{d}}[\mathcal{T}^{SU(N)}]}{24}} \mathrm{ch}^{SU(N)}\left(q^{\frac{p}{2}-1}, q^p\right)$$

$$\mathrm{ch}_0 = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}E_1\begin{bmatrix}-1\\b\end{bmatrix}, \mathrm{ch}_M = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}\left(1 - E_1\begin{bmatrix}-1\\b\end{bmatrix}\right)$$

$$\mathrm{ch}_{\log} = S\mathrm{ch}_0 = \frac{i\tau\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}E_1\begin{bmatrix}-1\\b\end{bmatrix} + \frac{i\mathfrak{b}\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}$$

$$\frac{\vartheta_4(i\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} \to \frac{i\vartheta_4\left(\frac{\tau}{2}\middle| 3\tau\right)}{\vartheta_1\left(\frac{\tau}{2}\middle| 3\tau\right)} = 1$$

$$\frac{2\pi\tau\vartheta_4\left(\frac{\tau}{2}\middle| 3\tau\right)}{q^{1/8}\vartheta_1(\tau \mid 3\tau)}\left(1 + E_6\begin{bmatrix}-1\\q^{1/2}\end{bmatrix}(3\tau)\right)$$
$$= \ln\, q(1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{19} + 12q^{21} + 6q^{25} + \cdots)$$

$$\mathrm{Res} = q^{\frac{\dim\mathfrak{g}}{8}}\prod_{i=1}^{r}\frac{\left(b^{d_i-1}q^{\frac{d_i+1}{2}};q\right)\left(b^{-d_i+1}q^{\frac{1-d_i}{2}};q\right)}{\left(b^{d_i}q^{\frac{d_i}{2}};q\right)\left(b^{-d_i}q^{1-\frac{d_i}{2}};q\right)}$$

$$\mathrm{Res} = \mathrm{ch}_0\left(\mathbb{V}_{bc\beta\gamma}^{SU(N)}\right)$$

$$\mathrm{ch}_0\left(\mathbb{V}_{bc\beta\gamma}^{SU(N)}\right) = \begin{cases}\dfrac{\vartheta_4(\mathfrak{b})}{\vartheta_4(N\mathfrak{b})}, N = \text{odd}\\ \dfrac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(N\mathfrak{b})}, N = \text{even}\end{cases},$$

$$q^{-\frac{c_{2\,\mathrm{d}}[\mathcal{T}_{p,N}] - pc_{2\,\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}}\mathrm{ch}_0\left(\mathbb{V}_{bc\beta\gamma}^{SU(N)}\right) \overset{b\to q^{\frac{p}{2}-1}, q\to q^p}{\to} 1.$$

$$\langle W_{[2,2]}\rangle_{SU(3)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\left[\frac{\sqrt{q}\left(b^3q^{\frac{1}{2}} - 1\right)\left(-b^5q - 2b^4q^{\frac{1}{2}}(q+1) - b^3(q(q+4)+2)\right)}{b^4(q^2-1)^2}\right]$$
$$+\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\left[\frac{\sqrt{q}\left(+\left(b^2(2q(q+2)+1)q^{\frac{1}{2}} + 2b(q+1)q + q^{3/2}\right)\right)}{b^4(q^2-1)^2}\right]$$
$$+\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\frac{\sqrt{q}(b^2-1)[(b^2+1)\sqrt{q} + 2bq + 2b]}{b^2(q^2-1)}\left(E_1\begin{bmatrix}-1\\b\end{bmatrix} + E_1\begin{bmatrix}-1\\b^2\sqrt{q}\end{bmatrix}\right)$$

$$\mathrm{ch}_1^{SU(3)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\left(E_1\begin{bmatrix}-1\\b\end{bmatrix} + E_1\begin{bmatrix}-1\\b^2\sqrt{q}\end{bmatrix}\right), \mathrm{ch}_2^{SU(3)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}$$

$$\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\left(E_1\begin{bmatrix}-1\\b\end{bmatrix} + E_1\begin{bmatrix}-1\\b^2\sqrt{q}\end{bmatrix}\right)$$

$$q^{-\frac{c_{2\,\mathrm{d}}[\mathcal{T}_{4,3}] - 4c_{2\,\mathrm{d}}[\mathcal{T}_{SU(3)}]}{24}}\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\left(E_1\begin{bmatrix}-1\\b\end{bmatrix} + E_1\begin{bmatrix}-1\\b^2\sqrt{q}\end{bmatrix}\right)\Bigg|_{b\to q^{\frac{p}{2}-1}, q\to q^p}$$
$$= \frac{1}{q}\frac{\vartheta_4(\tau \mid 4\tau)}{\vartheta_4(3\tau \mid 4\tau)}\left(E_1\begin{bmatrix}-1\\q\end{bmatrix}(4\tau) + E_1\begin{bmatrix}+1\\q^2\end{bmatrix}(4\tau)\right)$$
$$= \frac{1}{2} - q - q^2 - q^4 - 2q^5 - q^8 - q^9 - 2q^{10} + \cdots$$

$$= \frac{1}{50}e^{-i\arctan\frac{4}{3}}(720e^{i\arctan\frac{4}{3}}\mathrm{ch}_0 + 52\sqrt{10}e^{-i\arctan\, 3}\mathrm{ch}_3 + 104\sqrt{5}e^{-i\arctan\frac{1}{2}}\mathrm{ch}_4$$
$$+50\sqrt{10}e^{+i\arctan\, 3}\mathrm{ch}_5 + (27 + 114i)\mathrm{ch}_6$$
$$-(87 + 15i)\mathrm{ch}_7 - (18 + 77i)\mathrm{ch}_8 + 13i(\mathrm{ch}_{11} + \mathrm{ch}_{12}))$$

$$\Delta_{(N)}\mathrm{ch}(b,q) := b^{-(N^2-1)}q^{-\frac{N^2-1}{2}}\mathrm{ch}(bq,q) - \mathrm{ch}(b,q)$$

$$\Delta_{(2)}\mathcal{J}_{SU(2)}(b,q) = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}$$

$$\Delta_{(3)}\mathcal{J}_{SU(3)}(b,q) = -\frac{3}{2}\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})} + \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\left(E_1\left[\begin{matrix}-1\\ b\end{matrix}\right] + E_1\left[\begin{matrix}1\\ b^2\end{matrix}\right]\right),$$

$$\Delta^2_{(3)}\mathcal{J}_{SU(3)}(b,q) = -\frac{3\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}.$$

$$\begin{aligned}\Delta_{(5)}\mathcal{J}_{SU(5)} = -\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(5\mathfrak{b})}\Big(&-225 + 120E_1\left[\begin{matrix}1\\ b^4\end{matrix}\right] + 70E_1\left[\begin{matrix}-1\\ b\end{matrix}\right] \\ &-36E_1\left[\begin{matrix}-1\\ b\end{matrix}\right]^2 + 8E_1\left[\begin{matrix}-1\\ b\end{matrix}\right]^3 - 24E_1\left[\begin{matrix}-1\\ b^2\end{matrix}\right]^2 \\ &+E_1\left[\begin{matrix}-1\\ b\end{matrix}\right]\left(120 - 72E_1\left[\begin{matrix}-1\\ b^3\end{matrix}\right] - 48E_2\left[\begin{matrix}-1\\ b^3\end{matrix}\right]\right) + 48E_2\left[\begin{matrix}-1\\ b^3\end{matrix}\right] \\ &+E_1\left[\begin{matrix}1\\ b^2\end{matrix}\right]\left(70 - 48E_1\left[\begin{matrix}-1\\ b\end{matrix}\right] + 24E_1\left[\begin{matrix}-1\\ b\end{matrix}\right]^2 - 24E_2\left[\begin{matrix}-1\\ b^2\end{matrix}\right]\right) \\ &+E_2\left[\begin{matrix}1\\ b^2\end{matrix}\right]\left(36 - 24E_1\left[\begin{matrix}-1\\ b\end{matrix}\right]\right) + 96E_2\left[\begin{matrix}1\\ b^4\end{matrix}\right] + 16E_3\left[\begin{matrix}-1\\ b^3\end{matrix}\right] + 48E_3\left[\begin{matrix}1\\ b^4\end{matrix}\right]\Big)\end{aligned}$$

$$\Delta_{(5)}\mathrm{ch}_0^{SU(5)}(b,q)\Big|_{q\to q^2}\ \mathbb{V}[\mathcal{T}_{2,5}]$$

$$\Delta_{(5)}\mathrm{ch}_0^{SU(5)}(b,q^2)\Big|_{\mathfrak{b}^0} = -1000\mathrm{ch}_{15}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q) - \mathrm{ch}_{20}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q) - 5\mathrm{ch}_{21}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q) + 60\mathrm{ch}_{24}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q)$$

$$\begin{aligned}\Delta_{(5)}\mathrm{ch}_0^{SU(5)}(b,q^2)\Big|_{\mathfrak{b}^{-1}} = -\frac{i}{18\pi}\Big(&2600\mathrm{ch}_{15}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q) + 3\mathrm{ch}_{20}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q) \\ &+3\mathrm{ch}_{21}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q) - 161\mathrm{ch}_{24}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q)\Big).\end{aligned}$$

$$\begin{aligned}\Delta^2_{(5)}\mathrm{ch}_0^{SU(5)}(b,q^2)\Big|_{\mathfrak{b}^0} = -\frac{2}{3}\Big(&23000\mathrm{ch}_{15}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q) + 3\mathrm{ch}_{20}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q) \\ &+15\mathrm{ch}_{21}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q) - 1430\mathrm{ch}_{24}^{\mathbb{V}[\mathcal{T}_{2,5}]}(q)\Big)\end{aligned}$$

$$\Delta^2_{(5)}\mathrm{ch}_0^{SU(5)}(b,q^2)$$

$$\Delta^3_{(5)}\mathrm{ch}(b\to 1,q^2)$$

$$\begin{aligned}\Delta_{(7)}\mathrm{ch}_0^{SU(7)}(b,q^2)\Big|_{\mathfrak{b}^{-2}} &= \frac{665}{114\pi^2}\mathrm{ch}_{36} + \frac{25}{7104\pi^2}\mathrm{ch}_{41} + \frac{25}{7104\pi^2}\mathrm{ch}_{42} - \frac{24455}{85248\pi^2}\mathrm{ch}_{45} \\ \Delta^2_{(7)}\mathrm{ch}_0^{SU(7)}(b,q^2)\Big|_{\mathfrak{b}^{-2}} &= \frac{4865}{72\pi^2}\mathrm{ch}_{36} + \frac{25}{3552\pi^2}\mathrm{ch}_{41} + \frac{25}{3552\pi^2}\mathrm{ch}_{42} - \frac{179855}{42624\pi^2}\mathrm{ch}_{45} \\ \Delta^2_{(7)}\mathrm{ch}_0^{SU(7)}(b,q^2)\Big|_{\mathfrak{b}^{-1}} &= -\frac{53263i}{18\pi}\mathrm{ch}_{36} - \frac{325i}{444\pi}\mathrm{ch}_{41} - \frac{275i}{222\pi}\mathrm{ch}_{42} + \frac{1964131i}{10656\pi}\mathrm{ch}_{45} \\ \Delta^3_{(7)}\mathrm{ch}_0^{SU(7)}(b,q^2)\Big|_{\mathfrak{b}^{-2}} &= \frac{175}{\pi^2}\mathrm{ch}_{36} + \frac{175}{16\pi^2}\mathrm{ch}_{45} \\ \Delta^3_{(7)}\mathrm{ch}_0^{SU(7)}(b,q^2)\Big|_{\mathfrak{b}^{-1}} &= -\frac{271159i}{18\pi}\mathrm{ch}_{36} - \frac{325i}{444\pi}\mathrm{ch}_{41} - \frac{275i}{222\pi}\mathrm{ch}_{42} + \frac{10026283i}{10656\pi}\mathrm{ch}_{45} \\ \Delta^3_{(7)}\mathrm{ch}_0^{SU(7)}(b,q^2)\Big|_{\mathfrak{b}^0} &= -\frac{1394981}{3}\mathrm{ch}_{36} + \frac{5775}{148}\mathrm{ch}_{41} - \frac{29025}{148}\mathrm{ch}_{42} + \frac{51440147}{1776}\mathrm{ch}_{45}\end{aligned}$$

$$\begin{aligned}\Delta_{(4)}\mathrm{ch}_0^{SU(4)} &= -\frac{i\vartheta_4(\mathfrak{b})}{2\vartheta_1(4\mathfrak{b})}\Big(5 - 3E_1\left[\begin{matrix}-1\\ b\end{matrix}\right] - 3E_1\left[\begin{matrix}1\\ b^3\end{matrix}\right] - 2E_1\left[\begin{matrix}1\\ b^2\end{matrix}\right] + E_1\left[\begin{matrix}-1\\ b\end{matrix}\right]^2 \\ &\qquad +2E_1\left[\begin{matrix}-1\\ b\end{matrix}\right]E_1\left[\begin{matrix}1\\ b^2\end{matrix}\right] - 2E_2\left[\begin{matrix}-1\\ b^3\end{matrix}\right] - E_2\left[\begin{matrix}1\\ b^2\end{matrix}\right]\Big) \\ \Delta^2_{(4)}\mathrm{ch}_0^{SU(4)} &= -\frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(4\mathfrak{b})}\left(-16 + 3E_1\left[\begin{matrix}-1\\ b\end{matrix}\right] + 3E_1\left[\begin{matrix}-1\\ b^3\end{matrix}\right] + 2E_1\left[\begin{matrix}+1\\ b^2\end{matrix}\right]\right) \\ \Delta^3_{(4)}\mathrm{ch}_0^{SU(4)} &= -\frac{16i\vartheta_4(\mathfrak{b})}{\vartheta_1(4\mathfrak{b})}\end{aligned}$$

$$\Delta_{(4)}\mathrm{ch}_0^{SU(4)} \to -18\mathrm{ch}_0^{\mathbb{V}[\mathcal{T}_{3,4}]}(q) - \frac{1}{54}\mathrm{ch}_{16}^{\mathbb{V}[\mathcal{T}_{3,4}]} - \frac{134}{5}\mathrm{ch}_{28}^{\mathbb{V}[\mathcal{T}_{3,4}]}$$
$$-\frac{169}{5}\mathrm{ch}_{29}^{\mathbb{V}[\mathcal{T}_{3,4}]} - \frac{169}{5}e^{\frac{2\pi i}{3}}\mathrm{ch}_{30}^{\mathbb{V}[\mathcal{T}_{3,4}]} - \frac{169}{10}e^{\frac{4\pi i}{3}}\mathrm{ch}_{31}^{\mathbb{V}[\mathcal{T}_{3,4}]} + \frac{169}{5\sqrt{3}}e^{\frac{5\pi i}{6}}\mathrm{ch}_{32}^{\mathbb{V}[\mathcal{T}_{3,4}]}$$

$$\Delta_{(4)}^2\mathrm{ch}_0^{SU(4)} \to -\frac{873}{5}\mathrm{ch}_{28} + \frac{933}{5}\left(-\mathrm{ch}_{29} - e^{\frac{2\pi i}{3}}\mathrm{ch}_{30} + e^{\frac{\pi i}{3}}\mathrm{ch}_{31}\right) + \frac{311}{5}\sqrt{3}ie^{\frac{\pi i}{3}}\mathrm{ch}_{32}.$$

$$a = e^{2\pi i\mathfrak{a}}, b = e^{2\pi i\mathfrak{b}}, \tilde{b} = e^{2\pi i\tilde{\mathfrak{b}}}, \cdots, z = e^{2\pi i\mathfrak{z}},$$

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{+\infty}(1-q^n)$$

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau), \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)$$

$$(z;q) := \prod_{k=0}^{+\infty}(1-zq)$$

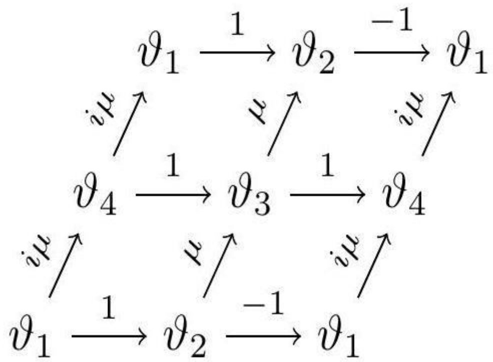
$$\vartheta_1(\mathfrak{z}\mid\tau) = iq^{\frac{1}{8}}z^{-\frac{1}{2}}(q;q)(z;q)(z^{-1}q;q) = -iq^{\frac{1}{8}}z^{\frac{1}{2}}(q;q)(zq;q)(z^{-1};q)$$
$$\vartheta_2(\mathfrak{z}\mid\tau) = q^{\frac{1}{8}}z^{-\frac{1}{2}}(q;q)(-z;q)(-z^{-1}q;q) = q^{\frac{1}{8}}z^{\frac{1}{2}}(q;q)(-zq;q)(-z^{-1};q)$$
$$\vartheta_3(\mathfrak{z}\mid\tau) = (q;q)\left(-zq^{1/2};q\right)\left(-z^{-1}q^{1/2};q\right)$$
$$\vartheta_4(\mathfrak{z}\mid\tau) = (q;q)\left(zq^{1/2};q\right)\left(z^{-1}q^{1/2};q\right)$$

$$\vartheta_i(0) = \vartheta_i(\mathfrak{z}=0\mid\tau), \vartheta_i^{(k)}(0) = \partial_{\mathfrak{z}}^n\big|_{\mathfrak{z}=0}\vartheta_i(\mathfrak{z}\mid\tau)$$

$$\vartheta_1(\mathfrak{z}\mid\tau) := -i\sum_{r\in\mathbb{Z}+\frac{1}{2}}(-1)^{r-\frac{1}{2}}e^{2\pi ir\mathfrak{z}}q^{\frac{r^2}{2}}, \quad \vartheta_2(\mathfrak{z}\mid\tau) := \sum_{r\in\mathbb{Z}+\frac{1}{2}}e^{2\pi ir\mathfrak{z}}q^{\frac{r^2}{2}}$$

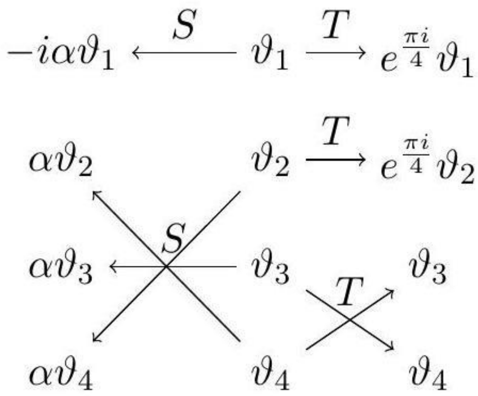
$$\vartheta_3(\mathfrak{z}\mid\tau) := \sum_{n\in\mathbb{Z}}e^{2\pi in\mathfrak{z}}q^{\frac{n^2}{2}}, \quad \vartheta_4(\mathfrak{z}\mid\tau) := \sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi in\mathfrak{z}}q^{\frac{n^2}{2}}.$$

$$\vartheta_1(\mathfrak{z}+m\tau+n) = (-1)^{m+n}e^{-2\pi im\mathfrak{z}}q^{-\frac{1}{2}m^2}\vartheta_1(\mathfrak{z}).$$



$$f\left(\mathfrak{z}+\frac{1}{2}\right) = ag(\mathfrak{z}) \text{ or } f\left(\mathfrak{z}+\frac{\tau}{2}\right) = ag(\mathfrak{z}),$$

$$\left(\frac{\mathfrak{z}}{\tau}, -\frac{1}{\tau}\right) \leftrightarrow S(\mathfrak{z},\tau) \xrightarrow{T} (\mathfrak{z},\tau+1)$$



$$\alpha := \sqrt{-i\tau}e^{\frac{\pi i}{\tau}\mathfrak{z}^2}$$

$$E_{k\geq 1}\begin{bmatrix}\phi\\ \theta\end{bmatrix} := -\frac{B_k(\lambda)}{k!}$$
$$+\frac{1}{(k-1)!}\sum_{r\geq 0}{}' \frac{(r+\lambda)^{k-1}\theta^{-1}q^{r+\lambda}}{1-\theta^{-1}q^{r+\lambda}} + \frac{(-1)^k}{(k-1)!}\sum_{r\geq 1}\frac{(r-\lambda)^{k-1}\theta q^{r-\lambda}}{1-\theta q^{r-\lambda}}$$

$$E_0\begin{bmatrix}\phi\\ \theta\end{bmatrix} = -1$$

$$E_2\begin{bmatrix}-1\\ z\end{bmatrix}(3\tau), E_3\begin{bmatrix}1\\ q^{1/4}b\end{bmatrix}(4\tau),\ \text{etc.}$$

$$E_{2n}\begin{bmatrix}+1\\ +1\end{bmatrix} = E_{2n}, E_1\begin{bmatrix}+1\\ z\end{bmatrix} = \frac{1}{2\pi i}\frac{\vartheta_1'(\mathfrak{z})}{\vartheta_1(\mathfrak{z})}, E_{2n+1\geq 3}\begin{bmatrix}+1\\ +1\end{bmatrix} = 0$$

$$E_k\begin{bmatrix}\pm 1\\ z^{-1}\end{bmatrix} = (-1)^k E_k\begin{bmatrix}\pm 1\\ z\end{bmatrix}$$

$$E_k\begin{bmatrix}+1\\ z\end{bmatrix} = \sum_{\ell=0}^{\lfloor k/2\rfloor}\frac{(-1)^{k+1}}{(k-2\ell)!}\left(\frac{1}{2\pi i}\right)^{k-2\ell}\mathbb{E}_{2\ell}\frac{\vartheta_1^{(k-2\ell)}(\mathfrak{z})}{\vartheta_1(\mathfrak{z})}$$
$$E_k\begin{bmatrix}\pm 1\\ zq^{\frac{n}{2}}\end{bmatrix} = \sum_{\ell=0}^{k}\left(\frac{n}{2}\right)^{\ell}\frac{1}{\ell!}E_{k-\ell}\begin{bmatrix}(-1)^n(\pm 1)\\ z\end{bmatrix}$$

$$S: (\tau,\mathfrak{z}) \to \left(-\frac{1}{\tau},\frac{\mathfrak{z}}{\tau}\right)$$

$$E_n\begin{bmatrix}+1\\ +z\end{bmatrix} \xrightarrow{S} E_n\begin{bmatrix}+1\\ e^{\frac{2\pi i i_{\mathfrak{z}}}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right) = \left(\frac{1}{2\pi i}\right)^n\sum_{\ell=0}^{k}\frac{(-\log z)^{n-\ell}(\log q)^{\ell}}{(n-\ell)!}E_\ell\begin{bmatrix}+1\\ z\end{bmatrix},$$
$$E_n\begin{bmatrix}-1\\ +z\end{bmatrix} \xrightarrow{S} E_n\begin{bmatrix}-1\\ e^{\frac{2\pi i i_{\mathfrak{z}}}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right) = \left(\frac{1}{2\pi i}\right)^n\sum_{\ell=0}^{k}\frac{(-\log z)^{n-\ell}(\log q)^{\ell}}{(n-\ell)!}E_\ell\begin{bmatrix}+1\\ -z\end{bmatrix},$$
$$E_n\begin{bmatrix}1\\ -z\end{bmatrix} \xrightarrow{S} E_n\begin{bmatrix}1\\ -e^{\frac{2\pi i i_{\mathfrak{z}}}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right) = \left(\frac{1}{2\pi i}\right)^n\sum_{\ell=0}^{k}\frac{(-\log z)^{n-\ell}(\log q)^{\ell}}{(n-\ell)!}E_\ell\begin{bmatrix}-1\\ z\end{bmatrix},$$
$$E_n\begin{bmatrix}-1\\ -z\end{bmatrix} \xrightarrow{S} E_n\begin{bmatrix}-1\\ -e^{\frac{2\pi i i_{\mathfrak{z}}}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right) = \left(\frac{1}{2\pi i}\right)^n\sum_{\ell=0}^{k}\frac{(-\log z)^{n-\ell}(\log q)^{\ell}}{(n-\ell)!}E_\ell\begin{bmatrix}-1\\ -z\end{bmatrix}.$$

$$E_n\begin{bmatrix}+1\\ +z\end{bmatrix} \xrightarrow{T} E_n\begin{bmatrix}+1\\ +z\end{bmatrix},\quad E_n\begin{bmatrix}-1\\ +z\end{bmatrix} \xrightarrow{T} E_n\begin{bmatrix}-1\\ -z\end{bmatrix}$$
$$E_n\begin{bmatrix}+1\\ -z\end{bmatrix} \xrightarrow{T} E_n\begin{bmatrix}+1\\ -z\end{bmatrix},\quad E_n\begin{bmatrix}-1\\ -z\end{bmatrix} \xrightarrow{T} E_n\begin{bmatrix}-1\\ +z\end{bmatrix}.$$

$$E_2(\tau) = E_2\begin{bmatrix}+1\\ +1\end{bmatrix} = \lim_{z\to 1}E_2\begin{bmatrix}+1\\ z\end{bmatrix}$$

$$(\tau, \mathfrak{z}) \to \left(-\frac{1}{\tau}, \frac{\mathfrak{b}}{\tau}\right)$$

$$E_2\begin{bmatrix}1\\e^{2\pi i\mathfrak{z}}\end{bmatrix}(\tau) \to -\frac{\mathfrak{z}^2}{2} - \mathfrak{z}\tau E_1\begin{bmatrix}1\\e^{2\pi i\mathfrak{z}}\end{bmatrix} + \tau^2 E_2\begin{bmatrix}1\\e^{2\pi i\mathfrak{z}}\end{bmatrix}$$

$$\lim_{\mathfrak{z}\to 0}\mathfrak{z}E_1\begin{bmatrix}1\\e^{2\pi i\mathfrak{z}}\end{bmatrix} = -\frac{i}{2\pi}$$

$$E_2(\tau) \xrightarrow{S} \tau^2 E_2(\tau) + \frac{i}{2\pi}\tau$$

$$E_2\begin{bmatrix}+1\\-1\end{bmatrix}(\tau) \xrightarrow{S} \left(\frac{1}{2\pi i}\right)^2 \sum_{\ell=0}^{k} \frac{(-\log z)^{2-\ell}(\log q)^{\ell}}{(2-\ell)!} E_\ell\begin{bmatrix}-1\\z\end{bmatrix}(\tau)\Bigg|_{z\to 1} = \tau^2 E_2\begin{bmatrix}-1\\+1\end{bmatrix}$$

$$E_2\begin{bmatrix}+1\\-1\end{bmatrix}(\tau) \xrightarrow{S} E_2\begin{bmatrix}+1\\e^{\frac{\pi i\tau}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right) = \left(\frac{1}{2\pi i}\right)^2 \sum_{\ell=0}^{2} \frac{(-\pi i\tau)^{2-\ell}(2\pi i\tau)^{\ell}}{(2-\ell)!} E_\ell\begin{bmatrix}-1\\q^{+\frac{1}{2}}\end{bmatrix}(\tau)$$
$$= \tau^2 E_2\begin{bmatrix}-1\\+1\end{bmatrix}(\tau)$$

$$E_n\begin{bmatrix}-1\\z\end{bmatrix} \xrightarrow{STS} \left(\frac{1}{2\pi i}\right)^n \left[\left(\sum_{k\geq 0}\frac{1}{k!}(-\log z)^k y^k\right)\left(\sum_{\ell\geq 0}(\log q - 2\pi i)^\ell y^\ell E_\ell\begin{bmatrix}-1\\+z\end{bmatrix}\right)\right]_n$$

$$E_n\begin{bmatrix}\pm 1\\b^p\end{bmatrix}(p\tau) = \frac{1}{p}\sum_{\ell=0}^{p-1} E_n\begin{bmatrix}\pm 1\\be^{2\pi i\ell/p}\end{bmatrix}(\tau), p = 1,2,3,\cdots$$

$$0 = E_1\begin{bmatrix}1\\q^{1/4}\end{bmatrix} - E_1\begin{bmatrix}1\\-q^{1/4}\end{bmatrix} + iE_1\begin{bmatrix}1\\iq^{1/4}\end{bmatrix} - iE_1\begin{bmatrix}1\\iq^{1/4}\end{bmatrix}$$

$$0 = E_2\begin{bmatrix}1\\-q^{1/4}\end{bmatrix} - E_2\begin{bmatrix}1\\q^{1/4}\end{bmatrix} - \frac{1}{4}E_1\begin{bmatrix}1\\-q^{1/4}\end{bmatrix} + \frac{1}{4}E_1\begin{bmatrix}1\\q^{1/4}\end{bmatrix}$$
$$+\frac{1}{2}E_1\begin{bmatrix}-1\\i\end{bmatrix}E_1\begin{bmatrix}1\\-iq^{1/4}\end{bmatrix} + \frac{1}{2}E_1\begin{bmatrix}1\\i\end{bmatrix}E_1\begin{bmatrix}1\\-iq^{1/4}\end{bmatrix}$$
$$-\frac{1}{2}E_1\begin{bmatrix}-1\\i\end{bmatrix}E_1\begin{bmatrix}1\\iq^{1/4}\end{bmatrix} - \frac{1}{2}E_1\begin{bmatrix}1\\i\end{bmatrix}E_1\begin{bmatrix}1\\iq^{1/4}\end{bmatrix}.$$

$$\mathcal{I}_{SU(2)} = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}E_1\begin{bmatrix}-1\\b\end{bmatrix}$$

$$\mathcal{I}_{SU(3)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\left(\frac{1}{24} - \frac{1}{2}E_1\begin{bmatrix}-1\\b\end{bmatrix}^2 + \frac{1}{2}E_2\begin{bmatrix}1\\b^2\end{bmatrix}\right)$$

$$\mathcal{I}_{SU(4)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_1(4\mathfrak{b})}\left(\frac{i}{24}E_1\begin{bmatrix}-1\\b\end{bmatrix} - \frac{i}{6}E_1\begin{bmatrix}-1\\b\end{bmatrix}^3 + \frac{i}{24}E_1\begin{bmatrix}-1\\b^3\end{bmatrix}\right.$$
$$\left. + \frac{i}{2}E_1\begin{bmatrix}-1\\b\end{bmatrix}E_2\begin{bmatrix}1\\b^2\end{bmatrix} - \frac{i}{3}E_3\begin{bmatrix}-1\\b^3\end{bmatrix}\right)$$

$$\mathcal{I}_{SU(5)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(5\mathfrak{b})}\left(\frac{3}{640} + \frac{1}{48}E_2\begin{bmatrix}1\\b^2\end{bmatrix} + \frac{1}{24}E_2\begin{bmatrix}1\\b^4\end{bmatrix} - \frac{1}{48}E_1\begin{bmatrix}-1\\b\end{bmatrix}^2 - \frac{1}{24}E_1\begin{bmatrix}-1\\b\end{bmatrix}E_1\begin{bmatrix}-1\\b^3\end{bmatrix}\right.$$
$$-\frac{1}{4}E_4\begin{bmatrix}1\\b^4\end{bmatrix} + \frac{1}{24}E_1\begin{bmatrix}-1\\b\end{bmatrix}^4 - \frac{1}{4}E_1\begin{bmatrix}-1\\b\end{bmatrix}^2 E_2\begin{bmatrix}1\\b^2\end{bmatrix}$$
$$\left. + \frac{1}{8}E_2\begin{bmatrix}1\\b^2\end{bmatrix}^2 + \frac{1}{3}E_1\begin{bmatrix}-1\\b\end{bmatrix}E_3\begin{bmatrix}-1\\b^3\end{bmatrix}\right)$$

$$
\begin{aligned}
\mathcal{J}_{SU(6)} =& \frac{i\vartheta_4(\mathfrak{b},q)}{\vartheta_1(6\mathfrak{b},q)}\Big(-\frac{3}{640}E_1\begin{bmatrix}-1\\b^5\end{bmatrix}-\frac{3}{640}E_1\begin{bmatrix}-1\\b\end{bmatrix}-\frac{1}{576}E_1\begin{bmatrix}-1\\b^3\end{bmatrix}\\
&-\frac{1}{48}E_2\begin{bmatrix}1\\b^4\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}-\frac{1}{48}E_2\begin{bmatrix}1\\b^2\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}-\frac{1}{48}E_1\begin{bmatrix}-1\\b^3\end{bmatrix}E_2\begin{bmatrix}1\\b^2\end{bmatrix}\\
&+\frac{1}{24}E_3\begin{bmatrix}-1\\b^5\end{bmatrix}+\frac{1}{72}E_3\begin{bmatrix}-1\\b^3\end{bmatrix}+\frac{1}{48}E_1\begin{bmatrix}-1\\b^3\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}^2+\frac{1}{144}E_1\begin{bmatrix}-1\\b\end{bmatrix}^3\\
&+\frac{1}{4}E_4\begin{bmatrix}1\\b^4\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}-\frac{1}{8}E_2\begin{bmatrix}\frac{1}{b^2}e^2E_1\begin{bmatrix}-1\\b\end{bmatrix}\end{bmatrix}-\frac{1}{6}E_3\begin{bmatrix}-1\\b^3\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}^2\\
&+\frac{1}{12}E_2\begin{bmatrix}1\\b^2\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}+\frac{1}{6}E_2\begin{bmatrix}1\\b^2\end{bmatrix}E_3\begin{bmatrix}-1\\b^3\end{bmatrix}\\
&-\frac{1}{5}E_5\begin{bmatrix}-1\\b^5\end{bmatrix}-\frac{1}{120}E_1\begin{bmatrix}-1\\b\end{bmatrix}^5\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{SU(7)} =& \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(7\mathfrak{b})}\Big(\frac{5}{7168}-\frac{3}{640}E_1\begin{bmatrix}-1\\b^5\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}-\frac{1}{576}E_1\begin{bmatrix}-1\\b^3\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}-\frac{3}{1280}E_1\begin{bmatrix}-1\\b\end{bmatrix}^2\\
&+\frac{3}{1280}E_2\begin{bmatrix}1\\b^2\end{bmatrix}+\frac{1}{576}E_2\begin{bmatrix}1\\b^4\end{bmatrix}+\frac{1}{180}E_2\begin{bmatrix}1\\b^6\end{bmatrix}+\frac{1}{192}E_2\begin{bmatrix}1\\b^2\end{bmatrix}^2\\
&+\frac{1}{72}E_3\begin{bmatrix}-1\\b^3\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}+\frac{1}{24}E_3\begin{bmatrix}-1\\b^5\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}-\frac{1}{18}E_3\begin{bmatrix}-1\\b^3\end{bmatrix}^2\\
&-\frac{1}{96}E_4\begin{bmatrix}1\\b^4\end{bmatrix}-\frac{1}{24}E_4\begin{bmatrix}1\\b^6\end{bmatrix}+\frac{1}{8}E_4\begin{bmatrix}1\\b^4\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}^2+\frac{1}{48}E_2\begin{bmatrix}1\\b^2\end{bmatrix}E_2\begin{bmatrix}1\\b^4\end{bmatrix}\\
&-\frac{1}{5}E_5\begin{bmatrix}-1\\b^5\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}+\frac{1}{6}E_2\begin{bmatrix}1\\b^2\end{bmatrix}E_3\begin{bmatrix}-1\\b^3\end{bmatrix}E_1\begin{bmatrix}-1\\b\end{bmatrix}\\
&+\frac{1}{6}E_6\begin{bmatrix}1\\b\end{bmatrix}+\frac{1}{48}E_2\begin{bmatrix}1\\b^2\end{bmatrix}-\frac{1}{720}E_1\begin{bmatrix}-1\\b\end{bmatrix}\\
&+\frac{1}{48}E_2^2E_2\begin{bmatrix}1\\b\end{bmatrix}-\frac{1}{24}E_4E_2\begin{bmatrix}1\\b\end{bmatrix}-\frac{1}{48}E_2^2E_2\begin{bmatrix}1\\b^6\end{bmatrix}\\
&+\frac{1}{24}E_4E_2\begin{bmatrix}1\\b^6\end{bmatrix}-\frac{1}{12}E_2E_4\begin{bmatrix}1\\b\end{bmatrix}+\frac{1}{12}E_2E_4\begin{bmatrix}1\\b^6\end{bmatrix}\Big).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_{3,N}&: x^3+y^3+z^3+w^N=0\\
\mathcal{T}_{4,N}&: x^2+y^4+z^4+w^N=0\\
\mathcal{T}_{6,N}&: x^2+y^3+z^6+w^N=0
\end{aligned}
$$

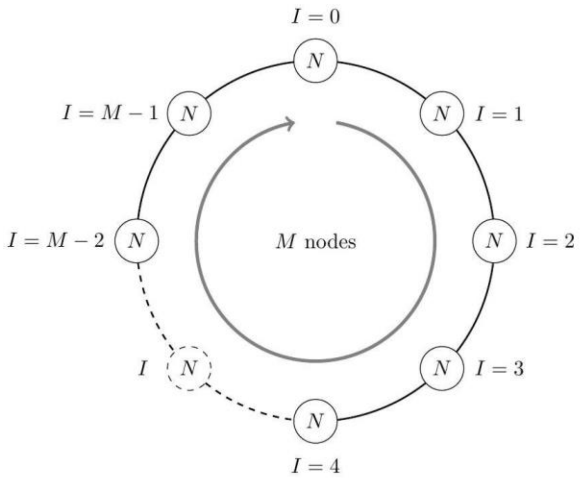
$$f(\mathfrak{a}+1)=f(\mathfrak{a}+\tau)=f(\mathfrak{a})$$

$$a_{4\mathrm{d}}=\frac{h_{\min}}{2}-\frac{5c_{2\,\mathrm{d}}}{48}$$

$$P_2(y,\tau):=-\sum_{n=1}^{\infty}\frac{1}{2n}E_{2n}(\tau)y^{2n}$$

$$\mathrm{f}\,\mathfrak{h}^f:=\{h\in\mathfrak{h}\mid[h,f]=0\}$$

$$\Delta_{\mathrm{l}}=\{\alpha\in\Delta|\alpha|_{\mathfrak{h}^f}=0\}$$



$$W^{(I)} \equiv \frac{1}{N}\mathrm{tr}\mathcal{P}\exp\left\{g\oint_{\mathcal{C}} d\tau\left[iA_\mu^I\dot{x}^\mu(\tau)+\frac{1}{2}(\varphi^I(x)+\bar{\varphi}^I(x))\right]\right\},$$

$$W_0 \equiv \frac{1}{\sqrt{M}}\left(W^{(0)}+W^{(1)}+\cdots+W^{(M-1)}\right)$$

$$W_\alpha \equiv \frac{1}{\sqrt{M}}\sum_{I=0}^{M-1}\rho^{I\alpha}W^{(I)}$$

$$\rho \equiv \mathrm{e}^{2\pi i/M}.$$

$$W_\alpha \equiv \frac{1}{\sqrt{M}}\sum_{I=0}^{M-1}\rho^{I\alpha}W^{(I)}$$

$$\langle W_{\alpha_1}W_{\alpha_2}\dots W_{\alpha_n}\rangle$$

$$\sum_{i=1}^{n}\alpha_i = 0\mathrm{mod}M$$

$$\mathcal{W} \equiv \partial_m^2\log\langle W\rangle|_{m=0} = \int d^4x_1d^4x_2\mu(x_1,x_2)\langle O_2(x_1)O_2(x_2)\rangle_W$$

$$\mathcal{Z} = \int\left(\prod_{I=0}^{M-1}da_I\right)\mathrm{e}^{-\mathrm{tr}a_I^2}\left|\mathcal{Z}_{1-\mathrm{loop}}\mathcal{Z}_{\mathrm{inst}}\right|^2$$

$$da_I = \prod_{b=1}^{N^2-1}\frac{da_I^b}{\sqrt{2\pi}}$$

$$\left|\mathcal{Z}_{1\text{-loop}}\right|^2 = \mathrm{e}^{-S_{\mathrm{int}}},$$

$$S_{\mathrm{int}} = \sum_{I=0}^{M-1}\left[\sum_{m=2}^{+\infty}\sum_{k=2}^{2m}(-1)^{m+k}\left(\frac{\lambda}{8\pi^2N}\right)^m\binom{2m}{k}\frac{\zeta_{2m-1}}{2m}\left(\mathrm{tr}a_I^{2m-k}-\mathrm{tr}a_{I+1}^{2m-k}\right)\left(\mathrm{tr}a_I^k-\mathrm{tr}a_{I+1}^k\right)\right]$$

$$\langle f(a)\rangle = \frac{\langle f(a)\mathrm{e}^{-S_{\mathrm{int}}}\rangle_0}{\langle\mathrm{e}^{-S_{\mathrm{int}}}\rangle_0},$$

$$t_{n_1,n_2,\dots,n_q} = \langle\mathrm{tr}a^{n_1}\mathrm{tr}a^{n_2}\cdots\mathrm{tr}a^{n_q}\rangle$$

$$A_{\alpha,k} = \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} \rho^{\alpha I} \mathrm{tr} a_I^k$$

$$\hat{A}_{\alpha,k} \equiv A_{\alpha,k} - \langle A_{\alpha,k} \rangle$$

$$\hat{A}_{\alpha,k} = A_{\alpha,k} \ \text{ for } \ \alpha \neq 0$$

$$\left\langle \hat{A}_{\alpha,k} \hat{A}^{\dagger}_{\beta,\ell} \right\rangle_0 \propto N^{\frac{k+\ell}{2}} \delta_{\alpha,M-\beta} \text{ with } k+\ell \text{ even,}$$
$$\langle \hat{A}_{\alpha,k} \hat{A}_{\beta,\ell} \hat{A}^{\dagger}_{\gamma,q} \rangle_0 \propto N^{\frac{k+\ell+q}{2}-1} \delta_{\alpha+\beta,M-\gamma}, \text{with } k+\ell+q \text{ even.}$$

$$\hat{A}_{\alpha,k} = \left(\frac{N}{2}\right)^{\frac{k}{2}} \sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \sqrt{k-2i} \binom{k}{i} \mathcal{P}_{\alpha,k-2i}$$

$$S_{\text{int}} = -\frac{1}{2} \sum_{\alpha=0}^{M-1} \sum_{k,\ell=2}^{\infty} s_\alpha \mathcal{P}^{\dagger}_{\alpha,k} \mathrm{X}_{k,\ell} \mathcal{P}_{\alpha,\ell}$$

$$s_\alpha \equiv \sin\left(\frac{\pi\alpha}{M}\right)^2$$

$$\mathrm{X}_{2k,2\ell+1} = 0$$

$$\mathrm{X}_{k,\ell} = -8(-1)^{\frac{k+\ell+2k\ell}{2}} \sqrt{k\ell} \int_0^\infty \frac{dt}{t} \frac{\mathrm{e}^t}{(\mathrm{e}^t-1)^2} J_k\left(\frac{t\sqrt{\lambda}}{2\pi}\right) J_\ell\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

$$\mathrm{X}^{\text{even}}_{k,\ell} \equiv \mathrm{X}_{2k,2\ell} = -\int_0^\infty dt U^{\text{even}}_k(t) U^{\text{even}}_\ell(t)$$
$$\mathrm{X}^{\text{odd}}_{k,\ell} \equiv \mathrm{X}_{2k+1,2\ell+1} = -\int_0^\infty dt U^{\text{odd}}_k(t) U^{\text{odd}}_\ell(t)$$

$$U^{\text{even}}_k(t) = (-1)^k \frac{2\sqrt{k}}{\sqrt{t}\sinh(t/2)} J_{2k}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$
$$U^{\text{odd}}_k(t) = (-1)^k \frac{\sqrt{2(2k+1)}}{\sqrt{t}\sinh(t/2)} J_{2k+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

$$\mathcal{Z} \simeq \det{}^{-1}(\mathbf{1} - s_\alpha \mathrm{X})$$

$$\left\langle \mathcal{P}_{\alpha,k} \mathcal{P}^{\dagger}_{\beta,\ell} \right\rangle \simeq \delta_{\alpha,\beta} \mathrm{D}^{(\alpha)}_{k,\ell}$$

$$\mathrm{D}^{(\alpha)}_{k,\ell} = \left(\frac{1}{1 - s_\alpha \mathrm{X}}\right)_{k,\ell}.$$

$$\mathrm{D}^{(\alpha)\,\text{even}}_{k,\ell} \equiv \mathrm{D}^{(\alpha)}_{2k,2\ell}, \mathrm{D}^{(\alpha)\,\text{odd}}_{k,\ell} \equiv \mathrm{D}^{(\alpha)}_{2k+1,2\ell+1}$$

$$\langle \mathcal{P}_{\alpha,k} \mathcal{P}_{\beta,\ell} \mathcal{P}^{\dagger}_{\gamma,q} \rangle \simeq \delta_{\alpha+\beta,\gamma} \frac{\mathrm{d}^{(\alpha)}_k \mathrm{d}^{(\beta)}_\ell \mathrm{d}^{(\gamma)}_q}{\sqrt{M}N} \ \text{ with } k+\ell+q \text{ even,}$$

$$\mathrm{d}^{(\alpha)}_k = \sum_{\ell=2}^{\infty} \mathrm{D}^{(\alpha)}_{k,\ell} \sqrt{\ell}.$$

$$W_\alpha = \frac{1}{\sqrt{M}N} \sum_{I=0}^{M-1} \rho^{\alpha I} \mathrm{tr}\exp\left[\sqrt{\frac{\lambda}{2N}} a_I\right] = \frac{1}{N} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\lambda}{2N}\right)^{\frac{k}{2}} A_{\alpha,k}.$$

$$\frac{\langle W_\alpha W_\alpha^\dagger \rangle}{W_{conn}^{(2)}(\lambda)} \equiv 1 + \Delta w^{(\alpha)}(M,\lambda)$$

$$\Delta w^{(\alpha)}(M,\lambda) \simeq \frac{2}{\sqrt{\lambda}} \sum_{k=2}^{\infty} \sum_{\ell=2}^{\infty} \frac{I_k(\sqrt{\lambda}) I_\ell(\sqrt{\lambda})}{I_1(\sqrt{\lambda}) I_2(\sqrt{\lambda})} \sqrt{k\ell} \left(\mathrm{D}_{k,\ell}^{(\alpha)} - \delta_{k,\ell} \right)$$

$$\frac{\left\langle W_\alpha W_\beta W_{\alpha+\beta}^\dagger \right\rangle}{\sqrt{M} W_{conn}^{(3)}(\lambda)} \equiv 1 + \Delta w^{(\alpha,\beta)}(M,\lambda)$$

$$1 + \Delta w^{(\alpha,\beta)} \simeq \frac{8}{\lambda^{3/2}} \left(\frac{I_1(\sqrt{\lambda})^2}{I_1(\sqrt{\lambda})^2 + 3 I_2(\sqrt{\lambda})^2} \right) \left[\prod_{p=1}^{3} \left(\mathcal{S}_{\text{even}}^{\alpha_p} + \frac{\sqrt{\lambda}}{2} \right) \right.$$
$$\left. + \sum_{\sigma \in \mathcal{Q}_3} \left(\mathcal{S}_{\text{even}}^{\alpha_{\sigma(1)}} + \frac{\sqrt{\lambda}}{2} \right) \left(\mathcal{S}_{\text{odd}}^{\alpha_{\sigma(2)}} + \frac{\sqrt{\lambda}}{2} \frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \right) \left(\mathcal{S}_{\text{odd}}^{\alpha_{\sigma(3)}} + \frac{\sqrt{\lambda}}{2} \frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \right) \right]$$

$$\mathcal{S}_{\text{even}}^{\alpha} = \sum_{k,\ell=1}^{\infty} \frac{I_{2k}(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \sqrt{(2k)(2\ell)} \left(\mathrm{D}_{2k,2\ell}^{(\alpha)} - \delta_{2k,2\ell} \right)$$

$$\mathcal{S}_{\text{odd}}^{\alpha} = \sum_{k,\ell=1}^{\infty} \frac{I_{2k+1}(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \sqrt{(2k+1)(2\ell+1)} \left(\mathrm{D}_{2k+1,2\ell+1}^{(\alpha)} - \delta_{2k+1,2\ell+1} \right)$$

$$\mathcal{Q}_3 = \{(1,2,3),(3,1,2),(2,3,1)\}$$

$$\mathcal{Z}(m_1,m_2) = \int da_0 \int da_1 \mathrm{e}^{-(\mathrm{tr} a_0^2 + \mathrm{tr} a_1^2)} \mathrm{e}^{-S_{\text{int}} + (m_1^2 + m_2^2)\mathcal{M} + O(m^4)}$$

$$\mathcal{M} = -\sum_{n=1}^{\infty} \sum_{\ell=0}^{2n} (-1)^{n+\ell} \frac{(2n+1)!\, \zeta_{2n+1}}{(2n-\ell)!\, \ell!} \left(\frac{\lambda}{8\pi^2 N} \right)^n \mathrm{tr} a_0^\ell \mathrm{tr} a_1^{2n-\ell}$$

$$\mathcal{W} = 2 \frac{\langle W_0 \mathcal{M} \rangle - \langle W_0 \rangle \langle \mathcal{M} \rangle}{\langle W_0 \rangle}$$

$$\mathcal{W} = \mathcal{W}^{(L)} + \frac{\mathcal{W}^{(NL)}}{N^2} + O\left(\frac{1}{N^4} \right)$$

$$\frac{2\pi\sqrt{\lambda}}{I_1(\sqrt{\lambda})} \int_0^\infty \frac{dt}{t} \frac{(t/2)^2}{\sinh(t/2)^2} \frac{1}{4\pi^2 + t^2} J_1\left(\frac{t\sqrt{\lambda}}{2\pi} \right) \left[2\pi I_0(\sqrt{\lambda}) J_1\left(\frac{t\sqrt{\lambda}}{2\pi} \right) - t I_1(\sqrt{\lambda}) J_0\left(\frac{t\sqrt{\lambda}}{2\pi} \right) \right]$$

$$\langle W_0 \mathcal{M} \rangle - \langle W_0 \rangle \langle \mathcal{M} \rangle \equiv \langle W_0 \mathcal{M} \rangle_c \simeq \langle W_0 \mathcal{M} \rangle_c^{(L)} + \frac{\langle W_0 \mathcal{M} \rangle_c^{(NL)}}{N^2} + O(N^{-4})$$

$$\langle W_0 \rangle \simeq W^{(L)} + \frac{W^{(NL)}}{N^2} + O(N^{-4})$$

$$W^{(L)} = \frac{2\sqrt{2} I_1(\sqrt{\lambda})}{\sqrt{\lambda}}$$

$$W^{(NL)} = \frac{\sqrt{2}}{48} \left(\lambda I_0(\sqrt{\lambda}) - 14\sqrt{\lambda} I_1(\sqrt{\lambda}) \right) - \frac{\lambda^{3/2} \partial_\lambda \mathcal{F}}{2\sqrt{2}} I_1(\sqrt{\lambda})$$

$$\sum_{k,\ell=1}^{\infty} a_k \mathrm{D}_{k,\ell}^{(\alpha)} b_\ell$$

$$\mathrm{D}_{k,\ell}^{(\alpha)\,\text{even}} = \delta_{k,\ell} - \int_0^\infty U_k^{\text{even}}(t) Z_\ell^{(\alpha)}(t)$$

$$Z_\ell^{(\alpha)}(t) + s_\alpha \int_0^\infty dt' \mathbb{K}^{\text{even}}(t,t') Z_\ell^{(\alpha)}(t') = s_\alpha U_\ell^{\text{even}}(t)$$

$$\mathbb{K}^{\text{even}}(t,t') = \sum_{k=1}^{\infty} U_k^{\text{even}}(t) U_k^{\text{even}}(t')$$

$$\sum_{k,\ell=1}^{\infty} a_k \mathrm{D}_{k,\ell}^{(\alpha)\,\text{even}} b_\ell = \sum_{k=1}^{\infty} a_k b_k - \int_0^\infty dt A(t) Z^{(\alpha)}(t)$$

$$Z^{(\alpha)}(t) + s_\alpha \int_0^\infty dt' \mathbb{K}^{\text{even}}(t,t') Z^{(\alpha)}(t') = s_\alpha B(t)$$

$$Z^{(\alpha)}(t) = \sum_{k=1}^{\infty} b_k Z_k^{(\alpha)}(t), A(t) = \sum_{k=1}^{\infty} a_k U_k^{\text{even}}(t) \text{ and } B(t) = \sum_{k=1}^{\infty} b_k U_k^{\text{even}}(t)$$

$$\int_0^\infty dx f(x) \approx \sum_{i=1}^{m} w_i f(x_i)$$

$$x_k = \cos\theta_k, \text{ with } \theta_k = (2k-1)\frac{\pi}{2m}, k = 1,2,\dots,m$$

$$w_k = \frac{2}{m}\left[1 - 2\sum_{r=1}^{\frac{m-1}{2}} \frac{\cos(2r\theta_k)}{4r^2-1}\right]$$

$$1 + \Delta w^{(\alpha)}(M,\lambda) \underset{\lambda\to\infty}{\sim} \kappa_0 \left(1 + \kappa_1 \frac{1}{\sqrt{\lambda}} + \kappa_2 \frac{1}{\lambda} + \kappa_3 \frac{1}{\lambda^{3/2}} + O(\lambda^{-2})\right)$$

$$\kappa_0 = \frac{1}{s_\alpha}\left(\frac{\mathcal{I}_0(s_\alpha)}{2}\right)^2$$
$$\kappa_1 = 2$$
$$\kappa_2 = 3 - \frac{\pi \mathcal{I}_1(s_\alpha)}{2}$$
$$\kappa_3 = \frac{15}{4} - \frac{3}{2}\pi \mathcal{I}_1(s_\alpha) - \pi^2 \mathcal{I}_1(s_\alpha)^2$$

$$\mathcal{I}_n(s_\alpha) = \int_0^{+\infty} \frac{dz}{\pi} z^{1-2n} \partial_z \log\left(1 + s_\alpha \sinh\left(\frac{z}{2}\right)^{-2}\right)$$

$$\mathcal{I}_0(s_\alpha) = -\frac{\alpha}{M}\left(1 - \frac{\alpha}{M}\right) 2\pi$$

$$\mathcal{I}_1(s_\alpha) = -\frac{1}{2\pi}\left[\psi\left(\frac{\alpha}{M}\right) + \psi\left(1 - \frac{\alpha}{M}\right) - 2\psi(1)\right]$$

$$\Delta w^{(\alpha)}(M,\lambda) = \Delta w^{(\alpha)\text{even}}(M,\lambda) + \Delta w^{(\alpha)\text{odd}}(M,\lambda)$$

$$\Delta w^{(\alpha)\text{even}}(M,\lambda) = \frac{2}{\sqrt{\lambda}} \sum_{k,\ell=1}^{\infty} \frac{I_{2k}(\sqrt{\lambda}) I_{2\ell}(\sqrt{\lambda})}{I_1(\sqrt{\lambda}) I_2(\sqrt{\lambda})} \sqrt{2k}\sqrt{2\ell}\left(\mathrm{D}_{k,\ell}^{(\alpha)\,\text{even}} - \delta_{k,\ell}\right)$$

$$\Delta w^{(\alpha)\,\text{odd}}(M,\lambda) = \frac{2}{\sqrt{\lambda}} \sum_{k,\ell=1}^{\infty} \frac{I_{2k+1}(\sqrt{\lambda}) I_{2\ell+1}(\sqrt{\lambda})}{I_1(\sqrt{\lambda}) I_2(\sqrt{\lambda})} \sqrt{2k+1}\sqrt{2\ell+1}\left(\mathrm{D}_{k,\ell}^{(\alpha)\,\text{odd}} - \delta_{k,\ell}\right)$$

$$\Delta w^{(\alpha)} \underset{\lambda\to\infty}{\sim} \frac{2}{\sqrt{\lambda}} \sum_{P=0}^{\infty} \frac{\mathcal{S}^{(P)}}{\lambda^{P/2}}$$

$$\mathcal{S}^{(P)} = \sum_{L+J=P} S^{(L,J)}_{\text{odd}} + S^{(L,J)}_{\text{even}}$$

$$S^{(L,J)}_{\text{odd}} = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \sqrt{2k+1}Q^{(1)\text{odd}}_{2L}(k)\sqrt{2\ell+1}Q^{(2)\text{odd}}_{2J}(\ell)\langle\psi_{2\ell+1}|\frac{s_\alpha \mathrm{X}}{1-s_\alpha \mathrm{X}}|\psi_{2k+1}\rangle$$

$$S^{(L,J)}_{\text{even}} = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \sqrt{2k}Q^{(1)\text{even}}_{2L}(k)\sqrt{2\ell}Q^{(2)\text{even}}_{2J}(\ell)\langle\psi_{2\ell}|\frac{s_\alpha \mathrm{X}}{1-s_\alpha \mathrm{X}}|\psi_{2k}\rangle$$

$$\frac{I_{2k+1}(\sqrt{\lambda})}{I_j(\sqrt{\lambda})} \equiv \sum_{s=0}^{\infty} \frac{Q^{(j)\text{odd}}_{2s}(k)}{\lambda^{s/2}}, \frac{I_{2k}(\sqrt{\lambda})}{I_j(\sqrt{\lambda})} \equiv \sum_{s=0}^{\infty} \frac{Q^{(j)\text{even}}_{2s}(k)}{\lambda^{s/2}}$$

$$\psi_k(x) = (-1)^{\frac{k}{2}(k-1)}\sqrt{k}\frac{J_k(\sqrt{x})}{\sqrt{x}}$$

$$\mathrm{X}_{k,\ell} = \langle\psi_k|\mathrm{X}|\psi_\ell\rangle$$

$$w^{(\ell)}_{n,m}(s_\alpha) = \left\langle(x\partial_x)^n\phi^{(\ell)}(x)\right|\frac{s_\alpha \mathrm{X}}{1-s_\alpha \mathrm{X}}\left|(x\partial_x)^m\phi^{(\ell)}(x)\right\rangle,$$

$$\phi^{(\ell)}(x) \equiv J_\ell(\sqrt{x})$$

$$\mathcal{S}^{(P)} = \mathcal{S}^{(P)}_0 + \mathcal{S}^{(P)}_1 + \mathcal{S}^{(P)}_2 + \mathcal{S}^{(P)}_3 + \cdots$$

$$\mathcal{S}^{(P)}_0 = (-2)^{P-2}\sum_{n+m=P}\left[w^{(1)}_{n,m} + w^{(2)}_{n,m}\right]$$

$$\mathcal{S}^{(P)}_1 = (-2)^{P-3}\sum_{n+m=P}\left[-(1+n^2+m^2)w^{(1)}_{n,m} + (5-n^2-m^2)w^{(2)}_{n,m}\right]$$

$$\mathcal{S}^{(P)}_2 = (-2)^{P-5}\sum_{n+m=P}\left[f^{(1)}_{n,m}w^{(1)}_{n,m} + f^{(2)}_{n,m}w^{(2)}_{n,m}\right]$$

$$f^{(1)}_{n,m} = \frac{1}{3}(-3m^4 - 8m^3 - 6m^2n^2 - 12m^2 + 20m - 3n^4 - 8n^3 - 12n^2 + 20n + 15)$$

$$f^{(2)}_{n,m} = \frac{1}{3}(-3m^4 - 8m^3 - 6m^2n^2 + 24m^2 + 56m - 3n^4 - 8n^3 + 24n^2 + 56n + 51)$$

$$w^{(\ell)}_{n,m} = \omega^{(\ell,0)}_{n,m}g^{n+m+1} + \omega^{(\ell,1)}_{n,m}g^{n+m} + \omega^{(\ell,2)}_{n,m}g^{n+m-1} + \cdots$$

$$g \equiv \frac{\sqrt{\lambda}}{4\pi}$$

$$\frac{2}{\sqrt{\lambda}}\sum_{P=0}^{\infty}\sum_{n+m=P}\frac{1}{(-2\pi)^n(-2\pi)^m}\left[\frac{(n^2+m^2-2)\omega^{(0)}_{n,m}}{16\pi} + \frac{1}{4}\sum_{\ell=1}^{2}\omega^{(\ell,1)}_{n,m}\right]$$

$$\frac{1}{\sqrt{\lambda}}\left(\frac{1}{8\pi}\left((x\partial_x)^2 + \left(y\partial_y\right)^2 - 2\right)G^{(0)}(s_\alpha,x,y) + \frac{1}{2}\sum_{\ell=1}^{2}G^{(1)}(\ell,s_\alpha,x,y)\right)\Bigg|_{x=y=-2\pi}$$

$$\frac{2}{s_\alpha\sqrt{\lambda}}\left(\frac{\mathcal{J}_0(s_\alpha)}{2}\right)^2,$$

$$2\pi\sum_{\ell=1}^{2}G^{(2)}(\ell,s_\alpha,x,y) + \frac{1}{4}\left((x\partial_x)^2 + \left(y\partial_y\right)^2\right)\sum_{\ell=1}^{2}G^{(1)}(\ell,s_\alpha,x,y)$$
$$+\frac{1}{4}\left(G^{(1)}(1,s_\alpha,x,y) - 5G^{(1)}(2,s_\alpha,x,y)\right) - \frac{1}{64\pi}\left(22 - \frac{76}{3}(x\partial_x + y\partial_y) +\right.$$
$$\left.4\left((x\partial_x)^2 + \left(y\partial_y\right)^2\right) + \frac{16}{3}\left((x\partial_x)^3 + \left(y\partial_y\right)^3\right) - 4(x\partial_x)^2\left(y\partial_y\right)^2 - 2\left((x\partial_x)^4 + \left(y\partial_y\right)^4\right)\right)G^{(0)}(s_\alpha,x,y)$$

$$\frac{1}{s_\alpha}\left(\frac{\mathcal{I}_0(s_\alpha)}{2}\right)^2\left(3-\frac{\pi\mathcal{I}_1(s_\alpha)}{2}\right)$$

$$\Delta w^{(\alpha)\,\mathrm{even}} = -\frac{2}{\sqrt{\lambda}}\frac{1}{I_1(\sqrt{\lambda})I_2(\sqrt{\lambda})}\int_0^\infty dt A(t) Z^{(\alpha)}(t)$$

$$Z^{(\alpha)}(t)+s_\alpha\int_0^\infty du\mathbb{K}^{\mathrm{even}}(t,u)Z^{(\alpha)}(u)=s_\alpha B(t)$$

$$A(t)=B(t)=\frac{\sqrt{2}t\pi}{(4\pi^2+t^2)\sinh(t/2)}\left(\frac{t\sqrt{\lambda}}{2\pi}I_1(\sqrt{\lambda})J_0\left(\frac{t\sqrt{\lambda}}{2\pi}\right)-\sqrt{\lambda}I_0(\sqrt{\lambda})J_1\left(\frac{t\sqrt{\lambda}}{2\pi}\right)\right)$$

$$\Delta w^{(\alpha)\mathrm{even}} = -\frac{3\zeta_3}{32\pi^4}s_\alpha\lambda^2+\frac{\pi^2\zeta_3+20\zeta_5}{256\pi^6}s_\alpha\lambda^3+O(\lambda^4)$$

$$1+\Delta w^{(1)} \underset{\lambda\to\infty}{\sim} \kappa_0\left(1+\kappa_1\frac{1}{\sqrt{\lambda}}+\kappa_2\frac{1}{\lambda}+\kappa_3\frac{1}{\lambda^{3/2}}+O(\lambda^{-2})\right)$$

$$\begin{aligned}\kappa_0 &= \frac{16\pi^2}{243}\\ \kappa_1 &= 2\\ \kappa_2 &= 3-\frac{3\log(3)}{4}\\ \kappa_3 &= -\frac{3}{4}(-5+3\log(3)+3\log^2(3)).\end{aligned}$$

$$\sum_{i=0}^{n} f_i\lambda^{-\frac{i}{2}}$$

$$\tilde{f}_{\mathrm{num}}(\lambda)=\left(\frac{f_{\mathrm{num}}(\lambda)}{\kappa_0}-1\right)\sqrt{\lambda}$$

$$\sum_{i=0}^{2} g_i\mathcal{I}_1(s_\alpha)^i.$$

$$S^\alpha_{\mathrm{even}} = c_0^{\mathrm{even}}\sqrt{\lambda}+c_1^{\mathrm{even}}+c_2^{\mathrm{even}}\frac{1}{\sqrt{\lambda}}+c_3^{\mathrm{even}}\frac{1}{\lambda}+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\begin{aligned}c_0^{\mathrm{even}} &= -\frac{1}{2}-\frac{1}{4}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}\\ c_1^{\mathrm{even}} &= -\frac{1}{8}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}\\ c_2^{\mathrm{even}} &= -\frac{3}{32}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}-\frac{\pi}{16}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}\mathcal{I}_1(s_\alpha)\\ c_3^{\mathrm{even}} &= -\frac{3}{32}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}-\frac{3\pi}{32}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}\mathcal{I}_1(s_\alpha)-\frac{\pi^2}{8}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}\mathcal{I}_1(s_\alpha)^2\end{aligned}$$

$$S^\alpha_{\mathrm{odd}} = c_0^{\mathrm{odd}}\sqrt{\lambda}+c_1^{\mathrm{odd}}+c_2^{\mathrm{odd}}\frac{1}{\sqrt{\lambda}}+c_3^{\mathrm{odd}}\frac{1}{\lambda}+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$c_0^{\mathrm{odd}} = -\frac{1}{2} - \frac{1}{4}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}$$
$$c_1^{\mathrm{odd}} = \frac{3}{4}$$
$$c_2^{\mathrm{odd}} = -\frac{3}{16} + \frac{3\pi}{16}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}\mathcal{I}_1(s_\alpha)$$
$$c_3^{\mathrm{odd}} = -\frac{3}{16} + \frac{3\pi}{16}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}\mathcal{I}_1(s_\alpha) + \frac{3\pi^2}{8}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}\mathcal{I}_1(s_\alpha)^2$$

$$1 + \Delta w^{(\alpha,\beta)}(M,\lambda) \underset{\lambda\to\infty}{\sim} c_0\left[1 + c_1\frac{1}{\sqrt{\lambda}} + c_2\frac{1}{\lambda} + c_3\frac{1}{\lambda^{3/2}} + O\left(\frac{1}{\lambda^2}\right)\right]$$

$$c_0 = -\frac{1}{8}\prod_{p=1}^{3}\frac{\mathcal{I}_0\left(s_{\alpha_p}\right)}{\sqrt{s_{\alpha_p}}}$$
$$c_1 = 3$$
$$c_2 = \frac{1}{4}\left(21 - \pi\sum_{p=1}^{3}\mathcal{I}_1\left(s_{\alpha_p}\right)\right)$$
$$c_3 = \frac{1}{32}\left(199 - 28\pi\sum_{p=1}^{3}\mathcal{I}_1\left(s_{\alpha_p}\right) - 16\pi^2\sum_{p=1}^{3}\mathcal{I}_1\left(s_{\alpha_p}\right)^2\right)$$

$$\mathcal{S}_{\mathrm{even}}^{\alpha} = \sum_{P=0}^{\infty}\frac{\mathcal{S}_{\mathrm{even}}^{(P)}}{\lambda^{P/2}}, \mathcal{S}_{\mathrm{odd}}^{\alpha} = \sum_{P=0}^{\infty}\frac{\mathcal{S}_{\mathrm{odd}}^{(P)}}{\lambda^{P/2}}$$

$$\mathcal{S}_{\mathrm{even}}^{(P)} = \mathcal{S}_{0,\mathrm{even}}^{(P)} + \mathcal{S}_{1,\mathrm{even}}^{(P)} + \mathcal{S}_{2,\mathrm{even}}^{(P)} + \cdots$$
$$\mathcal{S}_{\mathrm{odd}}^{(P)} = \mathcal{S}_{0,\mathrm{odd}}^{(P)} + \mathcal{S}_{1,\mathrm{odd}}^{(P)} + \mathcal{S}_{2,\mathrm{odd}}^{(P)} + \cdots$$

$$\mathcal{S}_{0,\mathrm{even}}^{(P)} = (-2)^{P-2}w_{P,0}^{(1)}$$
$$\mathcal{S}_{1,\mathrm{even}}^{(P)} = -(-2)^{P-4}(P^2 - 2P)(1-\delta_{P,0})w_{P-1,0}^{(1)}$$
$$\mathcal{S}_{2,\mathrm{even}}^{(P)} = -\frac{(-2)^{P-7}}{3}(3P^4 - 16P^3 + 24P^2 - 20P + 9)(1-\delta_{P,0})(1-\delta_{P,1})w_{P-2,0}^{(1)}$$

$$\mathcal{S}_{0,\mathrm{odd}}^{(P)} = (-2)^{P-2}w_{P,0}^{(2)}$$
$$\mathcal{S}_{1,\mathrm{odd}}^{(P)} = -(-2)^{P-4}(P^2 - 2P - 3)(1-\delta_{P,0})w_{P-1,0}^{(2)}$$
$$\mathcal{S}_{2,\mathrm{odd}}^{(P)} = -\frac{(-2)^{P-7}}{3}P(3P^3 - 16P^2 + 6P + 16)(1-\delta_{P,0})(1-\delta_{P,1})w_{P-2,0}^{(2)}$$

$$c_1^{\mathrm{even}} = \frac{1}{4}\sum_{P=0}^{\infty}\frac{\omega_{P,0}^{(1)}}{(-2\pi)^P} - \frac{1}{16}\sum_{P=0}^{\infty}(P^2 - 2P)\frac{\omega_{P-1,0}^{(1)}}{(-2\pi)^P}(1-\delta_{P,0})$$

$$c_1^{\mathrm{even}} = \left(\frac{1}{4}G^{(1)}(s_\alpha,1,x) + \frac{1}{32\pi}[(x\partial_x)^2 - 1]G^{(0)}(s_\alpha,x)\right)\Bigg|_{x=-2\pi} = -\frac{1}{8}\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}}.$$

$$c_1^{\mathrm{odd}} = \left(\frac{1}{4}G^{(1)}(s_\alpha,2,x) + \frac{1}{32\pi}[(x\partial_x)^2 - 4]G^{(0)}(s_\alpha,x)\right)\Bigg|_{x=-2\pi} = \frac{3}{4}$$

$$
\begin{aligned}
c_2^{\text{even}} &= \Big(\pi G^{(2)}(s_\alpha,1,x)+\frac{1}{8}((x\partial_x)^2-1)G^{(1)}(s_\alpha,1,x) \\
&\quad +\frac{1}{384\pi}(3(x\partial_x)^4-8(x\partial_x)^3+20(x\partial_x)-15)G^{(0)}(s_\alpha,x)\Big)\Big|_{x=-2\pi} \\
&= -\frac{3}{32}\frac{\mathcal{J}_0(s_\alpha)}{\sqrt{s_\alpha}}-\frac{\pi}{16}\frac{\mathcal{J}_0(s_\alpha)}{\sqrt{s_\alpha}}\mathcal{J}_1(s_\alpha) \\
c_2^{\text{odd}} &= \Big(\pi G^{(2)}(s_\alpha,2,x)+\frac{1}{8}((x\partial_x)^2-4)G^{(1)}(s_\alpha,2,x) \\
&\quad +\frac{1}{384\pi}(3(x\partial_x)^4-8(x\partial_x)^3-18(x\partial_x)^2+56(x\partial_x)-24)G^{(0)}(s_\alpha,x)\Big)\Big|_{x=-2\pi} \\
&= -\frac{3}{16}+\frac{3\pi}{16}\frac{\mathcal{J}_0(s_\alpha)}{\sqrt{s_\alpha}}\mathcal{J}_1(s_\alpha)
\end{aligned}
$$

$$
\mathcal{S}^{\alpha}_{\text{even}} = -\frac{1}{I_1(\sqrt{\lambda})}\int_0^\infty dt A(t) Z^{(\alpha)}(t)
$$

$$
Z^{(\alpha)}(t)+s_\alpha\int_0^\infty du\mathbb{K}^{\text{even}}(t,u)Z^{(\alpha)}(u) = -\frac{s_\alpha\sqrt{t\lambda}}{2\sqrt{2}\pi\sinh(t/2)}J_1\left(\frac{t\sqrt{\lambda}}{2\pi}\right)
$$

$$
\begin{aligned}
c_3^{\text{even}} &= \frac{\mathcal{J}_0(s_\alpha)}{32\sqrt{s_\alpha}}\sum_{i=0}^{2} c_{3,i}^{\text{even}}\left(\pi\mathcal{J}_1(s_\alpha)\right)^i \\
c_3^{\text{odd}} &= \frac{c_{3,0}^{\text{odd}}}{16}+\frac{\mathcal{J}_0(s_\alpha)}{16\sqrt{s_\alpha}}\sum_{i=1}^{2} c_{3,i}^{\text{odd}}\left(\pi\mathcal{J}_1(s_\alpha)\right)^i
\end{aligned}
$$

$$
\frac{\langle W_1W_1\dots W_1\rangle}{\left(W^{(2)}_{\text{conn}}(\lambda)\right)^n} \simeq \left(1+\Delta w^{(1)}(2,\lambda)\right)^n
$$

$$
\begin{aligned}
\frac{\langle W_1W_1\dots W_1\rangle}{\left(W^{(2)}_{\text{conn}}(\lambda)\right)^n} \underset{\lambda\to\infty}{\sim} \left(\frac{\pi^2}{16}\right)^n \Bigg(1+\frac{2n}{\sqrt{\lambda}}+\frac{n(2n+1-\log\,2)}{\lambda}+ \\
+\frac{n(16n^2+24n(1-\log\,2)+5-12\log\,2(1+4\log\,2)}{12\lambda^{3/2}}+O(\lambda^{-2})\Bigg)
\end{aligned}
$$

$$
1+\Delta w^{(\alpha_1,\dots,\alpha_{2n-1})}(M,\lambda) \equiv \frac{\langle W_{\alpha_1}W_{\alpha_2}\dots W_{\alpha_{2n-1}}W^\dagger_{\alpha_{2n}}\rangle}{\mathcal{N}^{\text{even}}(\alpha_1,\alpha_2,\dots,\alpha_{2n})\left(W^{(2)}_{\text{conn}}(\lambda)\right)^n}
$$

$$
\begin{aligned}
&1+\Delta w^{(\alpha_1,\dots,\alpha_{2n-1})}(M,\lambda) \simeq \\
&\frac{1}{\mathcal{N}^{\text{even}}(\alpha_1,\alpha_2,\dots,\alpha_{2n})}\left[\prod_{j=1}^{n}\delta_{\alpha_{2j-1},\alpha_{2j}}\left(1+\Delta w^{(\alpha_{2j-1})}(M,\lambda)\right)+\text{ permutations}\right]
\end{aligned}
$$

$$
1+\Delta w^{(\alpha_1,\dots,\alpha_{2n+1})}(M,\lambda) \equiv \frac{\langle W_{\alpha_1}W_{\alpha_2}\dots W_{\alpha_{2n}}W^\dagger_{\alpha_{2n+1}}\rangle}{\mathcal{N}^{odd}(\alpha_1,\alpha_2,\dots,\alpha_{2n+1})\sqrt{M}W^{(3)}_{conn}(\lambda)\left(W^{(2)}_{conn}(\lambda)\right)^{n-1}}
$$

$$
\begin{aligned}
&1+\Delta w^{(\alpha_1,\alpha_2,\dots,\alpha_{2n})}(M,\lambda) \simeq \frac{1}{\mathcal{N}^{odd}(\alpha_1,\alpha_2,\cdots,\alpha_{2n+1})} \\
&\left[\delta_{\alpha_1+\alpha_2,\alpha_3}(1+\Delta w^{(\alpha_1,\alpha_2)}(M,\lambda))\prod_{j=2}^{n}\delta_{\alpha_{2j},\alpha_{2j+1}}\left(1+\Delta w^{(\alpha_{2j})}(M,\lambda)\right)+\text{ permutations}\right]
\end{aligned}
$$

$$
\mathcal{W}^{(NL)} = 2\frac{\langle W_0\mathcal{M}\rangle_c^{(NL)}}{W^{(L)}}-\mathcal{W}^{(L)}\frac{W^{(NL)}}{W^{(L)}}
$$

$$
\langle W_0\mathcal{M}\rangle_c^{(NL)} = \left\langle W_0\mathcal{M}^{(1)}\right\rangle_c^{(NL)}+\left\langle W_0\mathcal{M}^{(2)}\right\rangle_c^{(NL)}
$$

$$\left\langle W_0 \mathcal{M}^{(1)} \right\rangle_c^{(NL)} \underset{\lambda\to\infty}{\sim} -\frac{\lambda^2}{768} - \frac{\lambda^{3/2}}{4608}(4\pi^2+45) + \frac{\lambda}{2048}(24\pi^2 - 115 - 16\log(2)) + O(\sqrt{\lambda})$$

$$\frac{I_1(\sqrt{\lambda})}{4\sqrt{2}} \sum_{k,\ell=2}^{\infty} (-)^{k-k\ell} \mathrm{M}_{k,\ell}\left(\sqrt{k\ell} - \mathrm{d}_k\, \mathrm{d}_\ell\right) - (\lambda\partial_\lambda \mathcal{F}) \sum_{q,p=1}^{\infty} \sqrt{p}\mathrm{M}_{2p,2q}\sqrt{2q} I_{2q}(\sqrt{\lambda})$$

$$\mathrm{M}_{k,\ell} = (-1)^{\frac{k+\ell+2k\ell}{2}} + 1\sqrt{k\ell}\int_0^\infty \frac{dt}{t}\frac{(t/2)^2}{\sinh(t/2)^2} J_k\left(\frac{t\sqrt{\lambda}}{2\pi}\right) J_\ell\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

$$\mathcal{R} = \sum_{k,\ell=2}^{\infty} (-)^{k-k\ell} \mathrm{M}_{k,\ell}\left(\sqrt{k\ell} - \mathrm{d}_k\, \mathrm{d}_\ell\right)$$

$$\mathcal{W}^{(NL)} \underset{\lambda\to\infty}{\sim} -\frac{\lambda^{3/2}}{128} + \frac{\sqrt{\lambda}}{512}(8\log(2) - 1) + \frac{1}{256}(2\zeta_3 + 32\log^2(2) - 1) + O\left(\lambda^{-1/2}\right)$$

$$\mathcal{R}(\lambda) = -\frac{\sqrt{\lambda}}{24} + \frac{5}{8}\frac{1}{\sqrt{\lambda}} + \frac{\log(2)}{2\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\mathcal{R}_1(\lambda) = \sum_{k,\ell=1}^{\infty} \mathrm{M}_{2k,2\ell}\left(\sqrt{(2k)(2\ell)} - \mathrm{d}_{2k}\, \mathrm{d}_{2\ell}\right)$$

$$\mathcal{R}_2(\lambda) = \sum_{k,\ell=1}^{\infty} \mathrm{M}_{2k,2\ell+1}\left(\sqrt{2k(2\ell+1)} - \mathrm{d}_{2k}\, \mathrm{d}_{2\ell+1}\right)$$

$$\mathcal{R}_3(\lambda) = -\sum_{k,\ell=1}^{\infty} \mathrm{M}_{2k+1,2\ell}\left(\sqrt{(2k+1)2\ell} - \mathrm{d}_{2k+1}\, \mathrm{d}_{2\ell}\right)$$

$$\mathcal{R}_4(\lambda) = \sum_{k,\ell=1}^{\infty} \mathrm{M}_{2k+1,2\ell+1}\left(\sqrt{(2k+1)(2\ell+1)} - \mathrm{d}_{2k+1}\, \mathrm{d}_{2\ell+1}\right)$$

$$\mathrm{d}_{2k} = \sqrt{2k} - \int_0^\infty dt U_k^{\mathrm{even}}(t) Z(t)$$

$$Z(t) + \int_0^\infty du \mathbb{K}^{\mathrm{even}}(t,u) Z(u) = -\frac{\sqrt{t\lambda}}{2\sqrt{2}\pi \sinh(t/2)} J_1\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

$$\begin{aligned}\mathcal{R}_1(\lambda) =& 2\sum_{k,\ell=1}^{\infty} \sqrt{2k}\mathrm{M}_{2k,2\ell} \int_0^\infty dt U_\ell^{\mathrm{even}}(t) Z(t) \\ & - \sum_{k,\ell=1}^{\infty} \mathrm{M}_{2k,2\ell} \int_0^\infty dt \int_0^\infty dt' U_k^{\mathrm{even}}(t) U_\ell^{\mathrm{even}}(t') Z(t) Z(t')\end{aligned}$$

$$\mathcal{R}(\lambda) \underset{\lambda\to\infty}{\sim} \sqrt{\lambda}\tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 + \frac{\tilde{\mathcal{R}}_3}{\sqrt{\lambda}} + \frac{\tilde{\mathcal{R}}_4}{\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right).$$

$$\lambda \mapsto \lambda' \equiv \lambda - 4\pi \mathcal{I}_1(s_\alpha)\sqrt{\lambda} + 4\pi^2 \mathcal{I}_1(s_\alpha)^2,$$

$$\begin{aligned} G(s_\alpha, \ell, x) &= \sum_{n=0}^{\infty} \frac{w_{0,n}^{(\ell)}}{(gx)^n} = g G^{(0)}(s_\alpha, x) + G^{(1)}(s_\alpha, \ell, x) + g^{-1} G^{(2)}(s_\alpha, \ell, x) + \cdots \\ G(s_\alpha, \ell, x, y) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{w_{n,m}^{(\ell)}}{(gx)^n (gy)^m} = g G^{(0)}(s_\alpha, x, y) + G^{(1)}(s_\alpha, \ell, x, y) + \cdots \end{aligned}$$

$$w_{0,n}^{(\ell)} = \sum_{i\geq 0} \omega_{0,n}^{(\ell,i)} g^{n+1-i}$$

$$w_{0,0}^{(\ell)} = 4g\mathcal{I}_0(s_\alpha) + (2\ell-1) - \frac{(2\ell-1)(2\ell-3)\mathcal{I}_1(s_\alpha)}{8g} - \frac{(2\ell-1)(2\ell-3)\mathcal{I}_1(s_\alpha)^2}{16g^2}$$
$$- \frac{(2\ell-1)(2\ell-3)(16\mathcal{I}_1(s_\alpha)^3 - 5\mathcal{I}_2(s_\alpha) - 8\mathcal{I}_2(s_\alpha)\ell + 4\mathcal{I}_2(s_\alpha)\ell^2)}{512g^3} + O\left(\frac{1}{g^4}\right)$$

$$\omega_{0,2n+1}^{(\ell,0)}(s_\alpha) = -\frac{1}{2(n+1)}\sum_{j=0}^{n} (-1)^j \mathcal{I}_{-j}(s_\alpha)\omega_{0,2n-2j}^{(\ell,0)}(s_\alpha)$$

$$\omega_{0,2n}^{(\ell,0)}(s_\alpha) = \frac{1}{2n+1}\left(\sum_{j=1}^{n} (-1)^{n-j-1}\mathcal{I}_{-n+j}(s_\alpha)\omega_{0,2j-1}^{(\ell,0)}(s_\alpha) + 4(-1)^n\mathcal{I}_{-n}(s_\alpha)\right)$$

$$\omega_{0,0}^{(\ell,1)} = 2\ell - 1$$
$$\omega_{0,1}^{(\ell,1)} = -\left(\omega_{0,0}^{(\ell,1)} + 1\right)\mathcal{I}_0(s_\alpha)$$
$$\omega_{0,2}^{(\ell,1)} = -\left[\frac{\omega_{0,0}^{(\ell,0)}}{8} - \frac{1}{2}\left(\omega_{0,0}^{(\ell,1)} + 1\right)\mathcal{I}_0(s_\alpha)\right]\mathcal{I}_0(s_\alpha)$$

$$\omega_{0,n}^{(\ell,1)} = \begin{cases} 2\ell - 1, & n = 0 \\ \left[-\frac{\ell}{2} + \left(\frac{n-1}{2}\right)^2\right]\omega_{0,n-1}^{(\ell,0)}, & n \geq 1 \end{cases}$$

$$G^{(1)}(s_\alpha,\ell,x) = \sum_{n=0}^{\infty}\frac{\omega_{0,n}^{(\ell,1)}}{x^n} = \omega_{0,0}^{(\ell,1)} + \frac{1}{4x}\sum_{n=1}^{\infty}\left[(n-1)^2 - 2\ell\right]\frac{\omega_{0,n-1}^{(\ell,0)}}{x^{n-1}}$$
$$= 2\ell - 1 + \frac{1}{4x}\left[-2\ell + (x\partial_x)^2\right]G^{(0)}(s_\alpha,x)$$

$$\omega_{0,n}^{(\ell,2)} = \begin{cases} -\frac{(2\ell-1)(2\ell-3)\mathcal{I}_1(s_\alpha)}{8}, & n = 0 \\ \left(\frac{3}{8} - \ell + \frac{\ell^2}{2}\right)\mathcal{I}_0(s_\alpha)\mathcal{I}_1(s_\alpha) - \frac{\ell^2}{2} - \frac{\ell}{2} + \frac{3}{8} & n = 1 \\ \left(\frac{3}{4} - 2\ell + \ell^2\right)\frac{\mathcal{I}_1(s_\alpha)\omega_{0,n-1}^{(\ell,0)}}{8} + \left(n\ell^2 - n(n-2)\ell + \frac{n(n-2)(20-14n+3n^2)}{12}\right)\frac{\omega_{0,n-2}^{(\ell,0)}}{8}, & n \geq 2 \end{cases}$$

$$G^{(2)}(s_\alpha,\ell,x) = \sum_{n=0}^{\infty}\frac{\omega_{0,n}^{(\ell,2)}}{x^n} = -\frac{(2\ell-1)(2\ell-3)\mathcal{I}_1(s_\alpha)}{8} + \frac{1}{x}\left(\frac{3}{8} - \frac{\ell}{2} - \frac{\ell^2}{2}\right) +$$
$$\left(\frac{3}{4} - 2\ell + \ell^2\right)\frac{\mathcal{I}_1(s_\alpha)}{8x}G^{(0)}(s_\alpha,x) + \frac{1}{8x^2}\left[\ell^2(2 - x\partial_x) - \ell((x\partial_x)^2 - 2x\partial_x) + \right.$$
$$\left.\frac{1}{4}(x\partial_x)^4 - \frac{1}{3}(x\partial_x)^3 - \frac{2}{3}(x\partial_x)\right]G^{(0)}(s_\alpha,x)$$

$$(gx + gy + 1)G(s_\alpha,\ell,x,y) + \frac{1}{2}\left(x\partial_x + y\partial_y - g\partial_g\right)G(s_\alpha,\ell,x,y) = -\frac{1}{4}G(s_\alpha,\ell,x)G(s_\alpha,\ell,y) +$$
$$gxG(s_\alpha,\ell,y) + gyG(s_\alpha,\ell,x)$$

$$(x+y)G^{(0)}(s_\alpha,x,y) = -\frac{1}{4}G^{(0)}(s_\alpha,x)G^{(0)}(s_\alpha,y) + xG^{(0)}(s_\alpha,y) + yG^{(0)}(s_\alpha,x)$$

$$4(x+y)G^{(1)}(s_\alpha,x,y) = \left(4y - G^{(0)}(s_\alpha,y)\right)G^{(1)}(s_\alpha,\ell,x) + \left(4x - G^{(0)}(s_\alpha,x)\right)G^{(1)}(s_\alpha,\ell,y)$$
$$-2\left(G^{(0)}(s_\alpha,x,y) + x\partial_x G^{(0)}(s_\alpha,x,y) + y\partial_y G^{(0)}(s_\alpha,x,y)\right)$$

$$
\begin{aligned}
G^{(1)}(s_\alpha,\ell,x,y) = &-\frac{xG^{(0)}(s_\alpha,y)\partial_x^2 G^{(0)}(s_\alpha,x)}{16(x+y)} + \frac{xy\partial_x^2 G^{(0)}(s_\alpha,x)}{4(x+y)} + \frac{xy\partial_y^2 G^{(0)}(s_\alpha,y)}{4(x+y)} \\
&-\frac{yG^{(0)}(s_\alpha,x)\partial_y^2 G^{(0)}(s_\alpha,y)}{16(x+y)} + \frac{xG^{(0)}(s_\alpha,y)\partial_x G^{(0)}(s_\alpha,x)}{8(x+y)^2} - \frac{xy\partial_x G^{(0)}(s_\alpha,x)}{2(x+y)^2} + \frac{x\partial_y G^{(0)}(s_\alpha,y)}{4(x+y)} \\
&-\frac{xy\partial_y G^{(0)}(s_\alpha,y)}{2(x+y)^2} - \frac{G^{(0)}(s_\alpha,y)\partial_x G^{(0)}(s_\alpha,x)}{16(x+y)} + \frac{y\partial_x G^{(0)}(s_\alpha,x)}{4(x+y)} - \frac{G^{(0)}(s_\alpha,x)\partial_y G^{(0)}(s_\alpha,y)}{16(x+y)} \\
&+\frac{yG^{(0)}(s_\alpha,x)\partial_y G^{(0)}(s_\alpha,y)}{8(x+y)^2} - \frac{\ell x G^{(0)}(s_\alpha y)}{2y(x+y)} - \frac{\ell G^{(0)}(s_\alpha,x)}{2(x+y)} + \frac{\ell G^{(0)}(s_\alpha,x)G^{(0)}(s_\alpha,y)}{8y(x+y)} \\
&-\frac{\ell G^{(0)}(s_\alpha,y)}{2(x+y)} - \frac{\ell y G^{(0)}(s_\alpha,x)}{2x(x+y)} + \frac{\ell G^{(0)}(s_\alpha,x)G^{(0)}(s_\alpha,y)}{8x(x+y)} + \frac{x^2 G^{(0)}(s_\alpha,y)}{2(x+y)^3} + \frac{y^2 G^{(0)}(s_\alpha,x)}{2(x+y)^3} \\
&+\frac{xyG^{(0)}(s_\alpha,x)}{2(x+y)^3} - \frac{xG^{(0)}(s_\alpha,x)G^{(0)}(s_\alpha,y)}{8(x+y)^3} - \frac{xG^{(0)}(s_\alpha,y)}{(x+y)^2} + \frac{xyG^{(0)}(s_\alpha,y)}{2(x+y)^3} + \frac{G^{(0)}(s_\alpha,x)}{4(x+y)} \\
&-\frac{yG^{(0)}(s_\alpha,x)}{(x+y)^2} + \frac{G^{(0)}(s_\alpha,x)G^{(0)}(s_\alpha,y)}{8(x+y)^2} - \frac{yG^{0}(s_\alpha,x)G^{(0)}(s_\alpha,y)}{8(x+y)^3} + \frac{G^{(0)}(s_\alpha,y)}{4(x+y)} + \frac{2\ell x}{x+y} \\
&+\frac{2\ell y}{x+y} - \frac{x}{x+y} - \frac{y}{x+y}.
\end{aligned}
$$

$$
\begin{aligned}
-4(x+y)G^{(2)}(s_\alpha,\ell,x,y) = &G^{(1)}(s_\alpha,\ell,x)G^{(1)}(s_\alpha,\ell,y) + \left(G^{(0)}(s_\alpha,x) - 4x\right)G^{(2)}(s_\alpha,\ell,y) \\
&+\left(G^{(0)}(s_\alpha,y) - 4y\right)G^{(2)}(s_\alpha,\ell,x) + 4G^{(1)}(s_\alpha,\ell,x,y) \\
&+2x\partial_x G^{(1)}(s_\alpha,\ell,x,y) + 2y\partial_y G^{(1)}(s_\alpha,\ell,x,y)
\end{aligned}
$$

$$\mathrm{tr}T_b T_c = \frac{1}{2}\delta_{b,c}, b,c = 1,\cdots,N^2-1$$

$$F_n(x_1,\ldots,x_n;x_0) := \pi^2 \frac{\langle 0|W_n[x_1,\ldots,x_n]\mathcal{L}(x_0)|0\rangle}{\langle 0|W_n[x_1,\ldots,x_n]|0\rangle},$$

$$F^{(L)} = \sum_{ij} c_{ij} R_i f_j^{(L)}$$

$$\sum_{i=1}^{6} p_i = 0, p_i^2 = 0, \forall i \in \{1,\ldots,6\}$$

$$\{s_{i,i+1} = (p_i + p_{i+1})^2\}_{i=1,\cdots,6} \cup \{s_{i,i+1,i+2} = (p_i + p_{i+1} + p_{i+2})^2\}_{i=1,2,3}$$

$$\mathcal{G}(p_1,p_2,p_3,p_4,p_5) \equiv \det G(p_1,p_2,p_3,p_4,p_5) = 0$$

$$\epsilon(i,j,k,l) := 4i\varepsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma, \forall i,j,k,l \in \{1,\ldots,6\}$$

$$\epsilon(i,j,k,l)^2 = \mathcal{G}\left(p_i,p_j,p_k,p_l\right)$$

$$(p_i)_\mu \to p_{\alpha\dot\beta} \equiv (p_i)_\mu (\sigma^\mu)_{\alpha\dot\beta} = (\lambda_i)_\alpha \left(\tilde\lambda_i\right)_{\dot\beta}$$

$$\langle ab\rangle = \varepsilon_{\alpha\beta}\lambda_a^\alpha \lambda_b^\beta, [ab] = \varepsilon_{\dot\alpha\dot\beta}\tilde\lambda_a^{\dot\alpha}\tilde\lambda_b^{\dot\beta}$$

$$p_i = x_{i+1} - x_i, x_7 \equiv x_1$$

$$s_{i,i+1} = x_{i,i+2}^2, s_{i,i+1,i+2} = x_{i,i+3}^2$$

$$Z_i^I := \left(\lambda_i^\alpha, x_i^{\dot\beta\gamma}\lambda_{i,\gamma}\right)^I, I = \{\alpha = 1,2, \dot\beta = 1,2\} = 1,\cdots,4$$

$$\langle ijkl\rangle := \det\left(Z_i Z_j Z_k Z_l\right) = \varepsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L,$$

$$s_{i,i+1} = \frac{\langle i-1 i i+1 i+2\rangle}{\langle i-1 i\rangle\langle i+1 i+2\rangle}, s_{i,i+1,i+2} = \frac{\langle i-1 i i+2 i+3\rangle}{\langle i-1 i\rangle\langle i+2 i+3\rangle},$$

$$I_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\langle ij \rangle = \langle ij I_\infty \rangle$$

$$\left\{ \frac{x_{i,i+2}^2 x_{1,0}^2 x_{3,0}^2}{x_{1,3}^2 x_{i,0}^2 x_{i+2,0}^2} \right\}_{i=2,\cdots 6} \cup \left\{ \frac{x_{i,i+3}^2 x_{1,0}^2 x_{3,0}^2}{x_{1,3}^2 x_{i,0}^2 x_{i+3,0}^2} \right\}_{i=2,\cdots 4}$$

$$\mathcal{G}(x_0, x_1, \cdots, x_6) := 0$$

$$B_{ijklm} := \frac{\langle AB(mij) \cap (jkl) \rangle^2}{\langle ABjm \rangle \langle ABij \rangle \langle ABjk \rangle \langle ABlj \rangle \langle ABmi \rangle \langle ABkl \rangle},$$

$$B_{ijkl} := \frac{\langle ijkl \rangle^2}{\langle ABij \rangle \langle ABjk \rangle \langle ABkl \rangle \langle ABli \rangle}$$

$$R_1 = b_{12346} + b_{12456} + b_{23456} - b_{1256} - b_{2356} - b_{3456}$$
$$R_2 = b_{12346} + b_{12456} - b_{1256}$$
$$R_8 = b_{12346} + b_{12456} + b_{23456} + b_{1234} + b_{1456} - b_{1256} - b_{2356} - b_{3456}$$
$$R_{11} = b_{2356}$$
$$R_{14} = b_{1234} - b_{1456}$$
$$R_{17} = b_{1246} + b_{1345}$$
$$R_{20} = b_{1246} - b_{1345}$$

$$0 = R_2 - R_3 + R_4 - R_5 + R_6 - R_7 - R_{20} + R_{21} - R_{22}$$
$$0 = 4R_1 - 2R_3 - 2R_5 - 2R_7 + R_{17} + R_{18} + R_{19} - R_{20} + R_{21} - R_{22}$$

$$\{1\}, \{2, \ldots, 7\}, \{17,18,19\}, \{20,21,22\}$$

$$G(a_1, \cdots, a_n, z) := \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \cdots, a_n; t)$$

$$\mathrm{d}\mathcal{F}^{(w)} = \sum_i \mathcal{F}_i^{(w-1)} \mathrm{dlog}\ x_i$$

$$\mathcal{S}(\mathcal{F}^{(w)}) = \sum_i \mathcal{S}\left(\mathcal{F}_i^{(w-1)}\right) \otimes x_i$$

$$\mathcal{S}(\mathcal{F}^{(w)}) = \sum_I c_I a_1^I \otimes \cdots \otimes a_w^I$$

$$\mathrm{d}\mathbf{I} = \epsilon \mathrm{d}A \cdot \mathbf{I}$$

$$\mathrm{d}A = \sum_i A_i \ \mathrm{dlog}\ W_i$$

$$W_{100} = -s_{23}s_{34}s_{56} + s_{23}s_{345}s_{56} - s_{12}s_{45}s_{61} - s_{34}s_{61}s_{123} + s_{12}s_{45}s_{234} + s_{34}s_{123}s_{234} + s_{61}s_{123}s_{345} - s_{123}s_{234}s_{345}$$
$$W_{100+i} = \tau^i(W_{100}), i = 1, \ldots, 5$$
$$W_{138} = \Delta_6 = \langle 12 \rangle [23] \langle 34 \rangle [45] \langle 56 \rangle [61] - [12] \langle 23 \rangle [34] \langle 45 \rangle [56] \langle 61 \rangle$$
$$W_{242} = \frac{-s_{12}(s_{45} + s_{61} - s_{234}) + s_{23}(s_{34} + s_{56} - s_{345}) + s_{123}(-s_{34} + s_{61} - s_{234} + s_{345}) - \epsilon(1,2,3,5)}{-s_{12}(s_{45} + s_{61} - s_{234}) + s_{23}(s_{34} + s_{56} - s_{345}) + s_{123}(-s_{34} + s_{61} - s_{234} + s_{345}) + \epsilon(1,2,3,5)},$$
$$W_{242+i} = \tau^i(W_{242}), i = 1, \ldots, 5$$

$$F_6^{(0)} = -R_1.$$

$$F_6^{(1)} = R_2 \,\mathrm{Pent}_{2,6} + R_3 \,\mathrm{Pent}_{1,3} + R_4 \,\mathrm{Pent}_{2,4} + R_5 \,\mathrm{Pent}_{3,5} + R_6 \,\mathrm{Pent}_{4,6} + R_7 \,\mathrm{Pent}_{1,5} + R_8 \,\mathrm{Pent}_{1,4} + R_9 \,\mathrm{Pent}_{2,5} + R_{10} \,\mathrm{Pent}_{3,6}$$

$$\text{Pent}_{ij} = \log u \log v + \text{Li}_2(1-u) + \text{Li}_2(1-v) + \text{Li}_2(1-w) - \text{Li}_2(1-uw) - \text{Li}_2(1-vw)$$

$$u = \frac{x_{i+1,j+1}^2}{x_{i,j+1}^2}, v = \frac{x_{i,j}^2}{x_{i,j+1}^2}, w = \frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{i,j}^2 x_{i+1,j+1}^2}$$

$$F_6^{(L)} = \sum_{i=1}^{22} R_i G_i^{(L)}$$

$$G_i^{(L)} = \sum_j c_{ij} f_j^{(L)}$$

$$R_4 \to \tilde{R}_{17} + \tilde{R}_{18}, R_{13} \to \tilde{R}_{13}, R_{17}, R_{20} \to \tilde{R}_{17}, R_{18} \to \tilde{R}_{18}, R_{21} \to -\tilde{R}_{18}$$

$$G_4 + G_{17} + G_{20}|_{s_{24}=0} = G_4 + G_{18} - G_{21}|_{s_{24}=0} = G_{13}|_{s_{24}=0} = 0.$$

$$G_9|_{s_{25}=0} = G_{15}|_{s_{25}=0} = 0$$

$$\begin{array}{lll} R_1 \to -r_0, & & \\ R_2 \to \bar{R}_2, & R_{i=3,4,5} \to r_{i-1} - r_0, & R_6 \to \bar{R}_6, R_7 \to 0 \\ R_8 \to r_5 - r_0, & R_9 \to r_1 - r_0, & R_{10} \to \bar{R}_{10}, \\ R_{11} \to \bar{R}_{11}, & R_{12} \to \bar{R}_{12}, & R_{13} \to r_3, \\ R_{14} \to r_5, & R_{15} \to r_1, & R_{16} \to \bar{R}_{16}, \\ R_{17} \to \bar{R}_{17}, & R_{18} \to \bar{R}_{18}, & R_{19} \to R^\star, \\ R_{20} \to \bar{R}_{17} - 2r_2, & R_{21} \to 2r_4 - \bar{R}_{18}, & R_{22} \to R^\star, \end{array}$$

$$\begin{array}{l} G_1 + G_3 + G_4 + G_5 + G_8 + G_9 + G_{19} + G_{22} \to -g_{5,0}^{(2)} \\ G_9 + G_{15} \to g_{5,1}^{(2)} \\ G_3 + G_{19} - 2G_{20} + G_{22} \to g_{5,2}^{(2)} \\ G_4 + G_{13} + G_{19} + G_{22} \to g_{5,3}^{(2)} \\ G_5 + G_{19} + 2G_{21} + G_{22} \to g_{5,4}^{(2)} \\ G_8 + G_{14} \to g_{5,5}^{(2)} \\ G_{i=10,11,12,16} \to 0, G_{17} - G_{19} + G_{20} - G_{22} \to 0 \\ G_{18} - G_{19} - G_{21} - G_{22} \to 0, G_2 + G_{19} + G_{22} \to 0 \\ G_6 + G_{19} + G_{22} \to 0 \end{array}$$

$$\begin{array}{lll} R_1 \to -r_0, & & \\ R_{i=2,3,4} \to r_{i-1} - r_0, & R_5 \to r_4 - r_0, & R_{i=6,7} \to 0 \\ R_8 \to r_5 - r_0, & R_9 \to r_1 - r_0, & R_{10} \to r_4 - r_0 \\ R_{11} \to 0, & R_{12} \to r_2, & R_{13} \to r_3 \\ R_{14} \to r_5, & R_{15} \to r_1, & R_{16} \to -r_4 \\ R_{17} \to r_2 + r_3, & R_{18} \to r_4, & R_{19} \to r_1 \\ R_{20} \to r_3 - r_2, & R_{21} \to r_4, & R_{22} \to r_1 \end{array}$$

$$\begin{array}{l} G_1 + G_2 + G_3 + G_4 + G_5 + G_8 + G_9 + G_{10} \to -g_{5,0}^{(2)} \\ G_2 + G_9 + G_{15} + G_{19} + G_{22} \to g_{5,1}^{(2)} \\ G_3 + G_{12} + G_{17} - G_{20} \to g_{5,2}^{(2)} \\ G_4 + G_{13} + G_{17} + G_{20} \to g_{5,3}^{(2)} \\ G_5 + G_{10} - G_{16} + G_{18} + G_{21} \to g_{5,4}^{(2)} \\ G_8 + G_{14} \to g_{5,5}^{(2)} \end{array}$$

$$p_4 \to z_1 P, p_5 \to z_2 P, p_6 \to z_3 P$$

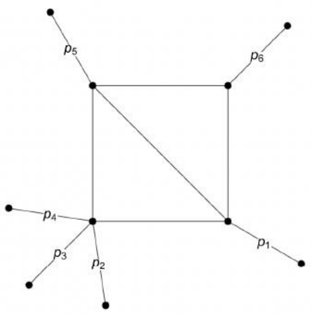
$$R_1, R_2, R_3, R_4, R_9, R_{16} \to b_{1234}, R_{14}, R_{18}, R_{21} \to -b_{1234},$$

$$G_1 + G_2 + G_3 + G_4 + G_9 - G_{14} + G_{16} - G_{18} - G_{21} \to g_{4,0}^{(2)}$$

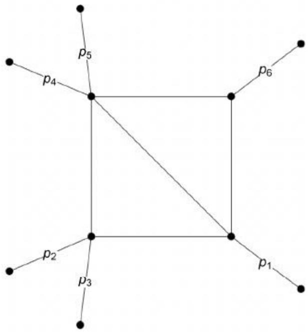
$$\{W_1, \cdots, W_9, W_{16}, \cdots, W_{33}, W_{46}, \cdots, W_{51}, W_{88}, \cdots, W_{93}\}.$$

$$\{W_{10+i}\}_{i=0,\cdots,5} \cup \{W_{34+i}\}_{i=0,\cdots,11} \cup \{W_{58+i}\}_{i=0,\cdots,11} \cup \{W_{76+i}\}_{i=0,\cdots,11} \cup \{W_{100+i}\}_{i=0,\cdots,5}$$

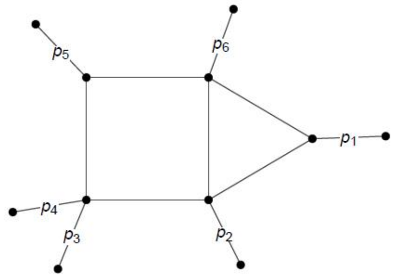
$$\left\{\mathcal{J}_{24}^{(pt)}, \mathcal{J}_{19}^{(pt)}, \mathcal{J}_{10}^{(pt)}, \mathcal{J}_{30}^{(pb)}, \mathcal{J}_{1}^{(pt)}\right\} \leftrightarrow \{W_{13}, W_{36}, W_{62}, W_{77}, W_{100}\}$$



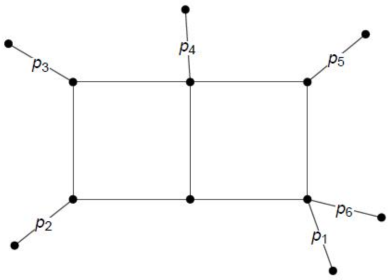
(a) $\{W_{i+10}\}_{i=0,\cdots,5}$



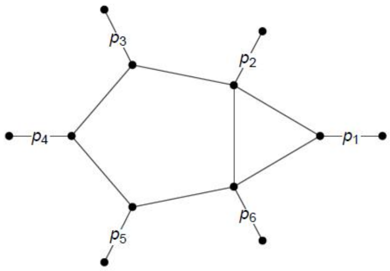
(b) $\{W_{i+34}\}_{i=0,\cdots,11}$



(c) $\{W_{i+58}\}_{i=0,\cdots,11}$



(d) $\{W_{i+76}\}_{i=0,\cdots,11}$



(e) $\{W_{i+100}\}_{i=0,\cdots,5}$

$$\{W_{182}, \cdots, W_{190}, W_{194}, \cdots, W_{211}, W_{218}, \cdots, W_{220}, W_{242}, \cdots, W_{247}\}.$$

$$r_1 = \sqrt{\lambda(s_{12}, s_{34}, s_{56})}, r_2 = \sqrt{\lambda(s_{23}, s_{45}, s_{16})},$$

$$\{W_{157}, \cdots, W_{166}, W_{275}, \cdots, W_{278}\}.$$

$$G_{11}^{(2)} = \text{Pent}_{2,6} \times \text{Pent}_{3,5}, G_{12}^{(2)} = \text{Pent}_{1,3} \times \text{Pent}_{4,6}, G_{13}^{(2)} = \text{Pent}_{1,5} \times \text{Pent}_{2,4}$$

$$\langle 0|W_n \mathcal{L}(x_0)|0\rangle_{x_0\to\infty} \sim \frac{\mathcal{A}_n^{\text{YM}}(1^+, \dots, n^+)}{\mathcal{A}_n^{\text{YM},(1)}(1^+, \dots, n^+)},$$

$$\frac{\mathcal{A}_n^{\text{YM}}}{\mathcal{A}_n^{\text{YM},(1)}} = \mathcal{Z}_{\text{IR}}^{\text{YM}}(\epsilon)\mathcal{H}_n^{\text{YM}}$$

$$\langle 0|W_n|0\rangle \sim \frac{\mathcal{A}_n^{\text{MHV}}}{\mathcal{A}_n^{\text{MHV, tree}}} = \mathcal{Z}_{\text{IR}}^{\text{MHV}}(\epsilon)\mathcal{H}_n^{\text{MHV}},$$

$$F_n \mathcal{H}^{\text{MHV}} \sim \mathcal{H}^{\text{YM}}.$$

$$F_6 \times \mathcal{H}_6 = g^2 F_6^{(0)} + g^4\left(F_6^{(1)} + F_6^{(0)}\mathcal{H}_6^{(1)}\right) + g^6\left(F_6^{(2)} + F_6^{(1)}\mathcal{H}_6^{(1)} + F_6^{(0)}\mathcal{H}_6^{(2)}\right) + O(g^8)$$

$$\underset{s_{i,i+1,i+2}=0}{\text{Disc}}\ \underset{s_{i-1,i,i+1}=0}{\text{Disc}}\left(F_6^{(2)} + F_6^{(1)}\mathcal{H}_6^{(1)}\right) = 0$$

$$F_4^{(L)} = b_{1234}\, g_{4,0}^{(L)}$$

$$g_{4,0}^{(0)} = -1$$

$$g_{4,0}^{(1)} = \log^2(z) + \pi^2$$

$$g_{4,0}^{(2)} = -\frac{1}{2}\log^4(z) + \log^2(z)\left[\frac{2}{3}\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \frac{2}{3}\mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{19\pi^2}{9}\right]$$
$$+\log(z)\left[4\mathrm{Li}_3\left(\frac{1}{z+1}\right) - 4\mathrm{Li}_3\left(\frac{z}{z+1}\right)\right] + \frac{2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right]^2$$
$$+\frac{8\pi^2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right] + 8\left[\mathrm{Li}_4\left(\frac{1}{z+1}\right) + \mathrm{Li}_4\left(\frac{z}{z+1}\right)\right] - \frac{23\pi^2}{18}$$

$$z = \frac{x_{1,3}^2}{x_{2,4}^2} = \frac{s}{t}$$

$$r_0 = b_{1245} + b_{2345} - b_{12345}, r_1 = b_{2345}$$

$$F_5^{(0)} = r_0$$

$$F_5^{(1)} = \sum_{i=1}^{5} (r_i - r_0)\,\mathrm{Pent}_{\,i-1,i+1}$$

$$F_5^{(2)} = \sum_{i=0}^{5} r_i g_{5,i}^{(2)}$$

$$\rho_5(r_0) = r_0, \rho_5(r_i) = r_{6-i}, i = 1, \ldots, 5$$

$$|\mathbf{p}_1| \simeq |\mathbf{p}_2| \simeq |\mathbf{p}_3| \simeq |\mathbf{p}_4|$$

$$|p_1^+| \gg |p_2^+| \gg |p_3^+| \gg |p_4^+|,$$
$$|p_1^-| \ll |p_2^-| \ll |p_3^-| \ll |p_4^-|,$$

$$s_{12} = \frac{s_1}{x}, s_{23} = \frac{s_2}{x}, s_{34} = \frac{s_3}{x}, s_{56} = \frac{s_1 s_2 s_3}{\kappa^2 |z_1 - z_2|^2 x^3}, s_{345} = -|z_2|^2\kappa$$
$$s_{123} = \frac{s_1 s_2}{\kappa |z_1 - z_2|^2 x^2}, s_{234} = \frac{s_2 s_3}{\kappa x^2}, s_{16} = -|z_1|^2\kappa, s_{45} = -|1 - z_2|^2\kappa$$

$$F_6^{(0)} = \frac{s_1 s_2 s_3 \bar{z}_1 (1 - z_2)}{\kappa |z_1 - z_2|^2 x^3} + O(1/x^2)$$

$$R_1, R_2, R_5, R_6, R_7, R_9, R_{10}, R_{14} = -F_6^{(0)} + O(1/x^2).$$

$$\frac{F_6^{(L)}}{F_6^{(0)}} = -G_1^{(L)} - G_2^{(L)} - G_5^{(L)} - G_6^{(L)} - G_7^{(L)} - G_9^{(L)} - G_{10}^{(L)} - G_{14}^{(L)} + O(\log^{2L}(x)/x^2)$$

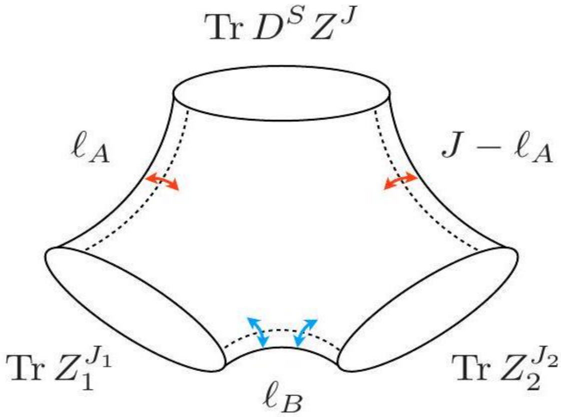
$$\frac{F_6^{(1)}}{F_6^{(0)}} = -9\log^2(x) + 2\log(x)\log\left(\frac{s_1^3 s_2^3 s_3^3}{\kappa^9 |z_1 z_2 (1 - z_2)|^2 |z_1 - z_2|^3}\right) - \log^2\left(\frac{s_1 s_2 s_3}{\kappa^3}\right)$$
$$+2\log(|z_1(z_1 - z_2)|^2)\log\left(\frac{s_1}{\kappa}\right) + 2\log(|z_2(z_1 - z_2)|^2)\log\left(\frac{s_2}{\kappa}\right)$$
$$+2\log(|(z_1 - z_2)(1 - z_2)|^2)\log\left(\frac{s_3}{\kappa}\right) + \log(|z_2|^2)\log\left(\frac{|z_2|^2}{|z_1(1 - z_2)|^2}\right)$$
$$-\log(|z_1 - z_2|^2)\log(|z_1 z_2 (z_1 - z_2)|^2) + O(x\log^2(x))$$

$$\frac{F_6^{(2)}}{F_6^{(0)}} - \frac{1}{2}\left(\frac{F_6^{(1)}}{F_6^{(0)}}\right)^2 \sim O(x\log^4(x))$$

$$\log\left(\frac{F_6}{g^2 F_6^{(0)}}\right) \sim g^2 \frac{F_6^{(1)}}{F_6^{(0)}} + O(g^6)$$

$$C_{123} \sim \langle \mathrm{Tr} Z_1^{J_1} \mathrm{Tr} Z_2^{J_2} \mathrm{Tr} D^S Z^J \rangle$$

$$\mathrm{Tr} D^S Z^J + \cdots$$



$$\frac{C_{123}}{C_{123}^{(0)}} = \mathcal{N}\mathcal{A}\mathcal{B},$$

$$\ell_A = \frac{|J_1 - J_2| + J}{2}, \ell_B = \frac{J_1 + J_2 - J}{2}$$

$$\mathcal{B} = 1 + \sum_{a=1}^{\infty} \int \frac{du}{2\pi} e^{-\frac{1}{2}(J_1+J_2)\tilde{E}_a(u)} \tilde{\mu}_a(u) \mathbf{t}_{a,1}(u) + \cdots,$$

$$\tilde{\mu}_a(u) = \frac{a}{g^2 (x^{[+a]} x^{[-a]})^2} \prod_{\sigma_1,\sigma_2=\pm} \left(1 - \frac{1}{x^{[\sigma_1 a]} x^{[\sigma_2 a]}}\right)^{-1},$$

$$\tilde{E}_a(u) = \log\left(x^{[+a]} x^{[-a]}\right)$$

$$x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2g}$$

$$\mathbf{t}_{a,1}^{\mathrm{phys}}(u) = -\sum_{b=1}^{4} \mathbf{P}_b^{[+a]}(u) \mathbf{P}^{b[-a]}(u) + \mathcal{O}(S^2)$$

$$\mathbf{P}_1 = \mathbf{P}^4 = \epsilon x^{-J/2}, \mathbf{P}_2 = -\mathbf{P}^3 = -\epsilon x^{\frac{J}{2}} \sum_{n=\frac{J}{2}+1}^{\infty} I_{2n-1} x^{1-2n},$$

$$\mathbf{P}_3 = \mathbf{P}^2 = \epsilon\left(x^{-J/2} - x^{J/2}\right),$$

$$\mathbf{P}_4 = -\mathbf{P}^1 = \epsilon x^{J/2} \sum_{n=J/2+1}^{\infty} I_{2n-1} x^{1-2n} - \epsilon x^{-J/2} \sum_{n=1-J/2}^{\infty} I_{2n-1} x^{2n-1},$$

$$\epsilon^2 = \frac{2\pi i S}{J I_J(4\pi g)}$$

$$\gamma = \Delta - S - J = \gamma_J^{(1)} S + \mathcal{O}(S^2), \gamma_J^{(1)} = \frac{4\pi g I_{J+1}(4\pi g)}{J I_J(4\pi g)}$$

$$\mathbf{t}_{a,1}(u) = -\sum_{b=1}^{4} \mathbf{P}_b^{[+a]}(u) \widetilde{\mathbf{P}}^{b[-a]}(u) + \mathcal{O}(S^2)$$

$$\mathbf{t}_{a,1}(u) \sim e^{2\pi u}$$

$$\mathcal{A} = 1 + \sum_{a\geqslant 1}\left(\oint\!\!\!\oint\!\!\!\oint_{\text{roots}} + \int\right)\frac{du}{2\pi}e^{-\frac{1}{2}(J_1-J_2)\tilde{E}_a(u)}\tilde{\mu}_a(u)\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^+_{a,0}(u)}$$
$$+\sum_{a\geqslant 1}\int\frac{du}{2\pi}e^{-\frac{1}{2}(J_2-J_1)\tilde{E}_a(u)}\tilde{\mu}_a(u)\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^-_{a,0}(u)} + \cdots$$

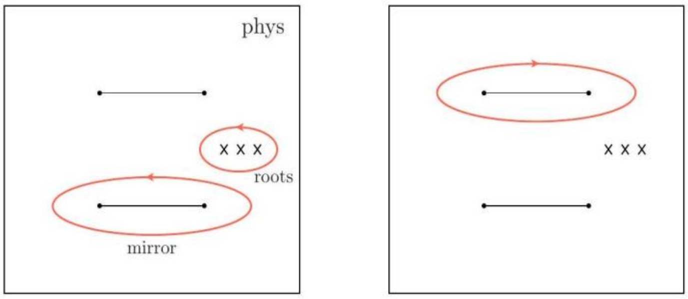
$$\sum_{a\geqslant 1}\oint\!\!\!\oint\!\!\!\oint\frac{du}{2\pi}\left[e^{-\frac{1}{2}(J_1-J_2)\tilde{E}_a(u)}\tilde{\mu}_a(u)\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^+_{a,0}(u)}\right]^{\circlearrowleft}$$
$$+\sum_{a\geqslant 1}\oint\!\!\!\oint\!\!\!\oint\frac{du}{2\pi}e^{-\frac{1}{2}(J_2-J_1)\tilde{E}_a(u)}\tilde{\mu}_a(u)\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^-_{a,0}(u)}$$

$$\left(x^{[\pm a]}\right)^{\circlearrowleft} = \frac{1}{x^{[\pm a]}}$$

$$\mathbf{t}_{a,1}(u) = \left[\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^+_{a,0}(u)}\right]^{\circlearrowleft} + \frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^-_{a,0}(u)} + \mathcal{O}(S^2)$$

$$\mathbf{T}_{a,1}(u) = -\left(\mathbf{P}_1^{[+a]}\mathbf{P}_2^{[-a]} - \mathbf{P}_2^{[+a]}\mathbf{P}_1^{[-a]}\right), \mathbf{T}_{a,0}(u) = 1$$

$$\mathcal{A} = 1 + SF_J(-\ell_A) + \mathcal{O}(S^2), \mathcal{B} = 1 + SF_J(\ell_B) + \mathcal{O}(S^2)$$



$$F_J(\ell) = \sum_{a=1}^{\infty}\oint\!\!\!\oint\!\!\!\oint\frac{du}{2\pi}e^{-\left(\ell+\frac{1}{2}J\right)\tilde{E}_a(u)}\tilde{\mu}_a(u)t_a(u),$$

$$F_J(\ell) = F_J(-\ell - J).$$

$$x = x^{[+a]}, y = x^{[-a]}$$

$$F_J(\ell) = \int_0^{\infty}\frac{dte^t}{(1-e^t)^2}\oint\!\!\!\oint\!\!\!\oint\frac{dxdy}{(2\pi)^2}\frac{e^{i(u-v)t}t_J(x,y)}{(xy)^{\ell+J/2}(xy-1)^2}$$

$$e^{i(u-v)t} \to e^{i(u-v)t} - 1 - i(u-v)t$$

$$t_J(x,y) = \frac{i}{(xy)^{J/2}}\left[\sum_{n=1}^{J/2}\mathrm{I}_{2n-1}\left(x^J(x^{1-2n} - y^{2n-1}) + y^J(x^{2n-1} - y^{1-2n})\right)\right.$$
$$\left. + \sum_{n=1}^{\infty}\mathrm{I}_{2n-1}(x^{1-2n} - y^{2n-1})(y^J - x^J)\right]$$

$$\mathrm{I}_n = \frac{2\pi I_n(4\pi g)}{J I_J(4\pi g)}.$$

ℓ_B	$F_J(\ell_B)$ for $J = 2$
1	$3g^4(4\zeta_2\zeta_3 + 5\zeta_5) - 48g^6(\zeta_3\zeta_4 + \zeta_2\zeta_5 + 7\zeta_7) + 4g^8(15\zeta_4\zeta_5 + 63\zeta_3\zeta_6 - 56\zeta_2\zeta_7 + 1470\zeta_9)$

2	$4g^6(-3\zeta_3\zeta_4+9\zeta_2\zeta_5+14\zeta_7)+84g^8(\zeta_3\zeta_6-4\zeta_2\zeta_7-20\zeta_9)$
3	$2g^8(-30\zeta_4\zeta_5+56\zeta_2\zeta_7+105\zeta_9)$

ℓ_B	$F_J(\ell_B)$ for $J=4$
1	$\frac{g^4}{\zeta_2}(9\zeta_3\zeta_4+10\zeta_2\zeta_5-7\zeta_7)+\frac{g^6}{\zeta_2}\left(-60\zeta_4\zeta_5-\frac{48}{5}\zeta_3\zeta_6-\frac{1152}{5}\zeta_2\zeta_7+240\zeta_9\right)$
2	$\frac{g^6}{\zeta_2}(30\zeta_4\zeta_5-3\zeta_3\zeta_6+36\zeta_2\zeta_7-30\zeta_9)$

$$\int_0^\infty \frac{dte^t}{(1-e^t)^2}t^n=\Gamma(n+1)\zeta_n,\ \text{for}\ n>1$$

$$F_{J=2}(-1)=-8g^2\zeta_3+g^4(-32\zeta_2\zeta_3+90\zeta_5)+g^6(160\zeta_3\zeta_4+288\zeta_2\zeta_5-1120\zeta_7)$$
$$+g^8(-1440\zeta_4\zeta_5-896\zeta_3\zeta_6-3360\zeta_2\zeta_7+14700\zeta_9)+\mathcal{O}(g^{10})$$

$$\int_0^\infty dt\frac{e^{i(u-v)t}-1}{e^t-1}=\psi(1)-\psi(1-iu+iv)$$

$$F_J(\ell)=-\frac{ig}{2}\oiiint\frac{dxdy}{(2\pi)^2xy}\frac{x-y}{xy-1}t_J(x,y)(\psi(1+iu-iv)-\psi(1)+(u\leftrightarrow v)).$$

$$\frac{1}{2}(\psi(1+iu-iv)-\psi(1)+(u\leftrightarrow v))=\sum_{k=1}^{\infty}(-1)^{k+1}g^{2k}\zeta_{2k+1}(x-y)^{2k}(1-1/xy)^{2k}$$

$$F_J(\ell)=f_J(\ell)+f_J(-J-\ell)$$

$$f_J(\ell)=\sum_{k=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^{k+\ell+1}g^{2k+1}\Gamma(2k)\Gamma(2k+2)\zeta_{2k+1}\varepsilon(n)\mathrm{I}_{2n-J-1}}{\Gamma(1+k+n)\Gamma(2+k-n)\Gamma(k+\ell+n)\Gamma(1+k-\ell-n)}$$

$$J_\mu(2gt)J_\nu(2gt)=\int\frac{dz}{2\pi i}\frac{\Gamma(-z)\Gamma(2z+\mu+\nu+1)(gt)^{2z+\mu+\nu}}{\Gamma(z+\mu+1)\Gamma(z+\nu+1)\Gamma(z+\mu+\nu+1)}$$

$$f_J(\ell)=g\sum_{n\in\mathbb{Z}}(-1)^{n+1}\varepsilon(n)\mathrm{I}_{2n-J-1}\sum_{m=0}^{\infty}\left(c_{\ell+m,\ell+2n+m}+c_{\ell+m+1,\ell+2n+m-1}\right),$$

$$c_{i,j}=\int_0^\infty\frac{dt}{e^t-1}\left(J_i(2gt)J_j(2gt)-\delta_{i,0}\delta_{j,0}\right)$$

$$\sum_{m=0}^{\infty}J_{\ell+m+\delta}(2gt)J_{\ell+2n+m-\delta}(2gt)$$
$$=\frac{gt}{2(n-\delta)}(J_{\ell+\delta}(2gt)J_{\ell+2n-\delta-1}(2gt)-J_{\ell+\delta-1}(2gt)J_{\ell+2n-\delta}(2gt))$$

$$\sum_{m=0}^{\infty}J_{\ell+m+\delta}(2gt)J_{\ell+m+\delta}(2gt)=\frac{1}{2}\left(1-\varepsilon(\ell+\delta)\sum_{m=\frac{1}{2}-\left|\ell+\delta-\frac{1}{2}\right|}^{\left|\ell+\delta-\frac{1}{2}\right|-\frac{1}{2}}J_m(2gt)J_m(2gt)\right).$$

$$f_J(\ell)=\frac{2\pi g}{J}[\delta_{\ell>0}(\log g-\psi(\ell))-\delta_{\ell<0}(\log g-\psi(-\ell))]+\mathcal{O}(1)$$

$$f_J(\ell) \approx \sum_{n=n_{\max}^{-}(g)}^{n_{\max}^{+}(g)} \tilde{f}_J(\ell, n),$$

$$\tilde{f}_J\left(\ell, n_{\max}^{\pm}(g)\right) \leq \text{ precision}$$

$$f_J(\ell) \approx \sum_{i=-1}^{i_{\max}} g^{-i}(c_i + d_i \log\ g)$$

$$\begin{aligned} f_J(\ell) =& \frac{\gamma_J^{(1)}}{4}(\log\ \lambda - 2\log\ (4\pi) - 2\psi(\ell)) - \frac{1}{2J} + \frac{1}{2}(\gamma_{\mathrm{E}} - \log\ (8\pi)) + \frac{3+4J}{8\sqrt{\lambda}} \\ &+ \frac{J+2\ell}{4J}(\log\ (8\sqrt{\lambda}) + \gamma_{\mathrm{E}}) + \frac{J+2\ell}{\sqrt{\lambda}J}\left(\frac{2J^2-1}{8} + \frac{1-J^2-\ell(J+\ell)}{6}\zeta_2\right) \\ &+ \mathcal{O}\left(\frac{1}{\lambda}\right) \end{aligned}$$

$$F_J(\ell) = \Psi_J(\ell) + P_J(\ell)$$

$$\begin{aligned} \Psi_J(\ell)\ &= \frac{\gamma_J^{(1)}}{2}(\psi(J+\ell) - \psi(\ell)) + \psi(J+\ell) + \ell) + C \\ \Psi_J(-\ell)\ &= \frac{\gamma_J^{(1)}}{2}(\psi(J-\ell) + \psi(\ell) - \log\ \lambda + 2\log\ (4\pi)) + \psi(J-\ell) + \psi(\ell) + C \end{aligned}$$

$$\begin{aligned} P_J(\ell) =& \frac{3}{4\sqrt{\lambda}} + \frac{1}{\lambda}\left(\frac{17-8J^2}{32} + \frac{11-8J^2-24\ell(J+\ell)}{48}\zeta_3\right) \\ &+ \frac{1}{\lambda^{3/2}}\left(\frac{11-9J^2}{16} + \frac{11-8J^2-18\ell(J+\ell)}{48}\zeta_3\right) \\ &+ \delta_{-J<\ell<0}\left(\frac{1}{J} - \frac{J}{\sqrt{\lambda}} - \frac{J}{2\lambda} - \frac{13J-4J^3}{24\lambda^{3/2}}\right) \\ &+ \mathcal{O}\left(\frac{1}{\lambda^2}\right) \end{aligned}$$

$$\begin{aligned} \mathcal{B}\ &= \left(\frac{e^{\gamma_{\mathrm{E}}S}}{2^S\lambda^{S/2}}\right)\frac{\Gamma(\ell_B - \gamma/2)\Gamma(J+\ell_B+S+\gamma/2)}{\Gamma(\ell_B)\Gamma(J+\ell_B)} \times \mathcal{D}_{\mathcal{B}} \\ \mathcal{A}\ &= \left(\frac{e^{\gamma_{\mathrm{E}}S}(4\pi)^{S+\gamma}}{2^S\lambda^{S+\gamma/2}}\right)\frac{\Gamma(\ell_A+\gamma/2+S)\Gamma(J-\ell_A+S+\gamma/2)}{\Gamma(\ell_A)\Gamma(J-\ell_A)} \times \mathcal{D}_{\mathcal{A}} \end{aligned}$$

$$\log\ \mathcal{D}_{\mathcal{B}} = SP_J(\ell_B) + \mathcal{O}(S^2), \log\ \mathcal{D}_{\mathcal{A}} = SP_J(-\ell_A) + \mathcal{O}(S^2),$$

$$\frac{C_{123}}{C_{123}^{(0)}} = \frac{\Gamma[\,\mathrm{AdS}\,]}{\Gamma[\,\mathrm{sphere}\,]} \times \frac{\mathcal{D}_{J_1J_2J}(S)}{\lambda^{S/4}\Gamma\left(1+\frac{S}{2}\right)},$$

$$\Gamma[AdS] = \frac{\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta+S}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta_1+\Delta+S}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta+S}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta+S}{2}\right)}{\sqrt{\Gamma(\Delta+S)\Gamma(\Delta+S-1)}},$$

$$\Gamma[\,\mathrm{sphere}\,] = \Gamma[\,\mathrm{AdS}\,]_{\Delta\to J, S\to 0}$$

$$\log\ \mathcal{D} = \mathcal{D}_1 S + \frac{\mathcal{D}_2}{\sqrt{\lambda}}S^2 + \frac{\mathcal{D}_3}{\lambda}S^3 + \mathcal{O}\left(\frac{S^4}{\lambda^{3/2}}\right)$$

$$\mathcal{D}_n = \mathcal{D}_n^{(0)} + \frac{\mathcal{D}_n^{(1)}}{\sqrt{\lambda}} + \frac{\mathcal{D}_n^{(2)}}{\lambda} + \cdots$$

$$\log\ \mathcal{D} = \sqrt{\lambda}\mathcal{D}^{\mathrm{cl}} + \mathcal{O}(1)$$

$$\log \mathcal{D} = \sum_{k=1}^{\infty} \frac{1}{(\sqrt{\lambda})^{k-1}} \sum_{n=1}^{k} \mathcal{D}_n^{(k-n)} \mathcal{S}^n$$

$$\Delta^2 = J^2 + \sqrt{\lambda}\left(A_1 \mathcal{S} + \frac{A_2}{\sqrt{\lambda}} \mathcal{S}^2 + \frac{A_3}{\lambda} \mathcal{S}^3 + \mathcal{O}\left(\frac{\mathcal{S}^4}{\lambda^{3/2}}\right)\right)$$

$$\begin{aligned}\log \tilde{\lambda}_{\Delta,\mathcal{S}}^2 = & \frac{\mathcal{S}\left(\frac{17}{6} + \mathcal{S} + \left(-\frac{7}{12} + \zeta_3\right)\mathcal{S}^2\right)}{\Delta^2} \\ & + \frac{\mathcal{S}\left(\frac{511}{60} + 6\mathcal{S} + \left(\frac{1}{12} - 2\zeta_3\right)\mathcal{S}^2 - \left(\frac{13}{8} + 6\zeta_3\right)\mathcal{S}^3 + \left(\frac{31}{40} - \frac{3}{2}\zeta_5\right)\mathcal{S}^4\right)}{\Delta^4} \\ & + \mathcal{O}\left(\frac{1}{\Delta^6}\right)\end{aligned}$$

$$\Delta^2 = 2\sqrt{\lambda}\mathcal{S} + \left(4 - \mathcal{S} + \frac{3}{2}\mathcal{S}^2\right) + \frac{1}{\sqrt{\lambda}}\left(\frac{15}{4}\mathcal{S} + \frac{3 - 24\zeta_3}{8}\mathcal{S}^2 - \frac{3}{8}\mathcal{S}^3\right) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

$$\mathcal{D}_{222}^2 = \mathcal{R} \times \tilde{\lambda}_{\Delta,\mathcal{S}}^2$$

$$\mathcal{R} = \left(\frac{\sqrt{\lambda}\mathcal{S}}{2}\right)^{\mathcal{S}} \frac{\Delta(\Delta - \mathcal{S})}{(\Delta + \mathcal{S})^2} \frac{\Gamma\left(\frac{1 + \Delta + \mathcal{S}}{2}\right)\Gamma\left(\frac{\Delta - \mathcal{S}}{2}\right)^3}{\Gamma\left(\frac{1 + \Delta - \mathcal{S}}{2}\right)\Gamma\left(\frac{\Delta + \mathcal{S}}{2}\right)^3}$$

$$\begin{aligned}\log \mathcal{D}_{222} = & \frac{1}{\sqrt{\lambda}}\left[\frac{5}{8}\mathcal{S} - \frac{7 - 4\zeta_3}{16}\mathcal{S}^2\right] \\ & + \frac{1}{\lambda}\left[-\frac{13 + 24\zeta_3}{32}\mathcal{S} - \frac{49 - 8\zeta_3}{64}\mathcal{S}^2 + \frac{25 - 12\zeta_3 - 12\zeta_5}{64}\mathcal{S}^3\right] \\ & + \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right)\end{aligned}$$

$$\log \mathcal{D}_{J_1 J_2 J} = \log \mathcal{D}_{222} + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

$$\log C_{123} = \sqrt{\lambda}\,\text{Area} + \mathcal{O}(1)$$

$$\text{Area} = \mathcal{A}^{\text{cl}} + \mathcal{B}^{\text{cl}} + \mathcal{N}^{\text{cl}},$$

$$\log \mathcal{A} = \sqrt{\lambda}\mathcal{A}^{\text{cl}} + \mathcal{O}(1)$$

$$V'(x) = R(x + i0) + R(x - i0), V'(x) = \text{sgn}(x) - \frac{2\mathcal{J}x}{x^2 - 1}$$

$$R(x) = 2x \int_a^b \frac{dyV'(y)}{x^2 - y^2} \sqrt{\frac{(x^2 - b^2)(x^2 - a^2)}{(b^2 - y^2)(y^2 - a^2)}}$$

$$R(x) \sim 2\pi x(\mathcal{S} - \mathcal{E} + \mathcal{J}), R(x) \sim \frac{2\pi}{x}(\mathcal{S} + \mathcal{E} - \mathcal{J})$$

$$\begin{aligned}\mathcal{J} &= \frac{\sqrt{(a^2 - 1)(b^2 - 1)}}{\pi b} K\left(1 - \frac{a^2}{b^2}\right) \\ \mathcal{S} &= \frac{ab + 1}{2\pi ab}\left[bE\left(1 - \frac{a^2}{b^2}\right) - aK\left(1 - \frac{a^2}{b^2}\right)\right] \\ \mathcal{E} &= \frac{ab - 1}{2\pi ab}\left[bE\left(1 - \frac{a^2}{b^2}\right) + aK\left(1 - \frac{a^2}{b^2}\right)\right]\end{aligned}$$

$$\rho(x) = \frac{R(x - i0) - R(x + i0)}{2\pi i} = -\frac{2x}{\pi} \int_a^b \frac{dyV'(y)}{x^2 - y^2} \sqrt{\frac{(b^2 - x^2)(x^2 - a^2)}{(b^2 - y^2)(y^2 - a^2)}},$$

$$\mathcal{A}^{\text{cl}} = \mathcal{A}^{\text{cl}}_{\text{asy}} + I_1[\mathcal{L}_A] + I_1[\mathcal{J} - \mathcal{L}_A]$$
$$\mathcal{B}^{\text{cl}} = I_{-1}[\mathcal{L}_B] + I_1[\mathcal{J} + \mathcal{L}_B]$$
$$\mathcal{N}^{\text{cl}} = \mathcal{N}^{\text{cl}}_{\text{asy}} - I_2[\mathcal{J}]$$

$$\mathcal{L}_A = \frac{\mathcal{J}_1 - \mathcal{J}_2 + \mathcal{J}}{2}, \mathcal{L}_B = \frac{\mathcal{J}_1 + \mathcal{J}_2 - \mathcal{J}}{2}.$$

$$I_q[\mathcal{L}] = \int_{U^-} \frac{dx(x - 1/x)}{8\pi^2 x}\left[\text{Li}_2\left(e^{\frac{4\pi i\mathcal{L}x}{x^2-1}+iqR(x)}\right) + \text{Li}_2\left(e^{\frac{4\pi i\mathcal{L}x}{x^2-1}-iqR(1/x)}\right)\right]$$
$$-(\text{ same with } R \to 0)$$

$$U^- = \{x \in \mathbb{C}: |x| = 1, \text{Im}x \leqslant 0\},$$

$$\mathcal{N}^{\text{cl}}_{\text{asy}} = -\int_a^b \frac{dx(x - 1/x)}{4\pi^2 x}\left[\text{Li}_2\left(e^{-2\pi\rho(x)}\right) + \pi^2\rho^2(x) - \zeta_2\right],$$

$$V'(\alpha) = 0 \;\Rightarrow\; \alpha = \mathcal{J} + \sqrt{1 + \mathcal{J}^2}.$$

$$a = \alpha - 2\alpha\sqrt{\frac{\alpha}{\alpha^2+1}}\sqrt{\mathcal{S}} + \frac{\alpha^2(3\alpha^4 + 6\alpha^2 - 1)}{(\alpha^2-1)(\alpha^2+1)^2}\mathcal{S} + \mathcal{O}\left(\mathcal{S}^{3/2}\right)$$
$$b = \alpha + 2\alpha\sqrt{\frac{\alpha}{\alpha^2+1}}\sqrt{\mathcal{S}} + \frac{\alpha^2(3\alpha^4 + 6\alpha^2 - 1)}{(\alpha^2-1)(\alpha^2+1)^2}\mathcal{S} + \mathcal{O}\left(\mathcal{S}^{3/2}\right)$$

$$R(x) = R^{(1)}(x)\mathcal{S} + R^{(2)}(x)\mathcal{S}^2 + \cdots$$

$$R^{(1)} = \frac{4\pi x\alpha^2}{(\alpha^2-1)(x^2-\alpha^2)}$$
$$R^{(2)} = -\frac{4\pi x\alpha^3\left((x^4+\alpha^6)(\alpha^4+6\alpha^2+1) - x^2\alpha^2(5\alpha^6+3\alpha^4+3\alpha^2+5)\right)}{(\alpha^2-1)^3(\alpha^2+1)^2(x^2-\alpha^2)^3}$$

$$\rho = \sqrt{1-t^2}\left[\frac{2\sqrt{\alpha(\alpha^2+1)}}{\alpha^2-1}\sqrt{\mathcal{S}} - \frac{4\alpha^3(\alpha^2+3)t}{(\alpha^2-1)^2(\alpha^2+1)}\mathcal{S}\right.$$
$$+\frac{\alpha^{3/2}\left(16\alpha^2(\alpha^4+6\alpha^2+1)(\alpha^2+1)t^2 - (9\alpha^8+48\alpha^6+70\alpha^4+1)\right)}{2(\alpha^2-1)^3(\alpha^2+1)^{5/2}}\mathcal{S}^{3/2}$$
$$\left.+\mathcal{O}(\mathcal{S}^2)\right]$$

$$I_q[\mathcal{L}] = qI[\mathcal{L}]\mathcal{S} + \mathcal{O}(\mathcal{S}^2)$$

$$I[\mathcal{L}] = \int_{U^-} \frac{dx}{2\pi i}\frac{x(x-1/x)^2\alpha^2(\alpha^2+1)}{(x^2-\alpha^2)(\alpha^2x^2-1)(\alpha^2-1)}\log\left(1 - e^{\frac{4\pi i\mathcal{L}x}{x^2-1}}\right)$$

$$x = \frac{-i \pm \sqrt{E^2-1}}{E} \;\Rightarrow\; E = \frac{-2ix}{x^2-1}$$

$$I[\mathcal{L}] = \oint \frac{i\sqrt{1+\mathcal{J}^2}dE}{\pi\mathcal{J}E\sqrt{1-E^2}(1+\mathcal{J}^2E^2)}\log\left(1 - e^{-2\pi\mathcal{L}E}\right)$$

$$I[\mathcal{J}] = I_0[\mathcal{J}] + I_\zeta[\mathcal{J}]$$

$$I_0[\mathcal{L}] = \log\Gamma\left(\frac{\mathcal{L}}{\mathcal{J}}\right) + \frac{\sqrt{1+\mathcal{J}^2}+\mathcal{J}}{2\mathcal{J}}\log\mathcal{L} + \frac{(2\mathcal{L}-\mathcal{J})}{2\mathcal{J}}\log\mathcal{J} + \frac{\mathcal{L}}{\mathcal{J}}\gamma_{\text{E}}$$
$$-\frac{\mathcal{L}}{\mathcal{J}}\log\left(1+\sqrt{1+\mathcal{J}^2}\right) - \frac{1}{2}\log\left(\frac{\sqrt{1+\mathcal{J}^2}+\mathcal{J}}{2}\right) + \frac{\sqrt{1+\mathcal{J}^2}-\mathcal{J}}{2\mathcal{J}}\log(4\pi)$$

$$I_\zeta[\mathcal{L}] = \sum_{m=1}^{\infty}\left[\frac{\sqrt{1+\mathcal{J}^2}}{\mathcal{J}}\log\left[\frac{\mathcal{L}+\sqrt{\mathcal{L}^2+m^2}}{m}\right] - \log\left[\frac{\mathcal{J}\sqrt{\mathcal{L}^2+m^2}+\mathcal{L}\sqrt{1+\mathcal{J}^2}}{\mathcal{J}m+\mathcal{L}}\right] - \frac{\mathcal{L}}{\mathcal{J}m}\right]$$

$$I_\zeta[\mathcal{L}] = \frac{1}{\mathcal{J}}\sum_{k=1}^{\infty} \frac{\zeta_{2k+1}}{2k+1} c_k(\mathcal{J})\mathcal{L}^{2k+1}$$

$$c_k(\mathcal{J}) = \sqrt{1+\mathcal{J}^2}\sum_{n=k}^{\infty} (-1)^n \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(1+n)}\mathcal{J}^{2(n-k)}$$

$$\mathcal{A}_{\text{asy}}^{\text{cl}} = I_{\text{asy}}[\mathcal{L}_A]\mathcal{S} + \mathcal{O}(\mathcal{S}^2)$$

$$I_{\text{asy}}[\mathcal{L}_A] = -\log\, \Gamma\left(\frac{\mathcal{L}_A}{\mathcal{J}}\right)\Gamma\left(1-\frac{\mathcal{L}_A}{\mathcal{J}}\right) + \log\,(2\pi)$$

$$\mathcal{A}^{\text{cl}} = \mathcal{A}_1^{\text{cl}}\mathcal{S} + \mathcal{O}(\mathcal{S}^2), \mathcal{B}^{\text{cl}} = \mathcal{B}_1^{\text{cl}}\mathcal{S} + \mathcal{O}(\mathcal{S}^2),$$

$$\mathcal{A}_1^{\text{cl}} = I_{\text{asy}}[\mathcal{L}_A] + I[\mathcal{L}_A] + I[\mathcal{J}-\mathcal{L}_A], \mathcal{B}_1^{\text{cl}} = I[\mathcal{J}+\mathcal{L}_B] - I[\mathcal{L}_B]$$

$$\begin{aligned}\mathcal{A}_1^{\text{cl}} =& \frac{1+\delta_1}{2}\log\left(\mathcal{L}_A(\mathcal{J}-\mathcal{L}_A)\right) + \delta_1\log\,(4\pi) + \gamma_{\text{E}}\\ &-\log\,[\mathcal{J}(1+\delta_1)(1+\mathcal{J}\delta_1)] + I_\zeta[\mathcal{L}_A] + I_\zeta[\mathcal{J}-\mathcal{L}_A]\\ \mathcal{B}_1^{\text{cl}} =& \frac{1+\delta_1}{2}\log\,(\mathcal{L}_B+\mathcal{J}) + \frac{1-\delta_1}{2}\log\,\mathcal{L}_B + \gamma_{\text{E}}\\ &-\log\,(1+\mathcal{J}\delta_1) - I_\zeta[\mathcal{L}_B] + I_\zeta[\mathcal{J}+\mathcal{L}_B]\end{aligned}$$

$$\mathcal{N}^{\text{cl}} = -\frac{1}{2}(\log\,(\mathcal{S}/2)-1)\mathcal{S} + \mathcal{N}_1^{\text{cl}}\mathcal{S} + \mathcal{O}(\mathcal{S}^2)$$

$$-\int_a^b \frac{(x-1/x)dx}{2\pi x}\rho(x)\log\left(\frac{2\pi\rho(x)}{e}\right) = -\frac{1}{2}\log\left[\frac{2\sqrt{1+\mathcal{J}^2}\pi^2\mathcal{S}}{e\mathcal{J}^2}\right]\mathcal{S} + \mathcal{O}(\mathcal{S}^2)$$

$$\begin{aligned}\mathcal{N}_1^{\text{cl}} =& -(1+\delta_1)\log\,\mathcal{J} - \delta_1\log\,(4\pi) - 2\gamma_{\text{E}}\\ &-\frac{1}{4}\log\,(1+\mathcal{J}^2) + \log\,[\mathcal{J}(1+\delta_1)(1+\mathcal{J}\delta_1)^2] - 2I_\zeta[\mathcal{J}]\end{aligned}$$

$$\mathcal{D}^{\text{cl}} = \mathcal{D}_1^{\text{cl}}\mathcal{S} + \mathcal{O}(\mathcal{S}^2)$$

$$\mathcal{D}_1^{\text{cl}} = -\frac{1}{4}\log\,(1+\mathcal{J}^2) + \sum_{\mathcal{L}\in L} I_\zeta[\mathcal{L}] - 2I_\zeta[\mathcal{J}]$$

$$\mathcal{D}_1^{\text{cl}} = -\frac{1}{4}\log\,(1+\mathcal{J}^2) + \sum_{k=1}^{\infty}\frac{\zeta_{2k+1}}{2k+1}c_k(\mathcal{J})P_k(\mathcal{J}_1,\mathcal{J}_2,\mathcal{J})$$

$$P_k = -2\mathcal{J}^{2k} + \frac{1}{2^{2k+1}\mathcal{J}}\sum_{\sigma_1,\sigma_2=\pm}(\mathcal{J}+\sigma_1\mathcal{J}_1+\sigma_2\mathcal{J}_2)^{2k+1}$$

$$F\left[\frac{4\pi\mathcal{L}x}{x^2-1}+R(x)\right] - F\left[\frac{4\pi\mathcal{L}x}{x^2-1}\right] = \Delta\left(\partial_{\mathcal{J}},\partial_{\mathcal{L}}\right)\left[R^{(1)}(x)F'\left[\frac{4\pi\mathcal{L}x}{x^2-1}\right]\right]$$

$$\Delta\left(\partial_{\mathcal{J}},\partial_{\mathcal{L}}\right) = \mathcal{S} + \sum_{n=2}^{\infty}\mathcal{S}^n\Delta^{(n)}\left(\partial_{\mathcal{J}},\partial_{\mathcal{L}}\right)$$

$$I_q[\mathcal{L}] = q\Delta\left(\partial_{\mathcal{J}},q\partial_{\mathcal{L}}\right)I[\mathcal{L}]$$

$$\Delta^{(n)}\left(\partial_{\mathcal{J}},\partial_{\mathcal{L}}\right) = \sum_{j+l=0}^{2(n-1)} c_{j,l}^{(n)}(\mathcal{J})\partial_{\mathcal{J}}^j\partial_{\mathcal{L}}^l$$

$$c_{j,l\geqslant n}^{(n)} = c_{0,0}^{(n)} = 0$$

$$\Delta^{(2)}(\partial_{\mathcal{J}},\partial_{\mathcal{L}})=\frac{\sqrt{1+\mathcal{J}^2}}{4}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})\partial_{\mathcal{J}}+\frac{\sqrt{1+\mathcal{J}^2}}{\mathcal{J}}\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right)$$

$$e^{\mathcal{J}\partial_{\mathcal{L}}}\Delta(\partial_{\mathcal{J}},\partial_{\mathcal{L}})=\Delta(\partial_{\mathcal{J}},-\partial_{\mathcal{L}})e^{\mathcal{J}\partial_{\mathcal{L}}}\Rightarrow\Delta(\partial_{\mathcal{J}},\partial_{\mathcal{L}})=\Delta(\partial_{\mathcal{J}}+\partial_{\mathcal{L}},-\partial_{\mathcal{L}})$$

$$\mathcal{A}^{\rm cl}=\Delta(\partial_{\mathcal{J}},\partial_{\mathcal{L}_A})\mathcal{A}_1^{\rm cl},\mathcal{B}^{\rm cl}=\Delta(\partial_{\mathcal{J}},-\partial_{\mathcal{L}_B})\mathcal{B}_1^{\rm cl},$$

$$\mathcal{N}^{\rm cl}=\mathcal{N}_{\rm asy}^{\rm cl}-2\big[\Delta(\partial_{\mathcal{J}},2\partial_{\mathcal{L}})I[\mathcal{L}]\big]_{\mathcal{L}=\mathcal{J}}$$

$$\Delta(\partial_{\mathcal{J}},\pm\partial_{\mathcal{L}})\left[\frac{1\pm\delta_1}{2}\log\,\mathcal{L}\right]$$
$$=\frac{2\mathcal{L}+\mathcal{S}\pm(\mathcal{E}-\mathcal{J})}{2}\log\left[\frac{2\mathcal{L}+\mathcal{S}\pm(\mathcal{E}-\mathcal{J})}{2}\right]-\mathcal{L}\log\,\mathcal{L}-\frac{\mathcal{S}\pm(\mathcal{E}-\mathcal{J})}{2}$$

$$\mathcal{D}^{\rm cl}=\mathcal{D}_\rho^{\rm cl}+\mathcal{D}_\zeta^{\rm cl},$$

$$\mathcal{D}_\rho^{\rm cl}=-\int_a^b\frac{(x-1/x)dx}{4\pi x}\rho(x)\log\left[\frac{(x^2-1)^2\rho^2(x)}{2e\mathcal{S}x^2}\right]$$

$$\mathcal{D}_\zeta^{\rm cl}=\sum_{\mathcal{L}\in L}\Delta(\partial_{\mathcal{J}},\partial_{\mathcal{L}})I_\zeta[\mathcal{L}]-2\big[\Delta(\partial_{\mathcal{J}},2\partial_{\mathcal{L}})I_\zeta[\mathcal{L}]\big]_{\mathcal{L}=\mathcal{J}}$$

$$\mathcal{D}_\zeta^{\rm cl}=\sum_{k=1}^{\infty}\zeta_{2k+1}\mathcal{D}_{\zeta_{2k+1}}^{\rm cl}$$

$$\mathcal{D}_\rho^{\rm cl}=-\frac{1}{4}\log\,(1+\mathcal{J}^2)\mathcal{S}-\frac{7+4\mathcal{J}^2}{16(1+\mathcal{J}^2)^{3/2}}\mathcal{S}^2+\frac{150+120\mathcal{J}^2+29\mathcal{J}^4}{384(1+\mathcal{J}^2)^3}\mathcal{S}^3$$
$$-\frac{1785+1748\mathcal{J}^2+640\mathcal{J}^4+86\mathcal{J}^6}{3072(1+\mathcal{J}^2)^{9/2}}\mathcal{S}^4+\mathcal{O}(\mathcal{S}^5)$$

$$\mathcal{D}_{\zeta_3}^{\rm cl}=\frac{\sqrt{1+\mathcal{J}^2}-1}{2\mathcal{J}^2}(\mathcal{J}^2-\vec{\mathcal{J}}^2)\mathcal{S}$$
$$+\left[\frac{\sqrt{1+\mathcal{J}^2}(3\mathcal{J}^2-\vec{\mathcal{J}}^2)}{4\mathcal{J}^4}-\frac{\mathcal{J}^2(3+2\mathcal{J}^2)(2+\mathcal{J}^2-\vec{\mathcal{J}}^2)-2\vec{\mathcal{J}}^2}{8\mathcal{J}^4(1+\mathcal{J}^2)}\right]\mathcal{S}^2$$
$$+\mathcal{O}(\mathcal{S}^3)$$

$$Z[R_{\mathcal{L}}]=\int_{-i\infty}^{i\infty}\frac{(x+1/x)}{(2\pi)^2i}\left[\frac{1}{2}\log\frac{\Gamma\left(1-\frac{R_{\mathcal{L}}(x)}{2\pi}\right)}{\Gamma\left(1+\frac{R_{\mathcal{L}}(x)}{2\pi}\right)}-\frac{\gamma_{\rm E}R_{\mathcal{L}}(x)}{2\pi}\right]dR_{\mathcal{L}}(x)-\big(R_{\mathcal{L}}\to\hat{R}_{\mathcal{L}}\big)$$

$$R_{\mathcal{L}}(x)=\frac{4\pi\mathcal{L}x}{x^2-1}+R(x),\hat{R}_{\mathcal{L}}(x)=\frac{4\pi\mathcal{L}x}{x^2-1}$$

$$Z[R_{\mathcal{L}}]=\Delta(\partial_{\mathcal{J}},\partial_{\mathcal{L}})I_\zeta[\mathcal{L}]$$

$$\mathcal{D}_\zeta^{\rm cl}=\sum_{\mathcal{L}\in L}Z[R_{\mathcal{L}}]-Z\big[2R_{\mathcal{J}/2}\big]$$

$$\frac{1}{2}\log\frac{\Gamma\left(1-\frac{R_{\mathcal{L}}(x)}{2\pi}\right)}{\Gamma\left(1+\frac{R_{\mathcal{L}}(x)}{2\pi}\right)}-\frac{\gamma_{\rm E}R_{\mathcal{L}}(x)}{2\pi}=\sum_{k=1}^{\infty}\frac{\zeta_{2k+1}}{2k+1}\left(\frac{R_{\mathcal{L}}(x)}{2\pi}\right)^{2k+1}$$

$$Z[R_{\mathcal{L}}]=\sum_{k=1}^{\infty}\frac{\zeta_{2k+1}}{2k+1}Z_k[R_{\mathcal{L}}]$$

$$Z_k[R_{\mathcal{L}}] = -\int_{-i\infty}^{i\infty} \frac{dx(x-1/x)}{4\pi i x(k+1)}\left[\left(\frac{R_{\mathcal{L}}(x)}{2\pi}\right)^{2k+2} - \left(\frac{\hat{R}_{\mathcal{L}}(x)}{2\pi}\right)^{2k+2}\right]$$

$$c_k(\mathcal{J}) = -\int_{-i\infty}^{i\infty} \frac{dx(x-1/x)}{2\pi i x}\frac{R^{(1)}(x)}{2\pi}\left(\frac{2x}{x^2-1}\right)^{2k+1}$$

$$\mathcal{D}^{\text{cl}} = \mathcal{D}_1^{\text{cl}}\mathcal{S} + \mathcal{D}_2^{\text{cl}}\mathcal{S}^2 + \mathcal{D}_3^{\text{cl}}\mathcal{S}^3 + \cdots$$

$$\mathcal{D}_1^{\text{cl}} = -\frac{\mathcal{J}^2}{4} + \frac{(\mathcal{J}^2 - \vec{\mathcal{J}}^2)}{4}\zeta_3 + \mathcal{O}(\mathcal{J}^4, \mathcal{J}_{1,2}^4, \mathcal{J}_1^2\mathcal{J}_2^2, \mathcal{J}^2\mathcal{J}_{1,2}^2)$$

$$\mathcal{D}_2^{\text{cl}} = -\frac{7}{16} + \frac{1}{4}\zeta_3 + \mathcal{O}(\mathcal{J}^2, \vec{\mathcal{J}}^2)$$
$$\mathcal{D}_3^{\text{cl}} = \frac{25}{64} - \frac{3}{16}\zeta_3 - \frac{3}{16}\zeta_5 + \mathcal{O}(\mathcal{J}^2, \vec{\mathcal{J}}^2)$$
$$\mathcal{D}_4^{\text{cl}} = -\frac{595}{1024} + \frac{29}{128}\zeta_3 + \frac{9}{32}\zeta_5 + \frac{45}{256}\zeta_7 + \mathcal{O}(\mathcal{J}^2, \vec{\mathcal{J}}^2)$$

$$1 = e^{ip_kJ}\prod_{j\neq k}^{S} S_{kj},$$

$$p_k = -i\log\left(\frac{x_k^+}{x_k^-}\right), x_k^{\pm} = x(u_k \pm i/2)$$

$$\mathcal{N} = \sqrt{\frac{H}{G}} \times e^W$$

$$G = \det_{1\leqslant j,k\leqslant S}\left[\frac{\partial}{\partial u_j}\left(Jp_k - i\sum_{l\neq k}^{S} \log S_{kl}\right)\right]$$

$$H = \prod_{k<j}^{S} \frac{(u_k-u_j)^2}{(u_k-u_j)^2+1}\exp\left[\sum_{k,j}^{S} \log\frac{(x_k^+x_j^- - 1)(x_k^-x_j^+ - 1)}{(x_k^+x_j^+ - 1)(x_k^-x_j^- - 1)}\right].$$

$$\mathcal{J} = J/\sqrt{\lambda} = \mathcal{O}(1)$$

$$n_k = \frac{2\mathcal{J}x_k}{x_k^2 - 1} + \sum_{j\neq k}^{S} \frac{4x_kx_j(x_kx_j - 1)}{\sqrt{\lambda}(x_k^2-1)(x_k-x_j)(x_j^2-1)} + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

$$x_k^{(0)} = \alpha = \mathcal{J} + \sqrt{1+\mathcal{J}^2}, k = 1, \ldots, S/2$$

$$x_k = x_k^{(0)} + \frac{x_k^{(1/2)}}{\lambda^{1/4}} + \frac{x_k^{(1)}}{\lambda^{1/2}} + \frac{x_k^{(3/2)}}{\lambda^{3/4}} + \cdots$$

$$H_{S/2}\left(\frac{x_k^{(1/2)}}{\sqrt{2}\beta}\right) = 0, k = 1, \ldots, S/2$$

$$G = G_+G_-\left(1 + \mathcal{O}\left(\frac{1}{\lambda}\right)\right)$$

$$\sqrt{G} \approx \left[\frac{4\pi^2\sqrt{1+\mathcal{J}^2}}{\mathcal{J}^2\sqrt{\lambda}}\right]^{S/2}$$
$$\times \det_{1\leqslant j,k\leqslant S/2}\left[\delta_{k=j}\left(1 + \sum_{l\neq k}^{S/2} \frac{2\beta^2}{\sqrt{\lambda}(x_k-x_l)^2}\right) - \delta_{k\neq j}\frac{2\beta^2}{\sqrt{\lambda}(x_k-x_j)^2}\right]$$

$$\sqrt{G} = \left[\frac{4\pi^2\sqrt{1+\mathcal{J}^2}}{\mathcal{J}^2\sqrt{\lambda}}\right]^{S/2} \Gamma\left(1+\frac{S}{2}\right)\exp\left(\sum_{n\geqslant 1}\frac{P_n^G(S)}{(\sqrt{\lambda})^n}\right),$$

$$\sqrt{H} = \exp\left(\sum_{n\geqslant 1}\frac{P_n^H(S)}{(\sqrt{\lambda})^n}\right)$$

$$W = \left(W^{(0)} + \frac{W^{(1)}}{\sqrt{\lambda}} + \frac{W^{(2)}}{\lambda} + \cdots\right)S + \mathcal{O}(S^2)$$

$$W^{(0)} = -2I[\mathcal{J}]$$

$$W^{(i)} = -2\left[\widehat{\Delta}^{(i)}I[\mathcal{L}]\right]_{\mathcal{L}=\mathcal{J}}$$

$$\widehat{\Delta}^{(1)} = -\frac{\sqrt{1+\mathcal{J}^2}}{2}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})\partial_{\mathcal{J}} - \frac{\sqrt{1+\mathcal{J}^2}}{\mathcal{J}}\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right) + \frac{1}{2\mathcal{J}}\left(\mathcal{J}\sqrt{1+\mathcal{J}^2}-\mathcal{R}\right)\partial_{\mathcal{L}}^2,$$

$$\mathcal{R}[f] = \text{res}_{\mathcal{J}=0}f(\mathcal{J})$$

$$\mathcal{N} = \frac{2^{2S}\lambda^{3S/2+\gamma/2}}{(4\pi)^{S+\gamma}e^{2\gamma_E S}}\sqrt{\frac{\Gamma(J)\Gamma(J-1)}{\Gamma(\Delta+S)\Gamma(\Delta+S-1)}} \times \frac{\mathcal{D}_{\mathcal{N}}}{\lambda^{S/4}\Gamma\left(1+\frac{S}{2}\right)},$$

$$\mathcal{D} = \mathcal{D}_{\mathcal{N}}\mathcal{D}_{\mathcal{A}}\mathcal{D}_{\mathcal{B}},$$

$$\Delta = \sqrt{\lambda}\mathcal{J} + S\left[\frac{\sqrt{1+\mathcal{J}^2}}{\mathcal{J}} - \frac{1}{2\mathcal{J}(1+\mathcal{J}^2)\sqrt{\lambda}} + \frac{4\mathcal{J}^2-1}{8\mathcal{J}(1+\mathcal{J}^2)^{5/2}\lambda}\right] + \mathcal{O}\left(S^2, \frac{S}{\lambda^{3/2}}\right),$$

$$\log \mathcal{D}_{\mathcal{N}} = S\left[c^{(0)} + \frac{c^{(1)}}{\sqrt{\lambda}} + \frac{c^{(2)}}{\lambda}\right] + \mathcal{O}\left(S^2, \frac{S}{\lambda^{3/2}}\right),$$

$$c^{(0)} = -\frac{1}{4}\log(1+\mathcal{J}^2) + K[\mathcal{J}]$$

$$c^{(1)} = \frac{5+2\mathcal{J}^2}{8(1+\mathcal{J}^2)^{3/2}} + \left[\widehat{\Delta}^{(1)}K[\mathcal{L}]\right]_{\mathcal{L}=\mathcal{J}}$$

$$c^{(2)} = \frac{57-12\mathcal{J}^2+4\mathcal{J}^4}{96(1+\mathcal{J}^2)^3} + \left[\widehat{\Delta}^{(2)}K[\mathcal{L}]\right]_{\mathcal{L}=\mathcal{J}}$$

$$K[\mathcal{L}] = \log\left[\frac{1}{4}\left(1+\sqrt{1+\mathcal{J}^2}\right)^2\left(\mathcal{J}+\sqrt{1+\mathcal{J}^2}\right)\right] - 2I_{\zeta}[\mathcal{L}]$$

$$\log \mathcal{D}_{\mathcal{N}} = S\left[-\frac{1}{J} - \frac{\frac{7}{8}-J}{\sqrt{\lambda}} - \frac{\left(\frac{15}{32}-\frac{J}{2}-\frac{J^2}{4}\right)+\left(\frac{5}{24}-\frac{J^2}{3}\right)\zeta_3}{\lambda}\right] + \mathcal{O}\left(S^2, \frac{S}{\lambda^{3/2}}\right)$$

$$\begin{aligned}\log \mathcal{D} &= \log \mathcal{D}_{\mathcal{N}} + \log \mathcal{D}_{\mathcal{A}} + \log \mathcal{D}_{\mathcal{B}} \\ &= S\left[\frac{5}{8\sqrt{\lambda}} + \frac{(19-8J^2)+8(1+J^2-J_1^2-J_2^2)\zeta_3}{32\lambda}\right] + \mathcal{O}\left(S^2, \frac{S}{\lambda^{3/2}}\right)\end{aligned}$$

$$\begin{aligned}\log \mathcal{D}_{J_1J_2J} &= \frac{1}{\sqrt{\lambda}}\left[\frac{5}{8}S - \frac{7-4\zeta_3}{16}S^2\right] \\ &+ \frac{1}{\lambda}\left[\frac{(19-8J^2)+8\left(1+J^2-\vec{J}^2\right)\zeta_3}{32}S - \frac{49-8\zeta_3}{64}S^2 + \frac{25-12\zeta_3-12\zeta_5}{64}S^3\right] \\ &+ \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right)\end{aligned}$$

$$\mathrm{S}_a(x)=\sum_{i=1}^{x}\frac{(\mathrm{sgn}a)^i}{i^a},\ \mathrm{S}_{a,b,c,\ldots}(x)=\sum_{i=1}^{x}\frac{(\mathrm{sgn}a)^i}{i^a}\,\mathrm{S}_{b,c,\ldots}(i),$$

$$\mathrm{S}_{2,3}(s)=\sum_{i_1\le i_2\le s}\frac{1}{i_2^2 i_1^3}=\sum_{i_1\le i_2\le s}\int_0^\infty\prod_{n=1}^{2}dt_n\,\frac{t_1^2t_2}{2}e^{-t_1i_1-t_2i_2}$$

$$\mathrm{S}_{2,3}(s)=\int_0^1\prod_{n=1}^{2}\frac{dx_n}{x_n}\frac{x_1x_2\left(1-x_2-(-1+x_1)x_1^sx_2^{1+s}+x_1^s(-1+x_1x_2)\right)\log\,x_1\log^2\,x_2}{2(-1+x_1)(-1+x_2)(-1+x_1x_2)}$$

$$\mathrm{S}_{2,3}(s)=\left(\frac{8}{7}\zeta_2^3-2\zeta_3^2\right)s+\mathcal{O}(s^2)$$

$$\mathcal{B}(\ell_B=1)=1+g^4(c_{1|4}+c_{2|4}\zeta_3)+g^6(c_{1|6}+c_{2|6}\zeta_3+c_{3|6}\zeta_5)+\mathcal{O}(g^8)$$

$$\begin{aligned}
c_{1|4}&=4\left(\mathrm{S}_{-2}^2-2\mathrm{S}_{-3}\mathrm{S}_1-2\mathrm{S}_{-2}\mathrm{S}_1^2-2\mathrm{S}_1\mathrm{S}_3-\mathrm{S}_4+2\mathrm{S}_{-3,1}+4\mathrm{S}_1\mathrm{S}_{-2,1}+2\mathrm{S}_{-2,2}+2\mathrm{S}_{3,1}-4\mathrm{S}_{-2,1,1}\right)\,,\\
c_{2|4}&=24\mathrm{S}_1\,,\\
c_{1|6}&=\frac{32}{3}\left(-6\mathrm{S}_{-6}+3\mathrm{S}_{-3}^2-30\mathrm{S}_{-5}\mathrm{S}_1-6\mathrm{S}_{-4}(\mathrm{S}_{-2}-3\mathrm{S}_1^2)+5\mathrm{S}_3^2+6\mathrm{S}_6+30\mathrm{S}_{-5,1}-12\mathrm{S}_{-4,2}-24\mathrm{S}_{-3,3}+\right.\\
&\mathrm{S}_1^3(\mathrm{S}_3-6\mathrm{S}_{-2,1})-48\mathrm{S}_3\mathrm{S}_{-2,1}+12\mathrm{S}_{-2,1}^2-3\mathrm{S}_{-3}(4\mathrm{S}_{-2}\mathrm{S}_1-3\mathrm{S}_1^3+11\mathrm{S}_1\mathrm{S}_2+4(-4\mathrm{S}_3+\mathrm{S}_{-2,1}))+\\
&54\mathrm{S}_{4,-2}-6\mathrm{S}_{4,2}+6\mathrm{S}_{5,1}-48\mathrm{S}_{-4,1,1}+6\mathrm{S}_{-2}(\mathrm{S}_1^2\mathrm{S}_2-9\mathrm{S}_4-10\mathrm{S}_1\mathrm{S}_{2,1}+2(\mathrm{S}_{-3,1}+\mathrm{S}_{-2,2}+4\mathrm{S}_{3,1}-\\
&2\mathrm{S}_{-2,1,1}))+12\mathrm{S}_2(3\mathrm{S}_{-3,1}+\mathrm{S}_{3,1}-6\mathrm{S}_{-2,1,1})+6\mathrm{S}_1^2(2\mathrm{S}_4-5\mathrm{S}_{-3,1}-4\mathrm{S}_{-2,2}-\mathrm{S}_{3,1}+6\mathrm{S}_{-2,1,1})+\\
&36\mathrm{S}_{-2,2,2}-48\mathrm{S}_{-2,3,1}-36\mathrm{S}_{2,-3,1}-12\mathrm{S}_{2,3,1}-48\mathrm{S}_{3,1,-2}+12\mathrm{S}_{4,1,1}-72\mathrm{S}_{-3,1,1,1}-3\mathrm{S}_1(3\mathrm{S}_2(\mathrm{S}_3-\\
&6\mathrm{S}_{-2,1})+2(\mathrm{S}_5+5\mathrm{S}_{-4,1}-6\mathrm{S}_{-2,3}+4\mathrm{S}_{2,-3}-2\mathrm{S}_{2,3}+3\mathrm{S}_{4,1}-11\mathrm{S}_{-3,1,1}-2\mathrm{S}_{-2,1,-2}-10\mathrm{S}_{-2,2,1}-\\
&\left.10\mathrm{S}_{2,1,-2}-3\mathrm{S}_{3,1,1}+18\mathrm{S}_{-2,1,1,1}))+72\mathrm{S}_{2,-2,1,1}-24\mathrm{S}_{3,1,1,1}+144\mathrm{S}_{-2,1,1,1,1}\right),\\
c_{2|6}&=-\frac{32}{3}\left(6\mathrm{S}_{-3}+15\mathrm{S}_{-2}\mathrm{S}_1-4\mathrm{S}_1^3+9\mathrm{S}_1\mathrm{S}_2+4\mathrm{S}_3-12\mathrm{S}_{-2,1}\right)\,,
\end{aligned}$$

$$c_{3|6}=-240\mathrm{S}_1\,.$$

$$\mathcal{B}(\ell_B=1)=1+s[3g^4(4\zeta_2\zeta_3+5\zeta_5)-48g^6(\zeta_3\zeta_4+\zeta_2\zeta_5+7\zeta_7)]+\mathcal{O}(sg^8,s^2),$$

$$\mathcal{B}(\ell_B=2)=1+g^6(c_{1|6}+c_{2|6}\zeta_3+c_{3|6}\zeta_5)+\mathcal{O}(g^8),$$

$$\begin{aligned}
c_{1|6}=&\frac{8}{3}(-6\mathrm{S}_{-6}+9\mathrm{S}_{-3}^2-12\mathrm{S}_{-5}\mathrm{S}_1-7\mathrm{S}_3^2+24\mathrm{S}_{-5,1}-6\mathrm{S}_{-4,2}-12\mathrm{S}_{-3,3}+2\mathrm{S}_{-3}(\mathrm{S}_1^3-9\mathrm{S}_1\mathrm{S}_2+\\
&11\mathrm{S}_3-6\mathrm{S}_{-2,1})-20\mathrm{S}_3\mathrm{S}_{-2,1}+12\mathrm{S}_{-2,1}^2-2\mathrm{S}_1^3(\mathrm{S}_3+2\mathrm{S}_{-2,1})+6\mathrm{S}_{-2}(-5\mathrm{S}_4+2\mathrm{S}_{-3,1}+\\
&\mathrm{S}_{-2,2}+4\mathrm{S}_{3,1})+30\mathrm{S}_{4,-2}+12\mathrm{S}_{4,2}-12\mathrm{S}_{5,1}-36\mathrm{S}_{-4,1,1}-12\mathrm{S}_{-3,-2,1}-12\mathrm{S}_{-3,1,-2}+\\
&6\mathrm{S}_1^2(\mathrm{S}_{-4}-\mathrm{S}_4-2(\mathrm{S}_{-3,1}+\mathrm{S}_{-2,2}-\mathrm{S}_{3,1}-2\mathrm{S}_{-2,1,1}))+24\mathrm{S}_2(\mathrm{S}_{-3,1}-\mathrm{S}_{3,1}-2\mathrm{S}_{-2,1,1})-\\
&6\mathrm{S}_{-2,2,-2}+24\mathrm{S}_{-2,2,2}-36\mathrm{S}_{-2,3,1}-24\mathrm{S}_{2,-3,1}+24\mathrm{S}_{2,3,1}-24\mathrm{S}_{3,1,-2}-12\mathrm{S}_{4,1,1}-\\
&48\mathrm{S}_{-3,1,1,1}-24\mathrm{S}_{-2,-2,1,1}+6\mathrm{S}_1(2\mathrm{S}_5-2\mathrm{S}_{-4,1}-\mathrm{S}_{-2}\mathrm{S}_{-2,1}+3\mathrm{S}_2(\mathrm{S}_3+2\mathrm{S}_{-2,1})+\\
&3\mathrm{S}_{-2,3}-3\mathrm{S}_{2,-3}-6\mathrm{S}_{-2}\mathrm{S}_{2,1}-3\mathrm{S}_{2,3}+3\mathrm{S}_{4,1}+6\mathrm{S}_{-3,1,1}+\mathrm{S}_{-2,1,-2}+6\mathrm{S}_{-2,2,1}+\\
&6\mathrm{S}_{2,1,-2}-6\mathrm{S}_{3,1,1}-12\mathrm{S}_{-2,1,1,1})+48\mathrm{S}_{2,-2,1,1}+48\mathrm{S}_{3,1,1,1}+96\mathrm{S}_{-2,1,1,1,1})\,,\\
c_{2|6}=&\frac{16}{3}(3\mathrm{S}_{-3}+3\mathrm{S}_{-2}\mathrm{S}_1+\mathrm{S}_1^3-\mathrm{S}_3-6\mathrm{S}_{-2,1})\,,
\end{aligned}$$

$$c_{3|6}=80\mathrm{S}_1\,.$$

$$\mathcal{B}(\ell_B=2)=1+4sg^6(-3\zeta_3\zeta_4+9\zeta_2\zeta_5+14\zeta_7)+\mathcal{O}(sg^8,s^2),$$

$$\mathcal{A}=\mathcal{A}_{\text{asy}}+\delta\mathcal{A},$$

$$\mathcal{A}_{\text{asy}} = \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-1)^{|\bar{\alpha}|} \prod_{j \in \bar{\alpha}} e^{ip_j \ell_A} \prod_{i \in \alpha, j \in \bar{\alpha}} \frac{1}{h_{ij}},$$

$$\mathcal{A}_{\text{asy}} = \frac{(2s)!}{(s!)^2}\Big(1 + g^2 c_{1|2} + g^4 c_{1|4} + g^6(c_{1|6} + c_{3|6}\zeta_3) + \mathcal{O}(g^8)\Big),$$

$$\begin{aligned}
c_{1|2} =& -4(\mathrm{S}_2 + 2\mathrm{S}_1(\mathrm{S}_1 - \tilde{\mathrm{S}}_1)),\\
c_{1|4} =& 2(4\mathrm{S}_{-4} + 4\mathrm{S}^2_{-2} + 4\mathrm{S}_{-3}\mathrm{S}_1 + 16\mathrm{S}_1^4 + 40\mathrm{S}_1^2\mathrm{S}_2 + 4\mathrm{S}_2^2 + 15\mathrm{S}_1\mathrm{S}_3 - \mathrm{S}_4 + 4\mathrm{S}_{-3,1} +\\
& 8\mathrm{S}_1\mathrm{S}_{-2,1} + 8\mathrm{S}_{-2,2} + 5\mathrm{S}_{1,3} + 9\mathrm{S}_{3,1} - 24\mathrm{S}_{-2,1,1} -\\
& 8\mathrm{S}_{-3}\tilde{\mathrm{S}}_1 - 32\mathrm{S}_1^3\tilde{\mathrm{S}}_1 - 32\mathrm{S}_1\mathrm{S}_2\tilde{\mathrm{S}}_1 - 8\mathrm{S}_3\tilde{\mathrm{S}}_1 + 16\mathrm{S}_{-2,1}\tilde{\mathrm{S}}_1 + 16\mathrm{S}_1^2\tilde{\mathrm{S}}_1^2 + 4\mathrm{S}_{-2}(\mathrm{S}_1^2 + \mathrm{S}_2 -\\
& 4\mathrm{S}_1\tilde{\mathrm{S}}_1) - 16\mathrm{S}_1^2\tilde{\mathrm{S}}_2),\\
c_{1|6} =& -\frac{32}{3}(12\mathrm{S}_{-6} - 6\mathrm{S}^2_{-3} + 12\mathrm{S}_{-4}\mathrm{S}_{-2} + 51\mathrm{S}_{-5}\mathrm{S}_1 + 24\mathrm{S}_{-4}\mathrm{S}_1^2 + 6\mathrm{S}^2_{-2}\mathrm{S}_1^2 + 6\mathrm{S}_{-2}\mathrm{S}_1^4 + 8\mathrm{S}_1^6 +\\
& 9\mathrm{S}_{-4}\mathrm{S}_2 + 3\mathrm{S}^2_{-2}\mathrm{S}_2 + 27\mathrm{S}_{-2}\mathrm{S}_1^2\mathrm{S}_2 + 48\mathrm{S}_1^4\mathrm{S}_2 + 3\mathrm{S}_{-2}\mathrm{S}_2^2 + 48\mathrm{S}_1^2\mathrm{S}_2^2 + \mathrm{S}_2^3 + 18\mathrm{S}_{-2}\mathrm{S}_1\mathrm{S}_3 +\\
& 45\mathrm{S}_1^3\mathrm{S}_3 + 48\mathrm{S}_1\mathrm{S}_2\mathrm{S}_3 + \mathrm{S}_3^2 + 114\mathrm{S}_{-2}\mathrm{S}_4 + 30\mathrm{S}_1^2\mathrm{S}_4 + 3\mathrm{S}_2\mathrm{S}_4 + 27\mathrm{S}_1\mathrm{S}_5 - \mathrm{S}_6 - 54\mathrm{S}_{-5,1} +\\
& 6\mathrm{S}_1\mathrm{S}_{-4,1} + 6\mathrm{S}_{-4,2} - 24\mathrm{S}_{-2}\mathrm{S}_{-3,1} + 12\mathrm{S}_1^2\mathrm{S}_{-3,1} - 57\mathrm{S}_2\mathrm{S}_{-3,1} + 66\mathrm{S}_{-3,3} - 6\mathrm{S}_{-2}\mathrm{S}_1\mathrm{S}_{-2,1} +\\
& 2\mathrm{S}_1^3\mathrm{S}_{-2,1} - 72\mathrm{S}_1\mathrm{S}_2\mathrm{S}_{-2,1} + 100\mathrm{S}_3\mathrm{S}_{-2,1} - 24\mathrm{S}^2_{-2,1} - 24\mathrm{S}_{-2}\mathrm{S}_{-2,2} + 12\mathrm{S}_1^2\mathrm{S}_{-2,2} -\\
& 6\mathrm{S}_2\mathrm{S}_{-2,2} - 42\mathrm{S}_1\mathrm{S}_{-2,3} + 36\mathrm{S}_1\mathrm{S}_{2,-3} + 60\mathrm{S}_{-2}\mathrm{S}_1\mathrm{S}_{2,1} - 12\mathrm{S}_1\mathrm{S}_{2,3} - 108\mathrm{S}_{-2}\mathrm{S}_{3,1} +\\
& 12\mathrm{S}_1^2\mathrm{S}_{3,1} - 9\mathrm{S}_2\mathrm{S}_{3,1} - 108\mathrm{S}_{4,-2} + 18\mathrm{S}_1\mathrm{S}_{4,1} + 6\mathrm{S}_{4,2} - 6\mathrm{S}_{5,1} + 120\mathrm{S}_{-4,1,1} - 66\mathrm{S}_1\mathrm{S}_{-3,1,1} -\\
& 6\mathrm{S}_1\mathrm{S}_{-2,1,-2} + 48\mathrm{S}_{-2}\mathrm{S}_{-2,1,1} - 24\mathrm{S}_1^2\mathrm{S}_{-2,1,1} + 150\mathrm{S}_2\mathrm{S}_{-2,1,1} - 60\mathrm{S}_1\mathrm{S}_{-2,2,1} - 60\mathrm{S}_{-2,2,2} +\\
& 108\mathrm{S}_{-2,3,1} + 48\mathrm{S}_{2,-3,1} - 60\mathrm{S}_1\mathrm{S}_{2,1,-2} + 12\mathrm{S}_{2,3,1} + 108\mathrm{S}_{3,1,-2} - 18\mathrm{S}_1\mathrm{S}_{3,1,1} - 12\mathrm{S}_{4,1,1} +\\
& 120\mathrm{S}_{-3,1,1,1} + 108\mathrm{S}_1\mathrm{S}_{-2,1,1,1} - 144\mathrm{S}_{2,-2,1,1} + 24\mathrm{S}_{3,1,1,1} - 336\mathrm{S}_{-2,1,1,1,1} + 18\mathrm{S}_{-5}\tilde{\mathrm{S}}_1 -\\
& 54\mathrm{S}_{-4}\mathrm{S}_1\tilde{\mathrm{S}}_1 - 12\mathrm{S}^2_{-2}\mathrm{S}_1\tilde{\mathrm{S}}_1 - 30\mathrm{S}_{-2}\mathrm{S}_1^3\tilde{\mathrm{S}}_1 - 24\mathrm{S}_1^5\tilde{\mathrm{S}}_1 - 42\mathrm{S}_{-2}\mathrm{S}_1\mathrm{S}_2\tilde{\mathrm{S}}_1 - 84\mathrm{S}_1^3\mathrm{S}_2\tilde{\mathrm{S}}_1 - 30\mathrm{S}_1\mathrm{S}_2^2\tilde{\mathrm{S}}_1 -\\
& 12\mathrm{S}_{-2}\mathrm{S}_3\tilde{\mathrm{S}}_1 - 54\mathrm{S}_1^2\mathrm{S}_3\tilde{\mathrm{S}}_1 - 12\mathrm{S}_2\mathrm{S}_3\tilde{\mathrm{S}}_1 - 24\mathrm{S}_1\mathrm{S}_4\tilde{\mathrm{S}}_1 - 6\mathrm{S}_5\tilde{\mathrm{S}}_1 + 36\mathrm{S}_{-4,1}\tilde{\mathrm{S}}_1 + 66\mathrm{S}_1\mathrm{S}_{-3,1}\tilde{\mathrm{S}}_1 +\\
& 36\mathrm{S}_1^2\mathrm{S}_{-2,1}\tilde{\mathrm{S}}_1 - 48\mathrm{S}_2\mathrm{S}_{-2,1}\tilde{\mathrm{S}}_1 + 48\mathrm{S}_1\mathrm{S}_{-2,2}\tilde{\mathrm{S}}_1 - 36\mathrm{S}_{-2,3}\tilde{\mathrm{S}}_1 + 36\mathrm{S}_{2,-3}\tilde{\mathrm{S}}_1 + 72\mathrm{S}_{-2}\mathrm{S}_{2,1}\tilde{\mathrm{S}}_1 -\\
& 6\mathrm{S}_1\mathrm{S}_{3,1}\tilde{\mathrm{S}}_1 - 72\mathrm{S}_{-3,1,1}\tilde{\mathrm{S}}_1 - 60\mathrm{S}_1\mathrm{S}_{-2,1,1}\tilde{\mathrm{S}}_1 - 72\mathrm{S}_{-2,2,1}\tilde{\mathrm{S}}_1 - 72\mathrm{S}_{2,1,-2}\tilde{\mathrm{S}}_1 + 144\mathrm{S}_{-2,1,1,1}\tilde{\mathrm{S}}_1 +\\
& 24\mathrm{S}_{-2}\mathrm{S}_1^2\tilde{\mathrm{S}}_1^2 + 24\mathrm{S}_1^4\tilde{\mathrm{S}}_1^2 + 36\mathrm{S}_1^2\mathrm{S}_2\tilde{\mathrm{S}}_1^2 + 12\mathrm{S}_1\mathrm{S}_3\tilde{\mathrm{S}}_1^2 - 24\mathrm{S}_1\mathrm{S}_{-2,1}\tilde{\mathrm{S}}_1^2 - 8\mathrm{S}_1^3\tilde{\mathrm{S}}_1^3 + \mathrm{S}_{-3}(24\mathrm{S}_{-2}\mathrm{S}_1 +\\
& 13\mathrm{S}_1^3 - 54\mathrm{S}_1^2\tilde{\mathrm{S}}_1 + 4(-25\mathrm{S}_3 + 6\mathrm{S}_{-2,1} + 6\mathrm{S}_2\tilde{\mathrm{S}}_1) + 12\mathrm{S}_1(4\mathrm{S}_2 + \tilde{\mathrm{S}}_1^2 - \tilde{\mathrm{S}}_2)) - 24\mathrm{S}_{-2}\mathrm{S}_1^2\tilde{\mathrm{S}}_2 -\\
& 24\mathrm{S}_1^4\tilde{\mathrm{S}}_2 - 36\mathrm{S}_1^2\mathrm{S}_2\tilde{\mathrm{S}}_2 - 12\mathrm{S}_1\mathrm{S}_3\tilde{\mathrm{S}}_2 + 24\mathrm{S}_1\mathrm{S}_{-2,1}\tilde{\mathrm{S}}_2 + 24\mathrm{S}_1^3\tilde{\mathrm{S}}_1\tilde{\mathrm{S}}_2 - 16\mathrm{S}_1^3\tilde{\mathrm{S}}_3),
\end{aligned}$$

$$c_{3|6} = -32\mathrm{S}_1(\mathrm{S}_{-2} + \mathrm{S}_2),$$

$$\mathcal{A}_{\text{asy}} = 1 + s[-8\zeta_3 g^2 + g^4(-32\zeta_2\zeta_3 + 90\zeta_5) + g^6(160\zeta_3\zeta_4 + 448\zeta_2\zeta_5 - 1120\zeta_7)] + \mathcal{O}(sg^8, s^2)$$

$$\frac{\delta\mathcal{A}}{\mathcal{A}_{\text{asy}}} = \frac{2\delta\gamma}{\gamma} + \mathcal{O}(g^8).$$

$$\frac{\delta\mathcal{A}}{\mathcal{A}_{\text{asy}}} = (-160\,\mathrm{S}_1\zeta_5 + \cdots)g^6 + \mathcal{O}(g^8)$$

$$\frac{\delta\mathcal{A}}{\mathcal{A}_{\text{asy}}} = -160\zeta_2\zeta_5 g^6 s + \mathcal{O}(g^8 s, g^6 s^2)$$

$$2\pi n_k = -iJ\log\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right) - i\sum_{j \neq k}^{S} \log\left(\frac{u_k - u_j + i}{u_k - u_j - i}\right)$$

$$u_k = \pm\frac{J}{2\pi}\left(1 + \frac{u_k^{(1/2)}}{\sqrt{J}} + \frac{u_k^{(1)}}{J} + \mathcal{O}\left(\frac{1}{J^{3/2}}\right)\right)$$

$$e^{ip_k}=\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}+\mathcal{O}(g^2),h(u_k,u_j)=\frac{u_k-u_j}{u_k-u_j-i}+\mathcal{O}(g^2)$$

$$\mathcal{A}_{\text{asy}}(\ell_A)=\left(\frac{2\pi}{J}\right)^S\frac{\Gamma(\ell_A+S)}{\Gamma(\ell_A)}\exp\left(\sum_{k=1}^{\infty}\frac{P_k(S,\ell_A)}{J^k}\right)$$

$$P_1(S,\ell_A)=\frac{S(2-3S)}{4},P_2(S,\ell_A)=\frac{S\left(4-24S+17S^2+4\pi^2\left(S-2\ell_A(\ell_A+1)\right)\right)}{48}$$

$$\hat{\mathcal{A}}_{\text{asy}}(z)=\sum_{\ell_A=1}^{\infty}z^{\ell_A}\mathcal{A}_{\text{asy}}(\ell_A)$$

$$\hat{\mathcal{A}}_{\text{asy}}(z)=\frac{z}{1-z}\left(1+Sa_J(z)+\mathcal{O}(S^2)\right)$$

$$a_J(z)=\log\left(\frac{2\pi}{J(1-z)}\right)+\psi(1)+\frac{1}{2J}+\frac{1}{J^2}\left(\frac{1}{12}-\frac{\pi^2}{3(1-z)^2}\right)+\mathcal{O}\left(\frac{1}{J^3}\right)$$

$$f_J(\ell)=\sum_{k=1}^{J/2-1}\frac{(-1)^{k+\ell+1}\Gamma(J)\Gamma(2k)\zeta_{2k+1}}{(2\pi)^{2k}\Gamma(-\ell)\Gamma(J-2k)\Gamma(1+\ell+2k)}$$

$$\sum_{\ell_A=1}^{\infty}z^{\ell_A}F_J(-\ell_A)=\hat{f}_J(z)+z^J\hat{f}_J(1/z)$$

$$\hat{f}_J(z)=\sum_{\ell_A=1}^{\infty}z^{\ell_A}f_J(-\ell_A)$$

$$\hat{f}_J(z)=\frac{z}{2(1-z)}\int_0^{\infty}\frac{dt}{e^t-1}\left[\left(1+(1-z)\frac{it}{2\pi}\right)^{J-1}+\left(1-(1-z)\frac{it}{2\pi}\right)^{J-1}-2\right]$$

$$\left(1\pm(1-z)\frac{it}{2\pi}\right)^{J-1}=D_\tau\cdot e^{\pm i\tau t}$$

$$\tau=\frac{(J-1)(1-z)}{2\pi},$$

$$D_\tau=\exp\left[-\sum_{n=2}^{\infty}\frac{(J-1)}{n}\left(-\frac{\tau}{J-1}\right)^n\partial_\tau^n\right].$$

$$\hat{f}_J(z)=\frac{z}{2(1-z)}D_\tau\cdot(2\psi(1)-\psi(1+i\tau)-\psi(1-i\tau)).$$

$$\hat{f}_J(z)=\frac{z}{1-z}a_J(z),$$

$$\left\langle \text{Tr}Z_1^2(x_1)\text{Tr}Z_2^2(x_2)\text{Tr}Z_3^p(x_3)\text{Tr}Z_4^p(x_4)\right\rangle_{\text{conn}}\propto\mathcal{G}_{22pp}(U,V),$$

$$U=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}=z\bar{z},V=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}=(1-z)(1-\bar{z}),$$

$$z=\sigma e^{\rho},\bar{z}=\sigma e^{-\rho}$$

$$\mathcal{G}_{22pp}\propto\sigma^2\to 0$$

$$\mathcal{G}_{22pp} = 2\pi i \int_{-\infty}^{\infty} d\nu \sigma^{-1-S(\nu)} \alpha(\nu) \Omega_{i\nu}(\rho) + \cdots$$

$$S = -\frac{\nu^2 + 4}{2\sqrt{\lambda}} \left(1 + \frac{1}{2\sqrt{\lambda}}\right) + \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\Omega_{i\nu}(\rho) = \frac{\nu \sin(\nu\rho)}{4\pi^2 \sinh \rho}$$

$$\alpha(\nu) = -\frac{2^{S-1}\pi^2 S' e^{\frac{i\pi S}{2}}}{\nu \sin\left(\frac{\pi S}{2}\right)} \gamma_S(\nu)\gamma_S(-\nu) K_{2+\Delta,2+S} b_{2+S},$$

$$\gamma_S(\nu) = \Gamma\left(2 + \frac{S + i\nu}{2}\right)\Gamma\left(p + \frac{S + i\nu}{2}\right)$$

$$K_{2+\Delta,2+S} = \frac{\Gamma(\Delta + S + 4)\Gamma(\Delta + S + 3)}{4^{S+1}\Gamma\left(2 + \frac{\Delta + S}{2}\right)^5 \Gamma\left(p + \frac{\Delta + S}{2}\right)\Gamma\left(2 + \frac{S - \Delta}{2}\right)\Gamma\left(p + \frac{S - \Delta}{2}\right)}.$$

$$b_{2+S} = \frac{2^{S-6}(\Delta + S)^2(\Delta + S + 2)^2}{(\Delta + S - 1)(\Delta + S + 1)^2(\Delta + S + 3)} \times C_{222}(S) C_{pp2}(S)$$

$$K_{2+\Delta,2+S} b_{2+S} = \frac{\mathcal{D}_{222}\mathcal{D}_{pp2}}{Z(p)(2\sqrt{\lambda})^S \Gamma\left(1 + \frac{S}{2}\right)^2}$$

$$Z(p) = \frac{(p-1)!\,(p-2)!}{2}$$

$$\mathcal{G}_{22pp} = \frac{\pi^2 i\sqrt{\lambda}}{Z(p)} \int_{-\infty}^{\infty} \frac{S' d\nu}{\nu} \frac{\gamma_S(\nu)\gamma_S(-\nu)\Omega_{i\nu}(\rho)}{(\sqrt{\lambda}\sigma)^{1+S}} \frac{e^{\frac{i\pi S}{2}}\Gamma\left(-\frac{S}{2}\right)}{\Gamma\left(1 + \frac{S}{2}\right)} \mathcal{D}_{222}\mathcal{D}_{pp2}$$

$$\mathcal{G}_{22pp} = \mathcal{G}_{22pp}^{\mathrm{LO}} + \frac{1}{\sqrt{\lambda}} \mathcal{G}_{22pp}^{\mathrm{NLO}} + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

$$\mathcal{G}_{22pp}^{\mathrm{LO/NLO}} = -\frac{(2\pi)^2 i}{Z(p)\sigma} \int d\nu \Omega_{i\nu}(\rho)\gamma_0(\nu)\gamma_0(-\nu) f^{\mathrm{LO/NLO}}$$

$$f^{\mathrm{LO}} = \frac{1}{\nu^2 + 4}$$

$$f^{\mathrm{NLO}} = \frac{1}{4}\left[\log(\sqrt{\lambda}\sigma) - \frac{i\pi}{2} - \gamma_{\mathrm{E}} - \psi\left(2 + \frac{i\nu}{2}\right) - \psi\left(p + \frac{i\nu}{2}\right) + (\nu \to -\nu)\right]$$

$$\mathcal{G}_{22pp}^{\mathrm{DS}} \approx -\frac{2\pi^2 i}{Z(p)} \sum_{k=1}^{\infty} \left(\frac{1}{\xi}\right)^{1+2k} \zeta(1 + 2k) \int_{-\infty}^{\infty} d\nu \Omega_{i\nu}(\rho)\gamma_{2k}(\nu)\gamma_{2k}(-\nu)$$

$$\sum_{k=1}^{\infty} f(k) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{i e^{i\pi k} dk}{2\sin \pi k} f(k)$$

$$R(x) = \sum_{k=1}^{\infty} \mathcal{S}^k R^{(k)}(x)$$

$$R^{(3)} = \frac{2\pi x\alpha^4}{(\alpha^2-1)^5(\alpha^2+1)^5(x^2-\alpha^2)^5}$$
$$\times \big[(\alpha^{10}+x^8)(3+34\alpha^2+157\alpha^4+124\alpha^6+157\alpha^8+34\alpha^{10}+3\alpha^{12})$$
$$- x^2\alpha^6(-13-5\alpha^2+275\alpha^4+835\alpha^6+417\alpha^8+297\alpha^{10}+217\alpha^{12}+25\alpha^{14})$$
$$+ x^4\alpha^4(73+111\alpha^2+189\alpha^4+1163\alpha^6+1163\alpha^8+189\alpha^{10}+111\alpha^{12}+73\alpha^{14})$$
$$- x^6\alpha^2(25+217\alpha^2+297\alpha^4+417\alpha^6+835\alpha^8+275\alpha^{10}-5\alpha^{12}-13\alpha^{14}) \big],$$

$$R^{(4)} = -\frac{\pi x\alpha^5}{2(\alpha^2-1)^7(\alpha^2+1)^8(x^2-\alpha^2)^7}$$
$$\times \big[(\alpha^{14}+x^{12})(21+342\alpha^2+2689\alpha^4+10536\alpha^6+13674\alpha^8+27396\alpha^{10}$$
$$+13674\alpha^{12}+10536\alpha^{14}+2689\alpha^{16}+342\alpha^{18}+21\alpha^{20})$$
$$- x^2\alpha^{10}(1+107\alpha^2+2147\alpha^4+20353\alpha^6+73514\alpha^8+94222\alpha^{10}$$
$$+161318\alpha^{12}+77986\alpha^{14}+38053\alpha^{16}+20087\alpha^{18}+3495\alpha^{20}+237\alpha^{22})$$
$$+ x^4\alpha^8(-1167-3288\alpha^2+13145\alpha^4+88042\alpha^6+208498\alpha^8+204872\alpha^{10}$$
$$+363178\alpha^{12}+257948\alpha^{14}+59581\alpha^{16}+24704\alpha^{18}+12125\alpha^{20}+1162\alpha^{22})$$
$$- 2x^6\alpha^6(1+\alpha^2)(1169+350\alpha^2+2573\alpha^4+55048\alpha^6+127330\alpha^8$$
$$+36660\alpha^{10}+127330\alpha^{12}+55048\alpha^{14}+2573\alpha^{16}+350\alpha^{18}+1169\alpha^{20})$$
$$+ x^8\alpha^4(1162+12125\alpha^2+24704\alpha^4+59581\alpha^6+257948\alpha^8+363178\alpha^{10}$$
$$+204872\alpha^{12}+208498\alpha^{14}+88042\alpha^{16}+13145\alpha^{18}-3288\alpha^{20}-1167\alpha^{22})$$
$$- x^{10}\alpha^2(237+3495\alpha^2+20087\alpha^4+38053\alpha^6+77986\alpha^8+161318\alpha^{10}$$
$$+94222\alpha^{12}+73514\alpha^{14}+20353\alpha^{16}+2147\alpha^{18}+107\alpha^{20}+\alpha^{22}) \big],$$

$$\Delta^{(3)}(\partial_{\mathcal{J}},\partial_{\mathcal{L}}) = \left[\frac{1+\mathcal{J}^2}{48}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})\partial_{\mathcal{J}} + \frac{6+7\mathcal{J}^2}{24\mathcal{J}}\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right) + \frac{8+22\mathcal{J}^2+13\mathcal{J}^4}{16\mathcal{J}^2(1+\mathcal{J}^2)}\right](\partial_{\mathcal{J}}+\partial_{\mathcal{L}})\partial_{\mathcal{J}}$$

$$\Delta^{(4)}(\partial_{\mathcal{J}},\partial_{\mathcal{L}}) + \frac{3+4\mathcal{J}^2}{24\mathcal{J}^2}\partial_{\mathcal{L}}^2 - \frac{2+\mathcal{J}^2}{4\mathcal{J}^3(1+\mathcal{J}^2)}\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right)$$
$$= \frac{(1+\mathcal{J}^2)^{3/2}}{1152}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})^3\partial_{\mathcal{J}}^3 + \frac{\sqrt{1+\mathcal{J}^2}(4+5\mathcal{J}^2)}{192\mathcal{J}}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})^2\partial_{\mathcal{J}}^2\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right)$$
$$+ \frac{48+137\mathcal{J}^2+86\mathcal{J}^4}{384\mathcal{J}^2\sqrt{1+\mathcal{J}^2}}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})^2\partial_{\mathcal{J}}^2 + \frac{12+31\mathcal{J}^2+19\mathcal{J}^4}{384\mathcal{J}^2\sqrt{1+\mathcal{J}^2}}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})\partial_{\mathcal{J}}\partial_{\mathcal{L}}^2$$
$$+ \frac{8+132\mathcal{J}^2+103\mathcal{J}^4}{192\mathcal{J}^3\sqrt{1+\mathcal{J}^2}}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})\partial_{\mathcal{J}}\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right) + \frac{\sqrt{1+\mathcal{J}^2}(1+2\mathcal{J}^2)}{24\mathcal{J}^3}\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right)\partial_{\mathcal{L}}^2$$
$$- \frac{64+160\mathcal{J}^2+133\mathcal{J}^4+32\mathcal{J}^6-\mathcal{J}^8}{128\mathcal{J}^4(1+\mathcal{J}^2)^{5/2}}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})\partial_{\mathcal{J}} - \frac{2+\mathcal{J}^2}{16\mathcal{J}^4\sqrt{1+\mathcal{J}^2}}\partial_{\mathcal{L}}^2$$
$$+ \frac{8+20\mathcal{J}^2+13\mathcal{J}^4+3\mathcal{J}^6}{16\mathcal{J}^5(1+\mathcal{J}^2)^{5/2}}\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right)$$

$$\mathcal{E} = \mathcal{J} + \delta_1\mathcal{S} + \delta_2\mathcal{S}^2 + \delta_3\mathcal{S}^3 + \delta_4\mathcal{S}^4 + \cdots$$

$$\delta_1 = \frac{\sqrt{1+\mathcal{J}^2}}{\mathcal{J}}, \delta_2 = -\frac{2+\mathcal{J}^2}{4\mathcal{J}^3(1+\mathcal{J}^2)},$$
$$\delta_3 = \frac{8+20\mathcal{J}^2+13\mathcal{J}^4+3\mathcal{J}^6}{16\mathcal{J}^5(1+\mathcal{J}^2)^{5/2}},$$
$$\delta_4 = -\frac{80+336\mathcal{J}^2+540\mathcal{J}^4+385\mathcal{J}^6+138\mathcal{J}^8+21\mathcal{J}^{10}}{128\mathcal{J}^7(1+\mathcal{J}^2)^4}.$$

$$p_k = \frac{4\pi x_k}{\sqrt{\lambda}(x_k^2-1)} - \frac{16\pi^3 x_k^3 (x_k^4+4x_k^2+1)}{3\lambda^{3/2}(x_k^2-1)^5} + \mathcal{O}\left(\frac{1}{\lambda^{5/2}}\right)$$

$$x_k^{\pm} = x_k \pm \frac{2\pi i x_k^2}{\sqrt{\lambda}(x_k^2-1)} + \frac{4\pi^2 x_k^3}{\lambda(x_k^2-1)^3} \mp \frac{8\pi^3 i x_k^4 (x_k^2+1)}{\lambda^{3/2}(x_k^2-1)^5} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

$$S_{kj} = \frac{u_k-u_j+i}{u_k-u_j-i}\left(\frac{1-1/x_k^- x_j^+}{1-1/x_k^+ x_j^-}\right)^2 \frac{1}{\sigma_{kj}^2},$$

$$-i\log\left[\frac{u_k-u_j+i}{u_k-u_j-i}\right] = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\left[\frac{4\pi}{\sqrt{\lambda}(x_k-x_j)(1-1/x_k x_j)}\right]^{2n-1}$$

$$\frac{1}{\sqrt{\lambda}(x_k-x_j)} = \mathcal{O}\left(\frac{1}{\lambda^{1/4}}\right)$$

$$-2i\log\left(\frac{1-1/x_k^- x_j^+}{1-1/x_k^+ x_j^-}\right) = \frac{8\pi(x_k-x_j)(x_k x_j+1)}{\sqrt{\lambda}(x_k^2-1)(x_k x_j-1)(x_j^2-1)} + \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right),$$

$$2i\log\sigma_{kj} = \frac{\delta_{kj}^{\mathrm{AFS}}}{\sqrt{\lambda}} + \frac{\delta_{kj}^{\mathrm{HL}}}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\delta_{kj}^{\mathrm{AFS}} = -\frac{8\pi(x_k-x_j)}{(x_k^2-1)(x_k x_j-1)(x_j^2-1)}$$

$$\delta_{kj}^{\mathrm{HL}} = -\frac{16\pi x_k^2 x_j^2}{(x_k^2-1)(x_j^2-1)}\left[\frac{2}{(x_k-x_j)(x_k x_j-1)} + \left(\frac{1}{(x_k-x_j)^2} + \frac{1}{(x_k x_j-1)^2}\right)\log\frac{(x_k+1)(x_j-1)}{(x_k-1)(x_j+1)}\right]$$

$$\frac{1}{2}\log\frac{(u_k-u_j)^2}{(u_k-u_j)^2+1} \approx \sum_{n=1}^{L}\frac{(-1)^n}{2n}\left[\frac{4\pi}{\sqrt{\lambda}(x_k-x_j)(1-1/x_k x_j)}\right]^{2n},$$

$$\log\frac{(x_k^+ x_j^- -1)(x_k^- x_j^+ -1)}{(x_k^+ x_j^+ -1)(x_k^- x_j^- -1)} = -\frac{16\pi^2 x_k^2 x_j^2}{\lambda(x_k^2-1)(x_k x_j-1)^2(x_j^2-1)} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

$$P_1^G(S) = \left[\frac{4+3\mathcal{J}^2+2\mathcal{J}^4}{8\mathcal{J}^2(1+\mathcal{J}^2)^{3/2}} + \frac{\sqrt{1+\mathcal{J}^2}\pi^2}{12\mathcal{J}^2}\right]S - \left[\frac{12+17\mathcal{J}^2+8\mathcal{J}^4}{16\mathcal{J}^2(1+\mathcal{J}^2)^{3/2}} + \frac{\sqrt{1+\mathcal{J}^2}\pi^2}{24\mathcal{J}^2}\right]S^2$$

$$\begin{aligned} P_2^G(S) = &-\left[\frac{24+88\mathcal{J}^2+153\mathcal{J}^4+12\mathcal{J}^6-4\mathcal{J}^8}{96\mathcal{J}^4(1+\mathcal{J}^2)^3} + \frac{(16+23\mathcal{J}^2+6\mathcal{J}^4)\pi^2}{24\mathcal{J}^4(1+\mathcal{J}^2)} - \frac{(1+\mathcal{J}^2)\pi^4}{360\mathcal{J}^4}\right]S \\ &-\left[\frac{32+104\mathcal{J}^2+79\mathcal{J}^4+76\mathcal{J}^6+25\mathcal{J}^8}{64\mathcal{J}^4(1+\mathcal{J}^2)^3} + \frac{\pi^2}{48\mathcal{J}^2} + \frac{7(1+\mathcal{J}^2)\pi^4}{720\mathcal{J}^4}\right]S^2 \\ &+\left[\frac{240+784\mathcal{J}^2+834\mathcal{J}^4+456\mathcal{J}^6+107\mathcal{J}^8}{384\mathcal{J}^4(1+\mathcal{J}^2)^3} + \frac{(4+6\mathcal{J}^2+3\mathcal{J}^4)\pi^2}{96\mathcal{J}^4(1+\mathcal{J}^2)} + \frac{(1+\mathcal{J}^2)\pi^4}{240\mathcal{J}^4}\right]S^3 \\ &+P_2^{\mathrm{HL}}(S) \end{aligned}$$

$$P_2^{\mathrm{HL}}(S) = \frac{S}{3\mathcal{J}^4\sqrt{1+\mathcal{J}^2}} + \left[-\frac{2+11\mathcal{J}^2+3\mathcal{J}^4}{12\mathcal{J}^4(1+\mathcal{J}^2)^{3/2}} + \frac{4+3\mathcal{J}^2+\mathcal{J}^4}{4\mathcal{J}^2(1+\mathcal{J}^2)^2}\log\left(\frac{1+\sqrt{1+\mathcal{J}^2}}{\mathcal{J}}\right)\right]S^2$$

$$P_1^H = \frac{\sqrt{1+\mathcal{J}^2}\pi^2}{4\mathcal{J}^2} S - \frac{\sqrt{1+\mathcal{J}^2}\pi^2}{8\mathcal{J}^2} S^2$$

$$P_2^H = -\left[\frac{(6+9\mathcal{J}^2+2\mathcal{J}^4)\pi^2}{8\mathcal{J}^4(1+\mathcal{J}^2)} - \frac{(1+\mathcal{J}^2)\pi^4}{72\mathcal{J}^4}\right] S - \left[\frac{(2+3\mathcal{J}^2)\pi^2}{16\mathcal{J}^4} + \frac{(1+\mathcal{J}^2)\pi^4}{48\mathcal{J}^4}\right] S^2 + \left[\frac{(4+6\mathcal{J}^2+3\mathcal{J}^4)\pi^2}{32\mathcal{J}^4(1+\mathcal{J}^2)} + \frac{(1+\mathcal{J}^2)\pi^4}{144\mathcal{J}^4}\right] S^3$$

$$W = W^F + W^\Phi$$

$$W^F = \frac{1}{2}\sum_{a=1}^{\infty} \int \frac{du}{2\pi} e^{-J\tilde{E}_a(u)} \mathrm{STr}_{aa}(u,u;\mathbf{u})$$

$$\mathbb{K}_{ab}(u,v;\mathbf{u}) = -i\mathbb{S}_{ba}(v,u)\partial_u \mathbb{S}_{ab}(u,v) \prod_{k=1}^{S} \mathbb{S}_{a1}(u,u_k)\mathbb{S}_{b1}(v,u_k)$$

$$W^F = W_1^F S + \mathcal{O}(S^2)$$

$$W_1^F = \sum_{a=1}^{\infty} \int \frac{du}{2\pi i} \frac{1}{(x^{[+a]}x^{[-a]})^J} \left[\frac{\Sigma_+}{a}\left(\frac{x^{[+a]} - 1/x^{[+a]}}{x^{[-a]} - 1/x^{[-a]}} + \frac{x^{[-a]} - 1/x^{[-a]}}{x^{[+a]} - 1/x^{[+a]}}\right) - \frac{iJ\Sigma_-}{g}\left(\frac{1}{x^{[+a]} - 1/x^{[+a]}} + \frac{1}{x^{[-a]} - 1/x^{[-a]}}\right)\right]$$

$$\Sigma_\pm = \sum_{n=1}^{\infty} \frac{2\pi I_{J+2n-1}(\sqrt{\lambda})}{JI_J(\sqrt{\lambda})}\left[\left(x^{[+a]}\right)^{1-2n} \mp \left(x^{[-a]}\right)^{1-2n}\right]$$

$$x^{[\pm a]} = x \pm \frac{2\pi i a x^2}{\sqrt{\lambda}(x^2-1)} + \frac{4\pi^2 a^2 x^3}{\lambda(x^2-1)^3} \mp \frac{8\pi^3 i a^3 x^4(x^2+1)}{\lambda^{3/2}(x^2-1)^5} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

$$\sum_{n=1}^{\infty} \frac{I_{J+2n-1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})} x^{1-2n} = \sum_{k=0}^{\infty} \frac{r^{(k)}(x)}{(\sqrt{\lambda})^k}$$

$$r^{(0)}(x) = \frac{\alpha x}{\alpha^2 x^2 - 1}$$

$$r^{(1)}(x) = -\frac{2\alpha^2 x}{(1+\alpha^2)^2(\alpha^2x^2-1)^3}[1+\alpha^6x^4+3\alpha^2x^2(1+\alpha^2)]$$

$$r^{(2)}(x) = \frac{4\alpha^3 x}{(1+\alpha^2)^5(\alpha^2x^2-1)^5} \times [(1+\alpha^{10}x^8)(1-3\alpha^2+\alpha^4) + \alpha^2x^2(20+37\alpha^2+9\alpha^4+2\alpha^6) + 25\alpha^4x^4(1+\alpha^2(2+\alpha^2)^2) + \alpha^6x^6(2+9\alpha^2+37\alpha^4+20\alpha^6)]$$

$$W_1^F = -\frac{1}{\mathcal{J}}\int_{U^-} \frac{dx(x^2-1)}{2\pi i x^2} \sum_{a=1}^{\infty} \frac{e^{-2\pi a \mathcal{J} E(x)}}{a}\left(r^{(0)}(1/x) - r^{(0)}(x)\right) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

$$W_1^F = -2I[\mathcal{J}] + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

$$W_1^F = -2\sum_{k=0}^{\infty} \frac{1}{(\sqrt{\lambda})^k}\left[\hat{\Delta}_F^{(k)} I[\mathcal{L}]\right]_{\mathcal{L}=\mathcal{J}}$$

$$\widehat{\Delta}_F^{(1)} = -\frac{\sqrt{1+\mathcal{J}^2}}{2}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})\partial_{\mathcal{J}} - \frac{\sqrt{1+\mathcal{J}^2}}{\mathcal{J}}\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right) + \frac{1}{2\mathcal{J}}\left(\mathcal{J}\sqrt{1+\mathcal{J}^2}-\mathcal{R}\right)\partial_{\mathcal{L}}^2$$

$$\widehat{\Delta}_F^{(2)} = \frac{1+\mathcal{J}^2}{8}\left(\partial_{\mathcal{J}}^3+2\partial_{\mathcal{J}}^2\partial_{\mathcal{L}}-\partial_{\mathcal{J}}\partial_{\mathcal{L}}^2-2\partial_{\mathcal{L}}^3\right)\partial_{\mathcal{J}} + \frac{3+5\mathcal{J}^2}{12\mathcal{J}}(2\partial_{\mathcal{J}}+3\partial_{\mathcal{L}})\partial_{\mathcal{J}}^2 + \frac{3}{8\mathcal{J}}\partial_{\mathcal{J}}\partial_{\mathcal{L}}^2$$
$$+\frac{3\mathcal{L}+2\mathcal{J}^2\mathcal{L}-6\mathcal{J}^3}{24\mathcal{J}^2}\partial_{\mathcal{L}}^3 + \frac{5+4\mathcal{J}^2}{4(1+\mathcal{J}^2)}(\partial_{\mathcal{J}}+\partial_{\mathcal{L}})\partial_{\mathcal{J}} + \frac{2+4\mathcal{J}^2+3\mathcal{J}^4}{4\mathcal{J}^2(1+\mathcal{J}^2)}\partial_{\mathcal{L}}^2$$
$$+\frac{1}{2\mathcal{J}(1+\mathcal{J}^2)}\left(\partial_{\mathcal{J}}+\frac{1}{2}\partial_{\mathcal{L}}\right) - \frac{\sqrt{1+\mathcal{J}^2}}{8\mathcal{J}^3}\left((2\mathcal{J}+\mathcal{L})\partial_{\mathcal{L}}-1\right)\partial_{\mathcal{L}}^2\mathcal{R}$$

$$\Phi_k = -\sum_{a=1}^{\infty}\int \frac{du}{2\pi} e^{-J\tilde{E}_a(u)}\mathrm{STr}[\mathbb{S}_{a1}(u,u_1)\ldots\partial_u\mathbb{S}_{a1}(u,u_k)\ldots\mathbb{S}_{a1}(u,u_S)]+\cdots$$

$$W^{\Phi} = W_1^{\Phi}S + \mathcal{O}(S^2)$$

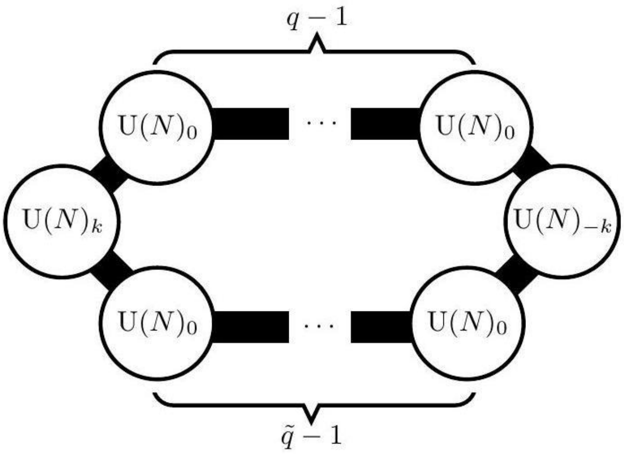
$$W_1^{\Phi} = \int_{U^-} dx \frac{4\alpha^4(x^2-1)\left((1+x^2)(1+\alpha^2)-4x\alpha\right)\left(\alpha(1+\alpha^2)(1+x^2)-2x(1+\alpha^4)\right)}{\lambda(x-\alpha)^4(x\alpha-1)^4(\alpha^2-1)^2(\alpha^2+1)(e^{2\pi\mathcal{J}E(x)}-1)} + \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right)$$

$$W_1^{\Phi} = -\frac{2}{\lambda}\left[\widehat{\Delta}_{\Phi}^{(2)} I[\mathcal{L}]\right]_{\mathcal{L}=\mathcal{J}} + \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right),$$

$$\widehat{\Delta}_{\Phi}^{(2)} = \left(-\frac{1+\mathcal{J}^2}{12}\partial_{\mathcal{J}}^3 - \frac{\mathcal{J}}{4}\partial_{\mathcal{J}}^2 + \frac{3+2\mathcal{J}^2}{4\mathcal{J}^2}\partial_{\mathcal{J}} + \frac{1+2\mathcal{J}^2}{4\mathcal{J}^3}\right)\partial_{\mathcal{L}}.$$

$$\widehat{\Delta}^{(1)} = \widehat{\Delta}_F^{(1)}, \widehat{\Delta}^{(2)} = \widehat{\Delta}_F^{(2)} + \widehat{\Delta}_{\Phi}^{(2)},$$

$$Z^{\mathcal{T}}(N;\boldsymbol{\xi}) = e^{A^{\mathcal{T}}(\boldsymbol{\xi})}C^{\mathcal{T}}(\boldsymbol{\xi})^{-\frac{1}{3}}\mathrm{Ai}\left[C^{\mathcal{T}}(\boldsymbol{\xi})^{-\frac{1}{3}}(N-B^{\mathcal{T}}(\boldsymbol{\xi}))\right]\left(1+\mathcal{O}\left(e^{-\#\sqrt{N}}\right)\right).$$



$$\mathrm{U}(N)_k \times \underbrace{\mathrm{U}(N)_0\times\cdots\times\mathrm{U}(N)_0}_{q-1}\times\mathrm{U}(N)_{-k}\times\underbrace{\mathrm{U}(N)_0\times\cdots\times\mathrm{U}(N)_0}_{\tilde{q}-1}.$$

$$Z_k^{(q,\tilde{q})}(N;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})$$
$$=\frac{1}{(N!)^{q+\tilde{q}}}\int_{-\infty}^{\infty}\left(\prod_{\alpha=1}^{q}\prod_{i=1}^{N}\frac{d\lambda_i^{(\alpha)}}{2\pi}\right)\left(\prod_{\alpha=1}^{\tilde{q}}\prod_{i=1}^{N}\frac{d\tilde{\lambda}_i^{(\alpha)}}{2\pi}\right)$$

$$\times e^{-\frac{ik}{4\pi}\sum_i^N\left(\left(\tilde{\lambda}_i^{(\tilde{q})}\right)^2-\left(\lambda_i^{(q)}\right)^2\right)} e^{-\frac{i}{2}\sum_{\alpha=1}^{q}\eta_\alpha\sum_i^N\left(\lambda_i^{(\alpha-1)}-\lambda_i^{(\alpha)}\right)} e^{-\frac{i}{2}\sum_{\alpha=1}^{\tilde{q}}\tilde{\eta}_\alpha\sum_i^N\left(\tilde{\lambda}_i^{(\alpha-1)}-\tilde{\lambda}_i^{(\alpha)}\right)}$$
$$\times\prod_{\alpha=1}^{q}\frac{\prod_{i<j}^N 2\sinh\frac{\lambda_{ij}^{(\alpha-1)}}{2}\prod_{i<j}^N 2\sinh\frac{\lambda_{ij}^{(\alpha)}}{2}}{\prod_{i,j}^N 2\cosh\frac{\lambda_i^{(\alpha-1)}-\lambda_j^{(\alpha)}+\pi M}{2}}\prod_{\alpha=1}^{\tilde{q}}\frac{\prod_{i<j}^N 2\sinh\frac{\tilde{\lambda}_{ij}^{(\alpha-1)}}{2}\prod_{i<j}^N 2\sinh\frac{\tilde{\lambda}_{ij}^{(\alpha)}}{2}}{\prod_{i,j}^N 2\cosh\frac{\tilde{\lambda}_i^{(\alpha-1)}-\tilde{\lambda}_j^{(\alpha)}+\pi\tilde{M}}{2}},$$

$$\sum_{\alpha=1}^{q}\eta_\alpha=\sum_{\alpha=1}^{\tilde{q}}\tilde{\eta}_\alpha=0$$
$$|\mathrm{Im}(\eta_\alpha)|<1, |\mathrm{Im}(\tilde{\eta}_\alpha)|<1,$$

$$|\mathrm{Im}(M)|<1, |\mathrm{Im}(\tilde{M})|<1.$$

$$\frac{\prod_{i<j}^N 2\sinh\frac{\mu_i-\mu_j}{2}\prod_{i<j}^N 2\sinh\frac{\nu_i-\nu_j}{2}}{\prod_{i,j}^N 2\cosh\frac{\mu_i-\nu_j+c}{2}}=\det\left(\left[\frac{1}{2\cosh\frac{\mu_i-\nu_j+c}{2}}\right]_{i,j}^{N\times N}\right)$$

$$\frac{1}{2\cosh\left(\frac{\mu-\nu}{2k}+c\right)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\frac{e^{\frac{i}{2\pi k}p(\mu-\nu+2kc)}}{2\cosh\frac{p}{2}}=k\langle\mu|\frac{e^{\frac{ic}{\pi}\hat{p}}}{2\cosh\left(\frac{\pi k}{\hbar}\hat{p}\right)}|\nu\rangle.$$

$$[\hat{x},\hat{p}]=i\hbar, \hbar=2\pi k$$

$$\langle x\mid y\rangle=2\pi\delta(x-y), \langle\langle p\mid p'\rangle\rangle=2\pi\delta(p-p'), \langle x\mid p\rangle\rangle=\frac{1}{\sqrt{k}}e^{\frac{ixp}{\hbar}}$$

$$Z_k^{(q,\tilde{q})}(N;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})$$
$$=\frac{1}{(N!)^{q+\tilde{q}}}\int_{-\infty}^{\infty}\left(\prod_{\alpha=1}^{q}\prod_{i=1}^{N}\frac{d\lambda_i^{(\alpha)}}{2\pi}\right)\left(\prod_{\alpha=1}^{\tilde{q}}\prod_{i=1}^{N}\frac{d\tilde{\lambda}_i^{(\alpha)}}{2\pi}\right)$$
$$\times\prod_{\alpha=1}^{q}\det\left(\left[\left\langle\lambda_i^{(\alpha-1)}\right|e^{-\frac{i}{4\pi k}\delta_{\alpha,1}\hat{x}^2}e^{-\frac{i}{2k}\eta_\alpha\hat{x}}\frac{e^{\frac{i}{2}M\hat{p}}}{2\cosh\frac{\hat{p}}{2}}e^{\frac{i}{2k}\eta_\alpha\hat{x}}e^{\frac{i}{4\pi k}\delta_{\alpha,q}\hat{x}^2}\left|\lambda_j^{(\alpha)}\right\rangle\right]_{i,j}^{N\times N}\right)$$
$$\times\prod_{\alpha=1}^{\tilde{q}}\det\left(\left[\left\langle\tilde{\lambda}_i^{(\alpha-1)}\right|e^{-\frac{i}{2k}\tilde{\eta}_\alpha\hat{x}}\frac{e^{\frac{i}{2}\tilde{M}\hat{p}}}{2\cosh\frac{\hat{p}}{2}}e^{\frac{i}{2k}\tilde{\eta}_\alpha\hat{x}}\left|\tilde{\lambda}_j^{(\alpha)}\right|\right]_{i,j}^{N\times N}\right)$$

$$\frac{1}{N!}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{N}\frac{d\nu_i}{2\pi}\right)\det\left([\langle\mu_i|\hat{A}|\nu_j\rangle]_{i,j}^{N\times N}\right)\det\left([\langle\nu_i|\hat{B}|\sigma_j\rangle]_{i,j}^{N\times N}\right)$$
$$=\det\left([\langle\mu_i|\hat{A}\hat{B}|\sigma_j\rangle]_{i,j}^{N\times N}\right)$$

$$Z_k^{(q,\tilde{q})}(N;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})$$
$$=\frac{1}{N!}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{N}\frac{d\lambda_i}{2\pi}\right)\det\left(\left[\langle\lambda_i|\prod_{\alpha=1}^{q}\frac{e^{\frac{i}{2}M(\hat{x}+\hat{p})}}{2\cosh\frac{\hat{x}+\hat{p}+\pi\eta_\alpha}{2}}\prod_{\alpha=1}^{\tilde{q}}\frac{e^{\frac{i}{2}\tilde{M}\hat{p}}}{2\cosh\frac{\hat{p}+\pi\tilde{\eta}_\alpha}{2}}|\lambda_j|\right]_{i,j}^{N\times N}\right)$$

$$e^{-\frac{ic}{\hbar}\hat{x}}f(\hat{p})e^{\frac{ic}{\hbar}\hat{x}}=f(\hat{p}+c), e^{-\frac{ic}{2\hbar}\hat{x}^2}f(\hat{p})e^{\frac{ic}{2\hbar}\hat{x}^2}=f(\hat{p}+c\hat{x})$$

$$e^{-\frac{ic}{2\hbar}\hat{p}^2}f(\hat{x})e^{\frac{ic}{2\hbar}\hat{p}^2}=f(\hat{x}-c\hat{p})$$

$$Z_k^{(q,\tilde{q})}(N;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})$$
$$=\frac{1}{N!}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{N}\frac{d\lambda_i}{2\pi}\right)\det\left(\left[\langle\lambda_i|\hat{\rho}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M,\tilde{\eta}_\alpha,\tilde{M})|\lambda_j\rangle\right]_{i,j}^{N\times N}\right)$$

$$\hat{\rho}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = \prod_{\alpha=1}^{q} \frac{e^{\frac{i}{2}M\hat{x}}}{2\cosh\frac{\hat{x}+\pi\eta_\alpha}{2}} \prod_{\alpha=1}^{\tilde{q}} \frac{e^{\frac{i}{2}\tilde{M}\hat{p}}}{2\cosh\frac{\hat{p}+\pi\tilde{\eta}_\alpha}{2}}$$

$$\Xi_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = 1 + \sum_{N=1}^{\infty} e^{\mu N} Z_k^{(q,\tilde{q})}(N;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})$$

$$\Xi_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = \mathrm{Det}\left(1 + e^{\mu}\hat{\rho}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})\right)$$

$$\Xi_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = \sum_{n=-\infty}^{\infty} e^{J_k^{(q,\tilde{q})}(\mu+2\pi i n;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})}$$

$$Z_k^{(q,\tilde{q})}(N;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} e^{J_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) - \mu N}$$

$$\begin{aligned} &J_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M,\tilde{\eta}_\alpha;\tilde{M}) \\ &= \frac{C_k^{(q,\tilde{q})}(M;\tilde{M})}{3}\mu^3 + B_k^{(q,\tilde{q})}(\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})\mu + A_k^{(q,\tilde{q})}(\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) + \mathcal{O}(e^{-\#\mu}), \end{aligned}$$

$$C_k^{(q,\tilde{q})}(M;\tilde{M}) = \frac{2}{\pi^2 k q\tilde{q}(1+M^2)(1+\tilde{M}^2)}$$

$$\begin{aligned} B_k^{(q,\tilde{q})}(\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = &-\frac{1}{2kq(1+M^2)}\left(\sum_{\alpha=1}^{\tilde{q}} \tilde{\eta}_\alpha^2 + \frac{\tilde{q}}{3}\right) - \frac{1}{2k\tilde{q}(1+\tilde{M}^2)}\left(\sum_{\alpha=1}^{q} \eta_\alpha^2 + \frac{q}{3}\right) \\ &+ \frac{2}{3kq\tilde{q}(1+M^2)(1+\tilde{M}^2)} + \frac{kq\tilde{q}}{24} \end{aligned}$$

$$A_k^{(q,\tilde{q})}(\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = \frac{1}{4}\sum_{\pm}\left[\sum_{\alpha,\beta=1}^{\tilde{q}} \mathcal{A}((1\pm iM)qk,\tilde{\eta}_{\alpha\beta}) + \sum_{\alpha,\beta=1}^{q} \mathcal{A}((1\pm i\tilde{M})\tilde{q}k,\eta_{\alpha\beta})\right],$$

$$\mathcal{A}(\kappa,\chi) = \frac{2\zeta(3)}{\pi^2\kappa} + \frac{\chi^2}{2\kappa} - \frac{\kappa}{12} + \frac{1}{\pi}\int_0^\infty dy \frac{1}{e^{2\pi y}-1}\frac{d}{dy}\left[\frac{\cos(\pi\chi y)}{y\tanh\frac{\pi\kappa y}{2}} - \frac{2}{\pi\kappa y^2}\right]$$

$C^{\mathcal{T}}(\boldsymbol{\xi}) = C_k^{(q,\tilde{q})}(M;\tilde{M}), B^{\mathcal{T}}(\boldsymbol{\xi}) = B_k^{(q,\tilde{q})}(\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})$ and $A^{\mathcal{T}}(\boldsymbol{\xi}) = A_k^{(q,\tilde{q})}(\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})$

$$\begin{aligned} &J_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) \\ &= \mathrm{Tr}\log\left(1 + e^{\mu}\hat{\rho}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})\right) \\ &-\log\left[1 + \sum_{n\neq 0} e^{J_k^{(q,\tilde{q})}}(\mu+2\pi i n;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) - J_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})\right] \end{aligned}$$

$$J_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) \approx \mathrm{Tr}\log\left(1 + e^{\mu}\hat{\rho}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})\right)$$

$$\hat{\rho}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = e^{-\frac{1}{2}U(\hat{x})}e^{-T(\hat{p})}e^{-\frac{1}{2}U(\hat{x})} = e^{-\hat{H}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})}$$

$$U(x) = \sum_{\alpha=1}^{q} \log\, 2\cosh\frac{x+\pi\eta_\alpha}{2} - \frac{iqMx}{2}$$
$$T(p) = \sum_{\alpha=1}^{\tilde{q}} \log\, 2\cosh\frac{p+\pi\tilde{\eta}_\alpha}{2} - \frac{i\tilde{q}\tilde{M}p}{2}$$

$$n(E) = \mathrm{Tr}\theta\left(E - \hat{H}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})\right)$$

$$\theta(z) = \begin{cases} 0 & (z < 0) \\ 1 & (z \geq 0) \end{cases}$$

$$J_k^{(q,\tilde{q})}(\mu; \eta_\alpha, M; \tilde{\eta}_\alpha, \tilde{M}) \approx \int_0^\infty dE \frac{dn(E)}{dE} \log\left(1 + e^{\mu - E}\right) = \int_0^\infty dE n(E) \frac{e^{\mu-E}}{1 + e^{\mu - E}}$$

$$n(E) = C_k^{(q,\tilde{q})}(M; \tilde{M})E^2 + B_k^{(q,\tilde{q})}(\eta_\alpha, M; \tilde{\eta}_\alpha, \tilde{M}) - \frac{\pi^2 C_k^{(q,\tilde{q})}(M; \tilde{M})}{3} + \mathcal{O}(e^{-\#E}),$$

$$\begin{aligned} \int_0^\infty dE E^a \frac{e^{\mu-E}}{1 + e^{\mu-E}} &= -\Gamma(a+1)\mathrm{Li}_{a+1}(-e^\mu) \\ &= \frac{(2\pi i)^{a+1}}{a+1} B_{a+1}\left(\frac{\mu}{2\pi i} + \frac{1}{2}\right) - (-1)^a \Gamma(a+1)\mathrm{Li}_{a+1}(-e^{-\mu}) \end{aligned}$$

$$\begin{aligned} J_k^{(q,\tilde{q})}(\mu; \eta_\alpha, M; \tilde{\eta}_\alpha, \tilde{M}) =& \frac{C_k^{(q,\tilde{q})}(M; \tilde{M})}{3}\mu^3 + B_k^{(q,\tilde{q})}(\eta_\alpha, M; \tilde{\eta}_\alpha, \tilde{M})\mu \\ &+(\boxtimes) + \mathcal{O}(e^{-\#\mu}) \end{aligned}$$

$$\mathcal{O}_W = \int_{-\infty}^\infty \frac{dy}{2\pi} e^{\frac{ipy}{\hbar}} \left\langle x - \frac{y}{2} \right| \hat{\mathcal{O}} \left| x + \frac{y}{2} \right\rangle,$$

$$(f(\hat{x}))_W = f(x), (f(\hat{p}))_W = f(p), (\hat{A}\hat{B})_W = A_W \star B_W, \mathrm{tr}\hat{\mathcal{O}} = \int \frac{dxdp}{2\pi\hbar} \mathcal{O}_W$$

$$\star = e^{\frac{i\hbar}{2}\left(\overleftarrow{\partial}_x \vec{\partial}_p - \overleftarrow{\partial}_p \vec{\partial}_x\right)}$$

$$n(E) = \int_{H_W \leq E} \frac{dxdp}{2\pi\hbar}$$

$$\begin{aligned} \hat{H}_k^{(q,\tilde{q})}(\hat{x}, \hat{p}; \eta_\alpha, M; \tilde{\eta}_\alpha, \tilde{M}) =& U(\hat{x}) + T(\hat{p}) - \frac{1}{24}[U(\hat{p}), [U(\hat{p}), T(\hat{p})]] \\ &- \frac{1}{12}[T(\hat{p}), [U(\hat{p}), T(\hat{p})]] + \cdots, \end{aligned}$$

$$H_W(x,p) = U(x) + T(p) + \frac{\hbar^2}{24}(\partial_x U(x))^2 \partial_p^2 T(p) - \frac{\hbar^2}{12}\partial_x^2 U(x)\left(\partial_p T(p)\right)^2 + \cdots.$$

$$\begin{aligned} U(x) &= \frac{q|x|}{2} - \frac{iqMx}{2} + \mathcal{O}\left(e^{-|x|}\right), \\ \partial_x U(x) &= \frac{q\mathrm{sgn}(x)}{2} - \frac{iqM}{2} + \mathcal{O}\left(e^{-|x|}\right), \\ \partial_x^2 U(x) &= \mathcal{O}\left(e^{-|x|}\right), \\ T(p) &= \frac{\tilde{q}|p|}{2} - \frac{i\tilde{q}\tilde{M}p}{2} + \mathcal{O}\left(e^{-|p|}\right), \\ \partial_p T(p) &= \frac{\tilde{q}\mathrm{sgn}(p)}{2} - \frac{i\tilde{q}\tilde{M}}{2} + \mathcal{O}\left(e^{-|p|}\right), \\ \partial_p^2 T(p) &= \mathcal{O}\left(e^{-|p|}\right). \end{aligned}$$

$$\frac{q|x|}{2} - \frac{iqMx}{2} + \frac{\tilde{q}|p|}{2} - \frac{i\tilde{q}\tilde{M}p}{2} = E$$

$$n(E) = \frac{1}{2\pi\hbar}\left[\frac{8E^2}{q\tilde{q}(1+M^2)(1+\tilde{M}^2)} - \mathrm{vol}_I - \mathrm{vol}_{II} - \mathrm{vol}_{III} - \mathrm{vol}_{IV}\right]$$

$$H_W = \frac{q(1-iM)x}{2} + T(p) + \frac{q^2(1-iM)^2\hbar^2}{96}\partial_p^2 T(p) + \frac{q(1-iM)}{2} f\left(\partial_p^3 T(p), \ldots\right),$$

$$x_{\mathrm{in}}(p) = \frac{2}{q(1-iM)}\left[E - T(p) - \frac{q^2(1-iM)^2\hbar^2}{96}\partial_p^2 T(p) - \frac{q(1-iM)}{2} f\left(\partial_p^3 T(p), \cdots\right)\right].$$

$$x_{\text{out}}(p) = \frac{2}{q(1-iM)}\left(E - \frac{\tilde{q}|p|}{2} + \frac{i\tilde{q}\tilde{M}p}{2}\right)$$

$$\begin{aligned}\text{vol}_{\text{I}} =& \int_{-\infty}^{\infty} dp(x_{\text{out}}(p) - x_{\text{in}}(p)) \\ =& \frac{2}{q(1-iM)}\int_{-\infty}^{\infty} dp\left[T(p) - \frac{\tilde{q}|p|}{2} + \frac{i\tilde{q}\tilde{M}p}{2} + \frac{q^2(1-iM)^2\hbar^2}{96}\partial_p^2 T(p)\right. \\ &\left. + \frac{q(1-iM)}{2} f\left(\partial_p^3 T(p), \cdots\right)\right]\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} dp\left[T(p) - \frac{\tilde{q}|p|}{2} + \frac{i\tilde{q}\tilde{M}p}{2}\right] =& \sum_{\alpha=1}^{\tilde{q}}\left(\int_0^{\infty} dp\log\left(1 + e^{-p-\pi\tilde{\eta}_\alpha}\right) + \int_{-\infty}^{0} dp\log\left(1 + e^{p+\pi\tilde{\eta}_\alpha}\right)\right) \\ =& -\sum_{\alpha=1}^{\tilde{q}}\left(\text{Li}_2\left(-e^{\pi\tilde{\eta}_\alpha}\right) + \text{Li}_2\left(-e^{-\pi\tilde{\eta}_\alpha}\right)\right) \\ =& \sum_{\alpha=1}^{\tilde{q}}\left(\frac{\pi^2\tilde{\eta}_\alpha^2}{2} + \frac{\pi^2}{6}\right)\end{aligned}$$

$$\int_{-\infty}^{\infty} dp\partial_p^2 T(p) = \left[\partial_p T(p)\right]_{p=-\infty}^{\infty} = \tilde{q}$$

$$\text{vol}_I = \frac{\pi^2}{q(1-iM)}\left(\sum_{\alpha=1}^{\tilde{q}}\tilde{\eta}_\alpha^2 + \frac{\tilde{q}}{3}\right) + \frac{\hbar^2 q\tilde{q}(1-iM)}{48}$$

$$\text{vol}_{II} = \frac{\pi^2}{\tilde{q}(1-i\tilde{M})}\left(\sum_{\alpha=1}^{q}\eta_\alpha^2 + \frac{q}{3}\right) - \frac{\hbar^2 q\tilde{q}(1-i\tilde{M})}{24}$$

$$\text{vol}_{III} = \frac{\pi^2}{q(1+iM)}\left(\sum_{\alpha=1}^{\tilde{q}}\tilde{\eta}_\alpha^2 + \frac{\tilde{q}}{3}\right) + \frac{\hbar^2 q\tilde{q}(1+iM)}{48}$$

$$\text{vol}_{IV} = \frac{\pi^2}{\tilde{q}(1+i\tilde{M})}\left(\sum_{\alpha=1}^{q}\eta_\alpha^2 + \frac{q}{3}\right) - \frac{\hbar^2 q\tilde{q}(1+i\tilde{M})}{24}$$

$$\begin{aligned}J_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) \approx& \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\text{Tr}\hat{\rho}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})^n e^{n\mu} \\ =& \int_{-i\infty+0^+}^{i\infty+0^+}\frac{ds}{2\pi i}\Gamma(s)\Gamma(-s)\mathcal{Z}^{(q,\tilde{q})}(s;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})e^{s\mu}\end{aligned}$$

$$\mathcal{Z}^{(q,\tilde{q})}(s;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = \text{Tr}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})^s$$

$$\begin{aligned}J_k^{(q,\tilde{q})}(\mu;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) =& -\text{Res}\left[\Gamma(s)\Gamma(-s)\mathcal{Z}^{(q,\tilde{q})}(s;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})e^{s\mu}, s \to 0\right] \\ &+\mathcal{O}(e^{-\#\mu})\end{aligned}$$

$$\mathcal{Z}^{(q,\tilde{q})}(s;\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M}) = \int\frac{dxdp}{2\pi\hbar}\left(\hat{\rho}_k^{(q,\tilde{q})}(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})^s\right)_W$$

$$\mathcal{Z}^{(q,\tilde{q})}\left(s;\frac{\hbar v_\alpha}{\pi},M;\frac{\hbar\tilde{v}_\alpha}{\pi},\tilde{M}\right) = \sum_{\ell=0}^{\infty}\hbar^{2\ell-1}\mathcal{Z}_{2\ell}^{(q,\tilde{q})}(s;v_\alpha,M;\tilde{v}_\alpha,\tilde{M})$$

$$\mathcal{Z}_0^{(q,\tilde{q})}(s;v_\alpha,M;\tilde{v}_\alpha,\tilde{M}) = \frac{\Gamma\left(\frac{sq(1+iM)}{2}\right)\Gamma\left(\frac{sq(1-iM)}{2}\right)\Gamma\left(\frac{s\tilde{q}(1+i\tilde{M})}{2}\right)\Gamma\left(\frac{s\tilde{q}(1-i\tilde{M})}{2}\right)}{2\pi\Gamma(sq)\Gamma(s\tilde{q})},$$

$$\mathcal{Z}_{2\ell}^{(q,\tilde{q})}(s;v_\alpha,M;\tilde{v}_\alpha,\tilde{M}) = D_{2\ell}^{(q,\tilde{q})}(s;v_\alpha,M;\tilde{v}_\alpha,\tilde{M})\mathcal{Z}_0^{(q,\tilde{q})}(s;v_\alpha,M;\tilde{v}_\alpha,\tilde{M})$$

$$\left(\hat{\rho}_k^{(q,\tilde{q})}\left(\hat{x},\hat{p};\frac{\hbar v_\alpha}{\pi},M;\frac{\hbar\tilde{v}_\alpha}{\pi},\tilde{M}\right)^{s+1}\right)_W$$
$$=\left(\hat{\rho}_k^{(q,\tilde{q})}\left(\hat{x},\hat{p};\frac{\hbar v_\alpha}{\pi},M;\frac{\hbar\tilde{v}_\alpha}{\pi},\tilde{M}\right)^{s}\right)_W\star\rho_{k,W}^{(q,\tilde{q})}\left(x,p;\frac{\hbar v_\alpha}{\pi},M,\frac{\hbar\tilde{v}_\alpha}{\pi},\tilde{M}\right)$$
$$\rho_{k,W}^{(q,\tilde{q})}\left(x,p;\frac{\hbar v_\alpha}{\pi},M;\frac{\hbar\tilde{v}_\alpha}{\pi},\tilde{M}\right)$$
$$=\prod_{\alpha=1}^{q}\frac{e^{\frac{iM}{2}x}}{F(x)\cosh\frac{\hbar v_\alpha}{2}+2F'(x)\sinh\frac{\hbar v_\alpha}{2}}\star\prod_{\alpha=1}^{\tilde{q}}\frac{e^{\frac{i\tilde{M}}{2}p}}{F(p)\cosh\frac{\hbar\tilde{v}_\alpha}{2}+2F'(p)\sinh\frac{\hbar\tilde{v}_\alpha}{2}},$$

$$\partial_x^{2n}F(x)=2^{-2n}F(x),\partial_x^{2n-1}F(x)=2^{-2n+2}F'(x),F'(x)^2=\frac{F(x)^2}{4}-1,$$

$$I_1(\alpha,n)=\int_{-\infty}^{\infty}dx\frac{e^{\alpha x}}{F(x)^n}=\frac{\Gamma\left(\frac{n}{2}+\alpha\right)\Gamma\left(\frac{n}{2}-\alpha\right)}{\Gamma(n)}$$
$$I_2(\alpha,n)=\int_{-\infty}^{\infty}dx\frac{e^{\alpha x}F'(x)}{F(x)^n}=\frac{\alpha\Gamma\left(\frac{n-1}{2}+\alpha\right)\Gamma\left(\frac{n-1}{2}-\alpha\right)}{\Gamma(n)}$$

$$I_1(\alpha,n+2)=\frac{n^2-4\alpha^2}{4n(n+1)}I_1(\alpha,n),I_2(\alpha,n)=\frac{\alpha I_1(n-1,\alpha)}{n-1}$$

$$J_k^{(q,\tilde{q})}\left(\mu;\frac{\hbar v_\alpha}{\pi},M;\frac{\hbar\tilde{v}_\alpha}{\pi},\tilde{M}\right)=\frac{C_k^{(q,\tilde{q})}(M;\tilde{M})}{3}\mu^3+B_k^{(q,\tilde{q})}\left(\frac{\hbar v_\alpha}{\pi},M;\frac{\hbar\tilde{v}_\alpha}{\pi},\tilde{M}\right)\mu$$
$$+\frac{1}{4}\sum_{\pm}\left[\sum_{\alpha=1}^{\tilde{q}}\mathcal{A}\left((1\pm iM)qk,\frac{\hbar\left(\tilde{v}_\alpha-\tilde{v}_\beta\right)}{\pi}\right)+\sum_{\alpha=1}^{q}\mathcal{A}\left((1\pm i\tilde{M})\tilde{q}k,\frac{\hbar\left(v_\alpha-v_\beta\right)}{\pi}\right)\right]$$

$$+\mathcal{O}(e^{-\#\mu})$$

$$\mathcal{A}(\kappa,\chi)=\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\frac{(-1)^{\ell+n}}{(2n)!\,(2\ell-2n)!}B_{2\ell-2}B_{2\ell-2n}\pi^{2\ell-2}\kappa^{2\ell-2n-1}\chi^{2n}$$

$$\kappa\neq0,|\mathrm{Im}(\chi)|<2.$$

$$\mathcal{A}(\kappa,\chi)=-\mathcal{A}(-\kappa,\chi),\mathcal{A}(\kappa,\chi)=\mathcal{A}(\kappa,-\chi).$$

$$A_k^{(1,1)}(M=0;\tilde{M}=0)=A^{\mathrm{ABJM}}(k)$$

$$\mathcal{A}(\kappa,0)=A^{\mathrm{ABJM}}(\kappa).$$

$$A^{\mathrm{ABJM}}(\kappa)=\frac{2\zeta(3)}{\pi^2\kappa}+\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}}{(2\ell)!}B_{2\ell-2}B_{2\ell}\pi^{2\ell-2}\kappa^{2\ell-1}$$

$$\sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}f(\ell,n)=\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}f(\ell+n,n)$$

$$\mathcal{A}(\kappa,\chi)=\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{(2n)!\,(2\ell)!}B_{2\ell+2n-2}B_{2\ell}\pi^{2\ell+2n-2}\kappa^{2\ell-1}\chi^{2n}$$

$$B_{2n}=\begin{cases}2\zeta(3) & (n=-1)\\ 1 & (n=0)\\ (-1)^{n-1}2\int_0^{\infty}dy\left(\frac{1}{e^{2\pi y}-1}\frac{d}{dy}y^{2n}\right) & (n\geq1)\end{cases}$$

$$\mathcal{A}(\kappa,\chi)=\frac{2\zeta(3)}{\pi^2\kappa}+\frac{\chi^2}{2\kappa}-\frac{\kappa}{12}$$
$$+\frac{2}{\pi^2\kappa}\int_0^\infty dy\frac{1}{e^{2\pi y}-1}\frac{d}{dy}\left[\frac{1}{y^2}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^n(\pi\chi y)^{2n}}{(2n)!}\frac{B_{2\ell}(\pi\kappa y)^{2\ell}}{(2\ell)!}-\frac{1}{y^2}\right]$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n}=\frac{z}{2\tanh\frac{z}{2}}$$

$$\mathcal{A}(\kappa,\chi)=A^{\mathrm{ABJM}}(\kappa)+\sum_{\ell=1}^{\infty}\sum_{n=1}^{\ell}\frac{(-1)^{\ell+n}}{(2n)!\,(2\ell-2n)!}B_{2\ell-2}B_{2\ell-2n}\kappa^{2\ell-2n-1}\chi^{2n}\pi^{2\ell-2}$$

$$\mathcal{A}(\kappa,\chi)=A^{\mathrm{ABJM}}(\kappa)+\frac{\chi^2}{2\kappa}-\frac{2}{\pi}\int_0^\infty dy\frac{1}{e^{2\pi y}-1}\frac{d}{dy}\left(\frac{\sin^2\frac{\pi\chi y}{2}}{y\tanh\frac{\pi\kappa y}{2}}\right)$$

$$A^{\mathrm{ABJM}}(\kappa)=\frac{2\zeta(3)}{\pi^2\kappa}\left(1-\frac{\kappa^3}{16}\right)+\frac{\kappa^2}{\pi^2}\int_0^\infty dy\frac{y}{e^{\kappa y}-1}\log\left(1-e^{-2y}\right)$$

$$\mathcal{A}(\kappa,\chi)=A^{\mathrm{ABJM}}(\kappa)+\frac{\chi^2}{2\kappa}-\int_0^\infty dy\frac{1}{\sinh^2\pi y}\left(\frac{\sin^2\frac{\pi\chi y}{2}}{y\tanh\frac{\pi\kappa y}{2}}-\frac{\pi\chi^2}{2\kappa}\right)$$

$$\partial_\chi\mathcal{A}(\kappa,\chi)=\frac{\chi}{\kappa}-\int_0^\infty dy\frac{1}{\sinh^2\pi y}\left(\frac{\pi\sin\pi\chi y}{2\tanh\frac{\pi\kappa y}{2}}-\frac{\pi\chi}{\kappa}\right)$$

$$\partial_\chi\mathcal{A}(\kappa,\chi)=\frac{\chi}{\kappa}-\frac{1}{2}⨍_{-\infty}^{\infty}dy\frac{1}{\sinh^2\pi y}\frac{\pi\sin\pi\chi y}{2\tanh\frac{\pi\kappa y}{2}}+\frac{1}{2}⨍_{-\infty}^{\infty}dy\frac{1}{2\sinh^2\pi y}\frac{\pi\chi}{\kappa}$$

$$⨍_{-\infty}^{\infty}dy\frac{e^{\alpha y}}{\prod_{a=1}^{n}\sinh\pi\nu_a(y-\beta_a)}$$
$$=\frac{1}{1-(-1)^{\sum_a\nu_a}e^{i\alpha}}⨍_\gamma dy\frac{e^{\alpha y}}{\prod_{a=1}^{n}\sinh\pi\nu_a(y-\beta_a)}$$
$$=\frac{2\pi i}{1-(-1)^{\sum_a\nu_a}e^{i\alpha}}\sum_{a=1}^{n}\left(\frac{1}{2}\sum_{j=0,\nu_a}\frac{(-1)^je^{\alpha\left(\beta_a+\frac{ij}{\nu_a}\right)}}{\pi\nu_a\prod_{a'(\neq a)}\sinh\pi\nu_{a'}\left(\beta_a+\frac{ij}{\nu_a}-\beta_{a'}\right)}\right.$$
$$\left.+\sum_{j=1}^{\nu_a-1}\frac{(-1)^je^{\alpha\left(\beta_a+\frac{ij}{\nu_a}\right)}}{\pi\nu_a\prod_{a'(\neq a)}\sinh\pi\nu_{a'}\left(\beta_a+\frac{ij}{\nu_a}-\beta_{a'}\right)}\right)$$

$$\frac{1}{2}⨍_{-\infty}^{\infty}dy\frac{1}{\sinh^2\pi y}\frac{\pi\chi}{\kappa}=\frac{\pi\chi}{2\kappa}\lim_{\epsilon_1,\epsilon_2\to0}⨍dy\frac{e^{\epsilon_1y}}{\sinh\pi y\sinh\pi(y-\epsilon_2)}=-\frac{\chi}{\kappa}$$

$$\partial_\chi\mathcal{A}(\kappa,\chi)$$
$$=-\frac{1}{2}⨍_{-\infty}^{\infty}dy\frac{1}{\sinh^2\pi y}\left(\frac{\pi\sin\pi\chi y}{2\tanh\frac{\pi\kappa y}{2}}\right)$$
$$=\sum_{\pm}\frac{\pi i}{8}\lim_{\epsilon_1,\epsilon_2\to0}⨍_{-\infty}^{\infty}dy\frac{e^{\pi\left(i\chi\pm\frac{\kappa}{2}\right)y}}{\sinh\pi y\sinh\pi(y-\epsilon_1)\sinh\frac{\pi\kappa(y-\epsilon_3)}{2}}$$
$$=\begin{cases}\dfrac{\pi}{1-e^{-\pi\chi}}\left[\dfrac{(1+e^{-\pi\chi})(4-\kappa^2+6\chi^2)}{24\kappa}+\displaystyle\sum_{a=1}^{\frac{\kappa}{2}-1}\frac{e^{-\frac{2\pi\chi a}{\kappa}}}{\kappa\sin^2\frac{2\pi a}{\kappa}}\right] & (\kappa\text{: even })\\ \dfrac{\pi}{1-e^{-2\pi\chi}}\left[\dfrac{(1+e^{-2\pi\chi})(4-\kappa^2+6\chi^2)}{24\kappa}-\dfrac{\kappa}{4}e^{-\pi\chi}+\displaystyle\sum_{a=1}^{\kappa-1}\frac{e^{-\frac{2\pi\chi a}{\kappa}}}{\kappa\sin^2\frac{2\pi a}{\kappa}}\right] & (\kappa\text{: odd })\end{cases}$$

$$\mathcal{A}(1,\chi) = A^{\mathrm{ABJM}}(1) + \frac{1}{8}\left(1+\frac{\pi^2}{4}\right)\chi^2 + \frac{\pi^2}{48}\left(1-\frac{\pi^2}{16}\right)\chi^4 + \cdots$$

$$\mathcal{A}(\kappa,\chi) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} B_{2\ell}(\pi\kappa)^{2\ell-1} a_\ell(\chi)$$

$$a_\ell(z) = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n)!} B_{2\ell+2n-2}(\pi z)^{2n}$$

$$a_\ell(z) = \left(\frac{1}{\pi}\frac{d}{dz}\right)^{2\ell} a_0(z)$$

$$\mathcal{A}(\kappa,\chi) = \frac{i}{2\pi\tanh\frac{i\kappa}{2}\frac{d}{d\chi}} \frac{d}{d\chi} a_0(\chi)$$

$$\mathcal{A}(\kappa,\chi) = -\frac{i}{2\pi}\frac{d}{d\chi}a_0(\chi) - \frac{2}{\pi\kappa}\int_{\mathbb{R}+i0^+} \frac{a_0\left(\chi+\frac{\kappa}{\pi}t\right)}{(2\sinh\, t)^2} dt$$

$$\sum_{n=0}^{\infty} \frac{B_{2n-2}}{(2n)!} z^{2n} = -\frac{\pi^2}{6}z - \frac{1}{12}z^3 - z\mathrm{Li}_2(e^z) + 2\mathrm{Li}_3(e^z)$$

$$a_0(z) = -\frac{\pi^2}{6}z - \frac{\pi^2}{12}z^3 - z\mathrm{Li}_2(e^{\pi z}) + \frac{2}{\pi}\mathrm{Li}_3(e^{\pi z})$$

$$\frac{d}{dz}a_0(z) = -2\pi i \log\, s_{b=1}\left(\frac{z}{2}\right)$$

$$\mathcal{A}(\kappa,\chi) = -\log\, s_{b=1}\left(\frac{\chi}{2}\right) - \frac{2}{\pi\kappa}\int_{\mathbb{R}+i0^+} \frac{a_0\left(\chi+\frac{\kappa}{\pi}t\right)}{(2\sinh\, t)^2} dt$$

$$C_k^{(q,\tilde{q})}(M;\tilde{M}) = \frac{2}{\pi^2 k q\tilde{q}(1+M^2)(1+\tilde{M}^2)}$$

$$B_k^{(q,\tilde{q})}(0,M;0,\tilde{M}) = \frac{\pi^2}{3}C_k^{(q,\tilde{q})}(M;\tilde{M}) - \frac{q\tilde{q}}{6k}\left(\frac{1}{q^2(1+M^2)} + \frac{1}{\tilde{q}^2(1+\tilde{M}^2)}\right) + \frac{kq\tilde{q}}{24}$$

$$A_k^{(q,\tilde{q})}(0,M;0,\tilde{M}) = \frac{1}{4}\sum_{\pm}\left[\tilde{q}^2 A^{\mathrm{ABJM}}((1\pm iM)qk) + q^2 A^{\mathrm{ABJM}}((1\pm i\tilde{M})\tilde{q}k)\right]$$

$$\eta_\alpha = \begin{cases} \frac{1}{3}i & (1\le\alpha\le N_{\mathrm{f}}) \\ -\frac{1}{3}i & (N_{\mathrm{f}}+1\le\alpha\le 2N_{\mathrm{f}}) \end{cases},$$

$$C_k^{(2N_{\mathrm{f}},1)}\left(0;\frac{i}{3}\right) = \frac{9}{8\pi^2 kN_{\mathrm{f}}},\ B_k^{(2N_{\mathrm{f}},1)}\left(\eta_\alpha,0;\frac{i}{3}\right) = \frac{7}{24kN_{\mathrm{f}}} + \frac{kN_{\mathrm{f}}}{12} - \frac{N_{\mathrm{f}}}{4k},$$

$$A_k^{(2N_{\mathrm{f}},1)}\left(\eta_\alpha,0;\frac{i}{3}\right) = \frac{1}{2\pi k}\left[\frac{\zeta(3)}{\pi N_{\mathrm{f}}} + N_{\mathrm{f}}^2\left(\frac{5\zeta(3)}{2\pi} + \frac{\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{2}{3}\right)}{4\sqrt{3}}\right)\right] + 2\pi k\left(-\frac{N_{\mathrm{f}}}{24\pi} - N_{\mathrm{f}}^2\left(\frac{1}{24\pi} + \frac{1}{72\sqrt{3}}\right)\right) + \mathcal{O}(k^3)$$

$$A_k^{(2N_{\mathrm{f}},1)}\left(\eta_\alpha,0;\frac{i}{3}\right) = \frac{1}{2}A^{\mathrm{ABJM}}(2N_{\mathrm{f}}k) + \frac{N_{\mathrm{f}}^2}{2}\sum_{\pm}\left[A^{\mathrm{ABJM}}\left(\left(1\pm\frac{1}{3}\right)k\right) + \mathcal{A}\left(\left(1\pm\frac{1}{3}\right)k, \frac{2i}{3}\right)\right].$$

$$a_1(z) = \frac{z}{2\tanh\left(\frac{\pi z}{2}\right)}$$

$$a_0\left(\pm\frac{2i}{3}\right) = -\frac{8\zeta(3)}{9\pi} + \frac{\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{2}{3}\right)}{9\sqrt{3}}, a_1\left(\pm\frac{2i}{3}\right) = \frac{1}{3\sqrt{3}}$$

$$C_{k=1}^{(q,1)}(0;\tilde{M}) = \frac{2}{\pi^2 q\left(1+\tilde{M}^2\right)}$$

$$B_{k=1}^{(q,1)}(\eta_\alpha,0;\tilde{M}) = \frac{2}{3q\left(1+\tilde{M}^2\right)} + \frac{q}{24} - \frac{1}{6q} - \frac{1}{1+\tilde{M}^2}\left(\frac{q}{6} + \sum_{I=1}^{q-1}\mu_I^2\right)$$

$$A_{k=1}^{(q,1)}(\eta_\alpha,0;\tilde{M}) = \frac{q^2}{4}\sum_{\pm} A^{\mathrm{ABJM}}(1\pm i\tilde{M}) + \frac{1}{2}A^{\mathrm{ABJM}}(q) + q\sum_{I=1}^{q-1}\mu_I^2\left[\frac{1}{1+\tilde{M}^2} - \frac{\pi^2}{72} - \frac{\pi^4\left(3\tilde{M}^2-1\right)}{21600} - \frac{\pi^6\left(5\tilde{M}^4-10\tilde{M}^2+1\right)}{1270080} + \cdots\right] + \mathcal{O}(\mu_I^4)$$

$$T^I = \frac{1}{\sqrt{2I(I+1)}}\mathrm{diag}(\underbrace{1,\ldots,1}_{I}, -I, \underbrace{0,\ldots,0}_{q-I-1}).$$

$$\mathrm{tr}(T^I) = 0, \mathrm{tr}(T^I T^J) = \frac{1}{2}\delta_{IJ}$$

$$\sum_{I=1}^{q-1}\mu_I (T^I)_{\alpha,\alpha} = \frac{1}{2}\eta_\alpha$$

$$\partial_{\mu_1}^4 A\Big|_{\mu_I,\tilde{M}=0} = 4(6+q)\mathfrak{b},\ \partial_{\mu_3}^4 A\Big|_{\mu_I,\tilde{M}=0} = 4\left(6+\frac{7}{6}q\right)\mathfrak{b}$$

$$\mathfrak{b} = -\frac{1}{2}\pi^2\left(\frac{\pi^2}{32} - \frac{1}{2}\right)$$

$$\sum_{\alpha=1}^{q}\eta_\alpha^2 = 2\sum_{I=1}^{q-1}\mu_I^2, \sum_{\alpha,\beta=1}^{q}\left(\eta_\alpha-\eta_\beta\right)^2 = 4q\sum_{I=1}^{q-1}\mu_I^2$$

$$\mathfrak{b} = -\frac{\pi^2}{2}\left(xA^{\mathrm{ABJM}}(x)\right)''\Big|_{x=1}$$

$$-\frac{1}{24}\pi^2\left(xA^{\mathrm{ABJM}}(x)\right)''\Big|_{x=1}$$

$$\prod_{n=1}^{L} 2\cosh\left(z + \frac{\pi i}{L}\left(\frac{L+1}{2} - n\right)\right) = 2\cosh(Lz)$$

$$(rq,\tilde{r}\tilde{q})_k \text{ with } \{\eta_\alpha = \xi'_\alpha, M; \tilde{\eta}_\alpha = \tilde{\xi}'_\alpha, \tilde{M}\} \leftrightarrow (q,\tilde{q})_{r\tilde{r}k} \text{ with } \{\eta_\alpha = \xi_\alpha, M; \tilde{\eta}_\alpha = \tilde{\xi}_\alpha, \tilde{M}\},$$

$$\xi'_\alpha = \begin{cases} \frac{\xi_1}{r} + \frac{2i}{r}\left[\frac{1+r}{2} - \alpha\right] & (1 \le \alpha \le r) \\ \frac{\xi_2}{r} + \frac{2i}{r}\left[\frac{1+r}{2} - (\alpha - r)\right] & (r+1 \le \alpha \le 2r) \\ \vdots & \vdots \\ \frac{\xi_q}{r} + \frac{2i}{r}\left[\frac{1+r}{2} - (\alpha - (q-1)r)\right] & ((q-1)r+1 \le \alpha \le qr) \end{cases}$$

$$\tilde{\xi}'_\alpha = \begin{cases} \dfrac{\tilde{\xi}_1}{\tilde{r}} + \dfrac{2i}{\tilde{r}}\left[\dfrac{1+\tilde{r}}{2} - \alpha\right] & (1 \le \alpha \le \tilde{r}) \\ \dfrac{\tilde{\xi}_2}{\tilde{r}} + \dfrac{2i}{\tilde{r}}\left[\dfrac{1+\tilde{r}}{2} - (\alpha - \tilde{r})\right] & (\tilde{r}+1 \le \alpha \le 2\tilde{r}) \\ \vdots & \vdots \\ \dfrac{\tilde{\xi}_{\tilde{q}}}{\tilde{r}} + \dfrac{2i}{\tilde{r}}\left[\dfrac{1+\tilde{r}}{2} - (\alpha - (\tilde{q}-1)\tilde{r})\right] & ((\tilde{q}-1)\tilde{r}+1 \le \alpha \le \tilde{q}\tilde{r}) \end{cases}$$

$$C_k^{(rq,\tilde{r}\tilde{q})}(M;\tilde{M}) = C_{r\tilde{r}k}^{(q,\tilde{q})}(M;\tilde{M}), B_k^{(rq,\tilde{r}\tilde{q})}(\xi'_\alpha, M; \tilde{\xi}'_\alpha, \tilde{M}) = B_{r\tilde{r}k}^{(q,\tilde{q})}(\xi_\alpha, M; \tilde{\xi}_\alpha, \tilde{M})$$

$$A_k^{(rq,\tilde{r}\tilde{q})}(\xi'_\alpha, M; \tilde{\xi}'_\alpha, \tilde{M}) = A_{r\tilde{r}k}^{(q,\tilde{q})}(\xi_\alpha, M; \tilde{\xi}_\alpha, \tilde{M})$$

$$\sum_{\alpha,\beta=1}^{L} \mathcal{A}\left(\kappa, \frac{\chi}{L} + \frac{2i}{L}(\alpha - \beta)\right) = \mathcal{A}(L\kappa, \chi)$$

$$\sum_{\alpha,\beta=1}^{L} a_0\left(z + \frac{2i}{L}(\alpha - \beta)\right) = \frac{1}{L} a_0(Lz)$$

$$\sum_{\alpha,\beta=1}^{L} i\log\, s_{b=1}\left(z + \frac{i}{L}(\alpha - \beta)\right) = i\log\, s_{b=1}(Lz)$$

$$\sum_{\alpha,\beta=1}^{L} \left[\left(z + \frac{2i}{L}(\alpha - \beta)\right) + \frac{1}{2}\left(z + \frac{2i}{L}(\alpha - \beta)\right)^3\right] = z + \frac{1}{2} L^2 z^3$$

$$\sum_{\alpha,\beta=1}^{L} (\alpha - \beta) f(\alpha - \beta) = \sum_{\alpha=1}^{L-1} \alpha(L - \alpha)(f(\alpha) - f(\alpha - L))$$

$$\sum_{\alpha,\beta=1}^{L} \left(z + \frac{2i}{L}(\alpha - \beta)\right) \mathrm{Li}_2\left(e^{\pi\left(z + \frac{2i}{L}(\alpha - \beta)\right)}\right) = z\mathrm{Li}_2(e^{\pi L z})$$

$$\sum_{\alpha,\beta=1}^{L} \mathrm{Li}_3\left(e^{\pi\left(z + \frac{2i}{L}(\alpha - \beta)\right)}\right) = L^{-1}\mathrm{Li}_3(e^{\pi L z})$$

$$\sum_{\alpha,\beta=1}^{L} a_0\left(\frac{1}{L} z + \frac{2i}{L}(\alpha - \beta)\right) = \frac{1}{L}\left(-\frac{\pi^3}{6} z - \frac{\pi^3}{12} z^3 - \pi z \mathrm{Li}_2(e^{\pi z}) + 2\mathrm{Li}_3(e^{\pi z})\right)$$

$$\sum_{\alpha,\beta=1}^{L} \left[\frac{1}{6} + \left(z + \frac{i}{L}(\alpha - \beta)\right)^2\right] = \frac{1}{6} + L^2 z^2$$

$$\sum_{\alpha,\beta=1}^{L} \left(z + \frac{2i}{L}(\alpha - \beta)\right) \log\left(1 - e^{2\pi\left(z + \frac{i}{L}(\alpha - \beta)\right)}\right) = Lz\log\,(1 - e^{2\pi L z})$$

$$\sum_{\alpha,\beta=1}^{L} \mathrm{Li}_2\left(e^{2\pi\left(z + \frac{i}{L}(\alpha - \beta)\right)}\right) = \mathrm{Li}_2(e^{2\pi L z})$$

$$\sum_{\alpha,\beta=1}^{L} i\log\, s_{b=1}\left(z + \frac{i}{L}(\alpha - \beta)\right) = \frac{\pi}{12} + \frac{\pi}{2} L^2 z^2 - Lz\log\,(1 - e^{2\pi L z}) - \frac{1}{2\pi}\mathrm{Li}_2(e^{2\pi L z})$$

$$Z^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(N;\zeta,m,y_\alpha)=\frac{1}{N!}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\frac{d\lambda_i}{2\pi}e^{i\zeta\sum_{i=1}^{N}\lambda_i}\prod_{i<j}^{N}2\sinh\frac{b\lambda_{ij}}{2}\prod_{i<j}^{N}2\sinh\frac{\lambda_{ij}}{2b}$$

$$\times\prod_{i,j}^{N}\mathcal{D}_b\left(\frac{\lambda_{ij}}{2\pi}+m\right)\prod_{\alpha=1}^{N_{\mathrm{f}}}\prod_{i=1}^{N}\mathcal{D}_b\left(\frac{\lambda_i}{2\pi}+y_\alpha\right)$$

$$\sum_{\alpha=1}^{N_{\mathrm{f}}}y_\alpha=0$$

$$m_b=\frac{b^2-3}{4b}i$$

$$Z^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(N;\zeta,m_b,y_\alpha)=\frac{1}{N!}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{N}\frac{d\lambda_i}{2\pi}\right)e^{i\zeta\sum_{i=1}^{N}\lambda_i}\prod_{i<j}^{N}2\sinh\frac{b\lambda_{ij}}{2}\prod_{i<j}^{N}2\sinh\frac{\lambda_{ij}}{2b}$$

$$\times\prod_{i,j}^{N}\frac{s_b\left(\frac{\lambda_{ij}}{2\pi}+\frac{i}{2}(b-b^{-1})\right)}{s_b\left(\frac{\lambda_{ij}}{2\pi}-ib^{-1}\right)}\prod_{\alpha=1}^{N_{\mathrm{f}}}\prod_{i=1}^{N}\mathcal{D}_b\left(\frac{\lambda_i}{2\pi}+y_\alpha\right)$$

$$\prod_{i,j}^{N}s_b\left(\frac{\lambda_{ij}}{2\pi}+\frac{i}{2}(b-b^{-1})\right)=\left(\sqrt{\frac{s_b\left(\frac{i}{2}(b-b^{-1})\right)}{s_b\left(-\frac{i}{2}(b-b^{-1})\right)}}\right)^{N}\prod_{i<j}^{N}\frac{s_b\left(\frac{\lambda_{ij}}{2\pi}+\frac{i}{2}(b-b^{-1})\right)}{s_b\left(\frac{\lambda_{ij}}{2\pi}-\frac{i}{2}(b-b^{-1})\right)}$$

$$=b^{-N}\prod_{i<j}^{N}\frac{2\sinh\left(\frac{\lambda_{ij}}{2b}\right)}{2\sinh\left(\frac{b\lambda_{ij}}{2}\right)}$$

$$\prod_{i,j}^{N}\frac{1}{s_b\left(\frac{\lambda_{ij}}{2\pi}-ib^{-1}\right)}=\left(\sqrt{\frac{s_b(ib^{-1})}{s_b(-ib^{-1})}}\right)^{N}\prod_{i<j}^{N}\frac{s_b\left(\frac{\lambda_{ij}}{2\pi}+ib^{-1}\right)}{s_b\left(\frac{\lambda_{ij}}{2\pi}-ib^{-1}\right)}=\prod_{i,j}^{N}\frac{1}{2\cosh\left(\frac{\lambda_{ij}}{2b}+\frac{i}{2b^2}\right)}.$$

$$Z^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(N;\zeta,m_b,y_\alpha)$$

$$=\frac{1}{N!}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{N}\frac{d\lambda_i}{2\pi}\right)e^{ib\zeta\sum_{i=1}^{N}\lambda_i}\frac{\prod_{i<j}^{N}\left(2\sinh\frac{\lambda_{ij}}{2}\right)^2}{\prod_{i,j}^{N}2\cosh\left(\frac{\lambda_{ij}}{2}+\frac{\pi i}{2b^2}\right)}\prod_{\alpha=1}^{N_{\mathrm{f}}}\prod_{i=1}^{N}\mathcal{D}_b\left(\frac{b}{2\pi}\lambda_i+y_\alpha\right)$$

$$Z^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(N;\zeta,m_b,y_\alpha)=\frac{1}{N!}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{N}\frac{d\lambda_i}{2\pi}\right)\det\left(\left[\langle\lambda_i|\hat{\rho}^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(\hat{x},\hat{p};\zeta,m_b,y_\alpha)|\lambda_j\rangle\right]_{i,j}^{N\times N}\right),$$

$$\hat{\rho}^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(\hat{x},\hat{p};\zeta,m_b,y_\alpha)=e^{ib\zeta\hat{x}}\left(\prod_{\alpha=1}^{N_{\mathrm{f}}}\mathcal{D}_b\left(\frac{b}{2\pi}\hat{x}+y_\alpha\right)\right)\frac{e^{-\frac{1}{2b^2}\hat{p}}}{2\cosh\frac{\hat{p}}{2}}.$$

$$\hat{\rho}^{\mathrm{SYM}}_{b=1,N_{\mathrm{f}}}(\hat{x},\hat{p};\zeta,m,y_\alpha)=\frac{e^{i\zeta\hat{x}}}{\prod_{\alpha=1}^{N_{\mathrm{f}}}2\cosh\frac{\hat{x}+2\pi y_\alpha}{2}}\frac{e^{im\hat{p}}}{2\cosh\frac{\hat{p}}{2}}$$

$$\frac{\zeta}{N_{\mathrm{f}}}=\frac{1}{2}M,m=\frac{1}{2}\tilde{M},y_\alpha=\frac{1}{2}\eta_\alpha$$

$$(nN_{\mathrm{f}},1)_{k=1}\text{ with }\left\{M=\frac{2b}{nN_{\mathrm{f}}}\zeta,\eta_\alpha=y'_\alpha;\tilde{M}=ib^{-2}\right\}\leftrightarrow\mathrm{SYM}_{b=\sqrt{2n-1},N_{\mathrm{f}}}\text{ with }\{\zeta,m_b,y_\alpha\}$$

$$y'_\alpha = \begin{cases} \frac{2}{b} y_1 + \frac{2i}{b^2}\left[\frac{1+n}{2} - \alpha\right] & (1 \le \alpha \le n) \\ \frac{2}{b} y_2 + \frac{2i}{b^2}\left[\frac{1+n}{2} - (\alpha - n)\right] & (n+1 \le \alpha \le 2n) \\ \vdots & \vdots \\ \frac{2}{b} y_{N_\mathrm{f}} + \frac{2i}{b^2}\left[\frac{1+n}{2} - (\alpha - (N_\mathrm{f}-1)n)\right] & ((N_\mathrm{f}-1)n + 1 \le \alpha \le N_\mathrm{f} n) \end{cases}$$

$$A^{\mathrm{SYM}}_{b=\sqrt{2n-1},N_\mathrm{f}}(\zeta, m_b, y_\alpha) = \frac{1}{4}\sum_{\pm}\left[A^{\mathrm{ABJM}}\left(\frac{(b^2+1)N_\mathrm{f} \pm 4ib\zeta}{2}\right)\right.$$

$$\left. + \sum_{\alpha,\beta=1}^{N_\mathrm{f}} \sum_{\alpha',\beta'=1}^{n} \mathcal{A}\left(1 \pm b^{-2}, \frac{2}{b}(y_\alpha - y_\beta) + \frac{2i}{b^2}(\alpha' - \beta')\right)\right].$$

$$A^{\mathrm{SYM}}_{b=\sqrt{2n-1},N_\mathrm{f}=1}(\zeta, m_b)$$
$$= \frac{1}{4}\left[\sum_{\pm} A^{\mathrm{ABJM}}\left(\frac{b^2+1 \pm 4ib\zeta}{2}\right) + A^{\mathrm{ABJM}}(2n-2) - A^{\mathrm{ABJM}}(4n-2)\right]$$

$$\sum_{\alpha,\beta=1}^{n}\left[\mathcal{A}\left(\frac{2n}{2n-1}, \frac{2i}{2n-1}(\alpha-\beta)\right) + \mathcal{A}\left(\frac{2n-2}{2n-1}, \frac{2i}{2n-1}(\alpha-\beta)\right)\right]$$
$$= A^{\mathrm{ABJM}}(2n-2) - A^{\mathrm{ABJM}}(4n-2)$$

$$\sum_{\alpha,\beta=1}^{n}\left[\mathcal{A}\left(\frac{2n}{2n-1}, \frac{\chi}{2n-1} + \frac{2i}{2n-1}(\alpha-\beta)\right) + \mathcal{A}\left(\frac{2n-2}{2n-1}, \frac{\chi}{2n-1} + \frac{2i}{2n-1}(\alpha-\beta)\right)\right]$$
$$= \mathcal{A}(2n-2, \chi) - \mathcal{A}(4n-2, \chi)$$

$$C^{\mathrm{SYM}}_{b,N_\mathrm{f}}(\zeta, m_b) = \frac{4b^4 N_\mathrm{f}}{\pi^2(b^2-1)\left((b^2+1)^2 N_\mathrm{f}^2 + 16b^2\zeta^2\right)}$$

$$B^{\mathrm{SYM}}_{b,N_\mathrm{f}}(\zeta, m_b, y_\alpha) = \frac{1}{b^2-1}\left[\frac{(3b^4+1)N_\mathrm{f}}{3\left((b^2+1)^2 N_\mathrm{f}^2 + 16b^2\zeta^2\right)} - b^2\sum_{\alpha=1}^{N_\mathrm{f}} y_\alpha^2 - \frac{N_\mathrm{f}}{24}(b^4 - b^2 + 2)\right]$$

$$A^{\mathrm{SYM}}_{b,N_\mathrm{f}}(\zeta, m_b, y_\alpha) = \frac{1}{4}\left[\sum_{\pm} A^{\mathrm{ABJM}}\left(\frac{(b^2+1)N_\mathrm{f} \pm 4ib\zeta}{2}\right)\right.$$

$$\left. + \sum_{\alpha,\beta=1}^{N_\mathrm{f}}\left(\mathcal{A}\left(b^2-1, 2b(y_\alpha - y_\beta)\right) - \mathcal{A}\left(2b^2, 2b(y_\alpha - y_\beta)\right)\right)\right]$$

$$C^{\mathrm{SYM}}_{b,N_\mathrm{f}}(\zeta, m) = \left(\frac{2}{\pi(b+b^{-1})^2\sqrt{2N_\mathrm{f}\Delta_1\Delta_2\Delta_3\Delta_4}}\right)^2$$

$$B^{\mathrm{SYM}}_{b,N_\mathrm{f}}(\zeta, m) = \frac{N_\mathrm{f}}{24} - \frac{N_\mathrm{f}}{12}\left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2}\right) - \frac{1}{12N_\mathrm{f}}\left(\frac{1}{\Delta_3} + \frac{1}{\Delta_4}\right)$$
$$- \frac{4}{3(b+b^{-1})^2}\left(-\frac{N_\mathrm{f}}{8\Delta_1\Delta_2} + \frac{\Delta_1^2 + \Delta_2^2 - 2(\Delta_1+\Delta_2) + \Delta_1\Delta_2}{8N_\mathrm{f}\Delta_1\Delta_2\Delta_3\Delta_4}\right)$$

$$A^{\mathrm{SYM}}_{b,N_\mathrm{f}}(\zeta, m_b) = \frac{1}{4}\sum_{\pm} A^{\mathrm{ABJM}}\left(\frac{(b^2+1)N_\mathrm{f} \pm 4ib\zeta}{2}\right)$$

y	$Z^{\mathrm{SYM,pert}}_{\sqrt{2},2}(\zeta=0)$	$Z^{\mathrm{SYM}}_{\sqrt{2},2}(\zeta=0)$
0	0.0596079	0.059275
0.1	0.0559911	0.0555738
0.2	0.046786	0.0461615

0.3	0.0354902	0.0346407
0.4	0.0250305	0.0240313

y	$Z^{\mathrm{SYM,pert}}_{\sqrt{2},2}(\zeta=0.2)$	$Z^{\mathrm{SYM}}_{\sqrt{2},2}(\zeta=0.2)$
0	0.0557911	0.0555738
0.1	0.0522504	0.0519804
0.2	0.0432631	0.0428701
0.3	0.0322991	0.031788
0.4	0.0222461	0.0216819

$$+\frac{N_{\mathrm{f}}^2}{4}\left(A^{\mathrm{ABJM}}(b^2-1)-A^{\mathrm{ABJM}}(2b^2)\right),$$

$$\Delta_1=\frac{1}{2}+\frac{2mi}{b+b^{-1}},\Delta_2=\frac{1}{2}-\frac{2mi}{b+b^{-1}},\Delta_3=\frac{1}{2}-\frac{2\zeta i}{(b+b^{-1})N_{\mathrm{f}}},\Delta_4=\frac{1}{2}+\frac{2\zeta i}{(b+b^{-1})N_{\mathrm{f}}}.$$

$$C^{\mathrm{SYM}}_{b,N_{\mathrm{f}}=1}(\zeta,m_b)=\frac{4b^4}{\pi^2(b^2-1)((b^2+1)^2+16b^2\zeta^2)}$$
$$B^{\mathrm{SYM}}_{b,N_{\mathrm{f}}=1}(\zeta,m_b)=-\frac{b^8+b^6(16\zeta^2+1)-b^4(16\zeta^2+23)+b^2(32\zeta^2+3)-6}{24(b^2-1)((b^2+1)^2+16b^2\zeta^2)}$$

$$Z^{\mathrm{SYM,\,pert}}_{b,N_{\mathrm{f}}=2}(1;\zeta,m_b,y_\alpha)$$
$$=e^{A^{\mathrm{SYM}}_{b,2}(\zeta,m_b,y_\alpha)}C^{\mathrm{SYM}}_{b,2}(\zeta,m_b)^{-\frac{1}{3}}\mathrm{Ai}\left[C^{\mathrm{SYM}}_{b,2}(\zeta,m_b)^{-\frac{1}{3}}\left(1-B^{\mathrm{SYM}}_{b,2}(\zeta,m_b,y_\alpha)\right)\right],$$

$$Z^{\mathrm{SYM}}_{b,N_{\mathrm{f}}=2}(1;\zeta,m_b,y_\alpha)=\mathcal{D}_b(m_b)\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}e^{i\zeta\sum_{i=1}^{N}\lambda_i}\prod_{\alpha=1}^{2}\mathcal{D}_b\left(\frac{\lambda}{2\pi}+y_\alpha\right).$$

$$A^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(\zeta,m_b,y_\alpha=y_\alpha(b))$$
$$=\frac{1}{4}\left[\sum_{\pm}A^{\mathrm{ABJM}}\left(\frac{(b^2+1)N_{\mathrm{f}}\pm 4ib\zeta}{2}\right)+A^{\mathrm{ABJM}}\left((b^2-1)N_{\mathrm{f}}\right)-A^{\mathrm{ABJM}}(2b^2N_{\mathrm{f}})\right],$$

$$y_\alpha(b)=\frac{i}{bN_{\mathrm{f}}}\left(\frac{1+N_{\mathrm{f}}}{2}-\alpha\right).$$

$$\hat{O}^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(\hat{x},\hat{p};\zeta,m_b,y_\alpha)$$
$$=e^{-\frac{ib\zeta}{2}\hat{x}}\prod_{\alpha=1}^{N_{\mathrm{f}}}s_b\left(\frac{b}{2\pi}\hat{x}+y_\alpha-\frac{i}{4}Q\right)\left(e^{\left(\frac{1}{2}+\frac{1}{2b^2}\right)\hat{p}}+e^{-\left(\frac{1}{2}-\frac{1}{2b^2}\right)\hat{p}}\right)\prod_{\alpha=1}^{N_{\mathrm{f}}}\frac{e^{-\frac{ib\zeta}{2}\hat{x}}}{s_b\left(\frac{b}{2\pi}\hat{x}+y_\alpha+\frac{i}{4}Q\right)}.$$

$$\hat{O}^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(\hat{x},\hat{p};\zeta,m_b,y_\alpha)=\hat{A}\hat{\rho}^{\mathrm{SYM}}_{b,N_{\mathrm{f}}}(\hat{x},\hat{p};\zeta,m_b,y_\alpha)^{-1}\hat{A}^{-1},$$

$$\hat{A}=e^{-\frac{ib\zeta}{2}\hat{x}}\prod_{\alpha=1}^{N_{\mathrm{f}}}s_b\left(\frac{b}{2\pi}\hat{x}+y_\alpha-\frac{i}{4}Q\right)$$

$$e^{c_1\hat{x}}e^{c_2\hat{p}}=e^{c_1\pi i}e^{c_1\hat{x}+c_2\hat{p}}=e^{2c_1c_2\pi i}e^{c_2\hat{p}}e^{c_1\hat{x}},$$

$$\hat{O}^{\mathrm{SYM}}_{b,N_\mathrm{f}}(\hat{x},\hat{p};\zeta,m_b,y_\alpha)$$
$$= e^{-ib\zeta\hat{x}+\left(\frac{1}{2}+\frac{1}{2b^2}\right)\hat{p}} + e^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2b^2}\right)\hat{p}} e^{-ib\zeta\hat{x}} \left(\prod_{\alpha=1}^{N_\mathrm{f}} 2\cosh\left(\frac{b^2}{2}\hat{x}+\pi b y_\alpha\right)\right) e^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2b^2}\right)\hat{p}}$$

$$\hat{X} = \left(\frac{b^2 N_\mathrm{f}}{2} - ib\zeta\right)\hat{x} - \left(\frac{1}{2}-\frac{1}{2b^2}\right)\hat{p}, \hat{P} = -ib\zeta\hat{x} + \left(\frac{1}{2}+\frac{1}{2b^2}\right)\hat{p}$$

$$\hat{O}^{\mathrm{SYM}}_{b,N_\mathrm{f}}(\hat{x},\hat{p};\zeta,m_b,y_\alpha(b)) = \hat{O}^{\mathbb{P}(1,\mathfrak{m},\mathfrak{n})}(\hat{X},\hat{P}),$$

$$\hat{O}^{\mathbb{P}(1,\mathfrak{m},\mathfrak{n})}(\hat{X},\hat{P}) = e^{\hat{X}} + e^{\hat{P}} + e^{-\mathfrak{m}\hat{X}-\mathfrak{n}\hat{P}}$$

$$C^{\mathbb{P}(1,\mathfrak{m},\mathfrak{n})}(\hbar') = \frac{(\mathfrak{m}+\mathfrak{n}+1)^2}{\mathfrak{m}\mathfrak{n}}\frac{1}{4\pi\hbar'}$$
$$B^{\mathbb{P}(1,\mathfrak{m},\mathfrak{n})}(\hbar') = \frac{\mathfrak{m}^2+\mathfrak{m}\mathfrak{n}+\mathfrak{n}^2+\mathfrak{m}+\mathfrak{n}+1}{12\mathfrak{m}\mathfrak{n}}\frac{\pi}{\hbar'} - \frac{\mathfrak{m}+\mathfrak{n}+1}{48\pi}\hbar'$$
$$A^{\mathbb{P}(1,\mathfrak{m},\mathfrak{n})}(\hbar') = \frac{1}{4}\Bigg[A^{\mathrm{ABJM}}\left(\frac{\hbar'}{\pi}\right) + A^{\mathrm{ABJM}}\left(\frac{\mathfrak{m}\hbar'}{\pi}\right)$$
$$+A^{\mathrm{ABJM}}\left(\frac{\mathfrak{n}\hbar'}{\pi}\right) - A^{\mathrm{ABJM}}\left(\frac{(\mathfrak{m}+\mathfrak{n}+1)\hbar'}{\pi}\right)\Bigg]$$

$$\mathfrak{m} = \frac{(b+b^{-1})N_\mathrm{f}+4i\zeta}{(b+b^{-1})N_\mathrm{f}-4i\zeta}, \mathfrak{n} = \frac{2(b-b^{-1})N_\mathrm{f}}{(b+b^{-1})N_\mathrm{f}-4i\zeta}, \hbar' = \frac{(b^2+1)N_\mathrm{f}-4ib\zeta}{2}\pi$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}dx$$

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}$$

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^s}$$

$$\mathrm{Li}_s(1) = \zeta(s)$$

$$\mathrm{Li}_1(z) = -\log(1-z)$$

$$\frac{\partial}{\partial z}\mathrm{Li}_s(z) = \frac{1}{z}\mathrm{Li}_{s-1}(z)$$

$$\sum_{n=1}^{p}\mathrm{Li}_s\left(ze^{2\pi i\frac{n}{p}}\right) = p^{1-s}\mathrm{Li}_s(z^p)$$

$$\mathrm{Li}_n(-e^x) + (-1)^n\mathrm{Li}_n(-e^{-x}) = -\frac{(2\pi i)^n}{n!}B_n\left(\frac{x}{2\pi i}+\frac{1}{2}\right)$$

$$s_b(z) = \prod_{\ell,m=0}^{\infty}\frac{\ell b + m b^{-1} + \frac{Q}{2} - iz}{\ell b + m b^{-1} + \frac{Q}{2} + iz}$$

$$Q = b + b^{-1}$$

$$i\log\, s_b(z) = \frac{\pi}{2}z^2 + \frac{\pi}{24}(b^2 + b^{-2}) + i\int_{\mathbb{R}+i0^+}\frac{dt}{t}\frac{e^{-2izt}}{4\sinh\,(bt)\sinh\,(b^{-1}t)}$$

$$s_b(0) = 1, s_b(z) = s_{b^{-1}}(z), s_b(z)s_b(-z) = 1, \overline{s_b(z)} = s_b(-\bar{z})$$

$$\frac{s_b\left(z + \frac{i}{2}b^{\pm 1}\right)}{s_b\left(z - \frac{i}{2}b^{\pm 1}\right)} = \frac{1}{2\cosh\,(\pi b^{\pm 1}z)}$$

$$i\log\, s_{b=1}(z) = \frac{\pi}{12} + \frac{\pi}{2}z^2 - z\log\,(1 - e^{2\pi z}) - \frac{1}{2\pi}\mathrm{Li}_2(e^{2\pi z})$$

$$\frac{d}{dz}i\log\, s_{b=1}(z) = -\frac{\pi z}{\tanh\,\pi z}$$

$$\mathcal{D}_b(z) = \frac{s_b\left(z + \frac{i}{4}Q\right)}{s_b\left(z - \frac{i}{4}Q\right)}$$

$$\mathcal{D}_{b=\sqrt{2n-1}}(\mu) = \prod_{j=1}^{n}\frac{1}{2\cosh\left(\frac{\pi}{b}\mu + \frac{\pi i}{b^2}\left(\frac{n+1}{2} - j\right)\right)}$$

$$\mathcal{D}_{\sqrt{2}}(\mu) = \frac{1}{2^{\frac{1}{4}}(2\cosh\,(2\sqrt{2}\pi\mu))^{\frac{1}{8}}(\sqrt{2}\cosh\,(\sqrt{2}\pi\mu) + 1)^{\frac{1}{2}}} \times \exp\left[-\sqrt{2}\mu\arctan\,\left(e^{-2\sqrt{2}\pi\mu}\right) + \frac{i}{4\pi}\left(\mathrm{Li}_2\left(ie^{-2\sqrt{2}\pi\mu}\right) - \mathrm{Li}_2\left(-ie^{-2\sqrt{2}\pi\mu}\right)\right)\right]$$

$$\mathcal{Z}_{2\ell}^{(q,\tilde{q})}(s; v_\alpha, M; \tilde{v}_\alpha, \tilde{M}) = D_{2\ell}^{(q,\tilde{q})}(s; v_\alpha, M; \tilde{v}_\alpha, \tilde{M})\mathcal{Z}_0^{(q,\tilde{q})}(s; v_\alpha, M; \tilde{v}_\alpha, \tilde{M})$$

$$\hat{\rho}_k^{(q,\tilde{q})}\left(\hat{x}, \hat{p}; \frac{\hbar v_\alpha}{\pi}, M; \frac{\hbar\tilde{v}_\alpha}{\pi}, \tilde{M}\right)^s = e^{-s\hat{H}_k^{(q,\tilde{q})}\left(\hat{x}, \hat{p}; \frac{\hbar v_\alpha}{\pi}, M; \frac{\hbar\tilde{v}_\alpha}{\pi}, \tilde{M}\right)}$$

$$\hat{H}_k^{(q,\tilde{q})}\left(\hat{x}, \hat{p}; \frac{\hbar v_\alpha}{\pi}, M; \frac{\hbar\tilde{v}_\alpha}{\pi}, \tilde{M}\right) = H_W(x, p)$$

$$\hat{\rho}_k^{(q,\tilde{q})}\left(\hat{x}, \hat{p}; \frac{\hbar v_\alpha}{\pi}, M; \frac{\hbar\tilde{v}_\alpha}{\pi}, \tilde{M}\right)^s$$

$$= \sum_{r=0}^{\infty} e^{-sH_W(x,p)}\frac{(-s)^r}{r!}\left(\hat{H}_k^{(q,\tilde{q})}\left(\hat{x}, \hat{p}; \frac{\hbar v_\alpha}{\pi}, M; \frac{\hbar\tilde{v}_\alpha}{\pi}, \tilde{M}\right) - H_W(x, p)\right)^r.$$

$$\left(\hat{\rho}_k^{(q,\tilde{q})}\left(\hat{x}, \hat{p}; \frac{\hbar v_\alpha}{\pi}, M; \frac{\hbar\tilde{v}_\alpha}{\pi}, \tilde{M}\right)^s\right)_W = \sum_{r=0}^{\infty} e^{-sH_W(x,p)}\frac{(-s)^r}{r!}\mathcal{G}_r(x, p),$$

$$\mathcal{G}_r(x, p) = \left(\left(\hat{H}_k^{(q,\tilde{q})}\left(\hat{x}, \hat{p}; \frac{\hbar v_\alpha}{\pi}, M; \frac{\hbar\tilde{v}_\alpha}{\pi}, \tilde{M}\right) - H_W(x, p)\right)^r\right)_W$$

$$\mathcal{G}_0(x, p) = 1$$
$$\mathcal{G}_1(x, p) = 0$$
$$\mathcal{G}_r(x, p) = (-1)^{r-1}(r-1)H_W(x, p)^r + \sum_{r'=2}^{r}\binom{r}{r'}(-1)^{r-r'}H_W(x, p)^{r-r'}(\underbrace{H_W(x, p) \star \cdots \star H_W(x, p)}_{r'}), (r \geq 2)$$

$$\mathcal{G}_r(x, p) = \sum_{a=2\left\lfloor\frac{r+2}{3}\right\rfloor}\hbar^a\mathcal{G}_r^{(a)}$$

$$H_W(x,p)=\sum_{a=0}^{\infty}\hbar^a H_W^{(a)}$$

$$\mathrm{Tr}_k^{(q,\tilde q)}\left(\hat x,\hat p;\frac{\hbar v_\alpha}{\pi},M;\frac{\hbar\tilde v_\alpha}{\pi},\tilde M\right)^s$$

$$\mathrm{Tr}\hat\rho_k^{(q,\tilde q)}\left(\hat x,\hat p;\frac{\hbar v_\alpha}{\pi},M;\frac{\hbar\tilde v_\alpha}{\pi},\tilde M\right)^s=\int\frac{dxdp}{2\pi\hbar}e^{-sH_W^{(0)}}\Delta_{\mathrm{WK}}$$

$$\Delta_{\mathrm{WK}}=\left(1+\sum_{r=1}^{\infty}\frac{(-s)^r}{r!}\left(\sum_{a=1}^{\infty}\hbar^a H_W^{(a)}\right)^r\right)\left(1+\sum_{r'=2}^{\infty}\frac{(-s)^{r'}}{r!}\sum_{a'=2\left\lceil\frac{r'+2}{3}\right\rceil}^{\infty}\hbar^{a'}\mathcal{G}_{r'}^{(a')}\right)$$

$$\mathcal{Z}_2^{(q,\tilde q)}(s;v_\alpha,M;\tilde v_\alpha,\tilde M)=\int\frac{dxdp}{2\pi}e^{-sH_W^{(0)}}\left(-sH_W^{(2)}+\frac{s^2}{2}\left(H_W^{(1)}\right)^2+\frac{s^2}{2}\mathcal{G}_2^{(2)}-\frac{s^3}{6}\mathcal{G}_3^{(2)}\right)$$

$$H_W^{(0)}=q\log\,2\cosh\frac{x}{2}-\frac{iqMx}{2}+\tilde q\log\,2\cosh\frac{p}{2}-\frac{i\tilde q\tilde Mx}{2}$$

$$H_W^{(1)}=0$$

$$H_W^{(2)}=\frac{1}{F(x)^2}\left(\frac{1}{2}\sum_{\alpha=1}^{q}v_\alpha^2-\frac{q\tilde q^2(1-\tilde M^2)}{48}\right)+\frac{1}{F(p)^2}\left(\frac{1}{2}\sum_{\alpha=1}^{\tilde q}\tilde v_\alpha^2+\frac{q^2\tilde q(1-M^2)}{96}\right)$$
$$+\frac{iq\tilde q^2\tilde MF'(p)}{12F(x)^2F(p)}-\frac{iq^2\tilde qMF'(x)}{24F(x)F(p)^2}+\frac{1}{F(x)^2F(p)^2}\left(-\frac{q^2\tilde q}{24}+\frac{q\tilde q^2}{12}\right)$$

$$\mathcal{G}_2^{(2)}=\frac{1}{4}\left[\left(\partial_x\partial_pH_W^{(0)}\right)^2-\partial_x^2H_W^{(0)}\partial_p^2H_W^{(0)}\right]=-\frac{q\tilde q}{4F(x)^2F(p)^2}$$

$$\mathcal{G}_3^{(2)}=\frac{1}{4}\left[2\partial_xH_W^{(0)}\partial_pH_W^{(0)}\partial_x\partial_pH_W^{(0)}-\partial_x^2H_W^{(0)}\left(\partial_pH_W^{(0)}\right)^2-\left(\partial_xH_W^{(0)}\right)^2\partial_p^2H_W^{(0)}\right]$$
$$=-\frac{q\tilde q^2(1-\tilde M^2)}{16F(x)^2}-\frac{q^2\tilde q(1-M^2)}{16F(p)^2}+\frac{iq\tilde q^2\tilde MF'(p)}{4F(x)^2F(p)}+\frac{iq^2\tilde qMF'(x)}{4F(x)F(p)^2}+\frac{q\tilde q(q+\tilde q)}{4F(x)^2F(p)^2}$$

$$\mathcal{Z}_2^{(q,\tilde q)}(s;v_\alpha,M;\tilde v_\alpha,\tilde M)$$
$$=\frac{1}{2\pi}\left[-s\left(\left(\frac{1}{2}\sum_{\alpha=1}^{q}v_\alpha^2-\frac{q\tilde q^2(1-\tilde M^2)}{48}\right)I_1\left(\frac{iqMs}{2},qs+2\right)I_1\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs\right)\right.\right.$$
$$+\left(\frac{1}{2}\sum_{\alpha=1}^{\tilde q}\tilde v_\alpha^2+\frac{q^2\tilde q(1-M^2)}{96}\right)I_1\left(\frac{iqMs}{2},qs\right)I_1\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs+2\right)$$
$$+\frac{iq\tilde q^2\tilde M}{12}I_1\left(\frac{iqMs}{2},qs+2\right)I_2\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs+1\right)$$
$$-\frac{iq^2\tilde qM}{24}I_2\left(\frac{iqMs}{2},qs+1\right)I_1\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs+2\right)$$
$$\left.+\left(-\frac{q^2\tilde q}{24}+\frac{q\tilde q^2}{12}\right)I_1\left(\frac{iqMs}{2},qs+2\right)I_1\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs+2\right)\right)$$
$$+\frac{s^2}{2}\left(-\frac{q\tilde q}{4}I_1\left(\frac{iqMs}{2},qs+2\right)I_1\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs+2\right)\right)$$
$$-\frac{s^3}{6}\left(-\frac{q\tilde q^2(1-\tilde M^2)}{16}I_1\left(\frac{iqMs}{2},qs+2\right)I_1\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs\right)\right.$$
$$-\frac{q^2\tilde q(1-M^2)}{16}I_1\left(\frac{iqMs}{2},qs\right)I_1\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs+2\right)$$
$$+\frac{iq\tilde q^2\tilde M}{4}I_1\left(\frac{iqMs}{2},qs+2\right)I_2\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs+1\right)$$
$$+\frac{iq^2\tilde qM}{4}I_2\left(\frac{iqMs}{2},qs+1\right)I_1\left(\frac{i\tilde q\tilde Ms}{2},\tilde qs+2\right)$$

$$+\frac{q\tilde{q}(q+\tilde{q})}{4} I_1\left(\frac{iqMs}{2}, qs+2\right) I_1\left(\frac{i\tilde{q}\tilde{M}s}{2}, \tilde{q}s+2\right)\Bigg)\Bigg]$$

$$\mathcal{Z}_2^{(q,\tilde{q})}(s; v_\alpha, M; \tilde{v}_\alpha, \tilde{M}) = \frac{I_1\left(\frac{iqMs}{2}, qs\right) I_1\left(\frac{i\tilde{q}\tilde{M}s}{2}, \tilde{q}s\right)}{2\pi} D_2^{(q,\tilde{q})}(s; v_\alpha, M; \tilde{v}_\alpha, \tilde{M})$$

$$D_2^{(q,\tilde{q})}(s; v_\alpha, M; \tilde{v}_\alpha, \tilde{M}) = \frac{s^2}{384(1+qs)(1+\tilde{q}s)}\Bigg(q^2\tilde{q}^2(1+M^2)(1+\tilde{M}^2)(1-s^2)$$
$$-48q(1+M^2)(1+\tilde{q}s)\sum_{\alpha=1}^{q} v_\alpha^2 - 48\tilde{q}(1+\tilde{M}^2)(1+qs)\sum_{\alpha=1}^{\tilde{q}} \tilde{v}_\alpha^2\Bigg)$$

$$\frac{1}{F(x)^{2a_1}F(p)^{2a_2}}, \frac{F'(x)}{F(x)^{2a_1+1}F(p)^{2a_2}}, \frac{F'(p)}{F(x)^{2a_1}F(p)^{2a_2+1}}, \frac{F'(x)F'(p)}{F(x)^{2a_1+1}F(p)^{2a_2+1}},$$

$$\text{Vec}\left(\frac{1}{F(x)^{2a_1}}, \frac{F'(x)}{F(x)^{2a_2+1}}; a_1, a_2 \geq 0\right) = \text{Pol}\left[\frac{F'(x)}{F(x)}\right].$$

$$\partial_x U(x) = \frac{qF'(x)}{F(x)} - \frac{iqM}{2} + \sum_{\alpha=1}^{q} \partial_x \log\left[\cosh\frac{\hbar v_\alpha}{2} + 2\sinh\frac{\hbar v_\alpha}{2}\frac{F'(x)}{F(x)}\right]$$

$$\partial_x \text{Pol}\left[\frac{F'(x)}{F(x)}\right] \subset \text{Pol}\left[\frac{F'(x)}{F(x)}\right]$$

$$V_\nu = \sum_{\alpha=1}^{q} v_\alpha^\nu, \tilde{V}_\nu = \sum_{\alpha=1}^{\tilde{q}} \tilde{v}_\alpha^\nu$$

$$\begin{aligned}D_4^{(q,\tilde{q})}(s; v_\alpha, 0; \tilde{v}_\alpha, 0) = &f_{4,\emptyset,\emptyset}^{(q,\tilde{q})}(s;0;0) + f_{4,\{2\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_2 + f_{4,\{2\},\emptyset}^{(\tilde{q},q)}(s;0;0)\tilde{V}_2 + f_{4,\{4\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_4\\ &+f_{4,\{2,2\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_2^2 + f_{4,\{2\},\{2\}}^{(q,\tilde{q})}(s;0;0)V_2\tilde{V}_2 + f_{4,\{4\},\emptyset}^{(\tilde{q},q)}(s;0;0)\tilde{V}_4\\ &+f_{4,\{2,2\},\emptyset}^{(\tilde{q},q)}(s;0;0)\tilde{V}_2^2\end{aligned}$$

$$\begin{aligned}f_{4,\emptyset,\emptyset}^{(q,\tilde{q})}(s;0;0) &= -\frac{q^3\tilde{q}^3s^3(1-s^2)(24q+24\tilde{q}+n(80+17q\tilde{q})+n^2(24q+24\tilde{q})+7q\tilde{q}n^3)}{1474560(1+qs)(3+qs)(1+\tilde{q}s)(3+\tilde{q}s)}\\ f_{4,\{2\},\emptyset}^{(q,\tilde{q})}(s;0;0) &= -\frac{q^2\tilde{q}^2s^3(1-s^2)(4+qs)}{3072(1+qs)(1+\tilde{q}s)(3+qs)}\\ f_{4,\{4\},\emptyset}^{(q,\tilde{q})}(s;0;0) &= \frac{q^2s^3}{192(1+qs)(3+qs)}\\ f_{4,\{2,2\},\emptyset}^{(q,\tilde{q})}(s;0;0) &= \frac{qs^3(2+qs)}{128(1+qs)(3+qs)}\\ f_{4,\{2\},\{2\}}^{(q,\tilde{q})}(s;0;0) &= \frac{q^2s^4}{64(1+\tilde{q}s)(1+qs)}\end{aligned}$$

$$\begin{aligned}D_6^{(q,\tilde{q})}(s; v_\alpha, 0; \tilde{v}_\alpha, 0) = &f_{6,\emptyset,\emptyset}^{(q,\tilde{q})}(s;0;0) + f_{6,\{2\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_2 + f_{6,\{2\},\emptyset}^{(\tilde{q},q)}(s;0;0)\tilde{V}_2 + f_{6,\{4\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_4\\ &+f_{6,\{2,2\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_2^2 + f_{6,\{2\},\{2\}}^{(q,\tilde{q})}(s;0;0)V_2\tilde{V}_2 + f_{6,\{4\},\emptyset}^{(\tilde{q},q)}(s;0;0)\tilde{V}_4\\ &+f_{6,\{2,2\},\emptyset}^{(\tilde{q},q)}(s;0;0)\tilde{V}_2^2 + f_{6,\{6\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_6 + f_{6,\{4,2\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_4V_2\\ &+f_{6,\{3,3\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_3^2 + f_{6,\{2,2,2\},\emptyset}^{(q,\tilde{q})}(s;0;0)V_2^3 + f_{6,\{4\},\{2\}}^{(q,\tilde{q})}(s;0;0)V_4\tilde{V}_2\\ &+f_{6,\{2,2\},\{2\}}^{(q,\tilde{q})}(s;0;0)V_2^2\tilde{V}_2 + f_{6,\{4\},\{2\}}^{(\tilde{q},q)}(s;0;0)V_2\tilde{V}_4 + f_{6,\{2,2\},\{2\}}^{(\tilde{q},q)}(s;0;0)V_2\tilde{V}_2^2\\ &+f_{6,\{6\},\emptyset}^{(\tilde{q},q)}(s;0;0)\tilde{V}_6 + f_{6,\{4,2\},\emptyset}^{(\tilde{q},q)}\tilde{V}_4\tilde{V}_2 + f_{6,\{3,3\},\emptyset}^{(\tilde{q},q)}\tilde{V}_3^2 + f_{6,\{2,2,2\},\emptyset}^{(\tilde{q},q)}\tilde{V}_2^3\end{aligned}$$

$$\begin{aligned}f_{6,\emptyset,\emptyset}^{(q,\tilde{q})}(s;0;0) = &\frac{q^3\tilde{q}^3s^3(1-s^2)}{3963617280(1+qs)(3+qs)(5+qs)(1+\tilde{q}s)(3+\tilde{q}s)(5+\tilde{q}s)}(1920q^3+1920\tilde{q}^3\\ &+(7168+5376q^2+14336q\tilde{q}+2944q^3\tilde{q}+5376\tilde{q}^2+2304q^2\tilde{q}^2+2944q\tilde{q}^3)s\\ &+(14336q+576q^3+14336\tilde{q}+17920q^2\tilde{q}+17920q\tilde{q}^2+1656q^3\tilde{q}^2+576\tilde{q}^3+1656q^2\tilde{q}^3)s^2\end{aligned}$$

$$+(5376q^2+25088q\tilde{q}+2272q^3\tilde{q}+5376\tilde{q}^2+12272q^2\tilde{q}^2+2272q\tilde{q}^3+367q^3\tilde{q}^3)s^3$$
$$+(576q^3+8960q^2\tilde{q}+8960q\tilde{q}^2+1488q^3\tilde{q}^2+576\tilde{q}^3+1488q^2\tilde{q}^3)s^4$$
$$+(928q^3\tilde{q}+3088q^2\tilde{q}^2+928q\tilde{q}^3+178q^3\tilde{q}^3)s^5+(312q^3\tilde{q}^2+312q^2\tilde{q}^3)s^6+31q^3\tilde{q}^3s^7),$$

$$f^{(q,\tilde{q})}_{6,\{2\},\emptyset}(s;0;0)=\frac{q^2\tilde{q}^3s^3(1-s^2)}{11796480(1+qs)(3+qs)(5+qs)(1+\tilde{q}s)(3+\tilde{q}s)}(192\tilde{q}+(640+192q^2+288q\tilde{q})s$$
$$+(960q+24q^3+192\tilde{q}+160q^2\tilde{q})s^2+(272q^2+288q\tilde{q}+17q^3\tilde{q})s^3+(24q^3+80q^2\tilde{q})s^4$$
$$+7q^3\tilde{q}s^5),$$

$$f^{(q,\tilde{q})}_{6,\{4\},\emptyset}(s;0;0)=\frac{q^2\tilde{q}^2s^3(1-s^2)(8+12qs+q^2s^2)}{73728(1+qs)(3+qs)(5+qs)(1+\tilde{q}s)},$$

$$f^{(q,\tilde{q})}_{6,\{2,2\},\emptyset}(s;0;0)=\frac{q\tilde{q}^2s^3(1-s^2)(2+qs)(8+8qs+q^2s^2)}{49152(1+qs)(3+qs)(5+qs)(1+\tilde{q}s)},$$

$$f^{(q,\tilde{q})}_{6,\{2\},\{2\}}(s;0;0)=\frac{q^2\tilde{q}^2s^4(1-s^2)(4+qs)(4+\tilde{q}s)}{24576(1+qs)(3+qs)(1+\tilde{q}s)(3+\tilde{q}s)},$$

$$f^{(q,\tilde{q})}_{6,\{6\},\emptyset}(s;0;0)=-\frac{q^2s^3(1+2qs)}{5760(1+qs)(3+qs)(5+qs)},$$

$$f^{(q,\tilde{q})}_{6,\{4,2\},\emptyset}(s;0;0)=-\frac{qs^3(2+qs)^2}{1536(1+qs)(3+qs)(5+qs)},$$

$$f^{(q,\tilde{q})}_{6,\{3,3\},\emptyset}(s;0;0)=\frac{qs^3(2+qs)}{1152(1+qs)(3+qs)(5+qs)},$$

$$f^{(q,\tilde{q})}_{6,\{2,2,2\},\emptyset}(s;0;0)=-\frac{qs^4(2+qs)(4+qs)}{3072(1+qs)(3+qs)(5+qs)},$$

$$f^{(q,\tilde{q})}_{6,\{4\},\{2\}}(s;0;0)=-\frac{q^2\tilde{q}s^5}{1536(1+qs)(3+qs)(1+\tilde{q}s)},$$

$$f^{(q,\tilde{q})}_{6,\{2,2\},\{2\}}(s;0;0)=-\frac{q\tilde{q}s^5(2+qs)}{1024(1+qs)(3+qs)(1+\tilde{q}s)}.$$

$$q\sum_{\alpha}v_\alpha^4+3\left(\sum_{\alpha}v_\alpha^2\right)^2=\sum_{\alpha<\beta}(v_\alpha-v_\beta)^4$$

$$q\sum_{\alpha}v_\alpha^6+15\sum_{\alpha}v_\alpha^4\sum_{\beta}v_\beta^2-10\left(\sum_{\alpha}v_\alpha^3\right)^2=\sum_{\alpha<\beta}(v_\alpha-v_\beta)^6.$$

$$D_4^{(q,1)}(s;v_\alpha,0;\tilde{M})=f^{(q,1)}_{4,\emptyset,\emptyset}(s;0;\tilde{M})+f^{(q,1)}_{4,\{2\},\emptyset}(s;0;\tilde{M})V_2+f^{(q,1)}_{4,\{4\},\emptyset}(s;0;\tilde{M})V_4+f^{(q,1)}_{4,\{2,2\},\emptyset}(s;0;\tilde{M})V_2^2,$$

$$f^{(q,1)}_{4,\emptyset,\emptyset}(s;0;\tilde{M})=\frac{(1+\tilde{M}^2)q^3s^3(1-s)\left(-8-8q+(-24-3q)s-7qs^2+\tilde{M}^2(24+(-24+5q)s-7qs^2)\right)}{1474560(1+qs)(3+qs)}$$

$$f^{(q,1)}_{4,\{2\},\emptyset}(s;0;\tilde{M})=(1+\tilde{M}^2)f^{(q,1)}_{4,\{2\},\emptyset}(s;0;0),$$

$$f^{(q,1)}_{4,\{4\},\emptyset}(s;0;\tilde{M})=f^{(q,1)}_{4,\{4\},\emptyset}(s;0;0),$$

$$f^{(q,1)}_{4,\{2,2\},\emptyset}(s;0;\tilde{M})=f^{(q,1)}_{4,\{2,2\},\emptyset}(s;0;0).$$

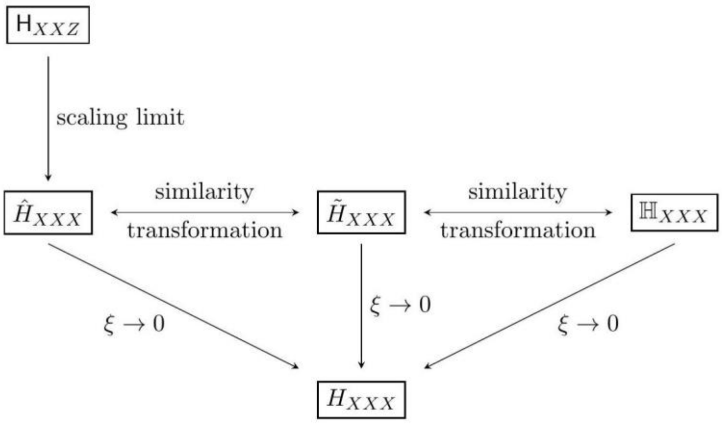
$$D_6^{(q,1)}(s;v_\alpha,0;\tilde{v}_\alpha,\tilde{M})=f^{(q,1)}_{6,\emptyset,\emptyset}(s;0;\tilde{M})+f^{(q,1)}_{6,\{2\},\emptyset}(s;0;\tilde{M})V_2+f^{(q,1)}_{6,\{4\},\emptyset}(s;0;\tilde{M})V_4$$
$$+f^{(q,1)}_{6,\{2,2\},\emptyset}(s;0;\tilde{M})V_2^2+f^{(q,1)}_{6,\{6\},\emptyset}(s;0;\tilde{M})V_6+f^{(q,1)}_{6,\{4,2\},\emptyset}(s;0;\tilde{M})V_4V_2$$
$$+f^{(q,1)}_{6,\{3,3\},\emptyset}(s;0;\tilde{M})V_3^2+f^{(q,1)}_{6,\{2,2,2\},\emptyset}(s;0;\tilde{M})V_2^3$$

$$f^{(q,1)}_{6,\emptyset,\emptyset}(s;0;\tilde{M}) = \frac{(1+\tilde{M}^2)q^3s^3(1-s)}{11890851840(1+qs)(3+qs)(5+qs)}(384+384q^3$$
$$+(2304+3456q+1536q^2+384q^3)s+(1728+4608q+3096q^2+216q^3)s^2$$
$$+(2784q+1776q^2+387q^3)s^3+(936q^2+192q^3)s^4+93q^3s^5$$
$$+\tilde{M}^2(-3840+(-1536-5888q-2304q^2)s+(3456-1280q-240q^2-168q^3)s^2$$
$$+(5568q+192q^2+38q^3)s^3+(1872q^2+56q^3)s^4+186q^3s^5)$$
$$+\tilde{M}^4(1920+(-3840+2944q)s+(1728-5888q+504q^2)s^2$$
$$+(2784q-1584q^2+35q^3)s^3+(936q^2-136q^3)s^4+93q^3s^5)\Big),$$

$$f^{(q,1)}_{6,\{2\},\emptyset}(s;0;\tilde{M}) = \frac{(1+\tilde{M}^2)q^2s^3(1-s)}{11796480(1+qs)(3+qs)(5+qs)}(64+(192+96q+64q^2)s$$
$$+(288q+32q^2+8q^3)s^2+(80q^2+3q^3)s^3+7q^3s^4$$
$$+\tilde{M}^2(-192+(192-288q)s+(288q-64q^2)s^2+(80q^2-5q^3)s^3+7q^3s^4)\Big),$$

$$f^{(q,1)}_{6,\{4\},\emptyset}(s;0;\tilde{M}) = (1+\tilde{M}^2)f^{(q,1)}_{6,\{4\},\emptyset}(s;0;0),$$
$$f^{(q,1)}_{6,\{2,2\},\emptyset}(s;0;\tilde{M}) = (1+\tilde{M}^2)f^{(q,1)}_{6,\{2,2\},\emptyset}(s;0;0),$$
$$f^{(q,1)}_{6,\{6\},\emptyset}(s;0;\tilde{M}) = f^{(q,1)}_{6,\{6\},\emptyset}(s;0;0),$$
$$f^{(q,1)}_{6,\{4,2\},\emptyset}(s;0;\tilde{M}) = f^{(q,1)}_{6,\{4,2\},\emptyset}(s;0;0),$$
$$f^{(q,1)}_{6,\{3,3\},\emptyset}(s;0;\tilde{M}) = f^{(q,1)}_{6,\{3,3\},\emptyset}(s;0;0),$$
$$f^{(q,1)}_{6,\{2,2,2\},\emptyset}(s;0;\tilde{M}) = f^{(q,1)}_{6,\{2,2,2\},\emptyset}(s;0;0).$$

$$\theta\left(E-\hat{H}^{(q,\bar{q})}_k(\hat{x},\hat{p};\eta_\alpha,M;\tilde{\eta}_\alpha,\tilde{M})\right)_W$$



$$[J^+,J^-]=-2J^3,[J^3,J^\pm]=\pm J^\pm$$
$$C=(J^3)^2-\tfrac{1}{2}\{J^+,J^-\}$$
$$J^3=\tfrac{1}{2}+a^\dagger a, J^+=a^\dagger+a^\dagger a^\dagger a, J^-=a$$
$$V_F=\text{span}\{(a^+)^n|0\rangle, n=0,1,\ldots\},\text{ where } a|0\rangle=0$$
$$\Delta(J)=J\otimes\mathbb{I}+\mathbb{I}\otimes J, J\in\{J^3,J^\pm\}$$
$$V_F\otimes V_F=\oplus_{j=0}^{\infty}V_j,$$
$$\Delta(J^-)|\Phi_j\rangle=0$$
$$\Delta(C)|\Phi_j\rangle=j(j+1)|\Phi_j\rangle$$
$$|\Phi_j\rangle=\left(a_1^\dagger-a_2^\dagger\right)^j|00\rangle,$$
$$H_{XXX}=\textstyle\sum_{m=1}^{L} h_{m,m+1}$$
$$h_{12}=\textstyle\sum_{j=0}^{\infty} 2h(j)P_{12,j}$$
$$P_{12,j}=\prod_{\substack{k=0\\k\neq j}}^{\infty}\frac{\Delta(C)-k(k+1)}{j(j+1)-k(k+1)}$$
$$\Delta(C)=-\left(a_1^\dagger-a_2^\dagger\right)^2a_1a_2+\left(a_1^\dagger-a_2^\dagger\right)(a_1-a_2).$$
$$h_{12}|n_1,n_2\rangle=\textstyle\sum_{k=0}^{n_1+n_2}\left(\delta_{kn_1}\left(h(n_1)+h(n_2)\right)-\frac{1-\delta_{kn_1}}{|n_1-k|}\right)|k,n_1+n_2-k\rangle,$$
$$|n,m\rangle=\left(a_1^\dagger\right)^n\left(a_2^\dagger\right)^m|0,0\rangle$$
$$R_{12}(u)=\textstyle\sum_{j=0}^{\infty} R_j(u)P_{12,j},\text{ with } R_j(u)=(-1)^{j+1}\frac{\Gamma(-j+u)}{\Gamma(-j-u)}\frac{\Gamma(1-u)}{\Gamma(1+u)},$$

$$T_a = \prod_{m=1}^{L} R_{am}$$

$(V_F)_a \otimes (V_F)_L \otimes \cdots \otimes (V_F)_1$

$$I_n = \left(\frac{d^n}{du^n}\log tr_a T_a\right)_{u=0}$$

$$I_1 = \sum_{m=1}^{L-1} P_{m,m+1}\left(\frac{\mathrm{d}}{\mathrm{du}} R_{m,m+1}(u)\right)_{u=0} + P_{L,1}\left(\frac{\mathrm{d}}{\mathrm{du}} R_{L,1}(u)\right)_{u=0}$$

$$N = \sum_{m=1}^{L} a_m^\dagger a_m$$

$$|\Psi_{xxx}\rangle_{(N)} = \sum_{1\le x_1\le\cdots\le x_N\le L}\left(\sum_{\pi\in S_N} A_\pi \exp\left(i\sum_{j=1}^{N} p_{\pi(j)}x_j\right)\right) \mid x_1,\cdots,x_N\big),$$

$$A_{id} = 1,$$

$$A_\pi = \prod_{\substack{1\le j<k\le N\\ \pi(j)>\pi(k)}} S_{xxx}(p_k,p_j), S_{xxx}(p,k) = -\frac{e^{i(p+k)}-2e^{ik}+1}{e^{i(p+k)}-2e^{ip}+1},$$

$$\mid x_1,\cdots,x_N\big) = a_{x_1}^\dagger \cdots a_{x_N}^\dagger |0\rangle^{\otimes L}.$$

$$e^{ip_kL} = \prod_{\substack{j=1\\ j\neq k}}^{N} S_{xxx}(p_k,p_j), k = 1,\cdots,N$$

$$E_{(N)} = \sum_{j=1}^{N} \epsilon(p_j), \text{ where } \epsilon(p_j) = 4\sin^2\left(\frac{p_j}{2}\right).$$

$$\lambda_j = \frac{1}{2}\cot\frac{p_j}{2}.$$

$$\epsilon\left(p_j(\lambda_j)\right) = \frac{1}{\lambda_j^2+\frac{1}{4}}, \left(\frac{\lambda_k+\frac{i}{2}}{\lambda_k-\frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k-\lambda_j-i}{\lambda_k-\lambda_j+i}.$$

$$F_{12} = e^{-J^3\otimes\log(1+2\xi J^-)}$$

$$J^3 \to -J^3, J^\pm \to J^\mp.$$

$$F_{12}' = e^{J^3\otimes\log(1+2\xi J^+)}$$

$$F_{12} = \sum_{k=0}^{\infty} \frac{(-2\xi)^k}{k!}(J^3)^{\langle k\rangle} \otimes (J^-)^k,$$

$$(J^3)^{\langle k\rangle} = \prod_{n=0}^{k-1} (J^3+n) = \sum_{n=0}^{k} \begin{bmatrix}k\\ n\end{bmatrix}(J^3)^n,$$

$$r = 2\xi(J^- \wedge J^3)$$

$$R_{12} \to \tilde{R}_{12} = F_{21}R_{12}F_{12}^{-1}$$

$$\tilde{H}_{XXX} = \sum_{m=1}^{L} \tilde{h}_{m,m+1}, \text{ with } \tilde{h}_{12} = P_{12}\left(\frac{\mathrm{d}}{\mathrm{du}}\tilde{R}_{12}(u)\right)_{u=0} \text{ and } \tilde{h}_{L,L+1} = \tilde{h}_{L,1}.$$

$$\tilde{h}_{12} = \sum_{j=0}^{\infty} 2h(j)\tilde{P}_{12,j},$$

$$\tilde{P}_{12,j} = F_{12}P_{12,j}F_{12}^{-1} = \prod_{\substack{k=0\\ k\neq j}}^{\infty} \frac{\tilde{\Delta}(C)-k(k+1)}{j(j+1)-k(k+1)}, \text{ with } \tilde{\Delta}(C) = F_{12}\Delta(C)F_{12}^{-1}.$$

$$\begin{aligned}
\tilde{\Delta}(J^3) &= J^3 \otimes e^{-w} + \mathbb{I}\otimes J^3,\\
\tilde{\Delta}(J^-) &= J^- \otimes e^{w} + \mathbb{I}\otimes J^-,\\
\tilde{\Delta}(J^+) &= J^+ \otimes e^{-w} + \mathbb{I}\otimes J^+ - 4\xi J^3 \otimes J^3 e^{-w} +\\
&\quad +2\xi(J^3+(J^3)^2)\otimes(e^{-w}-e^{-2w}),\\
\tilde{\Delta}(w) &= w\otimes\mathbb{I} + \mathbb{I}\otimes w,
\end{aligned}$$

$$V_F \otimes V_F = \oplus_{j=0}^{\infty} \tilde{V}_j,$$

$$\mathbb{H}_{XXX} = \sum_{m=1}^{L-1} h_{m,m+1} + S^{-1}h_{L,1}S,$$

$$\tilde{H}_{XXX} = \Omega\mathbb{H}_{XXX}\Omega^{-1},$$

$$\Omega = (F_{12}\ldots F_{1L})(F_{23}\ldots F_{2L})\ldots F_{L-1,L},$$

$$S = F_{L1}^{-1}\Omega.$$

$$\tilde{h}_{m,m+1} = \Omega h_{m,m+1}\Omega^{-1}, m = 1,\ldots,L-1.$$

$$H_{XXX}^{\text{bulk}} = \sum_{m=1}^{L-1} h_{m,m+1}$$

$$\tilde{H}_{XXX}^{\text{bulk}} = \sum_{m=1}^{L-1} \tilde{h}_{m,m+1}$$

$$\tilde{H}_{XXX}^{\text{bulk}} = \Omega H_{XXX}^{\text{bulk}}\Omega^{-1}.$$

$$\left[\Delta^{(L-1)}(x), H_{XXX}^{\text{bulk}}\right] = 0, x \in \mathfrak{sl}_2,$$

$$\Delta^{(n)}: V_F \to V_F^{\otimes n+1}$$

$$\Delta^{(n)} = (\Delta\otimes\mathbb{I})\Delta^{(n-1)}, \text{ with } \Delta^{(1)} = \Delta.$$

$$\tilde{\Delta}^{(L-1)} = F_{12\ldots L}\Delta^{(L-1)}F_{12\ldots L}^{-1},$$

$$F_{12\ldots L} = F_{12}F_{(12),3}\cdots F_{(12\ldots L-1),L}$$

$$F_{(12\ldots n),n+1} = \left(\Delta^{(n-1)}\otimes\mathbb{I}\right)(F)$$

$$\Delta(J^3) = J^3\otimes\mathbb{I} + \mathbb{I}\otimes J^3$$

$$(\Delta \otimes \mathbb{I})(F) = F_{13}F_{23}.$$
$$F_{12\ldots L} = F_{12}(F_{13}F_{23})\cdots(F_{1L}\cdots F_{L-1,L}).$$
$$[\tilde{\Delta}^{(L-1)}(x), \tilde{H}_{XXX}^{\text{bulk}}] = \Omega[\Delta^{(L-1)}(x), H_{XXX}^{\text{bulk}}]\Omega^{-1} = 0, x \in \mathfrak{sl}_2,$$
$$[\tilde{\Delta}^{(L-1)}(x), \tilde{H}_{XXX}] = [\tilde{\Delta}^{(L-1)}(x), \tilde{H}_{XXX}^{\text{bulk}}] + [\tilde{\Delta}^{(L-1)}(x), \tilde{h}_{L,L+1}] = [\tilde{\Delta}^{(L-1)}(x), \tilde{h}_{L,L+1}]$$
$$[\tilde{\Delta}^{(L-1)}(x), \tilde{h}_{L,L+1}] = U[U^{-1}\tilde{\Delta}^{(L-1)}(x)U, \tilde{h}_{L-1,L}]U^{-1}, x \in \mathfrak{sl}_2$$
$$[\tilde{\Delta}^{(L-1)}(x), \tilde{h}_{L-1,L}] = 0\ x \in \mathfrak{sl}_2.$$

$$U^{-1}\tilde{\Delta}^{(L-1)}(w)U = \tilde{\Delta}^{(L-1)}(w)$$

$$\tilde{T}_a = \Omega\mathbb{T}_a\Omega^{-1},$$
$$\tilde{T}_a = \Omega F_a T_a \Omega^{-1}(F_a)_{op}^{-1},$$
$$F_a = F_{1a}\cdots F_{La}$$
$$(F_a)_{op}^{-1} = F_{aL}^{-1}\cdots F_{a1}^{-1}$$
$$\mathbb{T}_a = F_a T_a \Omega^{-1}(F_a)_{op}^{-1}\Omega.$$
$$[F_{ab}, F_{cd}] = 0\ \forall a,b,c,d = 1,\ldots,L$$
$$\{|00\rangle, |10\rangle, |01\rangle, |20\rangle, |02\rangle, |11\rangle\}$$
$$\begin{pmatrix} 0 & 0 & 0 & \xi^2 & \xi^2 & 0 \\ 0 & 2 & -2 & 2\xi & 2\xi & -4\xi \\ 0 & -2 & 2 & 2\xi & 2\xi & -4\xi \\ 0 & 0 & 0 & 3 & -1 & -2 \\ 0 & 0 & 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & -2 & -2 & 4 \end{pmatrix}$$

Eigenvalue	Momenta	Eigenvector
0	-	$\|00\rangle$
0	(0)	$\|10\rangle + \|01\rangle$
0	$(0,0)$	$\|20\rangle + \|02\rangle + \|11\rangle$
4	(π)	$\|10\rangle - \|01\rangle$
4	$(\pi, 0)$	$\|20\rangle - \|02\rangle$
6	$(2\pi/3, 4\pi/3)$	$3(\|20\rangle + \|02\rangle) - 6\|11\rangle$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}.$$

Eigenvalue	(Generalised) eigenvector
0	$\|00\rangle$
0	$\frac{1}{2\xi^2}(\|20\rangle + \|02\rangle + \|11\rangle)$
0	$\|10\rangle + \|01\rangle$
4	$\|10\rangle - \|01\rangle$
4	$\|20\rangle - \|02\rangle$
6	$3(\|20\rangle + \|02\rangle) - 6\|11\rangle + 6\xi(\|10\rangle + \|01\rangle) + \xi^2\|00\rangle$

$$\frac{1}{2\xi^2}(|20\rangle + |02\rangle + |11\rangle)$$

$$\{|00\rangle, |10\rangle, |01\rangle, |20\rangle, |02\rangle, |11\rangle, |30\rangle, |03\rangle, |21\rangle, |12\rangle\}.$$

$$\begin{pmatrix} 0 & 0 & 0 & \xi^2 & \xi^2 & 0 & -6\xi^3 & -6\xi^3 & 0 & 0 \\ 0 & 2 & -2 & 2\xi & 2\xi & -4\xi & -3\xi^2 & -6\xi^2 & 0 & 5\xi^2 \\ 0 & -2 & 2 & 2\xi & 2\xi & -4\xi & -6\xi^2 & -3\xi^2 & 5\xi^2 & 0 \\ 0 & 0 & 0 & 3 & -1 & -2 & 4\xi & 2\xi & -8\xi & -2\xi \\ 0 & 0 & 0 & -1 & 3 & -2 & 2\xi & 4\xi & -2\xi & -8\xi \\ 0 & 0 & 0 & -2 & -2 & 4 & 6\xi & 6\xi & -2\xi & -2\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{3} & -\frac{2}{3} & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} & \frac{11}{3} & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -2 & 5 \end{pmatrix}.$$

Eigenvalue	Momenta	Eigenvector
0	$(0,0,0)$	$\lvert 30\rangle + \lvert 03\rangle + \lvert 21\rangle + \lvert 12\rangle$
4	$(\pi,0,0)$	$3(\lvert 30\rangle - \lvert 03\rangle) + \lvert 21\rangle - \lvert 12\rangle$
6	$(2\pi/3, 4\pi/3, 0)$	$3(\lvert 30\rangle + \lvert 03\rangle) - 3(\lvert 21\rangle + \lvert 12\rangle)$
$22/3$	$(\pi, \arctan\sqrt{35}, 2\pi - \arctan\sqrt{35})$	$5(\lvert 30\rangle - \lvert 03\rangle) - 15(\lvert 21\rangle - \lvert 12\rangle)$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{22}{3} \end{pmatrix}.$$

Eigenvalue	(Generalised) eigenvector
0	$\lvert 00\rangle$
0	$\frac{1}{2\xi^2}(\lvert 20\rangle + \lvert 02\rangle + \lvert 11\rangle)$
0	$\lvert 10\rangle + \lvert 01\rangle$
0	$\frac{1}{4\xi^2}(\lvert 30\rangle + \lvert 03\rangle + \lvert 21\rangle + \lvert 12\rangle) + \frac{1}{\xi}\left(\frac{3}{2}\lvert 20\rangle + \frac{3}{2}\lvert 02\rangle + \lvert 11\rangle\right)$
4	$\lvert 10\rangle - \lvert 01\rangle$
4	$\frac{1}{4\xi^2}(3\lvert 30\rangle - 3\lvert 03\rangle + \lvert 21\rangle - \lvert 12\rangle)$
4	$\lvert 20\rangle - \lvert 02\rangle$
6	$3(\lvert 20\rangle + \lvert 02\rangle) - 6\lvert 11\rangle + 6\xi(\lvert 10\rangle + \lvert 01\rangle) + \xi^2\lvert 00\rangle$
6	$3(\lvert 30\rangle + \lvert 03\rangle) - 3(\lvert 21\rangle + \lvert 12\rangle) + 12\xi(\lvert 20\rangle + \lvert 02\rangle) +$ $+\xi^2(\lvert 10\rangle + \lvert 01\rangle) - 2\xi^3\lvert 00\rangle$

22/3	$5(\lvert 30\rangle - \lvert 03\rangle) - 15(\lvert 21\rangle - \lvert 12\rangle) + 30\xi(\lvert 20\rangle - \lvert 02\rangle) +$ $+27\xi^2(\lvert 10\rangle - \lvert 01\rangle)$

$$\frac{1}{2\xi^2}(|20\rangle + |02\rangle + |11\rangle), \frac{1}{4\xi^2}(|30\rangle + |03\rangle + |21\rangle + |12\rangle) + \frac{1}{\xi}\left(\frac{3}{2}|20\rangle + \frac{3}{2}|02\rangle + |11\rangle\right)$$

$$\frac{1}{4\xi^2}(3|30\rangle - 3|03\rangle + |21\rangle - |12\rangle)$$

$$\frac{1}{2\xi^2}(|20\rangle + |02\rangle + |11\rangle)$$

$$\frac{1}{4\xi^2}(|30\rangle + |03\rangle + |21\rangle + |12\rangle) + \frac{1}{\xi}\left(\frac{3}{2}|20\rangle + \frac{3}{2}|02\rangle + |11\rangle\right),$$

$$\frac{1}{4\xi^2}(3|30\rangle - 3|03\rangle + |21\rangle - |12\rangle),$$

$$F_{12}|n_1, n_2\rangle = \sum_{k=0}^{n_2} (-2\xi)^k \binom{n_2}{k} \left(\frac{1}{2} + n_1\right)^{\langle k\rangle} |n_1, n_2 - k\rangle.$$

$$|\Psi\rangle_{(N)} = \sum_{r=0}^{N} \sum_{\vec{x}_{(r)}} \psi_{(r)}(\vec{x}_{(r)}) \mid \vec{x}_{(r)}\Big\rangle,$$

$$\mid \vec{x}_{(0)}\big\rangle = |0\rangle^{\otimes L}, \psi_{(0)}(\vec{x}_{(0)}) = \psi_{(0)} = g,$$

$$\sum_{\vec{x}_{(r)}} = \sum_{1 \le x_1 \le \cdots \le x_r \le L}, \mid \vec{x}_{(r)}\Big\rangle = \mid x_1, x_2, \dots, x_r\Big\rangle, r > 0.$$

$$\{\psi_{(r)}(\vec{x}_{(r)})\}$$

$$E_{(N)} \sum_{\vec{x}_{(r)}} \psi_{(r)}(\vec{x}_{(r)}) \mid \vec{x}_{(r)}\Big\rangle = \sum_{p=r}^{N} \sum_{\vec{x}_{(p)}} \psi_{(p)}(\vec{x}_{(p)}) \tilde{H}_{XXX}^{(p-r)} \mid \vec{x}_{(p)}\Big\rangle, r = 0, \dots, N,$$

$$\tilde{H}_{XXX}^{(q)} = \sum_{m=1}^{L} \tilde{h}_{m,m+1}^{(q)}$$

$$\tilde{h}_{12}^{(q)} = \sum_{k=0}^{q} \frac{(-2\xi)^q}{k!\,(q-k)!} \left((J^3)^{\langle q-k\rangle} \otimes (J^-)^{q-k}\right) h_{12} \left((-J^3)^{\langle k\rangle} \otimes (J^-)^k\right),$$

$$\psi_{(r)}(x_1, \dots, x_r) = \psi_{(r)}(x_2, \dots, x_r, x_1 + L), r = 1, \dots, N$$

$$\psi_r(\vec{x}_{(r)}) \propto \xi^{N-r}, r = 0, \dots, N$$

$$|\Psi\rangle_{(1)} = \psi_{(0)}|0\rangle^{\otimes L} + \sum_{1 \le x \le L} \psi_{(1)}(x) \mid x\Big\rangle$$

$$E_{(1)}\psi_{(0)}|0\rangle^{\otimes L} = \tilde{H}_{XXX}^{(1)} \sum_{1 \le x \le L} \psi_{(1)}(x) \mid x\Big\rangle$$

$$\tilde{h}_{12}^{(1)} = 2\xi[h_{12}, J^3 \otimes J^-]$$

$$\tilde{h}_{12}^{(1)}|01\rangle = -\xi|00\rangle,$$
$$\tilde{h}_{12}^{(1)}|10\rangle = \xi|00\rangle.$$

$$\tilde{H}^{(1)}_{XXX}\sum_{1\leq x\leq L}\psi_{(1)}(x)\|x\rangle=\sum_{1\leq x\leq L}\psi_{(1)}(x)\left(\tilde{h}^{(1)}_{x,x-1}+\tilde{h}^{(1)}_{x,x+1}\right)\|x\rangle=0,$$

$$\Psi\rangle_{(1)}=\sum_{1\leq x\leq L}e^{ipx}\|x\rangle, E_{(1)}=4\sin^2\left(\frac{p}{2}\right), p=\frac{2\pi n}{L}, n=0,\dots,L-1.$$

$$|\Psi\rangle_{(2)}=\psi_{(0)}|0\rangle^{\otimes L}+\sum_{1\leq x\leq L}\psi_{(1)}(x)\|x\rangle+\sum_{1\leq x_1\leq x_2\leq L}\psi_{(2)}(x_1,x_2)\|x_1,x_2\rangle.$$

$$\begin{aligned}\psi_{(2)}(x_1,x_2)&=e^{ip_1x_1+ip_2x_2}+S_{xxx}(p_2,p_1)e^{ip_2x_1+ip_1x_2}\\ E_{(2)}&=\epsilon(p_1)+\epsilon(p_2), \epsilon(p_j)=4\sin^2\frac{p_j}{2}\\ e^{ip_1L}&=S_{xxx}(p_1,p_2), e^{ip_2L}=S_{xxx}(p_2,p_1)\end{aligned}$$

$$\left(E_{(2)}-H_{XXX}\right)\sum_{1\leq x\leq L}\psi_{(1)}(x)\|x\rangle=\tilde{H}^{(1)}_{XXX}\sum_{1\leq x_1\leq x_2\leq L}\psi_{(2)}(x_1,x_2)\|x_1,x_2\rangle,$$
$$E_{(2)}\psi_{(0)}|0\rangle^{\otimes L}=\tilde{H}^{(1)}_{XXX}\sum_{1\leq x\leq L}\psi_{(1)}(x)\|x\rangle+\tilde{H}^{(2)}_{XXX}\sum_{1\leq x_1\leq x_2\leq L}\psi_{(2)}(x_1,x_2)\|x_1,x_2\rangle.$$

$$\begin{aligned}&\tilde{h}^{(1)}_{12}|10\rangle=-\tilde{h}^{(1)}_{12}|01\rangle=\xi|00\rangle, \quad \tilde{h}^{(1)}_{12}|11\rangle=-3\xi|10\rangle-\xi|01\rangle,\\ &\tilde{h}^{(1)}_{12}|20\rangle=3\xi|10\rangle+\xi|01\rangle, \quad \tilde{h}^{(1)}_{12}|02\rangle=\xi|10\rangle-\xi|01\rangle,\end{aligned}$$

$$\begin{aligned}\tilde{h}^{(2)}_{12}=&\,2\xi^2(J^3\otimes(J^-)^2+(J^3)^2\otimes(J^-)^2)h_{12}-4\xi^2(J^3\otimes J^-)h_{12}(J^3\otimes J^-)\\ &+2\xi^2h_{12}((J^3)^2\otimes(J^-)^2-J^3\otimes(J^-)^2)\end{aligned}$$

$$\tilde{h}^{(2)}_{12}|20\rangle=-\frac{3}{2}\xi^2|00\rangle, \tilde{h}^{(2)}_{12}|02\rangle=\frac{5}{2}\xi^2|00\rangle, \tilde{h}^{(2)}_{12}|11\rangle=0$$

$$\tilde{h}^{(2)}_{12}|10\rangle=\tilde{h}^{(2)}_{12}|01\rangle=0$$

$$\begin{aligned}(E_{(2)}-2)\psi_{(1)}(x)+\psi_{(1)}(x+1)+\psi_{(1)}(x-1)=&\\ =\xi\big[&2\psi_{(2)}(x,x)+\psi_{(2)}(x+1,x+1)\\ &+\psi_{(2)}(x-1,x-1)-2\psi_{(2)}(x,x+1)\\ &-2\psi_{(2)}(x-1,x)\big],\\ E_{(2)}\psi_{(0)}=\xi^2&\sum_{1\leq x\leq L}\psi_{(2)}(x,x).\end{aligned}$$

$$\psi_{(1)}(x)=A_{(1)}e^{ik^{(1)}x}$$

$$\begin{aligned}\left(E_{(2)}-\epsilon\left(k^{(1)}\right)\right)A_{(1)}e^{ik^{(1)}x}&=\xi e^{i(p_1+p_2)x}[B(p_1,p_2)+S_{xxx}(p_2,p_1)B(p_2,p_1)]\\ E_{(2)}\psi_{(0)}&=\xi^2(1+S_{xxx}(p_2,p_1))\left(\sum_{1\leq x\leq L}e^{i(p_1+p_2)x}\right)\end{aligned}$$

$$B(p_1,p_2)=2-2e^{-ip_1}-2e^{ip_2}+e^{i(p_1+p_2)}+e^{-i(p_1+p_2)}$$

$$k^{(1)}=p_1+p_2\ (\mathrm{mod}2\pi)=\frac{2\pi n}{L}, n=0,\dots,L-1$$

$$e^{i(p_1+p_2)L}=S_{xxx}(p_1,p_2)S_{xxx}(p_2,p_1)=1\Rightarrow p_1+p_2\ (\mathrm{mod}2\pi)=\frac{2\pi n}{L}, n=0,\dots,L-1$$

$$\begin{aligned}A_{(1)}&=4\xi\left(\frac{e^{ip_1}-e^{ip_2}}{1-2e^{ip_2}+e^{i(p_1+p_2)}}\right)\\ \psi_{(0)}&=\delta_{0,k^{(1)}}\frac{2\xi^2}{\epsilon(p_1)+\epsilon(p_2)}\left(\frac{e^{ip_1}-e^{ip_2}}{1-2e^{ip_2}+e^{i(p_1+p_2)}}\right)L\end{aligned}$$

$$\sum_{1\le x\le L} A_{(1)} e^{i(k_1+k_2)x} \| x\rangle = 2\xi \sum_{1\le x\le L} \Psi_{(2)}(x,x) \| x\rangle.$$

$$\sum_{1\le x\le L} e^{i(p_1+p_2)} \| x\rangle \sum_{1\le x_1\le x_2\le L} \left(e^{ip_1x_1+ip_2x_2} + S_{xxx}(p_2,p_1) e^{ip_2x_1+ip_1x_2}\right) \| x_1,x_2\rangle$$

$$0\cdot\psi_{(0)} = 2\xi^2 L \neq 0$$

$$\tilde{H}_{XXX} \sum_{1\le x_1\le x_2\le L} \| x_1,x_2\rangle = \xi^2 L |0\rangle^{\otimes L}$$

$$\tilde{H}^2_{XXX} \sum_{1\le x_1\le x_2\le L} \| x_1,x_2\rangle = 0$$

Momenta	(Generalised) eigenvector
–	$\|0\rangle^{\otimes L}$
$p=\frac{2\pi n}{L}$, $n=0,\ldots,L-1$	$\|\Psi_{xxx}\rangle_{(1)}$
$p_1+p_2 \pmod{2\pi} = \pi$ or $p_1=0, p_2\neq 0$	$\|\Psi_{xxx}\rangle_{(2)}$
$p_1=p_2=0$	$\frac{1}{\xi^2 L}\|\Psi_{xxx}\rangle_{(2)}$
$p_1+p_2 \pmod{2\pi} = 0$ with $p_1\neq 0, p_2\neq 0$	$\|\Psi_{xxx}\rangle_{(2)} + 2\xi\sum_{1\le x\le L}\psi_{(2)}(x,x)\|x\rangle + \frac{\xi^2 L}{4(1-\cos p_1)}\left(1+e^{ip_1}\right)\|0\rangle^{\otimes L}$
rest of p_1, p_2	$\|\Psi_{xxx}\rangle_{(2)} + 2\xi\sum_{1\le x\le L}\psi_{(2)}(x,x)\|x\rangle$

Eigenvalue	(Generalised) eigenvector
0	$\|00\rangle$
0	$\frac{1}{2\xi^2}\left(\|20\rangle+\|02\rangle+\|11\rangle\right)$
0	$\|10\rangle+\|01\rangle$
0	$\frac{1}{4\xi^2}\left(\|30\rangle+\|03\rangle+\|21\rangle+\|12\rangle\right)+\frac{1}{\xi}\left(\frac{3}{2}\|20\rangle+\frac{3}{2}\|02\rangle+\|11\rangle\right)$
4	$\|10\rangle-\|01\rangle$
4	$\frac{1}{4\xi^2}\left(3\|30\rangle-3\|03\rangle+\|21\rangle-\|12\rangle\right)$
4	$\|20\rangle-\|02\rangle$
6	$3\left(\|20\rangle+\|02\rangle\right)-6\|11\rangle+6\xi\left(\|10\rangle+\|01\rangle\right)+\xi^2\|00\rangle$
6	$3\left(\|30\rangle+\|03\rangle\right)-3\left(\|21\rangle+\|12\rangle\right)+12\xi\left(\|20\rangle+\|02\rangle\right)+ +\xi^2\left(\|10\rangle+\|01\rangle\right)-2\xi^3\|00\rangle$
22/3	$5\left(\|30\rangle-\|03\rangle\right)-15\left(\|21\rangle-\|12\rangle\right)+30\xi\left(\|20\rangle-\|02\rangle\right)+ +27\xi^2\left(\|10\rangle-\|01\rangle\right)$

Eigenvalue	Momenta	Eigenvector
0	–	$\|00\rangle$
0	(0)	$\|10\rangle + \|01\rangle$
0	(0, 0)	$\|20\rangle + \|02\rangle + \|11\rangle$
4	(π)	$\|10\rangle - \|01\rangle$
4	$(\pi, 0)$	$\|20\rangle - \|02\rangle$
6	$(2\pi/3, 4\pi/3)$	$3(\|20\rangle + \|02\rangle) - 6\|11\rangle$

Eigenvalue	(Generalised) eigenvector
0	$\|00\rangle$
0	$\frac{1}{2\xi^2}(\|20\rangle + \|02\rangle + \|11\rangle)$
0	$\|10\rangle + \|01\rangle$
4	$\|10\rangle - \|01\rangle$
4	$\|20\rangle - \|02\rangle$
6	$3(\|20\rangle + \|02\rangle) - 6\|11\rangle + 6\xi(\|10\rangle + \|01\rangle) + \xi^2\|00\rangle$

Eigenvalue	Momenta	Eigenvector
0	(0, 0, 0)	$\|30\rangle + \|03\rangle + \|21\rangle + \|12\rangle$
4	$(\pi, 0, 0)$	$3(\|30\rangle - \|03\rangle) + \|21\rangle - \|12\rangle$
6	$(2\pi/3, 4\pi/3, 0)$	$3(\|30\rangle + \|03\rangle) - 3(\|21\rangle + \|12\rangle)$
22/3	$(\pi, \arctan\sqrt{35}, 2\pi - \arctan\sqrt{35})$	$5(\|30\rangle - \|03\rangle) - 15(\|21\rangle - \|12\rangle)$

Eigenvalue	(Generalised) eigenvector
0	$\|00\rangle$
0	$\frac{1}{2\xi^2}(\|20\rangle + \|02\rangle + \|11\rangle)$
0	$\|10\rangle + \|01\rangle$
0	$\frac{1}{4\xi^2}(\|30\rangle + \|03\rangle + \|21\rangle + \|12\rangle) + \frac{1}{\xi}\left(\frac{3}{2}\|20\rangle + \frac{3}{2}\|02\rangle + \|11\rangle\right)$
4	$\|10\rangle - \|01\rangle$
4	$\frac{1}{4\xi^2}(3\|30\rangle - 3\|03\rangle + \|21\rangle - \|12\rangle)$
4	$\|20\rangle - \|02\rangle$
6	$3(\|20\rangle + \|02\rangle) - 6\|11\rangle + 6\xi(\|10\rangle + \|01\rangle) + \xi^2\|00\rangle$
6	$3(\|30\rangle + \|03\rangle) - 3(\|21\rangle + \|12\rangle) + 12\xi(\|20\rangle + \|02\rangle) +$ $+\xi^2(\|10\rangle + \|01\rangle) - 2\xi^3\|00\rangle$
22/3	$5(\|30\rangle - \|03\rangle) - 15(\|21\rangle - \|12\rangle) + 30\xi(\|20\rangle - \|02\rangle) +$ $+27\xi^2(\|10\rangle - \|01\rangle)$

Momenta	(Generalised) eigenvector
$-$	$\|0\rangle^{\otimes L}$
$p=\frac{2\pi n}{L},\ n=0,\ldots,L-1$	$\|\Psi_{xxx}\rangle_{(1)}$
$p_1+p_2 \pmod{2\pi}=\pi$ or $p_1=0, p_2\neq 0$	$\|\Psi_{xxx}\rangle_{(2)}$
$p_1=p_2=0$	$\frac{1}{\xi^2 L}\|\Psi_{xxx}\rangle_{(2)}$
$p_1+p_2 \pmod{2\pi}=0$ with $p_1\neq 0, p_2\neq 0$	$\|\Psi_{xxx}\rangle_{(2)}+2\xi\sum_{1\leq x\leq L}\psi_{(2)}(x,x)\|x\rangle + \frac{\xi^2 L}{4(1-\cos p_1)}\left(1+e^{ip_1}\right)\|0\rangle^{\otimes L}$
rest of p_1, p_2	$\|\Psi_{xxx}\rangle_{(2)}+2\xi\sum_{1\leq x\leq L}\psi_{(2)}(x,x)\|x\rangle$

$$[S^+,S^-]=-[2S^3]_q,[S^3,S^\pm]=\pm S^\pm$$

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$$

$$C=[S^3]_q[S^3-1]_q-S^+S^-$$

$$\begin{aligned}\Delta_q(S^3)&=S^3\otimes\mathbb{I}+\mathbb{I}\otimes S^3\\ \Delta_q(S^\pm)&=S^\pm\otimes q^{-S^3}+q^{S^3}\otimes S^\pm\end{aligned}$$

$$\begin{aligned}S^3&=\sum_{m=0}^{\infty}\left(\frac{1}{2}+m\right)|m\rangle\langle m|\\ S^+&=\sum_{m=0}^{\infty}[1+m]_q|m+1\rangle\langle m|\\ S^-&=\sum_{m=0}^{\infty}[1+m]_q|m\rangle\langle m+1|\end{aligned}$$

$$D_F\otimes D_F=\bigoplus_{j=0}^{\infty}D_j.$$

$$\Delta_q(S^-)|\chi_j\rangle=0,\Delta_q(S^3)|\chi_j\rangle=(j+1)|\chi_j\rangle.$$

$$\mathrm{h}_{12}=2\sinh\eta\sum_{j=0}^{\infty}\left(\prod_{k=1}^{j}\frac{\cosh k\eta}{\sinh k\eta}\right)\mathrm{P}_{12,j},$$

$$\mathrm{P}_{12,j}=\prod_{\substack{r=0\\ r\neq j}}^{\infty}\frac{\Delta_q(C)-[r+1]_q[r]_q}{[j+1]_q[j]_q-[r+1]_q[r]_q}.$$

$$\Delta_q(\mathrm{C})=e^{\eta S_1^3}\left(\frac{2\sinh(\eta S_1^3)\sinh(\eta S_2^3)}{\sinh^2(\eta)}\cosh^2\left(\frac{\eta}{2}\right)\right.\\ \left.-\frac{2\cosh(\eta S_1^3)\cosh(\eta S_2^3)}{\sinh^2(\eta)}\sinh^2\left(\frac{\eta}{2}\right)-S_1^+S_2^--S_1^-S_2^+\right)e^{-\eta S_2^3},$$

$$S^3\to J^3,S^\pm\to J^\pm+\mathcal{O}(q-1)^2$$

$$2\sinh\,\eta\prod_{k=1}^{j}\frac{\cosh\,k\eta}{\sinh\,k\eta}\to 2\prod_{k=1}^{j}\frac{1}{k}=2h(j),$$

$$S_{xxz}(\lambda_1,\lambda_2)\;=\frac{\sinh\,(\lambda_1-\lambda_2-\eta)}{\sinh\,(\lambda_1-\lambda_2+\eta)}$$

$$\epsilon_\eta(\lambda)\;=\frac{2(\cosh^2\,\eta-1)}{\cosh\,\eta-\cosh\,2\lambda}$$

$$p_\eta(\lambda)=\frac{1}{i}\log\frac{\sinh\,(\lambda+\eta/2)}{\sinh\,(\lambda-\eta/2)}$$

$$\mathrm{h}_{12}\to 2\sinh\,\eta\sum_{j=0}^{\infty}\left(\prod_{k=1}^{j}\frac{\cosh\,k\eta}{\sinh\,k\eta}\right)\mathrm{Ad}_{M(\xi)}^{\otimes 2}\mathrm{P}_{12,j}$$

$$\mathrm{Ad}_{M(\xi)}^{\otimes 2}\mathrm{P}_{12,j}=\prod_{\substack{r=0\\ r\neq j}}^{\infty}\frac{\mathrm{Ad}_{M(\xi)}^{\otimes 2}\Delta_q(\mathrm{C})-[r+1]_q[r]_q}{[j+1]_q[j]_q-[r+1]_q[r]_q}$$

$$\eta\to\epsilon\eta,\lambda\to\epsilon\lambda,\xi\to\frac{\xi}{\eta\epsilon}.$$

$$\mathrm{h}_{12}\to\hat{h}_{12}=\sum_{j=0}^{\infty}2h(j)\hat{P}_{12,j}$$

$$\hat{P}_{12,j}=\prod_{\substack{r=0\\ r\neq j}}^{\infty}\frac{\hat{\Delta}(C)-(r+1)r}{(j+1)j-(r+1)r},\ \text{with}\ \hat{\Delta}(C)=\lim_{\epsilon\to 0}\mathrm{Ad}_{M(\xi/\eta\epsilon)}^{\otimes 2}\Delta_q(\mathrm{C})$$

$$\hat{h}_{12}=\hat{F}_{12}h_{12}\hat{F}_{12}^{-1}\Leftrightarrow\hat{\Delta}(C)=\hat{F}_{12}\Delta(C)\hat{F}_{12}^{-1}$$

$$\hat{\Delta}^{(1)}(C)=\left[\hat{F}_{12}^{(1)},\Delta(C)\right]$$

$$\hat{F}_{12}=\mathbb{I}\otimes\mathbb{I}+\xi\hat{F}_{12}^{(1)}+O(\xi^2)\ \text{and}\ \hat{\Delta}(C)=\Delta(C)+\xi\hat{\Delta}^{(1)}(C)+O(\xi^2)$$

$$\mathrm{Ad}_{M(\xi)}^{\otimes 2}\Delta_q(\mathrm{C})=\sum_{n,m=0}^{\infty}\frac{(-1)^m\xi^{n+m}}{m!\,n!}\big(\Delta(J^-)\big)^n\Delta_q(\mathrm{C})\big(\Delta(J^-)\big)^m$$

$$\Delta_q(\mathrm{C})\to\Delta(C)+\eta(J_1^3\Delta(C)-\Delta(C)J_2^3)\epsilon+O(\epsilon^2).$$

$$\begin{aligned}\xi\big[\Delta(J^-),\Delta_q(\mathrm{C})\big]&\to\frac{\xi}{\eta\epsilon}[\Delta(J^-),\Delta(C)]+\xi[\Delta(J^-),J_1^3\Delta(C)-\Delta(C)J_2^3]+O(\epsilon)\\&=\xi(J_1^-\Delta(C)-\Delta(C)J_2^-)+O(\epsilon)\end{aligned}$$

$$(J^-\otimes\mathbb{I})\Delta(C)-\Delta(C)(\mathbb{I}\otimes J^-)=[J^-\wedge J^3,\Delta(C)].$$

$$\hat{F}_{12}=w^{-1}\otimes w^{-1}F_{12}\Delta(w).$$

$$\tilde{H}_{XXX}=\mathrm{Ad}_w^{\otimes L}\hat{H}_{XXX}$$

$$\hat{F}_{12}^{(1)}-F_{12}^{(1)}=\Delta\big(w^{(1)}\big)-\big(w^{(1)}\otimes\mathbb{I}+\mathbb{I}\otimes w^{(1)}\big)$$

$$w^{(1)}=J^3J^-$$

$$J^3=\frac{1}{2}\sigma^3,J^+=-\sigma^+,J^-=\sigma^-$$

$$\left[\sigma^{-}\wedge\frac{1}{2}\sigma^{3},\Delta(C)\right]=[-\sigma^{3}\otimes\sigma^{-},\Delta(C)].$$

$$|\Psi_{xxz}\rangle_{(1)}=\sum_{1\leq x\leq L}\Psi_{xxz}(x)\mid x\Big)$$

$$\Psi_{xxz}(x)=\left(\frac{\sinh\,(\lambda+\eta/2)}{\sinh\,(\lambda-\eta/2)}\right)^{x}.$$

$$|\Psi_{xxz}\rangle_{(1)}\rightarrow M(\xi/\epsilon\eta)^{\otimes L}|\Psi_{xxz}^{\epsilon}\rangle_{(1)}$$
$$=\sum_{1\leq x\leq L}\left(\frac{\sinh\,(\epsilon(\lambda+\eta/2))}{\sinh\,(\epsilon(\lambda-\eta/2))}\right)^{x}\left((\mid x)+\left(\frac{\xi}{\epsilon\eta}\right)|0\rangle^{\otimes L}\right)$$

$$\left(\hat{H}_{XXX}+\epsilon H_{XXZ_{\epsilon}}^{(1)}+O(\epsilon^{2})\right)\left(\left(\frac{\xi}{\epsilon\eta}\right)\left(\sum_{1\leq x\leq L}\Psi_{xxx}(x)\right)|0\rangle^{\otimes L}+|\Psi_{xxx}\rangle_{(1)}+O(\epsilon)\right)$$
$$=\left(E_{xxx}+O(\epsilon^{2})\right)\left(\left(\frac{\xi}{\epsilon\eta}\right)\left(\sum_{1\leq x\leq L}\Psi_{xxx}(x)\right)|0\rangle^{\otimes L}+|\Psi_{xxx}\rangle_{(1)}+O(\epsilon)\right)$$

$$O\left(\frac{1}{\epsilon}\right):\hat{H}_{XXX}\left(\sum_{1\leq x\leq L}\Psi_{xxx}(x)\right)|0\rangle^{\otimes L}=E_{xxx}\left(\sum_{1\leq x\leq L}\Psi_{xxx}(x)\right)|0\rangle^{\otimes L}$$

$$O(\epsilon^{0}):\hat{H}_{XXX}|\Psi_{xxx}\rangle_{(1)}+H_{XXZ_{\epsilon}}^{(1)}\left(\frac{\xi}{\eta}\right)\left(\sum_{1\leq x\leq L}\Psi_{xxx}(x)\right)|0\rangle^{\otimes L}$$
$$=E_{xxx}|\Psi_{xxx}\rangle_{(1)}$$

$$|\Psi_{xxx}\rangle_{(1)}=\lim_{\epsilon\to 0}\left(M(\xi/\epsilon\eta)^{\otimes L}|\Psi_{xxz}^{\epsilon}\rangle_{(1)}-\left(\frac{\xi}{\epsilon\eta}\right)\left(\sum_{1\leq x\leq L}\Psi_{xxx}(x)\right)|0\rangle^{\otimes L}\right)$$

$$0=E_{xxx}\left(\sum_{1\leq x\leq L}\Psi_{xxx}(x)\right)|0\rangle^{\otimes L}$$

$$\left(\sum_{1\leq x\leq L}e^{ikx}\right)=\frac{e^{ik}\left(e^{ikL}-1\right)}{e^{ik}-1}=0,\ \text{for}\ k\neq 0$$

$$\left(\sum_{1\leq x\leq L}\Psi_{xxx}(x)\right)|0\rangle^{\otimes L}|\Psi_{xxx}\rangle_{(1)}$$

$$|\Psi_{xxx}\rangle_{(1)}+\alpha\left(\sum_{1\leq x\leq L}\Psi_{xxx}(x)\right)|0\rangle^{\otimes L},\ \text{with}\ e^{ikL}=1$$

$$|\Psi_{xxz}\rangle_{(2)}=\sum_{1\leq x_{1}\leq x_{2}\leq L}\Psi_{xxz}(x_{1},x_{2})|x_{1},x_{2}\rangle$$

$$\Psi_{xxz}(x_{1},x_{2})=\left(\frac{\sinh\,(\lambda_{1}+\eta/2)}{\sinh\,(\lambda_{1}-\eta/2)}\right)^{x_{1}}\left(\frac{\sinh\,(\lambda_{2}+\eta/2)}{\sinh\,(\lambda_{2}-\eta/2)}\right)^{x_{2}}$$
$$+\frac{\sinh\,(\lambda_{2}-\lambda_{1}-\eta)}{\sinh\,(\lambda_{2}-\lambda_{1}+\eta)}\left(\frac{\sinh\,(\lambda_{2}+\eta/2)}{\sinh\,(\lambda_{2}-\eta/2)}\right)^{x_{1}}\left(\frac{\sinh\,(\lambda_{1}+\eta/2)}{\sinh\,(\lambda_{1}-\eta/2)}\right)^{x_{2}}$$

$$|\Psi_{xxz}\rangle_{(2)}\rightarrow M(\xi/\epsilon\eta)^{\otimes L}|\Psi_{xxz}^{\epsilon}\rangle_{(2)}$$
$$=\sum_{1\leq x_{1}\leq x_{2}\leq L}\Psi_{xxz}^{\epsilon}(x_{1},x_{2})|x_{1},x_{2}\rangle+\left(\frac{\xi}{\epsilon\eta}\right)|x_{1}\rangle+|x_{2}\rangle+\left(\frac{\xi}{\epsilon\eta}\right)^{2}|0\rangle^{\otimes L}$$

$$O(1/\epsilon^2): \hat{H}_{XXX} \sum_{1\le x_1 \le x_2 \le L} \Psi_{xxx}(x_1, x_2)|0\rangle^{\otimes L}$$

$$= E_{xxx}\left(\sum_{1\le x_1 \le x_2 \le L} \Psi_{xxx}(x_1, x_2)\right)|0\rangle^{\otimes L}$$

$$O(1/\epsilon): \hat{H}_{XXX} \sum_{1\le x_1 \le x_2 \le L} \Psi_{xxx}(x_1, x_2)|x_1\rangle + |x_2\rangle$$

$$= E_{xxx} \sum_{1\le x_1 < x_2 \le L} \Psi_{xxx}(x_1, x_2)|x_1\rangle + |x_2\rangle$$

$$O(\epsilon^0): \hat{H}_{XXX}\left(|\Psi_{xxx}\rangle_{(2)} + \left(\frac{\xi}{\eta}\right)^2\left(\sum_{1\le x_1 < x_2 \le L} \Psi_{xxz}^{(2)}(x_1, x_2)\right)|0\rangle^{\otimes L}\right)$$

$$+\left(\frac{\xi}{\eta}\right) H_{XXZ_\epsilon}^{(1)} \sum_{1\le x_1 \le x_2 \le L} \Psi_{xxx}(x_1, x_2)|x_1\rangle + |x_2\rangle$$

$$= E_{xxx}\left(|\Psi_{xxx}\rangle_{(2)} + \left(\frac{\xi}{\eta}\right)^2\left(\sum_{1\le x_1 < x_2 \le L} \Psi_{xxz}^{(2)}(x_1, x_2)\right)|0\rangle^{\otimes L}\right)$$

$$\Psi_{xxz}^{(2)}(x_1, x_2) = \left(\frac{d^2}{d\epsilon^2}\Psi_{xxz}^{\epsilon}(x_1, x_2)\right)_{\epsilon=0}$$

$$\mathrm{h}_{12}|10\rangle = q|10\rangle - |01\rangle$$
$$\mathrm{h}_{12}|01\rangle = \frac{1}{q}|01\rangle - |10\rangle$$

$$H_{XXZ}) = \left(q + \frac{1}{q}\right)\Big|\ x\Big) - |\ x - 1\Big) - |\ x + 1\Big)$$

$$\mathrm{Ad}_{M(\xi)}^{\otimes L} H_{XXZ}$$

$$\mathrm{Ad}_{M(\xi)}^{\otimes L} H_{XXZ}|x\rangle = H_{XXZ}|x\rangle + \xi\left(q + \frac{1}{q} - 2\right)|0\rangle^{\otimes L}.$$

$$\mathrm{Ad}_{M(\xi/\eta\epsilon)}^{\otimes L} H_{XXZ}|x\rangle = \hat{H}_{XXX}|x\rangle + \epsilon\xi\eta|0\rangle^{\otimes L} + O(\epsilon^2)$$

$$\hat{H}_{XXX}|x\rangle = 2|x\rangle - |\ x - 1\rangle\big) - |\ x + 1\rangle.$$

$$H_{XXZ_\epsilon}^{(1)}|x\rangle\big) = \xi\eta|0\rangle^{\otimes L}$$

$$H_{XXZ_\epsilon}^{(1)} \sum_{1\le x_1 \le x_2 \le L} \Psi_{xxx}(x_1, x_2)|x_1\rangle + |x_2\rangle$$

$$= 2\xi\eta\left(\sum_{1\le x_1 \le x_2 \le L} \Psi_{xxx}(x_1, x_2)\right)|0\rangle^{\otimes L}$$

$$|\Psi_{xxx}\rangle + \left(\frac{\xi}{\eta}\right)^2\left(\textstyle\sum_{1\le x_1 \le x_2 \le L} \Psi_{xxz}^{(2)}(x_1, x_2)\right)|0\rangle^{\otimes L}$$

$$\lim_{\epsilon\to 0}\Bigg(M(\xi/\epsilon\eta)^{\otimes L}|\Psi_{xxz}^{\epsilon}\rangle_{(2)} - \left(\frac{\xi}{\epsilon\eta}\right)\sum_{1\le x_1 < x_2 \le L} \Psi_{xxx}(x_1, x_2)|x_1\rangle + |x_2\rangle$$
$$- \left(\frac{\xi}{\epsilon\eta}\right)^2 \sum_{1\le x_1 \le x_2 \le L} \Psi_{xxz}(x_1, x_2)|0\rangle^{\otimes L}\Bigg)$$

$$e^{ik_1 L} = S_{xxx}(k_1, k_2), e^{ik_2 L} = S_{xxx}(k_2, k_1)$$

$$\sum_{1\leq x_1\leq x_2\leq L} \Psi_{xxx}(x_1,x_2)=0$$

$$\sum_{1\leq x_1\leq x_2\leq L} \Psi_{xxx}(x_1,x_2)=L(L+1)$$

$$\hat{H}_{XXX}|\Psi_{xxx}\rangle_{(2)}=-2\xi^2 L(L+1)|0\rangle^{\otimes L}.$$

$$0=E_{xxx}\left(\sum_{1\leq x_1\leq x_2\leq L} \Psi_{xxx}(x_1,x_2)\right)$$

$$\sum_{1\leq x_1\leq x_2\leq L} \Psi_{xxx}(x_1,x_2)|x_1\rangle+|x_2\rangle=\sum_{1\leq x_1\leq L} \Phi|x_1\rangle|x_2\rangle,$$

$$\Phi(x_1)=\sum_{x_2=x_1}^{L} \Psi_{xxx}(x_1,x_2)+\sum_{x_2=1}^{x_1} \Psi_{xxx}(x_2,x_1)$$

$$\sum_{1\leq x_1\leq L} \big(2\Phi(x_1)-\Phi(x_1+1)-\Phi(x_1-1)\big)|x_1\rangle.$$

$$\Phi(x_1+1)=\sum_{x_2=x_1}^{L-1} \Psi_{xxx}(x_1+1,x_2+1)+\sum_{x_2=0}^{x_1} \Psi_{xxx}(x_2+1,x_1+1)$$
$$=e^{i(k_1+k_2)}\Phi(x_1)$$

$$\Phi(x_1+1)=\sum_{x_2=x_1}^{L+1} \Psi_{xxx}(x_1-1,x_2-1)+\sum_{x_2=2}^{x_1} \Psi_{xxx}(x_2+1,x_1+1)$$
$$=e^{-i(k_1+k_2)}\Phi(x_1)$$

$$\hat{H}_{XXX}\sum_{1\leq x_1\leq L} \Phi|x_1\rangle|x_2\rangle=\epsilon(k_1+k_2)\sum_{1\leq x_1\leq L} \Phi|x_1\rangle|x_2\rangle,$$

$$\big(\epsilon(k_1+k_2)-\epsilon(k_1)-\epsilon(k_2)\big)\sum_{1\leq x_1\leq L} \Phi|x_1\rangle|x_2\rangle=0.$$

$$\epsilon(k_1+k_2)=\epsilon(k_1)+\epsilon(k_2)$$

$$\Phi(x_1)=0\ \forall x_1\in[1,L].$$

$$w^{\otimes L}|\Psi_{xxx}\rangle_{(2)}=|\Psi_{xxx}\rangle_{(2)}+3\xi\sum_{1\leq x\leq L} \Psi_{xxx}(x,x)|x\rangle$$
$$+\frac{\xi}{2}\sum_{1\leq x_1<x_2\leq L} \Psi_{xxx}(x,x)|x_1\rangle+|x_2\rangle+O(\xi^2)$$

$$\frac{\xi}{2}\sum_{1\leq x_1>x_2\leq L} \Psi_{xxx}(x,x)|x_1\rangle+|x_2\rangle=-\xi\sum_{1\leq x\leq L} \Psi_{xxx}(x,x)|x\rangle.$$

$$w^{\otimes L}|\Psi_{xxx}\rangle_{(2)}=|\Psi_{xxx}\rangle_{(2)}+2\xi\sum_{1\leq x\leq L} \Psi_{xxx}(x,x)|x\rangle+O(\xi^2)$$

$$\sum_{1\leq x_1<x_2\leq L} \Psi_{xxx}(x,x)|x_1\rangle+|x_2\rangle=L\sum_{1\leq x_1\leq L} e^{ik_1x_1}|x_1\rangle+|x_2\rangle.$$

$$w^{\otimes L}|\Psi_{xxx}\rangle_{(2)}=|\Psi_{xxx}\rangle_{(2)}+\xi\left(\frac{L}{2}+3\right)\sum_{1\leq x_1\leq L} e^{ik_1x_1}|x_1\rangle+O(\xi^2)$$

$$T_{1/2}(u) = R_{1/2,L}(u) \cdots R_{1/2,1}(u)$$

$$T_{1/2}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$\tilde{T}_{1/2}(u) = \tilde{R}_{1/2,L}(u) \cdots \tilde{R}_{1/2,1}(u) = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & \tilde{D}(u) \end{pmatrix},$$

$$\tilde{R}_{1/2,n}(u) = \begin{pmatrix} k_n^+ & \frac{1}{1-u} J_n^- e^{\frac{-w_n}{2}} \\ -2\xi J_n^3 k_n^+ - \frac{1}{1-u} J_n^+ e^{\frac{w_n}{2}} & -2\xi \frac{1}{1-u} J_n^3 J_n^- e^{\frac{-w_n}{2}} + k_n^- \end{pmatrix},$$

$$k_n^+ = \left(\frac{1-2u}{2(1-u)} + \frac{1}{1-u} J_n^3\right) e^{\frac{w_n}{2}}$$
$$k_n^- = \left(\frac{1-2u}{2(1-u)} - \frac{1}{1-u} J_n^3\right) e^{\frac{-w_n}{2}}$$
$$w_n = \log(1 + 2\xi J_n^-)$$

$$\begin{aligned}
\tilde{A}(u)\tilde{C}(v) =& \alpha(u-v)\tilde{C}(v)\tilde{A}(u) + \beta(u-v)\tilde{C}(u)\tilde{A}(v) + \xi(\tilde{D}(v)\tilde{A}(u) - \tilde{C}(v)\tilde{B}(u) \\
& - \tilde{A}(u)\tilde{A}(v)) + \xi^2 \tilde{D}(v)\tilde{B}(u) \\
\tilde{D}(u)\tilde{C}(v) =& \alpha(v-u)\tilde{C}(v)\tilde{D}(u) + \beta(v-u)\tilde{C}(u)\tilde{D}(v) + \xi(\tilde{A}(v)\tilde{D}(u) - \tilde{C}(v)\tilde{B}(u) \\
& - \tilde{D}(u)\tilde{D}(v)) + \xi^2 \tilde{A}(v)\tilde{B}(u) \\
\tilde{B}(u)\tilde{C}(v) =& \tilde{C}(v)\tilde{B}(u) + \beta(u-v)(\tilde{D}(u)\tilde{A}(v) - \tilde{D}(v)\tilde{A}(u)) + \xi(\tilde{D}(v)\tilde{B}(u) \\
& + \tilde{B}(u)\tilde{A}(v))
\end{aligned}$$

$$\begin{aligned}
\tilde{C}(u)\tilde{C}(v) =& \tilde{C}(v)\tilde{C}(u) + \frac{\xi}{\alpha(u-v)}(\tilde{D}(v)\tilde{C}(u) - \tilde{C}(v)\tilde{D}(u) + \tilde{C}(u)\tilde{A}(v) \\
& - \tilde{A}(u)\tilde{C}(v)) + \frac{\xi^2}{\alpha(u-v)}(\tilde{D}(v)\tilde{D}(u) - \tilde{A}(u)\tilde{A}(v))
\end{aligned}$$

$$\alpha(u-v) = 1 - \beta(u-v), \beta(u-v) = \frac{1}{u-v}.$$

$$\begin{aligned}
\tilde{A}(u)|0\rangle^L &= |0\rangle^L \\
\tilde{D}(u)|0\rangle^L &= d(u)|0\rangle^L \text{ with } d(u) = \left(\frac{u}{u-1}\right)^L \\
\tilde{B}(u)|0\rangle^L &= 0
\end{aligned}$$

$$\begin{aligned}
\tilde{\tau}(u)\tilde{C}(v)|0\rangle^L =& (\alpha(u-v) + \alpha(v-u)d(u))\tilde{C}(v)|0\rangle^L + \\
& + \beta(u-v)(1-d(v))\tilde{C}(u)|0\rangle^L + \xi(1-d(v))(d(u)-1)|0\rangle^L
\end{aligned}$$

$$\tilde{\tau}(u) = \tilde{A}(u) + \tilde{D}(u).$$

$$\left(\frac{v}{v-1}\right)^L = 1$$

$$\begin{aligned}
\tilde{\tau}(u)\tilde{C}(v_1)\tilde{C}(v_2)|0\rangle^L =& [\alpha(u-v_1)\alpha(u-v_2) + \alpha(v_1-u)\alpha(v_2-u)d(u)]\tilde{C}(v_1)\tilde{C}(v_2)|0\rangle^L \\
& + \gamma_1(v_1,v_2,u)\tilde{C}(u)\tilde{C}(v_1)|0\rangle^L + \gamma_2(v_1,v_2,u)\tilde{C}(u)\tilde{C}(v_2)|0\rangle^L + \xi\gamma_3(v_1,v_2,u)\tilde{C}(u)|0\rangle^L \\
& + \xi\gamma_4(v_1,v_2,u)\tilde{C}(v_1)|0\rangle^L + \xi\gamma_5(v_1,v_2,u)\tilde{C}(v_2)|0\rangle^L + \xi^2\gamma_6(v_1,v_2,u)|0\rangle^L
\end{aligned}$$

$$\left(\frac{v_1}{v_1-1}\right)^L = \frac{v_1-v_2-1}{v_1-v_2+1}, \left(\frac{v_2}{v_2-1}\right)^L = \frac{v_2-v_1-1}{v_2-v_1+1}$$

$$\begin{aligned}
\tilde{A} &= \Omega A \Omega^{-1} e^{\frac{1}{2}\tilde{\Delta}^{(L-1)}(w)} \\
\tilde{B} &= \Omega B \Omega^{-1} e^{-\frac{1}{2}\tilde{\Delta}^{(L-1)}(w)} \\
\tilde{C} &= \left(-2\xi\Omega\Delta^{(L-1)}(J^3)A\Omega^{-1} + \Omega C \Omega^{-1}\right) e^{\frac{1}{2}\tilde{\Delta}^{(L-1)}(w)} \\
\tilde{D} &= \left(-2\xi\Omega\Delta^{(L-1)}(J^3)B\Omega^{-1} + \Omega D \Omega^{-1}\right) e^{-\frac{1}{2}\tilde{\Delta}^{(L-1)}(w)}
\end{aligned}$$

$$\alpha(u-v_1)\alpha(u-v_2)+\alpha(v_1-u)\alpha(v_2-u)d(u),$$

$$\begin{pmatrix} \frac{2u^2-2u+1}{(u-1)^2} & 0 & 0 & \frac{\xi^2(u^2-4u+2)}{2(u-1)^2} & \frac{\xi^2(u^2-4u+2)}{2(u-1)^2} & \frac{2\xi^2 u}{u-1} \\ 0 & \frac{2(u^2-u+1)}{(u-1)^2} & -\frac{1}{(u-1)^2} & \frac{2\xi}{(u-1)^2} & \frac{2\xi}{(u-1)^2} & -\frac{4\xi}{(u-1)^2} \\ 0 & -\frac{1}{(u-1)^2} & \frac{2(u^2-u+1)}{(u-1)^2} & \frac{2\xi}{(u-1)^2} & \frac{2\xi}{(u-1)^2} & -\frac{4\xi}{(u-1)^2} \\ 0 & 0 & 0 & \frac{2u^2-2u+3}{(u-1)^2} & 0 & -\frac{2}{(u-1)^2} \\ 0 & 0 & 0 & 0 & \frac{2u^2-2u+3}{(u-1)^2} & -\frac{2}{(u-1)^2} \\ 0 & 0 & 0 & -\frac{2}{(u-1)^2} & -\frac{2}{(u-1)^2} & \frac{2u^2-2u+5}{(u-1)^2} \end{pmatrix}.$$

$$\begin{pmatrix} \frac{2u^2-2u+1}{(u-1)^2} & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{2u^2-2u+1}{(u-1)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2u^2-2u+1}{(u-1)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2u^2-2u+3}{(u-1)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2u^2-2u+3}{(u-1)^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2u^2-2u+7}{(u-1)^2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{2u^2-2u+1}{(u-1)^2} & 0 & 0 & \frac{\xi^2(u^2-4u+2)}{2(u-1)^2} & \frac{\xi^2(u^2-4u+2)}{2(u-1)^2} & \frac{2\xi^2 u}{u-1} \\ 0 & \frac{2(u^2-u+1)}{(u-1)^2} & -\frac{1}{(u-1)^2} & \frac{2\xi}{(u-1)^2} & \frac{2\xi}{(u-1)^2} & -\frac{4\xi}{(u-1)^2} \\ 0 & -\frac{1}{(u-1)^2} & \frac{2(u^2-u+1)}{(u-1)^2} & \frac{2\xi}{(u-1)^2} & \frac{2\xi}{(u-1)^2} & -\frac{4\xi}{(u-1)^2} \\ 0 & 0 & 0 & \frac{2u^2-2u+3}{(u-1)^2} & 0 & -\frac{2}{(u-1)^2} \\ 0 & 0 & 0 & 0 & \frac{2u^2-2u+3}{(u-1)^2} & -\frac{2}{(u-1)^2} \\ 0 & 0 & 0 & -\frac{2}{(u-1)^2} & -\frac{2}{(u-1)^2} & \frac{2u^2-2u+5}{(u-1)^2} \end{pmatrix}.$$

$$\begin{pmatrix} \frac{2u^2-2u+1}{(u-1)^2} & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{2u^2-2u+1}{(u-1)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2u^2-2u+1}{(u-1)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2u^2-2u+3}{(u-1)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2u^2-2u+3}{(u-1)^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2u^2-2u+7}{(u-1)^2} \end{pmatrix}.$$

Eigenvalue	(Generalised) eigenvector
$\frac{2u^2-2u+1}{(u-1)^2}$	$\vert 00\rangle$

$\frac{2u^2-2u+1}{(u-1)^2}$	$\frac{(u-1)^2}{\xi^2(3u^2-6u+2)}(\|20\rangle+\|02\rangle+\|11\rangle)$
$\frac{2u^2-2u+1}{(u-1)^2}$	$\|10\rangle+\|01\rangle$
$\frac{2u^2-2u+3}{(u-1)^2}$	$\|10\rangle-\|01\rangle$
$\frac{2u^2-2u+3}{(u-1)^2}$	$\|20\rangle-\|02\rangle$
$\frac{2u^2-2u+7}{(u-1)^2}$	$3(\|20\rangle+\|02\rangle)-6\|11\rangle+6\xi(\|10\rangle+\|01\rangle)+\xi^2\left(1-\frac{3}{2}u^2\right)\|00\rangle$

Eigenvalue	(Generalised) eigenvector
$\frac{2u^2-2u+1}{(u-1)^2}$	$\|00\rangle$
$\frac{2u^2-2u+1}{(u-1)^2}$	$\frac{(u-1)^2}{\xi^2(3u^2-6u+2)}(\|20\rangle+\|02\rangle+\|11\rangle)$
$\frac{2u^2-2u+1}{(u-1)^2}$	$\|10\rangle+\|01\rangle$
$\frac{2u^2-2u+3}{(u-1)^2}$	$\|10\rangle-\|01\rangle$
$\frac{2u^2-2u+3}{(u-1)^2}$	$\|20\rangle-\|02\rangle$
$\frac{2u^2-2u+7}{(u-1)^2}$	$3(\|20\rangle+\|02\rangle)-6\|11\rangle+6\xi(\|10\rangle+\|01\rangle)+\xi^2\left(1-\frac{3}{2}u^2\right)\|00\rangle$

$$\frac{(u-1)^2}{\xi^2(3u^2-6u+2)}(|20\rangle+|02\rangle+|11\rangle)$$

$$J^+=a^\dagger(-2j+a^\dagger a), J^-=a, J^3=-j+a^\dagger a,$$

$$\Delta(C)=-(a_1^\dagger-a_2^\dagger)^2a_1a_2-2j(a_1^\dagger-a_2^\dagger)(a_1-a_2)+2j(2j+1).$$

$$E=\sum_{p=1}^N\frac{2|j|}{j^2+\lambda_p^2},$$

$$\left(\frac{\lambda_p+|j|i}{\lambda_p-|j|i}\right)^L=\prod_{\substack{r=0\\r\neq p}}^N S_{xxx}(\lambda_p,\lambda_r)$$

$$F_{12}|n_1,n_2\rangle=\sum_{k=0}^{n_2}(-2\xi)^k\binom{n_2}{k}(|j|+n_1)^{\langle k\rangle}|n_1,n_2-k\rangle.$$

$$M=e^{\xi J^-}$$

$$(J^-\otimes\mathbb{I})\Delta(C)-\Delta(C)(\mathbb{I}\otimes J^-)$$

$$\tilde{H}_{XXX}=\Omega\mathbb{H}_{XXX}\Omega^{-1},$$

$$\tilde{h}_{m,m+1}=\Omega h_{m,m+1}\Omega^{-1}, m=1,\dots,L-1$$

$$(\Delta\otimes id)(F)=F_{13}F_{23}$$

$$(id\otimes\tilde{\Delta})(F)=F_{12}F_{13}$$

$$R_{12}F_{13}F_{23}=F_{23}F_{13}R_{12}$$

$$\tilde{R}_{23}F_{12}F_{13}=F_{13}F_{12}\tilde{R}_{23}$$

$$\Omega h_{m,m+1}\Omega^{-1}=(F_{12}\cdots F_{1L})\cdots(F_{m+1,m+2}\cdots F_{m+1,L})h_{m,m+1}\times(F_{m+1,L}^{-1}\cdots F_{m+1,m+2}^{-1})\cdots(F_{1L}^{-1}\cdots F_{12}^{-1})$$

$$h_{m,m+1}=P_{m,m+1}R'_{m,m+1}(0)$$

$$R'_{m,m+1}(0)=\left(\frac{\mathrm{d}}{\mathrm{d}u}R_{m,m+1}(u)\right)_{u=0}$$

$$\Omega h_{m,m+1}\Omega^{-1}=\cdots(F_{m-1,m}\cdots F_{m-1,L})\Lambda(F_{m-1,L}^{-1}\cdots F_{m-1,m}^{-1})\cdots$$

$$\Lambda = F_{m,m+1}P_{m,m+1}\left(F_{m+1,m+2}\cdots F_{m+1,L}\right)\left(F_{m,m+2}\cdots F_{mL}\right)$$
$$\times R'_{m,m+1}(0)\left(F^{-1}_{m+1,L}\cdots F^{-1}_{m+1,m+2}\right)\times\left(F^{-1}_{mL}\cdots F^{-1}_{m,m+1}\right)$$

$$\Lambda = F_{m,m+1}P_{m,m+1}\left(F_{m+1,m+2}\cdots F_{m+1,L-1}\right)\left(F_{m,m+2}\cdots F_{m,L-1}\right)F_{m+1,L}F_{mL}$$
$$\times R'_{m,m+1}(0)\left(F^{-1}_{m+1,L}\cdots F^{-1}_{m+1,m+2}\right)\left(F^{-1}_{mL}\cdots F^{-1}_{m,m+1}\right)$$

$$F_{m+1,L}F_{mL}R'_{m,m+1}(0) = R'_{m,m+1}(0)F_{mL}F_{m+1,L}$$

$$\Lambda = F_{m,m+1}P_{m,m+1}\left(F_{m+1,m+2}\cdots F_{m+1,L-1}\right)\left(F_{m,m+2}\cdots F_{m,L-1}\right)R'_{m,m+1}$$
$$\times\left(F^{-1}_{m+1,L-1}\cdots F^{-1}_{m+1,m+2}\right)\left(F^{-1}_{m,L-1}\cdots F^{-1}_{m,m+1}\right)$$

$$\Lambda = F_{m,m+1}P_{m,m+1}R'_{m,m+1}(0)F^{-1}_{m,m+1}(0) = \tilde{R}'_{m,m+1}(0)$$

$$\Omega h_{m,m+1}\Omega^{-1} = \cdots\left(F_{m-1,m}\cdots F_{m-1,L}\right)P_{m,m+1}\tilde{R}'_{m,m+1}(0)\left(F^{-1}_{m-1,L}\cdots F^{-1}_{m-1,m}\right)\cdots$$

$$\Omega h_{m,m+1}\Omega^{-1} = P_{m,m+1}\Gamma$$

$$\Gamma = \left(F_{12}\cdots F_{1,m+1}F_{1m}\cdots F_{1L}\right)\cdots\left(F_{m-1,m+1}F_{m-1,m}\cdots F_{m-1,L}\right)$$
$$\times\tilde{R}'_{m,m+1}(0)\left(F^{-1}_{m-1,L}\cdots F^{-1}_{m-1,m}\right)\cdots\left(F^{-1}_{1L}\cdots F^{-1}_{12}\right)$$

$$\Gamma = \left(F_{12}\cdots F_{1,m+1}F_{1m}\cdots F_{1L}\right)\cdots F_{m-1,m+1}F_{m-1,m}\tilde{R}'_{m,m+1}(0)$$
$$\times F^{-1}_{m-1,m+1}F^{-1}_{m-1,m}\left(F^{-1}_{m-2,L}\cdots F^{-1}_{m-2,m-1}\right)\cdots\left(F^{-1}_{1L}\cdots F^{-1}_{12}\right)$$

$$F_{m-1,m+1}F_{m-1,m}\tilde{R}'_{m,m+1}(0) = \tilde{R}'_{m,m+1}(0)F_{m-1,m}F_{m-1,m+1}$$

$$\Gamma = \left(F_{12}\cdots F_{1,m+1}F_{1,m}\cdots F_{1L}\right)\cdots\left(F_{m-2,m-1}F_{m-2,m+1}F_{m-2,m}\cdots\right.$$
$$\left.\cdots F_{m-2,L}\right)\tilde{R}'_{m,m+1}(0)\left(F^{-1}_{m-2,L}\cdots F^{-1}_{m-2,m-1}\right)\cdots\left(F^{-1}_{1L}\cdots F^{-1}_{12}\right)$$

$$\Gamma = \tilde{R}'_{m,m+1}(0)$$

$$\Omega h_{m,m+1}\Omega^{-1} = P_{m,m+1}\tilde{R}'_{m,m+1}(0) = \tilde{h}_{m,m+1}$$

$$\tilde{H}_{xxx} = \Omega\sum_{m=1}^{L-1} h_{m,m+1}\Omega^{-1} + \tilde{h}_{L1} = \Omega\left(\sum_{m=1}^{L-1} h_{m,m+1} + \Omega^{-1}F_{L1}h_{L1}F^{-1}_{L1}\Omega\right)\Omega^{-1}$$
$$= \Omega\mathbb{H}_{XXX}\Omega^{-1}$$

$$F^{-1}_{13}F^{-1}_{23}R_{12} = R_{12}F^{-1}_{23}F^{-1}_{13}$$
$$\tilde{R}_{23}F_{12} = F_{13}F_{12}\tilde{R}_{23}F^{-1}_{13}.$$
$$\tilde{R}_{a2}\tilde{R}_{a1} = \tilde{R}_{a2}F_{1a}R_{a1}F^{-1}_{a1}$$
$$\tilde{R}_{a2}\tilde{R}_{a1} = F_{12}F_{1a}F_{2a}R_{a2}F^{-1}_{a2}F^{-1}_{12}R_{a1}F^{-1}_{a1},$$
$$\tilde{R}_{a2}\tilde{R}_{a1} = F_{12}F_{1a}F_{2a}R_{a2}R_{a1}F^{-1}_{12}F^{-1}_{a2}F^{-1}_{a1}.$$
$$\tilde{R}_{a3}F_{1a}F_{2a}R_{a2}R_{a1} = F_{13}F_{23}F_{1a}F_{2a}F_{3a}R_{a3}F^{-1}_{a3}F^{-1}_{23}F^{-1}_{13}R_{a2}R_{a1}.$$
$$\tilde{R}_{a3}F_{1a}F_{2a}R_{a2}R_{a1} = F_{13}F_{23}F_{1a}F_{2a}F_{3a}R_{a3}R_{a2}R_{a1}F^{-1}_{13}F^{-1}_{23}F^{-1}_{a3},$$
$$\tilde{R}_{a3}\tilde{R}_{a2}\tilde{R}_{a1} = F_{12}F_{13}F_{23}F_{1a}F_{2a}F_{3a}R_{a3}R_{a2}R_{a1}F^{-1}_{23}F^{-1}_{13}F^{-1}_{12}F^{-1}_{a3}F^{-1}_{a2}F^{-1}_{a1}.$$

$$\left(F_{1j}F_{2j}\cdots F_{j-1,j}\right)F_{ja}\left(F^{-1}_{j-1,j}\cdots F^{-1}_{2j}F^{-1}_{1j}\right)F^{-1}_{aj}$$

$$\tilde{T}_a = \Omega F_a T_a\Omega^{-1}(F_a)^{-1}_{op},$$

$$\Omega = (F_{12}\cdots F_{1L})(F_{23}\cdots F_{2L})\cdots F_{L-1,L}$$
$$= (F_{12})(F_{13}F_{23})\cdots\left(F_{1L}\cdots F_{L-1,L}\right),$$
$$F_a = F_{1a}\cdots F_{La},$$
$$(F_a)^{-1}_{op} = F^{-1}_{aL}\cdots F^{-1}_{a1}.$$

$$\mathrm{h}_{12}|10\rangle = q|10\rangle - |01\rangle, \mathrm{h}_{12}|01\rangle = \frac{1}{q}|01\rangle - |10\rangle,$$
$$\mathrm{h}_{12}|11\rangle = \frac{1+q^2}{q}|11\rangle - |20\rangle - |02\rangle,$$
$$\mathrm{h}_{12}|20\rangle = \frac{q+2q^3}{1+q^2}|20\rangle - |11\rangle - \frac{q}{1+q^2}|02\rangle,$$
$$\mathrm{h}_{12}|02\rangle = \frac{2+q^2}{q+q^3}|02\rangle - |11\rangle - \frac{q}{1+q^2}|20\rangle.$$

$$|\Psi_{xxz}\rangle_{(1)} = \sum_{1\le x\le L} e^{ikx}\|x\rangle \text{ with } e^{ikL} = 1$$

$$H_{XXZ}|\Psi_{xxz}\rangle_{(1)} = \sum_{1\le x\le L} e^{ikx}\left(q+\frac{1}{q}\right)\|x\rangle - \|x+1\rangle - \|x-1\rangle$$
$$= 2(\cosh\,\eta - \cos\,k)\sum_{1\le x\le L} e^{ikx}\|x\rangle$$

$$\epsilon_\eta(k) = 2(\cosh\,\eta - \cos\,k)$$
$$\epsilon_\eta(\lambda) = 2\left(\frac{\cosh^2\,\eta - 1}{\cosh\,\eta - \cosh\,2\lambda}\right)$$

$$|\Psi_{xxz}\rangle_{(2)} = \sum_{1\le x_1\le x_2\le L} \Psi_{xxz}(x_1,x_2)\|x_1,x_2\rangle$$

$$\Psi_{xxz}(x_1,x_2) = e^{i(k_1x_1+k_2x_2)} + S_{xxz}(k_2,k_1)e^{i(k_2x_1+k_1x_2)}$$

$$\Psi_{xxz}(x_1,x_2+L) = \Psi_{xxz}(x_2+L,x_1)$$

if $x_1 = x_2 = x$:

$$E_{xxz}\Psi_{xxz}(x,x) = \frac{2(1+q^2+q^4)}{q+q^3}\Psi_{xxz}(x,x) - \frac{q}{1+q^2}\Psi_{xxz}(x+1,x+1)$$
$$-\frac{q}{1+q^2}\Psi_{xxz}(x-1,x-1) - \Psi_{xxz}(x-1,x) - \Psi_{xxz}(x,x+1)$$

if $x_1 < x_2$:

$$E_{xxz}\Psi_{xxz}(x_1,x_2) = 2\frac{1+q^2}{q}\Psi_{xxz}(x_1,x_2) - \Psi_{xxz}(x_1-1,x_2)$$
$$-\Psi_{xxz}(x_1+1,x_2) - \Psi_{xxz}(x_1,x_2-1) - \Psi_{xxz}(x_1,x_2+1)$$

$$E_{xxz} = \epsilon_\eta(k_1) + \epsilon_\eta(k_2)$$
$$C(k_1,k_2) + C(k_2,k_1)S_{xxz}(k_2,k_1) = 0$$
$$C(k_1,k_2) = \frac{q(1+e^{-i(k_1+k_2)})}{1+q^2}(1 + e^{i(k_1+k_2)} - 2\cosh\,\eta e^{ik_1}),$$
$$S_{xxz}(k_1,k_2) = -\frac{1+e^{i(k_1+k_2)} - 2\cosh\,\eta e^{ik_2}}{1+e^{i(k_1+k_2)} - 2\cosh\,\eta e^{ik_1}}$$
$$S_{xxz}(\lambda_1,\lambda_2) = \frac{\sinh\,(\lambda_1-\lambda_2-\eta)}{\sinh\,(\lambda_1-\lambda_2+\eta)}$$
$$|\chi_0\rangle = |00\rangle$$
$$|\chi_1\rangle = q|10\rangle - |01\rangle$$
$$\Delta_q(S^+)|\chi_0\rangle = q^{-1/2}|10\rangle + q^{1/2}|01\rangle.$$
$$|10\rangle = \frac{1}{q+q^{-1}}\left(q^{-1/2}\Delta_q(S^+)|\chi_0\rangle + |\chi_1\rangle\right)$$
$$|01\rangle = \frac{1}{q^2+1}\left(q^{3/2}\Delta_q(S^+)|\chi_0\rangle - |\chi_1\rangle\right)$$
$$\mathrm{h}_{12}|10\rangle = |\chi_1\rangle = q|10\rangle - |01\rangle,$$
$$\mathrm{h}_{12}|01\rangle = -\frac{1}{q}|\chi_1\rangle = \frac{1}{q}|01\rangle - |10\rangle.$$

$$\sum_{1\le x_1\le x_2\le L} \Psi_{xxx}(x_1,x_2) = \sum_{1\le x_1\le x_2\le L} \left(e^{ik_1x_1} + e^{ik_1x_2}\right)$$
$$= (L-1)\sum_{1\le x_1\le L} e^{ik_1x_1} = 0 \text{ with } e^{ik_1L} = 1,$$

$$\sum_{1\le x_1\le x_2\le L}\Psi_{xxx}(x_1,x_2)=\frac{G(k_1,k_2)+S_{xxx}(k_2,k_1)G(k_2,k_1)}{(e^{ik_1}-1)(e^{ik_2}-1)\left(e^{i(k_1+k_2)}-1\right)},$$

$$G(k_1,k_2)=e^{i(2k_1+k_2)}\left(1-e^{i(k_1+k_2)L}\right)+e^{i(k_1+k_2+k_2L)}-e^{i(2k_1+k_2(2+L))}$$
$$-e^{i(k_1+k_2)}+e^{i(k_1+k_2)(2+L)}$$

$$G(k_1,k_2)+S_{xxx}(k_2,k_1)G(k_2,k_1)=0$$

$$e^{ik_1L}=S_{xxx}(k_1,k_2)\ \text{and}\ e^{ik_2L}=S_{xxx}(k_2,k_1)$$

$$G(k_1,k_2)=e^{2i(k_1+k_2)}-e^{i(k_1+k_2)}+S_{xxx}(k_2,k_1)\left(e^{i(k_1+k_2)}-e^{2i(k_1+k_2)}\right).$$

$$S_{xxx}(k_1,k_2)S_{xxx}(k_2,k_1)=1$$

$$S_{xxx}(k_2,k_1)G(k_2,k_1)=e^{i(k_1+k_2)}-e^{2i(k_1+k_2)}+S_{xxx}(k_2,k_1)\left(e^{2i(k_1+k_2)}-e^{i(k_1+k_2)}\right)$$
$$=-G(k_1,k_2)$$

$$\Phi(x_1)=\frac{D(k_1,k_2)+S_{xxx}(k_2,k_1)D(k_2,k_1)}{(e^{ik_1}-1)(e^{ik_2}-1)},$$

$$D(k_1,k_2)=e^{i(k_1+k_2)x_1}+e^{i(k_1+k_2)(x_1+1)}+e^{i(k_1(1+x_1)+k_2(1+L))}+e^{i(k_1+k_2x_1)}$$
$$-e^{i(k_1x_1+k_2(1+L))}-e^{i(k_1+k_2(1+x_1))}-2e^{i(k_1(1+x_1)+k_2x_1)}$$

$$D(k_1,k_2)=e^{i(k_1+k_2)x_1}+e^{i(k_1+k_2)(x_1+1)}+e^{i(k_1+k_2x_1)}-e^{i\left(k_1+k_2(1+x_1)\right)}$$
$$-2e^{i(k_1(1+x_1)+k_2x_1)}+S_{xxx}(k_2,k_1)\left(e^{i(k_1(1+x_1)+k_2)}-e^{i(k_1x_1+k_2)}\right)$$

$$S_{xxx}(k_1,k_2)S_{xxx}(k_2,k_1)=1$$

$$D(k_1,k_2)+S_{xxx}(k_2,k_1)D(k_2,k_1)=e^{i(k_1+k_2)x_1}\left(1+e^{i(k_1+k_2)}-2e^{ik_1}\right)$$
$$+e^{i(k_1+k_2)x_1}S_{xxx}(k_2,k_1)\left(1+e^{i(k_1+k_2)}-2e^{ik_2}\right)=0$$

$$Z(x)=\frac{1}{\sqrt{2}}(\Phi^1(x)+i\Phi^2(x)),\bar{Z}(x)=\frac{1}{\sqrt{2}}(\Phi^1(x)-i\Phi^2(x))$$

$$\left\langle \bar{Z}^p{}_q(x)Z^r{}_s(y)\right\rangle=\frac{\delta^p{}_s\delta^r{}_q-\frac{1}{N}\delta^p{}_q\delta^r{}_s}{|x-y|^2}$$

$$\hat{T}_k=\mathrm{Tr}(Z^k)$$

$$T_k=\frac{\mathrm{Tr}(Z^k)}{\sqrt{k}N^{\frac{k}{2}}}$$

$$O_2=T_2,O_4=T_4-\frac{2}{N}(O_2)^2$$

$$J^-\equiv\sum_{a=3}^{6}J^{-a}J^{-a},J^{-a}\equiv(J^{1a}-iJ^{2a}),a=3,\dots,6$$

$$J^{ij}=-i\mathrm{Tr}\left(\Phi^i\frac{\partial}{\partial\Phi^j}-\Phi^j\frac{\partial}{\partial\Phi^i}\right),i,j=1,\dots,6$$

$$O_{2,0}=\frac{1}{\sqrt{6}N}\mathrm{Tr}\left(2Z\bar{Z}-\frac{1}{2}\Phi^a\Phi^a\right)=\frac{1}{\sqrt{6}N}\mathrm{Tr}(2Z\bar{Z}-(X\bar{X}+Y\bar{Y}))$$

$$T_{4,2}=\frac{1}{\sqrt{10}N^2}\mathrm{Tr}\left(2Z^3\bar{Z}-Z^2\Phi^a\Phi^a-\frac{1}{2}Z\Phi^aZ\Phi^a\right)$$

$$T_{4,0}=\frac{1}{2\sqrt{5}N^2}\mathrm{Tr}[2Z^2\bar{Z}^2+(Z\bar{Z})^2-((Z\bar{Z}+\bar{Z}Z)\Phi^a\Phi^a+Z\Phi^a\bar{Z}\Phi^a)$$
$$+\frac{1}{6}\left(\Phi^a\Phi^a\Phi^b\Phi^b+\frac{1}{2}\Phi^a\Phi^b\Phi^a\Phi^b\right)\Big]$$

$$O_{4,2}=T_{4,2}-\frac{4\sqrt{3}}{\sqrt{10}N}O_2O_{2,0}+\frac{2}{\sqrt{10}N^3}\mathrm{Tr}Z\Phi^a\mathrm{Tr}Z\Phi^a$$

$$O_{4,0}=T_{4,0}-\frac{3}{\sqrt{5}N}\left(O_{2,0}\right)^2-\frac{2}{\sqrt{5}N}O_2^{\dagger}O_2+\frac{2}{\sqrt{5}N^3}\mathrm{Tr}Z\Phi^a\mathrm{Tr}\bar{Z}\Phi^a$$
$$-\frac{1}{6\sqrt{5}N^3}\left(\mathrm{Tr}\Phi^a\Phi^b\mathrm{Tr}\Phi^a\Phi^b-\frac{1}{4}(\mathrm{Tr}\Phi^a\Phi^a)^2\right)$$

$$\hat{T}_I=C^I_{i_1\cdots i_k}\mathrm{Tr}\left(\Phi^{i_1}\cdots\Phi^{i_k}\right)$$

$$\left\langle T_{I_1}(x_1)T_{I_2}(x_2)T_{I_3}(x_3)\right\rangle=\frac{1}{N}\frac{\sqrt{(k_2+k_3)k_2k_3}\langle C^{I_1}C^{I_2}C^{I_3}\rangle}{|x_1-x_2|^{2k_2}|x_1-x_3|^{2k_3}},$$

$$a_{I_1I_2I_3}=\frac{1}{\left(\frac{1}{2}\Sigma+2\right)!\,2^{\frac{1}{2}(\Sigma-2)}}\frac{k_1!\,k_2!\,k_3!}{\alpha_1!\,\alpha_2!\,\alpha_3!}\langle C^{I_1}C^{I_2}C^{I_3}\rangle$$

$$a_{I_1I_2I_3}=\frac{1}{(k_1+1)(k_1+2)2^{k_1-1}}\langle C^{I_1}C^{I_2}C^{I_3}\rangle\equiv z(k_1)\langle C^{I_1}C^{I_2}C^{I_3}\rangle$$

$$\langle C^{I_1}C^{I_2}C^{I_3}\rangle=\frac{a_{I_1I_2I_3}}{z(k_1)},\ \text{for}\ k_1=k_2+k_3$$

$$\left\langle T_{I_1}(x_1)\left(T_{I_2}T_{I_3}\right)(x_2)\right\rangle=\frac{1}{N}\frac{a_{I_1I_2I_3}}{z(k_2+k_3)}\frac{\sqrt{(k_2+k_3)k_2k_3}}{|x_1-x_2|^{2(k_2+k_3)}}.$$

$$O_{I_1}=T_{I_1}-\sum_{I_2+I_3=I_1}\mathcal{C}^{I_1,I_2,I_3}T_{I_2}T_{I_3}$$

$$\mathcal{C}^{I_1,I_2,I_3}=\frac{\sqrt{(k_2+k_3)k_2k_3}}{2N}\langle C^{I_1}C^{I_2}C^{I_3}\rangle.$$

$$\mathcal{C}^{(4,m),(2,n),(2,p)}=\frac{2}{N}\frac{a_{(4m)(2n)(2p)}}{z(4)}$$

$$O_{4,m}=T_{4,m}-\frac{2}{N}\frac{a_{(4m)(2n)(2p)}}{z(4)}O_{2,n}O_{2,p}$$

$$\mathcal{T}_k=\mathcal{N}_kT_k,\mathcal{O}_k=\mathcal{N}_kO_k$$

$$\mathcal{N}_2=\frac{N}{\pi^2},\mathcal{N}_k=\frac{N}{\pi^2}(k-2)\sqrt{k-1}\ \text{for}\ k\neq 2$$

$$\mathcal{T}_{4,m}=\mathcal{O}_{4,m}+\frac{\mathcal{N}_4}{\mathcal{N}_2^2}\frac{2}{N}\frac{a_{(4m)(2n)(2p)}}{z(4)}\mathcal{O}_{2,n}\mathcal{O}_{2,p}$$

$$h_a^a(x,y)=\sum\pi_I(x)Y^I(y)$$
$$f_{abcde}(x,y)=\sum\Lambda^Ib_I(x)\epsilon_{abcde}Y^I(y)$$

$$s_I=\frac{1}{20(k+2)}(\pi_I-10(k+4)b_I)$$

$$S_I=s_I+\sum_{J,K}\left(J^{IJK}s_Js_K+L^{IJK}D_\mu s_JD^\mu s_K\right)+O([s]^3)$$

$$S_I = w(s_I)s_I, w(s_I) = \sqrt{\frac{8k(k-1)(k+2)z(k)}{k+1}}, k = 2,3$$

$$S_{(4,m)} = \frac{2\sqrt{3}}{5}\left(s_{(4,m)} - \frac{a_{(4m)(2n)(2p)}}{27z(4)}\left(83s_{(2,n)}s_{(2,p)} + 7\mathcal{D}_\mu s_{(2,n)}\mathcal{D}^\mu s_{(2,p)}\right)\right)$$

$$\pi_{2,m}^{(2)} = 2[S_{2,m}]_2, \pi_{k,m}^{(2k-4)} = (2k-4)[S_{k,m}]_k \ (k \neq 2)$$

$$\langle \mathcal{O}_{2,m} \rangle = \langle \mathcal{T}_{2,m} \rangle = \frac{N^2}{2\pi^2}\pi_{2,m}^{(2)}$$

$$\langle \mathcal{T}_{4,m} \rangle = \frac{N^2}{2\pi^2}\left(\pi_{4,m}^{(4)} + \frac{2\sqrt{3}}{z(4)}a_{(4m)(2n)(2p)}\pi_{2,n}^{(2)}\pi_{2,p}^{(2)}\right)$$

$$\langle \mathcal{O}_{4,m} \rangle = \frac{N^2}{2\pi^2}\pi_{4,m}^{(4)}$$

$$\langle \mathcal{T}_{4,m} \rangle = \langle \mathcal{O}_{4,m} \rangle + \frac{N^2}{2\pi^2}\frac{2\sqrt{3}}{z(4)}a_{(4m)(2n)(2p)}\pi_{2,n}^{(2)}\pi_{2,p}^{(2)}$$

$$\langle \mathcal{O}_{2,m}\mathcal{O}_{2,n} \rangle = \left(\frac{N^2}{2\pi^2}\right)^2 \pi_{2,m}^{(2)}\pi_{2,n}^{(2)} = \langle \mathcal{O}_{2,m} \rangle \langle \mathcal{O}_{2,n} \rangle$$

$$\langle \mathcal{O}_{2,m} \rangle = \frac{N^2}{2\pi^2}\frac{2\sqrt{8}}{3}[s_{(2,m)}]_2$$

$$\pi_{4,m}^{(4)} = 4[S_{4,m}]_4$$

$$[\mathcal{D}_\mu s_{(2,m)}\mathcal{D}^\mu s_{(2,n)}]_4 = 4[s_{(2,m)}s_{(2,n)}]_4$$

$$[S_{4,m}]_4 = \frac{2\sqrt{3}}{5}\left[s_{(4,m)} - \frac{37}{9z(4)}a_{(4m)(2n)(2p)}s_{(2,n)}s_{(2,p)}\right]_4$$

$$\langle \mathcal{O}_{4,m} \rangle = \frac{N^2}{2\pi^2}\frac{4\sqrt{3}}{5}\left[2s_{(4,m)} - \frac{74}{9z(4)}a_{(4m)(2n)(2p)}s_{(2,n)}s_{(2,p)}\right]_4$$

$$\langle \mathcal{T}_{4,m} \rangle = \frac{N^2}{2\pi^2}\frac{4\sqrt{3}}{5}\left[2s_{(4,m)} + \frac{2}{3z(4)}a_{(4m)(2n)(2p)}s_{(2,n)}s_{(2,p)}\right]_4$$

$$\langle \mathcal{O}_{2,m}\mathcal{O}_{2,n} \rangle = \frac{8N^4}{9\pi^4}[s_{2,m}]_2[s_{2,n}]_2$$

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G}\, d\tilde{\Omega}_3^2$$

$$h^{-2} = 2y\cosh\ G, z = \frac{1}{2}\tanh\ G$$

$$y\partial_y V_i = \epsilon_{ij}\partial_j z, y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z$$

$$F_5 = F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3$$

$$F = dB_t \wedge (dt + V) + B_t dV + d\hat{B}$$

$$\tilde{F} = d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\tilde{B}$$

$$B_t = -\frac{1}{4}y^2 e^{2G}, \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G}$$

$$d\hat{B} = -\frac{1}{4}y^3 *_3 d\left(\frac{z + \frac{1}{2}}{y^2}\right), d\tilde{B} = -\frac{1}{4}y^3 *_3 d\left(\frac{z - \frac{1}{2}}{y^2}\right)$$

$$z(x_1,x_2,y)=\frac{y^2}{\pi}\int_{R^2}\frac{z(x_1',x_2',0)dx_1'dx_2'}{((x-x')^2+y^2)^2}$$

$$V_i(x_1,x_2,y)=\frac{\epsilon_{ij}}{\pi}\int_{R^2}\frac{z(x_1',x_2',0)(x_j-x_j')dx_1'dx_2'}{((x-x')^2+y^2)^2}$$

$$r(\tilde{\phi})=r_1(\tilde{\phi}) \;=\sqrt{1+\alpha\cos(2\tilde{\phi})+\frac{\alpha^2}{2}\cos(4\phi)+O(\alpha^3)}$$

$$=1+\frac{\alpha}{2}\cos\,2\tilde{\phi}+\frac{\alpha^2}{16}(3\cos\,4\tilde{\phi}-1)+O(\alpha^3)$$

$$r(\tilde{\phi})=r_2(\tilde{\phi})=\sqrt{1+\alpha\cos(2\tilde{\phi})}$$

$$=1+\frac{\alpha}{2}\cos\,2\tilde{\phi}-\frac{\alpha^2}{16}(\cos\,4\tilde{\phi}+1)+O(\alpha^3)$$

$$g=g^{(0)}+\alpha g^{(1)}+\alpha^2 g^{(2)}.$$

$$y=R\cos\,\theta, r=\sqrt{R^2+1}\sin\,\theta, \tilde{\phi}=\phi-t,$$

$$ds^2\;=-(R^2+1)dt^2+\frac{dR^2}{R^2+1}+R^2d\widetilde{\Omega}_3^2+d\theta^2+\sin^2\,\theta d\phi^2+\cos^2\,\theta d\Omega_3^2$$

$$F_5\;=R^3dt\wedge dR\wedge d\widetilde{\Omega}_3+\cos^3\,\theta\sin\,\theta d\theta\wedge d\phi\wedge d\Omega_3$$

$$r(\tilde{\phi})=1+\frac{\alpha}{2}\cos(n\tilde{\phi})+O(\alpha^2),$$

$$D^a h_{(ab)}=D^a h_{a\mu}=0.$$

$$g_{\mu\nu}^{(1)}=\sum_{n=\pm2}\left(-\frac{6}{5}|n|\mathrm{s}_n\mathrm{Y}_n g_{\mu\nu}^{(0)}+\frac{4}{|n|+1}\mathrm{Y}_n\nabla_{(\mu}\nabla_{\nu)}\mathrm{s}_n\right), g_{\alpha\beta}^{(1)}=\sum_{n=\pm2}2|n|\mathrm{s}_n\mathrm{Y}_n g_{\alpha\beta}^{(0)}$$

$$A_4^{(1)}=\sum_{n=\pm2}(\mathrm{Y}_n\star AdS_5 d\,\mathrm{s}_n-\mathrm{s}_n\star{}_{S^5}d\mathrm{Y}_n)$$

$$\mathrm{s}_n=\frac{|n|+1}{8|n|(R^2+1)^{|n|/2}}e^{int}, \mathrm{Y}_n=e^{in\phi}\sin^{|n|}\,\theta$$

$$r_2(\tilde{\phi})-r_1(\tilde{\phi})=\frac{\beta}{2}\cos\,4\tilde{\phi}+O(\alpha^3), \beta=-\frac{\alpha^2}{2}$$

$$g_2^{(2)}-g_1^{(2)}=-\frac{1}{2}g_{(n=4)}^{(1)}$$

$$|\Phi\rangle=|0\rangle+\sum_n\delta_n O_n|0\rangle+O(\delta_n^2), \delta_n=\frac{N\alpha_n}{2\sqrt{n}}$$

$$|H\rangle=\mathcal{N}\left(|0\rangle+C\delta_2 O_2|0\rangle+B\frac{\delta_2^2}{2}(O_2)^2|0\rangle+A\delta_4 O_4|0\rangle+O(\delta_2^3,\delta_4^2)\right)$$

$$\delta_2=\frac{N\alpha}{2\sqrt{2}}, \delta_4=-\frac{N\alpha^2}{8}$$

$$\langle H|\mathcal{O}_2|H\rangle=\mathcal{N}_2 C\delta_2\langle O_2^\dagger O_2\rangle=\frac{N^2}{2\sqrt{2}\pi^2}C\alpha$$

$$\langle H|\mathcal{O}_4|H\rangle=\mathcal{N}_4 A\delta_4\langle O_4^\dagger O_4\rangle=-\frac{\sqrt{3}N^2}{4\pi^2}A\alpha^2$$

$$\langle H|(\mathcal{O}_2)^2|H\rangle=\mathcal{N}_2^2\frac{B\delta_2^2}{2}\left\langle\left(O_2^\dagger\right)^2(O_2)^2\right\rangle=\frac{N^4}{8\pi^4}B\alpha^2$$

$$\langle H|\mathcal{O}_{4,0}|H\rangle = \mathcal{N}_4 C^2 \delta_2^2 \langle O_2^\dagger O_{4,0} O_2\rangle = 0$$

$$\langle H|\mathcal{O}_{2,0}|H\rangle = \mathcal{N}_2 C^2 \delta_2^2 \langle O_2^\dagger O_{2,0} O_2\rangle = \frac{N^2\sqrt{2}}{4\sqrt{3}\pi^2} C^2 \alpha^2$$

$$s_{(2,2)} = \frac{3e^{-2it}}{8(R^2+1)}\alpha, s_{(2,-2)} = \frac{3e^{2it}}{8(R^2+1)}\alpha$$
$$s_{(2,0)} = \frac{\sqrt{3}(20R^4+57R^2+27)}{160(R^2+1)^3}\alpha^2$$
$$s_{(4,0)} = \frac{111R^2+55}{192\sqrt{5}(R^2+1)^3}\alpha^2$$
$$s_{(4,4)} = \frac{37e^{-4it}}{64(R^2+1)^2}\alpha^2, s_{(4,-4)} = \frac{37e^{4it}}{64(R^2+1)^2}\alpha^2$$

$$S_4 = 0, S_{4,0} = 0$$

$$g_{\mu\nu}^{(0)} + \tilde{h}_{\mu\nu}^0\, \tilde{h}_{\mu\nu}^0 = h_{\mu\nu}^0 + \tfrac{1}{3}\pi^0 g_{\mu\nu}^{(0)}$$

$$\alpha^2, g_{\mu\nu}^{(0)} + \tilde{h}_{\mu\nu}^0$$

$$\begin{aligned} ds_{(0)}^2 = -dt^2(R^2+1)&\left(1+\alpha^2\frac{24R^8+72R^6+77R^4+55R^2+24}{32(R^2+1)^4}\right)\\ &+\frac{dR^2}{R^2+1}\left(1-\alpha^2\frac{24R^6+76R^4+149R^2+15}{32(R^2+1)^4}\right)\\ &+R^2 d\Omega_3^2\left(1-\alpha^2\frac{72R^6+216R^4+199R^2+45}{96(R^2+1)^3}\right)\end{aligned}$$

$$R = \frac{1}{z} - \frac{z}{4} - \frac{3\alpha^2}{16}z - \frac{\alpha^2}{64}z^3 + \cdots$$

$$\langle\mathcal{O}_2\rangle = \frac{N^2}{2\sqrt{2}\pi^2}e^{-2it}\alpha, \langle\mathcal{O}_{2,0}\rangle = \frac{N^2\sqrt{2}}{4\sqrt{3}\pi^2}\alpha^2$$
$$\langle\mathcal{O}_4\rangle = 0, \langle\mathcal{O}_{4,0}\rangle = 0, \langle\mathcal{T}_4\rangle = \frac{N^2\sqrt{3}}{2\pi^2}e^{-4it}\alpha^2, \langle\mathcal{T}_{4,0}\rangle = \frac{N^2\sqrt{3}}{2\sqrt{5}\pi^2}\alpha^2$$

$$C = 1, B = 1, A = 0$$

$$|H\rangle = \mathcal{N}\left(|0\rangle + \delta_2 O_2|0\rangle + \frac{1}{2}\delta_2^2(O_2)^2|0\rangle + O(\delta_2^3)\right),$$

$$\delta_2 = \frac{N\alpha}{2\sqrt{2}}, \mathcal{N} = 1 - \frac{\delta_2^2}{2}$$

$$s_{(4,4)} = \frac{17e^{-4it}}{64(R^2+1)^2}\alpha^2, s_{(4,-4)} = \frac{17e^{4it}}{64(R^2+1)^2}\alpha^2$$

$$S_4 = -\frac{\sqrt{3}e^{-4it}}{8(R^2+1)^2}\alpha^2, S_{4,0} = 0$$

$$\langle\mathcal{O}_2\rangle = \frac{N^2}{2\sqrt{2}\pi^2}e^{-2it}\alpha, \langle\mathcal{O}_{2,0}\rangle = \frac{N^2\sqrt{2}}{4\sqrt{3}\pi^2}\alpha^2, \langle\mathcal{O}_{4,0}\rangle = 0$$
$$\langle\mathcal{O}_4\rangle = -\frac{\sqrt{3}N^2}{4\pi^2}e^{-4it}\alpha^2, \langle\mathcal{T}_4\rangle = \frac{N^2\sqrt{3}}{4\pi^2}e^{-4it}\alpha^2, \langle\mathcal{T}_{4,0}\rangle = \frac{N^2\sqrt{3}}{2\sqrt{5}\pi^2}\alpha^2$$

$$C = 1, B = 1, A = 1$$

$$|H\rangle = \mathcal{N}\left(|0\rangle + \delta_2 O_2|0\rangle + \frac{\delta_2^2}{2}(O_2)^2|0\rangle + \delta_4 O_4|0\rangle + O(\delta_2^3, \delta_4^2)\right),$$

$$\delta_2 = \frac{N\alpha}{2\sqrt{2}}, \delta_4 = -\frac{N\alpha^2}{8}, \mathcal{N} = 1 - \frac{\delta_2^2}{2}$$

$$s_{(4,2)} = \frac{e^{-2it}(119R^6 + 459R^4 + 477R^2 + 116)}{224\sqrt{10}(R^2+1)^5}\alpha^3$$
$$s_{(2,2)} = \frac{e^{-2it}(720R^8 + 2112R^6 + 2247R^4 + 836R^2 + 233)}{2560(R^2+1)^5}\alpha^3.$$

$$\langle \mathcal{O}_2\rangle = \frac{N^2}{2\sqrt{2}\pi^2}\left(\alpha + \frac{3}{4}\alpha^3\right)$$

$$S_{4,2} = -\sqrt{\frac{3}{10}}\frac{(2800R^6 + 10567R^4 + 8296R^2 + 2909)e^{-2it}}{11200(R^2+1)^5}\alpha^3$$

$$[S_{4,2}]_4 = -\frac{1}{4}\sqrt{\frac{3}{10}}e^{-2it}\alpha^3$$

$$\langle \mathcal{O}_{4,2}\rangle = -\frac{N^2}{2\pi^2}\sqrt{\frac{3}{10}}e^{-2it}\alpha^3$$

$$\langle H|\mathcal{O}_{4,2}|H\rangle = \mathcal{N}_4\delta_2\left(\frac{\delta_2^2}{2}\left\langle (O_2^\dagger)^2 O_{4,2}O_2\right\rangle + \delta_4\langle O_4^\dagger O_{4,2}O_2\rangle\right)$$

$$\left\langle (O_2^\dagger)^2 O_{4,2}O_2\right\rangle \langle O_4^\dagger O_{4,2}O_2\rangle$$

$$\langle O_4^\dagger O_{4,2}O_2\rangle O_4^\dagger$$

$$\langle O_4^\dagger O_{4,2}O_2\rangle = \langle T_4^\dagger O_{4,2}T_2\rangle - \frac{2}{N}\left\langle (O_2^\dagger)^2 O_{4,2}O_2\right\rangle$$

$$\langle T_4^\dagger O_{4,2}T_2\rangle = \langle T_4^\dagger T_{4,2}T_2\rangle - \frac{4\sqrt{3}}{\sqrt{10}N}\langle T_4^\dagger (T_{2,0}T_2)T_2\rangle$$

$$\langle O_4^\dagger O_{4,2}O_2\rangle = \langle T_4^\dagger T_{4,2}T_2\rangle = \frac{4}{\sqrt{5}N}$$

$$\langle H|\mathcal{O}_{4,2}|H\rangle = \mathcal{N}_4\frac{4\delta_2\delta_4}{\sqrt{5}N} = -\frac{N^2}{2\pi^2}\sqrt{\frac{3}{10}}\alpha^3$$

$$\hat{T}_I = C^I_{i_1\cdots i_k}\mathrm{Tr}\left(\Phi^{i_1}\cdots\Phi^{i_k}\right)$$

$$\langle C^I C^J\rangle \equiv C^I_{i_1\cdots i_k}C^J_{i_1\cdots i_k} = \delta^{IJ}$$

$$Y^I = C^I_{i_1\cdots i_k}x^{i_1}\cdots x^{i_k}\ x^{i_j} \in \mathbb{R}^6$$

$$\Box_{S^5} Y^I = \Lambda^I Y^I, \Lambda^I = -k(k+4), k = 0,1,2,\ldots$$

$$ds^2_{S^5} = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\Omega_3^2$$

$$\frac{1}{\omega_5}\int_{S^5} Y^{I_1}Y^{I_2} = z(k)\delta^{I_1I_2}, z(k) \equiv \frac{1}{2^{k-1}(k+1)(k+2)}$$

$$a_{I_1I_2I_3} \equiv \frac{1}{\omega_5}\int_{S^5} Y^{I_1}Y^{I_2}Y^{I_3}$$

$$\langle C^{I_1}C^{I_2}C^{I_3}\rangle \equiv C^{I_1}_{i_1\dots i_{\alpha_2}j_1\dots j_{\alpha_3}} C^{I_2}_{j_1\dots j_{\alpha_3}l_1\dots l_{\alpha_1}} C^{I_3}_{l_1\dots l_{\alpha_1}i_1\dots i_{\alpha_2}}$$

$$a_{I_1I_2I_3} = \frac{1}{\left(\frac{1}{2}\Sigma+2\right)!\,2^{\frac{1}{2}(\Sigma-2)}}\frac{k_1!\,k_2!\,k_3!}{\alpha_1!\,\alpha_2!\,\alpha_3!}\langle C^{I_1}C^{I_2}C^{I_3}\rangle,$$

$$\Sigma = k_1+k_2+k_3, \alpha_1 = \frac{1}{2}(k_2+k_3-k_1)$$

$$a_{(4,4)(2,-2)(2,-2)} = z(4), \qquad a_{(4,0)(2,0)(2,0)} = \frac{3z(4)}{2\sqrt{5}}$$

$$a_{(4,0)(2,2)(2,-2)} = \frac{z(4)}{2\sqrt{5}}, \qquad a_{(4,2)(2,0)(2,-2)} = z(4)\sqrt{\frac{3}{10}}$$

$$\langle H|O_{4,2}|H\rangle = \mathcal{N}_4\delta_2\left(\frac{\delta_2^2}{2}\left\langle\left(O_2^\dagger\right)^2O_{4,2}O_2\right\rangle + \delta_4\langle O_4^\dagger O_{4,2}O_2\rangle\right)$$

$$\left\langle\left(O_2^\dagger\right)^2O_{4,2}O_2\right\rangle \langle O_4^\dagger O_{4,2}O_2\rangle$$

$$\left\langle\left(O_2^\dagger\right)^2O_{4,2}O_2\right\rangle = \left\langle\left(T_2^\dagger\right)^2T_{4,2}T_2\right\rangle - \frac{4\sqrt{3}}{\sqrt{10N}}\left\langle\left(T_2^\dagger\right)^2(T_{2,0}T_2)T_2\right\rangle$$

$$\left\langle\left(T_2^\dagger\right)^2T_{4,2}T_2\right\rangle = \frac{1}{2\sqrt{5}N^5}\langle(\mathrm{Tr}\bar{Z}^2)^2\mathrm{Tr}Z^3\bar{Z}\mathrm{Tr}Z^2\rangle$$

$$\left\langle\left(T_2^\dagger\right)^2(T_{2,0}T_2)T_2\right\rangle = \frac{\sqrt{2}}{4\sqrt{3}N^5}\langle(\mathrm{Tr}\bar{Z}^2)^2(\mathrm{Tr}Z\bar{Z}\mathrm{Tr}Z^2)\mathrm{Tr}Z^2\rangle$$

$$\left\langle\left(T_2^\dagger\right)^2T_{4,2}T_2\right\rangle = \frac{16}{\sqrt{5}N^2} + O(N^{-4})$$

$$\left\langle\left(T_2^\dagger\right)^2(T_{2,0}T_2)T_2\right\rangle = \frac{4\sqrt{2}}{\sqrt{3}N} + O(N^{-4})$$

$$\left\langle\left(O_2^\dagger\right)^2O_{4,2}O_2\right\rangle = O(N^{-4})$$

$$\mathcal{N}_4\delta^3\left\langle\left(O_2^\dagger\right)^2O_{4,2}O_2\right\rangle = \alpha^3O(N^0)$$

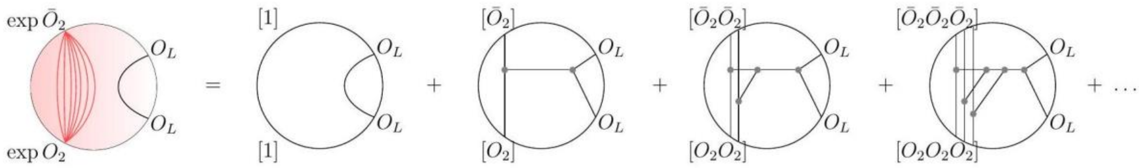
$$\langle T_4^\dagger(T_{2,0}T_2)T_2\rangle = \frac{\sqrt{2}}{4\sqrt{3}N^5}\langle\mathrm{Tr}\bar{Z}^4(\mathrm{Tr}Z\bar{Z}\mathrm{Tr}Z^2)\mathrm{Tr}Z^2\rangle$$

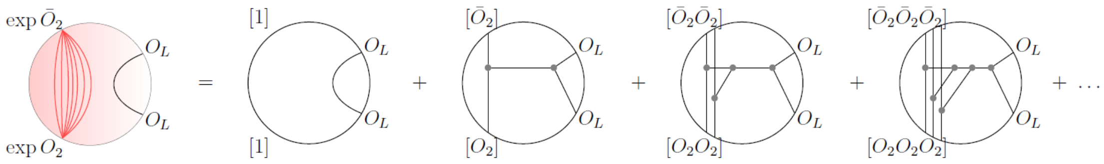
$$\langle\mathrm{Tr}\bar{Z}^4(\mathrm{Tr}Z\bar{Z}\mathrm{Tr}Z^2)\mathrm{Tr}Z^2\rangle = 8N^3 + O(N)$$

$$\frac{1}{N}\langle T_4^\dagger(T_{2,0}T_2)T_2\rangle = \frac{2\sqrt{2}}{\sqrt{3}N^3} + O(N^{-5})$$

$$\langle O_4^\dagger O_{4,2}O_2\rangle = \langle T_4^\dagger T_{4,2}T_2\rangle + O(N^{-3})$$

$$O_H = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha}{\sqrt{2}}\right)^n O_2^n = e^{\frac{\alpha}{\sqrt{2}}O_2}$$





$$O_H = \sum_{n=0}^{2N} \frac{1}{n!}\left(\frac{\alpha}{2\sqrt{2}}\right)^n O_1^n,$$

$$\Phi_B(z,\bar{z};\alpha) = \langle O_H(0)\bar{O}_H(\infty)O_L(1)O_L(z,\bar{z})\rangle$$

$$\Phi_B(z,\bar{z};\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!}\Phi_B^{(n)}(z,\bar{z}), \text{ and } \Phi_B^{(n)}(z,\bar{z}) \sim \langle O^n\bar{O}^n O_L O_L\rangle^c.$$

$$\langle O_2^2\bar{O}_2^2\bar{O}_2 O_2\rangle^c \ ; \ \langle O_2^3\bar{O}_2^3\bar{O}_2 O_2\rangle^c$$

$$\Psi_B(z,\bar{z};\alpha) = \sum_{l=0}^{\infty} \int_{\mathbb{R}} \frac{d\omega}{2\pi i}\mu_l^{(d)}\chi_l^{(d)}(z,\bar{z})E_{\text{gravity}}(\omega,l;\alpha)$$

$$\mu_l^{(4)} = \frac{l+1}{2} \ ; \ \chi_l^{(4)}(z,\bar{z}) = (z\bar{z})^{\frac{\omega-l}{2}}\left(\frac{z^{l+1}-\bar{z}^{l+1}}{z-\bar{z}}\right)$$

$$E_{\text{gravity}}(\omega,l,\alpha) = \sum_{k=1}^{\infty}\left(\frac{1}{k+\frac{l}{2}+a(\omega,l;\alpha)} + \frac{1}{k+\frac{l}{2}-a(\omega,l;\alpha)}\right)$$

$$E_{\text{ladder}}(\omega,l) = \frac{1}{\left(\frac{1+l+\omega}{2}\right)^{n+1}\left(\frac{1+l-\omega}{2}\right)^{n+1}}$$

$$\log^{m_1}(z\bar{z})\sum_l \frac{1}{(l+1)^{m_2}}\left(\frac{z^{l+1}-\bar{z}^{l+1}}{z-\bar{z}}\right) = \log^{m_1}(z\bar{z})\frac{\text{Li}_{m_2}(z)-\text{Li}_{m_2}(\bar{z})}{z-\bar{z}}$$

$$\log^{m_1}(z\bar{z})\sum_{k,l}(z\bar{z})^k\frac{p_{m_1,m_2}(k,l)}{(l+2k+m_3)^{m_2}}\left(\frac{z^{l+1}-\bar{z}^{l+1}}{z-\bar{z}}\right)$$

$$\Psi_B(z,\bar{z})|_{\alpha^2} = -4V^2\bar{D}_{2422}$$

$$\bar{D}_{2422} = (3+V\partial_V+U\partial_U)\partial_V\partial_U\left[\frac{2(\text{Li}_2(z)-\text{Li}_2(\bar{z}))-\log(z\bar{z})(\text{Li}_1(z)-\text{Li}_1(\bar{z}))}{z-\bar{z}}\right]$$

$$O = \sum_{r=1}^{N}\psi_{(r)}^{+A}\tilde{\psi}_{(r)}^{+B}\epsilon_{AB} \equiv \sum_{r=1}^{N} O_{(r)}$$

$$O' = \sum_{r=1}^{N}\psi_{(r)}^{+1}\tilde{\psi}_{(r)}^{+1}$$

$$O^2 = 2\left[\sum_{r<s}\psi_{(r)}^{+A}\tilde{\psi}_{(r)}^{+B}\epsilon_{AB}\psi_{(s)}^{+C}\tilde{\psi}_{(s)}^{+D}\epsilon_{CD} - \sum_r \psi_{(r)}^{+1}\psi_{(r)}^{+2}\tilde{\psi}_{(r)}^{+1}\tilde{\psi}_{(r)}^{+2}\right].$$

$$:O^2:=\sum_{r<s}\psi^{+A}_{(r)}\tilde{\psi}^{+B}_{(r)}\epsilon_{AB}\psi^{+C}_{(s)}\tilde{\psi}^{+D}_{(s)}\epsilon_{CD}$$

$$:O_H:=\sum_{n=0}^{N}\sqrt{\binom{N}{n}}\left(\frac{\alpha}{2}\right)^n\left(1-\frac{\alpha^2}{4}\right)^{\frac{N-n}{2}}[:O^n:]=\sum_{n=0}^{N}\left(\frac{\alpha}{2\sqrt{2}}\right)^n\left(1-\frac{\alpha^2}{4}\right)^{\frac{N-n}{2}}:O^n:,$$

$$[:O^n:]=\left(2^n\binom{N}{n}\right)^{-\frac{1}{2}}:O^n:$$

$$O_H=\sum_{n=0}^{2N}b_n\left(\frac{\alpha}{2}\right)^n N^{\frac{n}{2}}[O^n]$$

$$b_n=\frac{1}{\sqrt{n!}}\left(\frac{(2N)!}{(2N-n)!\,(2N)^n}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{n!}}+O(N^{-1})$$

$$[O^n]=\left(n!\frac{(2N)!}{(2N-n)!}\right)^{-\frac{1}{2}}O^n$$

$$O_H=\sum_{n=0}^{2N}\frac{1}{n!}\left(\frac{\alpha}{2\sqrt{2}}\right)^n O^n$$

$$ds_3^2=R_{\mathrm{AdS}}^2\left[-\Omega_1^2\left(dt+\frac{k}{(1-\xi^2)}d\psi\right)^2+\frac{\Omega_0^2}{(1-\xi^2)^2}(d\xi^2+\xi^2d\psi^2)\right]$$

$$\Omega_0=\sqrt{1-\frac{\alpha^2}{4}(1-\xi^2)},k=\frac{\xi^2}{\Omega_1},\Omega_1=1-\frac{\alpha^2}{4}$$

$$\Omega_0=1-\frac{8\alpha^2(1-\xi^2)}{64-\alpha^4\xi^2},k=\frac{\xi^2}{\Omega_1}\left(1-\frac{\alpha^4(1-\xi^2)}{64-\alpha^4\xi^2}\right),\Omega_1=\frac{8-\alpha^2}{8+\alpha^2}$$

$$\frac{\langle :O_H:|J|:O_H:\rangle}{\langle :O_H:|:O_H:\rangle}=\frac{N}{2}\frac{\alpha^2}{4},\qquad \frac{\langle O_H|J|O_H\rangle}{\langle O_H\mid O_H\rangle}=\frac{N\alpha^2}{8+\alpha^2}$$

$$\frac{\langle :O_H:|[O]|:O_H:\rangle}{\langle :O_H:|:O_H:\rangle}=\frac{\sqrt{N}}{\sqrt{2}}\frac{\alpha}{\sqrt{2}}\sqrt{1-\frac{\alpha^2}{4}},\quad \frac{\langle O_H|[O]|O_H\rangle}{\langle O_H\mid O_H\rangle}=\frac{\sqrt{N}}{\sqrt{2}}\frac{4\sqrt{2}\alpha}{8+\alpha^2}$$

$$O\leftrightarrow\frac{\sqrt{N}}{\sqrt{2}}s_1^{(7)}$$

$$\Box_3\,\Phi_0=0$$

$$\Phi_0=\sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}e^{il\theta}\phi(\xi)$$

$$\Phi_0\approx\delta(\tau)\delta(\theta)+w^2\Phi_B(\tau,\theta)$$

$$ds_3^2=\frac{dw^2}{w^2}+\frac{1}{w^2}\left(-d\tau^2+d\theta^2+O(w^2)\right)$$

$$\Phi_B(\tau,\theta)=\frac{1}{4}\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\theta^2}\right)\Psi_B(\tau,\theta)$$

$$\phi''(\xi)+\frac{\phi'(\xi)}{\xi}-\frac{(l^2(\alpha^2(\xi^2-1)+4)-4\xi^2\omega^2)}{(\alpha^2-4)\xi^2(\xi^2-1)}\phi(\xi)=0$$

$$\phi(\xi)\sim\xi^{|l|}\,{}_2F_1\left(\frac{|l|}{2}-\frac{\gamma}{2},\frac{|l|}{2}+\frac{\gamma}{2},|l|+1;\xi^2\right)$$

$$\gamma = \sqrt{\frac{\omega^2 - \frac{\alpha^2 l^2}{4}}{1 - \frac{\alpha^2}{4}}}$$

$$\xi^2 \approx 1 - \left(1 - \frac{\alpha^2}{4}\right) w^2 \to 1$$

$$\xi^{|l|}\,{}_2F_1\left(\hat{a}, \hat{b}, \hat{a} + \hat{b} + 1; \xi^2\right) \approx \frac{\Gamma(\hat{a} + \hat{b} + 1)}{\Gamma(\hat{a} + 1)\Gamma(\hat{b} + 1)}\left[\left(1 + \hat{a}\hat{b}(1 - \xi^2)\ln(1 - \xi^2) + \cdots\right)\right.$$
$$\left. + \hat{a}\hat{b}(1 - \xi^2)\left(H_{\hat{a}} + H_{\hat{b}} - 1 - \frac{|l|}{2\hat{a}\hat{b}} + \cdots\right)\right]$$

$$H_{\hat{a}} = \psi^{(0)}(\hat{a} + 1) + \gamma_E$$

$$H_{\hat{a}} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k + \hat{a}}\right)$$

$$\hat{a} = \frac{|l| + \gamma}{2}, \hat{b} = \frac{|l| - \gamma}{2}$$

$$\Psi_B = \sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}e^{il\theta}\left(\frac{1 - \frac{\alpha^2}{4}}{\frac{l^2 - \omega^2}{4}}\hat{a}\hat{b}\right)\left[\sum_{k=1}^{\infty}\left(\frac{1}{k + \hat{a}} + \frac{1}{k + \hat{b}}\right)\right]$$
$$= \sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}e^{il\theta}\sum_{k=1}^{\infty}\left[\frac{2}{\gamma + 2k + |l|} - \frac{2}{\gamma - 2k - |l|}\right]$$

$$\hat{a}\hat{b} = \frac{l^2}{4} - \frac{\gamma^2}{4} = \frac{l^2 - \omega^2}{4\left(1 - \frac{\alpha^2}{4}\right)}$$

$$\Psi_B(z, \bar{z}) = \langle[:O_H:](0)[:\bar{O}_H:](\infty)[\bar{O}](1)[O](z, \bar{z})\rangle$$

$$\Psi_B = \left(1 - \frac{\alpha^2}{4}\right)\sum_{l=-\infty}^{\infty}e^{il\theta}\sum_{k=1}^{\infty}\frac{\exp\left[-i\sqrt{\left(1 - \frac{\alpha^2}{4}\right)(|l| + 2k)^2 + \frac{\alpha^2 l^2}{4}}\,t\right]}{\sqrt{1 + \frac{\frac{\alpha^2}{4}}{1 - \frac{\alpha^2}{4}}\frac{l^2}{(|l| + 2k)^2}}}$$

$$\Psi_B = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha}{2}\right)^{2n}\Psi_B^{(n)}$$

$$x = \xi^2, \phi(\xi) = x^{-\frac{1}{2}}\tilde{\psi}(x)$$

$$\tilde{\psi}''(x) - \left[\frac{(l^2 - 1)}{4x^2} + \frac{(\alpha^2 + 8)^2(l^2 - \omega^2)}{4(x - 1)x(\alpha^4 x - 64)}\right]\tilde{\psi}(x) = 0.$$

$$\psi''(\hat{z}) + \left[(l^2 - \omega^2)\left(\frac{(\alpha^2 + 8)}{4(\alpha^2 - 8)(\hat{z} - 1)} - \frac{(\alpha^2 + 8)^2}{4\alpha^4\hat{z}} + \frac{1024(\alpha^2 + 8)}{\alpha^4(\alpha^2 - 8)(\alpha^4 - 64\hat{z})}\right) - \frac{l^2 - 1}{4\hat{z}^2}\right]\psi(\hat{z}) = 0$$

$$\hat{a} + \hat{b} = |l|, \hat{a}\hat{b} = \frac{l^2}{4} - a^2 \Leftrightarrow \hat{a} = -a + \frac{|l|}{2}, \hat{b} = a + \frac{|l|}{2}$$

$$\xi^2 \approx 1 - \frac{8 - \alpha^2}{8 + \alpha^2}w^2 \to 1 \Rightarrow \hat{z} \approx \frac{\alpha^4}{64}\left(1 - \frac{8 - \alpha^2}{8 + \alpha^2}w^2\right)$$

$$\Psi_B = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} e^{il\theta} \left(\frac{e^{-\partial_{a_t} F \frac{8-\alpha^2}{8+\alpha^2}}}{\frac{l^2-\omega^2}{4}} \hat{a}\hat{b} \right) \left[\sum_{k=1}^{\infty} \left(\frac{1}{k+\hat{a}} + \frac{1}{k+\hat{b}} \right) \right]$$

$$= \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} e^{il\theta} \left[\sum_{k=1}^{\infty} \left(\frac{1}{k+\hat{a}} + \frac{1}{k+\hat{b}} \right) \right]$$

$$\Psi_B(z,\bar{z}) = \langle [O_H](0)[\bar{O}_H](\infty)[\bar{O}](1)[O](z,\bar{z}) \rangle$$

$$\Psi_B^{(n)}(z,\bar{z}) = N^n \langle [O^n](0)[\bar{O}^n](\infty)[\bar{O}](1)[O](z,\bar{z}) \rangle^c$$

$$\langle [:O^2:](1)[:\bar{O}^2:](2)[\bar{O}](3)[O](4) \rangle^c = \langle [:O:]^2(1)[:\bar{O}^2:](2)[\bar{O}](3)[O](4) \rangle$$
$$-2\langle [O](1)[\bar{O}](2) \rangle \langle [O](1)[\bar{O}](2)[\bar{O}](3)[O](4) \rangle + (\langle [O](1)[\bar{O}](2)] \rangle)^2 \langle [\bar{O}](3)[O](4) \rangle$$

$$\langle O^2(1)\bar{O}^2(2)\bar{O}(3)O(4) \rangle^c = \langle O^2(1)\bar{O}^2(2)\bar{O}(3)O(4) \rangle$$
$$-4\langle O(1)\bar{O}(2) \rangle \langle O(1)\bar{O}(2)\bar{O}(3)O(4) \rangle + 4(\langle O(1)\bar{O}(2) \rangle)^2 \langle \bar{O}(3)O(4) \rangle$$

$$\Psi_B^{(1)}(z,\bar{z}) = -\frac{1}{U} + \frac{V}{U}\bar{D}_{1122}, \text{ with } U = (1-z)(1-\bar{z}), V = z\bar{z}$$

$$D^{(n)} \equiv \frac{\mathcal{P}_n}{z-\bar{z}}$$

$$\Psi_B^{(2)}(z,\bar{z}) = -2V\left[\partial_U D^{(2)} - (1+V-U)\partial_U^2 D^{(2)} - \partial_U D^{(1)}\right]$$

$$\delta\Psi_B^{(2)} \equiv \Psi_B^{(2)}(2.12) - \Psi_B^{(2)}(2.13) = \left[\frac{V}{U}\bar{D}_{1133} + \frac{1}{VU}\right]$$

$$\Psi_B^{(3)} = -6V\left[\partial_U D^{(3)} - 3(1+V-U)\partial_U^2 D^{(3)} + ((1+V-U)^2 + 2V)\partial_U^3 D^{(3)}\right.$$
$$\left. -\partial_U D^{(2)} + (1+V-U)\partial_U^2 D^{(2)}\right]$$

$$\mathcal{M}_{[1^2][1^2]11}(s,t) = \left[K_2(u,t)A(u,t) + \left(\psi^{(0)}(-u) - \psi^{(0)}(-t)\right)B(u,t) + C(u,t)\right]$$

$$A(u,t) = \frac{1 - \frac{2ut}{s+2}}{s+1}, B(u,t) = \frac{2(u-t)}{(s+1)(s+2)}, C(u,t) = \frac{2s}{(s+1)(s+2)}$$

$$\mathcal{M}_{[1^3][1^3]11}(s,t) = [K_4(u,t)a(u,t) + K_3(u,t)b(u,t) + \tilde{K}_2(u,t)c(u,t) +$$
$$+K_2(u,t)A(u,t) + K_1(u,t)B(u,t) + C(u,t)]$$

$$a(u,t) = \frac{4(s-t)(-1+s+t+3st)}{(-1+s+t)(s+t)(1+s+t)}$$
$$b(u,t) = \frac{4(s-t)(3st+s+t-1)}{(s+t-1)(s+t)(s+t+1)}, c(u,t) = \frac{6(s^2-4st-s+t^2-t)}{(s+t-1)(s+t)(s+t+1)}$$
$$A(u,t) = \frac{3(2s^2t - 3s^2 + 2st^2 + 16st + s - 3t^2 + t - 4)}{(s+t-1)(s+t)(s+t+1)}$$
$$B(u,t) = -\frac{6(s-t)(s+t+5)}{(s+t-1)(s+t)(s+t+1)}, C(u,t) = \frac{12(s+t+2)}{(s+t-1)(s+t)(s+t+1)}$$

$$T_p(x,y) = \mathrm{Tr}(y_I \phi^I(x))^p$$

$$\mathcal{O}_p = T_p + \sum_{q_1+\cdots+q_n=p} C_p^{q_1\ldots q_n} T_{q_1} \ldots T_{q_n}$$

$$\mathcal{O}_4 = T_4 - \frac{2N^2-3}{N(N^2+1)} T_2^2$$

$$ds_5^2 = -H_1^{-2/3} f dt^2 + H_1^{1/3}(f^{-1}dr^2 + r^2 d\Omega_3^2)$$

$$f = 1 + r^2 H_1 , H_1 = \sqrt{1 + \frac{2(1+q_1)}{r^2} + \frac{1}{r^4}} - \frac{1}{r^2}$$

$$q_1 = \cosh \epsilon - 1$$

$$O_2 \equiv \frac{1}{2}\mathrm{Tr}(\phi_1 + i\phi_2)^2 = \mathrm{Tr}(X^2)$$

$$[\mathcal{O}_2^n] = \mathcal{N}_n^{-1}\left(\frac{O_2}{\sqrt{2a}}\right)^n , \mathcal{N}_n = \sqrt{n!}\left[\left(\frac{a}{2}\right)_n \left(\frac{a}{2}\right)^{-n}\right]^{1/2}$$

$$a \equiv N^2 - 1$$

$$O_H = \sum_{n=0}^{\infty} b_n \alpha^n a^{n/2} [\mathcal{O}_2^n]$$

$$\alpha = \sum_{k=1}^{\infty} c_k \epsilon^k$$

$$\langle J\rangle_{\text{sugra}} = a\sinh^2\left(\frac{\epsilon}{2}\right) , \langle [O_2]\rangle_{\text{sugra}} = c_{\mathcal{O}_2} \sinh \epsilon$$

$$\langle J\rangle \equiv \frac{\langle O_H | J | O_H\rangle}{\langle O_H \mid O_H\rangle} = \alpha \partial_\alpha \log\left(\sum_{n=0} b_n^2 a^n \alpha^{2n}\right)$$

$$\langle [O_2]\rangle \equiv \frac{\langle O_H | [O_2] | O_H\rangle}{\langle O_H \mid O_H\rangle} = \frac{\sum_{n=0}^{\infty} b_{n+1} b_n \mathcal{N}_{n+1} \mathcal{N}_n^{-1} a^{n+\frac{1}{2}} \alpha^{2n+1}}{\sum_{n=0}^{\infty} b_n^2 a^n \alpha^{2n}},$$

$$b_n = \frac{\mathcal{N}_n}{n!} = \frac{1}{\sqrt{n!}}\left[\left(\frac{a}{2}\right)_n \left(\frac{a}{2}\right)^{-n}\right]^{1/2} , \alpha = \frac{1}{\sqrt{2}}\tanh\left(\frac{\epsilon}{2}\right) , c_{\mathcal{O}_2} = \frac{\sqrt{a}}{2\sqrt{2}}$$

$$O_H = \sum_{n=0}^{\infty} \frac{\mathcal{N}_n}{n!} \alpha^n a^{n/2} [O_2^n] = e^{\frac{\alpha}{\sqrt{2}} O_2} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(1 + O(a^{-1})\right) \alpha^n a^{n/2} [O_2^n]$$

$$\square_5 \Phi = 0$$

$$O_L = \bar{Q}^4 O_2$$

$$ds_5^2 = \frac{dw^2}{w^2} + \frac{1}{w^2}\left(-dt^2 + d\Omega_3^2 + O(w^2)\right)$$

$$d\Omega_3^2 = d\theta^2 + \sin^2 \theta d\Omega_2^2$$

$$\Phi \approx \delta_N + w^4 \Phi_B(t,\theta)$$

$$\Phi_B(t,\theta) = \langle O_H(t=-\infty) \bar{O}_H(t=\infty) O_L(t=0,\theta=0) O_L(t,\theta)\rangle$$

$$Y_l(\theta) = \frac{1}{\sqrt{2}\pi} \frac{\sin\left[(l+1)\theta\right]}{\sin \theta} \ (l = 0,1,\dots,\infty)$$

$$\int d\Omega_3 Y_l Y_l' = \delta_{l,l'}$$

$$\Phi = \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} Y_l(\theta) \phi(r)$$

$$\delta_N = \sum_{l=0}^{\infty} \frac{l+1}{\sqrt{2}\pi} Y_l(\theta)$$

$$r^{-3}\partial_r(r^3 f\partial_r\phi(r)) + \left(\frac{H_1}{f}\omega^2 - \frac{l(l+2)}{r^2}\right)\phi(r) = 0$$

$$q_1 = \frac{4\alpha^2}{1-2\alpha^2}$$

$$r^2 = \frac{\sqrt{2}\alpha\left(x+\frac{1}{x}\right) - (2\alpha^2+1)}{1-2\alpha^2}, \phi(r) = x^{\frac{1}{2}}\tilde{\psi}(x)$$

$$x = \frac{2\alpha^2 + (1-2\alpha^2)\hat{z}}{\sqrt{2}\alpha}, \tilde{\psi}(x) = \hat{z}^{-1}(1-\hat{z})^{-1}\psi(\hat{z})$$

$$\left(\partial_{\hat{z}}^2 + \frac{\frac{1}{4}-a_1^2}{(\hat{z}-1)^2} - \frac{\frac{1}{2}-a_0^2-a_1^2-a_t^2+a_\infty^2+u}{\hat{z}(\hat{z}-1)} + \frac{\frac{1}{4}-a_t^2}{\left(\hat{z}+\frac{q_1}{2}\right)^2} + \frac{u}{\hat{z}\left(\hat{z}+\frac{q_1}{2}\right)} + \frac{\frac{1}{4}-a_0^2}{\hat{z}^2}\right)\psi(\hat{z}) = 0$$

$$a_0^2 = \frac{1}{4}(l+1)^2, a_1^2 = \frac{1}{4}(-l(l+2)+2\omega^2+1), a_t^2 = 1, a_\infty^2 = 1$$
$$u = \frac{1}{4}(2\alpha^2\omega^2 + 8\alpha^2 - (2\alpha^2-1)l(l+2) - \omega^2 + 4)$$

$$\hat{z} = -\frac{2\alpha^2}{1-2\alpha^2} = -\frac{q_1}{2}$$

$$\hat{a}+\hat{b} = l, \hat{a}\hat{b} = \frac{l^2}{4} - a^2 \Leftrightarrow \hat{a} = -a + \frac{l}{2}, \hat{b} = a + \frac{l}{2}.$$

$$\Phi_B(t,\theta) = \frac{\pi}{12\sqrt{2}}\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}e^{i\omega t}Y_l(\theta)(l+1)\left[\frac{e^{-\partial_{a_t}F}}{(1-2\alpha^2)^2}\hat{a}\hat{b}(\hat{a}+1)(\hat{b}+1)\right]\left[\frac{1}{k+\hat{a}} + \frac{1}{k+\hat{b}}\right],$$

$$\left[\frac{e^{-\partial_{a_t}F}}{(1-2\alpha^2)^2}\hat{a}\hat{b}(\hat{a}+1)(\hat{b}+1)\right] = \frac{1}{16}(l^2-\omega^2)((l+2)^2-\omega^2).$$

$$z = e^{i(t+\theta)}, \bar{z} = e^{i(t-\theta)}, U = (1-z)(1-\bar{z}), V = z\bar{z}$$

$$\mathcal{D} \equiv U\partial_U^2 + V\partial_V^2 + (U+V-1)\partial_U\partial_V + 2\partial_U + 2\partial_V.$$

$$\mathcal{D} = \frac{1}{z-\bar{z}}\partial_z\partial_{\bar{z}}(z-\bar{z})$$

$$\mathcal{D}^2\left(e^{i\omega t}Y_l(\theta)\right) = \frac{e^{-4it}}{16}(l^2-\omega^2)((l+2)^2-\omega^2)e^{i\omega t}Y_l(\theta).$$

$$\Phi_B(z,\bar{z}) \equiv e^{-4it}\Phi_B(t,\theta) = \frac{1}{12}\mathcal{D}^2\Psi_B(z,\bar{z}),$$

$$\Psi_B(z,\bar{z}) = \frac{\pi}{\sqrt{2}}\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}e^{i\omega t}Y_l(\theta)(l+1)\left[\frac{1}{k+\hat{a}} + \frac{1}{k+\hat{b}}\right]$$
$$= \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}\frac{l+1}{2}(z\bar{z})^{\frac{\omega-l}{2}}\frac{z^{l+1}-\bar{z}^{l+1}}{z-\bar{z}}\left[\frac{1}{k+\frac{l}{2}+a(\omega,l;\alpha)} + \frac{1}{k+\frac{l}{2}-a(\omega,l;\alpha)}\right]$$

$$\langle O_H(z=0)\bar{O}_H(z\to\infty)O_L(z=1)O_L(z,\bar{z})\rangle = \frac{1}{12}\mathcal{D}^2\langle O_H(z=0)\bar{O}_H(z\to\infty)\bar{O}_2(z=1)O_2(z,\bar{z})\rangle$$

$$a(\omega,l;\alpha) = \frac{\omega}{2} + \gamma_1(\omega,l)\alpha^2 + \gamma_2(\omega,l)\alpha^4 + \cdots$$

$$\gamma_1(\omega,l)=\frac{(l^2-\omega^2)((l+2)^2-\omega^2)}{4(\omega^2-1)\omega}$$

$$\gamma_2(\omega,l)=\frac{4}{3}\gamma_1(\omega,l)\left[\frac{3l(l+2)(2\omega^2-l(l+2))(15\omega^4-35\omega^2+8)}{32(\omega^2-4)(\omega^2-1)^2\omega^2}-\frac{29\omega^4-73\omega^2+8}{32(\omega^2-1)^2}+1\right]$$

$$\Psi_B(z,\bar{z};\alpha)=\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{n!}\Psi_B^{(n)}(z,\bar{z})$$

$$E_{\text{gravity}}(\omega,l;\alpha)=\frac{1}{k+\frac{l}{2}+a(\omega,l;\alpha)}+\frac{1}{k+\frac{l}{2}-a(\omega,l;\alpha)}$$

$$E_{\text{gravity}}(\omega,l;\alpha)\Big|_{\alpha^{2n}}=\frac{(+\gamma_1(\omega,l))^n}{\left(k+\frac{l-\omega}{2}\right)^{n+1}}+\frac{(-\gamma_1(\omega,l))^n}{\left(k+\frac{l+\omega}{2}\right)^{n+1}}+\cdots$$

$$\Psi_B^{(n)}(z,\bar{z})\Big|_{\alpha^{2n}}=\log^n(z\bar{z})\sum_{l\geq 0}\sum_{k\geq 1}(l+1)\left[\frac{-4(k-1)k(k+l)(k+l+1)}{(-1+2k+l)(2k+l)(1+2k+l)}\right]^n(z\bar{z})^k\frac{z^{l+1}-\bar{z}^{l+1}}{z-\bar{z}}+\cdots$$

$$\Psi_B^{(0)}(z,\bar{z})=\frac{V}{U^2}$$

$$\Psi_B(z,\bar{z})=-\frac{1}{4\sin\,\theta}\partial_\theta\sum_{l\in\mathbb{Z}}^{\infty}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}e^{i\omega t+il\theta}\left[\frac{1}{k+\frac{|l|-1}{2}+a(\omega,|l|-1;\alpha)}+\frac{1}{k+\frac{|l|-1}{2}-a(\omega,|l|-1;\alpha)}\right]$$

$$\sum_{l\in\mathbb{Z}}^{\infty}\sum_{k=1}^{\infty}e^{-i(2k+|l|)t+il\theta}\frac{l^{2p}}{(2k+|l|\pm q)^n}$$

$$\Psi_B^{(1)}(z,\bar{z})=-4V^2\bar{D}_{2422}$$

$$\Psi_B(z,\bar{z})=\langle[O_H](0)[\bar{O}_H](\infty)[\bar{O}_2](1)[O_2](z,\bar{z})\rangle-\langle[O_H](0)[\bar{O}_H](\infty)[\bar{O}_2](1)\rangle\langle[O_H](0)[\bar{O}_H](\infty)[O_2](z,\bar{z})\rangle,$$

$$\Psi_B^{(0)}(z,\bar{z})=\frac{1}{U^2}\,,\Psi_B^{(n)}(z,\bar{z})=a^n\langle[O_2^n](0)[\bar{O}_2^n](\infty)[\bar{O}_2](1)[O_2](z,\bar{z})\rangle^c$$

$$\langle[O_2^n](0)[\bar{O}_2^n](\infty)[\bar{O}_2](1)[O_2](z,\bar{z})\rangle^c$$

$$\begin{aligned}
\langle\mathcal{O}_2(1)\mathcal{O}_2(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle^c &= \langle\mathcal{O}_2(1)\mathcal{O}_2(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle+\cdots\\
\langle\mathcal{O}_2^2(1)\mathcal{O}_2^2(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle^c &= \langle\mathcal{O}_2^2(1)\mathcal{O}_2^2(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle-4\langle\mathcal{O}_2(1)\mathcal{O}_2(2)\rangle\langle\mathcal{O}_2(1)\mathcal{O}_2(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle^c+\cdots\\
\langle\mathcal{O}_2^3(1)\mathcal{O}_2^3(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle^c &= \langle\mathcal{O}_2^3(1)\mathcal{O}_2^3(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle-9\langle\mathcal{O}_2(1)\mathcal{O}_2(2)\rangle\langle\mathcal{O}_2^2(1)\mathcal{O}_2^2(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle^c\\
&\quad -9\langle\mathcal{O}_2^2(1)\mathcal{O}_2^2(2)\rangle\langle\mathcal{O}_2(1)\mathcal{O}_2(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle^c+\cdots
\end{aligned}$$

$$\langle[O_2^n](0)[\bar{O}_2^n](\infty)[\bar{O}_2](1)[O_2](z,\bar{z})\rangle a^{-n},\Psi_B^{(k)}(z,\bar{z})$$

$$\langle[O_2](0)[\bar{O}_2](\infty)[\bar{O}_2](1)[O_2](z,\bar{z})\rangle^c=-\frac{4}{a}V^2\bar{D}_{2422}$$

$$\langle[O_2][\bar{O}_2][\bar{O}_2][O_2]\rangle\Psi_B^{(2)}(z,\bar{z})\langle[O_2^2][O_2^2][\bar{O}_2][O_2]\rangle 1/a^2\langle[O_2^3][O_2^3][\bar{O}_2][O_2]\rangle\langle[O_2][\bar{O}_2][\bar{O}_2][O_2]\rangle\langle[O_2^2][\bar{O}_2]^2[\bar{O}_2][O_2]\rangle$$

$$\left\langle\mathcal{O}_{p_1}(\vec{x}_1,\vec{y}_1)\mathcal{O}_{p_2}(\vec{x}_2,\vec{y}_2)\mathcal{O}_{p_3}(\vec{x}_3,\vec{y}_3)\mathcal{O}_{p_4}(\vec{x}_4,\vec{y}_4)\right\rangle=\mathcal{P}_{\vec{p}}\hat{\mathcal{C}}_{\vec{p}}(U,V,\sigma,\tau)$$

$$U=\frac{\vec{x}_{12}^2\vec{x}_{34}^2}{\vec{x}_{13}^2\vec{x}_{24}^2}\ ;\ V=\frac{\vec{x}_{14}^2\vec{x}_{23}^2}{\vec{x}_{13}^2\vec{x}_{24}^2}\ ;\ \sigma=\frac{\vec{y}_{13}^2\vec{y}_{24}^2}{\vec{y}_{12}^2\vec{y}_{34}^2}\ ;\ \tau=\frac{\vec{y}_{14}^2\vec{y}_{23}^2}{\vec{y}_{12}^2\vec{y}_{34}^2}$$

$$\hat{\mathcal{C}}_{\vec{p}}=\mathcal{G}_{\vec{p}}^{\text{free}}(U,V,\sigma,\tau)+\mathcal{I}(U,V,\sigma,\tau)\mathcal{H}_{\vec{p}}(U,V,\sigma,\tau),$$

$$\mathcal{I}=V+\sigma V(V-1-U)+\tau(1-U-V)+\sigma\tau U(U-1-V)+\sigma^2UV+\tau^2U$$

$$\kappa = \min(p_1 + p_2, p_3 + p_4) - \max(p_{43}, p_{12}) - 2$$

$$\langle \mathcal{O}_2^n \mathcal{O}_2^n \mathcal{O}_2 \mathcal{O}_2 \rangle$$

$$\mathcal{H}_{2222} = \mathcal{H}_{2222}^c$$
$$\mathcal{H}_{[2^2][2^2]22} = \mathcal{H}_{[2^2][2^2]22}^c + 2^2 \frac{\langle \mathcal{O}_2 \mathcal{O}_2 \rangle \langle \mathcal{O}_2 \mathcal{O}_2 \rangle}{\langle \mathcal{O}_2^2 \mathcal{O}_2^2 \rangle} \mathcal{H}_{2222}^c$$
$$\mathcal{H}_{[2^3][2^3]22} = \mathcal{H}_{[2^3][2^3]22}^c + 3^2 \frac{\langle \mathcal{O}_2 \mathcal{O}_2 \rangle \langle \mathcal{O}_2^2 \mathcal{O}_2^2 \rangle}{\langle \mathcal{O}_2^3 \mathcal{O}_2^3 \rangle} \mathcal{H}_{[2^2][2^2]22}^c + 3^2 \frac{\langle \mathcal{O}_2^2 \mathcal{O}_2^2 \rangle \langle \mathcal{O}_2 \mathcal{O}_2 \rangle}{\langle \mathcal{O}_2^3 \mathcal{O}_2^3 \rangle} \mathcal{H}_{2222}^c.$$

$$\mathcal{H}_{[2^n][2^n]22}^c \Big|_{\frac{1}{a^n}} = \frac{U^2}{V^2} \Psi_B^{(n)}$$

$$\mathcal{H}_{[2^n][2^n]22}(U,V) = \oint \frac{dsdt}{(2\pi i)^2} U^{s+2} V^t \Gamma^2(-s) \Gamma^2(-t) \Gamma^2(s+t+4) \mathcal{M}_{[2^n][2^n]22}(s,t)$$

$$\mathcal{P}_{2222} = 4a^2 g_{12}^2 g_{34}^2 \,;\, \mathcal{G}_{2222} = 1 + U^2\sigma^2 + \frac{U^2\tau^2}{V^2} + \frac{4}{a}\left[U\sigma + \frac{U\tau}{V} + \frac{U^2\sigma\tau}{V}\right]$$

$$\mathcal{H}_{2222}(U,V) = \frac{1}{V^2} \mathcal{H}_{2222}\left(\frac{U}{V}, \frac{1}{V}\right) = \frac{U^2}{V^2} \mathcal{H}_{2222}(V,U)$$

$$\mathcal{M}(s,t) = \mathcal{M}(s,u) = \mathcal{M}(t,s)$$

$$\mathcal{M}_{2222}\Big|_{\frac{1}{a}} = +\frac{4}{a} \frac{1}{(s+1)(t+1)(u+1)}$$

$$\mathcal{M}_{2222}\Big|_{\frac{1}{a^2}} = \frac{16}{a^2} [\mathcal{M}^{1l}(s,t) + \mathcal{M}^{1l}(s,u) + \mathcal{M}^{1l}(u,t)]$$

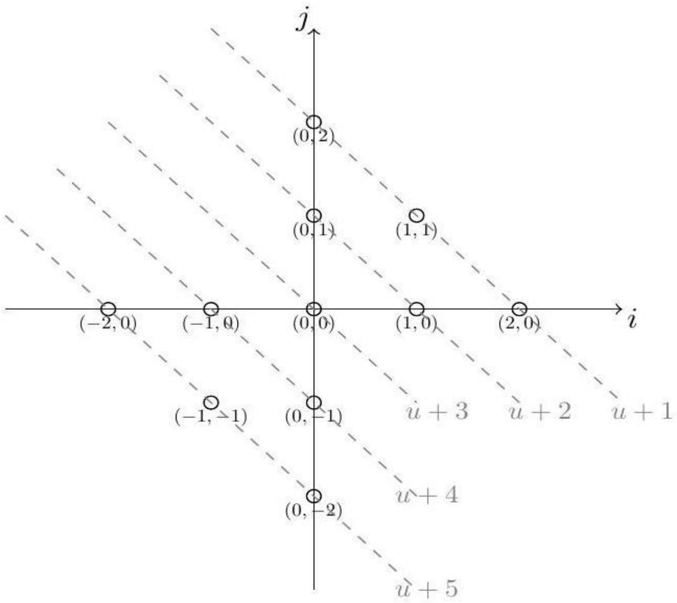
$$\mathcal{M}^{1l}(s,t) = \left[K(s,t) A(s,t) + \left(\psi^{(0)}(-s) - \psi^{(0)}(-t)\right) B(s,t) + C(s,t)\right]$$

$$K(s,t) = -\left(\psi^{(0)}(-s) - \psi^{(0)}(-t)\right)^2 + \left(\psi^{(1)}(-s) + \psi^{(1)}(-t)\right) - \pi^2$$

$$A(s,t) = \frac{1}{20}\left[\frac{s^2t^2\left(-15 - \frac{12}{4+u} + \frac{72}{5+u}\right)}{(1+u)(2+u)(3+u)} + \frac{st\left(25 - \frac{6}{3+u} - \frac{60}{4+u}\right)}{(1+u)(2+u)} - \frac{4\left(3 - \frac{2}{2+u} - \frac{1}{3+u}\right)}{1+u}\right]$$
$$B(s,t) = \frac{(s-t)}{20}\left[\frac{st\left(-15 - \frac{12}{4+u} + \frac{72}{5+u}\right)}{(1+u)(2+u)} + \frac{10}{4+u} + \frac{8}{2+u} + \frac{2}{1+u} - 10\right]$$
$$C(s,t) = \frac{1}{20}\left[\frac{st\left(-15 - \frac{12}{4+u} + \frac{72}{5+u}\right)}{1+u} + \frac{6}{5+u} - \frac{20}{4+u} + \frac{2}{1+u}\right] - 1 + B_0$$

$$K(s,t), \psi^{(0)}(-s) - \psi^{(0)}(-t), 1$$

$$\mathcal{M}^{1l}(s,t) = \sum_{ij \geq 0, |i|+|j| \leq 2} \frac{p_{ij}(s,t)}{(s+i)+(t+j)+1} K(s+i, t+j)$$



	$K(s+2,t)$	$K(s+1,t)$	$K(s-1,t)$	$K(s-2,t)$	$K(s,t)$	$K(s-1,t-1)$	$K(s+1,t+1)$
$K(s,t)$	1	1	1	1	1	1	1
$\Psi_1(s,t)$	$-\frac{2(3+2s)}{(s+1)(s+2)}$	$-\frac{2}{1+s}$	$\frac{2}{s}$	$\frac{2(-1+2s)}{(-1+s)s}$	0	$-\frac{2(s-t)}{st}$	$\frac{2(s-t)}{(s+1)(t+1)}$
1	$-\frac{2}{(s+1)(s+2)}$	0	$-\frac{2}{s^2}$	$-\frac{2(1-3s+3s^2)}{(-1+s)^2s^2}$	0	$-\frac{2(s^2-st+t^2)}{s^2t^2}$	$\frac{2}{(s+1)(t+1)}$

$$\mathcal{P}_{[2^2][2^2]22} = 16a^2(a+2)g_{12}^4 g_{34}^2$$

$$\mathcal{G}_{[2^2][2^2]22} = 1 + 2U^2\sigma^2 + \frac{2U^2\tau^2}{V^2} + \frac{8}{a}\left[U\sigma + \frac{U\tau}{V}\right] + \frac{24}{a}\frac{U^2\sigma\tau}{V} + \frac{4}{a}\left[U^2\sigma^2 + \frac{U^2\tau^2}{V^2}\right]$$

$$\mathcal{H}_{[2^2][2^2]22} = \mathcal{H}^c_{[2^2][2^2]22} + \frac{2a}{a+2}\mathcal{H}_{2222}$$

$$\mathcal{H}_{[2^2][2^2]22}\big|_{\frac{1}{a}} = +2\mathcal{H}_{2222}\big|_{\frac{1}{a}},$$

$$\mathcal{H}_{[2^2][2^2]22}\big|_{\frac{1}{a^2}} = \mathcal{H}^c_{[2^2][2^2]22}\big|_{\frac{1}{a^2}} - 4\mathcal{H}_{2222}\big|_{\frac{1}{a}} + 2\mathcal{H}_{2222}\big|_{\frac{1}{a^2}}.$$

$$\mathcal{H}^c_{[2^n][2^n]22}\big|_{\frac{1}{a^2}} = \frac{U^2}{V^2}\Psi_B^{(2)}$$

$$\mathcal{H}^c_{[2^2][2^2]22}\big|_{\frac{1}{a^2}} = \frac{U^2}{V^2}\Big[R_{11}^{(1)}\mathcal{P}_2 + R_{10}^{(2)}[z\partial_z - \bar{z}\partial_{\bar{z}}]\mathcal{P}_2 + R_{11}^{(3)}\mathcal{P}_1\log V + R_{10}^{(4)}\log^2 V + R_{10}^{(5)}\log V\log U$$
$$+R_{11}^{(6)}\mathcal{P}_1 + R_{10}^{(7)}\log V + R_{10}^{(8)}\log U + R_8^{(9)}\Big]$$

$$\mathcal{M}^c_{[2^2][2^2]22}(s,t) = \left[K(u,t)A(u,t) + \left(\psi^{(0)}(-u) - \psi^{(0)}(-t)\right)B(u,t) + C(u,t)\right]$$

$$A(u,t) = \frac{32}{(1+s)(2+s)(3+s)}\left[-16 + 3ut - 3s + \frac{18ut}{4+s} - \frac{18u^2t^2}{(4+s)(5+s)}\right]$$

$$B(u,t) = \frac{32(u-t)}{(1+s)(2+s)}\left[-\frac{1}{(1+u)(1+t)} - \frac{18ut - 18(5+s)}{(4+s)(5+s)}\right]$$

$$C(u,t) = \frac{32}{(1+s)}\left[\frac{3}{2} - \frac{\frac{1}{2}}{(1+u)(1+t)} + \frac{6 - 18ut + 12(5+s)}{(4+s)(5+s)}\right]$$

$$\mathcal{H}_{[2^2][2^2]22}(U,V) = \frac{1}{V^2}\mathcal{H}_{[2^2][2^2]22}\left(\frac{U}{V},\frac{1}{V}\right)$$

$$\mathcal{M}_{[2^2][2^2]22}(s,t) = \mathcal{M}_{[2^2][2^2]22}(s,u)$$

$$\mathcal{P}_{2[2^2]2[2^2]2} = 16a^2(a+2)g_{12}^2 g_{24}^2 g_{34}^2$$

$$\mathcal{H}_{2[2^2]2[2^2]2} = \mathcal{H}_{[2^2][2^2]22}\left(\frac{1}{U},\frac{V}{U}\right)\ ;\ \mathcal{M}_{2[2^2]2[2^2]}(s,t) = \mathcal{M}_{[2^2][2^2]22}(u,t)$$

$$\mathcal{M}^c_{2[2^2]2[2^2]}(s,t) = \left[K(s,t)A(s,t) + \left(\psi^{(0)}(-s) - \psi^{(0)}(-t)\right)B(s,t) + C(s,t)\right]$$

$$A(s,t) = \frac{32}{(1+u)(2+u)(3+u)}\left[-16 + 3st - 3u + \frac{18st}{4+u} - \frac{18s^2t^2}{(4+u)(5+u)}\right]$$
$$B(s,t) = \frac{32(s-t)}{(1+u)(2+u)}\left[-\frac{1}{(1+s)(1+t)} - \frac{18st - 18(5+u)}{(4+u)(5+u)}\right]$$
$$C(s,t) = \frac{32}{(1+u)}\left[\frac{3}{2} - \frac{\frac{1}{2}}{(1+s)(1+t)} + \frac{6 - 18st + 12(5+u)}{(4+u)(5+u)}\right]$$

$$\lim_{t\gg 1} A^c_{2[2^2]2[2^2]}(s,t) \sim \frac{(1+s)}{t^2}\ ;\ \lim_{t\gg 1} A^{1l}_{2222}(s,t) \sim \frac{(12+25s+15s^2)}{t},$$

$$\sum_{ij\geq 0,|i|+|j|\leq 2} \frac{P_{ij}(s,t)}{(s+i)+(t+j)+1}K(s+i,t+j)$$

$$\mathcal{H}^t_{2[2^2]2[2^2]2} \equiv \mathcal{H}_{2[2^2]2[2^2]2}\big|_{\frac{1}{a^2}} - 2\mathcal{H}_{2222}\big|_{\frac{1}{a^2}}$$

$$\mathcal{H}^t_{2[2^2]2[2^2]2} = \mathcal{H}^c_{[2^2][2^2]22}\big|_{\frac{1}{a^2}} - 4\mathcal{H}_{2222}\big|_{\frac{1}{a}}$$
$$= \mathcal{H}^c_{2[2^2]2[2^2]}\big|_{\frac{1}{a^2}} + 16U^2\bar{D}_{2422}(U,V)$$

$$\mathcal{M}^t_{2[2^2]2[2^2]} = \mathcal{M}^c_{2[2^2]2[2^2]} - \frac{16}{(s+1)(t+1)(u+1)}$$

$$\mathcal{M}^t_{2[2^2]2[2^2]2} = \sum_{ij\geq 0,|i|+|j|\leq 2} \frac{p_{ij}(s,t)}{(s+i)+(t+j)+1}K(s+i,t+j)$$

$$\mathcal{P}_{[2^3][2^3]22} = 96a^2(a+2)(a+4)g_{12}^6 g_{34}^2$$
$$\mathcal{G}_{[2^3][2^3]22} = 1 + 3\left[U^2\sigma^2 + \frac{U^2\tau^2}{V^2}\right] + \frac{12}{a}\left[U\sigma + \frac{U\tau}{V}\right] + \frac{60}{a}\frac{U^2\sigma\tau}{V} + \frac{12}{a}\left[U^2\sigma^2 + \frac{U^2\tau^2}{V^2}\right]$$
$$\mathcal{H}_{[2^3][2^3]22} = \mathcal{H}^c_{[2^3][2^3]22} + \frac{3a}{a+4}\mathcal{H}_{2222} + \frac{3a}{a+4}\mathcal{H}^c_{[2^2][2^2]22}$$

$$\mathcal{H}_{[2^3][2^3]22}\big|_{\frac{1}{a}} = 3\mathcal{H}_{2222}\big|_{\frac{1}{a}}$$
$$\mathcal{H}_{[2^3][2^3]22}\big|_{\frac{1}{a^2}} = 3\mathcal{H}^c_{[2^2][2^2]22}\big|_{\frac{1}{a^2}} - 12\mathcal{H}_{2222}\big|_{\frac{1}{a}} + 3\mathcal{H}_{2222}\big|_{\frac{1}{a}}{}^2$$

$$\mathcal{H}_{[2^3][2^3]22}\big|_{\frac{1}{a^2}}\mathcal{H}^c_{[2^2][2^2]22}\big|_{\frac{1}{a^2}}$$

$$\mathcal{H}_{[2^3][2^3]22}\big|_{\frac{1}{a^3}} = \mathcal{H}^c_{[2^3][2^3]22}\big|_{\frac{1}{a^3}} - 12\mathcal{H}^c_{[2^2][2^2]22}\big|_{\frac{1}{a^2}} + 3\mathcal{H}^c_{[2^2][2^2]22}\big|_{\frac{1}{a^3}} + 48\mathcal{H}_{2222}\big|_{\frac{1}{a}} - 12\mathcal{H}_{2222}\big|_{\frac{1}{a^2}} + 3\mathcal{H}_{2222}\big|_{\frac{1}{a^3}}$$

$$\mathcal{H}^c_{[2^n][2^n]22}\big|_{\frac{1}{a^3}} = \frac{U^2}{V^2}\Psi_B^{(3)}$$

$$\mathcal{H}^{c}_{[2^3][2^3]22}\Big|_{\frac{1}{a^3}} = \frac{U^2}{V^2}[R_{15}^{(1)}\mathcal{P}_3(z,\bar{z}) + R_{14}^{(2)}\partial_-\mathcal{P}_3(z,\bar{z}) + R_{15}^{(3)}\partial_+\mathcal{P}_3(z,\bar{z}) + R_{14}^{(4)}\partial_+\partial_-\mathcal{P}_3(z,\bar{z})$$
$$+R_{15}^{(5)}\mathcal{P}_2(z,\bar{z}) + R_{15}^{(6)}\mathcal{P}_1(z,\bar{z})\log^2 V +$$
$$+R_{14}^{(7)}\partial_-\mathcal{P}_2(z,\bar{z}) + R_{14}^{(8)}\log U\log^2 V + R_{14}^{(9)}\log^3 V + R_{15}^{(10)}\mathcal{P}_1(z,\bar{z})\log V$$
$$+R_{14}^{(11)}\log^2 V + R_{14}^{(12)}\log V\log U + R_{15}^{(13)}\mathcal{P}_1(z,\bar{z}) + R_{14}^{(14)}\log V + R_{14}^{(15)}\log U + R_{12}^{(16)}\Big]$$

$$\mathcal{M}^{c}_{[2^3][2^3]22}(s,t) = [K_4(u,t)a(u,t) + K_3(u,t)b(u,t) + \tilde{K}_2(u,t)c(u,t) +$$
$$+K_2(u,t)A(u,t) + K_1(u,t)B(u,t) + C(u,t)]$$

$$K_4(s,t) = \Psi_1^4 + 3\Psi_2^2 - 2\Psi_1\Psi_3 - 2\pi^2\Psi_2 - 2\pi^2K_2 - \pi^4$$
$$K_3(s,t) = \Psi_1^3(u,t) - \frac{1}{2}\Psi_3(u,t) + \pi^2\Psi_1$$
$$\tilde{K}_2(s,t) = \Psi_2 - \frac{2}{3}\pi^2\,;\; K_2(s,t) = -\Psi_1^2 + \Psi_2 - \pi^2\,;\; K_1(s,t) = \Psi_1$$

$$a(s,t) = \frac{256}{(u+1)(u+2)(u+3)}\Big[$$
$$14 - 15st + \frac{9}{2}u + \frac{15st(4+5st)}{4(u+4)_1} - \frac{3s^2t^2(58+15st)}{4(u+4)_2} + \frac{45s^2t^2(1+2st)}{2(u+4)_3} - \frac{135s^2t^2(-2+st)^2}{2(u+4)_4}\Big]$$

$$b(s,t) = \frac{384(s-t)}{(u+1)(u+2)(u+3)}\Big[$$
$$46 + \frac{2(10-77st)}{(u+4)_1} - \frac{st(116-195st)}{(u+4)_2} + \frac{15st(4+21st-9s^2t^2))}{(u+4)_3} - \frac{360st(2-st)(1-st)}{(u+4)_4}\Big]$$

$$\langle \mathcal{O}_2^2\mathcal{O}_2^2\mathcal{O}_2\mathcal{O}_2\rangle\,;\,\langle \mathcal{O}_2\mathcal{O}_2^2\mathcal{O}_2\mathcal{O}_2^2\rangle$$

$$\mathcal{H}^c|_{\log^0(U)}$$

$$\mathcal{H}^c|_{\log^n(U)}$$

$$\langle \mathcal{O}_2^2(\vec{x}_1,\vec{y}_1)\mathcal{O}_2^2(\vec{x}_2,\vec{y}_2)\mathcal{O}_2(\vec{x}_3,\vec{y}_3)\mathcal{O}_2(\vec{x}_4,\vec{y}_4)\rangle = \mathcal{P}_{[2^2][2^2]22}\sum_{\underline{R}} A_{\underline{R}}B_{\underline{R}}.$$

$$A_{\underline{R}} = \sum_{\mathcal{O},\mathcal{O}'\in\underline{R}} C_{[2^2][2^2]\mathcal{O}}\mathbb{G}^{\mathcal{O},\mathcal{O}'}C_{22\mathcal{O}'}$$

$$\mathcal{O}_2\mathcal{O}_2 \sim 1 \oplus \underbrace{\mathcal{O}_2 + \mathcal{K}_2 + \cdots}_{\tau=2} \oplus \underbrace{\mathcal{O}_2^2 + \cdots}_{\tau\geq 4}$$

$$C_{\mathcal{O}_2\mathcal{O}_2;\mathcal{K}_2} \sim \frac{1}{N}.$$

$$C_{\mathcal{O}_2\mathcal{O}_2;\mathcal{O}_2\partial^l\mathcal{O}_2} \sim 1$$

$$\mathcal{P}_{[2^2][2^2]22} = 16a^2(a+2)g_{12}^4g_{34}^2$$
$$\mathcal{G}_{[2^2][2^2]22} = 1 + 2U^2\sigma^2 + \frac{2U^2\tau^2}{V^2} + \frac{8}{a}\Big[U\sigma + \frac{U\tau}{V}\Big] + \frac{24}{a}\frac{U^2\sigma\tau}{V} + \frac{4}{a}\Big[U^2\sigma^2 + \frac{U^2\tau^2}{V^2}\Big].$$

$$\langle \mathcal{O}_2^2\mathcal{O}_2^2\mathcal{O}_2\mathcal{O}_2\rangle\Big|_{\substack{\tau=2\\ \text{OPE}}} = \mathcal{P}_{[2^2][2^2]22}\times\frac{8}{a}\Big[U\sigma + \frac{U\tau}{V}\Big]$$

$$A^{\text{free}}_{\mathcal{K},2,l,[000]} = \frac{8}{a}\times\frac{2(l+2)!^2}{(2l+4)!}\frac{1+(-1)^l}{2}.$$

$$A_{\mathcal{L},2,l,[000]} = A^{\text{free}}_{\mathcal{K},2,l,[000]} + \mathcal{H}_{[2^2][2^2]22}\Big|_{\tau=2,l,[000]} = 0$$

$$\mathcal{H}_{[2^2][2^2]22}\Big|_{\frac{1}{a},\tau=2} = -\frac{8}{a}\times\frac{2(l+2)!^2}{(2l+4)!}\frac{1+(-1)^l}{2}$$

$$\mathcal{H}_{[2^2][2^2]22}\Big|_{\frac{1}{a^n},\tau=2} = 0 \qquad\qquad n\geq 2$$

$$\mathcal{H}_{[2^2][2^2]22}\Big|_{\frac{1}{a}} = -8U\left[\frac{(1+V)}{(-1+V)^2V}-\frac{2\log(V)}{(-1+V)^3}\right]+U^2(\dots)$$

$$\mathcal{H}_{2222}\Big|_{\frac{1}{a}} = -4U^2\bar{D}_{2422} = -4U\left[\frac{(1+V)}{(-1+V)^2V}-\frac{2\log(V)}{(-1+V)^3}\right]+U^2(\dots)$$

$$\mathcal{H}_{[2^2][2^2]22}\Big|_{\frac{1}{a}} = 2\mathcal{H}_{2222}\Big|_{\frac{1}{a}}$$

$$\mathcal{H}^{t}_{[2^2][2^2]22}\Big|_{\log^0(U)} = \mathcal{H}^{c}_{[2^2][2^2]22}+16U^2\bar{D}_{2422}\big|_{\log^0(U)} = U^2(\dots),$$

$$\langle\mathcal{O}_2(\vec{x}_1,\vec{y}_1)\mathcal{O}_2^2(\vec{x}_2,\vec{y}_2)\mathcal{O}_2(\vec{x}_3,\vec{y}_3)\mathcal{O}_2^2(\vec{x}_4,\vec{y}_4)\rangle = \mathcal{P}_{2[2^2]2[2^2]}\sum_{\underline{R}} A_{\underline{R}}B_{\underline{R}}$$

$$A_{\underline{R}} = \sum_{\mathcal{O},\mathcal{O}'\in\underline{R}} C_{2[2^2]\mathcal{O}}\mathbb{G}^{\mathcal{O},\mathcal{O}'}C_{2[2^2]\mathcal{O}'}$$

$$\mathcal{O}_2\mathcal{O}_2^2 \sim \mathcal{O}_2\oplus\underbrace{\mathcal{O}_2^2+\mathcal{K}_4+\cdots}_{\tau=4}\oplus\underbrace{\mathcal{O}_2^3+\cdots}_{\tau\geq 6}$$

$$C_{\mathcal{O}_2\mathcal{O}_2^2;\mathcal{O}_2\partial^l\mathcal{O}_2}\sim\frac{1}{N}$$

$$\mathcal{O}_2\partial^l\mathcal{O}_2\big|_{[0,2,0],l=0,2,4,\dots};\ \mathcal{O}_2\partial^l\mathcal{O}_2\big|_{[1,0,1],l=1,3,5,\dots}$$

$$G = \begin{pmatrix}\langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle & \langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2^2\rangle\\ \langle\mathcal{O}_2\mathcal{O}_2^2\mathcal{O}_2\mathcal{O}_2\rangle & \langle\mathcal{O}_2\mathcal{O}_2^2\mathcal{O}_2\mathcal{O}_2^2\rangle\end{pmatrix},$$

$$\mathcal{S}^{\vec{p}}_{4,[020]}\equiv A^{\vec{p}}_{\mathcal{S},\tau=4,l,[aba]}$$

$$\mathcal{S}^{2[2^2]2[2^2]}_{4,[020]} = \frac{\left(\mathcal{S}^{[2^2]2;22}_{4,[020]}\right)^2}{\mathcal{S}^{22;22}_{4,[020]}}$$

$$\mathcal{S}^{2[2^2];22}_{4,[020]} = \mathcal{S}^{22;2[2^2]}_{4,[020]}$$

$$\mathcal{S}^{[2^2]2[2^2]2}_{4,[020]} = \frac{32}{a}\frac{\left(1+\frac{2}{a}\right)}{(l+3)(l+4)+\frac{4}{a}}\frac{(l+4)!^2}{(2l+6)!}\frac{1+(-1)^l}{2}$$

$$A^{\text{free}}_{\mathcal{K},4,l,[020]} = -\mathcal{S}_{4,[020]}+A^{\text{free}}_{4,l,[020]}$$

$$A^{\text{free}}_{4,l,[020]} = \frac{1}{a}\begin{cases}+4(l+1)(l+6)\dfrac{(l+2)!\,(l+4)!}{(2l+6)!} & \text{spin odd}\\ +4(18+7l+l^2)\dfrac{(l+2)!\,(l+4)!}{(2l+6)!} & \text{spin even}\end{cases}$$

$$A_{\mathcal{L},4,l,[020]} = A^{\text{free}}_{\mathcal{K},4,l,[020]}+\mathcal{H}_{2[2^2]2[2^2]}\big|_{\tau=4,l,[020]} = 0.$$

$$A_{\mathcal{L},4,l,[020]} = \begin{cases}+\dfrac{4}{a}(l+1)(l+6)\dfrac{(l+2)!\,(l+4)!}{(2l+6)!} & \text{spin odd}\\ +\dfrac{4(l+2)(l+5)}{a}\dfrac{(l+3)(l+4)-\frac{12}{a}}{(l+3)(l+4)+\frac{4}{a}}\dfrac{(l+2)!\,(l+4)!}{(2l+6)!} & \text{spin even}\end{cases}$$

$$\mathcal{H}_{2[2^2]2[2^2]}\big|_{\frac{1}{a},\tau=4} = \begin{cases} -4(l+1)(l+6)\dfrac{(l+2)!\,(l+4)!}{(2l+6)!}\dfrac{1-(-1)^l}{2} \\ -4(l+2)(l+5)\dfrac{(l+2)!\,(l+4)!}{(2l+6)!}\dfrac{1+(-1)^l}{2} \end{cases}$$

$$\mathcal{H}_{2[2^2]2[2^2]}\big|_{\frac{1}{a^2},\tau=4} = \begin{cases} 0 \\ +64(l+2)(l+5)\dfrac{(l+2)!^2}{(2l+6)!}\dfrac{1+(-1)^l}{2} \end{cases}$$

$$\mathcal{H}_{22^222^2} = -\frac{8}{a}U\left[\frac{(1+V)}{(1-V)^2V} + \frac{2\log(V)}{(1-V)^3}\right] + U^2(\dots)$$
$$+\frac{32}{a^2}U\left[\frac{1}{(1-V)^2V} - \frac{\log(V) + \log^2(V) + 2\mathrm{Li}_2(1-V)}{(1-V)^3}\right] + U^2(\dots)$$

$$C_{[2^n][2^n];\mathcal{D}} = C^{(0)}_{[2^n][2^n];\mathcal{D}} + \frac{1}{a}C^{(1)}_{[2^n][2^n];\mathcal{D}} + \cdots$$

$$\Delta_{\mathcal{D}} = \tau + l + \frac{2}{a}\eta^{(1)}_{\mathcal{D}} + \cdots$$

$$C^{(0)}_{[2^2][2^2];\mathcal{D}_{\tau,l}} = 2C^{(0)}_{22;\mathcal{D}_{\tau,l}}.$$

$$\hat{C}_{[2^2][2^2]22}\big|_{\frac{\log^n(U)}{a^n}} = \frac{1}{n!}\sum_{\tau,l}\left(\sum_{\mathcal{D}} C^{(0)}_{[2^2][2^2];\mathcal{D}}\eta^n_{\mathcal{D}}C^{(0)}_{22;\mathcal{D}}\right)B^{\mathcal{L}}_{\tau,l,[000]} = 2\hat{C}_{2222}\big|_{\frac{\log^n(U)}{a^n}}.$$

$$\mathcal{H}^t_{[2^2][2^2]22}\big|_{\frac{1}{a^2}} \equiv \mathcal{H}_{[2^2][2^2]22}\Big|_{\frac{1}{a^2}} - 2\mathcal{H}_{2222}\big|_{\frac{1}{a^2}}$$

$$\mathcal{H}^{1l}_{[2^2][2^2]22}\big|_{\frac{1}{a^3}} \equiv \mathcal{H}_{[2^2][2^2]22}\Big|_{\frac{1}{a^3}} - 2\mathcal{H}_{2222}\big|_{\frac{1}{a^3}}.$$

$$\hat{C}_{[2^2][2^2]22}\big|_{\frac{1}{a}} - 2\hat{C}_{2222}\big|_{\frac{1}{a}} = \sum_l\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_4} - 2C^{(1)}_{22;\mathcal{D}_4}\right]C^{(0)}_{22;\mathcal{D}_4}B^{\mathcal{L}}_{4,l,[000]} + (\text{twist } \geq 6)$$
$$\hat{C}_{[2^2][2^2]22}\big|_{\frac{\log(U)}{a^2}} - 2\hat{C}_{2222}\big|_{\frac{\log(U)}{a^2}} = \sum_l\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_4} - 2C^{(1)}_{22;\mathcal{D}_4}\right]\eta_{\mathcal{D}_4}C^{(0)}_{22;\mathcal{D}_4}B^{\mathcal{L}}_{4,l,[000]} + (\text{twist } \geq 6)$$

$$\hat{C}_{[2^2][2^2]22}\big|_{\frac{1}{a}} - 2\hat{C}_{2222}\big|_{\frac{1}{a}} = \frac{4}{a}U^2\sigma^2 + \frac{4}{a}\frac{U^2\tau^2}{V^2} + \frac{16}{a}\frac{U^2\sigma\tau}{V}$$

$$\frac{4}{a}U^2\sigma^2 + \frac{4}{a}\frac{U^2\tau^2}{V^2} + \frac{16}{a}\frac{U^2\sigma\tau}{V} = \sum_l\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_4} - 2C^{(1)}_{22;\mathcal{D}_4}\right]C^{(0)}_{22;\mathcal{D}_4}B^{\mathcal{L}}_{4,l,[000]} + (\text{twist } \geq 6)$$

$$\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_4} - 2C^{(1)}_{22;\mathcal{D}_4}\right]C^{(0)}_{22;\mathcal{D}_4} = \frac{4}{3}(l+2)(l+5)\frac{(l+3)!^2}{(2l+6)!}\frac{1+(-1)^l}{2}$$

$$\hat{C}_{[2^2][2^2]22}\big|_{\frac{\log(U)}{a^2}} - 2\hat{C}_{2222}\big|_{\frac{\log(U)}{a^2}} = \mathcal{I}\times\left[\mathcal{H}_{[2^2][2^2]22} - 2\mathcal{H}_{2222}\right]_{\frac{\log(U)}{a^2}}$$

$$\mathcal{H}^t_{[2^2][2^2]22}\big|_{\frac{\log(U)}{a^2}} = \sum_l\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_4} - 2C^{(1)}_{22;\mathcal{D}_4}\right]\eta_{\mathcal{D}_4}C^{(0)}_{22;\mathcal{D}_4}H_{4,l,[000]} + (\text{twist } \geq 6)$$

$$\eta_{\mathcal{D}_4} = -\frac{48}{(l+1)(l+6)}\frac{1+(-1)^l}{2}$$

$$\mathcal{H}^t_{[2^2][2^2]22}\big|_{\frac{\log(U)}{a^2}} = \sum_l\left[-64\frac{(l+2)(l+5)}{(l+1)(l+6)}\frac{(l+3)!^2}{(2l+6)!}\frac{1+(-1)^l}{2}\right]H_{4,l,[000]} + (\text{twist } \geq 6)$$

$$\mathcal{H}^{t}_{[2^2][2^2]22}\Big|_{\frac{\log(U)}{a^2}} =$$

$$-32U^2\left[\frac{(1+10V+V^2)}{(1-V)^4V}+\frac{12(1+V)\log(V)}{(1-V)^5}+\frac{2(1+4V+V^2)\log^2(V)}{(1-V)^6}+U^3(\dots)\right]$$

$$\mathcal{H}^{t}_{[2^2][2^2]22} = \mathcal{H}^{c}_{[2^2][2^2]22}\Big|_{\frac{1}{a^2}} - 4\mathcal{H}_{2222}\Big|_{\frac{1}{a}}$$

$$\mathcal{H}_{[2^3][2^3]22}\Big|_{\frac{1}{a^3}} = \mathcal{H}^{c}_{[2^3][2^3]22}\Big|_{\frac{1}{a^3}} - 12\mathcal{H}^{c}_{[2^2][2^2]22}\Big|_{\frac{1}{a^2}} + 3\mathcal{H}^{c}_{[2^2][2^2]22}\Big|_{\frac{1}{a^3}} + 48\mathcal{H}_{2222}\Big|_{\frac{1}{a}} - 12\mathcal{H}_{2222}\Big|_{\frac{1}{a^2}} + 3\mathcal{H}_{2222}\Big|_{\frac{1}{a^3}}$$

$$\mathcal{H}^{t}_{[2^3][2^3]22}\Big|_{\frac{1}{a^3}} = \mathcal{H}^{c}_{[2^3][2^3]22}\Big|_{\frac{1}{a^3}} - 12\mathcal{H}^{c}_{[2^2][2^2]22}\Big|_{\frac{1}{a^2}} + 24\mathcal{H}_{2222}\Big|_{\frac{1}{a}}$$

$$\left[\mathcal{H}^{t}_{[2^3][2^3]22}\right]_{\frac{\log^3(U)}{a^3}} = 0$$

$$\left[\mathcal{H}^{t}_{[2^3][2^3]22}\right]_{\frac{\log^0(U)}{a^3},\,\text{twist 2}} = 0$$

$$\left[\mathcal{H}^{t}_{[2^3][2^3]22}\right]_{\frac{\log^n(U)}{a^3},\,\text{twist 4}} = 0\,;\; n=1,2$$

$$\mathcal{H}_{[2^3][2^3]22}\Big|_{\frac{1}{a^3}} - 3\mathcal{H}_{2222}\Big|_{\frac{1}{a^3}} - 3\mathcal{H}^{1l}_{[2^2][2^2]22}\Big|_{\frac{1}{a^3}} = \left[\mathcal{H}^{c}_{[2^3][2^3]22}\Big|_{\frac{1}{a^3}} - 12\mathcal{H}^{c}_{[2^2][2^2]22}\Big|_{\frac{1}{a^2}} + 24\mathcal{H}_{2222}\Big|_{\frac{1}{a}}\right],$$

$$\mathcal{H}^{1l}_{[2^2][2^2]22}\Big|_{\frac{1}{a^3}} = \mathcal{H}^{c}_{[2^2][2^2]22}\Big|_{\frac{1}{a^3}} + 8\mathcal{H}_{2222}\Big|_{\frac{1}{a}} - 4\mathcal{H}_{2222}\Big|_{\frac{1}{a^2}}.$$

$$\mathcal{H}_{[2^3][2^3]22}\Big|_{\frac{1}{a^3}} - 3\mathcal{H}_{2222}\Big|_{\frac{1}{a^3}}\mathcal{H}^{1l}_{[2^2][2^2]22}\Big|_{\frac{1}{a^3}}$$

$$\mathcal{H}_{[2^3][2^3]22}\Big|_{\frac{1}{a^3}} - 3\mathcal{H}_{2222}\Big|_{\frac{1}{a^3}}$$

$$C^{(0)}_{[2^3][2^3];\mathcal{D}_{\tau,l}} = 3C^{(0)}_{22;\mathcal{D}_{\tau,l}}$$

$$C^{(0)}_{[2^2][2^2];\mathcal{D}_{\tau,l}} = 2C^{(0)}_{22;\mathcal{D}_{\tau,l}}$$

$$C^{(n)}_{[2^3][2^3];\mathcal{D}_4} = 3C^{(n)}_{[2^2][2^2]:\mathcal{D}_4} - 3C^{(n)}_{22;\mathcal{D}_4}\,;\; n=1,2$$

$$\hat{C}_{[2^3][2^3]22}\Big|_{\frac{1}{a}} - 3\hat{C}_{2222}\Big|_{\frac{1}{a}} = \frac{48}{a}\frac{U^2\sigma\tau}{V} + \frac{12}{a}\left[U^2\sigma^2 + \frac{U^2\tau^2}{V^2}\right] = 3\left[\hat{C}_{[2^2][2^2]22}\Big|_{\frac{1}{a}} - 2\hat{C}_{2222}\Big|_{\frac{1}{a}}\right]$$

$$\hat{C}_{[2^3][2^3]22}\Big|_{\frac{1}{a^2}} - 3\hat{C}_{2222}\Big|_{\frac{1}{a^2}} = \mathcal{I}\left[3\mathcal{H}^{c}_{[2^2][2^2]22}\Big|_{\frac{1}{a^2}} - 12\mathcal{H}_{2222}\Big|_{\frac{1}{a}}\right] = 3\left[\hat{C}_{[2^2][2^2]22}\Big|_{\frac{1}{a^2}} - 2\hat{C}_{2222}\Big|_{\frac{1}{a^2}}\right]$$

$$\mathcal{H}_{[2^3][2^3]22} - 3\mathcal{H}_{2222} - 3\mathcal{H}^{1l}_{[2^2][2^2]22}$$

$$\left[\mathcal{H}^{t}_{[2^3][2^3]22}\right]^{=}_{\frac{\log^2(U)}{a^3}} = \frac{1}{2}\sum_{l}\left[\left(C^{(1)}_{[2^3][2^3];\mathcal{D}_4} - 3C^{(1)}_{[2^2][2^2];\mathcal{D}_4} + 3C^{(1)}_{22;\mathcal{D}_4}\right)\left(\eta^{(1)}_{\mathcal{D}_4}\right)^2 C^{(0)}_{22\mathcal{D}_4}\right] H_{4,l,[000]} + (\text{twist } \geq 6)$$

$$C^{(1)}_{[2^3][2^3];\mathcal{D}_4} = 3C^{(1)}_{[2^2][2^2]:\mathcal{D}_4} - 3C^{(1)}_{22;\mathcal{D}_4}$$

$$\left[\mathcal{H}^{t}_{[2^3][2^3]22}\right] = \sum_{\frac{\log(U)}{a^3}}\left[\left(C^{(1)}_{[2^3][2^3];\mathcal{D}_4} - 3C^{(1)}_{[2^2][2^2];\mathcal{D}_4} + 3C^{(1)}_{22;\mathcal{D}_4}\right)\left(\eta^{(1)}_{\mathcal{D}_4}C^{(1)}_{22;\mathcal{D}_4} + \eta^{(2)}_{\mathcal{D}_4}C^{(0)}_{22\mathcal{D}_4}\right)\right.$$
$$\left. + \left(C^{(2)}_{[2^3][2^3];\mathcal{D}_4} - 3C^{(2)}_{[2^2][2^2];\mathcal{D}_4} + 3C^{(2)}_{22;\mathcal{D}_4}\right)\eta^{(1)}_{\mathcal{D}_4}C^{(0)}_{22\mathcal{D}_4}\right] H_{4,l,[000]} + \cdots + (\text{twist } \geq 6)$$

$$C_{[2^n][2^n];\mathcal{T}} = \frac{1}{\sqrt{a}}\left[C^{(\frac{1}{2})}_{[2^n][2^n];\mathcal{T}} + \cdots\right]\,;\; \Delta = \tau + l + \frac{2}{a}\eta^{(1)}_{\mathcal{T}} + \cdots$$

$$\left[\mathcal{H}^{t}_{[2^3][2^3]22}\right]_{\frac{\log^2(U)}{a^3}} = (\text{twist } 4) + \frac{1}{2}\sum_{\tau,l}\left[\sum_{\mathcal{D}}\left(C^{(1)}_{[2^3][2^3];\mathcal{D}} - 3C^{(1)}_{[2^2][2^2];\mathcal{D}} + 3C^{(1)}_{22;\mathcal{D}}\right)\left(\eta^{(1)}_{\mathcal{D}}\right)^2 C^{(0)}_{22\mathcal{D}} + \sum_{\mathcal{T}}\left(C^{\left(\frac{1}{2}\right)}_{[2^3][2^3];\mathcal{T}} - 3C^{\left(\frac{1}{2}\right)}_{[2^2][2^2];\mathcal{T}} + 3C^{\left(\frac{1}{2}\right)}_{22;\mathcal{T}}\right)\left(\eta^{(1)}_{\mathcal{T}}\right)^2 C^{\left(\frac{1}{2}\right)}_{22\mathcal{T}}\right] H_{\tau,l,[000]}.$$

$$C^{(1)}_{[2^3][2^3];\mathcal{D}} = 3C^{(1)}_{[2^2][2^2];\mathcal{D}} - 3C^{(1)}_{22;\mathcal{D}}\ ;\ C^{\left(\frac{1}{2}\right)}_{[2^3][2^3];\mathcal{T}} = 3C^{\left(\frac{1}{2}\right)}_{[2^2][2^2];\mathcal{T}} - 3C^{\left(\frac{1}{2}\right)}_{22;\mathcal{T}}$$

$$\langle \mathcal{O}_2^2\mathcal{O}_2^2\mathcal{D}_{\tau,l}\rangle = \langle \mathcal{O}_2^2\mathcal{O}_2^2\mathcal{D}_{\tau,l}\rangle_c + 4\langle \mathcal{O}_2\mathcal{O}_2\rangle\langle \mathcal{O}_2\mathcal{O}_2\mathcal{D}_{\tau,l}\rangle$$

$$\frac{\langle \mathcal{O}_2^2\mathcal{O}_2^2\mathcal{D}_{\tau,l}\rangle_c}{\langle \mathcal{O}_2^2\mathcal{O}_2^2\rangle} \sim \frac{1}{a}$$

$$\langle \mathcal{O}_2^3\mathcal{O}_2^3\mathcal{D}_{\tau,l}\rangle = \langle \mathcal{O}_2^3\mathcal{O}_2^3\mathcal{D}_{\tau,l}\rangle_c + 9\langle \mathcal{O}_2\mathcal{O}_2\rangle\langle \mathcal{O}_2^2\mathcal{O}_2^2\mathcal{D}_{\tau,l}\rangle + 9(\langle \mathcal{O}_2^2\mathcal{O}_2^2\rangle - 4\langle \mathcal{O}_2\mathcal{O}_2\rangle^2)\langle \mathcal{O}_2\mathcal{O}_2\mathcal{D}_{\tau,l}\rangle$$

$$\frac{\langle \mathcal{O}_2^3\mathcal{O}_2^3\mathcal{D}_{\tau,l}\rangle_c}{\langle \mathcal{O}_2^3\mathcal{O}_2^3\rangle} \sim \frac{1}{a^2}$$

$$\langle \mathcal{O}_2^3\mathcal{O}_2^3\mathcal{T}_{\tau,l}\rangle = \langle \mathcal{O}_2^3\mathcal{O}_2^3\mathcal{T}_{\tau,l}\rangle_c + 9\langle \mathcal{O}_2\mathcal{O}_2\rangle\langle \mathcal{O}_2^2\mathcal{O}_2^2\mathcal{T}_{\tau,l}\rangle + 9(\langle \mathcal{O}_2^2\mathcal{O}_2^2\rangle - 4\langle \mathcal{O}_2\mathcal{O}_2\rangle^2)\langle \mathcal{O}_2\mathcal{O}_2\mathcal{T}_{\tau,l}\rangle$$

$$\frac{\langle \mathcal{O}_2^3\mathcal{O}_2^3\mathcal{T}_{\tau,l}\rangle_c}{\langle \mathcal{O}_2^3\mathcal{O}_2^3\rangle} \sim \frac{1}{a^{\frac{3}{2}}}$$

$$C^{(0)}_{[2^n][2^n];\mathcal{D}} = nC^{(0)}_{22;\mathcal{D}}\ n = 2{,}3$$

$$C^{(1)}_{[2^3][2^3];\mathcal{D}} = 3C^{(1)}_{[2^2][2^2];\mathcal{D}} - 3C^{(1)}_{22;\mathcal{D}}\ ;\ C^{\left(\frac{1}{2}\right)}_{[2^3][2^3];\mathcal{T}} = 3C^{\left(\frac{1}{2}\right)}_{[2^2][2^2];\mathcal{T}} - 3C^{\left(\frac{1}{2}\right)}_{22;\mathcal{T}}$$

$$C_{22;\mathcal{T}} \sim \frac{1}{N}\ ;\ C_{[2^2][2^2];\mathcal{T}} \sim \frac{1}{N}$$

$$C_{22;\mathcal{T}} \neq 0\ ;\ C_{[2^2][2^2];\mathcal{T}} - 2C_{22;\mathcal{T}} \neq 0$$

$$C_{2[2^2],\mathcal{T}} \sim 1$$

$$C_{24,\mathcal{D}} \sim 1$$

fixed twist	$\langle \mathcal{O}_2\mathcal{O}_4\mathcal{O}_2\mathcal{O}_4\rangle$	$\langle \mathcal{O}_2\mathcal{O}_2^2\mathcal{O}_2\mathcal{O}_2^2\rangle$
$\frac{\log^0(U)}{N^0}\sum C^2$	$O(l^6)$	$O(l^6)$
$\frac{\log^1(U)}{N^2}\sum C\eta C$	$O(l^4)$	$O(l^4)$
$\frac{\log^2(U)}{N^4}\sum C\eta^2 C$	$O(l^2)$	$O(l^2)+O(l^2)\log(l)$

$$\left(\partial_{\hat{z}}^2 + \frac{\frac{1}{4} - a_1^2}{(\hat{z}-1)^2} - \frac{\frac{1}{2} - a_0^2 - a_1^2 - a_t^2 + a_\infty^2 + u}{\hat{z}(\hat{z}-1)} + \frac{\frac{1}{4} - a_t^2}{(\hat{z}-t)^2} + \frac{u}{\hat{z}(\hat{z}-t)} + \frac{\frac{1}{4} - a_0^2}{\hat{z}^2}\right)\psi(\hat{z}) = 0$$

$$a_0^2=\frac{l^2}{4}, a_1^2=a_t^2=\frac{1}{4}, u=-\frac{(\alpha^2+8)(\omega^2-l^2)}{4(8-\alpha^2)}, a_\infty^2=\frac{l^2}{4}, t=\frac{\alpha^4}{64}$$

$$F=c_1(a,a_0,a_t,a_1,a_\infty)t+\mathcal{O}(t^2)$$

$$c_1(a,a_0,a_t,a_1,a_\infty)=\frac{(4a^2-4a_0^2+4a_t^2-1)(4a^2+4a_1^2-4a_\infty^2-1)}{8-32a^2}$$

$$u=-a^2+a_t^2-\frac{1}{4}+a_0^2+t\partial_t F$$

$$a^2=\frac{\omega^2}{4}+\frac{\omega^2-l^2}{16}\left(\alpha^2+\frac{\alpha^4(l^2+3\omega^2-4)}{32(\omega^2-1)}+\frac{\alpha^6(l^4-2l^2+\omega^4-2\omega^2+2)}{128(\omega^2-1)^2}\right)+\mathcal{O}(\alpha^8).$$

$$w_-^{(0)}=\mathrm{HeunG}(t,\hat{q},\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta},\hat{z})$$

$$\psi(\hat{z})\approx\left[1+\frac{\Gamma(\hat{\epsilon}-1)\Gamma\left(\frac{1+\hat{\gamma}-\hat{\epsilon}}{2}-a\right)\Gamma\left(\frac{1+\hat{\gamma}-\hat{\epsilon}}{2}+a\right)}{\Gamma(1-\hat{\epsilon})\Gamma\left(\frac{-1+\hat{\gamma}+\hat{\epsilon}}{2}-a\right)\Gamma\left(\frac{-1+\hat{\gamma}+\hat{\epsilon}}{2}+a\right)}e^{-\partial_{a_t}F}t^{\hat{\epsilon}-1}(t-\hat{z})^{1-\hat{\epsilon}}+\cdots\right]$$

$$\hat{\epsilon}=1-2a_t,\ \text{and}\ \hat{\gamma}=1-2a_0=1+|l|.$$

$$a=\pm\left(k+\frac{|l|-\hat{\epsilon}}{2}\right),\ \text{with}\ k=1,2,\ldots$$

$$-k(k+|l|)\frac{e^{-\partial_{a_t}F}}{\hat{b}+k}\left(1-\frac{\hat{z}}{t}\right)\sim\left(\frac{l^2}{4}-a^2\right)\frac{e^{-\partial_{a_t}F}}{\hat{b}+k}\frac{8-\alpha^2}{8+\alpha^2}w^2=\frac{1}{\hat{b}+k}\left(\hat{a}\hat{b}e^{-\partial_{a_t}F}\frac{8-\alpha^2}{8+\alpha^2}\right)w^2.$$

$$\hat{a}\hat{b}e^{-\partial_{a_t}F}\frac{8-\alpha^2}{8+\alpha^2}=\frac{l^2-\omega^2}{4}$$

$$\hat{b}=\frac{l+\omega}{2}+\frac{\alpha^2(\omega^2-l^2)}{16\omega}+\frac{\alpha^4(-3l^4\omega^2+2l^4+2(l^2-1)\omega^4+\omega^6)}{512\omega^3(\omega^2-1)}+\frac{\alpha^6((\omega^2-l^2)(l^4(5\omega^4-5\omega^2+2)-2l^2(\omega^6+\omega^4)+(\omega^4-\omega^2+2)\omega^4))}{4096(\omega^5(\omega^2-1)^2)}+\mathcal{O}(\alpha^8)$$

$$\psi''(\hat{z})+\left[\frac{(l+2)(-l)}{4\hat{z}^2}+\frac{l(l+2)-2\omega^2}{4(\hat{z}-1)^2}+\frac{(3-2\alpha^2)\omega^2+(2\alpha^2-1)(l(l+2)-4)-8}{4(\hat{z}-1)}-\frac{3}{4\left(\frac{2\alpha^2}{1-2\alpha^2}+\hat{z}\right)^2}-\frac{(2\alpha^2+1)\omega^2-8\alpha^2-(1-2\alpha^2)l(l+2)-4}{8\alpha^2\hat{z}}+\frac{(1-2\alpha^2)(-2\alpha^2\omega^2-8\alpha^2+(2\alpha^2-1)(l+2)l+\omega^2-4)}{8\alpha^2\left(\frac{2\alpha^2}{1-2\alpha^2}+\hat{z}\right)}\right]\psi(\hat{z})=0$$

$$a_0^2=\frac{1}{4}(l+1)^2, a_1^2=\frac{1}{4}(-l(l+2)+2\omega^2+1), a_t^2=1, a_\infty^2=1$$
$$u=\frac{1}{4}(2\alpha^2\omega^2+8\alpha^2-(2\alpha^2-1)l(l+2)-\omega^2+4), t=-\frac{2\alpha^2}{1-2\alpha^2}$$

$$\hat{\epsilon}=-1, \hat{\gamma}=2+|l|$$

$$1-\frac{\hat{z}}{t}=\frac{w^2}{1-2\alpha^2}(1+O(w^2))$$

$$\hat{b}=\frac{l+\omega}{2}+\frac{\alpha^2((l+2)^2-\omega^2)(l^2-\omega^2)}{4\omega(\omega^2-1)}+\cdots$$

$$I_0[p,q]\equiv\sum_{l\in\mathbb{Z}}\sum_{k=1}^{\infty}\frac{l^{2p}}{(|l|+2k)^q}e^{-i(|l|+2k)\tau+il\sigma}$$

$$I_1[p,q] \equiv \sum_{l\in\mathbb{Z}}^{\infty}\sum_{k=1}^{\infty}\frac{l^{2p}}{(|l|+2k-1)^q}e^{-i(|l|+2k)\tau+il\sigma};$$
$$I_2[p,q] \equiv \sum_{l\in\mathbb{Z}}^{\infty}\sum_{k=1}^{\infty}\frac{l^{2p}}{(|l|+2k-2)^q}e^{-i(|l|+2k)\tau+il\sigma}\ (2p-q\geq 0);$$
$$I_3[p,q] \equiv \sum_{l\in\mathbb{Z}}^{\infty}\sum_{k=1}^{\infty}\frac{(l^2-1)^2 l^{2p}}{(|l|+2k-3)^q}e^{-i(|l|+2k)\tau+il\sigma}\ (q\leq 2);$$
$$I_4[p,q] \equiv \sum_{l\in\mathbb{Z}}^{\infty}\sum_{k=1}^{\infty}\frac{(l^2-4)^2 l^{2p}}{(|l|+2k-4)^q}e^{-i(|l|+2k)\tau+il\sigma}\ (q\leq 2);$$
$$\tilde{I}_1[p,q] \equiv \sum_{l\in\mathbb{Z}}^{\infty}\sum_{k=1}^{\infty}\frac{l^{2p}}{(|l|+2k+1)^q}e^{-i(|l|+2k)\tau+il\sigma};$$
$$\tilde{I}_2[p,q] \equiv \sum_{l\in\mathbb{Z}}^{\infty}\sum_{k=1}^{\infty}\frac{l^{2p}}{(|l|+2k+2)^q}e^{-i(|l|+2k)\tau+il\sigma}.$$

$$A^1 = -H_1^{-1}dt\ , X_1 = H_1^{-2/3}\ , \cosh\ \varphi_1 = \frac{d}{dr^2}(r^2 H_1) = \frac{1+\dfrac{1+q_1}{r^2}}{\sqrt{1+\dfrac{2(1+q_1)}{r^2}+\dfrac{1}{r^4}}}$$

$$ds_{10}^2 = \Delta^{1/2}ds_5^2 + \Delta^{-1/2}G_{\alpha\beta}(dy^\alpha + A^\alpha)(dy^\beta + A^\beta)$$

$$G_{\alpha\beta}dy^\alpha dy^\beta\ = \left(e^{2\mu}\left(e^{\lambda}\sin^2\ \phi + e^{-\lambda}\cos^2\ \phi\right)\sin^2\ \tilde{\theta} + e^{-2\mu}\cos^2\ \tilde{\theta}\right)d\tilde{\theta}^2$$
$$+e^{2\mu}\left(e^{\lambda}\cos^2\ \phi + e^{-\lambda}\sin^2\ \phi\right)\cos^2\ \tilde{\theta}d\phi^2 - e^{2\mu}\sinh\ \lambda\sin\ (2\phi)\sin\ (2\tilde{\theta})d\tilde{\theta}d\phi + e^{-\mu}\sin^2\ \tilde{\theta}d\tilde{\Omega}_3^2$$

$$A^\phi \equiv A^1 = -e^{-3\mu}dt$$

$$\Delta = \left(\frac{\det G}{\det G_0}\right)^{-2/3} = e^{-2\mu}\left(e^{\lambda}\cos^2\ \phi + e^{-\lambda}\sin^2\ \phi\right)\cos^2\ \tilde{\theta} + e^{\mu}\sin^2\ \tilde{\theta}$$

$$\lambda = \varphi_1\ , e^{3\mu} = H_1$$

$$ds_5^2 = \Omega_0^2\left[\frac{d\xi^2}{(1-\xi^2)^2} + \frac{\xi^2}{1-\xi^2}d\Omega_3^2\right] - \frac{\Omega_1^2}{1-\xi^2}dt^2$$

$$\frac{\Omega_1^2}{1-\xi^2} = H_1^{-2/3}f\ , \Omega_0^2\frac{\xi^2}{1-\xi^2} = H_1^{1/3}r^2\ , \frac{\Omega_0^2}{(1-\xi^2)^2} = H_1^{1/3}f^{-1}\left(\partial_\xi r\right)^2$$

$$(\partial_s r)^2 = r^2(1+r^2 H_1)\ \text{ with }\ \tanh\ s = \sqrt{1-\xi^2}$$

$$\Box_{10}\ \Phi = 0,$$

$$\Box_5\ \Phi_0 = 0$$

$$z \to e^{-2\pi i}z, \bar{z}\ \text{fixed}$$

$$z = 1-\sigma_R, \bar{z} = 1-\sigma_R\eta_R$$

$$\delta + \delta_{\text{AdS}} \equiv -p_t\Delta t - p_\varphi\Delta\varphi = 2|p_t|\int_{r_0}^{\infty}\frac{dr}{f}\sqrt{H_1 - \beta^2\frac{f}{r^2}}$$

$$r_0 = \sqrt{\frac{1}{(1-\beta^2)^2} + (q_1+2)q_1} - q_1 - 1$$

$$\Delta t = 2\int_{r_0}^{\infty} dr\frac{\dot{t}}{\dot{r}}, \Delta\varphi = 2\int_{r_0}^{\infty} dr\frac{\dot{\varphi}}{\dot{r}}$$

$$\delta+\delta_{\text{AdS}}=2|p_t|\int_0^1 dy\left[\frac{(1-\beta^2)\sqrt{1+2(1+q_1)\frac{y^2}{r_0^2}+\frac{y^4}{r_0^4}}-\frac{y^2}{r_0^2}}{1+2(1+q_1)\frac{y^2}{r_0^2}+\frac{y^4}{r_0^4}}\right]^{1/2}$$

$$\int_0^1\frac{(y^2)^{p_1}\sqrt{1-y^2}}{\left(y^2+\frac{\beta^2}{1-\beta^2}\right)^{p_2}}dy=\frac{\sqrt{\pi}}{4}\left(\frac{\beta^2}{1-\beta^2}\right)^{-p_2}\frac{\Gamma\left(p_1+\frac{1}{2}\right)}{\Gamma(p_1+2)}\,{}_2F_1\left(p_2,p_1+\frac{1}{2};p_1+2,-\frac{1-\beta^2}{\beta^2}\right)$$

$$\begin{aligned}\delta&=\pi|p_t|(1-\beta^2)^2\left\{\frac{\alpha^2}{2}+\alpha^4\left[1-\frac{15}{16}(\beta^2-1)^2\right]+\alpha^6\left[2-5(\beta^2-1)^2+\frac{105}{32}(\beta^2-1)^4\right]+\mathcal{O}(\alpha^8)\right\}\\&=8\pi\frac{h^2\bar{h}^2}{(h+\bar{h})^3}\alpha^2[1+2\alpha^2(1-15\nu^2)+4\alpha^4(1-40\nu^2+420\nu^4)]+\mathcal{O}(\alpha^8)\end{aligned}$$

$$|p_t|=h+\bar{h},\beta=\frac{h-\bar{h}}{h+\bar{h}},\nu=\left(\frac{h\bar{h}}{(h+\bar{h})^2}\right)$$

$$\Psi_B^R=\int_0^\infty dh\int_0^h d\bar{h}(h\bar{h})^{\Delta-2}\left[\frac{z^{h+1}\bar{z}^{\bar{h}}-z^{\bar{h}}\bar{z}^{h+1}}{z-\bar{z}}\right]e^{i\delta(h,\bar{h})}$$

$$\Psi_B^R=\frac{z}{z-\bar{z}}\int_0^\infty dh\int_0^\infty d\bar{h}(h\bar{h})^{\Delta-2}(h-\bar{h})z^h\bar{z}^{\bar{h}}e^{i\delta(h,\bar{h})}$$

$$\Psi_B^R=\left[\frac{z}{z-\bar{z}}(z\partial_z-\bar{z}\partial_{\bar{z}})\right]\int_0^\infty dh\int_0^\infty d\bar{h}z^h\bar{z}^{\bar{h}}e^{i\delta(h,\bar{h})}$$

$$\begin{aligned}I(a,c)\;&\equiv\int_0^\infty dh\int_0^\infty d\bar{h}\frac{h^{a+1}\bar{h}^{a+1}}{(h+\bar{h})^c}z^h\bar{z}^{\bar{h}}\\&=\frac{\Gamma^2(a+2)\Gamma(2a+4-c)}{\Gamma(2a+4)}\eta_R^{c-a-2}\sigma_R^{c-2a-4}\,{}_2F_1(a+2,c;2a+4;1-\eta_R)\end{aligned}$$

$$\begin{aligned}\Psi_B^R&\equiv\Psi_B^{R(0)}+\alpha^2\Psi_B^{R(2)}+\mathcal{O}(\alpha^4)=\Delta I(-1,0)+\alpha^2 8\pi i\Delta I(1,3)+\mathcal{O}(\alpha^4)\\&=\frac{1}{\eta_R^2\sigma_R^4}+\alpha^2 16\pi i\frac{(1-\eta_R^2)(1+28\eta_R+\eta_R^2)+12(1+3\eta_R+\eta_R^2)\eta_R\log(\eta_R)}{(1-\eta_R)^7\eta_R\sigma_R^5}+\mathcal{O}(\alpha^4)\end{aligned}$$

$$\Delta\equiv\frac{1}{\sigma_R(1-\eta_R)}\left(\partial_{\sigma_R}-\frac{1+\eta_R}{\sigma_R}\partial_{\eta_R}\right)$$

$$\Psi_B^{R(4)}=-32\pi^2\Delta I(3,6)+16\pi i\Delta[I(1,3)-15I(3,7)]$$

$$\Psi_B^{R(6)}=\Delta\left\{-i\frac{(8\pi)^3}{3!}I(5,9)-128\pi^2[I(3,6)-15I(5,10)]+32\pi i[I(1,3)-40I(3,7)+420I(5,11)]\right\}$$

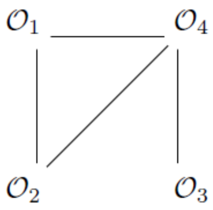
$$\langle O_{p_1}(\vec{x}_1,\vec{y}_1)O_{p_2}(\vec{x}_2,\vec{y}_2)O_{p_3}(\vec{x}_3,\vec{y}_3)O_{p_4}(\vec{x}_4,\vec{y}_4)\rangle$$

$$\gamma_{\min}=\max(p_{12},p_{43})\;;\;\gamma_{\max}=\min(p_1+p_2,p_3+p_4)$$

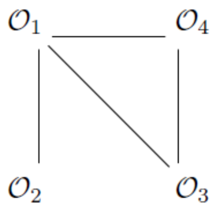
$$\langle O_{p_1}O_{p_2}O_{p_3}O_{p_4}\rangle=\mathcal{P}_{\vec{p}}\sum_{\underline{R}}A_{\underline{R}}B_{\underline{R}},$$

$$\mathcal{P}_{\vec{p}}=g_{12}^{\frac{p_1+p_2}{2}}g_{34}^{\frac{p_3+p_4}{2}}g_{13}^{\frac{p_3-p_4}{2}}g_{14}^{\frac{p_1+p_4-p_2-p_3}{2}}g_{24}^{\frac{p_2-p_1}{2}}\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{\max(p_{12},p_{43})}{2}}.$$

$$g_{ij}=\frac{\vec{y}_i\cdot\vec{y}_j}{(\vec{x}_{ij})^2}$$



$\max = p_{43}$



$\max = p_{12}$

$$\mathcal{B}^{\frac{1}{2}\frac{1}{2}}_{[0,p,0](0,0)} \,;\, \mathcal{B}^{\frac{1}{4}\frac{1}{4}}_{[q,p,q](0,0)}$$

$$\mathcal{C}^{\frac{1}{4}}_{[q,p,q](\frac{l}{2},\frac{l}{2})}$$

$\underline{\lambda}$	$\tau = \Delta - l$	l	$\mathfrak{R}$	multiplet
$[\varnothing]$	γ	0	$[0,\gamma,0]$	$\frac{1}{2}$-BPS
$[1^{\mu}]$	γ	0	$[\mu,\gamma-2\mu,\mu]$	$\frac{1}{4}$-BPS
$[\lambda,1^{\mu}]$ $(\lambda \geq 2)$	γ	$\lambda-2$	$[\mu,\gamma-2\mu-2,\mu]$	semi-short
$[\lambda_1,\lambda_2,2^{\mu_2},1^{\mu_1}]$ $(\lambda_2 \geq 2)$	$\gamma+2\lambda_2-4$	$\lambda_1-\lambda_2$	$[\mu_1,\gamma-2\mu_1-2\mu_2-4,\mu_1]$	long

$$\gamma = \gamma_{\min}, \gamma_{\min}+2, \dots, \gamma_{\max}$$

$$B_{\underline{R}} = \left[\left(\frac{g_{12}g_{34}}{g_{13}g_{24}}\right)^{\frac{\gamma-\gamma_{\min}}{2}} F_{\gamma,\underline{\lambda}}\right]$$

$$\frac{g_{12}g_{34}}{g_{13}g_{24}} = \frac{y_1y_2}{x_1x_2} \,;\, \frac{g_{14}g_{23}}{g_{13}g_{24}} = \frac{(1-y_1)(1-y_2)}{(1-x_1)(1-x_2)}.$$

$$U = x_1x_2 \,;\, V = (1-x_1)(1-x_2) \,;\, \sigma = \frac{1}{y_1y_2} \,;\, \tau = \frac{(1-y_1)(1-y_2)}{y_1y_2}.$$

$$F_{\gamma,\underline{\lambda}} = [P_{\underline{\lambda}} + P_{\underline{\lambda}+\square} + \cdots] = \sum_{\underline{\mu}:\underline{\lambda}\subseteq\underline{\mu}} (T_\gamma)^{\underline{\mu}}_{\underline{\underline{\mu}}} P_{\underline{\mu}}$$

$$B_{\gamma,\underline{\lambda}} = \mathcal{P}_{\vec{p}}\left[k + S(x_1,x_2,y_1,y_2) + \frac{\prod_{ij}(x_i-y_j)}{(y_1y_2)^2} H(x_1,x_2,y_1,y_2)\right]$$

$$S = \frac{\prod_{ij}(x_i-y_j)}{(x_1-x_2)(y_1-y_2)}\left[\frac{f(x_2,y_1)}{x_2y_1(x_1-y_2)} + \frac{f(x_1,y_2)}{x_1y_2(x_2-y_1)} - \frac{f(x_1,y_1)}{x_1y_1(x_2-y_2)} - \frac{f(x_2,y_2)}{x_2y_2(x_1-y_1)}\right]$$

$$\frac{\prod_{ij}(x_i-y_j)}{(y_1y_2)^2} = \mathcal{I}(U,V,\sigma,\tau)$$

$$H_{\gamma,\underline{\lambda}} = (-1)^{\lambda_1-\lambda_2}\left(\frac{x_1x_2}{y_1y_2}\right)^{\beta} \frac{\left(F_{\lambda_1}(x_1)F_{\lambda_2-1}(x_2) - x_1 \leftrightarrow x_2\right)}{x_1-x_2} \frac{\left(G_{\lambda_1'}(y_1)G_{\lambda_2'-1}(y_2) - y_1 \leftrightarrow y_2\right)}{y_1-y_2}$$

$$F_\lambda(z) = z^{\lambda-1}\,{}_2F_1\left[\begin{matrix}\lambda + \frac{\gamma-p_{12}}{2}, \lambda + \frac{\gamma-p_{43}}{2} \\ 2\lambda+\gamma\end{matrix}; z\right]$$

$$G_\lambda(z) = z^{\lambda+1}\,{}_2F_1\left[\begin{matrix}\lambda - \frac{\gamma-p_{12}}{2}, \lambda - \frac{\gamma-p_{43}}{2} \\ 2\lambda-\gamma\end{matrix}; z\right]$$

$$\lambda_1 = 2 + l + \frac{\tau-\gamma}{2} \,;\, \lambda_2 = 2 + \frac{\tau-\gamma}{2} \,;\, \lambda_1' = -\frac{b-\gamma}{2} \,;\, \lambda_2' = -a - \frac{b-\gamma}{2}$$

$$H_{\tau,l,[aba]} = (-1)^l (x_1x_2)^{\frac{\tau-\gamma_{\min}}{2}} \frac{x_1^{l+1}\ {}_2F_1\left[\begin{matrix}2+\frac{\tau}{2}+l-\frac{p_{12}}{2}, 2+\frac{\tau}{2}+l-\frac{p_{43}}{2}; x_1\\ 4+\tau+2l\end{matrix}\right]\ {}_2F_1\left[\begin{matrix}1+\frac{\tau}{2}-\frac{p_{12}}{2}, 1+\frac{\tau}{2}-\frac{p_{43}}{2}; x_2\\ 2+\tau\end{matrix}\right] - x_1 \leftrightarrow x_2}{(x_1-x_2)} \times$$

$$\frac{1}{(y_1y_2)^{\frac{b-\gamma_{\min}}{2}}} \frac{y_1 y_2^{-a}\ {}_2F_1\left[\begin{matrix}-1-\frac{b}{2}-a+\frac{p_{12}}{2}, -1-\frac{b}{2}-a+\frac{p_{43}}{2}; y_2\\ -2-b-2a\end{matrix}\right]\ {}_2F_1\left[\begin{matrix}-\frac{b}{2}+\frac{p_{12}}{2}, -\frac{b}{2}+\frac{p_{43}}{2}; y_1\\ -b\end{matrix}\right] - y_1 \leftrightarrow y_2}{(y_1-y_2)}$$

$$f_{\gamma,[\lambda,1^{\lambda'-1}]}(x,y) = (-1)^\lambda \left(\frac{x}{y}\right)^\beta \left[x^\lambda\ {}_2F_1\left[\lambda+\frac{\gamma-p_{12}}{2}, \lambda+\frac{\gamma-p_{43}}{2}; x\right] y^{\lambda'}\ {}_2F_1\left[\begin{matrix}\lambda'-\frac{\gamma-p_{12}}{2}, \lambda-\frac{\gamma-p_{43}}{2}\ ; y\\ 2\lambda'-\gamma\end{matrix}\right]\right]$$

$$H_{\gamma,[\lambda,1^{\lambda'-1}]} = (-1)^{\lambda-1} \sum_{h=1}^{\beta} \left(\frac{x_1x_2}{y_1y_2}\right)^\beta \frac{(F_{1-h}(x_1)F_\lambda(x_2) - x_1 \leftrightarrow x_2)}{x_1-x_2} \frac{(G_h(y_1)G_{\lambda'}(y_2) - y_1 \leftrightarrow y_2)}{y_1-y_2}$$

$$B_{\gamma,[\lambda,1^{\lambda'-1}]}\Big|_{x_2=y_2} = \frac{(x_1-y_1)}{x_1y_1} f_{\gamma,[\lambda,1^{\lambda'-1}]}$$

$$f_{\gamma,[\lambda,1^{\lambda'}-1]}(x,y) + f_{\gamma+2,[\lambda-1,1^{\lambda'}]}(x,y) = 0$$

$$B^{\mathcal{L}}_{\gamma,[\lambda,1^{\lambda'-1}]} = B_{\gamma,[\lambda,1^{\lambda'-1}]} + B_{\gamma+2,[\lambda-1,1^{\lambda'}]} = \frac{\prod_{ij}\ (x_i-y_j)}{(y_1y_2)^2} \mathcal{H}_{\tau=\gamma, l=\lambda-2, a=\lambda'-1, b=\gamma-2\lambda'}$$

$$B_{\gamma,[\lambda,1^{\lambda'-1}]} = B^{\mathcal{L}}_{\gamma,[\lambda,1^{\lambda'-1}]} - B_{\gamma+2,[\lambda-1,1^{\lambda'}]}$$

$$f_{\gamma,[\emptyset]}(x,y) = \left(\frac{x_1}{y_1}\right)^\beta \sum_{h=1}^{\beta} x^{1-h}\ {}_2F_1\left[1-h+\frac{\gamma-p_{12}}{2}, 1-h+\frac{\gamma-p_{43}}{2}; x\right] y^h\ {}_2F_1\left[\begin{matrix}h-\frac{\gamma-p_{12}}{2}, h-\frac{\gamma-p_{43}}{2}, y\\ 2h-\gamma\end{matrix}\right]$$

$$H_{\gamma,[\emptyset]} = \left(\frac{x_1x_2}{y_1y_2}\right)^\beta \sum_{1\le i<j\le\beta} \frac{[F_{1-i}(x_1)F_{1-j}(x_2) - x_1 \leftrightarrow x_2]}{x_1-x_2} \frac{[G_i(y_1)G_j(y_2) - y_1 \leftrightarrow y_2]}{y_1-y_2}$$

$$\langle \mathcal{O}_2^n \mathcal{O}_2^n \mathcal{O}_2 \mathcal{O}_2 \rangle$$

$$\partial H_{\tau,l,[aba]} \equiv U^{\tau/2} \partial_\tau U^{-\tau/2} H_{\tau,l,[aba]}$$

$$C_{[2^n][2^n];\mathcal{D}} = C^{(0)}_{[2^n][2^n];\mathcal{D}} + \frac{1}{a} C^{(1)}_{[2^n][2^n];\mathcal{D}} + \frac{1}{a^2} C^{(2)}_{[2^n][2^n];\mathcal{D}} + \cdots$$

$$\Delta_{\mathcal{D}_{\tau,l}} = \tau + l + \frac{2}{a}\eta_{(1);\mathcal{D}} + \frac{2}{a^2}\eta_{(2);\mathcal{D}} + \cdots$$

$$\log^1(U) \times \left[C^{(0)}_{[2^n][2^n]} \eta_{(1)} C^{(0)}_{22} H_{\tau,l,[aba]}\right]$$

$$\log^0(U) \times \left[2C^{(0)}_{[2^n][2^n]} \eta_{(1)} C^{(0)}_{22} \partial H_{\tau,l,[aba]} + \left[C^{(1)}_{[2^n][2^n]} C^{(0)}_{22} + C^{(0)}_{[2^n][2^n]} C^{(1)}_{22}\right] H_{\tau,l,[aba]}\right]$$

$$\log^2(U) \times \left[\frac{1}{2} C^{(0)}_{[2^n][2^n]} \eta^2_{(1)} C^{(0)}_{22} H_{\tau,l,[aba]}\right]$$

$$\log^1(U) \times \left[2C^{(0)}_{[2^n][2^n]} \eta^2_{(1)} C^{(0)}_{22} \partial H_{\tau,l,[aba]} + \left[C^{(0)}_{[2^n][2^n]} \eta_{(2)} C^{(0)}_{22} + C^{(1)}_{[2^n][2^n]} \eta_{(1)} C^{(0)}_{22} + C^{(0)}_{[2^n][2^n]} \eta_{(1)} C^{(1)}_{22}\right] H_{\tau,l,[aba]}\right]$$

$$\log^0(U) \times \left[\begin{matrix}2C^{(0)}_{[2^n][2^n]} \eta^2_{(1)} C^{(0)}_{22} \partial^2 H_{\tau,l,[aba]} + 2 \sum_{i+a+j=2} C^{(i)}_{[2^n][2^n]} \eta_{(a)} C^{(j)}_{22} \partial H_{\tau,l,[aba]} + \\ + \left[C^{(1)}_{[2^n][2^n]2} C^{(1)}_{22} + C^{(2)}_{[2^n][2^n]} C^{(0)}_{22} + C^{(0)}_{[2^n][2^n]} C^{(2)}_{22}\right] H_{\tau,l,[aba]}\end{matrix}\right.$$

$$\log^3(U)\times\left[\frac{1}{6}C^{(0)}_{[2^n][2^n]}\eta^3_{(1)}C^{(0)}_{22}H_{\tau,l,[aba]}\right]$$

$$\log^2(U)\times\left[C^{(0)}_{[2^n][2^n]}\eta^3_{(1)}C^{(0)}_{22}\partial H_{\tau,l,[aba]}+\frac{1}{2}\sum_{\substack{i+a+b+j=3\\ a,b\leq 2}}C^{(i)}_{[2^n][2^n]}\eta_{(a)}\eta_{(b)}C^{(j)}_{22}H_{\tau,l,[aba]}\right]$$

$$\log^1(U)\times\left[2C^{(0)}_{[2^n][2^n]}\eta^3_{(1)}C^{(0)}_{22}\partial^2 H_{\tau,l,[aba]}+2\sum_{\substack{i+a+b+j=3\\ a,b\leq 2}}C^{(i)}_{[2^n][2^n]}\eta_{(a)}\eta_{(b)}C^{(j)}_{22}\partial H_{\tau,l,[aba]}+\sum_{\substack{i+a+j=3\\ a\leq 3}}C^{(i)}_{[2^n][2^n]}\eta_{(a)}C^{(j)}_{22}H_{\tau,l,[aba]}\right]$$

$$\log^0(U)\times\left[\frac{4}{3}C^{(0)}_{[2^n][2^n]}\eta^3_{(1)}C^{(0)}_{22}\partial^3 H_{\tau,l,[aba]}+2\sum_{\substack{i+a+b+j=3\\ a,b\leq 2}}C^{(i)}_{[2^n][2^n]}\eta_{(a)}\eta_{(b)}C^{(j)}_{22}\partial H^2_{\tau,l,[aba]}\right.$$

$$\left.+\sum_{\substack{i+a+j=3\\ a\leq 3}}C^{(i)}_{[2^n][2^n]}\eta_{(a)}C^{(j)}_{22}\partial H_{\tau,l,[aba]}+\sum_{i+j=3}C^{(i)}_{[2^n][2^n]}C^{(j)}_{22}\partial H_{\tau,l,[aba]}\right]$$

$$C_{[2^n][2^n];\mathcal{T}}=\frac{1}{\sqrt{a}}\left[C^{\left(\frac{1}{2}\right)}_{[2^n][2^n];\mathcal{T}}+\frac{1}{a}C^{\left(\frac{3}{2}\right)}_{[2^n][2^n];\mathcal{T}}++\cdots\right]$$

$$\Delta_{\mathcal{T}_{\tau,l}}=\tau+l+\frac{2}{a}\eta_{(1);\mathcal{T}}+\frac{2}{a^2}\eta_{(2);\mathcal{T}}+\cdots$$

$$\log^1(U)\times 0+\log^0(U)\times\left[C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}C^{\left(\frac{1}{2}\right)}_{22}\right]H_{\tau,l,[aba]}$$

$$\log^2(U)\times 0+\log^1(U)\times\left[C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}\eta_{(1)}C^{\left(\frac{1}{2}\right)}_{22}H_{\tau,l,[aba]}\right]$$

$$\log^0(U)\times\left[2C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}\eta_{(1)}C^{\left(\frac{1}{2}\right)}_{22}\partial H_{\tau,l,[aba]}+\left[C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}C^{\left(\frac{3}{2}\right)}_{22}+C^{\left(\frac{3}{2}\right)}_{[2^n][2^n]}C^{\left(\frac{1}{2}\right)}_{22}\right]H_{\tau,l,[aba]}\right]$$

$$\log^3(U)\times 0+\log^2(U)\times\left[\frac{1}{2}C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}\eta^2_{(1)}C^{\left(\frac{1}{2}\right)}_{22}H_{\tau,l,[aba]}\right]$$

$$\log^1(U)\times\left[2C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}\eta^2_{(1)}C^{\left(\frac{1}{2}\right)}_{22}\partial H_{\tau,l,[aba]}+\left[C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}\eta_{(2)}C^{\left(\frac{1}{2}\right)}_{22}+C^{\left(\frac{3}{2}\right)}_{[2^n][2^n]}\eta_{(1)}C^{\left(\frac{1}{2}\right)}_{22}+C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}\eta_{(1)}C^{\left(\frac{3}{2}\right)}_{22}\right]H_{\tau,l,[aba]}\right]$$

$$\log^0(U)\times\left[2C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}\eta^2_{(1)}C^{\left(\frac{1}{2}\right)}_{22}\partial^2 H_{\tau,l,[aba]}+2\left[C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}\eta_{(2)}C^{\left(\frac{1}{2}\right)}_{22}+C^{\left(\frac{3}{2}\right)}_{[2^n][2^n]}\eta_{(1)}C^{\left(\frac{1}{2}\right)}_{22}+C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}\eta_{(1)}C^{\left(\frac{3}{2}\right)}_{22}\right]\partial H_{\tau,l,[aba]}\right.$$

$$\left.+\left[C^{\left(\frac{3}{2}\right)}_{[2^n][2^n]2}C^{\left(\frac{3}{2}\right)}_{22}+C^{\left(\frac{5}{2}\right)}_{[2^n][2^n]}C^{\left(\frac{1}{2}\right)}_{22}+C^{\left(\frac{1}{2}\right)}_{[2^n][2^n]}C^{\left(\frac{5}{2}\right)}_{22}\right]H_{\tau,l,[aba]}\right]$$

$$G=\mathcal{P}_{\vec{p}}\times\frac{U^a}{V^b}\sigma^i\tau^j\ ;\ a=\frac{\gamma-\gamma_{\min}}{2}\ ;\ b,i,j\geq 0$$

$$G|_{\substack{x_1=y_1\\ x_2=y_2}}=\mathcal{P}_{\vec{p}}\times k$$

$$G|_{x_2=y_2}=\mathcal{P}_{\vec{p}}\times\left[k+\frac{(x_1-y_1)}{x_1y_1}f_G(x_1,y_1)\right]$$

$$G=\mathcal{P}_{\vec{p}}\times\left[k+S(x_1,x_2,y_1,y_2)+\frac{\prod_{ij}(x_i-y_j)}{(y_1y_2)^2}H_G(x_1,x_2,y_1,y_2)\right]$$

$$f_G(x,y)=\sum_{h=1}^{\beta}f_{G,h}(x)\left[y^{h-\beta}\,{}_2F_1\left[\begin{matrix}h-\frac{\gamma-p_{12}}{2},h-\frac{\gamma-p_{43}}{2} & ,y\\ 2h-\gamma & \end{matrix}\right]\right]$$

$$f_{G,h}(x)=\sum_{\lambda\geq 1-h}c_{h,\lambda}\left[x^{\beta+\lambda}\,{}_2F_1\left[\begin{matrix}\lambda+\frac{\gamma-p_{12}}{2},\lambda+\frac{\gamma-p_{43}}{2};x_1\\ 2\lambda+\gamma\end{matrix}\right]\right]$$

$$\mathcal{G}_{2222} = U^2\sigma^2 + \frac{U^2\tau^2}{V^2} + \frac{4}{a}\frac{U^2\sigma\tau}{V}$$

$$f_{\lambda'=1}(x_1) = \frac{x_1^2(2-2x_1+x_1^2)}{(x_1-1)^2} - \frac{4}{a}\frac{x_1^2}{x_1-1}$$

$$f_{\lambda'=2}(x_1) = \frac{1}{2}\frac{(x_1-2)x_1(-2+2x_1+x_1^2)}{(x_1-1)^2} + \frac{2}{a}\frac{x_1(x_1-2)}{x_1-1}$$

$$A^{2222}_{\mathcal{S},4,l,[020]} = \left[(l+3)(l+4)+\frac{4}{a}\right]\frac{2(l+3)!^2}{(2l+6)}\frac{1+(-1)^l}{2}$$

$$\mathcal{P}_{2222^2} = 8a^2\sqrt{a+2}g_{12}g_{14}g_{24}g_{34}^2 \ ; \ \mathcal{G}_{2222^2} = \frac{4\sqrt{a+2}}{a}\left[1+U\sigma+\frac{U\tau}{V}\right]$$

$$|\mathcal{O}_2\mathcal{O}_2|\sqrt{|\mathcal{O}_2\mathcal{O}_2||\mathcal{O}_2^2\mathcal{O}_2^2|}\gamma_{\min} = 2$$

$$f_{\lambda'=1}(x_1) = \frac{4\sqrt{a+2}}{a}\left[x_1+\frac{x_1}{1-x_1}\right]$$

$$A^{22^222}_{\mathcal{S},4,[020]} = \frac{4\sqrt{a+2}}{a}(l+4)\frac{2(l+3)!^2}{(2l+6)!}\frac{1+(-1)^l}{2}$$

$$G = \mathcal{P}_{22^222^2}\left[\frac{8}{a}U\sigma+\frac{24}{a}\frac{U\tau}{V}\right]$$

$$f_{\lambda'=1}(x_1) = \frac{8}{a}x_1+\frac{24}{a}\frac{x_1}{1-x_1}$$

$$A_{4,l,[020]} = \frac{4}{a}[(l+3)(l+4)+(-1)^l 6]\frac{(l+2)!\,(l+4)!}{(2l+6)!}$$

$$\mathcal{G}_{[2^2][2^2]22} = \left[\frac{8}{a}U\sigma+\frac{8}{a}\frac{U\tau}{V}\right]$$

$$f_{\lambda'=1}(x_1) = \frac{8}{a}\left[x_1+\frac{x_1}{1-x_1}\right]$$

$$\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_4} - 2C^{(1)}_{22;\mathcal{D}_4}\right]C^{(0)}_{22;\mathcal{D}_4} = \frac{4}{3}(l+2)(l+5)\frac{(l+3)!^2}{(2l+6)!}\frac{1+(-1)^l}{2}$$

$$C^{(1)}_{[2^2][2^2];\mathcal{D}_4} = \frac{\frac{4}{3}(l+2)(l+5)\frac{(l+3)!^2}{(2l+6)!}}{C^{(0)}_{22;\mathcal{D}_4}}\frac{1+(-1)^l}{2} + 2C^{(1)}_{22;\mathcal{D}_4}$$

$$C^{(0)}_{22;\mathcal{D}_4} = \sqrt{\frac{(l+1)(l+6)}{3}}\frac{(l+3)!}{\sqrt{(2l+6)!}}\frac{1+(-1)^l}{2}$$

$$C^{(1)}_{22;\mathcal{D}_4} = \frac{\frac{2}{3}-8\left(1+2\psi^{(0)}(l+4)-2\psi^{(0)}(2l+7)\right)}{\sqrt{\frac{(l+1)(l+6)}{3}}}\frac{(l+3)!}{\sqrt{(2l+6)!}}\frac{1+(-1)^l}{2}$$

$$C^{(1)}_{[2^2][2^2];\mathcal{D}_4} = \frac{\frac{4}{3}(l+2)(l+5)+\frac{4}{3}-16\left(1+2\psi^{(0)}(l+4)-2\psi^{(0)}(2l+7)\right)}{\sqrt{\frac{(l+1)(l+6)}{3}}}\frac{(l+3)!}{\sqrt{(2l+6)!}}\frac{1+(-1)^l}{2}$$

$$C_{[2^2][2^2];\mathcal{O}_2^3} \sim \frac{1}{N} \ ; \ C_{22;\mathcal{O}_2^3} \sim \frac{1}{N}$$

$$\hat{C}_{[2^2][2^2]22}\big|_{\frac{1}{a}} - 2\hat{C}_{2222}\big|_{\frac{1}{a}} = \sum_{l}\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_4} - 2C^{(1)}_{22;\mathcal{D}_4}\right]C^{(0)}_{22;\mathcal{D}_4}B^{\mathcal{L}}_{4,l,[000]} + \sum_{l,\mathcal{D}_6}\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_6} - 2C^{(1)}_{22;\mathcal{D}_6}\right]C^{(0)}_{22;\mathcal{D}_6}B^{\mathcal{L}}_{6,l,[000]} + \sum_{l,\mathcal{T}_6}\left[C^{(\frac{1}{2})}_{[2^2][2^2];\mathcal{T}_6} - 2C^{(\frac{1}{2})}_{22;\mathcal{T}_6}\right]C^{(\frac{1}{2})}_{22;\mathcal{T}_6}B^{\mathcal{L}}_{6,l,[000]} + \mathfrak{H}$$

$$\hat{C}_{[2^2][2^2]22}\big|_{\frac{\log(U)}{a^2}} - 2\hat{C}_{2222}\big|_{\frac{\log(U)}{a^2}} = \sum_{l}\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_4} - 2C^{(1)}_{22;\mathcal{D}_4}\right]\eta_{\mathcal{D}_4}C^{(0)}_{22;\mathcal{D}_4}B^{\mathcal{L}}_{4,l,[000]} + \sum_{l,\mathcal{D}_6}\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_6} - 2C^{(1)}_{22;\mathcal{D}_6}\right]\eta_{\mathcal{D}_6}C^{(0)}_{22;\mathcal{D}_6}B^{\mathcal{L}}_{6,l,[000]} + \sum_{l,\mathcal{T}_6}\left[C^{(\frac{1}{2})}_{[2^2][2^2];\mathcal{T}_6} - 2C^{(\frac{1}{2})}_{22;\mathcal{T}_6}\right]\eta_{\mathcal{T}_6}C^{(\frac{1}{2})}_{22;\mathcal{T}_6}B^{\mathcal{L}}_{6,l,[000]} + \mathfrak{H}$$

$$C^{(\frac{1}{2})}_{22;\mathcal{T}_6} \neq 0 \text{ and } C^{(\frac{1}{2})}_{[2^2][2^2];\mathcal{T}_6} - 2C^{(\frac{1}{2})}_{22;\mathcal{T}_6} \neq 0$$

$$\mathcal{H}^{t}_{[2^2][2^2]22}\big|^{t}_{\frac{\log(U)}{a^2}} = \sum_{l}\left[-64\frac{(l+2)(l+5)}{(l+1)(l+6)}\frac{(l+3)!^2}{(2l+6)!}\frac{1+(-1)^l}{2}\right]H_{4,l,[000]} + \sum_{l}\left[-224\frac{(l+4)(l+5)}{(l+1)(l+8)}\frac{(l+4)!^2}{(2l+8)!}\frac{1+(-1)^l}{2}\right]H_{6,l,[000]} + \mathfrak{H}\ldots$$

$$C^{(0)}_{6} = \begin{pmatrix} C^{(0)}_{22;O_1} & C^{(0)}_{22;O_2} \\ C^{(0)}_{33;O_1} & C^{(0)}_{33;O_2}\end{pmatrix},$$

$$C^{(1)}_{pp,6} = \left[C^{(1)}_{pp;O_1},\quad C^{(1)}_{pp;O_2}\right]^t \ ;\ 2p > 6$$

$$V_{[2^2][2^2],6} = C^{(0)}_{6}\cdot\left[C^{(1)}_{[2^2][2^2],6} - 2C^{(1)}_{22,6}\right]$$

$$V^{(1)}_{[2^2][2^2],6} = \sum_{\mathcal{D}_6} C^{(0)}_{22;\mathcal{D}_6}\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_6} - 2C^{(1)}_{22;\mathcal{D}_6}\right]$$

$$V^{(2)}_{[2^2][2^2],6} = \sum_{\mathcal{D}_6} C^{(0)}_{33;\mathcal{D}_6}\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_6} - 2C^{(1)}_{22;\mathcal{D}_6}\right]$$

$$\hat{C}_{[2^2][2^2]22}\big|_{\frac{1}{a}} - 2\hat{C}_{2222}\big|_{\frac{1}{a}}\ ;\ \hat{C}_{[2^2][2^2]33}\big|_{\frac{1}{a}} - 2\hat{C}_{2233}\big|_{\frac{1}{a}}$$

$$A_6 = \left(C^{(0)}\right)\cdot\left(C^{(0)}\right)^t\ ;\ M_6 = \left(C^{(0)}\right)\cdot\begin{pmatrix}\eta_1 & 0\\ 0 & \eta_2\end{pmatrix}\cdot\left(C^{(0)}\right)^t$$

$$(M_6)\cdot(A_6)^{-1}\cdot V_{[2^2][2^2],6} = \begin{cases}\sum\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_6} - 2C^{(1)}_{22;\mathcal{D}_6}\right]\eta_{\mathcal{D}_6}C^{(0)}_{22;\mathcal{D}_6}\\ \sum\left[C^{(1)}_{[2^2][2^2];\mathcal{D}_6} - 2C^{(1)}_{22;\mathcal{D}_6}\right]\eta_{\mathcal{D}_6}C^{(0)}_{33;\mathcal{D}_6}\end{cases}$$

$$\mathcal{G}_{pppp}\big|_{\text{disco}} = 1 + U^p\sigma^p + \frac{U^p\tau^p}{V^p}\ ;\ \mathcal{G}_{2233} = 1 + \frac{6}{a}\left[U\sigma + \frac{U\tau}{V}\right] + \frac{12}{a}\frac{U^2\sigma\tau}{V}$$

$$\mathcal{H}_{22pp} = -\frac{2pU^p\bar{D}_{p,p+2,2,2}}{(p-2)!}\ ;\ \mathcal{H}_{3333}\big|_{[000]} = -\frac{3}{2}U^p\left[\bar{D}_{2,5,2,3} + \bar{D}_{2,5,3,2} + 6\bar{D}_{3,5,2,2} + 8\bar{D}_{3,5,3,3}\right]$$

$$A_6 = +\begin{pmatrix}\frac{1}{10} & 0\\ 0 & \frac{1}{40}(l+2)(l+7)\end{pmatrix}(l+1)(l+8)\frac{(l+4)!^2}{(2l+8)!}$$

$$M_6 = -6\begin{pmatrix}4 & 12\\ 12 & 44+9l+l^2\end{pmatrix}\frac{(l+4)!^2}{(2l+8)!}$$

$$C_{[2^2][2^2]33}\big|_{\frac{1}{a}} = 2C_{2233}\big|_{\frac{1}{a}}$$

$$\mathcal{P}_{[2^2][2^2]33} = g_{12}^4 g_{34}^3 \mathcal{N}_{[2^2]} \mathcal{N}_3$$

$$\mathcal{G}_{[2^2][2^2]33} = 1 + \frac{12}{a}\left[U\sigma + \frac{U\tau}{V}\right] + \frac{24}{a}\frac{U^2\sigma\tau}{V} + \frac{24}{a(a+2)}\left[U^2\sigma^2 + \frac{U^2\tau^2}{V^2} + \frac{4U^2\sigma\tau}{V} + 2\left[\frac{U^3\sigma^2\tau}{V} + \frac{U^3\sigma\tau^2}{V^2}\right]\right]$$

$$\mathcal{N}_{[2^2]}\mathcal{N}_3 = \frac{24(N^2-1)(N^2-4)(N^4-1)}{N}$$

$$\mathcal{C}_{[2^2][2^2]33}\Big|_{\frac{1}{a}} = 2\mathcal{C}_{2233}\Big|_{\frac{1}{a}}$$

$$\hat{\mathcal{C}}_{[2^2][2^2]22}\Big|_{\frac{1}{a}} - 2\hat{\mathcal{C}}_{2222}\Big|_{\frac{1}{a}} = \frac{4}{a}U^2\sigma^2 + \frac{4}{a}\frac{U^2\tau^2}{V^2} + \frac{16}{a}\frac{U^2\sigma\tau}{V},$$

$$V_{[2^2][2^2],6} = \left[\frac{4+9l+l^2}{5}\frac{(l+4)!^2}{(2l+8)!}, 0\right]\frac{1+(-1)^l}{2}.$$

$$\sum \left[C^{(1)}_{[2^2][2^2];\mathcal{D}_6} - 2C^{(1)}_{22;\mathcal{D}_6}\right]\eta_{\mathcal{D}_6} C^{(0)}_{22;\mathcal{D}_6} = -96\frac{(l^2+9l+4)}{(l+1)(l+8)}\frac{(l+4)!^2}{(2l+8)!}\frac{1+(-1)^l}{2}$$

$$\mathcal{H} = \sum_{\tau,l,[aba]} A_{\tau,l,[aba]} H_{\tau,l,[aba]}$$

$$\mathbb{F}_{\tau,l} = \frac{\left(\frac{\tau}{2}+1\right)!^2\left(l+\frac{\tau}{2}\right)!^2}{48\tau!\,(2l+\tau+2)!}\ ;\ T = \frac{\tau+1}{2}\ ;\ L = l + \frac{\tau+3}{2}$$

$$\mathcal{P}_{2[2^2]2[2^2]}\left(U^2\sigma^2 + \frac{2U^2\tau^2}{V^2}\right)$$

$$\mathbb{R}_{\tau,l} = (l+1)(2l+\tau+4)(2l+\tau+2)(\tau+2+l)\left((-1)^l\frac{96}{\tau(\tau+2)} - \frac{1}{4} + L^2\right)$$

$$A_{6,l} = \frac{2}{15}(l+1)(l+4)(l+5)(l+8)\left[\frac{1}{2}(l+3)(l+6) + 2\frac{1+(-1)^l}{2}\right]\frac{(l+3)!^2}{(2l+8)!}$$

$$\mathbb{R}_{\tau,l} = \mathbb{R}^-_{\tau,l}\frac{1-(-1)^l}{2} + \mathbb{R}^+_{\tau,l}\frac{1+(-1)^l}{2}$$

$$\mathbb{R}^-_{\tau,l} = -\frac{(-4+\tau)(-2+\tau)(\tau+4)(\tau+6)}{2}\left[(l+1)(l+\tau+2)\left(L^2 - \frac{96}{\tau(\tau+2)} - \frac{3}{5}T^2 + \frac{71}{10}\right)\right]$$

$$\mathbb{R}^+_{\tau,l} = \mathbb{R}^-_{\tau,l} - 24\frac{(-4+\tau)(-2+\tau)(\tau+4)(\tau+6)}{\tau(\tau+2)}(2+2l+\tau)(4+2l+\tau)$$

$$A_{6,l} = -8\left[(l+2)(l+7)(16+9l+l^2)\frac{1+(-1)^l}{2} + (l+1)(l+3)(l+6)(l+8)\frac{1-(-1)^l}{2}\right]\frac{(l+3)!^2}{(2l+8)!}$$

$$A_{6,l} = 32(-1)^{l+1}\left[\frac{3(l+1)(l+3)(l+6)(l+8)}{(l+4)(l+5)} - ((2l+9)^2-37)\frac{1+(-1)^l}{2}\right]\frac{(l+3)!^2}{(2l+8)!}$$

$$\mathbb{R}_{\tau,l} = 32\frac{(l+1)(l+\tau+2)}{\tau^2(\tau+2)^2}(-1)^{l+1}\left[2\left(\frac{\tau}{2}-2\right)_6\left[96 + \frac{8}{15}\left(\frac{\tau}{2}-2\right)_6\left(\frac{1}{2} - \frac{1+(-1)^l}{2}\left(1+\frac{3}{(l+1)(l+\tau+2)}\right)\right)\right] + \Sigma_{\tau,l}\right]$$

$$\Sigma_{\tau,l} = \sum_{i=0}^{\frac{\tau-6}{2}} \frac{(-1)^i(T-I-2)_5(T+I-2)_5 \times I \times \sum_{j=0}^{3} B_j(T-I)^{2j}}{L^2 - I^2}$$

$$B_0 = \frac{(-5+I+T)(-3+I+T)(3+I+T)(5+I+T)(1191-71(I+T)^2)}{92400}$$
$$B_1 = \frac{-508221+100379(I+T)^2-2979(I+T)^4-59(I+T)^6}{831600}$$
$$B_2 = \frac{32445-2979(I+T)^2-1330(I+T)^4+64(I+T)^6}{831600}$$
$$B_3 = \frac{-639-59(I+T)^2+64(I+T)^4+4(I+T)^6}{831600}$$

$$A_{6,l} = -192\frac{(l+3)!^2}{(2l+8)!}\Big[$$
$$\left[(82+45l+5l^2)-\frac{16}{(l+3)}+\frac{16}{(l+6)}-4(l+2)(l+7)\mathrm{HarmonicNumber}(l+4)\right]\frac{1+(-1)^l}{2}+$$
$$\left[(86+45l+5l^2)-\frac{24}{(l+4)}+\frac{24}{(l+5)}-4(l+2)(l+7)\mathrm{HarmonicNumber}(l+4)\right]\frac{1-(-1)^l}{2}\Big]$$

[n=0]	$\frac{1}{15}(l+1)(l+4)^2(l+5)^2(l+8)\frac{(l+3)!^2}{(2l+8)!}$
[n=1]	$-8(l+4)(l+5)\left((l+1)(l+8)+24\frac{1+(-1)^l}{2}\right)\frac{(l+3)!^2}{(2l+8)!}$
[n=2]	$480(l+1)(l+8)\left(84+\left(84+\frac{1008}{(l+1)(l+8)}\right)\frac{1+(-1)^l}{2}\right)\frac{(l+3)!^2}{(2l+8)!}$

[**n** = **0**]	$\frac{1}{15}(l+1)(l+4)^2(l+5)^2(l+8)\frac{(l+3)!^2}{(2l+8)!}$
[**n** = **1**]	$-8(l+4)(l+5)\left((l+1)(l+8)+24\frac{1+(-1)^l}{2}\right)\frac{(l+3)!^2}{(2l+8)!}$
[n = 2]	$480(l+1)(l+8)\left(84+\left(84+\frac{1008}{(l+1)(l+8)}\right)\frac{1+(-1)^l}{2}\right)\frac{(l+3)!^2}{(2l+8)!}$

$$\hat{C}_{[1^n][1^n]11} = \text{free } + \frac{(x_1-y_1)(x_2-y_2)}{y_1y_2}\mathcal{H}_{[1^n][1^n]11}(U,V)$$

$$\mathcal{H}_{[1^n][1^n]11} = \oint \frac{dsdt}{(2\pi i)^2}\Gamma[-s]^2\Gamma[-t]^2\Gamma[s+t+2]^2U^{1+s}V^t\mathcal{M}_{[1^n][1^n]11}(s,t)$$

$$\hat{C}_{[2^n][2^n]22} = \text{free } + \frac{(x_1-y_1)(x_1-y_2)(x_2-y_1)(x_2-y_2)}{(y_1y_2)^2}\mathcal{H}_{[2^n][2^n]22}(U,V)$$

$$\mathcal{H}_{[2^n][2^n]22} = \oint \frac{dsdt}{(2\pi i)^2}\Gamma[-s]^2\Gamma[-t]^2\Gamma[s+t+4]^2U^{2+s}V^t\mathcal{M}_{[2^n][2^n]22}(s,t)$$

$$\mathcal{P}_L(z,\bar{z}) = \sum_{r=0}^{L}\frac{(-1)^r}{r!}B\begin{bmatrix}2L-r\\L\end{bmatrix}\log^r(z\bar{z})(\mathrm{Li}_{2L-r}(z)-\mathrm{Li}_{2L-r}(\bar{z}))$$

$$\mathcal{P}_0(z,\bar{z}) = -\frac{z}{z-1}+\frac{\bar{z}}{\bar{z}-1}$$
$$\mathcal{P}_1(z,\bar{z}) = \log(z,\bar{z})(\log(1-z)-\log(1-\bar{z}))+2(\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar{z}))$$
$$\mathcal{P}_2(z,\bar{z}) = \frac{1}{2}\log^2(z,\bar{z})(\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar{z}))-3\log(z,\bar{z})(\mathrm{Li}_3(z)-\mathrm{Li}_3(\bar{z}))+6(\mathrm{Li}_4(z)-\mathrm{Li}_4(\bar{z}))$$

$$\mathcal{P}_L\left(\frac{1}{z},\frac{1}{\bar{z}}\right) = -\mathcal{P}_L(z,\bar{z})$$

$$\mathcal{P}_L(z,\bar{z})\ ;\ \mathcal{P}_L\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right)\ ;\ \mathcal{P}_L(1-z,1-\bar{z})$$

$$\Phi^{(L)}(z,\bar{z}) = -\frac{1}{z-\bar{z}}\mathcal{P}_L\left(\frac{z}{z-1},\frac{z}{\bar{z}-1}\right)$$

$$\frac{UV}{z-\bar{z}}\partial_z\partial_{\bar{z}}[(z-\bar{z})\Phi^L(z,\bar{z})] = -\Phi^{(L-1)}(z,\bar{z})\ ;\ \Phi^{(0)} = 1$$

$$U = (1-z)(1-\bar{z})\ ;\ V = z\bar{z}$$

$$\Phi^{(L)}(1-z,1-\bar{z}) = \Phi^{(L)}(z,\bar{z})$$

$$\Phi^{(L)}(z,\bar{z})\big|_{\log^L(U)\log^L(V)} = \frac{1}{(L!)^2}\frac{1}{z-\bar{z}}$$

$$\zeta = U + \sum_{n,m\geq 1}\frac{U^n}{n!}\frac{V^m}{m!}\frac{(m+n-1)!\,(m+n-2)!}{(n-1)!\,(m-1)!}$$
$$\bar{z} = V + \sum_{n,m\geq 1}\frac{U^n}{n!}\frac{V^m}{m!}\frac{(m+n-1)!\,(m+n-2)!}{(n-1)!\,(m-1)!}$$

$$\Phi^{(L)}(U,V)\big|_{\log^L(U)\log^L(V)} = \frac{1}{(L!)^2}\frac{1}{1-\zeta-\bar{z}} = \frac{1}{(L!)^2}\sum_{m,n,\geq 0}\frac{U^nV^m}{n!\,m!}\times\frac{(m+n)!^2}{n!\,m!}$$

$$\Phi^{(L)} = \oint\frac{U^sV^t}{(2\pi i)^2}\Gamma[-s]^2\Gamma[-t]^2\Gamma[-u]^2\mathcal{K}^{(L)}(s,t)\ ;\ u = -s-t-1\ ;\ L\geq 1$$

$$\Delta_{st}\big[\mathcal{K}^{(L)}\big] := -st\mathcal{K}^{(L)}[s,t] + \frac{s^2t}{s+t}\mathcal{K}^{(L)}[s-1,t] + \frac{t^2s}{s+t}\mathcal{K}^{(L)}[s,t-1]$$

$$\Delta_{st}\big[\mathcal{K}^{(L)}\big] = -\mathcal{K}^{(L-1)}(s,t)$$

$$\mathcal{K}^{(L)}(s,t) = \frac{1}{(s-n)^{L-1}(t-m)^{L-1}} + \int_{\rightsquigarrow}^{\leftsquigarrow}\langle\circlearrowleft\mid\circlearrowright\rangle n,m\in\mathbb{N}$$

$$\mathcal{K}^{(2)}(s,t) = \frac{2}{(2!)^2}[-\Psi_1^2+\Psi_2-\pi^2]$$
$$\mathcal{K}^{(3)}(s,t) = \frac{3}{(3!)^2}\big[+\Psi_1^4+3\Psi_2^2-2\Psi_1\Psi_3-2\pi^2\Psi_2-4\pi^2\mathcal{K}^{(2)}-\pi^4\big]$$
$$\mathcal{K}^{(4)}(s,t) = \frac{4}{(4!)^2}[-\Psi_1^6-3\Psi_1^4\Psi_2-9\Psi_1^2\Psi_2^2+9\Psi_2^3+4\Psi_1^3\Psi_3-12\Psi_1\Psi_2\Psi_3-4\Psi_3^2+3\Psi_1^2\Psi_4+3\Psi_2\Psi_4+$$
$$+\pi^2(6\Psi_1^2\Psi_2-12\Psi_2^2+6\Psi_1\Psi_3-\Psi_4)-9\times 4\times\pi^2\mathcal{K}^{(3)}-6\pi^4\mathcal{K}^{(2)}-\pi^6\big]$$

$$\Psi_n(s,t) = \mathrm{PolyGamma}[n-1,-s] + (-1)^n\mathrm{PolyGamma}[n-1,-t].$$

$$\mathcal{C}(z,\bar{z})\big|_{\frac{\log L(U)\log^L(V)}{(L!)^2}} = \frac{P(U,V)}{(z-\bar{z})^{2\gamma+1}}$$

$$P(U,V) = \sum_{a,b}c_{ab}U^aV^b\ ;\ c_{ab}\in\mathbb{R}$$

$$\frac{U^aV^b}{(z-\bar{z})^{2\gamma+1}} = \frac{\gamma!}{(2\gamma)!}\oint(-U)^s(-V)^t\frac{\Gamma[-s]\Gamma[-t]}{\Gamma[s+1]\Gamma[t+1]}\Gamma[s+t+X]^2\times m_{ab,\gamma}$$

$$m_{ab,\gamma} = \left[(-)^{a+b}\frac{(-s)_a(-t)_b}{(s+1)_{\gamma-a}(t+1)_{\gamma-b}}(s+t+X)_{2\gamma+1-a-b-X}(s+t+X)_{\gamma+1-a-b-X}\right]$$

$$A(s,t)=\sum_{a,b} c_{ab} m_{ab,\gamma}$$

$$\oint \frac{U^s V^t}{(2\pi i)^2}\Gamma[-s]^2\Gamma[-t]^2\Gamma[-u]^2[\mathcal{K}^{(L)}(s,t)A(s,t)+\cdots]=\frac{\log^L(U)\log^L(V)}{(L!)^2}\frac{P(U,V)}{(z-\bar{z})^{2\gamma+1}}+\cdots$$

weight	anti-symmetric functions	;	symmetric functions
0		;	1
1		;	$\log(V)$, $\log(U)$
2	$\mathcal{P}_1$	;	$\log^2(V)$, $\log(U)\log(V)$, $\log^2(U)$
3	$\mathcal{P}_1\log(V)$, $\mathcal{P}_1\log(U)$	;	$\log^3(V)$, $\log(U)\log(V)\log(UV)$, $\log^3(U)$ $\log(U)\log(V)\log(\frac{U}{V})$ $\partial_-\mathcal{P}_2(1-z)$, $\partial_-\mathcal{P}_2(z)$, $\partial_-\mathcal{P}_2(\frac{z}{z-1})$

weight	functions
4^-	$\mathcal{P}_1\log^2(V)$, $\mathcal{P}_1\log^2(U)$, $\mathcal{P}_1\log^2(\frac{U}{V})$ $\mathcal{P}_2(1-z)$, $\mathcal{P}_2(z)$, $\mathcal{P}_2(\frac{z}{z-1})$

weight	functions
4^+	$\mathcal{P}_1^2$, ~~$\log(U)\log(V)(\log^2(U)+\log^2(V))$~~ , ~~$\log^4(V)$~~ , ~~$\log^4(U)$~~ $\log(U)\log^2(V)\log(\frac{U}{V})$, $\log^2(U)\log(V)\log(\frac{U}{V})$ $\zeta_3\log(V)$, $\zeta_3\log(U)$, $\log(UV)\partial_-\mathcal{P}_2(z)$ $\partial_-\partial_+\mathcal{P}_3(1-z)$, $\partial_-\partial_+\mathcal{P}_3(z)$, $\partial_-\partial_+\mathcal{P}_3(\frac{z}{z-1})$

weight	functions
5^-	~~$\mathcal{P}_1 \log^3(V)$~~ , ~~$\mathcal{P}_1 \log(U)\log(V)\log(UV)$~~ , ~~$\mathcal{P}_1 \log^3(U)$~~
	$\mathcal{P}_1 \log(U)\log(V)\log(\frac{U}{V})$
	$\mathcal{P}_1 \partial_- \mathcal{P}_2(1-z)$, $\mathcal{P}_1 \partial_- \mathcal{P}_2(z)$, $\mathcal{P}_1 \partial_- \mathcal{P}_2(\frac{z}{z-1})$
	$\log(U)\mathcal{P}_2(1-z)$, $\log(V)\mathcal{P}_2(z)$, $\log(UV)\mathcal{P}_2(\frac{z}{z-1})$
	$\partial_+ \mathcal{P}_3(1-z)$, $\partial_+ \mathcal{P}_3(z)$, $\partial_+ \mathcal{P}_3(\frac{z}{z-1})$

$$\mathcal{H}^t_{[2^3][2^3]22} = \text{span}\{\mathcal{P}_3, \partial_-\mathcal{P}_3, \partial_+\mathcal{P}_3, \partial_+\partial_-\mathcal{P}_3\} \cup \{4^-, 3, 2, 1, 0\}$$

$$\partial_+\mathcal{P}_3\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right)$$

$$\frac{1}{(z-\bar{z})}\left[z(1-z)\frac{\partial}{\partial z} + \bar{z}(1-\bar{z})\frac{\partial}{\partial \bar{z}}\right]\left[(z-\bar{z})\oint \frac{U^s V^t}{(2\pi i)^2}\Gamma[-s]^2\Gamma[-t]^2\Gamma[s+t+1]^2\mathcal{K}^{(3)}(s,t)\right].$$

$$\oint \frac{1}{(2\pi i)^2}\Gamma[-s]^2\Gamma[-t]^2\Gamma[s+t+1]^2[(s-t)U^sV^t + (1+s+t)(U^sV^{t+1} - U^{s+1}V^t)]\mathcal{K}^{(3)}(s,t)$$

$$\oint \frac{U^s V^t}{(2\pi i)^2}\Gamma[-s]^2\Gamma[-t]^2\Gamma[s+t+1]^2\left[(s-t) + \frac{t^2\mathcal{K}^{(3)}(s,t-1) - s^2\mathcal{K}^{(3)}(s-1,t)}{s+t}\right]$$

$$\frac{1}{3}\left[\Psi_1^3(u,t) - \frac{1}{2}\Psi_3(u,t) + \pi^2\Psi_1\right]$$

$$\frac{1}{(z-\bar{z})}\left[z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right]\left[(z-\bar{z})\oint \frac{U^s V^t}{(2\pi i)^2}\Gamma[-s]^2\Gamma[-t]^2\Gamma[s+t+1]^2\mathcal{K}^{(3)}(s,t)\right]$$

$$\langle \mathcal{O}_2\mathcal{O}_2^2\mathcal{O}_2\mathcal{O}_2^2\rangle|_{\tau=4} = \mathcal{P}_{2[2^2]2[2^2]2} \times \left[\frac{8}{a}U\sigma + \frac{24}{a}\frac{U\tau}{V}\right]$$

$$\mathcal{P}_{2[2^2]2[2^2]2} = 16a^2(a+2)g_{12}^2 g_{24}^2 g_{34}^2$$

$$\{\mathcal{O}_2 \,\square^{\tau-4}\, \partial^l \mathcal{O}_2, \mathcal{O}_3 \,\square^{\tau-6}\, \partial^l \mathcal{O}_3, \ldots\}$$

$$\mathcal{H}_{[2^3][2^3]22}\big|_{\frac{1}{a^2}} = 3\mathcal{H}^t_{[2^2][2^2]22}\big|_{\frac{1}{a^2}} + 3\mathcal{H}_{2222}\big|_{\frac{1}{a^2}}$$

$$\Delta = mK = \frac{R}{2},$$

$$m \to \infty, \text{ with } \lambda = \frac{m g_{YM}^2}{16\pi^2} \text{ fixed}$$

$$\langle \Phi_I(x)\rangle_\theta = \frac{2\sqrt{\lambda}}{\sqrt{d_{12}}}\left(e^{i\theta}\Omega_K^{(N)}\frac{(Y_1)_I}{(x-x_1)^2} + e^{-i\theta}\overline{\Omega_K^{(N)}}\frac{(Y_2)_I}{(x-x_2)^2}\right), I = 1, \ldots, 6.$$

$$\Omega_K^{(N)} = \begin{pmatrix} \omega_1^{(K)} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \omega_2^{(K)} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \omega_K^{(K)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}, \text{ with } \omega_n^{(K)} = e^{\frac{2\pi i}{K}n}$$

$$SU(N) \longrightarrow U(1)^{K-1} \times U(N-K)$$

$$\langle \mathcal{O}_K^m(x_1, Y_1)\mathcal{O}_K^m(x_2, Y_2)\rangle = \left(\frac{d_{12}}{4\pi^2}\right)^{\Delta} \Gamma(\Delta+1)(\Delta)^{\alpha_{N,K}} A^{\Delta} B\left(1 + O\left(\frac{1}{m}\right)\right),$$

$$\alpha_{N,K} = \frac{K}{2}(2N - K - 1).$$

$$\langle \mathcal{O}_N^m(x_1, Y_1)\mathcal{O}_N^m(x_2, Y_2)\rangle = \left(\frac{d_{12}}{4\pi^2}\right)^{\Delta} f_N(m) 2^{-\Delta} N^{2m-\Delta} \frac{\Gamma(\Delta + \alpha_{N,N} + 1)}{\Gamma(\alpha_{N,N} + 1)}$$

$$\frac{\langle \mathcal{O}_K^m(x_1, Y_1)\mathcal{O}_K^m(x_2, Y_2)\mathcal{O}(x_3, Y_3)\rangle}{\langle \mathcal{O}_K^m(x_1, Y_1)\mathcal{O}_K^m(x_2, Y_2)\rangle} \simeq \int_0^{2\pi} \frac{d\theta}{2\pi} \langle \mathcal{O}(x_3, Y_3)\rangle_\theta$$

$$\mathcal{T}_{m,K}(u, v; \lambda) \simeq \frac{1}{2u} \sum_s \left(\left(\sum_{\ell=0}^{\infty} (-M_s^2)^{\ell} P^{(\ell)}(u, v)\right)^2 - 1\right),$$

$$M = \sqrt{\lambda},$$

$$M^{(ij)} = 2\sqrt{\lambda} \sin\left(\frac{i-j}{K}\pi\right), 0 \le i < j < K$$

$$T_p(x, Y) = \left(\frac{\tau_2}{4\pi}\right)^{p/2} Y_{I_1} \cdots Y_{I_p} \mathrm{Tr}(\Phi^{I_1}(x) \cdots \Phi^{I_p}(x))$$

$$\tau := \tau_1 + \mathrm{i}\tau_2 = \frac{\theta}{2\pi} + \mathrm{i}\frac{4\pi}{g_{\mathrm{YM}}^2}.$$

$$T_{\mathbf{m}}(x, Y) = \prod_{k \ge 1} T_{m_k}(x, Y)$$

$$\mathbf{m} = \{m_1, m_2 \cdots\}, \text{ with } m_i \ge m_{i+1}, m_i \in \{2, \ldots, N\}$$

$$R_{\mathbf{m}} = 2\Delta_{\mathbf{m}} = 2 \sum_{k \ge 1} m_k$$

$$\langle T_{\mathbf{m}}(x_1, Y_1) T_{\mathbf{m}'}(x_2, Y_2)\rangle = \mathcal{N}_{\mathbf{m}\mathbf{m}'}(N, K)\left(\frac{d_{12}}{4\pi^2}\right)^{\Delta_{\mathbf{m}}}, \sum_{k=1} m_k = \sum_{k=1} m_k'$$

$$d_{ij} = \frac{Y_i \cdot Y_j}{x_{ij}^2}, \text{ with } x_{ij} := x_i - x_j$$

$$\{T_{4,4}(x, Y), T_{5,3}(x, Y), T_{3,3,2}(x, Y), T_{4,2,2}(x, Y), T_{2,2,2,2}(x, Y)\}.$$

$$\mathcal{O}_{4,4}(x, Y) = T_{4,4}(x, Y) + \sum_{\mathbf{n} \neq (4,4)} c_{\mathbf{n}}^{(4,4)} T_{\mathbf{n}}(x, Y),$$
$$\mathcal{O}_{5,3}(x, Y) = T_{5,3}(x, Y) + \sum_{\mathbf{n} \neq (4,4),(5,3)} c_{\mathbf{n}}^{(5,3)} T_{\mathbf{n}}(x, Y),$$

$$\langle \mathcal{O}_{\mathbf{m}}(x_1, Y_1)\mathcal{O}_{\mathbf{m}'}(x_2, Y_2)\rangle \propto \delta_{\mathbf{m}\mathbf{m}'}$$

$$\underbrace{\mathcal{O}_{K,\dots,K}}_{m}(x,Y):=\mathcal{O}_K^m(x,Y), K>2.$$

$$\langle\mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)\rangle=\mathcal{N}_m(N,K)\left(\frac{d_{12}}{4\pi^2}\right)^{mK}$$

$$\frac{\langle\mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)T_{\mathbf{p}}(x_3,Y_3)\rangle}{\langle\mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)\rangle}=\mathfrak{C}_{mm\mathbf{p}}(N,K)\times\left(\frac{1}{2\pi^2}\frac{d_{23}d_{31}}{d_{12}}\right)^{\frac{\Delta_{\mathbf{p}}}{2}},$$

$$\frac{\langle\mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)T_2(x_3,Y_3)T_2(x_4,Y_4)\rangle}{\langle\mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)\rangle}=\mathcal{G}_{\text{free}}(x_i,Y_i)+\mathcal{I}_4(x_i,Y_i)\mathcal{T}_{m,K}(u,v;\tau,\bar{\tau}),$$

$$\mathcal{I}_4(x_i,Y_i)=\left(\frac{d_{34}}{4\pi^2}\right)^2\frac{(z-\alpha)(z-\bar{\alpha})(\bar{z}-\alpha)(\bar{z}-\bar{\alpha})z\bar{z}}{\alpha^2\bar{\alpha}^2(1-z)(1-\bar{z})}$$

$$u=z\bar{z}=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, v=(1-z)(1-\bar{z})=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2},$$

$$\alpha\bar{\alpha}=\frac{Y_1\cdot Y_2Y_3\cdot Y_4}{Y_1\cdot Y_3Y_2\cdot Y_4},(1-\alpha)(1-\bar{\alpha})=\frac{Y_1\cdot Y_4Y_2\cdot Y_3}{Y_1\cdot Y_3Y_2\cdot Y_4}$$

$$m\to\infty,\tau_2\to\infty,\lambda=\frac{g_{\text{YM}}^2m}{16\pi^2}=\frac{m}{4\pi\tau_2}\quad\text{fixed}$$

$$\langle\mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)\dots\rangle=\frac{1}{Z}\int\mathcal{D}[\text{ fields }]\mathrm{e}^{-S_{\text{SYM}}}\mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)\dots$$

$$\delta_{\text{fields}}\left(-S_{\text{SYM}}+\log\left(\mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)\right)\right)=0$$

$$\Phi_I^{\text{cl}}(x)=\frac{2\sqrt{\lambda}}{\sqrt{d_{12}}}\left(e^{i\theta}\Omega_K^{(N)}\frac{(Y_1)_I}{(x-x_1)^2}+e^{-i\theta}\overline{\Omega_K^{(N)}}\frac{(Y_2)_I}{(x-x_2)^2}\right),$$

$$\Omega_K^{(N)}=\begin{pmatrix}\omega_1^{(K)}&0&\cdots&0&\cdots&0\\0&\omega_2^{(K)}&\cdots&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots&\cdots&\vdots\\0&0&\cdots&\omega_K^{(K)}&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&0\\0&0&\cdots&0&\cdots&0\end{pmatrix},\text{ with }\omega_n^{(K)}=e^{\frac{2\pi i}{K}n}$$

$$SU(N)\longrightarrow U(1)^{K-1}\times U(N-K),2<K\leq N$$

$$\underbrace{(K-1)}_{U(1)^{K-1}}+\underbrace{(N-K)^2}_{U(N-K)}$$

$$Z=\frac{\Phi_1+i\Phi_2}{\sqrt{2}},Y=\frac{\Phi_3+i\Phi_4}{\sqrt{2}},X=\frac{\Phi_5+i\Phi_6}{\sqrt{2}}.$$

$$X^{\text{cl}}(x)=0,Y^{\text{cl}}(x)=0,Z^{\text{cl}}(x)=2\sqrt{\lambda}e^{i\theta}\Omega_K^{(N)}\frac{|x_1-x_2|}{(x-x_1)^2},$$

$$\bar{X}^{\text{cl}}(x)=0,\bar{Y}^{\text{cl}}(x)=0,\bar{Z}^{\text{cl}}(x)=2\sqrt{\lambda}e^{-i\theta}\bar{\Omega}_K^{(N)}\frac{|x_1-x_2|}{(x-x_2)^2}.$$

$$\Phi_I=\Phi_I^{\text{cl}}+\delta\Phi_I$$

$$S\supset\frac{2}{g_{\text{YM}}^2}\int d^4x\frac{1}{4}\text{Tr}\left(\left|[\delta X,Z^{\text{cl}}]\right|^2+\left|[\delta Y,Z^{\text{cl}}]\right|^2+\left|[\delta Z,Z^{\text{cl}}]\right|^2\right)$$

$$m_{ij}(x)^2=\left|Z_i^{\text{cl}}-Z_j^{\text{cl}}\right|^2=\frac{4(x_1-x_2)^2}{(x-x_1)^2(x-x_2)^2}M_{ij}^2,M_{ij}=\sqrt{\lambda}\left|\omega_i^{(K)}-\omega_j^{(K)}\right|.$$

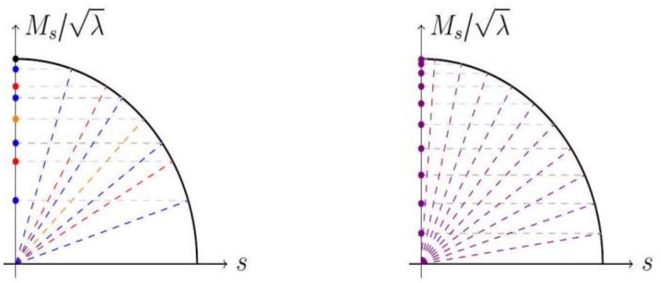
$$M_{ij} = \sqrt{\lambda}, i = 1, \dots, K, j = K+1, \dots, N.$$

$$M_{ij} = 2\sqrt{\lambda}\sin\left(\frac{i-j}{K}\pi\right), 1 \le i < j \le K$$

$$\underbrace{(K-1)+(N-K)^2}_{\text{massless}} + \underbrace{2K(N-K)}_{M_{ij}=\sqrt{\lambda}} + \underbrace{K(K-1)}_{M_{ij}=2\sqrt{\lambda}\sin\left(\frac{s}{K}\pi\right)} = N^2 - 1.$$

$$\begin{aligned} M_s &= 2\sqrt{\lambda}\sin\left(\frac{s}{K}\pi\right), && \text{with } 1 \le s < K, && \text{appear } K \text{ times,} \\ M_s &= \sqrt{\lambda}, && \text{with } s = K, && \text{appear } 2K(N-K) \text{ times.} \end{aligned}$$

$$\sum_s M_s^2 = 2KN\lambda$$
$$\sum_s \log(M_s) = K\left(\log K + \left(N - \frac{K+1}{2}\right)\log \lambda\right)$$



(a) K = 4, 6, 8 (b) K= 21

$$\log \mathcal{N}_m(N,K) = \log(\Gamma(\Delta+1)) + \alpha_{N,K}\log(\Delta) + \Delta\log(A) + B + O(\Delta^{-1}),$$

$$\alpha_{N,K} = 2\Delta a = 2(a_{\text{CFT}} - a_{\text{EFT}}),$$

$$a_{\text{CFT}} = \frac{1}{4}(N^2-1).$$

$$a_{\text{EFT}} = \frac{1}{4}(K-1+(N-K)^2).$$

$$\alpha_{N,K} = \frac{K}{2}(2N-K-1)$$

$$\langle T_2^m(x_1,Y_1)T_2^m(x_2,Y_2)\rangle = \left(\frac{d_{12}}{4\pi^2}\right)^{2m} 2^{2m}\Gamma(m+1)\frac{\Gamma\left(\frac{N^2-1}{2}+m\right)}{\Gamma\left(\frac{N^2-1}{2}\right)}$$

$$\alpha_{T_2^m} = \frac{N^2-2}{2} = 2\left(\frac{N^2-1}{4} - \frac{1}{4}\right).$$

$$\frac{\langle \mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)\mathcal{O}(x_3,Y_3)\rangle}{\langle \mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)\rangle} \simeq \int_0^{2\pi}\frac{d\theta}{2\pi}\langle \mathcal{O}(x_3,Y_3)\rangle_\theta$$

$$\langle \Phi_I(x)\rangle_\theta = \Phi_I^{\text{cl}}(x)$$

$$\langle T_2(x_3,Y_3)\rangle_\theta = \frac{\tau_2}{4\pi}Y_3^I Y_3^J \operatorname{Tr}\left(\Phi_I^{\text{cl}}(x_3)\Phi_J^{\text{cl}}(x_3)\right) = \frac{1}{4\pi^2}\frac{m}{d_{12}}\operatorname{Tr}\left(d_{13}\Omega_K^{(N)}e^{i\theta} + d_{23}\overline{\Omega_K^{(N)}}e^{-i\theta}\right)^2$$

$$\operatorname{Tr}\left(\Omega_K^{(N)}\overline{\Omega_K^{(N)}}\right) = K$$

$$\mathfrak{C}_{mm2} \simeq mK.$$

$$\langle T_{\mathbf{p}}(x_3,Y_3)\rangle_\theta = \left\langle \prod_{k\geq 1} T_{p_k}(x_3,Y_3)\right\rangle_\theta = \prod_{k\geq 1} \langle T_{p_k}(x_3,Y_3)\rangle_\theta,$$

$$\langle T_p(x_3,Y_3)\rangle_\theta = \left(\frac{\tau_2}{4\pi}\right)^{p/2} Y_3^{I_1}\dots Y_3^{I_p}\mathrm{Tr}\left(\langle \Phi_{I_1}(x_3)\rangle_\theta \dots \langle \Phi_{I_p}(x_3)\rangle_\theta\right).$$

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\langle T_{\mathbf{p}}(x_3,Y_3)\rangle_\theta = \left(\frac{1}{2\pi^2}\frac{d_{23}d_{31}}{d_{12}}\right)^{\frac{\Delta_{\mathbf{p}}}{2}}\left(\frac{m}{2}\right)^{\frac{\Delta_{\mathbf{p}}}{2}}\int_0^{2\pi}\frac{d\theta}{2\pi}\prod_k \mathrm{Tr}\left(e^{i\theta}\Omega_K^{(N)} + \text{c.c.}\right)^{p_k},$$

$$\mathrm{Tr}\left(\Omega_K^{(N)}\right)^\ell = 0, \ell = 1,\dots,K-1,$$

$$\mathrm{Tr}\left(e^{i\theta}\Omega_K^{(N)} + \text{c.c.}\right)^{p_k} = \begin{cases} K\left(e^{iK\theta}+e^{-iK\theta}\right)\delta_{p_k K}, & K \text{ odd},\\ K\binom{p_k}{p_k/2} + K\left(e^{iK\theta}+e^{-iK\theta}\right)\delta_{p_k K}, & K \text{ even}. \end{cases}$$

$$\mathfrak{C}_{mm\mathbf{p}}(N,K) \simeq \left(\frac{m}{2}\right)^{\frac{\Delta_{\mathbf{p}}}{2}} K^n \binom{n}{n/2}\frac{1+(-1)^n}{2}\left(\prod_{\text{even } p_k} K\binom{p_k}{\frac{p_k}{2}}\right).$$

$$K^n\left(\binom{K}{K/2} + \left(e^{iK\theta}+e^{-iK\theta}\right)\right)^n = K^n\sum_{b=0}^{n}\binom{n}{b}\binom{K}{K/2}^{n-b}\left(e^{iK\theta}+e^{-iK\theta}\right)^b.$$

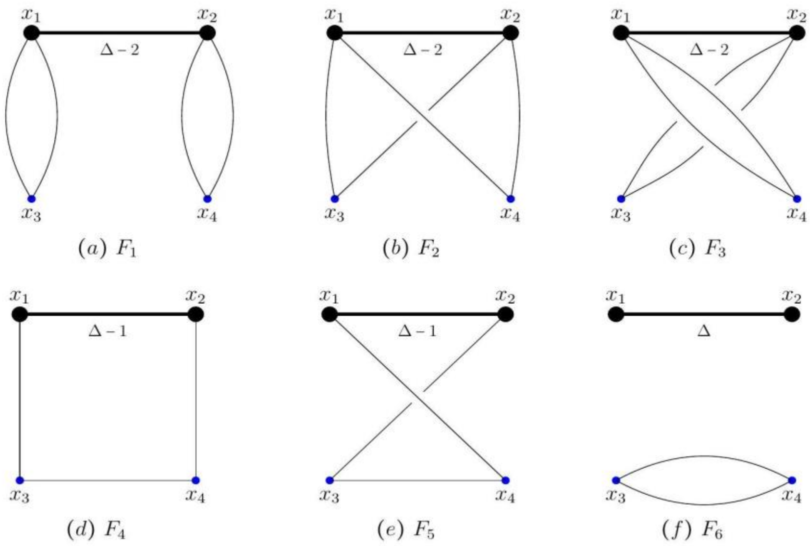
$$\int_0^{2\pi}\frac{d\theta}{2\pi}\sum_{b=0}^{n}\binom{n}{b}\binom{K}{K/2}^{n-b}\left(e^{iK\theta}+e^{-iK\theta}\right)^b = \sum_{b'=0}^{n/2}\binom{n}{2b'}\binom{K}{K/2}^{n-2b'}\binom{2b'}{b'}.$$

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\prod_{k=1}^{n}\mathrm{Tr}\left(e^{i\theta}\Omega_K^{(N)} + \text{c.c.}\right)^K = K^n\binom{K}{K/2}^n {}_2F_1\left(\frac{1-n}{2},-\frac{n}{2},1,\frac{4}{\binom{K}{K/2}^2}\right),$$

$$\mathfrak{C}_{mm\mathbf{p}}(N,K) \simeq \left(\frac{m}{2}\right)^{\frac{\Delta_{\mathbf{p}}}{2}}\left(\prod_{\text{even} p_k<K} K\binom{p_k}{\frac{p_k}{2}}\right)K^n\binom{K}{K/2}^n {}_2F_1\left(\frac{1-n}{2},-\frac{n}{2},1,\frac{4}{\binom{K}{K/2}^2}\right).$$

$$\mathcal{G}_{\text{free}} = \left(\frac{d_{34}}{4\pi^2}\right)^2\left(\frac{F_1}{\alpha^2\bar{\alpha}^2} + \frac{(1-\alpha)(1-\bar{\alpha})}{\alpha^2\bar{\alpha}^2}F_2 + \frac{(1-\alpha)^2(1-\bar{\alpha})^2}{\alpha^2\bar{\alpha}^2}F_3 + \frac{F_4}{\alpha\bar{\alpha}} + \frac{(1-\alpha)(1-\bar{\alpha})}{\alpha\bar{\alpha}}F_5 + F_6\right),$$

$$\langle \Phi_I(x)\rangle_\theta = \frac{2\sqrt{\lambda}}{\sqrt{d_{12}}}\left(e^{i\theta}\Omega_K^{(N)}\frac{(Y_1)_I}{(x-x_1)^2} + e^{-i\theta}\bar{\Omega}_K^{(N)}\frac{(Y_2)_I}{(x-x_2)^2}\right),$$



$$\mathcal{G}_{\text{free}}|_{O(m^2)} = \int_0^{2\pi}\frac{d\theta}{2\pi}\langle T_2(x_3,Y_3)\rangle_{\text{cl}}\langle T_2(x_4,Y_4)\rangle_{\text{cl}}$$

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\langle T_2\rangle_{\text{cl}}\langle T_2\rangle_{\text{cl}} = \left(\frac{1}{4\pi^2 d_{12}}\right)^2 m^2\left[(d_{13}^2d_{24}^2 + d_{14}^2d_{23}^2)\operatorname{Tr}\left[\left(\Omega_K^{(N)}\right)^2\right]\operatorname{Tr}\left[\left(\bar{\Omega}_K^{(N)}\right)^2\right]\right.$$
$$\left.+4d_{13}d_{24}d_{14}d_{23}\left[\operatorname{Tr}\left(\Omega_K^{(N)}\bar{\Omega}_K^{(N)}\right)\right]^2\right]$$

$$F_1(u,v) = F_3(u,v) = 0,$$

$$F_2(u,v) \simeq 4m^2K^2\frac{u^2}{v},$$

$$\mathcal{G}_{\text{free}}|_{O(m)} = \frac{\tau_2}{4\pi}\frac{d_{34}}{4\pi^2}(Y_3)_I(Y_4)_J\int_0^{2\pi}\frac{d\theta}{2\pi}\operatorname{Tr}(\langle\Phi^I(x_3)\rangle_{\text{cl}}\langle\Phi^J(x_4)\rangle_{\text{cl}})$$

$$\mathcal{G}_{\text{free}}|_{O(m)} = \frac{m}{(4\pi^2)^2}\frac{d_{34}}{d_{12}}(d_{13}d_{24} + d_{14}d_{23})\operatorname{Tr}\left(\Omega_K^{(N)}\bar{\Omega}_K^{(N)}\right)$$

$$F_4(u,v) \simeq mKu, F_5(u,v) \simeq mK\frac{u}{v},$$

$$F_6(u,v) = \frac{N^2-1}{2}$$

$$\mathcal{G}_{\text{free}} \simeq \left(\frac{d_{34}}{4\pi^2}\right)^2\left(\frac{(1-\alpha)(1-\bar{\alpha})}{\alpha^2\bar{\alpha}^2}\frac{4\Delta^2u^2}{v} + \frac{\Delta u}{\alpha\bar{\alpha}} + \frac{(1-\alpha)(1-\bar{\alpha})}{\alpha\bar{\alpha}}\frac{\Delta u}{v} + \frac{N^2-1}{2}\right),$$

$$Y_1 = \overline{Y_2} = Y = (1,-i,0,0,0,0), Y_3 = \overline{Y_4} = Y_X = (0,0,0,0,1,i)$$

$$\mathcal{G}_{\text{free}} = \left(\frac{1}{4\pi^2}\right)^2\frac{4}{x_{34}^4}F_6, \mathcal{I}_4(x_i,Y_i) = \left(\frac{1}{4\pi^2}\right)^2\frac{4u}{x_{34}^4},$$

$$\frac{\langle\mathcal{O}_K^m(x_1,Y)\mathcal{O}_K^m(x_2,\bar{Y})T_2(x_3,Y_X)T_2(x_4,\overline{Y_X})\rangle}{\langle\mathcal{O}_K^m(x_1,Y)\mathcal{O}_K^m(x_2,\bar{Y})\rangle} = \left(\frac{1}{4\pi^2}\right)^2\frac{4}{x_{34}^4}\left(F_6 + u\mathcal{T}_{m,K}(u,v;\tau,\bar{\tau})\right).$$

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\langle\operatorname{Tr}X^2(x_3)\operatorname{Tr}\bar{X}^2(x_4)\rangle_\theta.$$

$$\operatorname{Tr}X^2 = \frac{\tau_2}{4\pi}\left(\sum_i \mathrm{x}_i^2 + \sum_s \mathrm{x}_s^+\mathrm{x}_s^-\right),$$

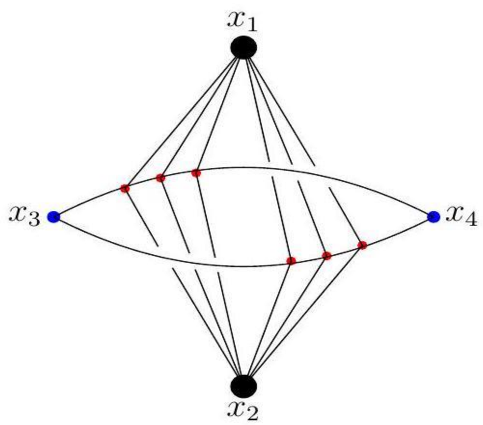
$$\frac{2}{g_{\rm YM}^2}\int d^4x(\partial_\mu\bar{\rm x}_i\partial^\mu{\rm x}_i+{\rm c.c.})+\frac{2}{g_{\rm YM}^2}\int d^4x\left(\sum_{\pm}\partial_\mu\bar{\rm x}_s^{\pm}\partial^\mu{\rm x}_s^{\pm}+\frac{4(x_1-x_2)^2}{(x-x_1)^2(x-x_2)^2}M_s^2{\rm x}_s^+{\rm x}_s^-+{\rm c.c.}\right),$$

$$\langle\bar{\rm x}_i(x_1){\rm x}_j(x_2)\rangle_\theta=\frac{g_{\rm YM}^2}{4\pi^2}\frac{\delta_{ij}}{x_{12}^2}.$$

$$\left(-\Box_x+\frac{4(x_1-x_2)^2}{(x-x_1)^2(x-x_2)^2}M_s^2\right)G_s(x,y)=\delta^{(4)}(x-y).$$

$$\left\langle\bar{\rm x}_s^{\pm}(x_3){\rm x}_s^{\mp}(x_4)\right\rangle_\theta=\frac{g_{\rm YM}^2}{4\pi^2x_{34}^2}\sum_s t_s(u,v;\lambda),\ \text{with}\ t_s(u,v;\lambda)=\sum_{\ell=0}^{\infty}(-M_s^2)^\ell P^{(\ell)}(u,v).$$

$$P^{(L)}(u,v)=\int\frac{d^4x_5}{\pi^2}\dots\frac{d^4x_{L+4}}{\pi^2}\frac{x_{13}^2x_{24}^2(x_{12}^2)^{L-1}}{x_{45}^2\prod_{i=5}^{L+4}x_{i,i+1}^2x_{1i}^2x_{2i}^2},$$



$$P^{(L)}(u,v)=\frac{u}{z-\bar{z}}\sum_{r=0}^{L}\frac{(-1)^r(2L-r)!}{r!(L-r)!L!}\log^r(v)(\mathrm{Li}_{2L-r}(1-z)-\mathrm{Li}_{2L-r}(1-\bar{z}))$$

$$\begin{aligned}\langle\mathrm{Tr}X^2(x_3)\mathrm{Tr}\bar{X}^2(x_4)\rangle_\theta&=2\left(\frac{\tau_2}{4\pi}\right)^2\sum_i\langle\bar{x}_i(x_1)x_j(x_2)\rangle_\theta^2+2\left(\frac{\tau_2}{4\pi}\right)^2\sum_s\langle\bar{x}_s^+(x_1)x_s^-(x_2)\rangle_\theta^2\\&=2\frac{(K-1)+(N-K)^2}{16\pi^4x_{34}^4}+\frac{2}{16\pi^4x_{34}^4}\sum_s t_s^2(u,v;\lambda)\end{aligned}$$

$$F_6(u,v)=\frac{N^2-1}{2}$$

$$\mathcal{T}_{m,K}(u,v;\lambda)=\frac{1}{2u}\sum_s(t_s^2(u,v;\lambda)-1)$$

$$t_s(u,v;\lambda)=\frac{u}{\sqrt{v}}\sum_{r=1}^{\infty}\frac{re^{-\sigma\sqrt{r^2+4M_s^2}}}{\sqrt{r^2+4M_s^2}}\frac{\sin(r\varphi)}{\sin(\varphi)}$$

$$t_s(u,v;\lambda)=\frac{u}{\sqrt{v}}\sum_{n\geq0}(W_s(\varphi+2\pi n,\sigma)+W_s(2\pi-\varphi+2\pi n,\sigma))$$

$$W_s(x,\sigma)=\frac{2M_sxK_1\left(2M_s\sqrt{x^2+\sigma^2}\right)}{\sin(x)\sqrt{x^2+\sigma^2}}$$

$$\mathcal{T}_{m,K}(u,v;\lambda)=\sum_{\Delta,S}\sum_i\left|C_{T_2\mathcal{O}_K^m\mathcal{O}_{\Delta,S}^{(i)}}\right|^2\mathcal{G}_{\Delta,S}(u,v),$$

$$\mathcal{G}_{\Delta,S}(u,v)=e^{-\sigma(\Delta-\Delta_H)}\frac{\sin(S+1)\varphi}{\sin\varphi}.$$

$$\mathcal{T}_{m,K}(u,v;\lambda)=\sum_{s}\sum_{S=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=1}^{S+1}C_{S,n,r}^{(s)}e^{-\sigma\left(\Delta_{S,n,r}^{(s)}-\Delta_{\mathcal{H}}\right)}\frac{\sin{(S+1)\varphi}}{\sin{\varphi}},$$

$$C_{S,n,r}^{(s)}=\frac{(r+n)(S+2+n-r)}{\sqrt{(r+n)^2+4M_s^2}\sqrt{(S+2+n-r)^2+4M_s^2}}$$

$$\Delta_{S,n,r}^{(s)}=\Delta_{\mathcal{O}_K^m}+\sqrt{(r+n)^2+4M_s^2}+\sqrt{(S+2+n-r)^2+4M_s^2}$$

$$\Delta_{S,n,r}^{(s)}=\Delta_{\mathcal{O}_K^m}+2+2n+S+O(\lambda).$$

$$\mathcal{O}_K^m\ \square^n\underbrace{\partial\ldots\partial}_{S}T_2.$$

$$\mathcal{T}_{m,K}(u,v;\lambda)\simeq C_{\mathcal{O}_K^m\mathcal{O}_K^m\mathcal{O}_\mathcal{K}}C_{T_2T_2\mathcal{O}_\mathcal{K}}u^{\gamma/\mathcal{K}/2}-\left(C_{\mathcal{O}_K^m\mathcal{O}_K^m\mathcal{O}_K}C_{T_2T_2\mathcal{O}_\mathcal{K}}\right)\Big|_{g_{\mathrm{YM}}=0},$$

$$\gamma_\mathcal{K}=3\frac{Ng_{\mathrm{YM}}^2}{4\pi^2}-3\left(\frac{Ng_{\mathrm{YM}}^2}{4\pi^2}\right)^2+\frac{21}{4}\left(\frac{Ng_{\mathrm{YM}}^2}{4\pi^2}\right)^3+\cdots.$$

$$\gamma_\mathcal{K}=12N\frac{\lambda}{m}+O(1/m^2),$$

$$C_{T_2T_2\mathcal{O}_K}=1-\frac{3}{2}\frac{Ng_{\mathrm{YM}}^2}{4\pi^2}+\cdots=1-6N\frac{\lambda}{m}+O(1/m^2).$$

$$C_{\mathcal{O}_K^m\mathcal{O}_K^m\mathcal{O}_K}=mKC_{\mathcal{O}_K^m\mathcal{O}_K^m\mathcal{O}_K}^{(0)}+\sum_{L\geq1}C_{\mathcal{O}_K^m\mathcal{O}_K^m\mathcal{O}_K}^{(L)}\lambda^L+O(1/m)$$

$$\mathcal{T}_{m,K}(u,v;\lambda)\simeq\sum_{L\geq1}C_{\mathcal{O}_K^m\mathcal{O}_K^m\mathcal{O}_K}^{(L)}\lambda^L+6N\lambda C_{\mathcal{O}_K^m\mathcal{O}_K^m\mathcal{O}_K}^{(0)}(\log u-1)$$

$$P^{(1)}(u,v)=u(2-\log u)+O(u^2,u(v-1)),$$
$$P^{(L\geq2)}(u,v)=u\binom{2L}{L}\zeta(2L-1)+O(u^2,u(v-1)).$$

$$\frac{1}{2u}(t_s^2(u,v;\lambda)-1)\simeq-M_s^2(2-\log u)+\sum_{L\geq2}(-M_s^2)^L\binom{2L}{L}\zeta(2L-1).$$

$$\mathcal{T}_{m,K}(u,v;\lambda)\simeq2KN\lambda(\log u-2)+\sum_{s}\sum_{L\geq2}(-M_s^2)^L\binom{2L}{L}\zeta(2L-1),$$

$$C_{\mathcal{O}_K^m\mathcal{O}_K^m\mathcal{O}_K}\simeq\frac{mK}{3}-2KN\lambda+\sum_{s}\sum_{L\geq2}(-M_s^2)^L\binom{2L}{L}\zeta(2L-1),$$

$$C_{\mathcal{O}_K^m\mathcal{O}_K^m\mathcal{O}_K}\simeq\frac{mK}{3}\underbrace{-2KN\lambda+\sum_{s}\int_0^\infty\frac{M_s dw}{\sinh^2(w)}[2M_s w-J_1(4M_s w)]}_{C_1(\lambda)},$$

$$C_{\mathcal{O}_N^m\mathcal{O}_N^m\mathcal{O}_K}(\lambda^*)=\frac{mK}{3}.$$

$$a=\mathrm{diag}(a_1,a_2,\cdots,a_N),\sum_{i=1}^N a_i=0.$$

$$\mathcal{Z}(\tau,\tau_p;\mu)=\int d\sigma(a_i)\left|\exp\left(i\pi\tau\sum_i a_i^2+\mathrm{i}\sum_{p>2}\pi^{p/2}\tau_p\sum_i a_i^p\right)\right|^2 Z_{1-\mathrm{loop}}(a;\mu)\left|Z_{\mathrm{inst}}(\tau,\tau_p,a;\mu)\right|^2,$$

$$d\sigma(a_i) = \prod_{i=1}^{N} da_i \prod_{i<j} a_{ij}^2 \delta\left(\sum_i a_i\right), a_{ij} := a_i - a_j$$

$$\mathcal{Z} \equiv \mathcal{Z}(\tau, 0; 0) = \int d\sigma(a_i) \exp\left(-2\pi\tau_2 \sum_i a_i^2\right)$$

$$a \to \frac{a}{\sqrt{2\pi\tau_2}},$$

$$\langle\langle f(a_i)\rangle\rangle = \frac{1}{\mathcal{Z}} \int d\sigma(a_i) \exp\left(-\sum_i a_i^2\right) f(a_i)$$

$$t_n(a) = \prod_{k\geq 1} (\mathrm{Tr}a^{n_k})$$

$$\langle\langle t_{\boldsymbol{n}}(a) t_{\boldsymbol{m}}(a)\rangle\rangle = \frac{1}{\mathcal{Z}} \prod_{k\geq 1} \prod_{l\geq 1} \left(-\mathrm{i}\pi^{1/2}\right)^{-n_k} \left(\mathrm{i}\pi^{1/2}\right)^{-m_l} \partial_{\tau_{n_k}} \partial_{\bar{\tau}_{m_l}} \mathcal{Z}(\tau, \tau_p; 0)\Big|_{\tau_p=0}.$$

$$T_{\boldsymbol{n}}(a) = t_{\boldsymbol{n}}(a) + \sum_{\boldsymbol{m} \vdash m < n} \alpha_{\boldsymbol{m},\boldsymbol{n}} t_{\boldsymbol{m}}(a),$$

$$\langle\langle T_{\boldsymbol{n}}(a) t_{\boldsymbol{m}}(a)\rangle\rangle = 0, \boldsymbol{m} \vdash m < n.$$

$$\mathcal{G}_{m,K}(\tau, \bar{\tau}) = I_2[\mathcal{T}_{m,K}(u, v; \tau, \bar{\tau})] = -\frac{8}{\pi} \int_0^\infty dr \int_0^\pi d\theta \frac{r^3 \sin^2\theta}{uv} \mathcal{T}_{m,K}(u, v; \tau, \bar{\tau})$$

$$Z_{1-\mathrm{loop}}(a; \mu) = \frac{1}{H(\mu)^N} \prod_{i<j} \frac{H^2(a_{ij})}{H(a_{ij} + \mu) H(a_{ij} - \mu)},$$

$$H(x) = \mathrm{e}^{-(1+\gamma)x^2} G(1 + \mathrm{i}x) G(1 - \mathrm{i}x).$$

$$\mathcal{G}_{m,K}(\tau, \bar{\tau}) = \frac{\partial_{\tau_{m,K}} \partial_{\bar{\tau}_{m,K}} \partial_\mu^2 \log \mathcal{Z}(\tau, \tau_p; \mu)\big|_{\tau_p, \mu=0}}{\partial_{\tau_{m,K}} \partial_{\bar{\tau}_{m,K}} \log \mathcal{Z}(\tau, \tau_p; 0)\big|_{\tau_p=0}},$$

$$\log Z_{1-\mathrm{loop}} = -\mu^2 \left[\sum_{L=1}^{\infty} \sum_{j=0}^{2L} (-1)^{L+j} \left(\frac{1}{2\pi\tau_2}\right)^L \binom{2L}{j} (2L+1) \zeta(2L+1) \mathrm{Tr}a^{2L-j} \mathrm{Tr}a^j\right] + O(\mu^4).$$

$$\{t_{4,4,4}, t_{5,4,3}, t_{3,3,3,3}, t_{5,5,2}, t_{4,3,3,2}, t_{4,4,2,2}, t_{5,3,2,2}, t_{3,3,2,2,2}, t_{4,2,2,2,2}, t_{2,2,2,2,2,2}\}.$$

$$\mathcal{O}_{\mathbf{m}}(a), \mathbf{m} = \{m_1, m_2, \ldots\}$$

$$\{t_2^n \mathcal{O}_{\mathbf{m}}(a)\}_{n,\mathbf{m}\geq 0}$$

$$\left\langle\left\langle t_2^n \mathcal{O}_{\mathbf{m}}(a), t_2^{n'} \mathcal{O}_{\mathbf{m}'}(a)\right\rangle\right\rangle \propto \delta_{\mathbf{m},\mathbf{m}'}$$

$$\mathcal{O}_{i_1,\ldots i_{k-1}, \underbrace{i_k, \ldots, i_k}_{m}, i_{k+1}, \ldots i_n} \text{ with } i_\ell \neq i_j \text{ if } \ell \neq j$$

$$\mathcal{O}_{i_1,\ldots i_{k-1}, \underbrace{i_k, \ldots, i_k}_{m}, i_{k+1}, \ldots i_n} \simeq \mathcal{O}_{\underbrace{i_k, \ldots, i_k}_{m}}.$$

$$\mathcal{O}_K^m(a) = t_{\underbrace{K, \ldots, K}_{m}}(a) + \sum_{\mathbf{n}} c_{\mathbf{n}} t_{\mathbf{n}}(a)$$

$$\mathcal{O}_{\underbrace{4,\ldots,4}_{m_4=\Delta/4}} = t_4^{m_4} + \text{GS with } \{t_4^{n_4}t_3^{n_3}\}_{n_3\geq 1} \cup \{t_4^{n_4}t_3^{n_3}t_2^{n_2}\}_{n_2\geq 1}$$

$$\vdots$$

$$\mathcal{O}_{\underbrace{4,\ldots,4,3,\ldots,3}_{m_4}} = t_4^{m_4}t_3^{m_3} + \text{GS with } \{t_4^{n_4}t_3^{n_3}\}_{n_3\geq m_3+1} \cup \{t_4^{n_4}t_3^{n_3}t_2^{n_2}\}_{n_2\geq 1}$$

$$\vdots$$

$$\mathcal{O}_{\underbrace{3,\ldots,3}_{m_3=\Delta/3}} = t_3^{m_3} + \text{GS with } \{t_4^{n_4}t_3^{n_3}t_2^{n_2}\}_{n_2\geq 1}$$

$$\mathcal{O}_3^5 = t_3^5 + \frac{35325}{39442}t_3^3t_2^3 - \frac{90}{41}t_4t_3^3t_2 + \frac{1050327}{9781616}t_3t_2^6 - \frac{22761}{39442}t_4t_3t_2^4 + \frac{405}{533}t_4^2t_3t_2^2$$

$$\mathcal{O}_4^3 = t_4^3 - \frac{6103}{5140}t_4^2t_2^2 + \frac{9551}{20560}t_4t_2^4 - \frac{162}{1285}t_4t_3^2t_2 + \frac{53}{6939}t_3^4 + \frac{1313}{23130}t_3^2t_2^3 - \frac{985}{16448}t_2^6.$$

$$\mathcal{O}^{5,\ldots,5}_{\underbrace{5,\ldots,5}_{m_5},\underbrace{4,\ldots,4}_{m_4},\underbrace{4,3,\ldots,3}_{m_3}},$$

$$= t_5^{m_5}t_4^{m_4}t_3^{m_3} + \text{ GS with } \{t_5^{n_5}t_4^{n_4}t_3^{m_3}\}_{n_4\geq m_4+1} \cup \{t_5^{n_5}t_4^{n_4}t_3^{n_3}\}_{n_3\geq m_3+1} \cup \{t_5^{n_5}t_4^{n_4}t_3^{n_3}t_2^{n_2}\}_{n_2\geq 1},$$

$$\mathcal{O}_5^m = t_5^m + \text{GS with } \{t_5^{n_5}t_4^{n_4}t_3^{n_3}\}_{n_3+n_4\geq 1} \cup \{t_5^{n_5}t_4^{n_4}t_3^{n_3}t_2^{n_2}\}_{n_2\geq 1}$$
$$\mathcal{O}_4^m = t_4^m + \text{GS with } \{t_5^{n_5}t_4^{n_4}t_3^{n_3}\}_{n_3\geq 1} \cup \{t_5^{n_5}t_4^{n_4}t_3^{n_3}t_2^{n_2}\}_{n_2\geq 1}$$
$$\mathcal{O}_3^m = t_3^m + \text{GS with } \{t_5^{n_5}t_4^{n_4}t_3^{n_3}t_2^{n_2}\}_{n_2\geq 1}$$

$$\mathcal{O}_5^2 = t_5^2 - \frac{1}{90}t_4t_3^2 - \frac{23}{800}t_4^2t_2 - \frac{443}{300}t_5t_3t_2 + \frac{2461}{4500}t_3^2t_2^2 + \frac{179}{8000}t_4t_2^3 - \frac{7}{1600}t_2^5$$

$$\mathcal{N}_m(N,K) = \langle\langle \mathcal{O}_K^m(a)\mathcal{O}_K^m(a)\rangle\rangle$$

$$\mathcal{N}_m(N,N) = f_N(m)(2^{-N}N^{2-N})^m \frac{\Gamma\left(Nm + \frac{N(N-1)}{2} + 1\right)}{\Gamma\left(\frac{N(N-1)}{2} + 1\right)},$$

$$f_2(m) = 1$$
$$f_3(m) = 1$$
$$f_4(m) = \frac{1-2^{-4}}{1-2^{-4-4m}} = \frac{15}{16} + O(e^{-m}),$$

$$f_5(m) = \frac{\frac{504}{625}}{\left(1 + 5^{-\frac{5}{2}m-4}(1+(-1)^m) - 2^{-m-2}5^{-3m-5}\left((25+11\sqrt{5})^{m+2} + (25-11\sqrt{5})^{m+2}\right)\right)}$$
$$= \frac{504}{625} + O(e^{-m})$$

$$\Gamma\left(\frac{R}{2} + \alpha_{N,N} + 1\right),$$

$$\langle\langle \mathcal{O}_3^m(a)\mathcal{O}_3^m(a)\rangle\rangle = 24^{-m}\frac{\Gamma(3m+7)}{\Gamma(7)}\left(\frac{20}{27} + \frac{250}{243m} + O\left(\frac{1}{m^2}\right)\right).$$

$$\langle\langle \mathcal{O}_3^m(a)\mathcal{O}_3^m(a)\rangle\rangle = 24^{-m}\frac{\Gamma(3m+10)}{\Gamma(10)}\left(\frac{140}{243} + O\left(\frac{1}{m}\right)\right).$$

$$\mathcal{N}_m(N,K) = (2^K K^{K-2})^{-m}\frac{\Gamma(\Delta + \alpha_{N,K} + 1)}{\Gamma(\alpha_{N,K} + 1)}A(N,K)\left(1 + O\left(\frac{1}{m}\right)\right).$$

$$\frac{\langle \mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)T_{\mathbf{p}}(x_3,Y_3)\rangle}{\langle \mathcal{O}_K^m(x_1,Y_1)\mathcal{O}_K^m(x_2,Y_2)\rangle} = \mathfrak{C}_{mm\mathbf{p}}(N,K) \times \left(\frac{1}{2\pi^2}\frac{d_{23}d_{31}}{d_{12}}\right)^{\frac{\Delta_{\mathbf{p}}}{2}},$$

$$\mathfrak{C}_{mm\mathbf{p}}(N,K)=\frac{\left\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)T_{\mathbf{p}}(a)\rangle\right\rangle}{\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)\rangle\rangle},$$

$$\frac{\left\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)T_{\mathbf{p}}(a)\rangle\right\rangle}{\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)\rangle\rangle}=\frac{\left\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)t_{\mathbf{p}}(a)\rangle\right\rangle}{\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)\rangle\rangle}\left(1+O\left(\frac{1}{m}\right)\right).$$

$$\frac{\left\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)t_{\mathbf{p}}(a)\rangle\right\rangle}{\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)\rangle}=\int_0^{2\pi}\frac{d\theta}{2\pi}\langle t_{\mathbf{p}}\rangle_\theta\left(1+O\left(\frac{1}{m}\right)\right)$$

$$\langle t_{\mathbf{p}}\rangle_\theta=\prod_{k\geq 1}\left(\mathrm{Tr}\left(a_{\mathrm{cl}}^{p_k}(\theta)\right)\right),\ \text{and}\ a_{\mathrm{cl}}(\theta)=\frac{\Omega_K^{(N)}\mathrm{e}^{\mathrm{i}\theta}+\bar{\Omega}_K^{(N)}\mathrm{e}^{-\mathrm{i}\theta}}{\sqrt{2}}.$$

$$\mathcal{T}_{\mathcal{H}}(u,v;\lambda)=\sum_{L=1}^{\infty}d_{\mathcal{H},N;L}\frac{(-\lambda)^L}{u}\sum_{\ell=0}^{L}P^{(\ell)}(u,v)P^{(L-\ell)}(u,v),$$

$$I_2\left[\frac{1}{u}\sum_{\ell=0}^{L}P^{(\ell)}(u,v)P^{(L-\ell)}(u,v)\right]=-4\frac{\Gamma(2L+2)}{\Gamma(L+1)^2}\zeta(2L+1)$$

$$\mathcal{G}_{\mathcal{H}}(\lambda)=-4\sum_{L=1}^{\infty}d_{\mathcal{H},N;L}(-\lambda)^L\frac{\Gamma(2L+2)}{\Gamma(L+1)^2}\zeta(2L+1)$$

$$\mathcal{G}_{m,K}(\tau_2)=\left.\frac{\int d^{N-1}a\prod_{1\leq i<j\leq N}a_{ij}^2\partial_\mu^2 Z_{1-\mathrm{loop}}\,e^{-\mathrm{Tr}a^2}\mathcal{O}_K^m\mathcal{O}_K^m}{\int d^{N-1}a\prod_{1\leq i<j\leq N}a_{ij}^2 e^{-\mathrm{Tr}a^2}\mathcal{O}_K^m\mathcal{O}_K^m}\right|_{\mu=0}$$

$$\mathcal{G}_{m,K}(\tau_2)=-2\sum_{L=1}^{\infty}\sum_{j=0}^{2L}(-1)^{L+j}\left(\frac{1}{2\pi\tau_2}\right)^L\binom{2L}{j}(2L+1)\zeta(2L+1)\frac{\left\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)t_{2L-j,j}(a)\rangle\right\rangle}{\langle\langle\mathcal{O}_K^m(a)\mathcal{O}_K^m(a)\rangle}.$$

$$\mathcal{G}_{m,K}(\lambda)\simeq 4\sum_{L=1}^{\infty}(-1)^{L+1}\frac{\Gamma(2L+2)}{\Gamma(L+1)^2}\zeta(2L+1)\left[\sum_{1\leq i<j\leq K}\left(4\sin^2\frac{\pi(i-j)}{K}\lambda\right)^L+K(N-K)\lambda^L\right].$$

$$\mathcal{G}_{m,K}(\lambda)\simeq -2\sum_{L=1}^{\infty}\frac{\Gamma(2L+2)}{\Gamma(L+1)^2}\zeta(2L+1)\sum_s(-M_s^2)^L.$$

$$d_{\mathcal{O}_K^m,N;L}=\frac{K}{2}\sum_{s=1}^{K-1}\left(4\sin^2\left(\frac{s\pi}{K}\right)\right)^L+K(N-K).$$

$$\mathcal{T}_{m,K}(u,v;\lambda)=K\sum_{s=1}^{K-1}L\left(u,v;4\lambda\sin^2\frac{\pi s}{K}\right)+2K(N-K)L(u,v;\lambda),$$

$$L(u,v;a)=\frac{1}{2u}\left[\left(\sum_{L=0}^{\infty}(-a)^L P^{(L)}(u,v)\right)^2-1\right]$$

$$-\sum_{L\geq 1}\frac{\Gamma(2L+2)}{\Gamma(L+1)^2}\zeta(2L+1)(-a)^L=\int_0^\infty dw\frac{w}{\sinh^2 w}\left(1-J_0(4w\sqrt{a})\right)$$
$$=1+\gamma_E+\frac{\log a}{2}+2\sum_{n\geq 1}\left(4\pi n\sqrt{a}K_1(4\pi n\sqrt{a})-K_0(4\pi n\sqrt{a})\right)$$

$$\mathcal{G}_{m,K}(\lambda) \simeq 2K(2N-K-1)(1+\gamma_E)+2K\left(\log K+\left(N-\frac{K+1}{2}\right)\log\lambda\right)$$
$$+4\sum_{s}\sum_{n\geq 1}\left(4\pi n M_s K_1(4\pi n M_s)-K_0(4\pi n M_s)\right)$$

$$K_\nu(4\pi n M_s)\sim \mathrm{e}^{-4\pi n M_s}$$

$$\mathcal{G}_{m,K}(\lambda)\big|_{(5.26)}\simeq 4NK\sum_{L\geq 1}(-1)^{L+1}\frac{\Gamma(2L+2)}{\Gamma(L+1)^2}\zeta(2L+1)\lambda^L$$

$$N,K\to\infty,\ \text{with}\ \kappa=\frac{K}{N},\ \text{fixed}$$

$$\frac{1}{K}\sum_{s=1}^{K-1}\left(4\sin^2\left(\frac{\pi s}{K}\right)\right)^L=\int_0^1 \mathrm{d}x(4\sin^2(\pi x))^L+\mathcal{O}\left(\frac{1}{K}\right)=\frac{2^{2L}\Gamma\left(L+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(L+1)}+\mathcal{O}\left(\frac{1}{K}\right)$$

$$\mathcal{G}_{m,K}(\lambda)\big|_{(5.28)}\simeq -4\kappa N^2\sum_{L=1}^{\infty}\frac{(-\lambda)^L\zeta(2L+1)\Gamma\left(L+\frac{3}{2}\right)\left[\sqrt{\pi}2^{2L+1}(1-\kappa)\Gamma(L+1)+2^{4L}\kappa\Gamma\left(L+\frac{1}{2}\right)\right]}{\pi\Gamma(L+1)^2}$$
$$=4\kappa N^2\int_0^\infty \mathrm{d}w\frac{w}{\sinh^2(w)}\left[(1-\kappa)(1-J_0(4w\sqrt{\lambda}))+\frac{\kappa}{2}(1-J_0(4w\sqrt{\lambda})^2)\right].$$

$$\mathcal{G}_{m,K}(\lambda)\big|_{(5.28)}\simeq(2-\kappa)\kappa N^2(\log(\lambda)+2\gamma_E+2)+16\kappa^2N^2\sum_{k\geq 1}\frac{a_k}{\lambda^{k-1/2}}$$
$$+8N^2\kappa(1-\kappa)\sum_{n\geq 1}\left(4\pi n\sqrt{\lambda}K_1(4\pi n\sqrt{\lambda})-K_0(4\pi n\sqrt{\lambda})\right)$$

$$a_k=\frac{\zeta(2k-1)\Gamma\left(k-\frac{1}{2}\right)^3}{\sqrt{\pi}(16\pi^2)^k\Gamma(k-1)}$$

$$a_k\sim\Gamma(2k-1)\pi^{-2k}(2^{4-10k}+2^{3-8k}+2^{2-6k}+2^{2-6k}3^{1-2k}+\mathcal{O}(1/k))$$

$$\sigma(x)=\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}.$$

$$\mathcal{G}_{m,3}^{SU(3)}(\lambda)=\int_0^1\left(\partial_\mu^2 Z_{1-\text{loop}}\right)\Big|_{\mu=0,\mathrm{Tr}a^2=6\lambda,\mathrm{Tr}a^3=6\lambda^{3/2}(2x-1)}\sigma(x)\mathrm{d}x.$$

$$\mathcal{G}_{m,K}(\lambda)=\int_0^1\left(\partial_\mu^2 Z_{1-\text{loop}}\big|_{\mu=0}\right)\Big|_{\text{locus}}\sigma(x)\mathrm{d}x,$$

$$\mathrm{Tr}a^m=2K\lambda^{\frac{m}{2}}\left(\binom{m-1}{m/2}\epsilon_m+(2x-1)\delta_{m,K}\right),m=2,\dots,K,$$

$$\mathrm{Tr}a^m=\lambda^{\frac{m}{2}}\sum_{s=0}^{K-1}\left(\mathrm{e}^{-\frac{2i\pi s}{K}}(\sqrt{x-1}-\sqrt{x})^{-2/K}+\mathrm{e}^{\frac{2i\pi s}{K}}(\sqrt{x-1}-\sqrt{x})^{2/K}\right)^m.$$

$$\mathcal{Z}=\int da\mathrm{e}^{-\mathrm{Tr}a^2},a=\sum_{b=1}^{N^2-1}a^bT_b,da=\prod_{b=1}^{N^2-1}\frac{da_b}{\sqrt{2\pi}}$$

$$\mathrm{Tr}T_bT_c=\frac{1}{2}\delta_{bc},\mathrm{Tr}T_b=0$$

$$\langle\langle f(a)\rangle\rangle=\int da\mathrm{e}^{-\mathrm{Tr}a^2}f(a)$$

$$u_{\boldsymbol{p}}=\langle\langle\mathrm{Tr}a^{p_1}\mathrm{Tr}a^{p_2}\dots\mathrm{Tr}a^{p_m}\rangle\rangle$$

$$\mathrm{Tr}T^bB_1T^bB_2 = \frac{1}{2}\mathrm{Tr}B_1\mathrm{Tr}B_2 - \frac{1}{2N}\mathrm{Tr}B_1B_2,$$
$$\mathrm{Tr}T^bB_1\mathrm{Tr}T^bB_2 = \frac{1}{2}\mathrm{Tr}B_1B_2 - \frac{1}{2N}\mathrm{Tr}B_1\mathrm{Tr}B_2.$$

$$u_p = 0,\ \text{for } p \text{ odd},\ \ \text{and}\ \ u_0 = N,$$

$$u_{p_1,p_2,\dots,p_m} = \frac{1}{2}\sum_{j=0}^{p_1-2}\left(u_{j,p_1-j-2,p_2,\dots,p_m}\right) - \frac{p_1-1}{2N}u_{p_1-2,p_2,\dots,p_m}$$
$$+\sum_{k=2}^{m}\frac{p_k}{2}\left(u_{p_1+p_k-2,p_2,\dots,p_k,\dots,p_m} - \frac{1}{N}u_{p_1-1,p_2,\dots,p_k-1,\dots,p_m}\right),$$

$$u_2 = \frac{N^2-1}{2}, u_4 = \frac{(N^2-1)(2N^2-3)}{4N},$$
$$u_{2,2} = \frac{N^4-1}{4}, u_6 = \frac{5(N^2-1)(N^4-3N^2+3)}{8N^2},$$
$$u_{4,2} = \frac{(N^2-1)(N^2+3)(2N^2-3)}{8N}, u_{3,3} = \frac{3(N^2-1)(N^2-4)}{8N}.$$

$$\frac{\left\langle\langle \mathcal{O}_5^m \mathcal{O}_5^m t_4\rangle\right\rangle}{\left\langle\langle \mathcal{O}_5^m \mathcal{O}_5^m\rangle\right\rangle} = \frac{30}{2^{m+2}5^{3m+5} + (1+(-1)^m)2^{m+2}5^{m/2+1} - (x_5^+)^{m+2} - (x_5^-)^{m+2}}$$
$$\Bigg[2^m5^{3m+4}(5m^2+27m-2) - \frac{(x_5^+)^m + (x_5^-)^m}{2}(325m^2+2219m-1290)$$
$$-\sqrt{5}\frac{(x_5^+)^m - (x_5^-)^m}{2}(145m^2+991m-578)$$
$$+2^m5^{m/2}\left((1+(-1)^m)(8m-20) - \frac{1-(-1)^m}{\sqrt{5}}(25m^2+95m+90)\right)\Bigg]$$

$$x^2 - 50x + 20 = 0,\ \text{with}\ x_5^+ = 25 + 11\sqrt{5}, x_5^- = 25 - 11\sqrt{5}.$$

$$\frac{\left\langle\langle \mathcal{O}_3^m \mathcal{O}_3^m t_{4^2,3^2}\rangle\right\rangle}{\left\langle\langle \mathcal{O}_3^m \mathcal{O}_3^m\rangle\right\rangle} = 45.5625m^7 + O(m^6),$$

$$\mathcal{G}_{m,K}(\lambda) \simeq 2\sum_{L=1}^{\infty}(-1)^{L+1}\frac{\Gamma(2L+2)}{\Gamma(L+1)^2}\zeta(2L+1)\left[K\sum_{s=1}^{K-1}\left(4\lambda\sin^2\frac{\pi s}{K}\right)^L + 2K(N-K)(\lambda)^L\right].$$

$$\mathcal{O}_{3^{m_3}2^{m_2}} = t_3^{m_3}t_2^{m_2} + \mathrm{GS\ with}\ \{t_3^{n_3}t_2^{n_2}\}_{\Delta=3n_3+2n_2,n_3\le m_3} \cup \{t_3^{n_3}t_2^{n_2}\}_{\Delta\le 3m_3+2m_2}$$

$$\mathcal{G}_{3^{m_3}2^{m_2}}(\tau_2) = 2\sum_{L=1}^{\infty}(-1)^L\frac{\Gamma(2L+2)}{\Gamma(L+1)}\left(\frac{1}{4\pi\tau_2}\right)^L\zeta(2L+1)((2L+1)(L^2+L+6)$$
$$+6\ {}_3F_2\left(-L, L+1, -m_2; 1, \frac{15}{2}+3m_3; 1\right)\frac{(-L)_{(3m_3+3)} - (L+1)_{(3m_3+3)}}{\Gamma(3m_3+4)}\Bigg)$$

$$\mathcal{G}_{m,3}(\lambda) \simeq 4\sum_{L=1}^{\infty}(-1)^{L+1}\frac{\Gamma(2L+2)}{\Gamma(L+1)^2}3(3\lambda)^L$$

$$\mathcal{G}_{m,5}(\tau_2)\big|_{L=1} = \frac{150\zeta(5)}{\pi\tau_2}m$$

$$\mathcal{G}_{m,5}(\tau_2)\big|_{L=2} = -\frac{4500\zeta(5)}{\pi^2\tau_2^2(2^{m+2}5^{3m+5} + 2^{m+2}5^{m/2+1}(1+(-1)^m) - (x_5^+)^{m+2} - (x_5^-)^{m+2})}$$
$$\Bigg[2^m5^{3m+4}\left(\frac{5m^2}{2} + \frac{77m}{6} - \frac{1}{3}\right) - ((x_5^+)^m + (x_5^-)^m)\frac{(1555m^2 + 8369m - 1290)}{12}$$
$$+2^m5^{\frac{m}{2}}\left((1+(-1)^m)\frac{(5m^2+29m-10)}{3} - \sqrt{5}(1-(-1)^m)\left(\frac{5m^2}{6} + \frac{19m}{6} + 3\right)\right)$$
$$-\sqrt{5}((x_5^+)^m - (x_5^-)^m)\left(\frac{695m^2}{12} + \frac{1247m}{4} - \frac{289}{6}\right)\Bigg]$$

$$\mathcal{G}_{m,5}(\tau_2)\big|_{L=3} = \frac{220500\zeta(7)}{\pi^3\tau_2^3(2^{m+2}5^{3m+5} + 2^{m+2}5^{m/2+1}(1+(-1)^m) - (x_5^+)^{m+2} - (x_5^-)^{m+2})}$$
$$\Bigg[2^m5^{3m+4}\frac{(50m^3 + 405m^2 + 1087m - 84)}{252} - ((x_5^+)^m + (x_5^-)^m)\frac{50m^3 + 1825 + 7623 - 5840}{56}$$
$$+2^m5^{\frac{m}{2}}\left((1+(-1)^m)\left(\frac{25m^3}{72} + \frac{125m^2}{56} + \frac{1279m}{252} - \frac{5}{42}\right) - 5\sqrt{5}(1-(-1)^m)\frac{5m^3 + 29m^2 + 56m + 36}{56}\right)$$
$$-5\sqrt{5}((x_5^+)^m - (x_5^-)^m)\left(\frac{5m^3}{63} + \frac{163m^2}{56} + \frac{6131m}{504} - \frac{28}{3}\right)\Bigg]$$

$$\mathcal{G}_{m,5}(\lambda) = 600\lambda\zeta(3) - 7200\lambda^2\zeta(5) + 140000\lambda^3\zeta(7) + \mathcal{O}(\lambda^4, 1/m)$$

$$\mathcal{G}_{m,3}(\lambda) = 288.0\lambda\zeta(3) - (3600.0\lambda^2\zeta(5) + \mathcal{O}(1/m)) + (47040.0\lambda^3\zeta(7) + \mathcal{O}(1/m))$$
$$-(619917.3\lambda^4\zeta(9) + \mathcal{O}(1/m)) + \lambda,$$

$$\mathcal{G}_{m,3}(\lambda) = 288\lambda\zeta(3) - 3600\lambda^2\zeta(5) + 47040\lambda^3\zeta(7) - 619920\lambda^4\zeta(9) + \mathcal{O}(\lambda^5)$$

$$AdS_2 \times S^2 \times S^4 \times \Sigma,$$

$$(a;q)_0 := 1, (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), (q)_n := \prod_{k=1}^{n}(1-q^k)$$
$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), (q)_\infty := \prod_{k=1}^{\infty}(1-q^k)$$

$$(ax^{\pm};q)_n := (ax;q)_n(ax^{-1};q)_n.$$

$$\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle^G(t;q)$$
$$= \frac{1}{|\mathrm{Weyl}(G)|}\frac{(q)_\infty^{2\mathrm{rank}(G)}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^{\mathrm{rank}(G)}}\oint\prod_{\alpha\in\mathrm{root}(G)}ds\frac{(s^\alpha;q)_\infty(qs^\alpha;q)_\infty}{\left(q^{\frac{1}{2}}t^2s^\alpha;q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s^\alpha;q\right)_\infty}\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathfrak{g}}$$

$$\mathcal{I}^G(t;q) := \mathrm{Tr}(-1)^F q^{J+\frac{H+C}{4}}t^{H-C}$$

$$\langle\mathcal{W}_{\mathcal{R}_1}\cdots\mathcal{W}_{\mathcal{R}_k}\rangle^G(t;q) = \frac{\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle^G(t;q)}{\mathcal{I}^G(t;q)}$$

$$\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle^G_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1}{|\mathrm{Weyl}(G)|}\frac{1}{(1-\mathfrak{q}^2)^{\mathrm{rank}(G)}}\oint\prod_{\alpha\in\mathrm{root}(G)}ds\frac{(1-s^\alpha)}{(1-\mathfrak{q}^2s^\alpha)}\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathfrak{g}}.$$

$$\frac{1}{N!}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\frac{\prod_{i\neq j}1-\frac{s_i}{s_j}}{\prod_{i,j}1-\mathrm{t}\frac{s_i}{s_j}}P_\mu(s;\mathrm{t})P_\lambda(s^{-1};\mathrm{t}) = \frac{\delta_{\mu\lambda}}{(\mathrm{t};\mathrm{t})_{N-l(\mu)}\prod_{j\geq 1}(\mathrm{t};\mathrm{t})_{m_j(\mu)}}$$

$$\frac{1}{|\mathrm{Weyl}(G)|}\oiiint \prod_{\alpha\in\mathrm{root}(G)} ds(1-s^{\alpha})\prod_{i=1}^{k}\chi^{\mathfrak{g}}_{\mathcal{R}_i}$$

$$\langle T_B T_B\rangle^G(t;q) = \sum_{\nu\in\mathrm{Rep}(B)} \frac{1}{|\mathrm{Weyl}(B)|}\frac{(q)_\infty^{2\mathrm{rank}(G)}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^{\mathrm{rank}(G)}}\oint \prod_{\alpha\in\mathrm{root}(G)} ds$$
$$\times \frac{\left(q^{\frac{|\alpha(B)|}{2}}s^{\alpha};q\right)_\infty \left(q^{1+\frac{|\alpha(B)|}{2}}s^{\alpha};q\right)_\infty}{\left(q^{\frac{1+|\alpha(B)|}{2}}t^{2}s^{\alpha};q\right)_\infty \left(q^{\frac{1+|\alpha(B)|}{2}}t^{-2}s^{\alpha};q\right)_\infty} Z^{(B,\nu)}_{\mathrm{bubb}}(t,s;q)$$

$$(z_e,z_m)\in\mathbb{Z}_2\times\mathbb{Z}_2.$$

$$(z_e,z_m)=(0,0),(z_e,z_m)=(1,0)$$

$$(z_e,z_m)=(0,0),(z_e,z_m)=(0,1)$$

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$$\chi_{\mathrm{sp}}^{\mathfrak{so}(2N+1)}=\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)$$

$$\chi_{\square}^{\mathfrak{so}(2N+1)}=1+\sum_{i=1}^{N}(s_i+s_i^{-1}).$$

$$\chi_{\lambda}^{\mathfrak{so}(2N+1)}=\frac{\det\left(s_j^{\lambda_i+N-i+1/2}-s_j^{-(\lambda_i+N-i+1/2)}\right)}{\det\left(s_j^{N-i+1/2}-s_j^{-(N-i+1/2)}\right)}.$$

$$\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle^{SO(2N+1)}$$
$$=\int d\mu^{SO(2N+1)}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)\frac{\bar{P}_n(s)^2-\bar{P}_{2n}(s)}{2}\right]\prod_{i=1}^{k}\chi^{\mathfrak{s}(2N+1)}_{\mathcal{R}_i}(s)$$

$$d\mu^{SO(2N+1)}=\frac{1}{2^N N!}\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}(1-s_i)(1-s_i^{-1})$$
$$\times\prod_{1\le i<j\le N}(1-s_is_j)(1-s_i^{-1}s_j^{-1})(1-s_is_j^{-1})(1-s_i^{-1}s_j)$$

$$f_n(q,t)=\frac{q^{n/2}(t^{2n}+t^{-2n})-2q^n}{1-q^n}$$

$$P_m(s):=\sum_{i=1}^{N}(s_i^m+s_i^{-m})$$

$$\bar{P}_m(s):=1+P_m(s)=1+\sum_{i=1}^{N}(s_i^m+s_i^{-m})$$

$$\bar{M}_n(s)=\frac{\bar{P}_n(s)^2-\bar{P}_{2n}(s)}{2}=P_n(s)+\frac{P_n(s)^2-P_{2n}(s)}{2}$$

$$\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)\bar{M}_n(s)\right)=\sum_{\lambda}\frac{1}{z_\lambda}f_\lambda(q,t)\bar{M}_\lambda(s),$$

$$\lambda=(\lambda_1,\lambda_2,\dots)=(1^{m_1}2^{m_2}\dots),z_\lambda,f_\lambda(q,t),\bar{M}_\lambda(s)$$

$$z_\lambda = \prod_{i=1}^{\infty} i^{m_i} m_i!, f_\lambda(q,t) = \prod_{i=1}^{\ell(\lambda)} f_{\lambda_i}(q,t), \bar{M}_\lambda(s) = \prod_{i=1}^{\ell(\lambda)} \bar{M}_{\lambda_i}(s)$$

$$\langle W_{\mathcal{R}_1} \cdots W_{\mathcal{R}_k} \rangle^{SO(2N+1)} = \sum_\lambda \frac{1}{z_\lambda} f_\lambda(q,t) \int d\mu^{SO(2N+1)} \bar{M}_\lambda(s) \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\mathfrak{so}(2N+1)}(s).$$

$$\int d\mu^{SO(2N+1)} \bar{M}_\lambda(s) \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\mathfrak{so}(2N+1)}(s)$$

$$\int d\mu^{SO(2N+1)} \bar{P}_\mu(s) = \sum_{\nu \in R_{2N+1}(|\mu|)} \chi_\nu^S(\mu) + \sum_{\nu \in W_{2N+1}(|\mu|)} \chi_\nu^S(\mu),$$

$$R_n(p) = \{\lambda \vdash p \mid \ell(\lambda) \leq n \text{ and } \forall \lambda_i \text{ is even }\},$$
$$W_n(p) = \{\lambda \vdash p \mid \ell(\lambda) = n \text{ and } \forall \lambda_i \text{ is odd }\}.$$

$$s_\lambda = \sum_{\mu \vdash \lambda} \frac{\chi_\lambda^S(\mu)}{z_\mu} p_\mu$$

$$\bar{M}_\lambda(s) \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\mathfrak{s}(2N+1)}(s) = \sum_\mu a_{\lambda,\mathcal{R}}^\mu \bar{P}_\mu(s),$$

$$\int d\mu^{SO(2N+1)} \bar{M}_\lambda(s) \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\mathfrak{s}(2N+1)}(s) = \sum_\mu a_{\lambda,\mathcal{R}}^\mu \int d\mu^{SO(2N+1)} \bar{P}_\mu(s)$$
$$= \sum_\mu a_{\lambda,\mathcal{R}}^\mu \left(\sum_{\nu \in R_{2N+1}(|\mu|)} \chi_\nu^S(\mu) + \sum_{\nu \in W_{2N+1}(|\mu|)} \chi_\nu^S(\mu) \right)$$

$$\langle W_{\mathcal{R}_1} \cdots W_{\mathcal{R}_k} \rangle^{SO(2N+1)}$$
$$= \sum_\lambda \frac{1}{z_\lambda} f_\lambda(q,t) \sum_\mu a_{\lambda,\mathcal{R}}^\mu \left(\sum_{\nu \in R_{2N+1}(|\mu|)} \chi_\nu^S(\mu) + \sum_{\nu \in W_{2N+1}(|\mu|)} \chi_\nu^S(\mu) \right)$$

$$\left(\chi_{\mathrm{sp}}^{\mathfrak{so}(2N+1)}\right)^2 = \prod_{i=1}^{N} (1+s_i)(1+s_i^{-1}),$$

$$d\mu^{SO(2N+1)} \left(\chi_{\mathrm{sp}}^{\mathfrak{so}(2N+1)}\right)^2 = d\mu^{USp(2N)}$$

$$d\mu^{USp(2N)} = \frac{1}{2^N N!} \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} (1-s_i^2)(1-s_i^{-2})$$
$$\times \prod_{1 \leq i < j \leq N} (1-s_i s_j)(1-s_i^{-1}s_j^{-1})(1-s_i s_j^{-1})(1-s_i^{-1}s_j)$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(2N+1)} = \int d\mu^{\mathrm{USp}(2N)} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f_n(q,t) \bar{M}_n(s) \right]$$
$$= \int d\mu^{\mathrm{USp}(2N)} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f_n(q,t) \left(P_n(s) + \frac{P_n(s)^2 - P_{2n}(s)}{2} \right) \right]$$

$$\mathcal{I}^{SO(3)}(t;q) = \mathcal{I}^{USp(2)}(t;q)$$
$$= -\frac{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty}{(q;q)_\infty^2} \sum_{\substack{p_1, p_2 \in \mathbb{Z} \\ p_1 < p_2}} \frac{\left(q^{\frac{1}{2}} t^{-2}\right)^{p_1+p_2-2}}{\left(1-q^{p_1-\frac{1}{2}} t^2\right)\left(1-q^{p_2-\frac{1}{2}} t^2\right)}$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(3)}(t;q) = \frac{1}{2} \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty} \oint \frac{ds}{2\pi i s} \frac{(s^{\pm};q)_\infty (q s^{\pm};q)_\infty}{\left(q^{\frac{1}{2}} t^2 s^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s^{\pm}; q\right)_\infty} \left(s^{\frac{1}{2}} + s^{-\frac{1}{2}}\right)^2$$

$$\left\langle T_{\left(\frac{1}{2}\right)} T_{\left(\frac{1}{2}\right)} \right\rangle^{USp(2)/\mathbb{Z}_2}(t;q) = \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty} \oint \frac{ds}{2\pi i s} \frac{\left(q^{\frac{1}{2}} s^{\pm 2}; q\right)_\infty \left(q^{\frac{3}{2}} s^{\pm 2}; q\right)_\infty}{(q t^2 s^{\pm 2}; q)_\infty (q t^{-2} s^{\pm 2}; q)_\infty}$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(3)}(t;q) = \langle T_{\left(\frac{1}{2}\right)} T_{\left(\frac{1}{2}\right)} \rangle^{\mathrm{USp}(2)/\mathbb{Z}_2}(t;q).$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(3)}(t;q) = \left\langle T_{\left(\frac{1}{2}\right)} T_{\left(\frac{1}{2}\right)} \right\rangle^{U\mathrm{Sp}(2)/\mathbb{Z}_2}(t;q) = \langle W_\square W_\square \rangle^{SU(2)}(t;q) = \frac{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty}{(q t^{\pm 4}; q)_\infty} \sum_{m \in \mathbb{Z} \setminus \{0,n\}} \frac{t^{2m} - t^{-2m}}{t^2 - t^{-2}} \frac{q^{\frac{m-1}{2}}}{1 - q^m}$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle T_{\left(\frac{1}{2}\right)} T_{\left(\frac{1}{2}\right)} \right\rangle^{U\mathrm{Sp}(2)/\mathbb{Z}_2}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle W_\square W_\square \rangle^{SU(2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1 + \mathfrak{q}^2}{1 - \mathfrak{q}^4} = \frac{1}{1 - \mathfrak{q}^2}$$

$$\langle \underbrace{W_{\mathrm{sp}} W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{2k} \rangle^{\mathrm{Sin}(3)}(t;q) = \langle \underbrace{W_\square W_\square \cdots W_\square}_{2k} \rangle^{\mathrm{SU}(2)}(t;q).$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Sin}(3)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{2 + 3\mathfrak{q}^2 + \mathfrak{q}^4}{1 - \mathfrak{q}^4}$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(3)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{5 + 9\mathfrak{q}^2 + 5\mathfrak{q}^4 + \mathfrak{q}^6}{1 - \mathfrak{q}^4}$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(3)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{14 + 28\mathfrak{q}^2 + 20\mathfrak{q}^4 + 7\mathfrak{q}^6 + \mathfrak{q}^8}{1 - \mathfrak{q}^4}$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(3)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{42 + 90\mathfrak{q}^2 + 75\mathfrak{q}^4 + 35\mathfrak{q}^6 + 9\mathfrak{q}^8 + \mathfrak{q}^{10}}{1 - \mathfrak{q}^4}.$$

$$\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{2k} \rangle^{\mathrm{Spin}(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \mathcal{I}^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) \sum_{i=0}^{k} a^{\mathfrak{so}(3)}_{k\mathrm{sp}}(i) \mathfrak{q}^{2i} = \frac{1}{1 - \mathfrak{q}^4} \sum_{i=0}^{k} a^{\mathfrak{so}(3)}_{k\mathrm{sp}}(i) \mathfrak{q}^{2i}$$

$$a^{\mathfrak{so}(3)}_{k\mathrm{sp}}(i) = (2i+1) \frac{(2k)!}{(k-i)!\,(k+i+1)!} = C_{k+i+1,2i+1}$$

$$C_{n,m} = \frac{m}{n} \binom{2n - m - 1}{n - 1}$$

$$C_k = \frac{1}{k+1} \binom{2k}{k} = \prod_{1 \le i \le j \le k-1} \frac{i + j + 2}{i + j}.$$

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{k=0}^{\infty} a_{ksp}^{\mathfrak{so}(3)}(0)x^k$$

$$\frac{1}{x^{i+1}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{2i+1} = \sum_{k=0}^{\infty} a_{ksp}^{\mathfrak{so}(3)}(i)x^k.$$

$$\sum_{k=0}^{\infty} x^k \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{2k} \rangle_{\frac{1}{2}\mathrm{BPS}}^{\mathrm{Spin}(3)}(\mathfrak{q}) = \frac{1}{1-\mathfrak{q}^4} \cdot \frac{2(1-\sqrt{1-4x})}{4x-(1-\sqrt{1-4x})^2\mathfrak{q}^2}$$

$$\begin{aligned}&\langle W_{\Box} W_{\Box}\rangle^{SO(3)}(t;q)\\ &= \frac{1}{2}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty} \oiiint \frac{ds}{2\pi i s} \frac{(s^{\pm};q)_\infty (qs^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2 s^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2}s^{\pm};q\right)_\infty}(1+s+s^{-1})^2\end{aligned}$$

$$\begin{aligned}&\langle W_{\Box} W_{\Box}\rangle^{SO(3)}(t;q) = \langle W_{\Box\Box} W_{\Box\Box}\rangle^{SU(2)}(t;q)\\ &= \frac{(q^{\frac{1}{2}}t^{\pm 2};q)_\infty}{(qt^{\pm 4};q)_\infty}\left[\frac{3}{2}\sum_{m\in\mathbb{Z}\setminus\{0\}}\left(\frac{t^{2m}-t^{-2m}}{t-t^{-2}}\frac{q^{\frac{m-1}{2}}}{1-q^m}\right) - \frac{2}{1-q} - \frac{q^{\frac{1}{2}}(t^2+t^{-2})}{1-q^2}\right].\end{aligned}$$

$$\begin{aligned}\langle W_{\Box} W_{\Box}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) &= \langle W_{\Box\Box} W_{\Box\Box}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SU(2)}(\mathfrak{q})\\ &= \frac{1+\mathfrak{q}^2+\mathfrak{q}^4}{1-\mathfrak{q}^4}\\ &= \frac{1-\mathfrak{q}^6}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}.\end{aligned}$$

$$\langle \underbrace{W_{\Box} \cdots W_{\Box}}_{k} \rangle^{SO(3)}(t;q) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{2(k-i)} \rangle^{SO(3)}(t;q),$$

$$\begin{aligned}\langle W_{\Box}\rangle^{SO(3)}(t;q) &= \left\langle W_{\mathrm{sp}} W_{\mathrm{sp}}\right\rangle^{SO(3)}(t;q) - \mathcal{I}^{SO(3)}(t;q)\\ &= -\frac{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}{(q;q)_\infty^2}\sum_{\substack{p_1,p_2\in\mathbb{Z}\\ p_1+1<p_2}} \frac{\left(q^{\frac{1}{2}}t^{-2}\right)^{p_1+p_2-2}}{\left(1-q^{p_1-\frac{1}{2}}t^2\right)\left(1-q^{p_2-\frac{1}{2}}t^2\right)}\end{aligned}$$

$$\begin{aligned}\langle W_{\Box}\rangle_{\frac{1}{2}BPS}^{SO(3)}(\mathfrak{q}) &= \langle W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle_{\frac{1}{2}BPS}^{SO(3)}(\mathfrak{q}) - \mathcal{I}_{\frac{1}{2}BPS}^{SO(3)}(\mathfrak{q})\\ &= \frac{\mathfrak{q}^2}{1-\mathfrak{q}^4}\end{aligned}$$

$$\begin{aligned}\langle W_{\Box} W_{\Box} W_{\Box}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) &= \langle W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) - 3\langle W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q})\\ &+3\langle W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) - \mathcal{I}_{\frac{1}{2}\mathrm{BPS}}^{SO(9)}(\mathfrak{q})\\ &= \frac{1+3\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^6}{1-\mathfrak{q}^4}\end{aligned}$$

$$\langle W_\square W_\square W_\square W_\square\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{3+6\mathfrak{q}^2+6\mathfrak{q}^4+3\mathfrak{q}^6+\mathfrak{q}^8}{1-\mathfrak{q}^4}$$

$$\langle W_\square W_\square W_\square W_\square W_\square\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{6+15\mathfrak{q}^2+15\mathfrak{q}^4+10\mathfrak{q}^6+4\mathfrak{q}^8+\mathfrak{q}^{10}}{1-\mathfrak{q}^4}$$

$$\langle W_\square W_\square W_\square W_\square W_\square W_\square\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{15+36\mathfrak{q}^2+40\mathfrak{q}^4+29\mathfrak{q}^6+15\mathfrak{q}^8+5\mathfrak{q}^{10}+\mathfrak{q}^{12}}{1-\mathfrak{q}^4}$$

$$\langle \underbrace{W_\square\cdots W_\square}_{k}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{k} a^{\mathfrak{so}(3)}_{k\square}(i)\mathfrak{q}^{2i}}{1-\mathfrak{q}^4},$$

$$a^{\mathfrak{so}(3)}_{k\square}(i) = c_k^{(i)} - c_k^{(i+1)}$$

$$(1+x+x^2)^n = \sum_{i=-k}^{k} c_k^{(i)} x^{k+i}$$

$$R_n = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} C_i,$$

$$\sum_{n=1}^{\infty} R_n x^n = \frac{1}{2x}\left(1-\frac{\sqrt{1-3x}}{\sqrt{1+x}}\right)$$

$$\left\langle W_{(k)}W_{(k)}\right\rangle^{SO(3)}(t;q) = \frac{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}{(qt^{\pm 4};q)_\infty}\left[\frac{2k+1}{2}\sum_{m\in\mathbb{Z}\setminus\{0\}}\left(\frac{t^{2m}-t^{-2m}}{t-t^{-2}}\frac{q^{\frac{m-1}{2}}}{1-q^m}\right) - \sum_{m=1}^{2k}(2k-m+1)\left(\frac{t^{2m}-t^{-2m}}{t-t^{-2}}\frac{q^{\frac{m-1}{2}}}{1-q^m}\right)\right]$$

$$\begin{aligned}\langle W_{(k)}W_{(k)}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) &= \frac{1+\mathfrak{q}^2+\cdots+\mathfrak{q}^{4k}}{1-\mathfrak{q}^4}\\ &= \frac{1-\mathfrak{q}^{4k+2}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}\end{aligned}$$

$$\langle W_{(\infty)}W_{(\infty)}\rangle^{SO(\mathfrak{q})}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\langle\underbrace{W_{\square\square}\cdots W_{\square\square}}_{k}\rangle^{SO(3)}(t;q) = \sum_{k_1+k_2+k_3=k}\binom{k}{k_1,k_2,k_3}(-3)^{k_2}\langle\underbrace{W_{\mathrm{sp}}\cdots W_{\mathrm{sp}}}_{4k_1+2k_2}\rangle^{SO(3)}(t;q)$$

$$\begin{aligned}\langle W_{\square\square}\rangle^{SO(\mathfrak{q})}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) &= \langle W_{\mathrm{sp}}W_{\mathrm{sp}}W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 3\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + \mathcal{I}^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\\ &= \frac{\mathfrak{q}^4}{1-\mathfrak{q}^4},\end{aligned}$$

$$\begin{aligned}\langle W_{\square\square}W_{\square\square}W_{\square\square}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) &= \langle\underbrace{W_{\mathrm{sp}}\cdots W_{\mathrm{sp}}}_{12}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 9\langle\underbrace{W_{\mathrm{sp}}\cdots W_{\mathrm{sp}}}_{10}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\\ &+30\langle\underbrace{W_{\mathrm{sp}}\cdots W_{\mathrm{sp}}}_{8}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 45\langle\underbrace{W_{\mathrm{sp}}\cdots W_{\mathrm{sp}}}_{6}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\\ &+30\langle\underbrace{W_{\mathrm{sp}}\cdots W_{\mathrm{sp}}}_{4}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 9\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + \mathcal{I}^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\\ &= \frac{1+3\mathfrak{q}^2+5\mathfrak{q}^4+4\mathfrak{q}^6+3\mathfrak{q}^8+2\mathfrak{q}^{10}+\mathfrak{q}^{12}}{1-\mathfrak{q}^4}\end{aligned}$$

$$
\begin{aligned}
&\langle \underbrace{W_{\square\square\square\ldots} \cdots W_{\square\square\square}}_{k} \rangle^{SO(3)}(t;q) \\
&= \sum_{k_1+k_2+k_3+k_4=k} \binom{k}{k_1,k_2,k_3,k_4} (-1)^{k_2+k_4} 5^{k_2} 6^{k_3} \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{6k_2+4k_1+2k_2} \rangle^{SO(3)}(t;q).
\end{aligned}
$$

$$
\begin{aligned}
&\langle W_{\square\square\square} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) \\
&= \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{6} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 5\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{4} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + 6\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - \mathcal{I}^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) \\
&= \frac{\mathfrak{q}^6}{1-\mathfrak{q}^4}, \\
&\langle W_{\square\square\square} W_{\square\square\square} W_{\square\square\square} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) \\
&= \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{18} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 15\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{16} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + 93\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{14} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) \\
&-308\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{12} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + 588\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{10} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 651\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{8} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) \\
&+399\langle \underbrace{\langle_{\underbrace{}_{\mathrm{sp}}} \cdots W_{\mathrm{sp}}}_{6} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 123\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{4} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + 18\left\langle W_{\mathrm{sp}}^{W_{\mathrm{sp}}} \right\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - \mathcal{I}^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) \\
&= \frac{1+3\mathfrak{q}^2+5\mathfrak{q}^4+7\mathfrak{q}^6+6\mathfrak{q}^8+5\mathfrak{q}^{10}+4\mathfrak{q}^{12}+3\mathfrak{q}^{14}+2\mathfrak{q}^{16}+\mathfrak{q}^{18}}{1-\mathfrak{q}^4}
\end{aligned}
$$

$$
\chi^{\mathfrak{so}(3)}_{(k)} = \sum_{n=0}^{k} (-1)^n \binom{2k-n}{n} \chi_{\mathrm{sp}}^{\mathfrak{so}(3)^{2k-2n}}.
$$

$$
\langle W_{(k)} \rangle^{SO(3)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{\mathfrak{q}^{2k}}{1-\mathfrak{q}^4}.
$$

$$
\langle W_{(k)} W_{(l)} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{2(l-k)}(1-\mathfrak{q}^{4k+2})}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}.
$$

$$
\langle W_{(k)} W_{(k)} W_{(k)} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2-3\mathfrak{q}^{2k+2}+\mathfrak{q}^{6k+4}}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)}.
$$

$$
\langle W_{(\infty)} W_{(\infty)} W_{(\infty)} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)^3}
$$

$$
\langle W_{(k)} W_{(k)} W_{(k)} W_{(k)} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{2k+1-3\mathfrak{q}^2-(2k+1)\mathfrak{q}^4+4\mathfrak{q}^{4k+4}-\mathfrak{q}^{8k+6}}{(1-\mathfrak{q}^2)^3(1-\mathfrak{q}^4)}
$$

$$
\begin{aligned}
&\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Sin}(5)}(t;q) \\
&= \frac{1}{8} \frac{(q)^4_\infty}{\left(q^{\frac{1}{2}} t^{\pm}; q\right)^2_\infty} \oiiint \prod_{i=1}^{2} \frac{ds_i}{2\pi i s_i} \frac{(s_i^{\pm}; q)_\infty (q s_i^{\pm}; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_i^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm}; q\right)_\infty} \\
&\times \frac{(s_1^{\pm} s_2^{\mp}; q)_\infty (s_1^{\pm} s_2^{\pm}; q)_\infty (q s_1^{\pm} s_2^{\mp}; q)_\infty (q s_1^{\pm} s_2^{\pm}; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^2 s_1^{\pm} s_2^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_1^{\pm} s_2^{\pm}; q\right)_\infty} \prod_{i=1}^{2} \left(s_i^{\frac{1}{2}} + s_i^{-\frac{1}{2}}\right)^2
\end{aligned}
$$

$$\left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2}\right)} \right\rangle^{USp(4)/\mathbb{Z}_2}(t;q)$$

$$= \frac{1}{2} \frac{(q)_\infty^4}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^2} \oint\!\!\oint \prod_{i=1}^{2} \frac{ds_i}{2\pi i s_i} \frac{\left(q^{\frac{1}{2}} s_i^{\pm 2}; q\right)_\infty \left(q^{\frac{3}{2}} s_i^{\pm 2}; q\right)_\infty}{\left(q t^2 s_i^{\pm 2}; q\right)_\infty \left(q t^{-2} s_i^{\pm 2}; q\right)_\infty}$$

$$\times \frac{\left(s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} s_1^{\pm} s_2^{\pm}; q\right)_\infty \left(q s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{3}{2}} s_1^{\pm} s_2^{\pm}; q\right)_\infty}{\left(q^{\frac{1}{2}} t^2 s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q t^2 s_1^{\pm} s_2^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q t^{-2} s_1^{\pm} s_2^{\pm}; q\right)_\infty}$$

$$\langle W_{\rm sp} W_{\rm sp} \rangle^{\rm Spin(5)}(t;q) = \left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2}\right)} \right\rangle^{\rm USp(4)/\mathbb{Z}_2}(t;q)$$

$$\langle W_{\rm sp} W_{\rm sp} \rangle^{\rm Spin(5)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2}\right)} \right\rangle^{\rm USp(4)/\mathbb{Z}_2}_{\frac{1}{2}\rm BPS}(\mathfrak{q})$$

$$= \frac{1 + \mathfrak{q}^2 + \mathfrak{q}^4 + \mathfrak{q}^6}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\langle \underbrace{W_{\rm sp} \cdots W_{\rm sp}}_{4} \rangle^{\rm Spin(5)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{3 + 6\mathfrak{q}^2 + 8\mathfrak{q}^4 + 9\mathfrak{q}^6 + 6\mathfrak{q}^8 + 3\mathfrak{q}^{10} + \mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle \underbrace{W_{\rm sp} \cdots W_{\rm sp}}_{6} \rangle^{\rm Spin(5)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(14 + 40\mathfrak{q}^2 + 66\mathfrak{q}^4 + 85\mathfrak{q}^6$$
$$+81\mathfrak{q}^8 + 59\mathfrak{q}^{10} + 34\mathfrak{q}^{12} + 15\mathfrak{q}^{14} + 5\mathfrak{q}^{16} + \mathfrak{q}^{18})$$

$$\langle \underbrace{W_{\rm sp} \cdots W_{\rm sp}}_{8} \rangle^{\rm Spin(5)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(84 + 300\mathfrak{q}^2 + 581\mathfrak{q}^4 + 840\mathfrak{q}^6 + 945\mathfrak{q}^8 + 842\mathfrak{q}^{10}$$
$$+616\mathfrak{q}^{12} + 378\mathfrak{q}^{14} + 195\mathfrak{q}^{16} + 83\mathfrak{q}^{18} + 28\mathfrak{q}^{20} + 7\mathfrak{q}^{22} + \mathfrak{q}^{24})$$

$$\langle \underbrace{W_{\rm sp} \cdots W_{\rm sp}}_{2k} \rangle^{\rm Spin(5)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{\sum_{i=0}^{3k} a_{k\rm sp}^{\mathfrak{so}(5)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}.$$

$$a_{k\rm sp}^{\mathfrak{so}(5)}(0) = C_k C_{k+2} - C_{k+1}^2$$
$$= \frac{24(2k+1)!\,(2k-1)!}{(k-1)!\,k!\,(k+2)!\,(k+3)!}$$
$$= \prod_{1\le i\le j\le k-1} \frac{i+j+4}{i+j}$$

$${}_3F_2\left(1, \frac{1}{2}, \frac{3}{2}; 3, 4; 16x\right) = \sum_{k=0}^{\infty} a_{k\rm sp}^{\mathfrak{so}(5)}(0) x^k$$

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}$$

$$a_{k\rm sp}^{\mathfrak{so}(5)}(1) = \frac{60(2k)!\,(2k+2)!}{(k-1)!\,k!\,(k+3)!\,(k+4)!}$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{k} \rangle^{SO(5)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{\sum_{i=0}^{2k} a_{k\square}^{\mathfrak{so}(5)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square}\rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^4}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_{\square}W_{\square}\rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$= \frac{1-\mathfrak{q}^{10}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_{\square}W_{\square}W_{\square}\rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2+3\mathfrak{q}^4+3\mathfrak{q}^6+3\mathfrak{q}^8+2\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square}\rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(3+3\mathfrak{q}^2+9\mathfrak{q}^4+15\mathfrak{q}^6+12\mathfrak{q}^8$$

$$+12\mathfrak{q}^{10}+6\mathfrak{q}^{12}+\mathfrak{q}^{16})$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square}W_{\square}\rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(1+10\mathfrak{q}^2+24\mathfrak{q}^4+36\mathfrak{q}^6+44\mathfrak{q}^8$$

$$+41\mathfrak{q}^{10}+31\mathfrak{q}^{12}+19\mathfrak{q}^{14}+10\mathfrak{q}^{16}+4\mathfrak{q}^{18}+\mathfrak{q}^{20})$$

$$a^{\mathfrak{so}(5)}_{k\square}(0) = \sum_{i=0}^{\lfloor\frac{k}{2}\rfloor} C_i C_{i+1}\binom{k}{2i} - \sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor} C_i^2\binom{k}{2i-1}$$

$$= -k\,{}_3F_2\left(\frac{3}{2},\frac{1}{2}-\frac{k}{2};3,3;16\right) + {}_3F_2\left(\frac{3}{2},\frac{1}{2}-\frac{k}{2};2,3;16\right) \quad (4.96)$$

$$\langle \underbrace{W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\cdots W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}_{k}\rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{3k} a^{\mathfrak{so}(5)}_{k\begin{smallmatrix}\square\\\square\end{smallmatrix}}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\left\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right\rangle^{SO(\mathfrak{q})}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{\mathfrak{q}^2+\mathfrak{q}^6}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$= \frac{\mathfrak{q}^2}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right\rangle^{SO(\mathfrak{q})}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+3\mathfrak{q}^4+2\mathfrak{q}^6+3\mathfrak{q}^8+\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$= \frac{(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^3}$$

$$\left\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right\rangle^{SO(5)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(1+6\mathfrak{q}^2+9\mathfrak{q}^4+16\mathfrak{q}^6+15\mathfrak{q}^8$$

$$+15\mathfrak{q}^{10}+9\mathfrak{q}^{12}+6\mathfrak{q}^{14}+2\mathfrak{q}^{16}+\mathfrak{q}^{18})$$

$$\left\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right\rangle^{SO(5)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(6+22\mathfrak{q}^2+54\mathfrak{q}^4+82\mathfrak{q}^6+15\mathfrak{q}^8$$

$$+15\mathfrak{q}^{10}+9\mathfrak{q}^{12}+6\mathfrak{q}^{14}+2\mathfrak{q}^{16}+\mathfrak{q}^{18})$$

$$\langle \underbrace{W_{(2)}\cdots W_{(2)}}_{k}\rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{4k} a^{\mathfrak{so}(5)}_{k\square\square}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square\square}\rangle^{SO(5)}_{\frac{1}{2}BPS}(\mathfrak{q})=\frac{\mathfrak{q}^4+\mathfrak{q}^8}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$=\frac{\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2}$$
$$\langle W_{\square\square}W_{\square\square}\rangle^{SO(5)}_{\frac{1}{2}BPS}(\mathfrak{q})=\frac{1+\mathfrak{q}^2+2\mathfrak{q}^4+2\mathfrak{q}^6+3\mathfrak{q}^8+2\mathfrak{q}^{10}+3\mathfrak{q}^{12}+\mathfrak{q}^{14}+\mathfrak{q}^{16}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$\langle W_{\square\square}W_{\square\square}W_{\square\square}\rangle^{SO(5)}_{\frac{1}{2}BPS}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$\times(1+3\mathfrak{q}^2+9\mathfrak{q}^4+13\mathfrak{q}^6+20\mathfrak{q}^8+21\mathfrak{q}^{10}$$
$$+22\mathfrak{q}^{12}+18\mathfrak{q}^{14}+15\mathfrak{q}^{16}+9\mathfrak{q}^{18}+6\mathfrak{q}^{20}+2\mathfrak{q}^{22}+\mathfrak{q}^{24})$$

$$\langle W_{(k)}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\mathfrak{q}^{4k-4}+\mathfrak{q}^{4k}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$=\frac{\mathfrak{q}^{4k-4}}{(1-\mathfrak{q}^4)^2}$$

$$\langle W_{(k)}W_{(k)}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\sum_{i=0}^{8k}a_2^{\mathfrak{s}(5)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square\square\square}W_{\square\square\square}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(1+\mathfrak{q}^2+2\mathfrak{q}^4+3\mathfrak{q}^6+4\mathfrak{q}^8+4\mathfrak{q}^{10}$$
$$+6\mathfrak{q}^{12}+5\mathfrak{q}^{14}+5\mathfrak{q}^{16}+4\mathfrak{q}^{18}+3\mathfrak{q}^{20}+\mathfrak{q}^{22}+\mathfrak{q}^{24})$$
$$\langle W_{\square\square\square}W_{\square\square\square}W_{\square\square\square}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(1+\mathfrak{q}^2+2\mathfrak{q}^4+3\mathfrak{q}^6+5\mathfrak{q}^8+5\mathfrak{q}^{10}$$
$$+8\mathfrak{q}^{12}+8\mathfrak{q}^{14}+10\mathfrak{q}^{16}+9\mathfrak{q}^{18}+10\mathfrak{q}^{20}+7\mathfrak{q}^{22}+7\mathfrak{q}^{24}$$
$$+4\mathfrak{q}^{26}+3\mathfrak{q}^{28}+\mathfrak{q}^{30}+\mathfrak{q}^{32})$$

$$\langle W_{(\infty)}W_{(\infty)}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$=1+\mathfrak{q}^2+3\mathfrak{q}^4+4\mathfrak{q}^6+9\mathfrak{q}^8+11\mathfrak{q}^{10}+21\mathfrak{q}^{12}+26\mathfrak{q}^{14}+44\mathfrak{q}^{16}+54\mathfrak{q}^{18}+84\mathfrak{q}^{20}+\cdots$$

$$\langle W_{(\infty)}W_{(\infty)}\rangle^{SO(\mathfrak{q})}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1-\mathfrak{q}^{24}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^2(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)^2(1-\mathfrak{q}^{12})}.$$

$$\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(7)}(t;q)$$
$$=\frac{1}{48}\frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}t^{\pm};q\right)_\infty^3}\oint\!\!\oint\!\!\oint\prod_{i=1}^{3}\frac{ds_i}{2\pi is_i}\frac{(s_i^{\pm};q)_\infty(qs_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty(qs_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}\prod_{i=1}^{3}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)^2$$

$$\left\langle T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{USP(6)/\mathbb{Z}_2}(t;q)$$
$$=\frac{1}{6}\frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}t^{\pm2};q\right)_\infty^3}\oint\!\!\oint\!\!\oint\prod_{i=1}^{3}\frac{ds_i}{2\pi is_i}\frac{\left(q^{\frac{1}{2}}s_i^{\pm2};q\right)_\infty\left(q^{\frac{3}{2}}s_i^{\pm2};q\right)_\infty}{(qt^2s_i^{\pm2};q)_\infty(qt^{-2}s_i^{\pm2};q)_\infty}$$
$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty\left(q^{\frac{1}{2}}s_i^{\pm}s_j^{\pm};q\right)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty\left(q^{\frac{3}{2}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty(qt^2s_i^{\pm}s_j^{\pm};q)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty(qt^{-2}s_i^{\pm}s_j^{\pm};q)_\infty}$$

$$
\begin{aligned}
\langle W_{\rm sp} W_{\rm sp}\rangle^{Spin(7)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) &= \langle T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} T_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}\rangle^{USp(6)/\mathbb{Z}_2}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) \\
&= \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+2\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})} \\
&= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)}.
\end{aligned}
$$

$$
\langle \underbrace{W_{\rm sp}\cdots W_{\rm sp}}_{2k}\rangle^{\rm Spin(7)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{\sum_{i=0}^{6k} a^{\mathfrak{so}(7)}_{k\rm sp}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}.
$$

$$
\begin{aligned}
\langle W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}\rangle^{\rm Spin(7)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}\ &(4+9\mathfrak{q}^2+15\mathfrak{q}^4 \\
&+25\mathfrak{q}^6+29\mathfrak{q}^8+32\mathfrak{q}^{10}+33\mathfrak{q}^{12}+26\mathfrak{q}^{14}+20\mathfrak{q}^{16} \\
&+13\mathfrak{q}^{18}+6\mathfrak{q}^{20}+3\mathfrak{q}^{22}+\mathfrak{q}^{24}) \\
\langle W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}\rangle^{\rm Spin(7)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}\ &(30+105\mathfrak{q}^2+235\mathfrak{q}^4 \\
&+435\mathfrak{q}^6+650\mathfrak{q}^8+855\mathfrak{q}^{10}+1010\mathfrak{q}^{12}+1055\mathfrak{q}^{14} \\
&+1006\mathfrak{q}^{16}+865\mathfrak{q}^{18}+665\mathfrak{q}^{20}+470\mathfrak{q}^{22}+299\mathfrak{q}^{24} \\
&+170\mathfrak{q}^{26}+89\mathfrak{q}^{28}+40\mathfrak{q}^{30}+15\mathfrak{q}^{32}+5\mathfrak{q}^{34}+\mathfrak{q}^{36})
\end{aligned}
$$

$$
a^{\mathfrak{so}(7)}_{ksp}(0) = \prod_{1\le i\le j\le k-1} \frac{i+j+6}{i+j}
$$

$$
{}_4F_3\left(1,\frac{1}{2},\frac{5}{2},\frac{3}{2};4,5,6;64x\right) = \sum_{k=0}^{\infty} a^{\mathfrak{so}(7)}_{ksp}(0)x^k
$$

$$
\langle \underbrace{W_{\square}\cdots W_{\square}}_{k}\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{\sum_{i=0}^{3k} a^{\mathfrak{so}(7)}_{k\square}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})},
$$

$$
\begin{aligned}
\langle W_{\square}\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) &= \frac{\mathfrak{q}^6}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}, \\
\langle W_{\square}W_{\square}\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) &= \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})} \\
&= \frac{1-\mathfrak{q}^{14}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}, \\
\langle W_{\square}W_{\square}W_{\square}\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) &= \frac{\mathfrak{q}^4+3\mathfrak{q}^6+3\mathfrak{q}^8+3\mathfrak{q}^{10}+3\mathfrak{q}^{12}+3\mathfrak{q}^{14}+2\mathfrak{q}^{16}+\mathfrak{q}^{18}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}.
\end{aligned}
$$

$$
\begin{aligned}
\langle \underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k}\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) &= \frac{\sum_{i=0}^{5k} a^{\mathfrak{s}(7)}_{k\substack{\square\\\square}}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}, \\
\underbrace{\left\langle W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}\right\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}}_{k}(\mathfrak{q}) &= \frac{\sum_{i=0}^{6k} a^{\mathfrak{s}(7)}_{k}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}.
\end{aligned}
$$

$$
\begin{aligned}
\left\langle W_{\substack{\square\\\square}}\right\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) &= \frac{\mathfrak{q}^2+\mathfrak{q}^6+\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}, \\
\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) &= \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})} \\
&\times(1+\mathfrak{q}^2+3\mathfrak{q}^4+2\mathfrak{q}^6+5\mathfrak{q}^8+3\mathfrak{q}^{10} \\
&+5\mathfrak{q}^{12}+2\mathfrak{q}^{14}+3\mathfrak{q}^{16}+\mathfrak{q}^{18}+\mathfrak{q}^{20}).
\end{aligned}
$$

$$\left\langle W_{\substack{\square\\\square\\\square}} \right\rangle^{SO(7)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^4+\mathfrak{q}^8+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})},$$

$$\left\langle W_{\substack{\square\\\square\\\square}} W_{\substack{\square\\\square\\\square}} \right\rangle^{SO(7)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})} \times (1+\mathfrak{q}^2+3\mathfrak{q}^4+4\mathfrak{q}^6+7\mathfrak{q}^8+6\mathfrak{q}^{10}+9\mathfrak{q}^{12} +6\mathfrak{q}^{14}+7\mathfrak{q}^{16}+4\mathfrak{q}^{18}+3\mathfrak{q}^{20}+\mathfrak{q}^{22}+\mathfrak{q}^{24}).$$

$$\langle \underbrace{W_{(l)}\cdots W_{(l)}}_{k} \rangle^{SO(7)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{3lk} a_{k(l)}^{\mathfrak{so}(7)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}.$$

$$\langle W_{\square\square} \rangle^{SO(7)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^4+\mathfrak{q}^8+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})},$$

$$\langle W_{\square\square} W_{\square\square} \rangle^{SO(7)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})} \times (1+\mathfrak{q}^2+2\mathfrak{q}^4+2\mathfrak{q}^6+4\mathfrak{q}^8+3\mathfrak{q}^{10}+5\mathfrak{q}^{12} +3\mathfrak{q}^{14}+5\mathfrak{q}^{16}+2\mathfrak{q}^{18}+3\mathfrak{q}^{20}+\mathfrak{q}^{22}+\mathfrak{q}^{24}).$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(2N+1)}(t;q) = \frac{1}{2^N N!} \frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm};q\right)_\infty^N} \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \frac{(s_i^{\pm};q)_\infty (q s_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2} s_i^{\pm};q\right)_\infty} \times \prod_{i<j} \frac{(s_i^{\pm}s_j^{\mp};q)_\infty (s_i^{\pm}s_j^{\pm};q)_\infty (q s_i^{\pm}s_j^{\mp};q)_\infty (q s_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\pm};q\right)_\infty} \prod_{i=1}^N \left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)^2$$

$$\langle T_{(\frac{1}{2}^N)} T_{(\frac{1}{2}^N)} \rangle^{USp(2N)/\mathbb{Z}_2}(t;q) = \frac{1}{N!} \frac{(q)_\infty^{2N}}{(q^{\frac{1}{2}}t^{\pm 2};q)_\infty^N} \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \frac{(q^{\frac{1}{2}} s_i^{\pm 2};q)_\infty (q^{\frac{3}{2}} s_i^{\pm 2};q)_\infty}{(q t^2 s_i^{\pm 2};q)_\infty (q t^{-2} s_i^{\pm 2};q)_\infty} \times \prod_{i<j} \frac{(s_i^{\pm}s_j^{\mp};q)_\infty (q^{\frac{1}{2}} s_i^{\pm}s_j^{\pm};q)_\infty (q s_i^{\pm}s_j^{\mp};q)_\infty (q^{\frac{3}{2}} s_i^{\pm}s_j^{\pm};q)_\infty}{(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\mp};q)_\infty (q t^2 s_i^{\pm}s_j^{\pm};q)_\infty (q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\mp};q)_\infty (q t^{-2} s_i^{\pm}s_j^{\pm};q)_\infty}.$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(2N+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle T_{(\frac{1}{2}^N)^\blacksquare} T_{(\frac{1}{2}^N)^\blacksquare} \right\rangle^{\mathrm{USp}(2N)/\mathbb{Z}_2}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \prod_{n=1}^N \frac{1}{(1-\mathfrak{q}^{2n})}$$

$$\mathcal{I}^{\mathrm{Spin}(2N+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \prod_{n=1}^N \frac{1}{1-\mathfrak{q}^{4n}},$$

$$\langle \mathcal{W}_{\mathrm{sp}} \mathcal{W}_{\mathrm{sp}} \rangle^{\mathrm{Spin}(2N+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \prod_{n=1}^N (1+\mathfrak{q}^{2n})$$

$$\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{2k} \rangle_{\frac{1}{2}\mathrm{BPS}}^{\mathrm{Spin}(2N+1)}(\mathfrak{q}) = \frac{\sum_{i=0}^{\frac{N(N+1)k}{2}} a_{k\mathrm{sp}}^{\mathfrak{so}(2N+1)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})},$$

$$a_{k\,\mathrm{sp}}^{\mathfrak{so}(2N+1)}(0) = \det(C_{2N-i-j+k}) = \prod_{1\le i\le j\le k-1} \frac{i+j+2N}{i+j},$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{k} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{\sum_{i=0}^{Nk} a_{k\square}^{\mathfrak{so}(2N+1)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})},$$

$$\langle W_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{\mathfrak{q}^{2N}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})},$$

$$\langle W_{\square} W_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\cdots+\mathfrak{q}^{4N}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})} = \frac{1-\mathfrak{q}^{4N+2}}{(1-\mathfrak{q}^2)\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})}.$$

$$\langle \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \mathfrak{q}^{2N},$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{4N+2}}{1-\mathfrak{q}^2}.$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS},c}^{SO(2N+1)}(\mathfrak{q}) = \langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) - \langle \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q})^2 = \frac{1-\mathfrak{q}^{4N}}{1-\mathfrak{q}^2}$$

$$\langle \underbrace{W_{\substack{\square\\\square}} \cdots W_{\substack{\square\\\square}}}_{k} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{\sum_{i=0}^{(2N-1)k} a_{k\substack{\square\\\square}}^{\mathfrak{so}(2N+1)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})},$$

$$\left\langle W_{\substack{\square\\\square}} W_{\substack{\square\\\square}} \right\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)} = \frac{\mathfrak{q}^2+\mathfrak{q}^6+\cdots+\mathfrak{q}^{4N-2}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})} = \frac{\mathfrak{q}^2(1-\mathfrak{q}^{4N})}{(1-\mathfrak{q}^4)\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})}$$

$$\left\langle \mathcal{W}_{\substack{\square\\\square}} \right\rangle_{\frac{1}{2}\mathrm{BPS}}^{\mathrm{SO}(2N+1)}(\mathfrak{q}) = \frac{\mathfrak{q}^2(1-\mathfrak{q}^{4N})}{1-\mathfrak{q}^4}.$$

$$\langle \underbrace{W_{(l)} \cdots W_{(l)}}_{k} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{\sum_{i=0}^{Nlk} a_{k(l)}^{\mathfrak{so}(2N+1)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})},$$

$$(z_e, z_m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$(z_e, z_m) = (0,0), (z_e, z_m) = (1,0)$$

$$(z_e, z_m) = (0,0), (z_e, z_m) = (0,1)$$

$$(z_e, z_m) = (0,0), (z_e, z_m) = (1,1)$$

$$\chi_{\square}^{\mathfrak{usp}(2N)} = \sum_{i=1}^{N} (s_i + s_i^{-1}).$$

$$\chi_{\square}^{\mathfrak{usp}(2N)} = \frac{\det\left(s_j^{\lambda_i+N-i+1} - s_j^{-\lambda_i-N+i-1}\right)}{\det\left(s_j^{N-i+1} - s_j^{-N+i-1}\right)}.$$

$$\begin{aligned}&\left\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\right\rangle^{\mathrm{USp}(2N)}\\&=\int d\mu^{\mathrm{USp}(2N)}\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)L_n(s)\right)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathrm{up}(2N)}(s),\end{aligned}$$

$$L_n(s)=\frac{P_n(s)^2+P_{2n}(s)}{2}$$

$$\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)L_n(s)\right)=\sum_{\lambda}\frac{1}{z_\lambda}f_\lambda(q,t)L_\lambda(s)$$

$$\left\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\right\rangle^{\mathrm{USp}(2N)}=\sum_{\lambda}\frac{1}{z_\lambda}f_\lambda(q,t)\int d\mu^{\mathrm{USp}(2N)}L_\lambda(s)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathrm{usp}(2N)}(s)$$

$$L_\lambda(s)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathrm{up}(2N)}(s)=\sum_{\mu}b_{\lambda,\mathcal{R}}^{\mu}P_\mu(s),$$

$$\int d\mu^{USp(2N)}P_\mu(s)=\sum_{\nu\in R_{2N}^{c}(|\mu|)}\chi_\nu^{S}(\mu)$$

$$R_n^c(p)=\{\lambda\vdash p\mid \ell(\lambda)\le n \text{ and } \forall\lambda_i' \text{ is even }\}.$$

$$\begin{aligned}\int d\mu^{USp(2N)}L_\lambda(s)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathrm{usp}(2N)}(s)&=\sum_{\mu}b_{\lambda,\mathcal{R}}^{\mu}\int d\mu^{USp(2N)}P_\mu(s)\\&=\sum_{\mu}b_{\lambda,\mathcal{R}}^{\mu}\sum_{\nu\in R_{2N}^{c}(|\mu|)}\chi_\nu^{S}(\mu)\end{aligned}$$

$$\left\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\right\rangle^{USp(2N)}=\sum_{\lambda}\frac{1}{z_\lambda}f_\lambda(q,t)\sum_{\mu}b_{\lambda,\mathcal{R}}^{\mu}\sum_{\nu\in R_{2N}^{c}(|\mu|)}\chi_\nu^{S}(\mu).$$

$$\begin{aligned}&\langle W_\square W_\square\rangle^{USp(2)}(t;q)\\&=\frac{1}{2}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm2};q\right)_\infty}\oint\frac{ds}{2\pi is}\frac{(s^{\pm2};q)_\infty(qs^{\pm2};q)_\infty}{\left(q^{\frac{1}{2}}t^{2}s^{\pm2};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s^{\pm2};q\right)_\infty}(s+s^{-1})^2\end{aligned}$$

$$\begin{aligned}&\left\langle T_{(1)}T_{(1)}\right\rangle^{SO(3)}(t;q)\\&=\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm2};q\right)_\infty}\oint\frac{ds}{2\pi is}\frac{\left(q^{\frac{1}{2}}s^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}s^{\pm};q\right)_\infty}{(qt^{2}s^{\pm};q)_\infty(qt^{-2}s^{\pm};q)_\infty}\end{aligned}$$

$$\begin{aligned}&\langle W_\square W_\square\rangle^{\mathrm{USp}(2)}(t;q)=\left\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\right\rangle^{\mathrm{Spin}(3)}(t;q)\\&=\left\langle T_{(1)}T_{(1)}\right\rangle^{SO(3)}(t;q)=\left\langle T_{\left(\frac{1}{2}\right)}T_{\left(\frac{1}{2}\right)}\right\rangle^{USp(2)/\mathbb{Z}_2}(t;q).\end{aligned}$$

$$\begin{aligned}\langle W_\square W_\square\rangle_{\frac{1}{2}\mathrm{BPS}}^{\mathrm{USp}(2)}(\mathfrak{q})&=\left\langle T_{(1)}T_{(1)}\right\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q})\\&=\frac{1+\mathfrak{q}^2}{1-\mathfrak{q}^4}\\&=\frac{1}{1-\mathfrak{q}^2}\end{aligned}$$

$$\langle W_{\square\square} W_{\square\square}\rangle^{USp(2)}(t;q)$$
$$= \frac{1}{2} \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty} \oint \frac{ds}{2\pi i s} \frac{(s^{\pm 2}; q)_\infty (q s^{\pm 2}; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s^{\pm 2}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s^{\pm 2}; q\right)_\infty} (1 + s^2 + s^{-2})^2$$

$$\langle W_{(2k)}\rangle^{USp(2)}(t;q) = \langle W_{(k)}\rangle^{SO(3)}(t;q).$$

$$\langle W_{(2k)}\rangle^{USp(2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{2k}}{(1-\mathfrak{q}^4)}.$$

$$\langle W_{(k)} W_{(k)}\rangle^{USp(2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{2k+2}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\langle W_{(\infty)} W_{(\infty)}\rangle^{USp(2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\langle W_{\square} W_{\square}\rangle^{USp(4)}(t;q)$$
$$= \frac{1}{8} \frac{(q)_\infty^4}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^2} \oint\oint \prod_{i=1}^{2} \frac{ds_i}{2\pi i s_i} \frac{(s_i^{\pm 2}; q)_\infty (q s_i^{\pm 2}; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_i^{\pm 2}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm 2}; q\right)_\infty}$$
$$\times \frac{(s_1^{\pm} s_2^{\mp}; q)_\infty (s_1^{\pm} s_2^{\pm}; q)_\infty (q s_1^{\pm} s_2^{\mp}; q)_\infty (q s_1^{\pm} s_2^{\pm}; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^2 s_1^{\pm} s_2^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_1^{\pm} s_2^{\pm}; q\right)_\infty} \left[\sum_{i=1}^{2} (s_i + s_i^{-1})\right]^2$$

$$\langle T_{(1,0)} T_{(1,0)}\rangle^{SO(5)}(t;q)$$
$$= \frac{1}{2} \frac{(q)_\infty^4}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^2} \oint\oint \prod_{i=1}^{2} \frac{ds_i}{2\pi i s_i} \frac{\left(q^{\frac{1}{2}} s_1^{\pm}; q\right)_\infty (s_2^{\pm}; q)_\infty \left(q^{\frac{3}{2}} s_1^{\pm}; q\right)_\infty (q s_2^{\pm}; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_2^{\pm}; q\right)_\infty (q t^{-2} s_1^{\pm}; q)_\infty \left(q^{\frac{1}{2}} t^{-2} s_2^{\pm}; q\right)_\infty}$$
$$\times \frac{\left(q^{\frac{1}{2}} s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} s_1^{\pm} s_2^{\pm}; q\right)_\infty \left(q^{\frac{3}{2}} s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{3}{2}} s_1^{\pm} s_2^{\pm}; q\right)_\infty}{(q t^2 s_1^{\pm} s_2^{\mp}; q)_\infty (q t^2 s_1^{\pm} s_2^{\pm}; q)_\infty (q t^{-2} s_1^{\pm} s_2^{\mp}; q)_\infty (q t^{-2} s_1^{\pm} s_2^{\pm}; q)_\infty}$$

$$\langle W_{\square} W_{\square}\rangle^{USp(4)}(t;q) = \langle W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(5)}(t;q)$$
$$= \langle T_{(1,0)} T_{(1,0)}\rangle^{SO(5)}(t;q) = \left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{USp(4)/\mathbb{Z}_2}(t;q).$$

$$\langle W_{\square} W_{\square}\rangle^{USp(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle T_{(1,0)} T_{(1,0)}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\left\langle W_{\substack{\square\\\square}}\right\rangle^{USp(4)}(t;q) = \langle W_{\square}\rangle^{SO(5)}(t;q).$$

$$\langle \underbrace{W_{\substack{\square\\\square}} \cdots W_{\substack{\square\\\square}}}_{k}\rangle^{USp(4)}(t;q) = \langle \underbrace{W_{\square} \cdots W_{\square}}_{k}\rangle^{SO(5)}(t;q).$$

$$\langle \underbrace{W_{\square\square} \cdots W_{\square\square}}_{k}\rangle^{USp(4)}(t;q) = \langle \underbrace{W_{\substack{\square\\\square}} \cdots W_{\substack{\square\\\square}}}_{k}\rangle^{SO(5)}(t;q).$$

$$\langle \underbrace{W_{(2l)} \cdots W_{(2l)}}_{k}\rangle^{USp(4)}(t;q) = \langle \underbrace{W_{(l^2)} \cdots W_{(l^2)}}_{k}\rangle^{SO(5)}(t;q).$$

$$\langle \underbrace{W_{(2l)} \cdots W_{(2l)}}_{k}\rangle^{USP(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{3lk} a^{\mathrm{usp}(4)}_{k(2l)}(i)\, \mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{(\infty)}W_{(\infty)}\rangle^{USp(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle W_{(\infty^2)}W_{(\infty^2)}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= 1 + \mathfrak{q}^2 + 4\mathfrak{q}^4 + 5\mathfrak{q}^6 + 13\mathfrak{q}^8 + 16\mathfrak{q}^{10} + 33\mathfrak{q}^{12} + 41\mathfrak{q}^{14} + 73\mathfrak{q}^{16} + 90\mathfrak{q}^{18} + 145\mathfrak{q}^{20} + \cdots$$

$$\langle W_{(\infty)}W_{(\infty)}\rangle^{USp(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle W_{(\infty^2)}W_{(\infty^2)}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{16}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^3(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)^2}$$

$$\langle \underbrace{W_{(l^2)}\cdots W_{(l^2)}}_{k}\rangle^{USp(4)}(t;q) = \langle \underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle^{SO(5)}(t;q).$$

$$\langle W_{\square}W_{\square}\rangle^{USp(6)}(t;q)$$
$$= \frac{1}{48}\frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^3}\oint\prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}\frac{\left(s_i^{\pm 2};q\right)_\infty\left(qs_i^{\pm 2};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm 2};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm 2};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty\left(qs_i^{\pm}s_j^{\mp};q\right)_\infty\left(qs_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}\left[\sum_{i=1}^{3}\left(s_i+s_i^{-1}\right)\right]^2.$$

$$\langle T_{(1,0,0)}T_{(1,0,0)}\rangle^{SO(7)}(t;q)$$
$$= \frac{1}{8}\frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^3}\oint\prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}\frac{\left(q^{\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty}{\left(q^{\frac{1+\delta_{i,1}}{2}}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1+\delta_{i,1}}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1+\delta_{i+j,1}}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1+\delta_{i+j,1}}{2}}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1+\delta_{i+j,1}}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1+\delta_{i+j,1}}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\langle W_{\square}W_{\square}\rangle^{USp(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle T_{(1,0,0)}T_{(1,0,0)}\rangle^{SO(7)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)\left(1-\mathfrak{q}^8(1-\mathfrak{q}^{12})\right)} = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle \underbrace{W_{\square}\cdots W_{\square}}_{2k}\rangle^{USp(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{5k} a_{k\square}^{\mathfrak{usp}(6)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})},$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square}\rangle^{USp(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}(3+6\mathfrak{q}^2+9\mathfrak{q}^4+12\mathfrak{q}^6+14\mathfrak{q}^8$$
$$+15\mathfrak{q}^{10}+12\mathfrak{q}^{12}+9\mathfrak{q}^{14}+6\mathfrak{q}^{16}+3\mathfrak{q}^{18}+\mathfrak{q}^{20})$$

$$\det\begin{pmatrix}F_0 & F_1 & F_2\\ F_1 & F_2 & F_3\\ F_2 & F_3 & F_4\end{pmatrix} = \sum_{k=0}^{\infty}\frac{a_{k\square}^{\mathfrak{usp}(6)}(0)}{(2k)!}x^{2k},$$

$$F_m(x) := \sum_{j=0}^{m}\binom{m}{j}\left(I_{2j-m}(2x) - I_{2j-m+2}(2x)\right)$$

$$I_k(2x) := \sum_{n=0}^{\infty}\frac{x^{2n+k}}{n!\,(n+k)!}$$

$$\left\langle W_{\substack{\square\\\square}}\right\rangle^{USP(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^4+\mathfrak{q}^8}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})},$$

$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{USP(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+2\mathfrak{q}^4+2\mathfrak{q}^6+3\mathfrak{q}^8+2\mathfrak{q}^{10}+3\mathfrak{q}^{12}+\mathfrak{q}^{14}+\mathfrak{q}^{16}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}.$$

$$\left\langle W_{\substack{\square\\\square\\\square}}W_{\substack{\square\\\square\\\square}}\right\rangle^{USp(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+2\mathfrak{q}^6+2\mathfrak{q}^8+2\mathfrak{q}^{10}+2\mathfrak{q}^{12}+\mathfrak{q}^{14}+\mathfrak{q}^{16}+\mathfrak{q}^{18}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$

$$\langle \underbrace{W_{\square\square}\cdots W_{\square\square}}_{k}\rangle^{USp(6)}(t;q) = \langle \underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k}\rangle^{SO(7)}(t;q).$$

$$\langle W_{\square\square}\rangle^{USp(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2+\mathfrak{q}^6+\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})},$$

$$\begin{aligned}\langle W_{\square\square}W_{\square\square}\rangle^{USP(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) &= \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}\\ &\times(1+\mathfrak{q}^2+3\mathfrak{q}^4+2\mathfrak{q}^6+5\mathfrak{q}^8+3\mathfrak{q}^{10}\\ &+5\mathfrak{q}^{12}+2\mathfrak{q}^{14}+3\mathfrak{q}^{16}+\mathfrak{q}^{18}+\mathfrak{q}^{20}),\end{aligned}$$

$$\langle \underbrace{W_{(2l)}\cdots W_{(2l)}}_{k}\rangle^{USp(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{5lk} a_{k(2l)}^{\mathrm{usp}(6)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$

$$\langle W_{\square\square\square\square}\rangle^{USp(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^4+\mathfrak{q}^8+2\mathfrak{q}^{12}+\mathfrak{q}^{16}+\mathfrak{q}^{20}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}.$$

$$\begin{aligned}&\langle W_{\square}W_{\square}\rangle^{USp(2N)}(t;q)\\ &= \frac{1}{2^N N!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\frac{\left(s_i^{\pm 2};q\right)_\infty\left(qs_i^{\pm 2};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm 2};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm 2};q\right)_\infty}\\ &\times\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty\left(qs_i^{\pm}s_j^{\mp};q\right)_\infty\left(qs_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}\left[\sum_{i=1}^{N}\left(s_i+s_i^{-1}\right)\right]^2\end{aligned}$$

$$\begin{aligned}&\left\langle T_{(1,0^{N-1})}T_{(1,0^{N-1})}\right\rangle^{SO(2N+1)}(t;q)\\ &= \frac{1}{2^{N-1}(N-1)!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}\oiiint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\frac{\left(q^{\frac{1}{2}}\delta_{i,1}s_i^{\pm};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty}{\frac{1+\delta_{i,1}}{2}t^2 s_i^{\pm};q\Big)_\infty\left(q^{\frac{1+\delta_{i,1}}{2}}t^{-2}s_i^{\pm};q\right)_\infty}\\ &\times\prod_{i<j}\frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1+\delta_{i+j,1}}{2}}t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1+\delta_{i+j,1}}{2}}t^2 s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1+\delta_{i+j,1}}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1+\delta_{i+j,1}}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}.\end{aligned}$$

$$\begin{aligned}\langle W_{\square}W_{\square}\rangle^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) &= \left\langle T_{(1,0^{N-1})}T_{(1,0^{N-1})}\right\rangle^{SO(2N+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\\ &= \frac{1}{(1-\mathfrak{q}^2)\prod_{n=1}^{N-1}(1-\mathfrak{q}^{4n})}.\end{aligned}$$

$$\mathcal{J}^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \prod_{n=1}^{N}\frac{1}{1-\mathfrak{q}^{4n}},$$

$$\langle W_{\square}W_{\square}\rangle^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{4N}}{1-\mathfrak{q}^2},$$

$$\langle \underbrace{W_{\square}\cdots W_{\square}}_{2k} \rangle^{USp(2N)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{(2N-1)k} a_{k\square}^{\mathrm{usp}(2N)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})},$$

$$\det\big(F_{i+j-2}(x)\big) = \sum_{k=0}^{\infty} \frac{a_{k\square}^{\mathrm{usp}(2N)}(0)}{(2k)!} x^{2k}$$

$$\langle \underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k} \rangle^{USp(2N)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{2(N-1)k} a_{k\substack{\square\\\square}}^{\mathrm{usp}(2N)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})},$$

$$\langle \underbrace{W_{\square\square}\cdots W_{\square\square}}_{k} \rangle^{USp(2N)}(t;q) = \langle \underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k} \rangle^{SO(2N+1)}(t;q).$$

$$\langle W_{\square\square} \rangle^{USp(2N)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2 + \mathfrak{q}^6 + \cdots + \mathfrak{q}^{4N-2}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})} = \frac{\mathfrak{q}^2(1-\mathfrak{q}^{4N})}{(1-\mathfrak{q}^4)\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})}$$

$$\langle \underbrace{W_{(2l)}\cdots W_{(2l)}}_{k} \rangle^{USp(2N)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{(2N-1)lk} a_{k(2l)}^{\mathrm{usp}(2N)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})},$$

$$\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) \in (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

$$\begin{aligned} &\mathrm{Spin}(2N)\\ &\mathrm{SO}(2N) = \mathrm{Spin}(2N)/\mathbb{Z}_2^V\\ &\mathrm{Ss}(2N) = \mathrm{Spin}(2N)/\mathbb{Z}_2^S\\ &\mathrm{Sc}(2N) = \mathrm{Spin}(2N)/\mathbb{Z}_2^C\\ &\mathrm{SO}(2N)/\mathbb{Z}_2 = \mathrm{Spin}(2N)/(\mathbb{Z}_2^S \times \mathbb{Z}_2^C) \end{aligned}$$

$$\begin{aligned} Ss(2N)_+:\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (0,0;0,0),\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (1,0;0,0),\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (0,0;0,1),\\ Ss(2N)_-:\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (0,0;0,0),\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (1,0;0,0),\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (0,1;0,1). \end{aligned}$$

$$\begin{aligned} Ss(2N)_+:\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (0,0;0,0),\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (1,0;0,0),\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (0,0;1,0),\\ Ss(2N)_-:\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (0,0;0,0),\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (1,0;0,0),\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (0,1;1,0). \end{aligned}$$

$$\begin{aligned} &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (0,0;0,0)\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (n_{SS}, n_{SC};1,0)\\ &\big(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\big) = (n_{CS}, n_{CC};0,1) \end{aligned}$$

$$(z_e, z_m) \in \mathbb{Z}_4 \times \mathbb{Z}_4.$$

$$\begin{aligned} &\mathrm{Spin}(2N)\\ &\mathrm{SO}(2N) = \mathrm{Spin}(2N)/\mathbb{Z}_2\\ &\mathrm{SO}(2N)/\mathbb{Z}_2 = \mathrm{Spin}(2N)/\mathbb{Z}_4 \end{aligned}$$

$$(z_e, z_m) = (0,0), (z_e, z_m) = (n,1)$$

$$\chi_{\mathrm{sp}}^{\mathfrak{so}(2N)}=\frac{1}{2}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)+\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right],$$

$$\chi_{\overline{\mathrm{sp}}}^{\mathfrak{so}(2N)}=\frac{1}{2}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)-\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right].$$

$$\chi_{\square}^{\mathfrak{so}(2N)}=\sum_{i=1}^{N}\left(s_i+s_i^{-1}\right).$$

$$\chi_{\lambda}^{\mathfrak{so}(2N)}=\frac{\det\left(s_j^{\lambda_i+N-i}+s_j^{-\lambda_i-N+i}\right)+\det\left(s_j^{\lambda_i+N-i}-s_j^{-\lambda_i-N+i}\right)}{\det\left(s_j^{N-i}+s_j^{-N+i}\right)}.$$

$$\begin{aligned}&\left\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\right\rangle^{SO(2N)}\\&=\int d\mu^{SO(2N)}\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)M_n(s)\right)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathfrak{so}(2N)}(s)\end{aligned}$$

$$d\mu^{SO(2N)}=\frac{1}{2^{N-1}N!}\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{1\le i<j\le N}\left(1-s_is_j\right)\left(1-s_i^{-1}s_j^{-1}\right)\left(1-s_is_j^{-1}\right)\left(1-s_i^{-1}s_j\right),$$

$$M_n(s)=\frac{P_n(s)^2+P_{2n}(s)}{2},$$

$$\begin{aligned}\mathcal{I}^{SO(4)}(t;q)&=\mathcal{I}^{SU(2)}(t;q)\times\mathcal{I}^{SU(2)}(t;q)\\&=\frac{\left(q^{\frac{1}{2}}t^{\pm2};q\right)_\infty^2}{(q;q)_\infty^4}\left(\sum_{\substack{p_1,p_2\in\mathbb{Z}\\p_1<p_2}}\frac{\left(q^{\frac{1}{2}}t^{-2}\right)^{p_1+p_2-2}}{\left(1-q^{p_1-\frac{1}{2}}t^2\right)\left(1-q^{p_2-\frac{1}{2}}t^2\right)}\right)^2\end{aligned}$$

$$\begin{aligned}\left\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\right\rangle^{\mathrm{Spin}(4)}(t;q)&=\frac{1}{4}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\\&\times\frac{\left(s_1^{\pm}s_2^{\mp};q\right)_\infty\left(s_1^{\pm}s_2^{\pm};q\right)_\infty\left(qs_1^{\pm}s_2^{\mp};q\right)_\infty\left(qs_1^{\pm}s_2^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}\\&\times\left(s_1^{\frac{1}{2}}s_2^{\frac{1}{2}}+s_1^{-\frac{1}{2}}s_2^{-\frac{1}{2}}\right)^2\end{aligned}$$

$$\begin{aligned}\left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{SO(4)/\mathbb{Z}_2}(t;q)&=\frac{1}{2}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm2};q\right)_\infty^2}\oiiint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\\&\times\frac{\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(qs_1^{\pm}s_2^{\pm};q\right)_\infty}{\left(qt^2s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\pm};q\right)_\infty\left(qt^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}\end{aligned}$$

$$\begin{aligned}\left\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\right\rangle^{\mathrm{Spin}(4)}(t;q)&=\left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{SO(4)/\mathbb{Z}_2}(t;q)\\&=\mathcal{I}^{SU(2)}(t;q)\left\langle W_{\square}W_{\square}\right\rangle^{SU(2)}(t;q),\end{aligned}$$

$$\langle W_{\rm sp}W_{\rm sp}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})=\left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{\rm SO(4)/\mathbb{Z}_2}_{\frac{1}{2}\rm BPS}(\mathfrak{q})$$

$$=\frac{1+\mathfrak{q}^2}{(1-\mathfrak{q}^4)^2}$$

$$=\frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\langle\underbrace{W_{\rm sp}\cdots W_{\rm sp}}_{2k}\rangle^{\rm Spin(4)}(t;q)=\mathcal{I}^{SU(2)}(t;q)\langle\underbrace{W_{\Box}\cdots W_{\Box}}_{2k}\rangle^{SU(2)}(t;q).$$

$$\langle\underbrace{W_{\rm sp}\cdots W_{\rm sp}}_{2k}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})=\mathcal{I}^{SO(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})\sum_{i=0}^{k}a^{\mathfrak{so}(4)}_{k\rm sp}(i)\mathfrak{q}^{2i}$$

$$=\frac{1}{(1-\mathfrak{q}^4)^2}\sum_{i=0}^{k}a^{\mathfrak{so}(4)}_{k\rm sp}(i)\mathfrak{q}^{2i}$$

$$a^{\mathfrak{so}(4)}_{k\rm sp}(i)=\frac{(2i+1)(2k)!}{(k-i)!\,(k+i+1)!}.$$

$$\langle W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})=\frac{2+3\mathfrak{q}^2+\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2},$$

$$\langle W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})=\frac{5+9\mathfrak{q}^2+5\mathfrak{q}^4+\mathfrak{q}^6}{(1-\mathfrak{q}^4)^2},$$

$$\langle W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})=\frac{14+28\mathfrak{q}^2+20\mathfrak{q}^4+7\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2}.$$

$$\langle W_{\rm sp}^{2k}W_{\rm sp}^{2m}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})=\left(\mathcal{I}^{\rm SO(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})\right)^{-1}\langle W_{\rm sp}^{2k}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})\langle W_{\rm sp}^{2m}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})$$

$$=(1-\mathfrak{q}^4)^2\langle W_{\rm sp}^{2k}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})\langle W_{\rm sp}^{2m}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})$$

$$\langle W_{\overline{\rm sp}}^{2m}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})=\langle W_{\rm sp}^{2m}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})$$

$$\langle W_{\rm sp}^{2k}W_{\overline{\rm sp}}^{2m}\rangle^{\rm Spin(4)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)^2}\left(\sum_{i=0}^{k}a^{\mathfrak{so}(4)}_{k\rm sp}(i)\mathfrak{q}^{2i}\right)\left(\sum_{j=0}^{m}a^{\mathfrak{so}(4)}_{m\rm sp}(j)\mathfrak{q}^{2j}\right)$$

$$\langle W_{\Box}W_{\Box}\rangle^{SO(4)}(t;q)=\frac{1}{4}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi is_i}$$
$$\times\frac{(s_1^{\pm}s_2^{\mp};q)_\infty(s_1^{\pm}s_2^{\pm};q)_\infty(qs_1^{\pm}s_2^{\mp};q)_\infty(qs_1^{\pm}s_2^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}$$
$$\times(s_1+s_2+s_1^{-1}+s_2^{-1})^2$$

$$\left\langle T_{(1,0)}T_{(1,0)}\right\rangle^{SO(4)}(t;q)=\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi is_i}$$
$$\times\frac{\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty}{(qt^2s_1^{\pm}s_2^{\mp};q)_\infty(qt^2s_1^{\pm}s_2^{\pm};q)_\infty(qt^{-2}s_1^{\pm}s_2^{\mp};q)_\infty(qt^{-2}s_1^{\pm}s_2^{\pm};q)_\infty}$$

$$\langle W_{\Box}W_{\Box}\rangle^{SO(4)}(t;q)=\langle W_{\Box}W_{\Box}\rangle^{SU(2)}(t;q)^2,$$

$$\langle W_\square W_\square \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)^2}$$

$$\langle \underbrace{W_\square \cdots W_\square}_{2k} \rangle^{SO(4)}(t;q) = \langle \underbrace{W_\square \cdots W_\square}_{k} \rangle^{SU(2)}(t;q)^2,$$

$$\langle \underbrace{W_\square \cdots W_\square}_{2k} \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{2k} a_{k\square}^{\mathfrak{so}(4)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)^2},$$

$$\langle W_\square W_\square W_\square W_\square \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{4+12\mathfrak{q}^2+13\mathfrak{q}^4+6\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2},$$

$$\langle W_\square W_\square W_\square W_\square W_\square W_\square \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{25+90\mathfrak{q}^2+131\mathfrak{q}^4+100\mathfrak{q}^6+43\mathfrak{q}^8+10\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)^2},$$

$$\langle W_\square W_\square W_\square W_\square W_\square W_\square W_\square W_\square \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)^2}(196+784\mathfrak{q}^2+1344\mathfrak{q}^4+1316\mathfrak{q}^6$$
$$+820\mathfrak{q}^8+336\mathfrak{q}^{10}+89\mathfrak{q}^{12}+14\mathfrak{q}^{14}+\mathfrak{q}^{16}).$$

$$a_{k\square}^{\mathfrak{so}(4)}(0) = C_k^2.$$

$$\langle W_\square W_\square \rangle^{SO(4)^-}(t;q) = \frac{1}{2}\frac{(q)_\infty^2(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty \left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint \frac{ds}{2\pi i s}$$
$$\times \frac{(s^\pm;q)_\infty(-s;q)_\infty(qs^\pm;q)_\infty(-qs^\pm;q)_\infty}{\left(q^{\frac{1}{2}}t^2s^\pm;q\right)_\infty \left(-q^{\frac{1}{2}}t^2s^\pm;q\right)_\infty \left(q^{\frac{1}{2}}t^{-2}s^\pm;q\right)_\infty \left(-q^{\frac{1}{2}}t^{-2}s^\pm;q\right)_\infty}(s+s^{-1})^2$$

$$\langle T_{(1)}T_{(1)} \rangle^{SO(4)^-}(t;q) = \frac{(q)_\infty^2(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty \left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint \frac{ds}{2\pi i s}$$
$$\times \frac{\left(q^{\frac{1}{2}}s^\pm;q\right)_\infty \left(-q^{\frac{1}{2}}s;q\right)_\infty \left(q^{\frac{3}{2}}s^\pm;q\right)_\infty \left(-q^{\frac{3}{2}}s^\pm;q\right)_\infty}{(qt^2s^\pm;q)_\infty(-qt^2s^\pm;q)_\infty(qt^{-2}s^\pm;q)_\infty(-qt^{-2}s^\pm;q)_\infty}$$

$$\langle W_\square W_\square \rangle^{SO(4)^-}(t;q) = \langle T_{(1)}T_{(1)} \rangle^{SO(4)^-}(t;q)$$
$$= \langle W_\square W_\square \rangle^{SU(2)}(t^2;q^2).$$

$$\langle W_\square W_\square \rangle^{SO(4)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle T_{(1)}T_{(1)} \rangle^{SO(4)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1}{1-\mathfrak{q}^4}$$

$$\langle \underbrace{W_\square \cdots W_\square}_{2k} \rangle^{SO(4)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{k} a_{k\square}^{\mathfrak{so}(4)^-}(i)\mathfrak{q}^{4i}}{1-\mathfrak{q}^8}$$

$$a_{k\square}^{\mathfrak{so}(4)^-}(i) = (2i+1)\frac{(2k)!}{(k-i)!(k+i+1)!}$$

$$\langle W_{\lambda_1} \cdots W_{\lambda_k} \rangle^{O(4)^+}(t;q) = \frac{1}{2}\left[\langle W_{\lambda_1} \cdots W_{\lambda_k} \rangle^{SO(4)}(t;q) + \langle W_{\lambda_1} \cdots W_{\lambda_k} \rangle^{SO(4)^-}(t;q)\right]$$

$$\langle W_\square W_\square \rangle^{O(4)^+}(t;q) = \langle T_{(1)}T_{(1)} \rangle^{O(4)^+}(t;q),$$

$$\langle \underbrace{W_\square \cdots W_\square}_{2k} \rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{2k+1} a_k^{\mathfrak{o}(4)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_{\square}W_{\square}\rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$=\frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)},$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square}\rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{3+6\mathfrak{q}^2+9\mathfrak{q}^4+9\mathfrak{q}^6+6\mathfrak{q}^8+3\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square}W_{\square}W_{\square}\rangle^{O(4)+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(15+45\mathfrak{q}^2+80\mathfrak{q}^4+95\mathfrak{q}^6+85\mathfrak{q}^8$$

$$+55\mathfrak{q}^{10}+20\mathfrak{q}^{12}+5\mathfrak{q}^{14})$$

$$a^{\mathfrak{o}(4)}_{k\square}(0)=\frac{1}{2}\left(C_k^2+C_k\right),$$

$$\substack{\square\\\square}=(1,1),\ \substack{\overline{\square}\\\overline{\square}}=(1,-1).$$

$$\langle\underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k}\rangle^{SO(4)}(t;q)=\langle\underbrace{W_{\substack{\overline{\square}\\\square}}\cdots W_{\substack{\overline{\square}\\\square}}}_{k}\rangle^{SO(4)}(t;q)$$

$$=\mathcal{J}^{SU(2)}(t;q)\langle\underbrace{W_{\square\square}\cdots W_{\square\square}}_{k}\rangle^{SU(2)}(t;q).$$

$$\langle\underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\sum_{i=0}^{k}\ a^{\mathfrak{so}(4)}_{k\substack{\square\\\square}}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)^2},$$

$$\left\langle W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\ =\frac{\mathfrak{q}^2}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\ =\frac{1+\mathfrak{q}^2+\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\ =\frac{1+3\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^6}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\ =\frac{3+6\mathfrak{q}^2+6\mathfrak{q}^4+3\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\ =\frac{6+15\mathfrak{q}^2+15\mathfrak{q}^4+10\mathfrak{q}^6+4\mathfrak{q}^8+\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\ =\frac{15+36\mathfrak{q}^2+40\mathfrak{q}^4+29\mathfrak{q}^6+15\mathfrak{q}^8+5\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle\left(W_{\substack{\square\\\square}}\right)^k\left(W_{\substack{\overline{\square}\\\square}}\right)^m\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=(1-\mathfrak{q}^4)^2\left\langle\left(W_{\substack{\square\\\square}}\right)^k\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\left\langle\left(W_{\substack{\overline{\square}\\\square}}\right)^m\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}).$$

$$\langle\underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle^{SO(4)}(t;q)=\langle\underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle^{SU(2)}(t;q)^2,$$

$$\langle\underbrace{W_{\square\square}\cdots W_{\square\square}}_{k}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\sum_{i=0}^{2k}\ a^{\mathfrak{so}(4)^-}_{k\square\square}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)^2},$$

$$\langle W_{\square\square}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2},$$

$$\langle W_{\square\square}W_{\square\square}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{(1+\mathfrak{q}^2+\mathfrak{q}^4)^2}{(1-\mathfrak{q}^4)^2},$$

$$\langle W_{\square\square}W_{\square\square}W_{\square\square}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{(1+3\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^6)^2}{(1-\mathfrak{q}^4)^2}.$$

$$a^{\mathfrak{so}(4)^-}_{k\square\square}(0) = R_k^2,$$
$$a^{\mathfrak{so}(4)^-}_{k\square\square}(1) = 2R_kR_{k+1},$$

$$\langle (W_{\square\square})^k\rangle^{SO(4)} = \left\langle \left(W_{\substack{\square\\ \square}}\right)^k \left(W_{\substack{\bar\square\\ \square}}\right)^k \right\rangle^{SO(4)}.$$

$$\langle W_{(2k)}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{4k}}{(1-\mathfrak{q}^4)^2}$$

$$\langle W_{(k)}W_{(k)}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{(1-\mathfrak{q}^{2k+2})^2}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)^2}$$

$$\langle W_{(\infty)}W_{(\infty)}\rangle^{SO(4)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)^2}$$

$$\langle \underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle^{SO(4)^-}(t;q) = \langle \underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle^{SU(2)}(t^2;q^2).$$

$$\langle W_{(2l)}\rangle^{SO(4)^-}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{\mathfrak{q}^{4l}}{1-\mathfrak{q}^8},$$
$$\langle W_{(l)}W_{(l)}\rangle^{SO(4)^-}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{4l+4}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}.$$

$$\langle W_{(2l)}\rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{4l}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$
$$\langle W_{(l)}W_{(l)}\rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^2+\mathfrak{q}^4-\mathfrak{q}^{2l+2}-\mathfrak{q}^{2l+6}+\mathfrak{q}^{4l+6}}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}.$$

$$\langle W_{(\infty)}W_{(\infty)}\rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^2+\mathfrak{q}^4}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle \underbrace{W_{(l,l)}\cdots W_{(l,l)}}_{k}\rangle^{SO(4)}(t;q) = \langle \underbrace{W_{(l,-l)}\cdots W_{(l,-l)}}_{k}\rangle^{SO(4)}(t;q)$$
$$= \mathcal{I}^{SU(2)}(t;q)\langle \underbrace{W_{(2l)}\cdots W_{(2l)}}_{k}\rangle^{SU(2)}(t;q).$$

$$\langle W_{(l,l)}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle W_{(l,-l)}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{2l}}{(1-\mathfrak{q}^4)^2},$$
$$\langle W_{(l,l)}W_{(l,l)}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle W_{(l,-l)}W_{(l,-l)}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{4l+2}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^2}.$$

$$\chi^{\mathfrak{so}(4)}_{(l,l)}\chi^{\mathfrak{so}(4)}_{(l,-l)} = \chi^{\mathfrak{so}(4)}_{(2l)}$$

$$\langle W^k_{(l,l)}W^k_{(l,-l)}\rangle^{SO(4)} = \langle W^k_{(2l)}\rangle^{SO(4)}$$

$$\mathcal{I}^{SO(6)}(t;q) = \mathcal{I}^{SU(4)}(t;q)$$
$$= -\frac{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}{(q;q)^2_\infty}\sum_{\substack{p_1,p_2,p_3,p_4\in\mathbb{Z}\\ p_1<p_2<p_3<p_4}}\frac{\left(q^{\frac{1}{2}}t^{-2}\right)^{p_1+p_2+p_3+p_4-8}}{(1-q^{p_1-1}t^4)(1-q^{p_2-1}t^4)(1-q^{p_3-1}t^4)(1-q^{p_4-1}t^4)}$$

$$\langle W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}\rangle^{\mathrm{Spin}(6)}(t;q) = \frac{1}{24}\frac{(q)^6_\infty}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)^3_\infty}\oint\prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}$$
$$\times\prod_{i\neq j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty(qs_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$
$$\times(1+s_1s_2+s_1s_3+s_2s_3)(1+s_1^{-1}s_2^{-1}+s_1^{-1}s_3^{-1}+s_2^{-1}s_3^{-1})$$

$$\left\langle T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} \right\rangle^{SO(6)/\mathbb{Z}_2}(t;q) = \frac{1}{6} \frac{(q)_\infty^6}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^3} \oiiint \prod_{i=1}^{3} \frac{ds_i}{2\pi i s_i}$$

$$\times \prod_{i<j} \frac{\left(s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} s_i^{\pm} s_j^{\pm}; q\right)_\infty \left(q s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{3}{2}} s_i^{\pm} s_j^{\pm}; q\right)_\infty}{\left(q^{\frac{1}{2}} t^2 s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q t^2 s_i^{\pm} s_j^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q t^{-2} s_i^{\pm} s_j^{\pm}; q\right)_\infty}$$

$$\langle W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} \rangle^{\mathrm{Spin}(6)}(t;q) = \left\langle T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} \right\rangle^{SO(6)/\mathbb{Z}_2}(t;q)$$

$$= \langle W_{\square} W_{\overline{\square}} \rangle^{SU(4)}(t;q).$$

$$\langle W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} \rangle^{\mathrm{Spin}(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} \right\rangle^{\mathrm{SO}(6)/\mathbb{Z}_2}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)}$$

$$\langle \underbrace{W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} \cdots W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}}}_{2k} \rangle^{\mathrm{Spin}(6)}(t;q) = \langle \underbrace{W_{\square} W_{\overline{\square}} \cdots W_{\square} W_{\overline{\square}}}_{2k} \rangle^{SU(4)}(t;q).$$

$$\langle \underbrace{W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} \cdots W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}}}_{2k} \rangle^{\mathrm{Spin}(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{3k} a_{k\mathrm{sp}}^{\mathfrak{so}(6)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)},$$

$$\langle W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} \rangle^{\mathrm{Spin}(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{2+4\mathfrak{q}^2+6\mathfrak{q}^4+7\mathfrak{q}^6+5\mathfrak{q}^8+3\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}$$

$$\langle W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} |^{\mathrm{Spin}(6)}_{\frac{1}{2}\mathrm{BPS}} \rangle(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}(6+18\mathfrak{q}^2+35\mathfrak{q}^4+50\mathfrak{q}^6$$

$$+53\mathfrak{q}^8+45\mathfrak{q}^{10}+29\mathfrak{q}^{12}+14\mathfrak{q}^{14}+5\mathfrak{q}^{16}+\mathfrak{q}^{18})$$

$$\det\begin{pmatrix} I_0(2x) & I_1(2x) & I_2(2x) & I_3(2x) \\ I_1(2x) & I_0(2x) & I_1(2x) & I_2(2x) \\ I_2(2x) & I_1(2x) & I_0(2x) & I_1(2x) \\ I_3(2x) & I_2(2x) & I_1(2x) & I_0(2x) \end{pmatrix} = \sum_{k=0}^{\infty} \frac{a_{k\mathrm{sp}}^{\mathfrak{so}(6)}(0)}{(k!)^2} x^{2k}$$

$$\langle W_{\square} W_{\square} \rangle^{SO(6)}(t;q) = \frac{1}{24} \frac{(q)_\infty^6}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^3} \oint \prod_{i=1}^{3} \frac{ds_i}{2\pi i s_i}$$

$$\times \prod_{i\neq j} \frac{\left(s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(s_i^{\pm} s_j^{\pm}; q\right)_\infty \left(q s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q s_i^{\pm} s_j^{\pm}; q\right)_\infty}{\left(q^{\frac{1}{2}} t^2 s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^2 s_i^{\pm} s_j^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm} s_j^{\pm}; q\right)_\infty}$$

$$\times (s_1+s_2+s_3+s_1^{-1}+s_2^{-1}+s_3^{-1})^2.$$

$$\langle T_{(1,0,0)} T_{(1,0,0)} \rangle^{SO(6)}(t;q) = \frac{1}{4} \frac{(q)_\infty^6}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^3} \oiint \prod_{i=1}^{3} \frac{ds_i}{2\pi i s_i}$$

$$\times \prod_{i<j} \frac{\left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\pm}; q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^2 s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^2 s_i^{\pm} s_j^{\pm}; q\right)_\infty}$$

$$\times \frac{\left(q^{1+\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\pm}; q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^{-2} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^{-2} s_i^{\pm} s_j^{\pm}; q\right)_\infty}$$

$$\langle W_{\square} W_{\square}\rangle^{SO(6)}(t;q) = \left\langle T_{(1,0,0)} T_{(1,0,0)}\right\rangle^{SO(6)}(t;q) = \left\langle W_{\substack{\square\\\square}} W_{\substack{\square\\\square}}\right\rangle^{SU(4)}(t;q)$$

$$\langle W_{\square} W_{\square}\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle T_{(1,0,0)} T_{(1,0,0)}\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)} = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^2}$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{2k}\rangle^{SO(6)}(t;q) = \langle \underbrace{W_{\substack{\square\\\square}} \cdots W_{\substack{\square\\\square}}}_{2k}\rangle^{SU(4)}(t;q).$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{2k}\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{4k} a_{k\square}^{\mathrm{so}(6)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square} W_{\square} W_{\square} W_{\square}\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}(3+7\mathfrak{q}^2+15\mathfrak{q}^4+18\mathfrak{q}^6+20\mathfrak{q}^8 +14\mathfrak{q}^{10}+9\mathfrak{q}^{12}+3\mathfrak{q}^{14}+\mathfrak{q}^{16})$$

$$\langle W_{\square} W_{\square} W_{\square} W_{\square} W_{\square} W_{\square}\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}(16+60\mathfrak{q}^2+149\mathfrak{q}^4+249\mathfrak{q}^6 +334\mathfrak{q}^8+347\mathfrak{q}^{10}+301\mathfrak{q}^{12}+206\mathfrak{q}^{14}+119\mathfrak{q}^{16} +53\mathfrak{q}^{18}+20\mathfrak{q}^{20}+5\mathfrak{q}^{22}+\mathfrak{q}^{24})$$

$$\langle W_{\square} W_{\square}\rangle^{SO(6)^-}(t;q) = \frac{1}{8}\frac{(q)_\infty^4(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2\left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}$$
$$\times \oint \prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm};q)_\infty(-s_i^{\pm};q)_\infty(qs_i^{\pm};q)_\infty(-qs_i^{\pm};q)_\infty}{(q^2s_i^{\pm};q)_\infty\left(-q^{\frac{1}{2}}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$
$$\times\frac{(s_1^{\pm}s_2^{\mp};q)_\infty(s_1^{\pm}s_2^{\pm};q)_\infty(qs_1^{\pm}s_2^{\mp};q)_\infty(qs_1^{\pm}s_2^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}$$
$$\times(s_1+s_2+s_1^{-1}+s_2^{-1})^2$$

$$\left\langle T_{(1,0)}T_{(1,0)}\right\rangle^{SO(6)^-}(t;q) = \frac{1}{2}\frac{(q)_\infty^4(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2\left(-q^{\frac{1}{2}}t^{\pm};q\right)_\infty}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}$$
$$\times\frac{\left(q^{\frac{1}{2}}s_1^{\pm};q\right)_\infty(s_2^{\pm};q)_\infty\left(-q^{\frac{1}{2}}s_1^{\pm};q\right)_\infty(-s_2^{\pm};q)_\infty}{(qt^2s_1^{\pm};q)_\infty\left(q^{\frac{1}{2}}t^2s_2^{\pm};q\right)_\infty(-qt^2s_1^{\pm};q)_\infty\left(-q^{\frac{1}{2}}t^2s_2^{\pm};q\right)_\infty}$$
$$\times\frac{\left(q^{\frac{3}{2}}s_1^{\pm};q\right)_\infty(qs_2^{\pm};q)_\infty\left(-q^{\frac{3}{2}}s_1^{\pm};q\right)_\infty(-qs_2^{\pm};q)_\infty}{(qt^{-2}s_1^{\pm};q)_\infty\left(q^{\frac{1}{2}}t^{-2}s_2^{\pm};q\right)_\infty(-qt^{-2}s_1^{\pm};q)_\infty\left(-q^{\frac{1}{2}}t^{-2}s_2^{\pm};q\right)_\infty}$$
$$\times\frac{\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty}{(qt^2s_1^{\pm}s_2^{\mp};q)_\infty(qt^2s_1^{\pm}s_2^{\pm};q)_\infty(qt^{-2}s_1^{\pm}s_2^{\mp};q)_\infty(qt^{-2}s_1^{\pm}s_2^{\pm};q)_\infty}$$

$$\langle W_{\square} W_{\square}\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle T_{(1,0)} T_{(1,0)}\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$= \frac{1+\mathfrak{q}^2+\mathfrak{q}^6+\mathfrak{q}^8}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^8)}$$

$$\langle \underbrace{W_{\square}\cdots W_{\square}}_{2k}\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{4k} a_{k\square}^{\mathfrak{s}(6)^-}(i)\,\mathfrak{q}^{2i}}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square}\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{3+5\mathfrak{q}^2+3\mathfrak{q}^4+6\mathfrak{q}^6+8\mathfrak{q}^8+4\mathfrak{q}^{10}+3\mathfrak{q}^{12}+3\mathfrak{q}^{14}+\mathfrak{q}^{16}}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square}W_{\square}W_{\square}\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(14+30\mathfrak{q}^2+31\mathfrak{q}^4+49\mathfrak{q}^6$$
$$+66\mathfrak{q}^8+55\mathfrak{q}^{10}+49\mathfrak{q}^{12}+46\mathfrak{q}^{14}+29\mathfrak{q}^{16}+15\mathfrak{q}^{18}$$
$$+10\mathfrak{q}^{20}+5\mathfrak{q}^{22}+\mathfrak{q}^{24}).$$

$$a_k^{\mathfrak{so}(6)^-}(0) = C_k C_{k+2} - C_{k+1}^2,$$

$$\langle W_{\lambda_1}\cdots W_{\lambda_k}\rangle^{O(6)^+}(t;q) = \frac{1}{2}\Big[\langle W_{\lambda_1}\cdots W_{\lambda_k}\rangle^{SO(6)^+}(t;q) + \langle W_{\lambda_1}\cdots W_{\lambda_k}\rangle^{SO(6)^-}(t;q)\Big].$$

$$\langle W_{\square}W_{\square}\rangle^{O(6)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$

$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square}\rangle^{O(6)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(3+6\mathfrak{q}^2+9\mathfrak{q}^4+12\mathfrak{q}^6$$
$$+15\mathfrak{q}^8+15\mathfrak{q}^{10}+12\mathfrak{q}^{12}+9\mathfrak{q}^{14}+6\mathfrak{q}^{16}+3\mathfrak{q}^{18}).$$

$$\langle \underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k}\rangle^{SO(6)}(t;q) = \langle \underbrace{W_{\substack{\square\square\\\square\\\square}} W_{\substack{\square\square\\\square\\\square}}}_{k}\rangle^{SU(4)}(t;q),$$

$$\langle \underbrace{W_{\substack{\square\\\square\\\square}}\cdots W_{\substack{\square\\\square\\\square}}}_{2k}\rangle^{SO(6)}(t;q) = \langle \underbrace{W_{\square\square}\cdots W_{\square\square}}_{2k}\rangle^{SU(4)}(t;q).$$

$$\left\langle W_{\substack{\square\\\square}}\right\rangle^{SO(6)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{\mathfrak{q}^2}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(6)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$
$$\times(1+2\mathfrak{q}^2+4\mathfrak{q}^4+6\mathfrak{q}^6+7\mathfrak{q}^8+7\mathfrak{q}^{10}$$
$$+6\mathfrak{q}^{12}+5\mathfrak{q}^{14}+3\mathfrak{q}^{16}+\mathfrak{q}^{18}),$$

$$\left\langle W_{\substack{\square\\\square\\\square}}W_{\substack{\square\\\square\\\square}}\right\rangle^{SO(6)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$
$$\times(1+\mathfrak{q}^2+2\mathfrak{q}^4+3\mathfrak{q}^6+3\mathfrak{q}^8+3\mathfrak{q}^{10}$$
$$+3\mathfrak{q}^{12}+2\mathfrak{q}^{14}+\mathfrak{q}^{16}+\mathfrak{q}^{18}).$$

$$\left\langle W_{\substack{\square\\\square}}\right\rangle^{SO(6)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{\mathfrak{q}^2}{(1+\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(6)}_{\frac{1}{2}BPS}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^8}{(1+\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}.$$

$$\left\langle W_{\substack{\square\\\square}}\right\rangle^{O(6)^{+}}_{\frac{1}{2}BPS}(\mathrm{q}) = \frac{\mathrm{q}^{2}}{(1-\mathrm{q}^{4})^{2}(1-\mathrm{q}^{8})},$$

$$\left\langle W_{\substack{\square\\\square}} W_{\substack{\square\\\square}}\right\rangle^{O(6)^{+}}_{\frac{1}{2}\mathrm{BPS}}(\mathrm{q}) = \frac{1+\mathrm{q}^{2}+2\mathrm{q}^{4}+\mathrm{q}^{6}+2\mathrm{q}^{8}}{(1-\mathrm{q}^{4})^{2}(1-\mathrm{q}^{8})}.$$

$$\langle \underbrace{W_{(l)}\cdots W_{(l)}}_{2k}\rangle^{SO(6)}(t;q) = \langle \underbrace{W_{(l^{2})}\cdots W_{\overline{(l^{2})}}}_{k}\rangle^{SU(4)}(t;q).$$

$$\langle W_{\square\square}\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathrm{q}) = \frac{\mathrm{q}^{4}+\mathrm{q}^{8}}{(1+\mathrm{q}^{4})(1-\mathrm{q}^{6})(1-\mathrm{q}^{8})},$$

$$\langle W_{\square\square}W_{\square\square}\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathrm{q}) = \frac{1}{(1-\mathrm{q}^{4})(1-\mathrm{q}^{8})(1-\mathrm{q}^{12})}$$

$$\times(1+\mathrm{q}^{2}+3\mathrm{q}^{4}+4\mathrm{q}^{6}+6\mathrm{q}^{8}+6\mathrm{q}^{10}+7\mathrm{q}^{12} +6\mathrm{q}^{14}+4\mathrm{q}^{16}+4\mathrm{q}^{18}+\mathrm{q}^{20}+\mathrm{q}^{22}).$$

$$\langle W_{\square\square}\rangle^{SO(6)^{-}}_{\frac{1}{2}\mathrm{BPS}}(\mathrm{q}) = \frac{\mathrm{q}^{4}+\mathrm{q}^{8}}{(1+\mathrm{q}^{6})(1-\mathrm{q}^{4})(1-\mathrm{q}^{8})},$$

$$\langle W_{\square\square}W_{\square\square}\rangle^{SO(6)^{-}}_{\frac{1}{2}\mathrm{BPS}}(\mathrm{q}) = \frac{1+2\mathrm{q}^{8}-2\mathrm{q}^{10}+\mathrm{q}^{12}-\mathrm{q}^{14}-\mathrm{q}^{18}}{(1+\mathrm{q}^{6})(1-\mathrm{q}^{2})(1-\mathrm{q}^{4})(1-\mathrm{q}^{8})}.$$

$$\langle W_{\square\square}\rangle^{O(6)^{+}}_{\frac{1}{2}\mathrm{BPS}}(\mathrm{q}) = \frac{\mathrm{q}^{4}+\mathrm{q}^{8}}{(1+\mathrm{q}^{6})(1-\mathrm{q}^{4})(1-\mathrm{q}^{8})}$$

$$\langle W_{\square\square}W_{\square\square}\rangle^{O(6)^{+}}_{\frac{1}{2}\mathrm{BPS}}(\mathrm{q}) = \frac{1+\mathrm{q}^{2}+2\mathrm{q}^{4}+2\mathrm{q}^{6}+4\mathrm{q}^{8}+3\mathrm{q}^{10}+4\mathrm{q}^{12}+2\mathrm{q}^{14}+2\mathrm{q}^{16}+\mathrm{q}^{18}}{(1-\mathrm{q}^{4})(1-\mathrm{q}^{8})(1-\mathrm{q}^{12})}$$

$$\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(2N)}(t;q) = \frac{1}{2^{N-1}N!}\frac{(q)_{\infty}^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_{\infty}^{N}} \oint \prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}$$

$$\times\prod_{i\neq j}\frac{(s_i^{\pm}s_j^{\mp};q)_{\infty}(s_i^{\pm}s_j^{\pm};q)_{\infty}(qs_i^{\pm}s_j^{\mp};q)_{\infty}(qs_i^{\pm}s_j^{\pm};q)_{\infty}}{\left(q^{\frac{1}{2}}t^{2}s_i^{\pm}s_j^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}t^{2}s_i^{\pm}s_j^{\pm};q\right)_{\infty}\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_{\infty}}$$

$$\times\frac{1}{4}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)+\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right]^{2}$$

$$\langle W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}\rangle^{\mathrm{Spin}(2N)}(t;q) = \frac{1}{2^{N-1}N!}\frac{(q)_{\infty}^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_{\infty}^{N}} \oint \prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}$$

$$\times\prod_{i\neq j}\frac{(s_i^{\pm}s_j^{\mp};q)_{\infty}(s_i^{\pm}s_j^{\pm};q)_{\infty}(qs_i^{\pm}s_j^{\mp};q)_{\infty}(qs_i^{\pm}s_j^{\pm};q)_{\infty}}{\left(q^{\frac{1}{2}}t^{2}s_i^{\pm}s_j^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}t^{2}s_i^{\pm}s_j^{\pm};q\right)_{\infty}\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_{\infty}}$$

$$\times\frac{1}{4}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)+\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right]\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)-\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right].$$

$$\left\langle T_{\left(\frac{1}{2}^{N}\right)^{T}}T_{\frac{1}{2}^{N}}\right\rangle^{SO(2N)/\mathbb{Z}_2}(t;q) = \frac{1}{N!}\frac{(q)_{\infty}^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_{\infty}^{N}}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}$$

$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_{\infty}\left(q^{\frac{1}{2}}s_i^{\pm}s_j^{\pm};q\right)_{\infty}(qs_i^{\pm}s_j^{\mp};q)_{\infty}\left(q^{\frac{3}{2}}s_i^{\pm}s_j^{\pm};q\right)_{\infty}}{\left(q^{\frac{1}{2}}t^{2}s_i^{\pm}s_j^{\mp};q\right)_{\infty}(qt^{2}s_i^{\pm}s_j^{\pm};q)_{\infty}\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_{\infty}(qt^{-2}s_i^{\pm}s_j^{\pm};q)_{\infty}}$$

$$\langle W_{\rm sp} W_{\rm sp} \rangle^{\rm Spin(2N)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \left\langle T_{\left(\frac{1}{2}^N\right)} {}^{\tau}T_{\left(\frac{1}{2}^N\right)} \right\rangle^{\rm SO(2N)/\mathbb{Z}_2}_{\frac{1}{2}\rm BPS} (\mathfrak{q})$$

$$= \prod_{n=1}^{N} \frac{1}{1-\mathfrak{q}^{2n}}$$

$$\langle W_{\rm sp} W_{\overline{\rm sp}} |^{\rm Spin(2N)}_{\frac{1}{2}\rm BPS} \rangle (\mathfrak{q}) = \left\langle T_{\left(\frac{1}{2}^N\right)} {}^{\tau}T_{\left(\frac{1}{2}^N\right)} \right\rangle^{\rm SO(2N)/\mathbb{Z}_2}_{\frac{1}{2}\rm BPS} (\mathfrak{q})$$

$$= \prod_{n=1}^{N} \frac{1}{1-\mathfrak{q}^{2n}}$$

$$\mathcal{I}^{\rm Spin(2N)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{1}{1-\mathfrak{q}^{2N}} \prod_{n=1}^{N-1} \frac{1}{1-\mathfrak{q}^{4n}},$$

$$\langle \mathcal{W}_{\rm sp} \mathcal{W}_{\rm sp} \rangle^{\rm Spin(2N)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \langle \mathcal{W}_{\rm sp} \mathcal{W}_{\overline{\rm sp}} \rangle^{\rm Spin(2N)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})$$

$$= \prod_{n=1}^{N-1} (1+\mathfrak{q}^{2n})$$

$$\langle \underbrace{W_{\rm sp} \cdots W_{\rm sp}}_{2k} \rangle^{Spin(2N=4n)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \langle \underbrace{W_{\rm sp} \cdots W_{\overline{\rm sp}}}_{2k} \rangle^{Spin(2N=4n+2)}_{\frac{1}{2}\rm BPS}(\mathfrak{q})$$

$$= \frac{\sum_{i=0}^{\frac{N(N+1)k}{2}} a_{k\ \rm sp}^{\mathfrak{so}(2N)}(i)\mathfrak{q}^{2i}}{1-\mathfrak{q}^{2N}\prod_{n=1}^{N-1} 1-\mathfrak{q}^{4n}},$$

$$\langle W_{\Box} W_{\Box} \rangle^{SO(2N)}(t;q) = \frac{1}{2^{N-1}N!} \frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i}$$

$$\times \prod_{i\neq j} \frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty \left(s_i^{\pm}s_j^{\pm};q\right)_\infty \left(qs_i^{\pm}s_j^{\mp};q\right)_\infty \left(qs_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\pm};q\right)_\infty} \left[\sum_{i=1}^{N} (s_i + s_i^{-1})\right]^2$$

$$\left\langle T_{(1,0^{N-1})} T_{(1,0^{N-1})} \right\rangle^{SO(2N)} (t;q) = \frac{1}{2^{N-2}(N-1)!} \frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}$$

$$\times \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \prod_{i<j} \frac{\left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^2 s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\times \frac{\left(q^{1+\frac{1}{2}\delta_{i+j,1}} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i+j,1}} s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^{-2} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^{-2} s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\langle W_{\Box} W_{\Box} \rangle^{SO(2N)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\cdots \mathfrak{q}^{2N-4}+2\mathfrak{q}^{2N-2}+\mathfrak{q}^{2N}+\cdots \mathfrak{q}^{4N-4}}{(1-\mathfrak{q}^{2N})\prod_{n=1}^{N-1}(1-\mathfrak{q}^{4n})}$$

$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^{2(N-1)})\prod_{n=1}^{N-2}(1-\mathfrak{q}^{4n})}$$

$$\langle \underbrace{W_{\Box} \cdots W_{\Box}}_{2k} \rangle^{SO(2N)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \frac{\sum_{i=0}^{(2N-2)k} a_{k\Box}^{\mathfrak{so}(2N)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^{2N})\prod_{n=1}^{N-1}(1-\mathfrak{q}^{4n})}.$$

$$\langle W_\square W_\square\rangle^{SO(2N)^-}(t;q)=\frac{1}{2^{N-1}(N-1)!}\frac{(q)_\infty^{2N-2}(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^{N-1}\left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}$$

$$\times\oint\prod_{i=1}^{N-1}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm};q)_\infty(-s_i^{\pm};q)_\infty(qs_i^{\pm};q)_\infty(-qs_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$

$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty(qs_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}\left[\sum_{i=1}^{N-1}(s_i+s_i^{-1})\right]^2$$

$$\left\langle T_{(1,0^{N-2})}T_{(1,0^{N-2})}\right\rangle^{SO(2N)^-}(t;q)$$

$$=\frac{1}{2^{N-2}(N-2)!}\frac{(q)_\infty^{2N-2}(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^{N-1}\left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\prod_{i=1}^{N-1}\frac{ds_i}{2\pi i s_i}$$

$$\times\frac{\left(q^{\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty\left(-q^{1+\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i,1})}t^2s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}(1+\delta_{i,1})}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i,1})}t^{-2}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}(1+\delta_{i,1})}t^{-2}s_i^{\pm};q\right)_\infty}$$

$$\times\prod_{i<j}\frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\times\frac{\left(q^{1+\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\langle W_\square W_\square\rangle^{SO(2N)^-}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\left\langle T_{(1,0^{N-2})}T_{(1,0^{N-2})}\right\rangle^{SO(2N)^-}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})$$

$$=\frac{1+\mathfrak{q}^2+\cdots+\mathfrak{q}^{2N-4}}{\prod_{n=1}^{N-1}(1-\mathfrak{q}^{4n})}$$

$$=\frac{1-\mathfrak{q}^{2(N-1)}}{1-\mathfrak{q}^2}\prod_{n=1}^{N-1}\frac{1}{1-\mathfrak{q}^{4n}}$$

$$\langle W_\square W_\square\rangle^{O(2N)^+}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{1+\mathfrak{q}^2+\cdots+\mathfrak{q}^{4N-2}}{\prod_{n=1}^{N}(1-\mathfrak{q}^{4n})}$$

$$=\frac{1}{1-\mathfrak{q}^2}\prod_{n=1}^{N-1}\frac{1}{1-\mathfrak{q}^{4n}}.$$

$$\mathcal{I}^{O(2N)^+}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\prod_{n=1}^{N}\frac{1}{1-\mathfrak{q}^{4n}},$$

$$\langle W_\square W_\square\rangle^{O(2N)^+}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{1-\mathfrak{q}^{4N}}{1-\mathfrak{q}^2}$$

$$\mathcal{I}^{SO(2\infty+1)}(t;q)=\mathcal{I}^{USp(2\infty)}(t;q)=\mathcal{I}^{SO(2\infty)}(t;q)=\mathcal{I}^{O(2\infty)^+}(t;q)$$

$$=\prod_{n,m,l=0}^{\infty}\frac{\left(1-q^{n+m+l+\frac{3}{2}}t^{-4m+4l\pm 2}\right)^2}{(1-q^{n+m+l+1}t^{-4m+4l\pm 4})(1-q^{n+m++1}t^{-4m+4l})(1-q^{n+m+l+3}t^{-4m+4l})}$$

$$i^{AdS_5\times\mathbb{RP}^5}(t;q)=\frac{q^{\frac{1}{2}}(t^2+t^{-2})-q-q^2}{(1-qt^4)(1-qt^{-4})}-\frac{q^{\frac{1}{2}}(t^2+t^{-2})}{\left(1+q^{\frac{1}{2}}t^2\right)\left(1+q^{\frac{1}{2}}t^{-2}\right)(1-q)}$$

$$i^X(t;q) := \mathrm{Tr}(-1)^F q^{\frac{h+j}{2}} t^{2(q_2-q_3)}$$

$$i^X_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) := \mathrm{Tr}(-1)^F \mathfrak{q}^{2(q_2-q_3)}$$

$$\langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) = \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{SO(2\infty)}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) = \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{1-\mathfrak{q}^2}.$$

$$\langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{SO(2\infty+1)}(t;q) = \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{USp(2\infty)}(t;q)$$
$$= \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{SO(2\infty)}(t;q) = \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{O(2\infty)^+}(t;q) = \frac{1-q}{\left(1-q^{\frac{1}{2}}t^2\right)\left(1-q^{\frac{1}{2}}t^{-2}\right)}$$

$$i^{\mathrm{curvature}}(t;q) = -q + q^{\frac{1}{2}}t^2 + q^{\frac{1}{2}}t^{-2}$$

$$\left\langle \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{\mathfrak{q}^2}{(1-\mathfrak{q}^4)}$$
$$= \mathfrak{q}^2 + \mathfrak{q}^6 + \mathfrak{q}^{10} + \mathfrak{q}^{14} + \mathfrak{q}^{18} + \cdots$$

$$\left\langle \mathcal{W}_{\substack{\square\\\square}} \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle \mathcal{W}_{\substack{\square\\\square}} \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1+\mathfrak{q}^2+\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2}$$
$$= 1 + \mathfrak{q}^2 + 3\mathfrak{q}^4 + 2\mathfrak{q}^6 + 5\mathfrak{q}^8 + 3\mathfrak{q}^{10} + 7\mathfrak{q}^{12} + 4\mathfrak{q}^{14} + 9\mathfrak{q}^{16} + \cdots$$

$$\left\langle \mathcal{W}_{\substack{\square\\\square}} \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) = \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) = \left\langle \mathcal{W}_{\substack{\square\\\square}} \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q})$$
$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$
$$= 1 + \mathfrak{q}^2 + 2\mathfrak{q}^4 + 2\mathfrak{q}^6 + 3\mathfrak{q}^8 + 3\mathfrak{q}^{10} + 4\mathfrak{q}^{12} + 4\mathfrak{q}^{14} + 5\mathfrak{q}^{16} + 5\mathfrak{q}^{18} + \cdots,$$

$$\langle \mathcal{W}_\lambda \mathcal{W}_\lambda \rangle^G_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) := \langle \mathcal{W}_\lambda \mathcal{W}_\lambda \rangle^G_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - \langle \mathcal{W}_\lambda \rangle^G_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})^2.$$

$$\left\langle \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{SO(2\infty+1)}(t;q) = \langle \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}(t;q) = \left\langle \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{O(2\infty)^+}(t;q)$$
$$= \frac{q^{\frac{1}{2}}(t^2+t^{-2}) - q - q^2}{(1-qt^4)(1-qt^4)}$$

$$\left\langle \mathcal{W}_{\substack{\square\\\square}} \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{SO(2\infty+1)}(t;q) = \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}(t;q) = \left\langle \mathcal{W}_{\substack{\square\\\square}} \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{O(2\infty)^+}(t;q)$$
$$= \frac{1}{(1-qt^4)(1-qt^{-4})}\left(1 + (t^2+t^{-2})q^{\frac{1}{2}} + (3+t^4+t^{-4})q - 3(t^2+t^{-2})q^{\frac{3}{2}}\right.$$
$$\left. -(t^2+t^{-2})q^2 - 3(t^2+t^{-2})q^{\frac{5}{2}} + (3+t^4+t^{-4})q^3 + (t^2+t^{-2})q^{\frac{7}{2}} + q^4\right)$$

$$\left\langle \mathcal{W}_{\substack{\square\\\square}} \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{SO(2\infty+1)}_c(t;q) = \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}_c(t;q) = \left\langle \mathcal{W}_{\substack{\square\\\square}} \mathcal{W}_{\substack{\square\\\square}} \right\rangle^{O(2\infty)^+}_c(t;q)$$
$$= \frac{(1-q)\left(1+q-q^{\frac{3}{2}}(t^2+t^{-2})\right)}{\left(1-q^{\frac{1}{2}}t^2\right)\left(1-q^{\frac{1}{2}}t^{-2}\right)(1-qt^4)(1-qt^{-4})}$$

$$\langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\mathrm{sp}}\rangle^{\mathrm{Spin}(2\infty+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\mathrm{sp}}\rangle^{\mathrm{Spin}(4\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\overline{\mathrm{sp}}}\rangle^{\mathrm{Spin}(4\infty+2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \prod_{n=1}^{\infty} \frac{1}{1-\mathfrak{q}^{4n-2}}$$

$$\langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\mathrm{sp}}|^{\mathrm{Spin}(\infty)}_{\frac{1}{2}\mathrm{BPS}}\rangle(\mathfrak{q}) = \sum_{n\geq 0} d^{(H)}_{\{\mathrm{sp},\mathrm{sp}\}}(n)\mathfrak{q}^{2n}$$

$$i^{\mathrm{supercurvature}}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2}{1-\mathfrak{q}^4} = \mathfrak{q}^2 + \mathfrak{q}^6 + \mathfrak{q}^{10} + \cdots.$$

$$S_{\mathrm{D5}} = T_5 \int d^6\sigma \sqrt{\det(g + 2\pi\alpha' F)} - iT_5 \int 2\pi\alpha' F \wedge C_{(4)}$$

$$S_{\mathrm{D5}AdS_2\times\mathbb{R}^4} = T_5 \int d^6\sigma\sqrt{\det g} = T_5 \mathrm{vol}(AdS_2)\mathrm{vol}(\mathbb{RP}^4)$$

$$ds^2_{AdS_2} = \frac{1}{r^2}(-dt^2 + dr^2), ds^2_4 = g_{ij}d\sigma^i d\sigma^j, i,j = 1,2,3,4$$

$$S = T_5 \int d^6\sigma\sqrt{g^{(4)}}\frac{1}{2}\frac{1}{r^2}\left[r^2(\partial_t\phi)^2 - r^2(\partial_r\phi)^2 + \left(\nabla_i\phi\nabla^i\phi - 4\phi^2\right)\right]$$

$$\phi(t,r,\Theta) = \sum_w \phi_w(t,r)Y^w(\Theta)$$

$$S = T_5\sum_w \frac{2}{3}\pi^2 \int d^2\sigma\frac{1}{r^2}(r^2(\partial_t\phi_w)^2 - r^2(\partial_r\phi_w)^2 - w(w+1)\phi_w^2)$$

$$h = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2},$$

$$\langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\mathrm{sp}}\rangle^{\mathrm{Spin}(2\infty+1)}(t;q) = \langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\mathrm{sp}}\rangle^{\mathrm{Spin}(4\infty)}(t;q) = \langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\overline{\mathrm{sp}}}\rangle^{\mathrm{Spin}(4\infty+2)}(t;q)$$
$$= \prod_{n=0}^{\infty}\prod_{m=0}^{\infty} \frac{(1-q^{1+n+m}t^{4n-4m})(1-q^{2+n+m}t^{4n-4m})}{\left(1-q^{\frac{1}{2}+n+m}t^{2+4n-4m}\right)\left(1-q^{\frac{1}{2}+n+m}t^{-2+4n+4m}\right)}$$

$$\langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\mathrm{sp}}\rangle^{\mathrm{Spin}(2\infty+1)}(q) = \langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\mathrm{sp}}\rangle^{\mathrm{Spin}(4\infty)}(q) = \langle \mathcal{W}_{\mathrm{sp}}\mathcal{W}_{\overline{\mathrm{sp}}}\rangle^{\mathrm{Spin}(4\infty+2)}(q)$$
$$= \prod_{n=1}^{\infty}\frac{(1-q^n)^{2n-1}}{\left(1-q^{n-\frac{1}{2}}\right)^{2n}}$$
$$= 1 + 2q^{1/2} + 2q^2 + 6q^{3/2} + 7q^2 + 10q^{5/2} + 21q^3 + 22q^{7/2} + \cdots$$

$$i^{\mathrm{supercurvature}}(t;q) = \frac{q^{\frac{1}{2}}(t^2+t^{-2}) - q - q^2}{(1-qt^4)(1-qt^{-4})}$$

$$\langle \mathcal{W}_{\Box\Box}\rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle \mathcal{W}_{\substack{\Box\\\Box}}\right\rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\Box\Box}\rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{\mathfrak{q}^4}{(1-\mathfrak{q}^4)}$$

$$\langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\substack{\Box\\\Box}}\mathcal{W}_{\substack{\Box\\\Box}}\rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1+\mathfrak{q}^2+\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2}.$$

$$\langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) = \langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) = \langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q})$$
$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)},$$

$$\langle \mathcal{W}_{\square\square}\rangle^{SO(2\infty+1)}(t;q) = \left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right\rangle^{USp(2\infty)}(t;q) = \langle \mathcal{W}_{\square\square}\rangle^{O(2\infty)^+}(t;q)$$
$$= \frac{q(1+t^4+t^{-4}) - q^{\frac{3}{2}}(t^2+t^{-2}) - q^2}{(1-qt^4)(1-qt^{-4})}$$

$$\langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{SO(2\infty+1)}(t;q) = \left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right\rangle^{USp(2\infty)}(t;q) = \langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{O(2\infty)^+}(t;q)$$
$$= \frac{1}{(1-qt^4)(1-qt^{-4})}\Big(1+(t^2+t^{-2})q^{\frac{1}{2}} + q - (t^2+t^{-2})q^{\frac{3}{2}} + (t^8+t^4+t^{-4}+t^{-8})q^2$$
$$-(2t^6+5t^2+5t^{-2}+2t^{-6})q^{\frac{5}{2}} + q^3 + 3(t^2+t^{-2})q^{\frac{7}{2}} + q^4\Big)$$

$$\langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{SO(2\infty+1)}_c(t;q) = \left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right\rangle^{USp(2\infty)}_c(t;q) = \langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{O(2\infty)^+}_c(t;q)$$
$$= \frac{(1-q)\left(1+q-q^{\frac{3}{2}}(t^2+t^{-2})\right)}{\left(1-q^{\frac{1}{2}}t^2\right)\left(1-q^{\frac{1}{2}}t^{-2}\right)(1-qt^4)(1-qt^{-4})}$$

$$\chi^{\mathfrak{usp}(2N)}_\lambda = \det\left(E_{\lambda'_i-i+j} - E_{\lambda'_i-i-j}\right)_{1\le i,j\le l(\lambda')}$$
$$\chi^{\mathfrak{so}(2N+1)}_\lambda = \det\left(\bar{H}_{\lambda_i-i+j} - \bar{H}_{\lambda_i-i-j}\right)_{1\le i,j\le l(\lambda)}$$

$$E_k = e_k(s_1,\dots,s_N,s_1^{-1},\dots,s_N^{-1})$$
$$\bar{H}_k = h_k(s_1,\dots,s_N,s_1^{-1},\dots,s_N^{-1},1)$$

$$P_k = p_k(s_1,\dots,s_N,s_1^{-1},\dots,s_N^{-1})$$
$$\bar{P}_k = p_k(s_1,\dots,s_N,s_1^{-1},\dots,s_N^{-1},1)$$

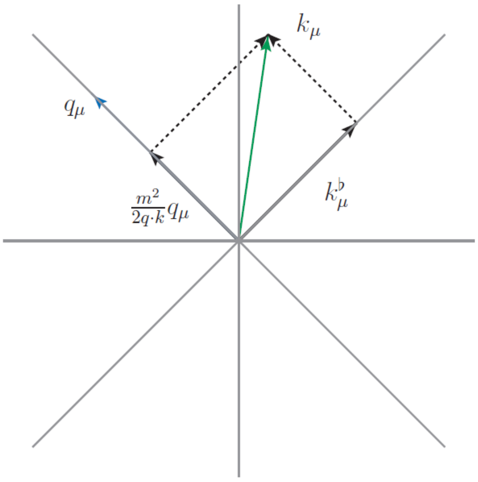
$$p_k(x_1,\dots,x_n) = \sum_{i=1}^{n} x_i^k$$

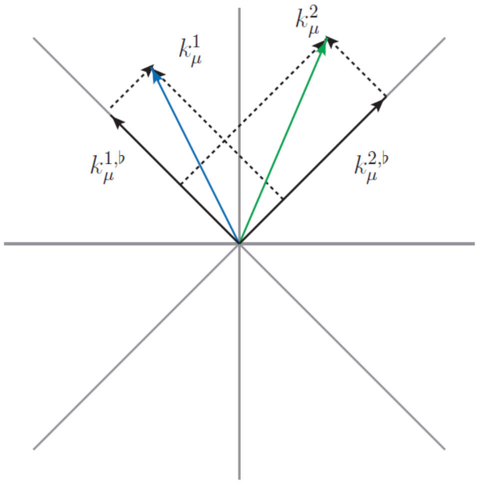
$$\chi^{\mathfrak{usp}(2N)}_{\square} = P_1, \qquad \chi^{\mathfrak{so}(2N+1)}_{\square} = \overline{P}_1,$$
$$\chi^{\mathfrak{usp}(2N)}_{\square\square} = \frac{P_2}{2} + \frac{P_1^2}{2}, \qquad \chi^{\mathfrak{so}(2N+1)}_{\square\square} = \frac{\overline{P}_2}{2} + \frac{\overline{P}_1^2}{2} - 1,$$
$$\chi^{\mathfrak{usp}(2N)}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = -\frac{P_2}{2} + \frac{P_1^2}{2} - 1, \qquad \chi^{\mathfrak{so}(2N+1)}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = -\frac{\overline{P}_2}{2} + \frac{\overline{P}_1^2}{2},$$
$$\chi^{\mathfrak{usp}(2N)}_{\square\square\square} = \frac{P_3}{3} + \frac{P_2P_1}{2} + \frac{P_1^3}{6}, \qquad \chi^{\mathfrak{so}(2N+1)}_{\square\square\square} = \frac{\overline{P}_3}{3} + \frac{\overline{P}_2\overline{P}_1}{2} + \frac{\overline{P}_1^3}{6} - \overline{P}_1,$$
$$\chi^{\mathfrak{usp}(2N)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = -\frac{P_3}{3} + \frac{P_1^3}{3} - P_1, \qquad \chi^{\mathfrak{so}(2N+1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = -\frac{\overline{P}_3}{3} + \frac{\overline{P}_1^3}{3} - \overline{P}_1,$$
$$\chi^{\mathfrak{usp}(2N)}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \frac{P_3}{3} - \frac{P_2P_1}{2} + \frac{P_1^3}{6} - P_1, \qquad \chi^{\mathfrak{so}(2N+1)}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \frac{\overline{P}_3}{3} - \frac{\overline{P}_2\overline{P}_1}{2} + \frac{\overline{P}_1^3}{6}.$$

SUPLEMENTO. ECUACIONES COMPLEMENTARIAS.

$$k_\mu = k_\mu^b + \frac{k^2}{2q \cdot k} q_\mu$$

$$q \cdot k = q \cdot k^b$$





$$s_\mu = k_\mu - \frac{k^2}{q \cdot k} q_\mu$$

$$R_z = \frac{q_\mu W^\mu}{2q \cdot k} \equiv \frac{q_\mu k_\nu \Sigma_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}}{2q \cdot k}$$

$$k_1 = k_1^b + \frac{m_1^2}{\gamma_{12}} k_2^b \; k_2 = k_2^b + \frac{m_2^2}{\gamma_{12}} k_1^b$$

$$\gamma_{12} = 2(k_1^b \cdot k_2^b) = \left((k_1 \cdot k_2) \pm \sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2} \right)$$

$$\begin{aligned} a^{1,\alpha\dot{\alpha}} &= k_1^{b,\alpha} k_1^{b,\dot{\alpha}} \quad & a^{3,\alpha\dot{\alpha}} &= k_1^{b,\alpha} k_2^{b,\dot{\alpha}} \\ a^{2,\alpha\dot{\alpha}} &= k_2^{b,\alpha} k_2^{b,\dot{\alpha}} \quad & a^{4,\alpha\dot{\alpha}} &= k_2^{b,\alpha} k_1^{b,\dot{\alpha}} \end{aligned}$$

$$a^{3,\mu} a_\mu^4 = -a^{1,\mu} a_\mu^2$$

$$k_1^{b,\dot{\alpha}} \equiv 1^{\dot{\alpha}} \; k_1^{b,\alpha} \equiv 1^\alpha \; k_2^{b,\dot{\alpha}} \equiv 2^{\dot{\alpha}} \; k_2^{b,\alpha} \equiv 2^\alpha$$

$$a_\mu^1 \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} a_\mu^2 \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} a_\mu^3 \sim \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} a_\mu^4 \sim \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$$

$$\begin{aligned} l_\mu &= c_1 a_\mu^1 + c_2 a_\mu^2 + c_3 a_\mu^3 + c_4 a_\mu^4 \\ &= c_1 a_\mu^1 + c_2 a_\mu^2 + c_4 a_\mu^4 + \left(c_3 - \frac{l^2}{2a^3 \cdot l} \right) a_\mu^3 + \frac{l^2}{2a^3 \cdot l} a_\mu^3 \\ &\equiv c_1 a_\mu^1 + c_2 a_\mu^2 + c_4 a_\mu^4 + \tilde{c}_3 a_\mu^3 + \frac{l^2}{2a^3 \cdot l} a_\mu^3 \\ &\equiv l_\mu^b + \frac{l^2}{2a^3 \cdot l} a_\mu^3 \end{aligned}$$

$$l_{\mu}^{b}\sigma^{\mu} = \left(\begin{pmatrix} c_1 \\ c_3 - \frac{l^2}{2a^3 \cdot l} \end{pmatrix} \begin{matrix} c_4 \\ c_2 \end{matrix}\right)$$

$$l_{\alpha\dot{\alpha}}^{b} = \lambda_{\alpha}\lambda_{\dot{\alpha}}$$

$$c_1 = \frac{\langle 2\lambda\rangle[2\lambda]}{\gamma_{12}} c_2 = \frac{\langle 1\lambda\rangle[1\lambda]}{\gamma_{12}}$$
$$c_4 = \frac{\langle 2\lambda\rangle[1\lambda]}{\gamma_{12}} \tilde{c}_3 = \frac{\langle 1\lambda\rangle[2\lambda]}{\gamma_{12}}$$

$$\gamma_{12} = \langle 12\rangle[12]$$

$$\lambda_{\alpha} = \frac{\langle\lambda 2\rangle 1_{\alpha} - \langle\lambda 1\rangle 2_{\alpha}}{\langle 12\rangle}$$
$$\lambda_{\dot{\alpha}} = \frac{[\lambda 1]2_{\dot{\alpha}} - [\lambda 2]1_{\dot{\alpha}}}{[12]}$$

$$e_{\alpha\dot{\alpha}}^{+} = \frac{1}{\sqrt{2}}\frac{k_{\alpha}^{b}q_{\dot{\alpha}}}{[qk^{b}]} \; e_{\alpha\dot{\alpha}}^{-} = \frac{1}{\sqrt{2}}\frac{q_{\alpha}k_{\dot{\alpha}}^{b}}{\langle qk^{b}\rangle} \; e_{\alpha\dot{\alpha}}^{0} = \frac{s_{\alpha\dot{\alpha}}}{m}$$

$$q^1 = k_{\mu}^{2,b} \; q^2 = k_{\mu}^{1,b}$$

$$e_1^{\pm} = -e_2^{\mp}$$

$$k_i \rightarrow k_i + nz \; k_i \rightarrow k_i - nz$$

$$k_i \cdot n = k_j \cdot n = n \cdot n = 0$$

$$\mathrm{Res}_{z=\infty}\frac{A(z)}{z} = 0$$

$$n_{\alpha\dot{\alpha}} = k_{\alpha}^{1,b}k_{\dot{\alpha}}^{2,b} \text{ or } n_{\alpha\dot{\alpha}} = k_{\alpha}^{2,b}k_{\dot{\alpha}}^{1,b}$$

$$(-e_j^{-} =)e_i^{+} \rightarrow e_i^{+} + z\frac{\left(k_i^{b}\right)_{\alpha}^{b}\left(k_j^{b}\right)_{\dot{\alpha}}}{\langle k_i^{b}k_j^{b}\rangle}$$

$$\langle(k_{\mu}A^{\mu} + m\Phi)X\rangle = 0$$

$$\left\langle\left(\left(k_{\mu} + zn\right)A^{\mu} + m\Phi\right)\right)X\right\rangle = 0$$

$$R_q^1 = q^{\mu}n_1^{\nu}n_2^{\rho}W_{\mu\nu\rho} \equiv R_z$$

$$W_{\mu\nu\rho} = k_{[\mu}\Sigma_{\nu\rho]}$$

$$k \cdot n_3 = k \cdot n_4 \rightarrow 0$$

$$m \equiv Z_1 + \mathrm{i}Z_2 = (k \cdot n_3) + \mathrm{i}(k \cdot n_4)$$

$$|\eta_I, \iota_I\rangle = e^{\sum_I \eta_I Q_-^I + \iota_I \bar{Q}_-^I}|\uparrow\rangle$$

$$|\bar{\eta}_I, \bar{\iota}_I\rangle = e^{\sum_I \bar{\eta}_I Q_+^I + \bar{\iota}_I \bar{Q}_+^I}|\downarrow\rangle$$

$$
\begin{aligned}
e^{\bar{\xi}_I Q_I}|\eta_I, \iota_I\rangle &= e^{\iota_I\left(m\frac{[\bar{\xi}_I q]}{[qk^b]}+\langle\bar{\xi}_I k^b\rangle\right)}\left|\eta_I + \left([\bar{\xi}_I k^b] + \bar{m}\frac{\langle\bar{\xi}_I q\rangle}{\langle qk^b\rangle}\right), \iota_I\right\rangle \\
e^{\bar{Q}_I \xi_I}|\eta_I, \iota_I\rangle &= e^{\eta_I\left(m\frac{[q\xi_I]}{[qk^b]}+\langle k^b \xi_I\rangle\right)}\left|\eta_I, \iota_I + \left([k^b \xi_I] + \bar{m}\frac{\langle q\xi_I\rangle}{\langle qk^b\rangle}\right)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
e^{\bar{\xi}_I Q^I}|\bar{\eta}_I, \bar{\iota}_I\rangle &= e^{\bar{\iota}_I\left([\bar{\xi}_I k^b]+\bar{m}\frac{\langle\bar{\xi}_I q\rangle}{\langle qk^b\rangle}\right)}\left|\bar{\eta}_I + \left(m\frac{[\bar{\xi}_I q]}{[qk^b]} + \langle\bar{\xi}_I k^b\rangle\right), \bar{\iota}_I\right\rangle \\
e^{\bar{Q}_I \xi^I}|\bar{\eta}_I, \bar{\iota}_I\rangle &= e^{\bar{\eta}_I\left([k^b \xi_I]+\bar{m}\frac{\langle q\xi_I\rangle}{\langle qk^b\rangle}\right)}\left|\bar{\eta}_I, \bar{\iota}_I + \left(m\frac{[q\xi_I]}{[qk^b]} + \langle k^b \xi_I\rangle\right)\right\rangle
\end{aligned}
$$

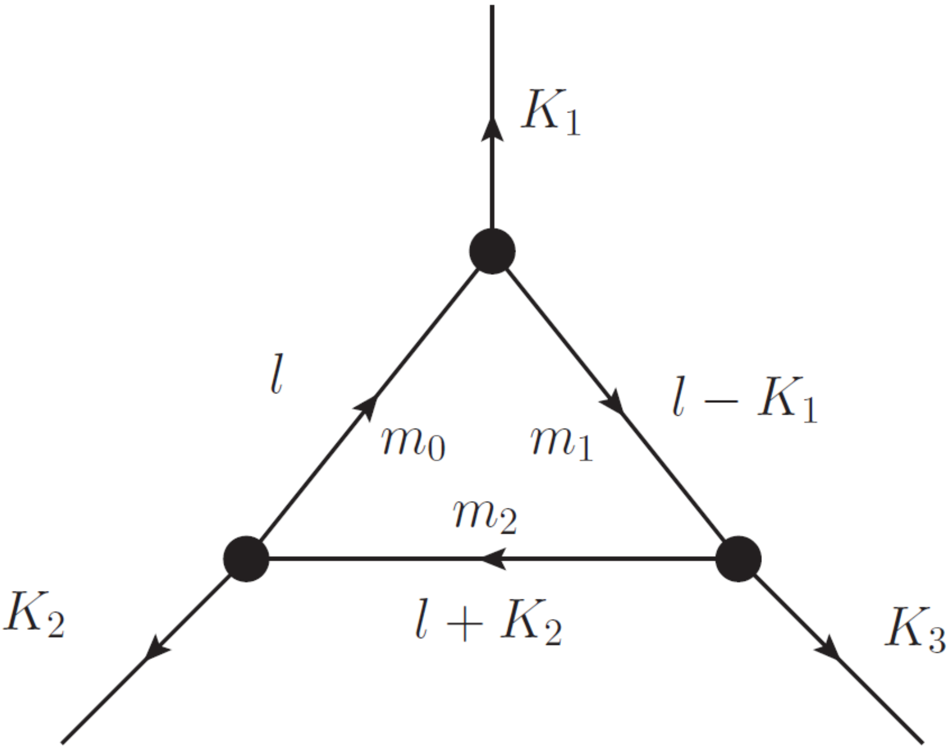
$$\sum_{s\in \text{multiplet}} A(\ldots,\{s,l_\mu\})A(\{s,-l_\mu\},\ldots) \to \int (dF)A(\ldots,\{\eta,\iota,l_\mu\})A(\{\eta,\iota,-l_\mu\},\ldots)$$

$$(dF) = d\eta_1 d\iota_1 d\eta_2 d\iota_2$$

$$(\overline{dF}) = d\bar{\eta}_1 d\bar{\iota}_1 d\bar{\eta}_2 d\bar{\iota}_2$$

$$l_\alpha^b \to -l_\alpha^b \text{ or } l_{\dot{\alpha}}^b \to -l_{\dot{\alpha}}^b$$

$$A^{1-\text{loop}} = \sum a_b + a_t + a_{bb} + a_{tp} + \mathbb{R}$$



$$l^2 = m_0^2 \; (l-K_1)^2 = m_1^2 \; (l+K_2)^2 = m_2^2$$

$$l_\mu = \alpha_1 a_\mu^1 + \alpha_2 a_\mu^2 + \frac{\alpha_1\alpha_2 - \frac{m_0^2}{\gamma_{12}}}{t} a_\mu^3 + t a_\mu^4$$

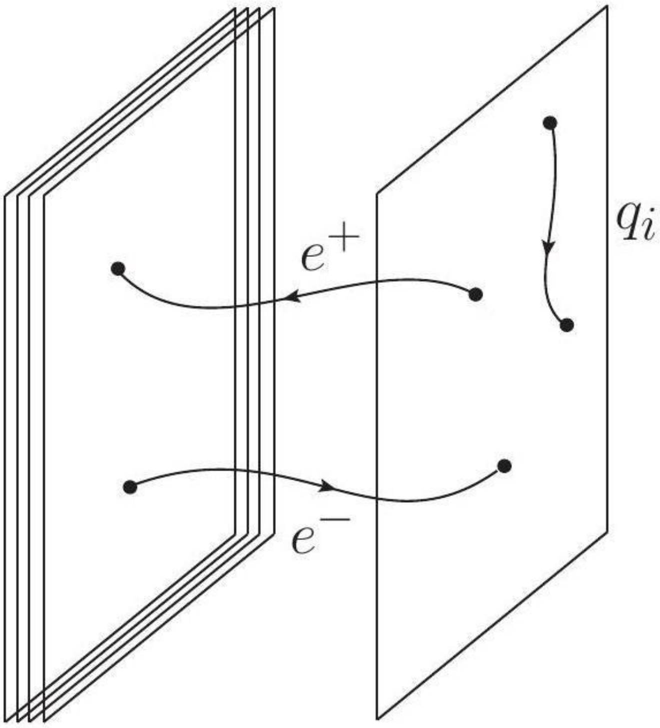
$$\alpha_1 = \frac{K_2^2K_1^2 - (m_2^2 + m_0^2)\gamma_{12} - (m_0^2 + m_1^2 + \gamma_{12})K_1^2}{K_1^2K_2^2 - \gamma_{12}^2}$$
$$\alpha_2 = \frac{-K_2^2K_1^2 + (m_1^2 + m_0^2)\gamma_{12} + (m_2^2 + m_0^2 - \gamma_{12})K_2^2}{K_1^2K_2^2 - \gamma_{12}^2}$$

$$\left\langle \begin{aligned} l_\mu &= l_\mu^b + \frac{m_0^2}{2(l \cdot a^3)} a_\mu^3 \\ l^\mu - K_1^\mu &= (l - k_1)_\mu^b + \frac{m_1^2}{2(l - k_1) \cdot a^3} a_\mu^3 \\ l^\mu + K_2^\mu &= (l + k_2)_\mu^b + \frac{m_2^2}{2(l + k_2) \cdot a^3} a_\mu^3 \end{aligned} \right|$$

$$\left| \begin{aligned} l_\mu^b &= \left(\alpha_1 a_\mu^1 + \alpha_2 a_\mu^2 + \frac{\alpha_1\alpha_2}{t} a_\mu^3 + t a_\mu^4\right) \\ (l - K_1)_\mu^b &= \left((\alpha_1 - 1)a_\mu^1 + \left(\alpha_2 - \frac{K_2^2}{\gamma_{12}}\right)a_\mu^2 + \frac{(\alpha_1 - 1)\left(\alpha_2 - \frac{K_2^2}{\gamma_{12}}\right)}{t} a_\mu^3 + t a_\mu^4\right) \\ (l + K_2)_\mu^b &= \left(\left(\alpha_1 + \frac{K_1^2}{\gamma_{12}}\right)a_\mu^1 + (\alpha_2 + 1)a_\mu^2 + \frac{\left(\alpha_1 + \frac{K_1^2}{\gamma_{12}}\right)(\alpha_2 + 1)}{t} a_\mu^3 + t a_\mu^4\right) \end{aligned} \right\rangle$$

$$A^1|_{\text{triple}} = \int dtJ_t \left(\sum_{s_1,s_2,s_3} A(\{s_1, -l\}, \{s_2, l - K_1\}, X_1) A(\{s_2, -l + K_1\}, \{s_3, l + K_2\}, X_2) \right.$$
$$\left. A(\{s_3, -l - K_2\}, \{s_1, l\}, X_3) \right)$$

$$a_t = \left(\lim_{t\to\infty} \sum_{s_1,s_2,s_3} A(\{s_1, -l\}, \{s_2, l - K_1\}, X_1) A(\{s_2, -l + K_1\}, \{s_3, l + K_2\}, X_2) \right.$$
$$\left. A(\{s_3, -l - K_2\}, \{s_1, l\}, X_3) \right)_{t=0}$$



$$|0\rangle_1 \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix} \; |0\rangle_2 \leftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e^+|0\rangle_1 = e^-|0\rangle_2 = 0$$

$$A \sim \sum \; {}_i\langle 0|V_1 \ldots V_n|0\rangle_i$$

$$q_1 = \frac{1}{2}(h+i) \; q_2 = \frac{1}{2}(h-i)$$

$$\mathrm{Tr}\left(T^{(1)} \ldots T^{(n)}\right)$$

$$(e^+)^2 = (e^-)^2 = 0$$

$$(e^+_{km})_{ij} = \delta_{i,k}\delta_{j,m} \; k < m$$

$$(e^-_{km})_{ij} = \delta_{i,k}\delta_{j,m} \; k > m$$

$$(q_k)_{ij} = \delta_{i,k}\delta_{j,k}$$

$$Pe^{\oint A^\mu(X)dX_\mu(+F_{\mu\nu}\psi^\mu\psi^\nu)}$$

$$\langle A_\mu \rangle = n_\mu \left(\sum_i c_i q^i \right)$$

$$V_a(z_1)T^a \to e^{n_\mu(\sum_i c_i q^i)X^\mu(z_1)}(V_a(z_1)T^a)e^{-n_\mu(\sum_i c_i q^i)X^\mu(z_1)}$$

$$e^{n_\mu(\sum_i c_i q^i)X^\mu(z_1)}\left(V_i(z_1)T^i\right)e^{-n_\mu(\sum_i c_i q^i)X^\mu(z_1)} = \begin{cases} \left(V_i(z_1)T^i\right)e^{(c_k-c_m)n_\mu X^\mu(z_1)} & \text{if } T = e^+_{km} \\ \left(V_i(z_1)T^i\right)e^{(c_m-c_k)n_\mu X^\mu(z_1)} & \text{if } T = e^-_{km} \\ \left(V_i(z_1)T^i\right) & \text{else} \end{cases}$$

$$k^2 = (c_k - c_m)^2$$

$$p^\mu \xi_\mu = 0 \; \xi_\mu \sim \xi_\mu + g p_\mu$$

$$\xi_\mu(\partial X^\mu + \psi^\mu\psi^\nu k_\nu)e^{ikX}$$

$$e_\mu \to e_\mu + f k_\mu$$

$$k^\mu = m\epsilon_0^\mu + \frac{m^2}{q \cdot k} q^\mu$$

$$\sim z^{\alpha'(k_1+k_n)^2}$$

$$:V_1(z_1)::V_2(z_2):=:V_2(z_2)::V_1(z_1):e^{2\alpha' k_1 k_2 \frac{(z_1-z_2)}{|z_1-z_2|}}$$

$$\alpha' \to 0 \; \sqrt{\alpha'}(c_k - c_m) \text{ fixed}$$

$$n_i^\mu(\partial_\mu + A_\mu) = (n_i^\mu \partial_\mu + \phi)$$

$$\phi \to \tilde{\phi} + \langle \phi \rangle$$

$$\psi = \sum_{k<m} \psi^{+,km} e^+_{km} + \sum_{k>m} \psi^{-,km} e^-_{km} + \sum_k \psi^{0,k} e^0_k$$

$$\langle \phi \rangle = \sum c_k e_k^0$$

$$([\langle \phi \rangle, \psi]) = \sum_{k<m} (c_k - c_m)\psi^{+,km} e_{km}^{+} - \sum_{k>m} (c_k - c_m)\psi^{-,km} e_{km}^{-}$$

$$n^\mu \partial_\mu \psi^{+,km} = -n^\mu \partial_\mu \psi^{-,km} = c_k - c_m$$

$$\psi = \sum_i \psi_i \tilde{e}^i$$

$$e_k^0 \tilde{e}^i = \delta_k^i \tilde{e}^i$$

$$n^\mu \partial_\mu \psi_i = c_i$$

$$F_{\mu\nu} \sim [D_\mu, D_\nu]$$

$$\mathcal{L}_{\text{rel}}^{D=4} = \text{Tr}\left(D_\mu \phi D_\mu \bar{\phi} + [\phi, \bar{\phi}]^2 + (F_4)^2\right)$$

$$\mathcal{L}_{\text{rel}}^{D=4} = \text{Tr}\left(F_{6,\mu\nu} F_{6,\mu\nu}\right)_{\text{red}}$$

$$F_6 = \begin{pmatrix} F_{\mu\nu}^4 & D_\mu A_5 & D_\mu A_6 \\ \cdots & 0 & [A_5, A_6] \\ \cdots & \cdots & 0 \end{pmatrix}$$

$$\langle A_5 + \text{i}A_6 \rangle = \sum_i (c_5 + \text{i}c_6)^i q_i$$

$$A_5 + \text{i}A_6 \to \left(\sum_i (c_5 + \text{i}c_6)^i q_i\right) + \tilde{A}_5 + \text{i}\tilde{A}_6$$

$$F_6 = \left(\begin{array}{c|cc} F_{\mu\nu}^4 & D_\mu \tilde{A}_5 + \sum_i c_5^i [A_\mu, q_i] & D_\mu \tilde{A}_6 + \sum_i c_6^i [A_\mu, q_i] \\ \hline \cdots & 0 & [\tilde{A}_5, \tilde{A}_6] + \sum_i c_5^i [q_i, \tilde{A}_6] - \sum_i c_6^i [q_i, \tilde{A}_5] \\ \cdots & \cdots & 0 \end{array}\right)$$

$$\frac{1}{g_4^2} \equiv \frac{1}{g_6^2} \int dx^2$$

$$(a^3)^{\alpha\dot{\alpha}} = k_1^{b,\alpha} k_2^{b,\dot{\alpha}} \equiv 1^\alpha 2^{\dot{\alpha}}$$

$$a_t = \lim_{t\to\infty} \int (dF)_1 (dF)_2 (\overline{dF})_3 A(\{\eta_1, \iota_1, l\}, \{\eta_2, \iota_2, l - k_1\}, X_1)$$
$$A(\{\eta_2, \iota_2, -(l - k_1)\}, \{\bar{\eta}_3, \bar{\iota}_3, l + k_2\}, X_2) A(\{\bar{\eta}_3, \bar{\iota}_3, -(l + k_2)\}, \{\eta_1, \iota_1, -l\}, X_3)$$

$$l^{b,\alpha\dot{\alpha}} = \lambda_1^\alpha \lambda_1^{\dot{\alpha}}\ (l - k_1)^{b,\alpha\dot{\alpha}} = \lambda_2^\alpha \lambda_2^{\dot{\alpha}}\ (l + k_2)^{b,\alpha\dot{\alpha}} = \lambda_3^\alpha \lambda_3^{\dot{\alpha}}$$

$$-l^{b,\alpha\dot{\alpha}} = (-\lambda_1^\alpha)\lambda_1^{\dot{\alpha}}\ -(l - k_1)^{b,\alpha\dot{\alpha}} = (-\lambda_2^\alpha)\lambda_2^{\dot{\alpha}}\ -(l + k_2)^{b,\alpha\dot{\alpha}} = \lambda_3^\alpha(-\lambda_3^{\dot{\alpha}})$$

$$\langle 1\lambda_i \rangle [2\lambda_i] = t$$

$$[1\lambda_i] \sim \frac{1}{t}\ \langle 1\lambda_i \rangle \sim t\ [2\lambda_i] \sim 1\ \langle 2\lambda_i \rangle \sim 1$$

$$\langle 2\lambda_i \rangle \sim \frac{1}{t}\ [2\lambda_i] \sim t\ \langle 1\lambda_i \rangle \sim 1\ [1\lambda_i] \sim 1$$

$$\left|\begin{matrix}\chi^I_{\dot\alpha} = \dfrac{\lambda_1^{\dot\alpha}\iota_2^I - \lambda_2^{\dot\alpha}\iota_1^I}{[\lambda_1\lambda_2]} & \bar\chi^{I,\dot\alpha} = \dfrac{\lambda_1^{\dot\alpha}\eta_2^I - \lambda_2^{\dot\alpha}\eta_1^I}{[\lambda_1\lambda_2]} \\ \chi^{I,\alpha} = 0 & \bar\chi^I_\alpha = 0\end{matrix}\right\rangle$$

$$\sim \left(e^{m_1\iota_I\frac{[\bar\chi^I 2]}{[2\lambda_1]}}\right)^2 \left(e^{m_2\iota_I\frac{[\bar\chi^I 2]}{2\lambda_2]}}\right)^2$$

$$\bar\eta_3 \to \bar\eta_3 + m_3\frac{[\bar\chi 2]}{[2\lambda_3]}$$
$$\bar\iota_3 \to \bar\iota_3 + m_3\frac{[2\chi]}{[2\lambda_3]}$$

$$\bar\eta_3 \to \bar\eta_3 - m_3\frac{[\bar\chi 2]}{[2\lambda_3]}$$
$$\bar\iota_3 \to \bar\iota_3 - m_3\frac{[2\chi]}{[2\lambda_3]}$$

$$\left\langle\begin{matrix}\xi^I_{\dot\alpha} = 0 \\ \xi^{I,\alpha} = -\dfrac{1^\alpha}{\langle 1\lambda_3\rangle}\bar\iota_3\end{matrix}\right|\begin{matrix}\bar\xi^{I,\dot\alpha} = 0 \\ \bar\xi^I_\alpha = -\dfrac{1^\alpha}{\langle 1\lambda_3\rangle}\bar\eta_3\end{matrix}\right\rangle$$

$$a_t = \lim_{t\to\infty}\iint (dF)_1(dF)_2(\overline{dF})_3 f(X_1,X_2,X_3)A \quad A(\{0,0,l\},\{0,0,l-k_1\},X_1)$$
$$A(\{0,0,-(l-k_1)\}, \quad \left\{m_3\frac{[\bar\chi 2]}{[2\lambda_3]}, m_3\frac{[2\chi]}{[2\lambda_3]}, l+k_2\right\}, X_2)$$
$$A\quad\left(\left\{-m_3\frac{[\bar\chi 2]}{[2\lambda_3]}, -m_3\frac{[2\chi]}{[2\lambda_3]}, -(l+k_2)\right\}, \{0,0,-l\}, X_3\right)$$

$$\chi^{I,\dot\alpha} = \frac{\lambda_1^{\dot\alpha}\iota_2^I - \lambda_2^{\dot\alpha}\iota_1^I}{[\lambda_1\lambda_2]}$$
$$= \frac{\kappa^{\dot\alpha}\iota_2^I(t) - \rho^{\dot\alpha}\iota_1^I(t)}{[\kappa\rho]}$$

$$\iota_1^I(t) = \frac{[\kappa\lambda_1]\iota_2^I - [\rho\lambda_2]\iota_1^I}{[\lambda_1\lambda_2]}$$
$$\iota_2^I(t) = \frac{[\rho\lambda_1]\iota_2^I - [\kappa\lambda_2]\iota_1^I}{[\lambda_1\lambda_2]}$$

$$(dF)_1(dF)_2 \to \left(\frac{[\kappa\rho]}{[\lambda_1\lambda_2]}\right)^4 (dF(t))_1(dF(t))_2$$

$$[\lambda_1\lambda_2] = \frac{[\lambda_1 1][2\lambda_2] - [\lambda_1 2][1\lambda_2]}{[12]}$$

$$J(t) \sim t^4 + \mathcal{O}(t^3)$$

$$\pm m_3\frac{[\bar\chi 2]}{[2\lambda_3]} \text{ and } \pm m_3\frac{[2\chi]}{[2\lambda_3]}$$

$$M^{ab}(t) = tg^{ab}h_0\left(\frac{1}{t}\right) + A^{ab}h_1\left(\frac{1}{t}\right) + \left(a_4^a K^b + K^a a_4^b\right)h_2\left(\frac{1}{t}\right) + \frac{1}{t}B^{ab}h_3\left(\frac{1}{t}\right) + \mathcal{O}\left(\frac{1}{t}\right)$$

$$e_i^+ \sim \frac{1}{t}2_\alpha 2_{\dot\alpha}\, e_i^- \to t$$

$$e^- \sim \frac{1}{t}1_\alpha 1_{\dot\alpha}\, e_i^+ \to t$$

$$A_{(--)} \to \frac{1}{t^3}$$

$$A_{(+-)} \to \frac{1}{t}$$

$$\lim_{t\to\infty} (t^4)\left(\frac{1}{t^3}\right)\left(\frac{1}{t}\right)\left(\frac{1}{t}\right) = 0$$

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{tree}} + 4\mathcal{L}_{\text{scalar}} + 2\mathcal{L}_{\text{white particle}} + \mathcal{L}_{\text{dark particle}}$$

$$\mathcal{L}_{\text{scalar}} = \log\ \det{}_{s=0}^{-1}(D_\mu D^\mu)_{D=6-2\epsilon}$$

$$\mathcal{L}_{\text{white particle}} = \frac{1}{2}\log\ \det{}_{s=\frac{1}{2}}^{\frac{1}{2}}(D_\mu D^\mu + \sigma_{\mu\nu}F^{\mu\nu})_{D=6-2\epsilon}$$

$$\mathcal{L}_{\text{dark particle}} = \log\ \det{}_{s=1}^{-\frac{1}{2}}(D_\mu D^\mu + \Sigma_{\mu\nu}F^{\mu\nu})_{D=6-2\epsilon} + \log\ \det{}_{s=0}^{1}(D_\mu D^\mu)_{D=6-2\epsilon}$$

$$\int\ [da_\mu]\exp\ \mathrm{Tr}\left(\int\ d^{6-\epsilon}x a_\mu\left(g^{\mu\nu}D^2 + (\Sigma_{\rho\sigma}F^{\rho\sigma})^{\mu\nu}\right)a_\nu\right)$$

$$\frac{1}{\xi^2}(D_\mu a^\mu)^2$$

$$4\mathrm{Tr}_{s=0}(1) - \frac{1}{2}\mathrm{Tr}_{s=\frac{1}{2}}(1) = 4 - \frac{1}{2}8 = 0$$

$$\frac{1}{2}\mathrm{Tr}_{s=\frac{1}{2}}(1) - \mathrm{Tr}_{s=1}(1) + 2\mathrm{Tr}_{s=0}(1) = \frac{1}{2}8 - 6 + 2 = 0$$

$$\mathrm{Tr}(\sigma) = \mathrm{Tr}(\Sigma) = 0$$

$$\mathrm{Tr}_V(\Sigma^{\mu\nu}\Sigma^{\rho\kappa}) = \frac{8}{D_{\text{spin}}}\mathrm{Tr}_S(\sigma^{\mu\nu}\sigma^{\rho\kappa}) = 2(g^{\mu\rho}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\rho})$$

$$\mathrm{Tr}_V\left(\Sigma^{\lambda\sigma}\Sigma^{\mu\nu}\Sigma^{\rho\kappa}\right) = \frac{8}{D_{\text{spin}}}\mathrm{Tr}_S\left(\sigma^{\lambda\sigma}\sigma^{\mu\nu}\sigma^{\rho\kappa}\right)$$

$$k_i = \left(E(m_i), 0, \ldots, k_3^i\right)$$
$$k_j = \left(E(m_j), 0, \ldots, -k_3^j\right)$$

$$q = (q_0, q_1, 0, q_3)$$

$$\tilde{k}_\alpha^i \tilde{k}_{\dot\alpha}^i = k_{\alpha\dot\alpha}^i + zn_{\alpha\dot\alpha} - \frac{k_i^2}{2q\cdot(k_i + zn)}q_\alpha q_{\dot\alpha}.$$

$$\tilde{k}_\alpha^i \to \begin{pmatrix}0\\1\end{pmatrix} \qquad \tilde{k}_{\dot\alpha}^i \to \begin{pmatrix}2z\\k_0^i - k_3\end{pmatrix}$$
$$\tilde{k}_\alpha^j \to \begin{pmatrix}2z\\k_0^j + k_3\end{pmatrix} \qquad \tilde{k}_{\dot\alpha}^j \to \begin{pmatrix}0\\1\end{pmatrix}$$

$$e_i^+ \sim e_i^- \sim \frac{1}{z}$$

$$e_i^- \sim e_i^+ \sim z$$

$$\{Q_I, \bar{Q}_J\} = \gamma_\mu k^\mu \delta_{IJ} + \epsilon_{IJ}(Z_1\mathrm{I} + \mathrm{i}Z_2\gamma_5)$$

$$Q = \frac{1}{\sqrt{2}}(Q_1 + \mathrm{i}Q_2)$$

$$\{Q, \bar{Q}\} = \gamma_\mu k^\mu + (Z_1\mathrm{I} + \mathrm{i}Z_2\gamma_5)$$

$$u\left(k,\tfrac{1}{2}\right) = \begin{pmatrix} k^{\flat}_{\dot{\alpha}} \\ (Z_1 - \mathrm{i}Z_2)\frac{q^{\alpha}}{\langle q k^{\flat}\rangle} \end{pmatrix} \qquad \overline{u\left(k,\tfrac{1}{2}\right)} = \left((Z_1 + \mathrm{i}Z_2)\frac{q^{\dot{\alpha}}}{[q k^{\flat}]} \;\; k^{\flat}_{\alpha} \right)$$

$$u\left(k,-\tfrac{1}{2}\right) = \begin{pmatrix} (Z_1 + \mathrm{i}Z_2)\frac{q_{\dot{\alpha}}}{[q k^{\flat}]} \\ k^{\flat,\alpha} \end{pmatrix} \qquad \overline{u\left(k,-\tfrac{1}{2}\right)} = \left(k^{\flat,\dot{\alpha}} \;\; (Z_1 - \mathrm{i}Z_2)\frac{q_{\alpha}}{\langle q k^{\flat}\rangle} \right)$$

$$u\left(q,-\frac{1}{2}\right) = \begin{pmatrix} q_{\dot{\alpha}} \\ 0 \\ 0 \end{pmatrix} \quad \overline{u\left(q,-\frac{1}{2}\right)} = \begin{pmatrix} 0 & 0 & q_{\alpha} \end{pmatrix}$$

$$u\left(q,\frac{1}{2}\right) = \begin{pmatrix} 0 \\ 0 \\ q^{\alpha} \end{pmatrix} \quad \overline{u\left(q,\frac{1}{2}\right)} = \begin{pmatrix} q^{\dot{\alpha}} & 0 & 0 \end{pmatrix}$$

$$u\left(k,\frac{1}{2}\right), u\left(k,-\frac{1}{2}\right), u\left(q,\frac{1}{2}\right), u\left(q,-\frac{1}{2}\right)$$

$$\overline{u\left(k,\frac{1}{2}\right)}, \overline{u\left(k,-\frac{1}{2}\right)}, \overline{u\left(q,\frac{1}{2}\right)}, \overline{u\left(q,-\frac{1}{2}\right)}$$

$$u\left(k,\frac{1}{2}\right)\overline{u\left(k,\frac{1}{2}\right)} + u\left(k,-\frac{1}{2}\right)\overline{u\left(k,-\frac{1}{2}\right)} = \gamma_{\mu}k^{\mu} + (Z_1\mathrm{I} + \mathrm{i}Z_2\gamma_5)$$

$$u\left(q,\frac{1}{2}\right)\overline{u\left(q,\frac{1}{2}\right)} + u\left(q,-\frac{1}{2}\right)\overline{u\left(q,-\frac{1}{2}\right)} = \gamma_{\mu}q^{\mu}$$

$$\overline{u\left(k,\tfrac{1}{2}\right)}u\left(q,\tfrac{1}{2}\right) = -\overline{u\left(q,-\tfrac{1}{2}\right)}u\left(k,-\tfrac{1}{2}\right) = \langle k^{\flat} q\rangle$$

$$\overline{u\left(k,-\tfrac{1}{2}\right)}u\left(q,-\tfrac{1}{2}\right) = -\overline{u\left(q,\tfrac{1}{2}\right)}u\left(k,\tfrac{1}{2}\right) = [k^{\flat} q]$$

$$Q \equiv u\left(k,\frac{1}{2}\right)Q_{-} + u\left(k,-\frac{1}{2}\right)Q_{+} + u\left(q,\frac{1}{2}\right)\tilde{Q}_{-} + \left(q,-\frac{1}{2}\right)\tilde{Q}_{+}$$

$$\bar{Q} \equiv \bar{Q}_{+}\overline{u\left(k,\frac{1}{2}\right)} + \bar{Q}_{-}\overline{u\left(k,-\frac{1}{2}\right)} + \bar{\tilde{Q}}_{+}\overline{u\left(q,\frac{1}{2}\right)} + \bar{\tilde{Q}}_{-}\overline{\left(q,-\frac{1}{2}\right)}$$

$$Q_{-} = \frac{\overline{u\left(q,\frac{1}{2}\right)}Q}{\overline{u\left(q,\frac{1}{2}\right)}u\left(k,\frac{1}{2}\right)} \quad \bar{Q}_{-} = \frac{\bar{Q}u\left(q,-\frac{1}{2}\right)}{\overline{u\left(k,-\frac{1}{2}\right)}u\left(q,-\frac{1}{2}\right)}$$

$$Q_{+} = \frac{\overline{u\left(q,-\frac{1}{2}\right)}Q}{\overline{u\left(q,-\frac{1}{2}\right)}u\left(k,-\frac{1}{2}\right)} \quad \bar{Q}_{+} = \frac{\bar{Q}u\left(q,\frac{1}{2}\right)}{\overline{u\left(k,\frac{1}{2}\right)}u\left(q,\frac{1}{2}\right)}$$

$$\tilde{Q}_{-} = \frac{\overline{u\left(k,\frac{1}{2}\right)}Q}{\overline{u\left(k,\frac{1}{2}\right)}u\left(q,\frac{1}{2}\right)} \quad \bar{Q}_{-} = \frac{\bar{Q}u\left(k,-\frac{1}{2}\right)}{\overline{u\left(q,-\frac{1}{2}\right)}u\left(k,-\frac{1}{2}\right)}$$

$$\tilde{Q}_{+} = \frac{\overline{u\left(k,-\frac{1}{2}\right)}Q}{\overline{u\left(k,-\frac{1}{2}\right)}u\left(q,-\frac{1}{2}\right)} \quad \bar{Q}_{+} = \frac{\bar{Q}u\left(k,\frac{1}{2}\right)}{\overline{u\left(q,\frac{1}{2}\right)}u\left(k,\frac{1}{2}\right)}$$

$$\{Q,\bar{Q}\} = u\left(k,\frac{1}{2}\right)\overline{u\left(k,\frac{1}{2}\right)} + u\left(k,-\frac{1}{2}\right)\overline{u\left(k,-\frac{1}{2}\right)}$$

$$\{Q_{-},\bar{Q}_{+}\} = 1$$
$$\{Q_{+},\bar{Q}_{-}\} = 1$$

$$\tilde{Q}_- = \tilde{Q}_+ = \overline{\tilde{Q}}_- = \overline{\tilde{Q}}_+ = 0$$

$$|\eta_I, \iota_I\rangle = e^{\sum_I \eta_I Q_-^I + \iota_I \bar{Q}_-^I} | \uparrow\rangle$$

$$|\bar{\eta}_I, \bar{\iota}_I\rangle = e^{\sum_I \bar{\eta}_I Q_+^I + \bar{\iota}_I \bar{Q}_+^I} | \downarrow\rangle$$

$$|\eta_I, \iota_I\rangle = \int \ (\overline{dF}) e^{\sum_I (\eta_I \bar{\eta}_I + \iota_I \bar{\iota}_I)} |\bar{\eta}_I, \bar{\iota}_I\rangle$$

$$(\overline{dF}) = d\bar{\eta}_1 d\bar{\iota}_1 d\bar{\eta}_2 d\bar{\iota}_2$$

$$e^{\bar{\xi}_I Q^I} |\eta_I, \iota_I\rangle \ = e^{\iota_I \left(\bar{\xi}^I u\left(k, -\frac{1}{2}\right)\right)} \left|\eta_I + \left(\bar{\xi}_I u\left(k, \frac{1}{2}\right)\right), \iota_I\right\rangle$$

$$e^{\bar{Q}_I \xi^I} |\eta_I, \iota_I\rangle \ = e^{\eta_I \left(\overline{u\left(k, \frac{1}{2}\right)} \xi_I\right)} \left|\eta_I, \iota_I + \left(\overline{u\left(k, -\frac{1}{2}\right)} \xi_I\right)\right\rangle$$

$$e^{\bar{\xi}_I Q^I} |\bar{\eta}_I, \bar{\iota}_I\rangle \ = e^{\bar{\iota}_I \left(\bar{\xi}^I u\left(k, \frac{1}{2}\right)\right)} \left|\bar{\eta}_I + \left(\bar{\xi}_I u\left(k, -\frac{1}{2}\right)\right), \bar{\iota}_I\right\rangle$$

$$e^{\bar{Q}_I \xi^I} |\bar{\eta}_I, \bar{\iota}_I\rangle \ = e^{\bar{\eta}_I \left(\overline{u\left(k, -\frac{1}{2}\right)} \xi_I\right)} \left|\bar{\eta}_I, \bar{\iota}_I + \left(\overline{u\left(k, \frac{1}{2}\right)} \xi_I\right)\right\rangle$$

$$\mathcal{L} = \phi \,\Box\, \bar{\phi} + \chi \,\Box\, \bar{\chi} + m_\chi^2 \chi \bar{\chi} - \sqrt{\lambda} m_\chi \chi \bar{\phi}^2 + \sqrt{\lambda} m_\chi \bar{\chi} \phi^2$$

$$A_n(z) \sim \mathcal{O}\left(\frac{1}{z}\right)$$

$$S = \frac{1}{g^2} \int_{\mathbb{R}^3 \times \mathbb{S}_\beta^1} \frac{1}{2} \mathrm{tr}_F\left(F_{\mu\nu} F_{\mu\nu}\right) + i \mathrm{tr}_F\left(\bar{\lambda} \bar{\sigma}^\mu D_\mu \lambda\right)$$

$$S_{3-D}^{\text{dark particle}} \ = \frac{\beta}{g^2(\beta)} \int_{\mathbb{R}^3} \left(\frac{1}{2} \mathrm{tr}_F\left(F_{ij} F_{ij}\right) + \mathrm{tr}_F (D_i A_4)^2 + g^2 V(\boldsymbol{A}_4) + \mathcal{O}(g^4)\right),$$

$$V(\boldsymbol{A}_4) = \frac{4T^4}{\pi^2} \sum_{\boldsymbol{\beta}^+} \sum_{n=1} \frac{-1 + n_f (-1)^n}{n^4} \cos\left[\frac{n \boldsymbol{A}_4 \cdot \boldsymbol{\beta}}{T}\right]$$

$$\langle \boldsymbol{A}_4 \rangle_{(a)} \beta \equiv \boldsymbol{\Phi}_0 = 2\pi \boldsymbol{\omega}_a, a = 0,1,2, \dots, N-1$$

$$\mathrm{tr}_F\left[e^{i \boldsymbol{\Phi}_0 \cdot \boldsymbol{H}}\right]\Big|_{\boldsymbol{\Phi}_0 = 2\pi \boldsymbol{\omega}_a} = N e^{-i \frac{2\pi a}{N}},$$

$$\boldsymbol{A}_4^{DW}(z) = T \boldsymbol{\Phi}^{DW}(z), \boldsymbol{\Phi}^{DW}(z \to -\infty) = 0, \boldsymbol{\Phi}^{DW}(z \to +\infty) = 2\pi \boldsymbol{\omega}_k, k > 0.$$

$$A_4^{DW(k)}(z) = T Q^{(k)}(z) \tilde{H}^{N-k}$$

$$\tilde{H}^{N-k} \ = \frac{1}{\sqrt{kN(N-k)}} \mathrm{diag}[\underbrace{k, k, \dots, k}_{k-k \text{ times}}, \underbrace{k-N, k-N, \dots, k-N}_{k \text{ times}}],$$

$$Q^{(k)}(z \to -\infty) = 0, Q^{(k)}(z \to +\infty) = -2\pi \sqrt{\frac{k(N-k)}{N}}$$

$$q(z) \equiv -\frac{1}{2\pi} \sqrt{\frac{N}{k(N-k)}} Q^{(k)}(z), z' \equiv \frac{T\sqrt{g^2 N}}{\pi^2} z$$

$$S_{k-\text{wall}} = 4\mathcal{A}T^2\frac{(N-k)k}{\sqrt{g^2N}}\int_{-\infty}^{\infty}dz'\left\{\left(\frac{\partial q(z')}{\partial z'}\right)^2 + \sum_{n=1}\frac{-1+(-1)^n}{n^4}\cos\left(2\pi nq(z')\right)\right\}$$

$$A_4^{DW(k)}(-\infty) = \text{diag}[0,0,\dots,0]$$
$$A_4^{DW(k)}(+\infty) = 2\pi T\text{diag}[\underbrace{-\frac{k}{N},-\frac{k}{N},\dots,-\frac{k}{N}}_{N-k\text{ times}},\underbrace{1-\frac{k}{N},1-\frac{k}{N},\dots,1-\frac{k}{N}}_{k\text{ times}}]$$

$$\text{tr}e^{i\beta A_4^{DW(k)}(-\infty)} = N$$

$$\text{tr}e^{i\beta A_4^{DW(k)}(\infty)} = Ne^{-i\frac{2\pi k}{N}}$$

$$T = \begin{bmatrix}\mathcal{T}^a_{(N-k)\times(N-k)} & E_{\boldsymbol{\beta}(N-k)\times k}\\ E_{-\boldsymbol{\beta}k\times(N-k)} & \mathcal{T}^A_{k\times k}\end{bmatrix}$$

$$\lambda = \lambda^{N-k}\tilde{H}^{N-k} + \lambda^a\mathcal{T}^a + \lambda^A\mathcal{T}^A + \sum_{cc'}\lambda^{\boldsymbol{\beta}_{cc'}}E_{\boldsymbol{\beta}_{cc'}} + \lambda^{-\boldsymbol{\beta}_{cc'}}E_{-\boldsymbol{\beta}_{c'c}}$$
$$A_\mu = A_\mu^{N-k}\tilde{H}^{N-k} + A_\mu^a\mathcal{T}^a + A_\mu^A\mathcal{T}^A + \sum_{cc'}A_\mu^{\boldsymbol{\beta}_{cc'}}E_{\boldsymbol{\beta}_{cc'}} + A_\mu^{-\boldsymbol{\beta}_{cc'}}E_{-\boldsymbol{\beta}_{cc'}}$$

$$T\gamma^{(N-k)} = T\sqrt{\frac{N}{k(N-k)}}$$

$$L_{k-\text{wall}} = i\text{tr}\bar{\psi}_+\left(\partial_-\psi_+ - i\gamma^{(N-k)}A_-^{N-k}\psi_+ - iA_-^a\mathcal{T}^a\psi_+ + i\psi_+A_-^A\mathcal{T}^A\right)$$
$$+i\text{tr}\bar{\psi}_-\left(\partial_+\psi_- + i\gamma^{(N-k)}A_+^{N-k}\psi_- - iA_+^A\mathcal{T}^A\psi_- + i\psi_-A_+^a\mathcal{T}^a\right)$$

$$\mathcal{D}\psi \to \mathcal{J}\mathcal{D}\psi,\text{ where }\mathcal{J} \equiv \exp\left[i2\chi(N-k)k\gamma^{(N-k)}\oint\frac{F_{12}^{N-k}dx^1dx^2}{2\pi}\right]$$

$$\oint\frac{F_{12}^{N-k}dx^1dx^2}{2\pi} = \gamma^{(N-k)}n, n\in\mathbb{Z}$$

$$q_1 \sim \left(\frac{k}{N}\gamma^{(N-k)},\square,\mathbf{1}\right)$$

$$q_2 \sim \left(\left(\frac{k-N}{N}\right)\gamma^{(N-k)},\mathbf{1},\square\right)$$

$$\mathcal{J} = e^{2i\chi Nn}$$

$$\mathbb{Z}_{2N}^{d\chi}: \psi_\pm \to e^{i\frac{\pi}{N}}\psi_\pm.$$

$$\mathcal{A}_i(x_1,x_2) = \frac{\pi n_{12}\epsilon_{ij}x_j}{L_1L_2}\boldsymbol{u}\cdot\boldsymbol{H}$$

$$\mathcal{A}_i(x_1+L_1,x_2) = \Omega_1(x_2)[\mathcal{A}_i(x_1,x_2)+i\partial_i]\Omega_1^\dagger(x_2),\quad \Omega_1(x_2)\equiv e^{i\frac{\pi n_{12}x_2}{L_2}\boldsymbol{u}\cdot\boldsymbol{H}}$$
$$\mathcal{A}_i(x_1,x_2+L_2) = \Omega_2(x_1)[\mathcal{A}_i(x_1,x_2)+i\partial_i]\Omega_2^\dagger(x_1),\quad \Omega_2(x_1)\equiv e^{-i\frac{\pi n_{12}x_1}{L_1}\boldsymbol{u}\cdot\boldsymbol{H}}$$

$$\Omega_1(L_2)\Omega_2(0) = z\Omega_2(L_1)\Omega_1(0)$$
$$\Longrightarrow e^{i2\pi n_{12}\boldsymbol{u}\cdot\boldsymbol{H}} = z \Longrightarrow \begin{cases}\boldsymbol{u}\in\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_{N-1}\},\text{ if }z=1\\ \boldsymbol{u}\in\{\boldsymbol{\omega}_1,\dots,\boldsymbol{\omega}_{N-1}\},\text{ if }z\in\mathbb{Z}_N, z\neq 1\end{cases}$$

$$\oint\frac{\mathcal{F}_{12}dx^1dx^2}{2\pi} = -n_{12}\boldsymbol{u}\cdot\boldsymbol{H}$$

$$\mathrm{tr}H^a\tilde{H}^{N-k} = -\sqrt{\frac{N}{k(N-k)}}\sum_{A=N-k+1}^{N}\lambda^{aA}.$$

$$\oint\frac{F_{12}^{N-k}dx^1dx^2}{2\pi} = -n_{12}\boldsymbol{u}\cdot\mathrm{tr}\boldsymbol{H}\tilde{H}^{N-k}$$
$$= n_{12}\sqrt{\frac{N}{k(N-k)}}\sum_{A=N-k+1}^{N}\sum_{a=1}^{N-1}(\boldsymbol{v}^A)^a(\boldsymbol{u})^a = n_{12}\gamma^{(N-k)}\sum_{A=N-k+1}^{N}\boldsymbol{v}^A\cdot\boldsymbol{u}$$

$$\oint\frac{F_{12}^{N-k}dx^1dx^2}{2\pi} = \gamma^{(N-k)}\times n, n\in\mathbb{Z}$$

$$\mathcal{J} = e^{i\frac{2\pi}{N}k(N-k)\gamma^{(N-k)}\alpha}$$

$$\alpha = \oint\frac{F_{12}^{N-k}dx^1dx^2}{2\pi}$$

$$\mathcal{J} = e^{i\frac{2\pi}{N}p(N-k)} = e^{-i\frac{2\pi}{N}kp}$$

$$S_{5-D} = i\frac{2\pi}{N}\int_{M_5(\partial M_5=M_4)}\frac{2NA^{(1)}}{2\pi}\wedge\frac{NB^{(2)}}{2\pi}\wedge\frac{NB^{(2)}}{2\pi},$$

$$\delta_\chi S_{5-D} = i\frac{2\pi}{N}\frac{2N\phi^{(0)}|_{M_4}}{2\pi}\int_{M_4}\frac{NB^{(2)}}{2\pi}\wedge\frac{NB^{(2)}}{2\pi} = i\frac{2\pi}{N}m,$$

$$\int_{M_4}\frac{NB^{(2)}}{2\pi}\wedge\frac{NB^{(2)}}{2\pi} = m\in\mathbb{Z}$$

$$S_{3-D} = i\frac{2\pi k}{N}\int_{M_3(\partial M_3=M_2)}\frac{2NA^{(1)}}{2\pi}\wedge\frac{NB^{(2)}}{2\pi}.$$

$$\delta_\chi S_{3-D} = i\frac{2\pi k}{N}\frac{2N\phi^{(0)}|_{M_2}}{2\pi}\int_{M_2}\frac{NB^{(2)}}{2\pi} = i\frac{2\pi kp}{N},$$

$$\int_{M_2}\frac{NB^{(2)}}{2\pi} = p$$

$$q_1\sim\left(\frac{k}{N}\gamma^{(N-k)},\square,\mathbf{1}\right), q_2\sim\left(\left(\frac{k-N}{N}\right)\gamma^{(N-k)},\mathbf{1},\square\right).$$

$$W_{SU(N)}\simeq W_{q_1}+W_{q_2}$$

$$W_{q_1}(x_2) = \mathrm{tr}\left[e^{i\int_0^{L_1}A_{1[N-k]}dx^1}\Omega_{1[N-k]}(x_2)\right]e^{i\frac{k}{N}\gamma^{(N-k)}}\int_0^{L_1}A_1^{N-k}dx^1e^{i\omega_1(x_2)\frac{k}{N}\gamma^{(N-k)}}$$
$$W_{q_2}(x_2) = \mathrm{tr}\left[e^{i\int_0^{L_1}A_{1[k]}dx^1}\Omega_{1[k]}(x_2)\right]e^{i\frac{k-N}{N}\gamma^{(N-k)}\int_0^{L_1}A_1^{N-k}dx^1}e^{i\omega_1(x_2)\frac{k-N}{N}\gamma^{(N-k)}}$$

$$W_{q_1(q_2)}(x_2)\rightarrow e^{i\frac{2\pi}{N}p_{(1)}}W_{q_1(q_2)}(x_2)$$

$$\bar{\psi}_+^a\psi_{-b} = \mu h_b^a e^{-i\sqrt{\frac{4\pi}{N-1}}\phi},$$

$$\left\langle e^{-i\sqrt{\frac{4\pi}{N-1}}\phi(x)}e^{i\sqrt{\frac{4\pi}{N-1}}\phi(y)}\right\rangle$$

$$\langle\mathrm{tr}h^\dagger(x)\mathrm{tr}h(y)\rangle$$

$$\langle \mathrm{tr}\bar{\psi}_+(x)\psi_-(x)\mathrm{tr}\bar{\psi}_-(y)\psi_+(y)\rangle \sim \varphi$$

$$\langle \mathrm{tr}\bar{\psi}_+\psi_-\rangle \neq 0: \mathbb{Z}_{2N}^{d\chi} \to \mathbb{Z}_2$$

$$S_{3-D} = i\frac{2\pi}{N}\int_{M_3(\partial M_3 = M_2)} \frac{2NA^{(1)}}{2\pi} \wedge \frac{NB^{(2)}}{2\pi}$$

$$\delta_{\mathbb{Z}_{2N}} A^{(1)} = d\phi^{(0)}\big|_{M_2} = \frac{2\pi}{2N}\int_{M_2} \frac{NB^{(2)}}{2\pi} = p$$

$$S_{2-D} = i\frac{N}{2\pi}\int_{M_2} \varphi^{(0)} da^{(1)}$$

$$\left\langle e^{i\varphi(x)} e^{i\oint_C a^{(1)}} \right\rangle = e^{i\frac{2\pi}{N} l_{x,C}}$$

$$e^{i\varphi} \to e^{i\frac{2\pi}{N}} e^{i\varphi},\ e^{i\oint a^{(1)}} \to e^{i\frac{1}{N}\oint \epsilon^{(1)}} e^{i\oint a^{(1)}} = e^{i\frac{2\pi\mathbb{Z}}{N}} e^{i\oint a^{(1)}}$$

$$a(t) \equiv \oint_{\mathbb{S}_1} a^{(1)}$$

$$S_{\mathbb{R}_t \times \mathbb{S}_1} = \frac{N}{2\pi}\int dt\varphi\frac{da}{dt}$$

$$e^{i\int_{\mathbb{R}_t} a^{(1)}} \left\langle e^{i\varphi(x)} e^{i\oint\ c a^{(1)}} \right\rangle = e^{i\frac{2\pi}{N} l_{x,C}}$$

$$S_{2-D} = i\frac{2\pi}{N}\int_{M_2} \frac{N\varphi^{(0)}}{2\pi} \wedge \frac{N\left(da^{(1)} - B^{(2)}\right)}{2\pi}$$

$$\delta_{\mathbb{Z}_N^{d\chi}} S_{2-D} = i\frac{2\pi}{N}\int_{M_2} \frac{N\left(da^{(1)} - B^{(2)}\right)}{2\pi} = -i\frac{2\pi p}{N},$$

$$\mathcal{J} \equiv \exp\left[i2n_f\chi(N-k)k\gamma^{(N-k)} \oint \frac{F_{12}^{N-k}dx^1dx^2}{2\pi}\right]$$

$$\bar{\psi}_+^{ai}\psi_{-bj} = \mu h_b^a g_j^i e^{-i\sqrt{\frac{4\pi}{n_f(N-1)}}\phi}$$

$$\left\langle e^{-i\sqrt{\frac{4\pi}{n_f(N-1)}}\phi(x)} e^{i\sqrt{\frac{4\pi}{n_f(N-1)}}\phi(y)} \right\rangle$$

$$\langle \mathrm{tr}h^\dagger(x)\mathrm{tr}h(y)\rangle$$

$$\langle g_i^j(x) g_k^l(y)\rangle$$

$$\langle g_i^j(x) g_k^l(y)\rangle = \frac{\delta_i^l \delta_k^j}{[M|x-y|]^{\frac{n_f^2-1}{n_f(n_f+N-1)}}}$$

$$\mathcal{O}_j^{(1)i} \equiv \bar{\psi}_+^{ai}\psi_{-aj}$$

$$\left\langle \mathcal{O}_j^{(1)\dagger i}(x)\mathcal{O}_j^{(1)i}(y)\right\rangle \to 0 \text{ as } |x-y| \to \infty$$

$$\langle \bar{\psi}_+^{a\ i}\psi_{-aj}\rangle = 0$$

$$\mathcal{O}^{(2)}(x) \equiv \det_{i,j}\bar{\psi}_+^{ai}(x)\psi_{-aj}(x)$$

$$H^a \ = \mathrm{diag}[\lambda^{a1},\dots,\lambda^{aN}] = \frac{1}{\sqrt{a(a+1)}}\mathrm{diag}[\underbrace{1,1,\dots,1}_{a\text{ times}},-a,\ \underbrace{0,0,\dots0}_{N-1-a\text{ times}}\ ]$$

$$\lambda^{aA} \ \equiv \frac{1}{\sqrt{a(a+1)}}(\theta^{aA}-a\delta_{a+1,A}), a=1,\dots,N-1, A=1,\dots,N, \theta^{aA} \equiv \begin{cases}1, a \geq A\\ 0, a < A\end{cases}$$

$$(\boldsymbol{\nu}^A)_a = \lambda^{aA}, \boldsymbol{\nu}^A\cdot\boldsymbol{\nu}^B \equiv \sum_{a=1}^{N-1}\lambda^{aA}\lambda^{aB} = \delta^{AB}-\frac{1}{N}, \sum_{A=1}^{N}\lambda^{aA}\lambda^{bA} = \delta^{ab}$$

$$\boldsymbol{\omega}^a \ = \sum_{A=1}^{a}\boldsymbol{\nu}^A, a=1,\dots N-1,$$

$$\boldsymbol{\alpha}^a \ = \boldsymbol{\nu}^a-\boldsymbol{\nu}^{a+1}, a=1,\dots N-1.$$

$$\boldsymbol{\beta}^{AB} \equiv \boldsymbol{\nu}^A-\boldsymbol{\nu}^B, A,B=1,\dots N$$

$$\boldsymbol{\omega}^a\cdot\boldsymbol{\omega}^b \ = \min(a,b)-\frac{ab}{N}$$

$$\boldsymbol{\omega}^b\cdot\boldsymbol{H} \ = \sum_{a=1}^{N-1}(\boldsymbol{\omega}^b)_a H^a = \mathrm{diag}[\underbrace{1-\frac{b}{N},1-\frac{b}{N},\dots,1-\frac{b}{N}}_{b\text{ times}},\underbrace{-\frac{b}{N},-\frac{b}{N},\dots-\frac{b}{N}}_{N-b\text{ times}}]$$

$$\boldsymbol{\omega}^b\cdot\boldsymbol{\beta}^{AB} \ = \sum_{k=1}^{b}\delta^{kA}-\delta^{kB} = \begin{cases}0, & b<A\\ 1, & A\leq b<B\\ 0, & b\geq B\end{cases}$$

$$\tilde{H}^{N-k} = \frac{1}{\sqrt{kN(N-k)}}\mathrm{diag}[\underbrace{k,k,\dots,k}_{k-k\text{ times}},\underbrace{k-N,k-N,\dots,k-N}_{k\text{ times}}]$$

$$T = \begin{bmatrix}\mathcal{T}^a_{(N-k)\times(N-k)} & E_{\boldsymbol{\beta}(N-k)\times k}\\ E_{-\boldsymbol{\beta}k\times(N-k)} & \mathcal{T}^A_{k\times k}\end{bmatrix}$$

$$\left(E_{\boldsymbol{\beta}_{AA'}}\right)_{BB'} = \delta_{AB}\delta_{A'B'}, B=1,\dots,N-k, B'=1,\dots,k.$$

$$\left(E_{-\boldsymbol{\beta}_{AA'}}\right)_{B'B} = \delta_{AB}\delta_{A'B'}, B=1,\dots,N-k, B'=1,\dots,k$$

$$\mathrm{tr}\left(E_{\boldsymbol{\beta}_{CC'}}E_{-\boldsymbol{\beta}_{BB'}}\right) = \delta_{CB}\delta_{C'B'}$$

$$\mathrm{tr}\left(E_{\boldsymbol{\beta}_{CC'}}\mathcal{T}^A E_{-\boldsymbol{\beta}_{BB'}}\right) = \delta_{CB}\mathcal{T}^A_{C'B'}$$

$$\mathrm{tr}\left(E_{-\boldsymbol{\beta}_{CC'}}\mathcal{T}^a E_{\boldsymbol{\beta}_{BB'}}\right) = \delta_{C'B'}\mathcal{T}^a_{CB}.$$

$$\left[\tilde{H}^{N-k},E_{\pm\boldsymbol{\beta}_{AA'}}\right] = \pm\gamma^{(N-k)}E_{\pm\boldsymbol{\beta}_{AA'}}, \gamma^{(N-k)} \equiv \sqrt{\frac{N}{k(N-k)}}$$

$$\left(\left[\mathcal{T}^a,E_{\boldsymbol{\beta}_{AA'}}\right]\right)_{BB'} \equiv \left(\mathcal{T}^a E_{\boldsymbol{\beta}_{AA'}}\right)_{BB'} = \sum_C(\mathcal{T}^a)_{BC}\left(E_{\boldsymbol{\gamma}_{AA'}}\right)_{CB'}$$

$$\left(\left[\mathcal{T}^a,E_{-\boldsymbol{\beta}_{AA'}}\right]\right)_{B'B} = -\left(E_{-\boldsymbol{\beta}_{AA'}}\mathcal{T}^a\right)_{B'B} = -\sum_C\left(E_{-\boldsymbol{\beta}_{AA'}}\right)_{B'C}(\mathcal{T}^a)_{CB}$$

$$\left(\left[\mathcal{T}^A,E_{\boldsymbol{\beta}_{AA'}}\right]\right)_{BB'} = -\left(E_{\boldsymbol{\beta}_{AA'}}\mathcal{T}^A\right)_{BB'} = -\sum_{C'}\left(E_{\boldsymbol{\beta}_{AA'}}\right)_{BC'}(\mathcal{T}^A)_{C'B'}$$

$$\left(\left[\mathcal{T}^A,E_{-\boldsymbol{\beta}}\right]\right)_{B'B} = \left(\mathcal{T}^A E_{-\boldsymbol{\beta}}\right)_{B'B} = \sum_{C'}(\mathcal{T}^A)_{B'C'}\left(E_{-\boldsymbol{\beta}_{AA'}}\right)_{C'B}$$

$$
\begin{aligned}
\partial_\mu\lambda - i[A_\mu,\lambda] =& \partial_\mu\lambda^{\boldsymbol{\beta}_{cc'}}E_{\boldsymbol{\beta}_{cc'}} + \partial_\mu\lambda^{-\boldsymbol{\beta}_{cc'}}E_{-\boldsymbol{\beta}_{cc'}} \\
& -i\gamma^{(N-k)}A_\mu^{N-k}\left(\lambda^{\boldsymbol{\beta}_{cc'}}E_{\boldsymbol{\beta}_{cc'}} - \lambda^{-\boldsymbol{\beta}_{cc'}}E_{-\boldsymbol{\beta}_{cc'}}\right) \\
& -iA_\mu^a\left(\mathcal{T}^a E_{\boldsymbol{\beta}_{cc'}}\lambda^{\boldsymbol{\beta}_{cc'}} - E_{-\boldsymbol{\beta}_{cc'}}\mathcal{T}^a\lambda^{-\boldsymbol{\beta}_{cc'}}\right) \\
& +iA_\mu^A\left(E_{\boldsymbol{\beta}_{cc'}}\mathcal{T}^A\lambda^{\boldsymbol{\beta}_{cc'}} - \mathcal{T}^A E_{-\boldsymbol{\beta}_{cc'}}\lambda^{-\boldsymbol{\beta}_{cc'}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\bar{\sigma}^\mu(\partial_\mu\lambda^{\boldsymbol{\beta}_{cc'}}E_{\boldsymbol{\beta}_{cc'}} - i\gamma^{(N-k)}A_\mu^{N-k}\lambda^{\boldsymbol{\beta}_{cc'}}E_{\boldsymbol{\beta}_{cc'}} \\
&\quad -iA_\mu^a\mathcal{T}^a E_{\boldsymbol{\beta}_{cc'}}\lambda^{\boldsymbol{\beta}_{cc'}} + iA_\mu^A E_{\boldsymbol{\beta}_{cc'}}\mathcal{T}^A\lambda^{\boldsymbol{\beta}_{cc'}}\Big) = 0 \\
&\bar{\sigma}^\mu\Big(\partial_\mu\lambda^{-\boldsymbol{\beta}_{cc'}}E_{-\boldsymbol{\beta}_{cc'}} + i\gamma^{(N-k)}A_\mu^{N-k}\lambda^{-\boldsymbol{\beta}_{cc'}}E_{-\boldsymbol{\beta}_{cc'}} \\
&\quad +iA_\mu^a E_{-\boldsymbol{\beta}_{cc'}}\mathcal{T}^a\lambda^{-\boldsymbol{\beta}_{cc'}} - iA_\mu^A\mathcal{T}^A E_{-\boldsymbol{\beta}_{cc'}}\lambda^{-\boldsymbol{\beta}_{cc'}}\Big) = 0
\end{aligned}
$$

$$0 = \bar{\sigma}^\mu\left(\partial_\mu\lambda^{\boldsymbol{\beta}_{BB'}} - iA_\mu^{N-k}\gamma^{(N-k)}\lambda^{\boldsymbol{\beta}_{BB'}} - iA_\mu^a\mathcal{T}_{BD}^a\lambda^{\boldsymbol{\beta}_{DB'}} + iA_\mu^A\lambda^{\boldsymbol{\beta}_{BD'}}\mathcal{T}_{D'B'}^A\right)$$

$$U_{N-k} = e^{i\omega^a\mathcal{T}^a}, U_k = e^{i\omega^A\mathcal{T}^A}$$

$$A_\mu^a\mathcal{T}^a \to U_{N-k}\left(A_\mu^a\mathcal{T}^a + i\partial_\mu\right)U_{N-k}^\dagger$$

$$A_\mu^A\mathcal{T}^A \to U_k\left(A_\mu^A\mathcal{T}^A + i\partial_\mu\right)U_k^\dagger$$

$$0 = \bar{\sigma}^\mu\left(\partial_\mu\lambda^{-\boldsymbol{\beta}_{BB'}} + iA_\mu^{N-k}\gamma^{(N-k)}\lambda^{-\boldsymbol{\beta}_{BB'}} + i\lambda^{-\boldsymbol{\beta}_{DB'}}\mathcal{T}_{DB}^a A_\mu^a - iA_\mu^A\mathcal{T}_{B'D'}^A\lambda^{-\boldsymbol{\beta}_{BD'}}\right)$$

$$U(1) \text{ as } \lambda^{\pm\beta} \to e^{\pm i\omega\gamma^{(N-k)}}\lambda^{\pm\beta}$$

$$A_\mu^{N-k} \to e^{i\omega}\left(A_\mu^{N-k} + i\partial_\mu\right)e^{-i\omega}$$

$$\lambda^{\pm\beta}\left(x^4, x^i\right) = \sum_{p\in\mathbb{Z}} \lambda_p^{\pm\beta}\left(x^i\right)e^{i2\pi Tp'x^4}$$

$$0 = \left[\sigma^3\partial_z + \sigma^0\left(2\pi p'T \mp TQ^{(k)}(z)\gamma^{(N-k)}\right)\right]\lambda_p^{\pm\boldsymbol{\beta}}$$

$$\lambda_p^{\pm\boldsymbol{\beta}}(z) = e^{\sigma^3\left[-2\pi p'Tz \pm T\gamma^{(N-k)}\int_0^z Q^{(k)}(z')dz'\right]}\lambda_p^{\pm\boldsymbol{\beta}}(0)$$

$$\int_0^{z\to-\infty} Q^{(k)}(z')dz'$$

$$\gamma^{(N-k)}\int_0^{z\to+\infty} Q^{(k)}(z')dz' \to -2\pi z$$

$$
\begin{aligned}
&\lambda_{p=-1,1}^{\boldsymbol{\beta}}:\psi_+,\circlearrowleft\left(\gamma^{(N-k)},\square,\bar{\square}\right)\leftsquigarrow(U(1),SU(N-k),SU(k)) \\
&\lambda_{p=0,2}^{-\boldsymbol{\beta}}:\psi_-,\circlearrowright\left(-\gamma^{(N-k)},\bar{\square},\square\right)\rightsquigarrow(U(1),SU(N-k),SU(k)),
\end{aligned}
$$

	λ^{β}	$\lambda^{-\beta}$
gauge $U(1)$	$\gamma^{(N-k)} = \sqrt{\frac{N}{k(N-k)}}$	$-\gamma^{(N-k)} = -\sqrt{\frac{N}{k(N-k)}}$
gauge $SU(k)$	$\bar{\square}$	$\square$
gauge $SU(N-k)$	$\square$	$\bar{\square}$
2-D chirality	left mover	right mover
global $U(1)_R$	1	1
2-D field	ψ_+	ψ_-

$$A_{[b]}(L_1, x_2) = \Omega_{1[b]}(x_2)\big(A_{[b]}(0, x_2) + id\big)\Omega^{\dagger}_{1[b]}(x_2),$$
$$A_{[b]}(x_1, L_2) = \Omega_{2[b]}(x_1)\big(A_{[b]}(x_1, 0) + id\big)\Omega^{\dagger}_{2[b]}(x_1),$$

$$\hat{\Omega}_{i[N-k]} = \begin{bmatrix} \Omega_{i[N-k]} & 0 \\ 0 & I_k \end{bmatrix},$$

$$\hat{\Omega}_{i[k]} = \begin{bmatrix} I_{N-k} & 0 \\ 0 & \Omega_{i[k]} \end{bmatrix},$$

$$\Omega_{i[1]} = \begin{bmatrix} e^{i2\pi\omega_i \frac{k}{N}\gamma^{(N-k)}} I_{N-k} & 0 \\ 0 & e^{i2\pi\omega_i \frac{k-N}{N}\gamma^{(N-k)}} I_k \end{bmatrix}.$$

$$A_{[b]} \rightarrow g_{[b]}\big(A_{[b]} + id\big)g^{\dagger}_{[b]}$$
$$\Omega_{1[b]}(x_2) \rightarrow g_{[b]}(L_1, x_2)\Omega_{1[b]}(x_2)g^{\dagger}_{[b]}(0, x_2)$$
$$\Omega_{2[b]}(x_1) \rightarrow g_{[b]}(x_1, L_2)\Omega_{2[b]}(x_1)g^{\dagger}_{[b]}(x_1, 0)$$

$$W_{[b]}(x_2) = \mathrm{tr}\left(e^{i\int_0^{L_1} A_{1[b]}(x_1,x_2)dx^1}\Omega_{1[b]}(x_2)\right)$$

$$\Omega_{1[N-k]}(L_2)\Omega_{2[N-k]}(0) = \begin{bmatrix} e^{i\frac{2\pi}{N-k}p_{N-k}} I_{N-k} & 0 \\ 0 & I_k \end{bmatrix} \Omega_{2[N-k]}(L_1)\Omega_{1[N-k]}(0)$$
$$\Omega_{1[k]}(L_2)\Omega_{2[k]}(0) = \begin{bmatrix} I_{N-k} & 0 \\ 0 & e^{i\frac{2\pi}{k}p_k} I_k \end{bmatrix} \Omega_{2[k]}(L_1)\Omega_{1[k]}(0)$$

$$\Omega_{i[N-k]} \rightarrow z_{i[N-k]}\Omega_{i[N-k]}, i = 1{,}2$$

$$A^{N-k}(L_1, x_2) = e^{i\omega_1(x_2)}\big(A^{N-k}(0, x_2) + id\big)e^{-i\omega_1(x_2)}$$
$$A^{N-k}(x_1, L_2) = e^{i\omega_2(x_1)}\big(A^{N-k}(x_1, 0) + id\big)e^{-i\omega_2(x_1)}$$
$$e^{i\omega_1(L_2)}e^{i\omega_2(0)} = e^{i2\pi\alpha}e^{i\omega_2(L_1)}e^{i\omega_1(0)}$$

$$\Omega_{1[1]}(L_2)\Omega_{2[1]}(0) = \begin{bmatrix} e^{i2\pi\alpha\frac{k}{N}\gamma^{(N-k)}} I_{N-k} & 0 \\ 0 & e^{i2\pi\alpha\frac{k-N}{N}\gamma^{(N-k)}} I_k \end{bmatrix} \Omega_{2[1]}(L_1)\Omega_{1[1]}(0)$$

$$e^{i\frac{2\pi}{N}p_N} I_N = \begin{bmatrix} e^{i\alpha\frac{2\pi k}{N}\gamma^{(N-k)}} I_{N-k} & 0 \\ 0 & e^{i\alpha\frac{2\pi(k-N)}{N}\gamma^{(N-k)}} I_k \end{bmatrix} \begin{bmatrix} e^{i\frac{2\pi}{N-k}p_{N-k}} I_{N-k} & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} I_{N-k} & 0 \\ 0 & e^{i\frac{2\pi}{k}p_k} I_k \end{bmatrix}$$
$$= \begin{bmatrix} e^{i\alpha\frac{2\pi k}{N}\gamma^{(N-k)}} e^{i\frac{2\pi}{N-k}p_{N-k}} I_{N-k} & 0 \\ 0 & e^{i\alpha\frac{2\pi(k-N)}{N}\gamma^{(N-k)}} e^{i\frac{2\pi}{k}p_k} I_k \end{bmatrix}$$

$$\alpha\gamma^{(N-k)}k(N-k) = p_N(N-k) - Np_{N-k} - N(N-k)m_4$$
$$p_k = p_N - p_{N-k} - km_5 - (N-k)m_4 (\mathrm{mod} N)$$

$$\alpha\gamma^{(N-k)}k(N-k) = (N-k)p_k - kp_{N-k} - m_4 + m_5$$
$$p_k + p_{N-k} = p_N - km_5 - (N-k)m_4 (\mathrm{mod} N)$$

$$\oint \frac{F^{N-k}_{12}dx^1dx^2}{2\pi} = \frac{1}{2\pi}[\omega_1(L_2) - \omega_1(0) - \omega_2(L_1) + \omega_2(0)] = \alpha$$

$$\alpha = -\gamma^{(N-k)}(p_{N-k} + (N-k)m_4),$$

$$\psi_{\pm}(L_1, L_2) = \Omega_1(L_2)\Omega_2(0) \circ \psi_{\pm}(0{,}0) = \Omega_2(L_1)\Omega_1(0) \circ \psi_{\pm}(0{,}0)$$

$$\Omega\circ\psi_+ = e^{i\omega\gamma^{(N-k)}}\Omega_{[N-k]}\psi_+\Omega^\dagger_{[k]}, \Omega\circ\psi_- = \epsilon^{-i\omega\gamma^{(N-k)}}\Omega_{[k]}\psi_+\Omega^\dagger_{[N-k]}$$

$$\begin{gathered}\psi_+(L_1,L_2)\\ = e^{i\omega_1(L_2)\gamma^{(N-k)}}e^{i\omega_2(0)\gamma^{(N-k)}}\Omega_{1[N-k]}(L_2)\Omega_{2[N-k]}(0)\psi_+(0,0)\Omega^\dagger_{2[k]}(0)\Omega^\dagger_{1[k]}(L_2)\\ = e^{i\omega_2(L_1)\gamma^{(N-k)}}e^{i\omega_1(0)\gamma^{(N-k)}}\Omega_{2[N-k]}(L_1)\Omega_{1[N-k]}(0)\psi_+(0,0)\Omega^\dagger_{1[k]}(0)\Omega^\dagger_{2[k]}(L_1)\end{gathered}$$

$$\Omega_i = \begin{bmatrix} e^{i2\pi\omega_i\frac{k}{N}\gamma^{(N-k)}}\Omega_{i[N-k]} & 0\\ 0 & e^{i2\pi\omega_i\frac{k-N}{N}\gamma^{(N-k)}}\Omega_{i[k]}\end{bmatrix}, i = 1,2$$

$$\begin{gathered}e^{i\omega_i}\rightarrow e^{i2\pi\zeta_{(i)}}e^{i\omega_i},\\ \Omega_i[N-k]\rightarrow e^{i\frac{2\pi}{N-k}q_{(i)N-k}}\Omega_{i[N-k]},\\ \Omega_{i[k]}\rightarrow e^{i\frac{2\pi}{k}q_{(i)k}}\Omega_{i[k]}\end{gathered}$$

$$z_{i[N]}\equiv e^{i\frac{2\pi}{N}p_{(i)}}$$

$$\Omega_i\rightarrow e^{i\frac{2\pi}{N}p_{(i)}}\Omega_i$$

$$\begin{aligned}e^{i2\pi\zeta_{(i)}\frac{k}{N}\gamma^{(N-k)}}e^{i\frac{2\pi}{N-k}q_{(i)N-k}} &= e^{i\frac{2\pi}{N}p_{(i)}}\\ e^{i2\pi\zeta_{(i)}\frac{k-N}{N}\gamma^{(N-k)}}e^{i\frac{2\pi}{k}q_{(i)k}} &= e^{i\frac{2\pi}{N}p_{(i)}}\end{aligned}$$

$$\alpha\rightarrow\zeta_{(i)}, p_{N-k}\rightarrow q_{(i)N-k}, p_k\rightarrow q_{(i)k}, p_N\rightarrow p_{(i)}$$

$$\begin{aligned}\zeta_{(i)}\gamma^{(N-k)} &= \frac{q_{(i)k}}{k}-\frac{q_{(i)N-k}}{N-k}\\ q_{(i)k}+q_{(i)N-k} &= p_{(i)}(\mathrm{mod}N)\end{aligned}$$

$$q_1\sim\left(,\frac{k}{N}\gamma^{(N-k)},\square,\mathbf{1}\right)$$

$$q_2\sim\left(\left(\frac{k-N}{N}\right)\gamma^{(N-k)},\mathbf{1},\square\right)$$

$$W_{q_1}(x_2) = \mathrm{tr}\left(e^{i\int_0^{L_1}A_{1[N-k]}(x_1,x_2)dx^1}\Omega_{1[N-k]}(x_2)\right)e^{i\frac{k}{N}\gamma^{(N-k)}\int_0^{L_1}A_1^{N-k}(x_1,x_2)dx^1}e^{i\omega_1(x_2)\frac{k}{N}\gamma^{(N-k)}}$$

$$\Omega_{1[N-k]}, e^{i\omega_1\frac{k}{N}\gamma^{(N-k)}}$$

$$W_{q_1}(x_2)\rightarrow e^{i2\pi\zeta_{(1)}\frac{k}{N}\gamma^{(N-k)}}e^{i\frac{2\pi}{N-k}q_{(1)N-k}}W_{q_1}(x_2) = e^{i\frac{2\pi}{N}p_{(1)}}W_{q_1}(x_2)$$

$$t(\boldsymbol{u}) = e^{i2\pi\boldsymbol{u}\cdot\boldsymbol{H}}$$

$$t(\boldsymbol{u})_{U(1)} = \left(e^{-i\frac{2\pi}{N-k}\sum_{A=N-k+1}^{N}\boldsymbol{u}\cdot\boldsymbol{v}^A}I_{N-k}, e^{i\frac{2\pi}{k}\sum_{A=N-k+1}^{N}\boldsymbol{u}\cdot\boldsymbol{v}^A}I_k\right).$$

$$\sum_{A=N-k+1}^{N}\boldsymbol{\alpha}^b\cdot\boldsymbol{v}^A = -\delta^{b,N-k}$$

$$t(\boldsymbol{\alpha}^{N-k})_{U(1)} = \left(e^{i\frac{2\pi}{N-k}}I_{N-k}, e^{-i\frac{2\pi}{k}}I_k\right), t(\boldsymbol{\alpha}^{b\neq N-k})_{U(1)} = I_N$$

$$t(\boldsymbol{u})_{SU(N-k)}\equiv e^{i2\pi u^a\mathrm{tr}(H^a\tilde{H}^{\tilde{a}})H^{\tilde{a}}}$$

$$t(\boldsymbol{u})^{\tilde{A}}_{SU(N-k)}\equiv e^{i2\pi u^a\mathrm{tr}(H^a\tilde{H}^{\tilde{a}})(H^{\tilde{a}})^{\tilde{A}}} = e^{i2\pi u^a\left(\boldsymbol{v}^{\tilde{B}}\right)^a\left(\boldsymbol{v}^{\tilde{B}}\right)^{\tilde{a}}\left(\boldsymbol{v}^{\tilde{A}}\right)^{\tilde{a}}}$$

$$\left(\boldsymbol{\nu}^{\tilde{B}}\right)^{\tilde{a}}\left(\boldsymbol{\nu}^{\tilde{A}}\right)^{\tilde{a}} = \delta^{\tilde{A}\tilde{B}} - \frac{1}{N-k}\left(\boldsymbol{\nu}^{\tilde{A}}\right)^{\tilde{a}}$$

$$t(\boldsymbol{u})^{\tilde{A}}_{SU(N-k)} = e^{i2\pi u^a\left(\boldsymbol{\nu}^{\tilde{A}}\right)^a - i\frac{2\pi}{N-k}u^a\sum_{\tilde{B}=1}^{N-k}\left(\boldsymbol{\nu}^{\tilde{B}}\right)^a},$$

$$t(\boldsymbol{\alpha}^b)^{\tilde{A}}_{SU(N-k)} = e^{i2\pi(\boldsymbol{\alpha}^b)^a\left(\boldsymbol{\nu}^{\tilde{A}}\right)^a - i\frac{2\pi}{N-k}(\boldsymbol{\alpha}_b)^a\sum_{\tilde{B}=1}^{N-k}\left(\boldsymbol{\nu}^{\tilde{B}}\right)^a} = e^{-i\frac{2\pi}{N-k}\sum_{\tilde{B}=1}^{N-k}(\boldsymbol{\nu}^b-\boldsymbol{\nu}^{b+1})\cdot\boldsymbol{\nu}^{\tilde{B}}}$$
$$= e^{-i\frac{2\pi}{N-k}\sum_{\tilde{B}=1}^{N-k}\left(\delta^{b\tilde{B}}-\delta^{b+1\tilde{B}}\right)} = e^{-i\frac{2\pi}{N-k}\delta^{bN-k}}$$

$$t(\boldsymbol{\omega}^b)^{\tilde{A}}_{SU(N-k)} = e^{i2\pi\sum_{B=1}^{b}\boldsymbol{\nu}^B\cdot\boldsymbol{\nu}^{\tilde{A}} - i\frac{2\pi}{N-k}\sum_{\tilde{B}=1}^{N-k}\sum_{B=1}^{b}\boldsymbol{\nu}^B\cdot\boldsymbol{\nu}^{\tilde{B}}}$$
$$= e^{i2\pi\sum_{B=1}^{b}\left(\delta^{B\tilde{A}}-\frac{1}{N}\right) - i\frac{2\pi}{N-k}\sum_{\tilde{B}=1}^{N-k}\sum_{B=1}^{b}\left(\delta^{B\tilde{B}}-\frac{1}{N}\right)}$$
$$= e^{-i\frac{2\pi}{N-k}\sum_{\tilde{B}=1}^{N-k}\sum_{B=1}^{b}\delta^{B\tilde{B}}} = e^{-i\frac{2\pi}{N-k}\min(b,N-k)}$$

$$\sum_{A=N-k+1}^{N}\sum_{B=1}^{b}\boldsymbol{\nu}^B\cdot\boldsymbol{\nu}^A = \sum_{A=N-k+1}^{N}\sum_{B=1}^{b}\left(\delta^{AB}-\frac{1}{N}\right) = \begin{cases} -\dfrac{kb}{N} & ,b \le N-k \\ (N-k)\left(\dfrac{b}{N}-1\right) & ,b > N-k \end{cases}$$

$$t(\boldsymbol{\omega}^b)_{U(1)} = \begin{cases} \left(e^{i\frac{2\pi}{N-k}\frac{k}{N}b}I_{N-k}, \quad e^{-i\frac{2\pi b}{N}}I_k\right) \\ \left(e^{-i\frac{2\pi}{N}b}I_{N-k}, \quad e^{i\frac{2\pi}{k}\frac{N-k}{N}(b-N)}I_k\right), \end{cases} ,b > N-k.$$

$$t(\boldsymbol{u})_{SU(k)} \equiv e^{i2\pi u^a \mathrm{tr}(H^a H' a')H^{a'}},$$

$$t(\boldsymbol{u})^{A'}_{SU(k)} = e^{i2\pi\left(\boldsymbol{u}\cdot\boldsymbol{\nu}^{N-k+B'}\right)\left(\boldsymbol{\nu}^{B'}\right)^{a'}\left(\boldsymbol{\nu}^{A'}\right)^{a'}} = e^{i2\pi\sum_{B'=1}^{k}\left(\boldsymbol{u}\cdot\boldsymbol{\nu}^{N-k+B'}\right)\left(\delta^{A'B'}-\frac{1}{k}\right)}$$
$$= e^{i2\pi\boldsymbol{u}\cdot\boldsymbol{\nu}^{N-k+A'} - i\frac{2\pi}{k}\sum_{B'=1}^{k}\boldsymbol{u}\cdot\boldsymbol{\nu}^{N-k+B'}}$$

$$t(\boldsymbol{\alpha}^b)^{A'}_{SU(k)} = e^{i2\pi(\boldsymbol{\nu}^b-\boldsymbol{\nu}^{b+1})\cdot\boldsymbol{\nu}^{N-k+A'} - i\frac{2\pi}{k}\sum_{B'=1}^{k}(\boldsymbol{\nu}^b-\boldsymbol{\nu}^{b+1})\cdot\boldsymbol{\nu}^{N-k+B'}}$$
$$= e^{-i\frac{2\pi}{k}\sum_{A=N-k+1}^{N}(\delta^{b,A}-\delta^{b+1,A})} = e^{i\frac{2\pi}{k}\delta^{b,N-k}}$$

$$t(\boldsymbol{\omega}^b)^{A'}_{SU(k)} = e^{i2\pi\sum_{p=1}^{b}\boldsymbol{\nu}^p\cdot\boldsymbol{\nu}^{N-k+A'} - i\frac{2\pi}{k}\sum_{B'=1}^{k}\sum_{p=1}^{b}\boldsymbol{\nu}^p\cdot\boldsymbol{\nu}^{N-k+B'}}$$
$$= e^{i2\pi\sum_{p=1}^{b}\left(\delta^{p,N-k+A'}-\frac{1}{N}\right) - i\frac{2\pi}{k}\sum_{B'=1}^{k}\sum_{p=1}^{b}\left(\delta^{p,N-k+B'}-\frac{1}{N}\right)} = \begin{cases} 1, b \le N-k \\ e^{-i\frac{2\pi}{k}(b-N)}, b > N-k \end{cases}$$
$$= e^{-i\frac{2\pi}{k}\sum_{A=N-k+1}^{N}\sum_{p=1}^{b}\delta^{p,A}}$$

$e^{i\left(\oint_C a^{(1)} - \int_\Sigma B^{(2)}\right)}$ $SU(N) \to U(1)\times SU(N-k)\times SU(k)$ $\dfrac{SU(N-k)\times SU(k)\times U(1)}{\mathbb{Z}_{N-k}\times\mathbb{Z}_k}$

$$\mathrm{tr}\left(H^a\tilde{H}^{\tilde{a}}\right) = \sum_{\tilde{B}=1}^{N-k}\left(\boldsymbol{\nu}^{\tilde{B}}\right)^a\left(\boldsymbol{\nu}^{\tilde{B}}\right)^{\tilde{a}}$$

$$\{D^a_\alpha, D^b_\beta\} = 0, \{D^a_\alpha, \bar{D}_{b\dot{\alpha}}\} = \delta^a_b i\partial_{\alpha\dot{\alpha}}$$

$$\nabla_\alpha = D_\alpha + \zeta Q_\alpha, \bar{\nabla}_{\dot{\alpha}} = \bar{Q}_{\dot{\alpha}} - \zeta\bar{D}_{\dot{\alpha}}$$

$$\Delta_\alpha = Q_\alpha - \frac{1}{\zeta}D_\alpha, \bar{\Delta}_{\dot{\alpha}} = \bar{D}_\alpha + \frac{1}{\zeta}\bar{Q}_{\dot{\alpha}}$$

$$\{\nabla_\alpha, \bar{\Delta}_{\dot{\alpha}}\} = \{\Delta_\alpha, \bar{\nabla}_{\dot{\alpha}}\} = 2i\partial_{\alpha\dot{\alpha}}$$

$$\overline{f(\zeta)} = \zeta^p f^*\left(-\frac{1}{\zeta}\right)$$

$$\oiiint \frac{d\zeta}{2\pi i} \to \oiiint d\zeta$$

$$\Delta_\alpha = \frac{1}{\zeta}\nabla_\alpha - \frac{2}{\zeta}D_\alpha, \bar{\Delta}_\alpha = \frac{1}{\zeta}\bar{\nabla}_\alpha + 2\bar{D}_\alpha$$

$$\oiiint d\zeta \int d^4x \int [d^4\theta]_P = \oiiint d\zeta \int d^4x D^2\bar{D}^2$$

$$\int d^4x \int [d^4\theta]_P \oiiint \frac{d\zeta}{\zeta}\bar{\Upsilon}\Upsilon$$

$$\oiiint d\zeta \int d^4x \int [d^4\theta]_C f(F) = \oiiint d\zeta \int d^4x D^2Q^2 f(F)$$

$$\frac{1}{2}\int d^4x \int [d^4\theta]_C \oiiint \frac{d\zeta}{\zeta}\mathrm{Tr}(\mathbb{W}\mathbb{W})$$

$$\{\mathbb{D}_\alpha, \mathbb{Q}_\beta\} = iC_{\alpha\beta}\overline{\mathbb{W}}, \{\mathbb{D}_\alpha, \overline{\mathbb{D}}_{\dot{\alpha}}\} = i\mathbb{V}_{\alpha\dot{\alpha}}, \{\mathbb{Q}_\alpha, \overline{\mathbb{Q}}_{\dot{\alpha}}\} = i\nabla_{\alpha\dot{\alpha}}$$

$$\nabla_\alpha = \mathbb{D}_\alpha + \zeta\mathbb{Q}_\alpha, \quad \bar{\nabla}_{\dot{\alpha}} = \overline{\mathbb{Q}}_{\dot{\alpha}} - \zeta\overline{\mathbb{D}}_{\dot{\alpha}}$$
$$\triangle_\alpha = \mathbb{Q}_\alpha - \frac{1}{\zeta}\mathbb{D}_\alpha, \quad \bar{\triangle}_{\dot{\alpha}} = \overline{\mathbb{D}}_\alpha + \frac{1}{\zeta}\overline{\mathbb{Q}}_{\dot{\alpha}}$$
$$\{\nabla_\alpha(\zeta_1), \nabla_\beta(\zeta_2)\} = i(\zeta_2 - \zeta_1)C_{\alpha\beta}\overline{\mathbb{W}}$$
$$\{\mathbb{V}_\alpha, [\partial_\zeta, \nabla_\beta]\} = iC_{\alpha\beta}\overline{\mathbb{W}}$$

$$\Upsilon \to e^{i\Lambda}\Upsilon$$
$$\bar{\Upsilon} \to \bar{\Upsilon}e^{-i\bar{\Lambda}}$$

$$e^V \to e^{i\bar{\Lambda}}e^V e^{-i\Lambda}$$

$$\tilde{\Upsilon} \equiv \Upsilon \to e^{i\Lambda}\tilde{\Upsilon}$$
$$\bar{\tilde{\Upsilon}} \equiv \bar{\Upsilon}e^V \to \bar{\tilde{\Upsilon}}e^{-i\Lambda}$$

$$\tilde{\Upsilon} \equiv e^V\Upsilon \to e^{i\bar{\Lambda}}\tilde{\Upsilon}$$
$$\bar{\tilde{\Upsilon}} \equiv \bar{\Upsilon} \to \bar{\tilde{\Upsilon}}e^{-i\bar{\Lambda}}.$$

$$e^V = e^{\bar{U}}e^U$$

$$e^U \to e^{iK}e^U e^{-i\Lambda}$$
$$e^{\bar{U}} \to e^{i\bar{\Lambda}}e^{\bar{U}}e^{-iK}$$

$$\tilde{\Upsilon} \equiv e^U\Upsilon \to e^{iK}\tilde{\Upsilon},$$
$$\bar{\tilde{\Upsilon}} \equiv \bar{\tilde{\Upsilon}}e^{\bar{U}} \to \bar{\tilde{\Upsilon}}e^{-iK}.$$

$$\nabla_\alpha = e^U\nabla_\alpha e^{-U} = e^{-\bar{U}}\nabla_\alpha e^{\bar{U}}$$
$$\bar{\nabla}_{\dot{\alpha}} = e^U\bar{\nabla}_{\dot{\alpha}}e^{-U} = e^{-\bar{U}}\bar{\nabla}_{\dot{\alpha}}e^{\bar{U}}$$

$$0 = \nabla_\alpha e^V = (\nabla_\alpha e^{\bar{U}})e^U + e^{\bar{U}}(\nabla_\alpha e^U)$$

$$\Gamma_\alpha(\zeta) = e^U(\nabla_\alpha e^{-U}) = e^{-\bar{U}}(\nabla_\alpha e^{\bar{U}}).$$

$$\mathbb{D}_\alpha = D_\alpha + \Gamma^1_\alpha,$$
$$\mathbb{Q}_\alpha = Q_\alpha + \Gamma^2_\alpha.$$

$$\overline{\mathbb{W}} = \frac{i}{2}(D^\alpha\Gamma^2_\alpha - Q^\alpha\Gamma^1_\alpha + \{\Gamma^{1\alpha}, \Gamma^2_\alpha\})$$

$$\{\nabla_\alpha, [e^{-U}\partial_\zeta e^{U}, \nabla_\beta]\} = iC_{\alpha\beta}e^{-U}\overline{\mathbb{W}}e^{U}$$
$$\{\nabla_\alpha, [e^{\bar{U}}\partial_\zeta e^{-\bar{U}}, \nabla_\beta]\} = iC_{\alpha\beta}e^{\bar{U}}\overline{\mathbb{W}}e^{-\bar{U}}$$

$$\mathcal{D}_\zeta = \partial_\zeta + A_\zeta = e^{-U}\partial_\zeta e^{U}$$
$$\overline{\mathcal{W}}(\zeta) = e^{-U}\overline{\mathbb{W}}e^{U}$$

$$\widetilde{\mathcal{D}}_\zeta = \partial_\zeta + \tilde{A}_\zeta = e^{\bar{U}}\partial_\zeta e^{-\bar{U}}$$
$$\overline{\widetilde{\mathcal{W}}}(\zeta) = e^{\bar{U}}\overline{\mathbb{W}}e^{-\bar{U}}$$

$$\overline{\mathcal{W}} = -i\nabla^2 A_\zeta,$$
$$\overline{\widetilde{\mathcal{W}}} = -i\nabla^2 \tilde{A}_\zeta,$$

$$\mathcal{W} = -i\bar{\nabla}^2 A_\zeta$$
$$\widetilde{\mathcal{W}} = -i\bar{\nabla}^2 \tilde{A}_\zeta$$

$$e^{-V}(\partial_\zeta e^{V}) = e^{-U}(\partial_\zeta e^{U}) + e^{-U}e^{-\bar{U}}(\partial_\zeta e^{\bar{U}})e^{U} = A_\zeta - e^{-V}\tilde{A}_\zeta e^{V}$$

$$\partial_\zeta(e^{V} - 1) = A_\zeta - \tilde{A}_\zeta + (e^{V} - 1)A_\zeta - \tilde{A}_\zeta(e^{V} - 1)$$

$$A_\zeta^{(1)} - \tilde{A}_\zeta^{(1)} = \partial_\zeta X$$
$$A_\zeta^{(n+1)} - \tilde{A}_\zeta^{(n+1)} = -XA_\zeta^{(n)} + \tilde{A}_\zeta^{(n)}X n \geq 1$$

$$A_\zeta^{(1)} = \Pi_+(\partial_\zeta X)$$
$$A_\zeta^{(2)} = -\Pi_+(X\Pi_+(\partial_\zeta X) + \Pi_-(\partial_\zeta X)X)$$
$$A_\zeta^{(3)} = \Pi_+\Big[X\Pi_+\big(X\Pi_+(\partial_\zeta X)\big) + \Pi_-\big(X\Pi_+(\partial_\zeta X)\big)X$$
$$+X\Pi_+\big(\Pi_-(\partial_\zeta X)X\big)\big) + \Pi_-(\Pi_-(\partial_\zeta X)X)X\Big]$$
$$A_\zeta^{(4)} = \cdots$$

$$\tilde{A}_\zeta^{(1)} = -\Pi_-(\partial_\zeta X)$$
$$\tilde{A}_\zeta^{(2)} = \Pi_-(X\Pi_+(\partial_\zeta X) + \Pi_-(\partial_\zeta X)X)$$
$$\tilde{A}_\zeta^{(3)} = -\Pi_-\Big[X\Pi_+\big(X\Pi_+(\partial_\zeta X)\big) + \Pi_-\big(X\Pi_+(\partial_\zeta X)\big)X$$
$$+X\Pi_+\big(\Pi_-(\partial_\zeta X)X)\big) + \Pi_-(\Pi_-(\partial_\zeta X)X)X\Big]$$
$$\tilde{A}_\zeta^{(4)} = \cdots$$

$$\Pi_+(\partial_\zeta X)(\zeta_0) = \int d\zeta_1 \frac{X_1}{\zeta_{10}^2}$$
$$\Pi_-(\partial_\zeta X)(\zeta_0) = -\int d\zeta_1 \frac{X_1}{\zeta_{01}^2}$$

$$A_\zeta^{(1)} = \int d\zeta_1 \frac{X_1}{\zeta_{10}^2}$$
$$\tilde{A}_\zeta^{(1)} = \int d\zeta_1 \frac{X_1}{\zeta_{01}^2}$$

$$A_\zeta^{(n)}(\zeta_0) = (-1)^{n+1}\int d\zeta_1 \ldots \int d\zeta_n \frac{X_1 \ldots X_n}{\zeta_{10}\zeta_{21}\cdots\zeta_{n,n-1}\zeta_{n0}}$$
$$\tilde{A}_\zeta^{(n)}(\zeta_0) = (-1)^{n+1}\int d\zeta_1 \ldots \int d\zeta_n \frac{X_1 \ldots X_n}{\zeta_{01}\zeta_{21}\cdots\zeta_{n,n-1}\zeta_{0n}}$$

$$A_\zeta^{(n+1)}(\zeta_0) = (-1)^{n+1}\int d\zeta_1 \frac{1}{\zeta_{10}}\Big[-X_1\int d\zeta_2 \ldots \int d\zeta_{n+1} \frac{X_2 \ldots X_{n+1}}{\zeta_{21}\zeta_{32}\cdots\zeta_{n+1,n}\zeta_{n+1,1}}$$
$$+\int d\zeta_2 \ldots \int d\zeta_{n+1} \frac{X_2 \ldots X_{n+1}}{\zeta_{12}\zeta_{32}\cdots\zeta_{n+1,n}\zeta_{1,n+1}} X_1\Big]$$

$$A_\zeta^{(n+1)}(\zeta_0) = (-1)^{n+1}\int d\zeta_1 \ldots \int d\zeta_{n+1}\frac{X_1 \ldots X_{n+1}}{\zeta_{21}\ldots\zeta_{n+1,n}}\left[-\frac{1}{\zeta_{10}\zeta_{n+1,1}}+\frac{1}{\zeta_{n+1,1}\zeta_{n+1,0}}\right]$$

$$A_\zeta^{(n+1)}(\zeta_0) = (-1)^{n+2}\int d\zeta_1 \ldots \int d\zeta_{n+1}\frac{X_1 \ldots X_{n+1}}{\zeta_{21}\ldots\zeta_{n+1,n}}\frac{1}{\zeta_{10}\zeta_{n+1,0}}$$

$$A_\zeta \rightarrow -i\partial_\zeta\Lambda + [i\Lambda, A_\zeta]$$
$$\tilde{A}_\zeta \rightarrow -i\partial_\zeta\bar{\Lambda} + [i\bar{\Lambda}, \tilde{A}_\zeta]$$

$$X \rightarrow i\bar{\Lambda} - i\Lambda + i\bar{\Lambda}X - Xi\Lambda$$

$$\delta A_\zeta^{(1)} = \Pi_+\big(i\partial_\zeta\bar{\Lambda} - i\partial_\zeta\Lambda + \partial_\zeta(i\bar{\Lambda}X - Xi\Lambda)\big) = -i\partial_\zeta\Lambda + \Pi_+\big(\partial_\zeta(i\bar{\Lambda}X) - \partial_\zeta(Xi\Lambda)\big)$$
$$\delta\tilde{A}_\zeta^{(1)} = -\Pi_-\big(i\partial_\zeta\bar{\Lambda} - i\partial_\zeta\Lambda + \partial_\zeta(i\bar{\Lambda}X - Xi\Lambda)\big) = -i\partial_\zeta\bar{\Lambda} - \Pi_-\big(\partial_\zeta(i\bar{\Lambda}X - Xi\Lambda)\big)$$
$$\delta A_\zeta^{(2)} = -\Pi_+\left[(i\bar{\Lambda} - i\Lambda)A^{(1)} - \tilde{A}_\zeta^{(1)}(i\bar{\Lambda} - i\Lambda) - Xi\partial_\zeta\Lambda + i\partial_\zeta\bar{\Lambda}X\right] + \mathcal{O}(X^2)$$
$$\delta\tilde{A}_\zeta^{(2)} = \Pi_-\left[(i\bar{\Lambda} - i\Lambda)A^{(1)} - \tilde{A}_\zeta^{(1)}(i\bar{\Lambda} - i\Lambda) - Xi\partial_\zeta\Lambda + i\partial_\zeta\bar{\Lambda}X\right] + \mathcal{O}(X^2)$$

$$\delta A_\zeta^{(2)} + \delta A_\zeta^{(1)} = \left[i\Lambda, A_\zeta^{(1)}\right] + \mathcal{O}(X^2)$$
$$\delta\tilde{A}_\zeta^{(2)} + \delta\tilde{A}_\zeta^{(1)} = \left[i\Lambda, A_\zeta^{(1)}\right] + \mathcal{O}(X^2)$$

$$\left[\delta A_\zeta^{(n)} + \delta A_\zeta^{(n-1)}\right]^{(n-1)} = \left[i\Lambda, A_\zeta^{(n-1)}\right]$$
$$\left[\delta\tilde{A}_\zeta^{(n)} + \delta\tilde{A}_\zeta^{(n-1)}\right]^{(n-1)} = \left[i\bar{\Lambda}, \tilde{A}_\zeta^{(n-1)}\right]$$

$$\delta A_\zeta^{(n)} - \delta\tilde{A}_\zeta^{(n)} = -\delta X A_\zeta^{(n-1)} - X\delta A_\zeta^{(n-1)} + \delta\tilde{A}_\zeta^{(n-1)}X + \tilde{A}_\zeta^{(n-1)}\delta X$$

$$\left[\delta A_\zeta^{(n)} - \delta\tilde{A}_\zeta^{(n)}\right]^{(n)} = -(i\bar{\Lambda}X - Xi\Lambda)A_\zeta^{(n-1)} + \tilde{A}_\zeta^{(n-1)}(i\bar{\Lambda}X - Xi\Lambda) + \left[-X\delta A_\zeta^{(n-1)} + \delta\tilde{A}_\zeta^{(n-1)}X\right]^{(n)}$$
$$\left[\delta A_\zeta^{(n+1)} - \delta\tilde{A}_\zeta^{(n+1)}\right]^{(n)} = -(i\bar{\Lambda} - i\Lambda)A_\zeta^{(n)} + \tilde{A}_\zeta^{(n)}(i\bar{\Lambda} - i\Lambda) + \left[-X\delta A_\zeta^{(n)} + \delta\tilde{A}_\zeta^{(n)}X\right]^{(n)}$$

$$\left[\delta A_\zeta^{(n+1)} + \delta A_\zeta^{(n)} - \delta\tilde{A}_\zeta^{(n+1)} - \delta\tilde{A}_\zeta^{(n)}\right]^{(n)}$$
$$= -(i\bar{\Lambda}X - Xi\Lambda)A_\zeta^{(n-1)} + \tilde{A}_\zeta^{(n-1)}(i\bar{\Lambda}X - Xi\Lambda) - (i\bar{\Lambda} - i\Lambda)A_\zeta^{(n)} + \tilde{A}_\zeta^{(n)}(i\bar{\Lambda} - i\Lambda) - X\left[\delta A_\zeta^{(n)} + \delta A_\zeta^{(n-1)}\right]^{(n)} + \left[\delta\tilde{A}_\zeta^{(n)} + \delta\tilde{A}_\zeta^{(n-1)}\right]^{(n)}X$$

$$-(i\bar{\Lambda}X - Xi\Lambda)A_\zeta^{(n-1)} + \tilde{A}_\zeta^{(n-1)}(i\bar{\Lambda}X - Xi\Lambda) - (i\bar{\Lambda} - i\Lambda)A_\zeta^{(n)} + \tilde{A}_\zeta^{(n)}(i\bar{\Lambda} - i\Lambda) - X\left[i\Lambda, A_\zeta^{(n-1)}\right] + \left[i\bar{\Lambda}, \tilde{A}_\zeta^{(n-1)}\right]X$$
$$= \left\{XA_\zeta^{(n-1)} - \tilde{A}_\zeta^{(n-1)}X - \tilde{A}_\zeta^{(n)}\right\}i\Lambda + i\Lambda\left\{-XA_\zeta^{(n-1)} + \tilde{A}_\zeta^{(n-1)}X - A_\zeta^{(n)}\right\} + i\Lambda A_\zeta^{(n)} + \tilde{A}_\zeta^{(n)}i\bar{\Lambda}$$

$$\left[\delta A_\zeta^{(n+1)} + \delta A_\zeta^{(n)} - \delta\tilde{A}_\zeta^{(n+1)} - \delta\tilde{A}_\zeta^{(n)}\right]^{(n)} = \left[i\Lambda, A_\zeta^{(n)}\right] - \left[i\bar{\Lambda}, \tilde{A}_\zeta^{(n)}\right]$$

$$\delta A_\zeta = \sum_{n=1}^{\infty}\delta A_\zeta^{(n)} = \sum_{n=1}^{\infty}\left[\delta A_\zeta^{(n)}\right]^{(n-1)} + \sum_{n=1}^{\infty}\left[\delta A_\zeta^{(n)}\right]^{(n)} = \left[\delta A_\zeta^{(1)}\right]^{(0)} + \sum_{n=1}^{\infty}\left[\delta A_\zeta^{(n+1)} + \delta A_\zeta^{(n)}\right]^{(n)}$$
$$= -i\partial_\zeta\Lambda + [i\Lambda, A_\zeta]$$

$$\bar{\nabla}^2\overline{\mathcal{W}} = 0$$

$$\delta S=\sum_{n=2}^{\infty}(-1)^n\int d^8\theta \oiiint d\zeta_1\dots d\zeta_n\frac{\mathrm{Tr}(\delta X_1X_2\dots X_n)}{\zeta_{21}\cdots\zeta_{n,n-1}\zeta_{1n}}$$

$$\delta S=\sum_{n=2}^{\infty}(-1)^n\int d^8\theta \oiiint d\zeta_1\dots d\zeta_n\frac{\mathrm{Tr}(\delta X_1X_2\dots X_n)}{\zeta_{21}\cdots\zeta_{n,n-1}}\left(\delta_{n1}-\frac{1}{\zeta_{n1}}\right)=$$
$$\sum_{n=2}^{\infty}(-1)^n\int d^8\theta \oiiint d\zeta_1\dots d\zeta_{n-1}\frac{\mathrm{Tr}(\delta X_1X_2\dots X_{n-1}X_1)}{\zeta_{21}\cdots\zeta_{1,n-1}}$$
$$+\sum_{n=2}^{\infty}\int d^8\theta \oiiint d\zeta_1\mathrm{Tr}\left(\delta X_1A_1^{(n-1)}\right)$$

$$\delta S=\sum_{n=2}^{\infty}\int d^8\theta \oiiint d\zeta_1\sum_{k=1}^{n-1}(-1)^{k+1}\mathrm{Tr}\left(\delta X_1A_1^{(n-k)}X_1^{k-1}\right)+$$
$$\sum_{n=2}^{\infty}(-1)^n\int d^8\theta\oint d\zeta_1d\zeta_2\frac{\mathrm{Tr}(\delta X_1X_1^{n-1})}{\zeta_{21}}\delta_{12}$$

$$\delta S=\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\int d^8\theta \oiiint d\zeta_1(-1)^{k+1}\mathrm{Tr}\left(\delta X_1A_1^{(n-k)}X_1^{k-1}\right)=$$
$$\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\int d^8\theta \oiiint d\zeta_1(-1)^{k}\mathrm{Tr}\left(\delta X_1A_1^{(n)}X_1^{k}\right)=\int d^8\theta \oiiint d\zeta_1\mathrm{Tr}(\delta X_1A_1(1+X_1)^{-1})$$

$$\delta S=\int d^8\theta \oiiint d\zeta_1\mathrm{Tr}(e^{-V_1}\delta e^{V_1}A_1)$$

$$S=\frac{1}{2}\int[d^4\bar{\theta}]_C\mathrm{Tr}(\overline{\mathbb{W}\mathbb{W}})=\frac{1}{2}\int[d^4\bar{\theta}]_C\int d\zeta_0\frac{\mathrm{Tr}(\overline{W}_0\overline{W}_0)}{\zeta_0}=$$
$$-\frac{1}{2}\int[d^4\bar{\theta}]_C\int d\zeta_0\frac{\mathrm{Tr}(\nabla_0^2A_0\nabla_0^2A_0)}{\zeta_0}=\frac{1}{4}\int[d^4\bar{\theta}]_C\int d\zeta_0\nabla_0^2\frac{\mathrm{Tr}(\nabla_0^\alpha A_0\nabla_{0\alpha}A_0)}{\zeta_0}=$$
$$\frac{1}{4}\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\int[d^4\bar{\theta}]_C\int d\zeta_0\nabla_0^2\frac{\mathrm{Tr}\left(\nabla_0^\alpha A_0^{(k)}\nabla_{0\alpha}A_0^{(n-k)}\right)}{\zeta_0}$$

$$\frac{1}{4}\sum_{n=2}^{\infty}(-1)^n\sum_{k=1}^{n-1}\int[d^4\bar{\theta}]_C\oiiint d\zeta_1\dots d\zeta_nD^2\mathrm{Tr}\left(\frac{D^\alpha(X_1\dots X_k)}{\zeta_1\zeta_{21}\cdots\zeta_{k,k-1}\zeta_k}\frac{D_\alpha(X_{k+1}\dots X_n)}{\zeta_{k+1}\zeta_{k+2,k+1}\cdots\zeta_{n,n-1}\zeta_n}\right)$$

$$\frac{1}{\zeta_{21}\cdots\zeta_{n,n-1}\zeta_{1n}}\frac{(\zeta_1-\zeta_n)}{\zeta_1\zeta_n}\sum_{r=k}^{m+k-2}\frac{(\zeta_{r+1}-\zeta_r)}{\zeta_{r+1}\zeta_r}=\frac{1}{\zeta_{21}\cdots\zeta_{n,n-1}\zeta_{1n}}\frac{(\zeta_1-\zeta_n)}{\zeta_1\zeta_n}\frac{(\zeta_{m+k-1}-\zeta_k)}{\zeta_{m+k-1}\zeta_k}$$

$$S=-\frac{1}{4}\sum_{n=2}^{\infty}(-1)^n\sum_{m=2}^{n}\int[d^4\bar{\theta}]_C\oiiint d\zeta_1\dots d\zeta_nD^2\frac{\mathrm{Tr}(D^\alpha X_1\dots D_\alpha X_m\dots X_n))}{\zeta_{21}\cdots\zeta_{n,n-1}\zeta_{1n}}\frac{(\zeta_1-\zeta_m)^2}{\zeta_1^2\zeta_m^2}$$

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\int[d^4\bar{\theta}]_C\oiiint d\zeta_1\dots d\zeta_nD^2Q^2\frac{\mathrm{Tr}(X_1\dots X_n)}{\zeta_{21}\cdots\zeta_{1n}}=$$
$$\sum_{n=2}^{\infty}(-1)^n\int[d^4\bar{\theta}]_C\oiiint d\zeta_1\dots d\zeta_n\frac{D^2}{\zeta_{21}\cdots\zeta_{1n}}[\mathrm{Tr}(Q^2X_1\dots X_n)$$
$$+\sum_{m=2}^{n}\frac{n-m+1}{n}\mathrm{Tr}(Q^\alpha X_1\dots Q_\alpha X_m\dots X_n)\Big]$$

$$\sum_{n=2}^{\infty}(-1)^n\int[d^4\bar{\theta}]_C\oint d\zeta_1\dots d\zeta_n\frac{D^2}{\zeta_{21}\cdots\zeta_{1n}}\Big[\frac{1}{\zeta_1^2}\mathrm{Tr}(D^2X_1\dots X_n)$$
$$+\sum_{m=2}^{n}\frac{n-m+1}{n}\frac{1}{\zeta_1\zeta_m}\mathrm{Tr}(D^\alpha X_1\dots D_\alpha X_m\dots X_n)\Big]$$

$$\sum_{n=2}^{\infty} (-1)^n \int [d^4\bar{\theta}]_C \oint d\zeta_1 \dots d\zeta_n$$

$$\frac{D^2}{\zeta_{21} \dots \zeta_{1n}} \sum_{m=2}^{n} \frac{2(n-m+1)\zeta_1 - n\zeta_m}{2n} \frac{1}{\zeta_1^2 \zeta_m} \mathrm{Tr}(D^\alpha X_1 \dots D_\alpha X_m \dots X_n)$$

$$-\frac{1}{4}\sum_{n=2}^{\infty} (-1)^n \sum_{m=2}^{n} \int [d^4\bar{\theta}]_C \oiiint d\zeta_1 \dots d\zeta_n \frac{D^2}{\zeta_{21} \dots \zeta_{1n}} \frac{(\zeta_1 - \zeta_m)^2}{\zeta_1^2 \zeta_m^2} \mathrm{Tr}(D^\alpha X_1 \dots D_\alpha X_m \dots X_n)$$

$$A_\zeta^{(1)} - \tilde{A}_\zeta^{(1)} = \partial_\zeta X$$

$$A_\zeta^{(n+1)} - \tilde{A}_\zeta^{(n+1)} = -X\left(A_\zeta^{(n)} - \tilde{A}_\zeta^{(n)}\right) n \geq 1$$

$$A_\zeta^{(n+1)} - \tilde{A}_\zeta^{(n+1)} = (-1)^n X^n \partial_\zeta X$$

$$A_\zeta - \tilde{A}_\zeta = (1+X)^{-1}\partial_\zeta X = \partial_\zeta \ln(1+X) = \partial_\zeta V$$

$$S = \sum_n \frac{(-1)^n}{n} \int d^8\theta \oiiint d\zeta_1 \dots d\zeta_n \frac{(X_1 \dots X_n)}{\zeta_{21} \dots \zeta_{1n}} =$$

$$\int [d^4\bar{\theta}]_C \oiiint d\zeta_1 \frac{1}{2} D^2 Q^\alpha (A_1 (e^{-V} Q_\alpha e^V)_1) =$$

$$\int [d^4\bar{\theta}]_C \oiiint d\zeta_1 d\zeta_2 \frac{1}{2} D^2 Q^\alpha \frac{V_2 Q_\alpha V_1}{\zeta_{21}^2} =$$

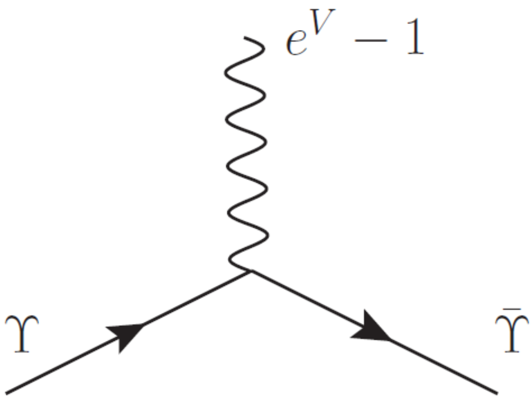
$$\int [d^4\bar{\theta}]_C \oiiint d\zeta_1 d\zeta_2 \frac{1}{2} D^2 Q^\alpha \frac{V_2 Q_\alpha V_1}{\zeta_{21}} \left(\delta_{12} - \frac{1}{\zeta_{12}}\right) =$$

$$\int [d^4\bar{\theta}]_C \oiiint d\zeta_1 d\zeta_2 \frac{1}{2} D^2 Q^2 \frac{V_1^2}{\zeta_{21}} \delta_{12} - \int [d^4\bar{\theta}]_C \oiiint d\zeta_1 d\zeta_2 \frac{1}{2} D^2 Q^\alpha \frac{V_2 Q_\alpha V_1}{\zeta_{21}\zeta_{12}}$$

$$-\frac{1}{4}\int [d^4\bar{\theta}]_C \oiiint d\zeta_1 d\zeta_2 D^2 Q^\alpha \frac{V_2 Q_\alpha V_1 + V_1 Q_\alpha V_2}{\zeta_{21}\zeta_{12}}$$

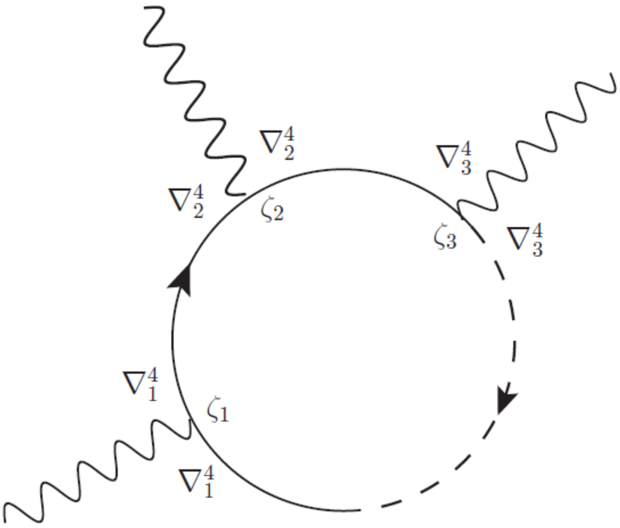
$$-\frac{1}{2}\int [d^4\bar{\theta}]_C \oiiint d\zeta_1 d\zeta_2 D^2 Q^2 \frac{V_1 V_2}{\zeta_{21}\zeta_{12}}$$

$$\int [d^4\theta]_P \frac{d\zeta}{\zeta} \bar{\Upsilon} e^V \Upsilon = \int [d^4\theta]_P \frac{d\zeta}{\zeta} (\bar{\Upsilon}\Upsilon + \bar{\Upsilon}(e^V - 1)\Upsilon)$$



$$< \bar{\Upsilon}(1)\Upsilon(2) >= -\frac{1}{\zeta_1^2}\sum_{n=0}^{\infty}\left(\frac{\zeta_2}{\zeta_1}\right)^n \frac{\nabla_1^4 \nabla_2^4}{(\zeta_1 - \zeta_2)^2 \,\square} \delta^4(x_1 - x_2)\delta^8(\theta_1 - \theta_2)$$

$$< \Upsilon(1)\bar{\Upsilon}(2) >= -\frac{1}{\zeta_2^2}\sum_{n=0}^{\infty}\left(\frac{\zeta_1}{\zeta_2}\right)^n \frac{\nabla_1^4 \nabla_2^4}{(\zeta_1 - \zeta_2)^2 \,\square} \delta^4(x_1 - x_2)\delta^8(\theta_1 - \theta_2)$$



$$\delta^8(\theta_{12})\nabla_1^4 \dots \nabla_k^4 \delta^8(\theta_{21}) = \Box^{k-2}\,(\zeta_1-\zeta_2)^2 \dots (\zeta_k-\zeta_1)^2 \delta^8(\theta_1-\theta_2),$$

$$\frac{1}{\Box^2}\frac{(-1)^n}{n}\int\, d^8\theta \frac{d\zeta_1}{\zeta_1} \dots \frac{d\zeta_n}{\zeta_n} \sum_{k_1=0}^{\infty}\left(\frac{\zeta_1}{\zeta_2}\right)^{k_1} \dots \sum_{k_n=0}^{\infty}\left(\frac{\zeta_n}{\zeta_1}\right)^{k_n} \mathrm{Tr}((e^V-1)_1 \dots (e^V-1)_n),$$

$$\frac{1}{\zeta_2}\sum_{k_1=0}^{\infty}\left(\frac{\zeta_1}{\zeta_2}\right)^{k_1} = \frac{1}{\zeta_2-\zeta_1}.$$

$$\frac{1}{\zeta_{12}} \equiv \frac{1}{\zeta_1-\zeta_2+\epsilon(\zeta_1+\zeta_2)} = \frac{1}{(1+\epsilon)\zeta_1-(1-\epsilon)\zeta_2} = \frac{1}{(1+\epsilon)\zeta_1}\sum_{n=0}^{\infty}\left(\frac{(1-\epsilon)\zeta_2}{(1+\epsilon)\zeta_1}\right)^n \to \frac{1}{\zeta_1}\sum_{n=0}^{\infty}\left(\frac{\zeta_2}{\zeta_1}\right)^n$$

$$\frac{1}{\zeta_{21}} \equiv \frac{1}{\zeta_2-\zeta_1+\epsilon(\zeta_2+\zeta_1)} = \frac{1}{(1+\epsilon)\zeta_2-(1-\epsilon)\zeta_1} = \frac{1}{(1+\epsilon)\zeta_2}\sum_{n=0}^{\infty}\left(\frac{(1-\epsilon)\zeta_1}{(1+\epsilon)\zeta_2}\right)^n \to \frac{1}{\zeta_2}\sum_{n=0}^{\infty}\left(\frac{\zeta_1}{\zeta_2}\right)^n$$

$$\frac{1}{\Box^2}\frac{(-1)^n}{n}\int\, d^8\theta \frac{d\zeta_1 \dots d\zeta_n}{\zeta_{21} \dots \zeta_{1n}} \mathrm{Tr}((e^V-1)_1 \dots (e^V-1)_n)$$

$$S = \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\int\, d^8\theta \frac{d\zeta_1 \dots d\zeta_n}{\zeta_{21} \dots \zeta_{1n}} \mathrm{Tr}((e^V-1)_1 \dots (e^V-1)_n)$$

$$\int\,[d^4\theta]_P \frac{d\zeta}{\zeta}\Upsilon e^{-V}\bar{\Upsilon}$$

$$S = \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\int\, d^8\theta \frac{d\zeta_1 \dots d\zeta_n}{\zeta_{12} \dots \zeta_{n1}} \mathrm{Tr}((e^{-V}-1)_1 \dots (e^{-V}-1)_n)$$

$$\delta(e^V-1) = i\bar{\Lambda} - i\Lambda + i\bar{\Lambda}(e^V-1) - (e^V-1)i\Lambda$$

$$(-1)^n\mathrm{Tr}\prod_{i=1}^{n}\oint d\zeta_i \frac{(i\bar{\Lambda}_1 - i\Lambda_1)(e^V-1)_2 \cdot \dots \cdot (e^V-1)_n}{\zeta_{21}\cdot \dots \cdot \zeta_{1n}} +$$
$$(-1)^n\mathrm{Tr}\prod_{i=1}^{n}\oint d\zeta_i \frac{(i\bar{\Lambda}_1(e^V-1)_1 - (e^V-1)_1 i\Lambda_1)(e^V-1)_2 \cdot \dots \cdot (e^V-1)_n}{\zeta_{21}\cdot \dots \cdot \zeta_{1n}}$$

$$-\frac{i\Lambda_1(e^V-1)_2\cdot\ldots\cdot(e^V-1)_n}{\zeta_{21}\cdot\ldots\cdot\zeta_{n,n-1}}$$

$$\frac{i\bar{\Lambda}_1(e^V-1)_2\cdot\ldots\cdot(e^V-1)_n}{\zeta_{32}\cdot\ldots\cdot\zeta_{1n}}$$

$$(-1)^{n-1}\mathrm{Tr}\prod_{i=1}^{n-1}\oint d\zeta_i\frac{((e^V-1)_1 i\Lambda_1 - i\bar{\Lambda}_1(e^V-1)_1)(e^V-1)_2\cdot\ldots\cdot(e^V-1)_{n-1}}{\zeta_{21}\cdot\ldots\cdot\zeta_{1,n-1}}$$

$$\mathrm{Tr}\int d^8\theta\oint d\zeta_1 d\zeta_2\frac{(i\bar{\Lambda}_1-i\Lambda_1)(e^V-1)_2}{\zeta_{21}\zeta_{12}}$$

$$\mathrm{Tr}\int d^8\theta\oiint d\zeta_1 d\zeta_2\frac{i\Lambda_1(e^V-1)_2}{\zeta_{21}\zeta_{12}} = \mathrm{Tr}\int d^8\theta\oint d\zeta_1 d\zeta_2\frac{i\Lambda_1(e^V-1)_2}{\zeta_{21}}\left(\delta_{12}-\frac{1}{\zeta_{21}}\right)$$

$$\frac{1}{\zeta_1}\sum_{n=0}^{\infty}\left(\frac{\zeta_2}{\zeta_1}\right)^n \to \frac{1}{\zeta_{12}}$$

$$\frac{1}{\zeta_{12}} = \frac{1}{\zeta_1-\zeta_2+\epsilon(\zeta_1+\zeta_2)}$$

$$\sum_{n=0}^{\infty} X_n\zeta_2^n = \oint d\zeta_1\frac{X(\zeta_1)}{\zeta_{12}}$$
$$\sum_{n=-\infty}^{-1} X_n\zeta_2^n = \oint d\zeta_1\frac{X(\zeta_1)}{\zeta_{21}}$$

$$\delta_{12} = \frac{1}{\zeta_{12}}+\frac{1}{\zeta_{21}}$$
$$\frac{1}{\zeta_{12}\zeta_{23}} = \frac{1}{\zeta_{13}\zeta_{23}}+\frac{1}{\zeta_{12}\zeta_{13}}$$

$$T^{(p,q)} = \sum_{k=p}^{q} T_k\zeta^k$$

$$\oiint d\zeta_1\frac{1}{\zeta_{12}}T^{(0,\infty)}(\zeta_1) = \frac{1}{1+\epsilon}T^{(0,\infty)}\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)$$
$$\oiint d\zeta_1\frac{1}{\zeta_{21}}T^{(-\infty,-1)}(\zeta_1) = \frac{1}{1-\epsilon}T^{(-\infty,-1)}\left(\zeta_2\frac{1+\epsilon}{1-\epsilon}\right)$$

$$\oint d\zeta_1 d\zeta_2\frac{1}{\zeta_{12}\zeta_{21}}(\zeta_1-\zeta_2)^2 X(\zeta_1)Y(\zeta_2)$$

$$(\zeta_1-\zeta_2)^2X(\zeta_1)=\Big[\zeta_1^2\left(X^{(-\infty,-3)}(\zeta_1)+X^{(-2,\infty)}(\zeta_1)\right)-2\zeta_1\zeta_2\left(X^{(-\infty,-2)}(\zeta_1)+X^{(-1,\infty)}(\zeta_1)\right)$$
$$+\zeta_2^2\left(X^{(-\infty,-1)}(\zeta_1)+X^{(0,\infty)}(\zeta_1)\right)\Big]$$

$$T^{(0,\infty)} = \frac{1}{\zeta_{21}}\left[\zeta_1^2X^{(-2,\infty)}(\zeta_1)-2\zeta_1\zeta_2X^{(-1,\infty)}(\zeta_1)+\zeta_2^2X^{(0,\infty)}(\zeta_1)\right]$$

$$T^{(-\infty,-2)} = \frac{1}{\zeta_{12}}\left[\zeta_1^2X^{(-\infty,-3)}(\zeta_1)-2\zeta_1\zeta_2X^{(-\infty,-2)}(\zeta_1)+\zeta_2^2X^{(-\infty,-1)}(\zeta_1)\right]$$

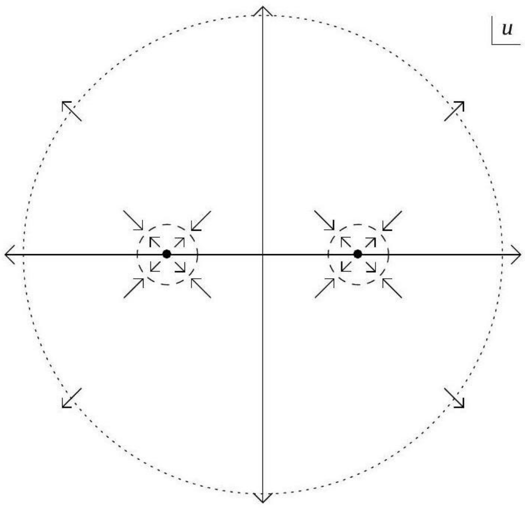
$$\frac{1}{\zeta_{21}} \to \frac{1}{\zeta_2-\zeta_2\frac{1-\epsilon}{1+\epsilon}+\epsilon\left(\zeta_2+\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)} = \frac{1+\epsilon}{4\epsilon\zeta_2}$$
$$\frac{1}{\zeta_{12}} \to \frac{1}{\zeta_2\frac{1+\epsilon}{1-\epsilon}-\zeta_2+\epsilon\left(\zeta_2+\zeta_2\frac{1+\epsilon}{1-\epsilon}\right)} = \frac{1-\epsilon}{4\epsilon\zeta_2}$$

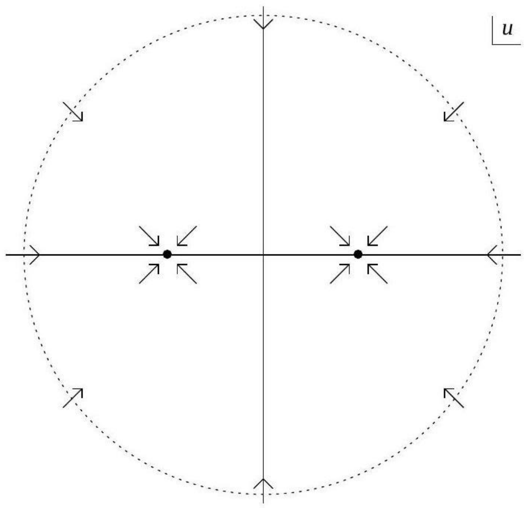
$$\frac{1}{4\epsilon\zeta_2}\left[\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)^2 X^{(-2,\infty)}\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)\right.$$
$$-2\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)\zeta_2 X^{(-1,\infty)}\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)$$
$$\left.+\zeta_2^2 X^{(0,\infty)}\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)\right]=$$
$$=\frac{\zeta_2}{4\epsilon}\left[\left(\frac{2\epsilon}{1+\epsilon}\right)^2 X^{(0,\infty)}\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)+X_{-2}\frac{1}{\zeta_2^2}-\frac{1+3\epsilon}{1+\epsilon}X_{-1}\frac{1}{\zeta_2}\right]$$

$$\frac{1}{4\epsilon\zeta_2}\left[\left(\zeta_2\frac{1+\epsilon}{1-\epsilon}\right)^2 X^{(-\infty,-3)}\left(\zeta_2\frac{1+\epsilon}{1-\epsilon}\right)\right.$$
$$-2\left(\zeta_2\frac{1+\epsilon}{1-\epsilon}\right)\zeta_2 X^{(-\infty,-2)}\left(\zeta_2\frac{1+\epsilon}{1-\epsilon}\right)$$
$$\left.+\zeta_2^2 X^{(-\infty,-1)}\left(\zeta_2\frac{1+\epsilon}{1-\epsilon}\right)\right]$$
$$=\frac{\zeta_2}{4\epsilon}\left[\left(\frac{2\epsilon}{1-\epsilon}\right)^2 X^{(-\infty,-3)}\left(\zeta_2\frac{1+\epsilon}{1-\epsilon}\right)-\frac{1+2\epsilon-3\epsilon^2}{(1+\epsilon)^2}X_{-2}\frac{1}{\zeta_2^2}+\frac{1-\epsilon}{1+\epsilon}X_{-1}\frac{1}{\zeta_2}\right]$$

$$\frac{\zeta_2}{4\epsilon}\left[\left(\frac{2\epsilon}{1-\epsilon}\right)^2 X^{(-\infty,-3)}\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)+\left(\frac{2\epsilon}{1+\epsilon}\right)^2 X^{(0,\infty)}\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)\right.$$
$$\left.+\left(1-\frac{1+2\epsilon-3\epsilon^2}{(1+\epsilon)^2}\right)X_{-2}\frac{1}{\zeta_2^2}+\left(\frac{1-\epsilon}{1+\epsilon}-\frac{1+3\epsilon}{1+\epsilon}\right)X_{-1}\frac{1}{\zeta_2}\right]=$$
$$\frac{\zeta_2}{4\epsilon}\left[\left(\frac{2\epsilon}{1-\epsilon}\right)^2 X^{(-\infty,-3)}\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)+\left(\frac{2\epsilon}{1+\epsilon}\right)^2 X^{(0,\infty)}\left(\zeta_2\frac{1-\epsilon}{1+\epsilon}\right)\right.$$
$$\left.+\left(\frac{2\epsilon}{1+\epsilon}\right)^2 X_{-2}\frac{1}{\zeta_2^2}-\frac{4\epsilon}{1+\epsilon}X_{-1}\frac{1}{\zeta_2}\right]$$

$$\oiiint d\zeta_1 d\zeta_2\frac{1}{\zeta_{12}\zeta_{21}}(\zeta_1-\zeta_2)^2 X(\zeta_1)Y(\zeta_2)=-\oiiint d\zeta_2 X_{-1}Y(\zeta_2)=-X_{-1}Y_{-1}$$





$$S_{(\text{Eucl.})}=\int dx_{\text{E}}^0 d^3x\mathcal{L}_{(\text{Eucl.})}$$

$$-g^2\mathcal{L}=\left(\int d^2\theta\frac{1}{2}\text{tr}(W^\alpha W_\alpha)+\text{ H.c.}\right)+2\int d^2\theta d^2\bar{\theta}\text{tr}\left(\Phi^\dagger e^{[2V,\cdot]}\Phi\right)$$

$$W_\alpha=-\frac{1}{8}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\left(e^{-2V}D_\alpha e^{2V}\right)$$

$$g^2\mathcal{L}=2\mathrm{tr}\Big\{\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+i\bar{\lambda}\bar{\sigma}_{\mathrm{E}}^{\mu}D_\mu\lambda+i\bar{\psi}\bar{\sigma}_{\mathrm{E}}^{\mu}D_\mu\psi+\left(D_\mu\phi\right)^{\dagger}D_\mu\phi$$
$$-i\sqrt{2}[\lambda,\psi]\phi^{\dagger}-i\sqrt{2}[\bar{\lambda},\bar{\psi}]\phi+\frac{1}{2}\left[\phi^{\dagger},\phi\right]^2\Big\}$$

$$M_W=\sqrt{2}|\langle a\rangle|.$$

$$\mathcal{L}_{\mathrm{eff}}=-\frac{1}{4\pi}\mathrm{Im}\int d^4\theta\mathcal{F}(\mathcal{A})-\int d^4\theta d^4\bar{\theta}\mathcal{K}(\mathcal{A},\overline{\mathcal{A}})+O(n\geq 6)$$

$$\mathcal{L}_{\mathrm{eff}}=\mathcal{L}_{\mathrm{eff}}^{n=2}+\mathcal{L}_{\mathrm{eff}}^{n=4}+O(n\geq 6),$$

$$\mathcal{L}_{\mathrm{eff}}^{n=2}=-\frac{1}{4\pi}\mathrm{Im}\left[\int d^2\theta\frac{1}{2}\mathcal{F}''(A)W^{\alpha}W_{\alpha}+\int d^2\theta d^2\bar{\theta}\mathcal{F}'(A)\bar{A}\right]$$

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}^{n=4}=-\int d^2\theta d^2\bar{\theta}\{&\mathcal{K}_{A\bar{A}}(A,\bar{A})\left[(D^{\alpha}D_{\alpha}A)\left(\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\bar{A}\right)+2\left(\bar{D}_{\dot{\alpha}}D^{\alpha}A\right)\left(D_{\alpha}\bar{D}^{\dot{\alpha}}\bar{A}\right)\right.\\&+4\left(D^{\alpha}W_{\alpha}\right)\left(\bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right)\\&-4\left(D^{(\alpha}W^{\beta)}\right)\left(D_{(\alpha}W_{\beta)}\right)-2D^{\alpha}D_{\alpha}\left(W^{\beta}W_{\beta}\right)\\&\left.-4\left(\bar{D}_{(\dot{\alpha}}\bar{W}_{\dot{\beta})}\right)\left(\bar{D}^{(\dot{\alpha}}\bar{W}^{\dot{\beta})}\right)-2\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\left(\bar{W}_{\dot{\beta}}\bar{W}^{\dot{\beta}}\right)\right]\\&-2\mathcal{K}_{AA\bar{A}}(A,\bar{A})W^{\alpha}W_{\alpha}D^{\beta}D_{\beta}A-2\mathcal{K}_{A\bar{A}\bar{A}}(A,\bar{A})\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}^{\dot{\beta}}\bar{A}\\&+\mathcal{K}_{AA\bar{A}\bar{A}}(A,\bar{A})\left[-8\left(W^{\alpha}D_{\alpha}A\right)\left(\bar{W}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\bar{A}\right)+4W^{\alpha}W_{\alpha}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right]\}.\end{aligned}$$

$$\mathcal{L}_{\mathrm{eff}}=\gamma(a,\bar{a})\partial_\mu a\partial_\mu\bar{a}+\cdots,$$

$$\gamma(a,\bar{a})=\partial_a\partial_{\bar{a}}K|_{\theta=\bar{\theta}=0}=\frac{1}{4\pi}\mathrm{Im}\mathcal{F}''(a)$$

$$\mathcal{L}_{\mathrm{eff}}=\frac{1}{4}g_{\mathrm{eff}}^{-2}(a,\bar{a})F_{\mu\nu}F_{\mu\nu}+\frac{1}{32\pi^2}\theta_{\mathrm{eff}}(a,\bar{a})F_{\mu\nu}\tilde{F}_{\mu\nu}+\cdots$$

$$g_{\mathrm{eff}}^{-2}(a,\bar{a})=\frac{1}{4\pi}\mathrm{Im}\tau(a),\theta_{\mathrm{eff}}(a,\bar{a})=2\pi\mathrm{Re}\tau(a)$$

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}^{n=2}=&g_{\mathrm{eff}}^{-2}(a,\bar{a})\left\{\left|\partial_\mu a\right|^2+\frac{1}{4}F_{\mu\nu}^2+\left(\frac{i}{2}\psi\sigma_{\mathrm{E}}^{\mu}D_\mu\bar{\psi}+\frac{i}{2}\lambda\sigma_{\mathrm{E}}^{\mu}D_\mu\bar{\lambda}+\text{ H.c}\right)-|F|^2-\frac{1}{2}D^2\right.\\&+\left[\Gamma(a,\bar{a})\left(\frac{1}{2}\psi^2F^*+\frac{1}{2}\lambda^2F-\frac{i}{\sqrt{2}}\lambda\psi D-\frac{1}{\sqrt{2}}\lambda\sigma_{\mathrm{E}}^{\mu\nu}\psi F_{\mu\nu}\right)+\text{ H.c.}\right]\\&\left.-\left[R(a,\bar{a})\frac{1}{4}\lambda^2\psi^2+\text{ H.c}\right]\right\}+\frac{1}{32\pi^2}\theta_{\mathrm{eff}}(a,\bar{a})F_{\mu\nu}\tilde{F}_{\mu\nu}\end{aligned}$$

$$\begin{aligned}D_\mu&=\partial_\mu-\Gamma(a,\bar{a})\partial_\mu a\\\Gamma(a,\bar{a})&=\gamma(a,\bar{a})^{-1}\partial_a\gamma(a,\bar{a})\\R(a,\bar{a})&=\partial_a\Gamma(a,\bar{a})+\Gamma(a,\bar{a})^2\\&=\gamma(a,\bar{a})^{-1}\partial_a^2\gamma(a,\bar{a})\end{aligned}$$

$$\mu\frac{dg^2}{d\mu}=-\frac{1}{2\pi^2}g^4\left[1+O\left(e^{-8\pi^2/g^2}\right)\right]$$

$$g^{-2}(\mu)=\frac{1}{4\pi^2}\ln\left(\mu^2/\Lambda^2\right)+\aleph+O\left(\Lambda^4/\mu^4\right)$$

$$\mathrm{Im}\tau(a)\approx\frac{1}{\pi}\ln\left(\frac{|a|^2}{\Lambda^2}\right)$$

$$\mathcal{F}(a)\approx\frac{i}{2\pi}a^2\ln\left(\frac{a^2}{\Lambda^2}\right)$$

$$ds^2=-\frac{i}{2}\epsilon_{mn}\frac{da^m}{du}\frac{d\bar{a}^n}{d\bar{u}}dud\bar{u}$$

$$S=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$\mathcal{L}_{D,\mathrm{eff}}^{n=2}=-\frac{1}{4\pi}\mathrm{Im}\left[\iint d^2\theta\frac{1}{2}\mathcal{F}_D''(A_D)W_D^{\alpha}W_{D\alpha}+\int d^2\theta d^2\bar{\theta}\mathcal{F}_D'(A_D)\bar{A}_D\right]$$

$$\mathcal{K}_D(-A_D,-\bar{A}_D)\equiv\mathcal{K}(A,\bar{A})$$

$$M=\sqrt{2}|Z|,$$

$$M_{\mathrm{m}}=\sqrt{2}|a_D|.$$

$$\mu\frac{dg_D^2}{d\mu}=\frac{1}{4\pi^2}g_D^4$$

$$g_D^{-2}(M_{\mathrm{m}})=\frac{1}{8\pi^2}\ln\left(\Lambda^2/M_{\mathrm{m}}^2\right)+\Re$$

$$\mathrm{Im}\tau_D(a_D)\approx-\frac{1}{2\pi}\ln\left(\frac{|a_D|^2}{\Lambda^2}\right).$$

$$\mathcal{F}_D(a_D)\approx-\frac{i}{4\pi}a_D^2\ln\left(\frac{a_D^2}{\Lambda^2}\right)$$

$$T\gg gT\gg g^2T\gg\Lambda$$

$$m_{\mathrm{D}}^2=2g^2T^2+O(g^4T^2)$$

$$m_{\phi}^2=g^2T^2+O(g^4T^2)$$

$$F/V=3T^4\left[-\frac{\pi^2}{12}+\frac{g^2}{8}-\frac{1+\sqrt{2}}{6\pi}g^3+O(g^4)\right].$$

$$\mathrm{r}\,D(\vec{x}-\vec{y})\equiv\left\langle\mathrm{tr}\left(\phi(\vec{x})^{\dagger}\right)^2\mathrm{tr}\left(\phi(\vec{y})^2\right)\right\rangle$$

$$(F/V)_{\mathrm{charged}}\approx-16T^4\left(\frac{M_W}{2\pi T}\right)^{3/2}e^{-M_W/T}$$

$$a(x)=a_0+\tilde{a}(x)$$

$$\mathcal{L}_{\mathrm{eff}}^{n=2}=\mathcal{L}_{\mathrm{free}}^{(0)}+\mathcal{L}_{\mathrm{int}}^{(1)}+\mathcal{L}_{\mathrm{int}}^{(2)}+\cdots$$

$$g_0^2=4\pi^2/(\ln|a_0/\Lambda|^2+3)$$

$$\mathcal{L}_{\mathrm{free}}^{(0)}=\left|\partial_\mu\tilde{a}\right|^2+\frac{1}{4}F_{\mu\nu}^2+\left(\frac{i}{2}\psi\sigma_{\mathrm{E}}^{\mu}\partial_\mu\bar{\psi}+\frac{i}{2}\lambda\sigma_{\mathrm{E}}^{\mu}\partial_\mu\bar{\lambda}+\mathrm{H.c.}\right)-|F|^2-\frac{1}{2}D^2$$

$$\frac{4\pi^2}{g_0^3}\mathcal{L}_{\mathrm{int}}^{(1)}=\left(\frac{\tilde{a}}{a_0}+\frac{\tilde{a}^*}{\bar{a}_0}\right)\mathcal{L}_{\mathrm{free}}^{(0)}+\frac{i}{4}\left(\frac{\tilde{a}}{a_0}-\frac{\tilde{a}^*}{\bar{a}_0}\right)F_{\mu\nu}\tilde{F}_{\mu\nu}+\left(\frac{1}{a_0}\mathcal{O}_{\mathrm{fermi}}+\mathrm{H.c.}\right)$$

$$\frac{4\pi^2}{g_0^4}\mathcal{L}_{\mathrm{int}}^{(2)}=-\frac{1}{2}\left(\frac{\tilde{a}^2}{a_0^2}+\frac{\tilde{a}^{*2}}{\bar{a}_0^2}\right)\mathcal{L}_{\mathrm{free}}^{(0)}-\frac{i}{8}\left(\frac{\tilde{a}^2}{a_0^2}-\frac{\tilde{a}^{*2}}{\bar{a}_0^2}\right)F_{\mu\nu}\tilde{F}_{\mu\nu}+\left(-\frac{\tilde{a}}{a_0^2}\mathcal{O}_{\mathrm{fermi}}+\frac{1}{4a_0^2}\lambda^2\psi^2+\mathrm{H.c.}\right)$$

$$\mathcal{O}_{\text{white particle}}\equiv-\frac{i}{2}\left(\psi\sigma_{\mathrm{E}}^{\mu}\bar{\psi}+\lambda\sigma_{\mathrm{E}}^{\mu}\bar{\lambda}\right)\partial_\mu\tilde{a}+\frac{1}{2}\psi^2F^*+\frac{1}{2}\lambda^2F-\frac{i}{\sqrt{2}}\lambda\psi D-\frac{1}{\sqrt{2}}\lambda\sigma_{\mathrm{E}}^{\mu\nu}\psi F_{\mu\nu}$$

$$V_{\mathrm{eff}}=-\frac{\pi^2}{12}T^4+\left\langle\mathcal{L}_{\mathrm{int}}^{(1)}\right\rangle_0^{1\mathrm{PI}}+\left\langle\mathcal{L}_{\mathrm{int}}^{(2)}+\left(\mathcal{L}_{\mathrm{int}}^{(1)}\right)^2\right\rangle_0^{1\mathrm{PI}}+\left\langle\mathcal{L}_{\mathrm{int}}^{(3)}+\mathcal{L}_{\mathrm{int}}^{(1)}\mathcal{L}_{\mathrm{int}}^{(2)}+\left(\mathcal{L}_{\mathrm{int}}^{(1)}\right)^3\right\rangle_0^{1\mathrm{PI}}$$
$$+\left\langle\mathcal{L}_{\mathrm{int}}^{(4)}+\mathcal{L}_{\mathrm{int}}^{(1)}\mathcal{L}_{\mathrm{int}}^{(3)}+\left(\mathcal{L}_{\mathrm{int}}^{(2)}\right)^2+\left(\mathcal{L}_{\mathrm{int}}^{(1)}\right)^2\mathcal{L}_{\mathrm{int}}^{(2)}+\left(\mathcal{L}_{\mathrm{int}}^{(1)}\right)^4\right\rangle_0^{1\mathrm{PI}}+\cdots$$

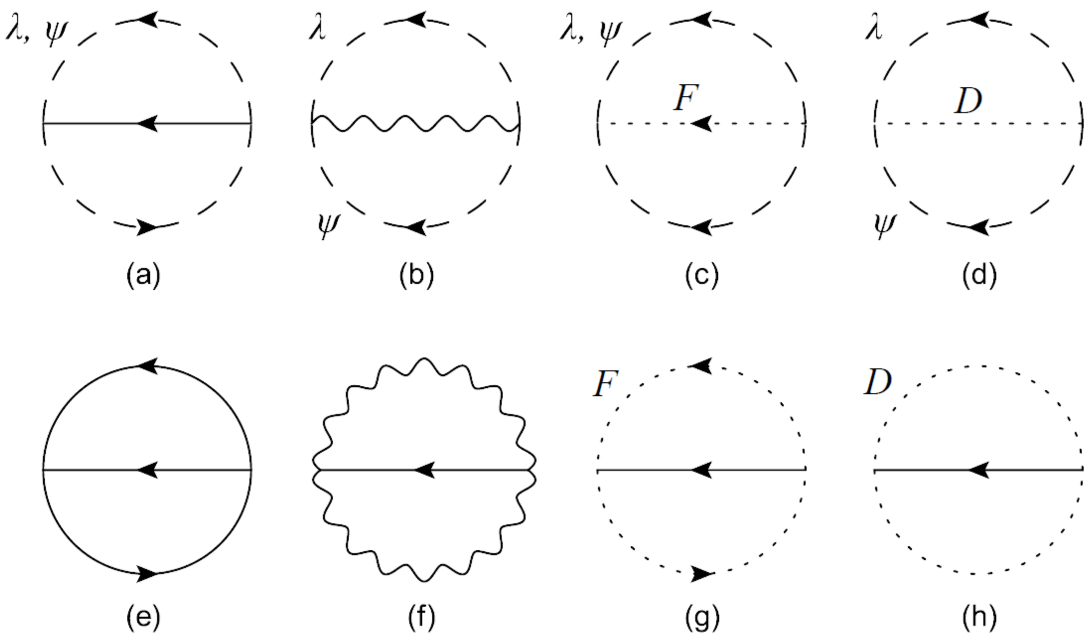
$$\left\langle \prod_{i=1}^{m} \mathcal{L}_{\text{int}}^{(p_i)} \right\rangle_0^{1\,\text{PI}}$$

$$\left\langle \mathcal{L}_{\text{int}}^{(2)} \right\rangle_0^{1\,\text{PI}}$$

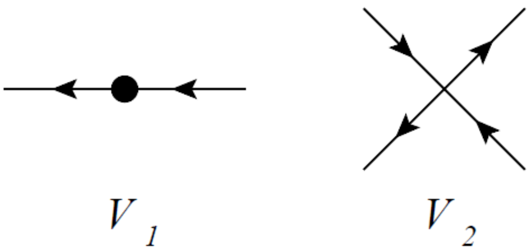
$$\left\langle \left(\mathcal{L}_{\text{int}}^{(1)}\right)^2 \right\rangle_0^{1\,\text{PI}}$$

$$\sum\!\!\!\!\!\!\int_{p,\pm} \sum\!\!\!\!\!\!\int_{q,\pm} \frac{p \cdot q}{p^2\, q^2} = 0\,,$$

$$\sum\!\!\!\!\!\!\int_{p,\pm} 1 = 0\,.$$

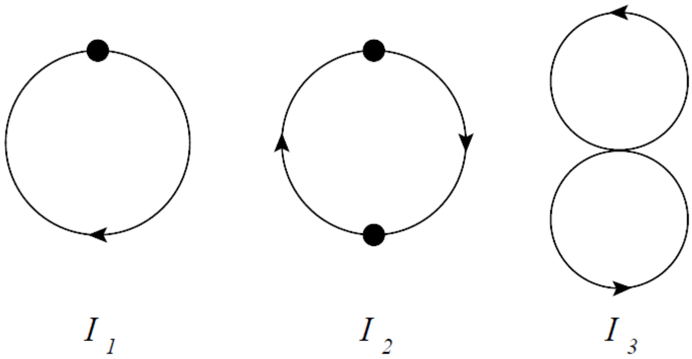


$$\mathcal{K}(A, \bar{A}) \approx \frac{c}{64} \ln\left(\frac{A^2}{\Lambda^2}\right) \ln\left(\frac{\bar{A}^2}{\Lambda^2}\right),$$



$$\mathcal{L}_{\text{eff}}^{n=4} \supset -\int d^2\theta d^2\bar{\theta} \mathcal{K}_{A\bar{A}}(A, \bar{A})\left[(D^\alpha D_\alpha A)\left(\bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{A}\right) + 2\left(\bar{D}_{\dot{\alpha}} D^\alpha A\right)\left(D_\alpha \bar{D}^{\dot{\alpha}} \bar{A}\right)\right].$$

$$\mathcal{L}_{\text{eff}}^{n=4} \supset -c\left[\frac{g_0^2}{|a_0|^2}\left(\left|\partial_\mu\partial_\nu\tilde{a}\right|^2 + |\partial^2\tilde{a}|^2\right) + \frac{g_0^4}{|a_0|^4}\left\{\left(\left|\partial_\mu\tilde{a}\right|^2\right)^2 + |\tilde{a}|^2\left|\partial_\mu\partial_\nu\tilde{a}\right|^2 + |\tilde{a}|^2|\partial^2\tilde{a}|^2 \right.\right.$$
$$\left.\left. +\left[\tilde{a}\left(\partial_\mu\partial_\nu\tilde{a}\right)\left(\partial_\mu\tilde{a}^*\right)(\partial_\nu\tilde{a}^*) + \text{ H.c.}\right]\right\}\right]$$



$$V_1 = \frac{2cg_0^2}{|a_0|^2}(p^2)^2$$

$$V_2 = \frac{cg_0^4}{|a_0|^4}[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3) + p_1^2p_2^2 + p_2^2p_3^2 + p_3^2p_4^2 + p_4^2p_1^2$$
$$+(p_1\cdot p_2)^2 + (p_2\cdot p_3)^2 + (p_3\cdot p_4)^2 + (p_4\cdot p_1)^2$$
$$+2(p_1\cdot p_2)(p_1\cdot p_4) + 2(p_2\cdot p_3)(p_3\cdot p_4) + 2(p_1\cdot p_2)(p_2\cdot p_3) + 2(p_1\cdot p_4)(p_3\cdot p_4)]$$

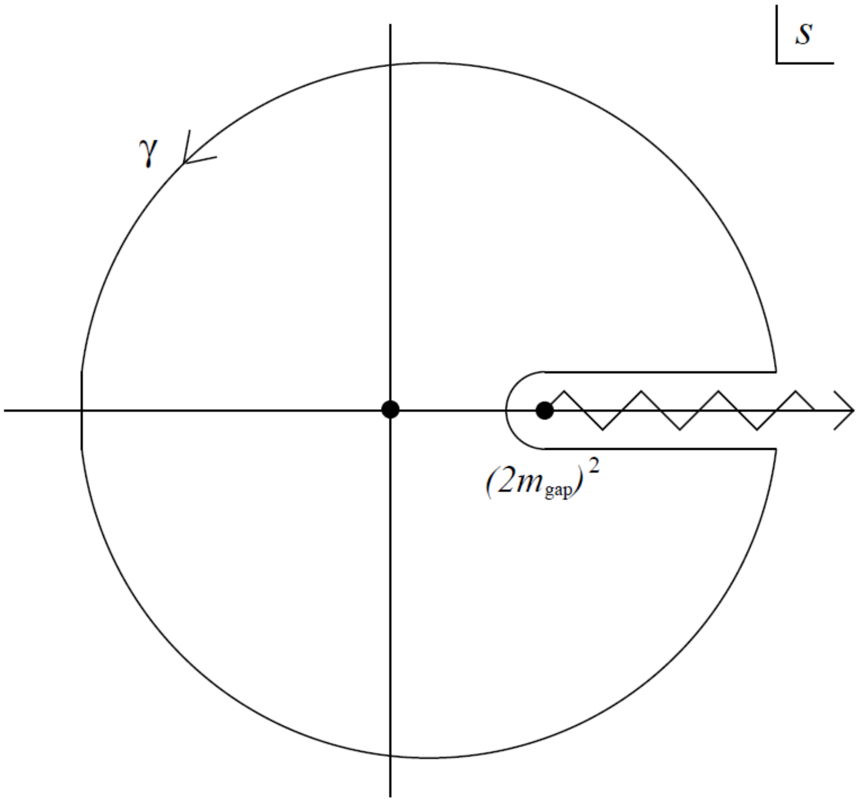
$$I_3 = \beta V\,\frac{1}{2}\sum\!\!\!\!\!\!\int_{p,+}\sum\!\!\!\!\!\!\int_{q,+}\frac{c\,g_0^4}{|a_0|^4}\left[\frac{4(p\cdot q)^2}{p^2q^2}\right] = \beta V\,\frac{8c\,g_0^4}{3|a_0|^4}\left(\sum\!\!\!\!\!\!\int_{p,+}\frac{\vec{p}^2}{p^2}\right)^2$$

$$\frac{(p\cdot q)^2}{p^2q^2}\left(1+\frac{1}{d-1}\right)\frac{\vec{p}^2\vec{q}^2}{p^2q^2}$$

$$V_{\text{eff}}(a_0)|_{\text{scalar-scalar}} = -c\frac{2\pi^4}{675}\frac{g_0^4T^4}{|a_0|^4}T^4.$$

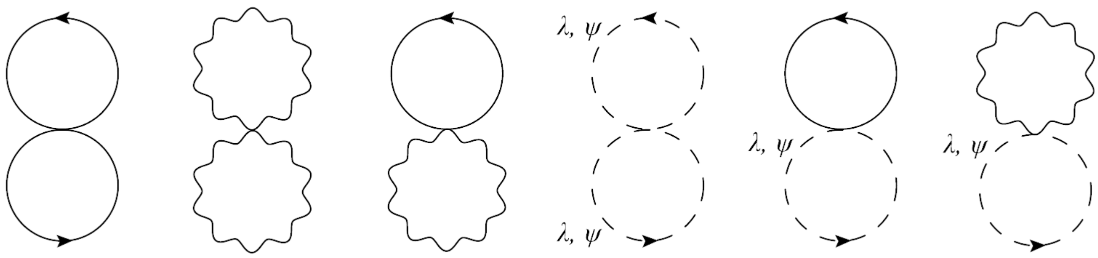
$$-i\mathcal{M}(s,t) = i\frac{cg_0^4}{|a_0|^4}(s+t)^2 + O(g_0^6)$$

$$\mathcal{A}(s) = \lim_{t\to 0^-}\mathcal{M}(s,t) = -cg_0^4\frac{s^2}{|a_0|^4} + O(g_0^6)$$



$$I = \oint_{\gamma} \frac{ds}{2\pi i} \frac{\mathcal{A}(s)}{s^3}$$

$$\frac{1}{2}\mathcal{A}''(0) = \frac{1}{\pi}\int_{4m_{\mathrm{gap}}^2}^{\infty} ds \frac{\mathrm{Im}[\mathcal{A}(s+i\epsilon)]}{s^3}$$



I_{ss} I_{gg} I_{sg} $I_{\psi\psi}, I_{\lambda\lambda}, I_{\psi\lambda}$ $I_{s\psi}, I_{s\lambda}$ $I_{g\psi}, I_{g\lambda}$

$$\mathcal{L}_{\mathrm{eff}}^{n=4} \supset -\frac{cg_0^4}{16|a_0|^4}\left[\left(F_{\mu\nu}F_{\mu\nu}\right)^2 - \left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right)^2\right]$$

$$I_{\mathrm{gg}} = \frac{d}{2}\frac{cg_0^4}{|a_0|^4}\beta V \Omega_{++}$$

$$\mathcal{L}_{\mathrm{eff}}^{n=4} \supset -\frac{cg_0^4}{|a_0|^4}\left[(\bar{\psi}\bar{\sigma}_{\mathrm{E}}^{\mu}\partial_{\mu}\psi)(\partial_{\nu}\bar{\psi}\bar{\sigma}_{\mathrm{E}}^{\nu}\psi) - (\psi\partial_{\mu}\psi)(\bar{\psi}\partial_{\mu}\bar{\psi})\right]$$

$$I_{\psi\psi} = I_{\lambda\lambda} = 2\frac{cg_0^4}{|a_0|^4}\beta V \Omega_{--}$$

$$\mathcal{L}_{\text{eff}}^{n=4} \supset -\frac{2cg_0^4}{|a_0|^4}\left[(\partial_\mu \tilde{a}^*)(\partial_\nu \tilde{a})F_{\mu\rho}F_{\nu\rho} - (\bar{\lambda}\partial_\mu\bar{\psi})(\psi\sigma_{\text{E}}^{\mu\nu}\partial_\nu\lambda) - (\lambda\partial_\mu\psi)(\bar{\psi}\bar{\sigma}_{\text{E}}^{\mu\nu}\partial_\nu\bar{\lambda})\right]$$

$$I_{\text{sg}} = 2(d-2)\frac{cg_0^4}{|a_0|^4}\beta V\Omega_{++}, I_{\psi\lambda} = 4\frac{cg_0^4}{|a_0|^4}\beta V\Omega_{--}$$

$$\begin{aligned} 0 &= \left[I_{\text{ss}} + I_{\text{gg}} + I_{\psi\psi} + I_{\lambda\lambda} + I_{\text{sg}} + I_{\psi\lambda} + I_{\text{s}\psi} + I_{\text{s}\lambda} + I_{\text{g}\psi} + I_{\text{g}\lambda}\right]\Big|_{\text{p.b.c.}} \\ &= 16\frac{cg_0^4}{|a_0|^4}\beta V\Omega_{++} + 4I_{\text{s}\psi}\Big|_{\text{p.b.c.}} \end{aligned}$$

$$I_{\text{s}\psi}\Big|_{\text{p.b.c.}} = -4\frac{cg_0^4}{|a_0|^4}\beta V\Omega_{++}$$

$$I_{\text{s}\psi} = I_{\text{s}\lambda} = I_{\text{g}\psi} = I_{\text{g}\lambda} = -4\frac{cg_0^4}{|a_0|^4}\beta V\Omega_{+-}$$

$$\Omega_{+-} \equiv \mathcal{F}_{p,+}\mathcal{F}_{q,-}\frac{(p\cdot q)^2}{p^2q^2} = \frac{4}{3}J_+J_-$$

$$(F(a_0)/V)_{\text{neutral}} = \left[-\frac{\pi^2}{12} - \frac{\pi^4}{24}\frac{cg_0^4T^4}{|a_0|^4} + O\left(\frac{g_0^6T^4}{|a_0|^4}\right)\right]T^4$$

$$-\frac{cg_0^4}{64|a_0|^4}\int d^3x\left(F_{\mu\nu} + \tilde{F}_{\mu\nu}\right)^2\left(F_{\rho\sigma} - \tilde{F}_{\rho\sigma}\right)^2$$

$$\begin{aligned} -g_D^2\mathcal{L}_{\text{QED}} =& \left(\int d^2\theta\frac{1}{4}W_D^\alpha W_{D\alpha} + \text{ H.c.}\right) + \int d^2\theta d^2\bar{\theta}A_D^\dagger A_D \\ &+ \int d^2\theta d^2\bar{\theta}\left(Q^\dagger e^{2V_D}Q + Q'^\dagger e^{-2V_D}Q'\right) + \left(-i\sqrt{2}\int d^2\theta Q' A_D Q + \text{ H.c.}\right) \end{aligned}$$

$$\begin{aligned} g_D^2\mathcal{L}_{\text{QED}} =& \frac{1}{4}F_D^{\mu\nu}F_D^{\mu\nu} + i\bar{\lambda}_D\bar{\sigma}_{\text{E}}^\mu\partial_\mu\lambda_D + i\bar{\psi}_D\bar{\sigma}_{\text{E}}^\mu\partial_\mu\psi_D + \left|\partial_\mu a_D\right|^2 \\ &+ \left|D_\mu^+ q\right|^2 + \left|D_\mu^- q'\right|^2 + i\bar{\psi}_q\bar{\sigma}_{\text{E}}^\mu D_\mu^+\psi_q + i\bar{\psi}_{q'}\bar{\sigma}_{\text{E}}^\mu D_\mu^-\psi_{q'} \\ &+ \left[i\sqrt{2}\left(q\bar{\lambda}_D\bar{\psi}_q + q'^*\lambda_D\psi_{q'} - q\psi_D\psi_{q'} - q'\psi_D\psi_q - a_D\psi_q\psi_{q'}\right) + \text{ H.c.}\right] \\ &+ 2|a_D|^2(|q|^2 + |q'|^2) + \frac{1}{2}(|q|^2 + |q'|^2)^2 \end{aligned}$$

$$Z = \int \mathcal{D}A_D^i\mathcal{D}A_D^0\mathcal{D}a_D\mathcal{D}q\mathcal{D}q'\exp\left[-\frac{1}{g_{D,3}^2}\int_V d^3x\mathcal{L}_{\text{QED}_3}\right]$$

$$\begin{aligned} \mathcal{L}_{\text{QED}_3} =& f + \frac{1}{4}\left(F_D^{ij}\right)^2 + \frac{1}{2}(\partial_i A_D^0)^2 + \frac{1}{2}m_{\text{E}}^2(A_D^0)^2 + |\partial_i a_D|^2 + m_{\text{s}}^2|a_D|^2 \\ &+ |D_i^+ q|^2 + |D_i^- q'|^2 + \left(m_{\text{h}}^2 + (A_D^0)^2 + 2|a_D|^2\right)(|q|^2 + |q'|^2) + \frac{1}{2}(|q|^2 + |q'|^2)^2 \\ &+ \delta U_{\text{thermal}}\left(F_D^{ij}, A_D^0, a_D, q, q'\right) \end{aligned}$$

$$m_{\text{E}}^2 = m_{\text{h}}^2 = 2m_{\text{s}}^2 = g_D^2T^2 + O(g_D^4T^2)$$

$$\begin{aligned} &(T/g_{D,3}^2)U_{\text{thermal}}(a_{D0})\Big|_{\text{all other fields zero}} \\ &= -\frac{\pi^2}{6}T^4 + \frac{\pi^2}{4}\left[\frac{M_{\text{m}}^2}{\pi^2T^2} + \ln 2\left(\frac{M_{\text{m}}^2}{\pi^2T^2}\right)^2 + \sum_{n=3}^{\infty} c_n\left(\frac{M_{\text{m}}^2}{\pi^2T^2}\right)^n\right]T^4 + O(g_D^2T^4) \end{aligned}$$

$$M_{\text{m}}^2 = 2|a_{D0}|^2$$

$$\begin{aligned} \mathcal{L}_{\text{QED}_3} =& a_D(-\nabla^2 + m_{\text{s}}^2)a_D^* + \frac{1}{2}A_D^0(-\nabla^2 + m_{\text{E}}^2)A_D^0 \\ &+ (q^*, q')\begin{pmatrix} -\nabla^2 + m_{\text{h}}^2 + M_{\text{m}}^2 & 0 \\ 0 & -\nabla^2 + m_{\text{h}}^2 + M_{\text{m}}^2 \end{pmatrix}\begin{pmatrix} q \\ q'^* \end{pmatrix} + \cdots \end{aligned}$$

$$(T/g_{D,3}^2)U_{\text{static}} = 4I(M_{\text{m}}^2)T[1 + O(m_{\text{h}}^2/M_{\text{m}}^2)],$$

$$F(a_{D0})/V = T^4\left\{-\frac{\pi^2}{6} + \frac{\pi^2}{4}\left[\frac{M_{\text{m}}^2}{\pi^2 T^2} + \ln\,2\left(\frac{M_{\text{m}}^2}{\pi^2 T^2}\right)^2 + \sum_{n=3}^{\infty} c_n\left(\frac{M_{\text{m}}^2}{\pi^2 T^2}\right)^n\right] + O(g_D^2)\right\}$$
$$+M_{\text{m}}^3 T\left[-\frac{1}{3\pi} + O(g_D^2 T^2/M_{\text{m}}^2)\right]$$
$$+O((g_D T)^3 T)$$

$$\rho = \frac{M_{\text{m}}}{\pi T}$$

$$F(a_{D0})/V = \left[-\frac{\pi^2}{12} + \frac{\pi^2}{4}h(\rho) + O(g_D^2)\right]T^4$$

$$F/V = -\frac{\pi^2}{6}T^4(1 + O(T/\Lambda))$$

$$\mathcal{L}_{D,\text{eff}} = \mathcal{L}_{D,\text{eff}}^{n=2} + \mathcal{L}_{D,\text{eff}}^{n=4} + O(n \geq 6)$$

$$F(a_{D0})/V = \left[-\frac{\pi^2}{12} - \frac{\pi^4}{24}\frac{c g_{D0}^4 T^4}{|a_{D0}|^4} + O\left(\frac{g_{D0}^6 T^4}{|a_{D0}|^4}\right)\right]T^4$$

$$W = -i\frac{2}{g^2}\text{tr}\left(\sqrt{2}\Phi[Q, Q'] + mQQ'\right)$$

$$\Lambda_0 \sim \left|q_0^{1/4} m\right|$$

$$\mathcal{F}(a) \sim \frac{1}{2}\tau_0 a^2 + \frac{i}{4\pi}a^2 \ln\left(\frac{a^2}{\Lambda_0^2}\right) - \frac{i}{4\pi}(a - m/\sqrt{2})^2 \ln\left(\frac{(a - m/\sqrt{2})^2}{\Lambda_0^2}\right).$$

$$a_0 = m/\sqrt{2} + \Delta a_0$$

$$O(\bar{g}_0^8 T^8/|\Delta a_0|^4)$$

$$1/\bar{g}_0^2 = 1/g_0^2 + \frac{1}{4\pi^2}\ln\left|\frac{m/\sqrt{2}}{\Delta a_0}\right|$$

$$\frac{F}{V} = -\frac{\pi^2}{6}T^4\left(1 + O\left(\frac{T}{\Lambda_0}, \frac{T}{m}\right)\right).$$

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\lambda \mathcal{D}\psi \mathcal{D}\phi \exp\left[-\frac{1}{g^2}\int_0^\beta dx_{\text{E}}^0 \int_V d^3x \mathcal{L}\right]$$

$$Z = \int \mathcal{D}A_i \mathcal{D}A_0 \mathcal{D}\phi \exp\left[-\frac{1}{g_3^2}\int_V d^3x \mathcal{L}_{\text{ESYM}}\right]$$

$$\mathcal{L}_{\text{ESYM}} = f + 2\text{tr}\left\{\frac{1}{4}F_{ij}^2 + \frac{1}{2}(D_i A_0)^2 + \frac{1}{2}m_{\text{E}}^2 A_0^2 + |D_i\phi|^2 + m_\phi^2|\phi|^2 + |[A_0, \phi]|^2 + \frac{1}{2}\left[\phi, \phi^\dagger\right]^2\right\}$$
$$+\delta U_{\text{thermal}}\left(F_{ij}, A_0, \phi\right)$$

$$(\pi T)^{-1} \ll |\vec{x} - \vec{y}| \ll (gT)^{-1}$$

$$f = 3g_3^2 T^3\left[-\frac{\pi^2}{12} + \frac{g^2}{8} + O(g^4)\right]$$

$$m_{\text{E}}^2 = 2g^2 T^2 + O(g^4 T^2)$$

$$m_\phi^2 = g^2 T^2 + O(g^4 T^2).$$

$$A_0^{\text{ren}}(\vec{x}) \equiv Z_{A_0}^{1/2} A_0^{(0)}(\vec{x})$$

$$(T/g_3^2)U_{\text{thermal}}\,(\phi_{\text{curvature}} = a\sigma^3/2)|_{\text{all other fields zero}}$$
$$= -\frac{\pi^2}{4}T^4 + \frac{\pi^2}{2}\left[\frac{M_W^2}{\pi^2T^2} + (\ln\ 2)\left(\frac{M_W^2}{\pi^2T^2}\right)^2 + \sum_{n=3}^{\infty} c_n\left(\frac{M_W^2}{\pi^2T^2}\right)^n\right]T^4 + O(g^2T^4)$$

$$M_W^2 = 2|a|^2$$

$$c_n = 8(-1)^n(1-4^{2-n})\frac{(2n-5)!!}{(2n)!!}\zeta(2n-3)$$

$$\rho \equiv \frac{M_W}{\pi T}$$

$$\mathcal{L}_{\text{ESYM}} \supset \frac{1}{2}A_i^3\Delta_{ij}A_j^3 + W_i\big(\Delta_{ij} + M_W^2\delta_{ij}\big)W_j^*$$

$$\Delta_{ij} = -\delta_{ij}\nabla^2 + (1-\alpha^{-1})\partial_i\partial_j$$

$$W_i \equiv \big(A_i^1 - iA_i^2\big)/\sqrt{2}$$

$$\mathcal{L}_{\text{ESYM}} \supset \frac{1}{2}A_0^3\big(-\nabla^2 + m_{\text{E}}^2\big)A_0^3 + W_0\big(-\nabla^2 + m_{\text{E}}^2 + M_W^2\big)W_0^*$$

$$W_0 \equiv (A_0^1 - iA_0^2)/\sqrt{2}$$

$$\mathcal{L}_{\text{ESYM}} \supset \phi^3\big(-\nabla^2 + m_\phi^2\big)\phi^{3*} + \Sigma^\dagger\begin{pmatrix} -\nabla^2 + m_\phi^2 + |a|^2 & -a^2 \\ -(a^*)^2 & -\nabla^2 + m_\phi^2 + |a|^2 \end{pmatrix}\Sigma$$

$$\Sigma = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^1 - i\sigma^2 \\ \sigma^{1*} - i\sigma^{2*} \end{pmatrix}$$

$$I(m^2) \equiv \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\ln\big(\vec{k}^2 + m^2\big) = -\frac{1}{12\pi}(m^2)^{3/2}$$

$$(T/g_3^2)U_{\text{static}} = 2\cdot 4I(M_W^2)T[1 + O(g^2T^2/M_W^2)]$$

$$F(a)/V = T^4\left\{-\frac{\pi^2}{4} + \frac{\pi^2}{2}\left[\frac{M_W^2}{\pi^2T^2} + \ln\ 2\left(\frac{M_W^2}{\pi^2T^2}\right)^2 + \sum_{n=3}^{\infty} c_n\left(\frac{M_W^2}{\pi^2T^2}\right)^n\right] + O(g^2)\right\}$$
$$+M_W^3T\left[-\frac{2}{3\pi} + O\left(\frac{g^2T^2}{M_W^2}\right)\right]$$
$$+O((gT)^3T)$$

$$F(a)/V = \left[-\frac{\pi^2}{12} + \frac{\pi^2}{2}h(\rho) + O(g^2)\right]T^4$$

$$h(\rho) \equiv -\frac{1}{3} + \rho^2 - \frac{4}{3}\rho^3 + (\ln\ 2)\rho^4 + \sum_{n\geq 3} c_n\rho^{2n}$$

$$\Psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$$

$$D = -\tfrac{1}{\sqrt{2}}\Gamma(a,\bar{a})i\lambda\psi + \text{H.c.}$$

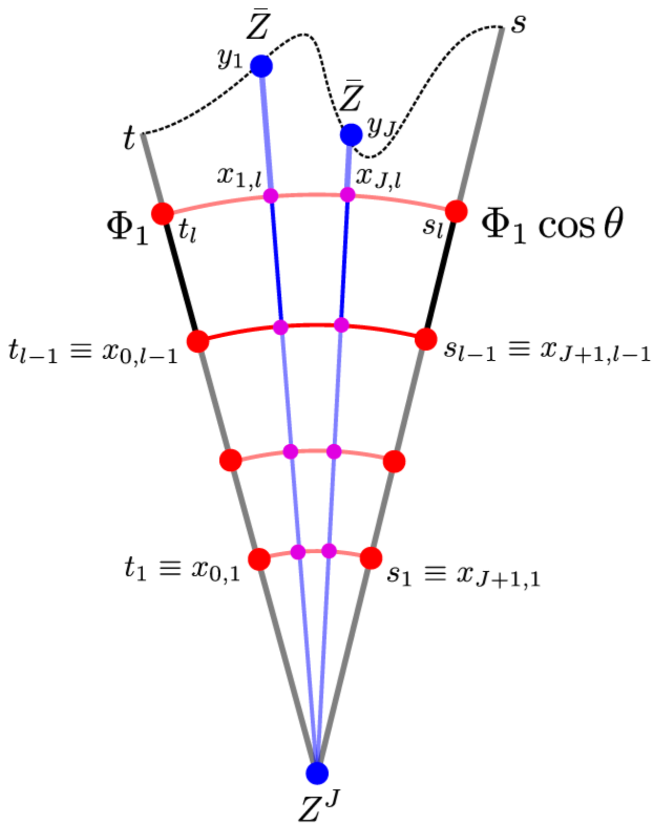
$$F = \frac{1}{2}\Gamma(a,\bar{a})\psi^2 + \frac{1}{2}\Gamma(a,\bar{a})^*\bar{\lambda}^2$$

$$f_{\pm} = \pm\frac{1}{2}T\sum_{n\in\mathbb{Z}}\int \frac{d^3p}{(2\pi)^3}\ln\left[\left(\omega_n^{\pm}\right)^2 + \vec{p}^2 + M_W^2\right]$$

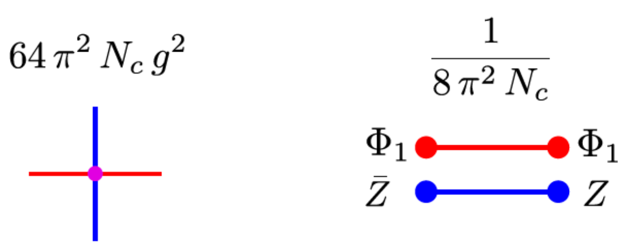
$$f_{\pm} = (\mathfrak{T}) \pm \frac{T^4}{2\pi^2}\int_0^{\infty} dx x^2 \ln\left(1 \mp e^{-\sqrt{x^2+M_W^2/T^2}}\right)$$

$$\mathcal{H}_{\ell,\pm} = T\sum_{\ell_0}\int \frac{d^3\ell}{(2\pi)^3}$$

$$A_0^{(0)}(\vec{x}) = \beta^{-1}\int_0^{\beta} dx_{\mathrm{E}}^0 A_0(x)$$



$$\hat{g} \equiv g\left(\frac{\exp\left(-i\theta/2\right)}{2}\right)^{\frac{1}{J+1}}$$



$$Z = \frac{\Phi_5 + i\Phi_6}{\sqrt{2}}$$

$$g = \frac{\sqrt{\lambda}}{4\pi}$$

$$\lambda = g_{YM}^2 N$$

$$W = \frac{1}{N}\text{trPexp}\int_0^\infty dt4\pi g(iA\cdot x'(t) + \Phi_1|x'(t)|)$$
$$\times Z(0)^J \times \text{Pexp}\int_0^\infty ds4\pi g[-iA\cdot x'(s) + (\Phi_1\cos\,\theta + \Phi_2\sin\,\theta)|x'(s)|]$$

$$x'(t) \equiv \frac{\partial x(t)}{\partial t}$$

$$x'(s) \equiv \frac{\partial x(s)}{\partial s}$$

$$\langle W\rangle \sim \left(\frac{R^{IR}}{\epsilon^{UV}}\right)^{-\Delta},$$

$$\hat{g} \equiv g\left(\frac{\exp\,(-i\theta/2)}{2}\right)^{\frac{1}{J+1}},$$

$$\psi(t, y_1, \dots, y_J, s) \equiv \frac{1}{N}\left\langle \text{tr}\prod_{j=1}^{J} \bar{Z}(y_j)\times \text{Pexp}\int_0^t dt'(4\pi g)|x'(t')|\Phi_1 \right.$$
$$\left. \times Z(0)^J \times \text{Pexp}\int_0^s ds'(4\pi g)|x'(s')|\Phi_1\cos\,\theta\right\rangle$$

$$\psi(t, y_1, \dots, y_J, s) =$$
$$\sum_{l=0}^{\infty}\psi_l(t, y_1, \dots, y_J, s) = \sum_{l=0}^{\infty}\text{tr}\int_0^t dt_l|x'(t_l)|\int_0^{t_l} dt_{l-1}|x'(t_{l-1})|\cdots\int_0^{t_2} dt_1|x'(t_1)|$$
$$\int_0^s ds_l|x'(s_l)|\int_0^{s_l} ds_{l-1}|x'(s_{l-1})|\cdots\int_0^{s_2} ds_1|x'(s_1)|F_l(y_j, t_i, s_i)$$

$$F_l(y_j, t_i, s_i) = \frac{1}{N}\left(\frac{1}{8\pi^2N}\right)^{l(J+1)+J(l+1)}(64\pi^2Ng^2)^{lJ}(16\pi^2g^2\cos\,\theta)^l N^{(l+1)(J+1)+1}$$
$$\int\left(\prod_{i=1}^{J}\prod_{j=1}^{l} d^4x_{i,j}\right)\left(\prod_{r=0}^{J}\prod_{k=1}^{l}\frac{1}{(x_{r+1,k} - x_{r,k})^2}\right)\left(\prod_{m=1}^{J}\prod_{n=0}^{l}\frac{1}{(x_{m,n+1} - x_{m,n})^2}\right)$$

$$x_{k,0} \equiv y_0 \equiv (e^{t_k}, 0{,}0{,}0), x_{k,J+1} \equiv y_{J+1} \equiv (e^{s_k}\cos\,\varphi, e^{s_k}\sin\,\varphi, 0{,}0)\forall k = 1\dots l$$

$$x_{l+1,j} \equiv y_j, x_{0,j} \equiv 0\forall j = 1\dots J$$

$$\Box_{y_j}\frac{1}{(y_j - x_{j,l})^2} = -4\pi^2\delta(y_j - x_{j,l}).$$

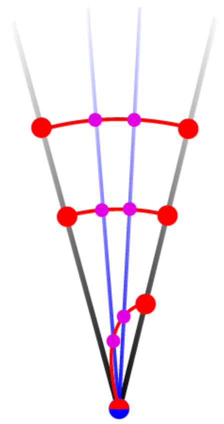
$$|y_0'(t)||y_{J+1}'(s)|$$

$$\partial_t\partial_s\prod_{j=1}^{J}\Box_{y_j}\psi_l = (-1)^J(4\hat{g}^2)^{J+1}\frac{|y_0'||y_{J+1}'|}{\prod_{i=0}^{J}(y_i - y_{i+1})^2}\psi_{l-1}$$

$$(\hat{B}^{-1} - 1)\psi = 0\,, \hat{B}^{-1} \equiv \frac{(-1)^J}{(4\hat{g}^2)^{J+1}}\frac{\prod_{i=0}^{J}(y_i - y_{i+1})^2}{|y_0'||y_{J+1}'|}\partial_t\partial_s\prod_{j=1}^{J}\Box_{y_j}$$

$$\hat{D} = -i\left(\partial_t + \partial_s + \sum_{i=1}^{J}(y_i\cdot\partial_{y_i} + 1)\right)$$

$$\hat{S} = i\sum_{i=1}^{J}\left(y_i^3\partial_{y_i^4} - y_i^4\partial_{y_i^3}\right)$$



$$p_i = -i\partial_{y_i}, \pi_t = -i\partial_t, \pi_s = -i\partial_s$$

$$\hat{H} = \pi_t\pi_s\prod_{i=1}^{J} p_i^2 + (4\hat{g}^2)^{J+1}\frac{|y_0'(t)||y_{J+1}'(s)|}{\prod_{i=0}^{J}(y_i - y_{i+1})^2}$$

$$y_0(t) = (e^t, 0,0,0), y_{J+1}(s) = (e^s\cos\,\varphi, e^s\sin\,\varphi, 0,0)$$

$$L = 2^{\frac{-2J}{2J+1}}(2J+1)\left(\dot{t}\dot{s}\prod_{i=1}^{J}\dot{y}_i^2\right)^{\frac{1}{2J+1}} - (4\hat{g}^2)^{J+1}\frac{|y_0'(t)||y_{J+1}'(s)|}{\prod_{i=0}^{J}(y_i - y_{i+1})^2}$$

$$\dot{f} \equiv \frac{d}{d\tau}f$$

$$S = \int L d\tau$$

$$L = 2^{\frac{-2J}{2J+1}}(2J+1)\left(\frac{1}{\gamma}\dot{t}\dot{s}\prod_{i=1}^{J}\dot{y}_i^2\right)^{\frac{1}{2J+1}} - \gamma(4\hat{g}^2)^{J+1}\frac{|y_0'(t)||y_{J+1}'(s)|}{\prod_{i=0}^{J}(y_i - y_{i+1})^2}$$

$$L = (2J+2)(2i)^{\frac{1}{J+1}}\hat{g}\left[\frac{|\dot{y}_0||\dot{y}_{J+1}|\prod_{i=1}^{J}\dot{x}_i^2}{\prod_{i=0}^{J}|y_i - y_{i+1}|^2}\right]^{\frac{1}{2(J+1)}}$$

$$y_i^\mu = \frac{X_i^\mu}{X_i^+}, X_i^2 = 0, X_i^+ = X_i^0 + X_i^{-1}$$

$$L = (2J+2)(2i)^{\frac{1}{J+1}}\hat{g}\left[\frac{|\dot{X}_0||\dot{X}_{J+1}|\prod_{i=1}^{J}\dot{X}_i^2}{\prod_{i=0}^{J}(-2X_i\cdot X_{i+1})}\right]^{\frac{1}{2(J+1)}}$$

$$L = \xi\left(\alpha_0\frac{|\dot{X}_0||\dot{X}_{J+1}|}{2} + \sum_{i=1}^{J}\left(\alpha_i\frac{\dot{X}_i^2}{2} + \eta_i X_i^2\right) + (J+1)\prod_{k=0}^{J}(-\alpha_k X_k\cdot X_{k+1})^{-\frac{1}{J+1}}\right),$$

$$\xi \equiv (2i)^{\frac{1}{J+1}}\hat{g}$$

$$X_i \to g_i(\tau)X_i, \alpha_i \to \alpha_i g_i^{-1/2}(\tau), \eta_i \to \eta_i g_i^{-1/2}(\tau), i = 0\ldots J+1$$

$$X_i \to \frac{X_i}{\dot{f}}, \alpha_i \to \dot{f}\alpha_i, \eta_i \to \frac{\eta_i}{\dot{f}}$$

$$\dot{X}_k^2 = \mathcal{L}$$

$$\mathcal{L} \equiv 2\prod_{i=0}^{J} (-X_i \cdot X_{i+1})^{-\frac{1}{J+1}}$$

$$|\dot{X}_0||\dot{X}_{J+1}| = \mathcal{L}$$

$$X_{J+1} \to \frac{1}{h(\tau)} X_{J+1}$$

$$|\dot{X}_0| = |\dot{X}_{J+1}|$$

$$\mathcal{L} = 1,$$

$$L = \xi\left(\frac{|\dot{X}_0||\dot{X}_{J+1}|}{2} + \sum_{i=1}^{J} \frac{\dot{X}_i^2}{2} + (J+1)\prod_{k=0}^{J} (-X_k \cdot X_{k+1})^{-\frac{1}{J+1}}\right),$$

$$2|\dot{X}_0||\dot{X}_{J+1}| = \dot{X}_0^2 + \dot{X}_{J+1}^2 - (|\dot{X}_0| - |\dot{X}_{J+1}|)^2$$

$$|\dot{X}_0||\dot{X}_{J+1}| \to \frac{\dot{X}_0^2}{2} + \frac{\dot{X}_{J+1}^2}{2}$$

$$y = 2\prod_{i=0}^{J} (-X_i \cdot X_{i+1})^{-\frac{1}{J+1}} \simeq 1$$

$$y = e^{\log y} = 1 + \log y + \mathcal{O}(\log^2 y)$$

$$\sum_{k=0}^{J} \frac{1}{2} \log \frac{-X_k \cdot X_{k+1}}{2e}$$

$$L = \xi\left(\frac{\dot{X}_0^2}{4} + \sum_{i=1}^{J} \frac{\dot{X}_i^2}{2} + \frac{\dot{X}_{J+1}^2}{4} - \sum_{i=0}^{J} \frac{1}{2} \log \frac{-X_i \cdot X_{i+1}}{2e}\right),$$

$$\prod_{i=0}^{J} \frac{-X_i \cdot X_{i+1}}{2} = 1$$

$$X_i^2 = 0, \dot{X}_i^2 = 1, i = 0, \dots, J+1$$

$$X_i(\tau) = r_i(\tau)(\cosh w_i(\tau), -\sinh w_i(\tau), \cos \phi_i, \sin \phi_i, 0,0) \; i = 0, J+1$$

$$\ddot{X}_j = 2\eta_j X_j - \frac{1}{2}\left(\frac{X_{j+1}}{X_{j+1} \cdot X_j} + \frac{X_{j-1}}{X_j \cdot X_{j-1}}\right), j = 1, \dots, J$$

$$\frac{\ddot{t}}{\dot{t}^2} = \frac{X_1 \cdot \partial_{t(\tau)} X_0}{X_1 \cdot X_0}, \frac{\ddot{s}}{\dot{s}^2} = \frac{X_J \cdot \partial_{s(\tau)} X_{J+1}}{X_J \cdot X_{J+1}}$$

$$C_N^M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{MN}, G^M{}_N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi & 0 & 0 \\ 0 & 0 & \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{MN}$$

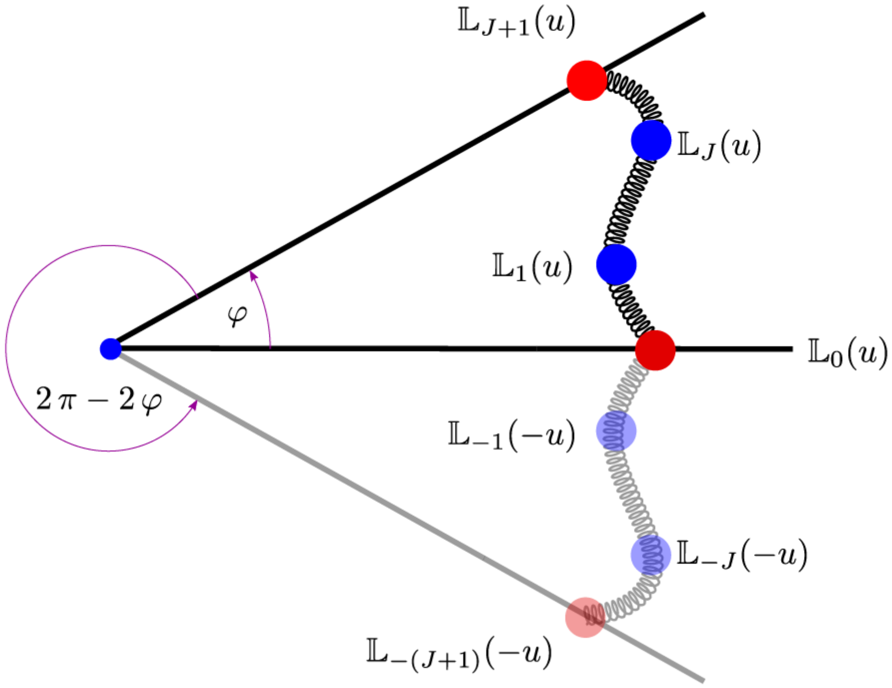
$$X_{-1} = C \cdot X_1$$

$$X_{J+2} = G \cdot C \cdot G^{-1} \cdot X_J = G^2 \cdot C \cdot X_J$$

$$q_j^{MN} \equiv \dot{X}_j^M X_j^N - \dot{X}_j^N X_j^N = 2\dot{X}_j^{[M} X_j^{N]}$$

$$Q^{MN} = \xi \left(\frac{q_0^{MN}}{2} + \sum_{i=0}^{J} q_j^{MN} + \frac{q_{J+1}^{MN}}{2} \right)$$

$$S = Q_{3,4}\,, D = Q_{-1,0} = i\Delta.$$



$$X_a(\tau) = \frac{1}{\sqrt{\dot{w}_a^2(\tau) + \dot{\phi}_a^2(\tau)}} (\cosh\ w_a(\tau), -\sinh\ w_a(\tau), \cos\ \phi_a(\tau), \sin\ \phi_a(\tau), 0, 0).$$

$$\phi_0(\tau) \ = 0,$$
$$\phi_{J+1}(\tau) \ = \varphi.$$

$$\phi_k(\tau) = \frac{k}{J+1}\varphi$$

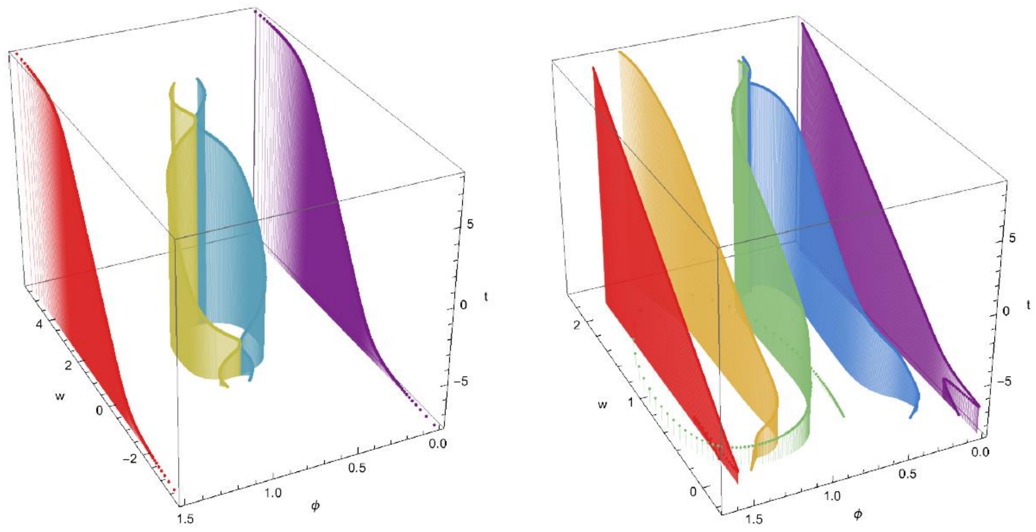
$$w_k(\tau) = \beta\tau$$

$$\left(\frac{\sin\left(\frac{\varphi}{2J+2}\right)}{\beta} \right)^{2J+2} = 1,$$

$$\beta = e^{2\pi i \frac{n}{2J+2}} \sin\left(\frac{\varphi}{2J+2}\right), n = 1 \dots 2J+2$$

$$\Delta = -\frac{(J+1)i}{\beta}\xi$$

$$\Delta = \pm \frac{2\hat{g}}{\sin\frac{\varphi}{2}}$$



$$P_i^M = \xi \dot{X}_i^M,$$

$$q_i^{MN} = \frac{1}{\xi}\left(X_i^N P_i^M - X_i^M P_i^N\right) = X_i^N \dot{X}_i^M - X_i^M \dot{X}_i^N,$$

$$\{q_k^{MN}, q_k^{KL}\} = \frac{1}{\xi}\left(-\eta^{MK} q_k^{NL} + \eta^{NK} q_k^{ML} + \eta^{ML} q_k^{NK} - \eta^{NL} q_k^{MK}\right), k = 1, \dots, J.$$

$$\Pi_t = \frac{\xi}{2t'(\tau)}, \Pi_s = \frac{\xi}{2s'(\tau)}$$

$$q_0^{NM} = \frac{2}{\xi}\left(Y_0^M Y_0'^N - Y_0^N Y_0'^M\right)\Pi_t, q_{J+1}^{NM} = \frac{2}{\xi}\left(Y_{J+1}^M Y_{J+1}'^N - Y_{J+1}^N Y_{J+1}'^M\right)\Pi_s,$$

$$Y_0 = \{\cosh\ t, -\sinh\ t, 1,0,0,0\}, Y_{J+1} = \{\cosh\ s, -\sinh\ s, \cos\ \varphi, \sin\ \varphi, 0,0\}.$$

$$\{q_0^{MN}, q_0^{KL}\} = \frac{1}{\xi}\left(-\tilde{\eta}^{MK} q_0^{NL} + \tilde{\eta}^{NK} q_0^{ML} + \tilde{\eta}^{ML} q_0^{NK} - \tilde{\eta}^{NL} q_0^{MK}\right), \tilde{\eta} = \eta(\mathbb{1} + C),$$

$$\{q_{J+1}^{MN}, q_{J+1}^{KL}\} = \frac{1}{\xi}\left(-\tilde{\eta}_\phi^{MK} q_{J+1}^{NL} + \tilde{\eta}_\phi^{NK} q_{J+1}^{ML} + \tilde{\eta}_\phi^{ML} q_{J+1}^{NK} - \tilde{\eta}_\phi^{NL} q_{J+1}^{MK}\right), \tilde{\eta}_\phi = \eta(\mathbb{1} + G.G.C),$$

$$H_q \equiv \frac{1}{2^{2J+2}} \mathrm{tr}\left(q_0^2 \cdot q_1^2 \dots q_J^2 \cdot q_{J+1}^2 \cdot G \cdot G \cdot C \cdot q_J^2 \dots q_1^2 \cdot C\right) - 1.$$

$$H_q = \exp\left(\frac{4}{\xi} H\right) - 1 \simeq \frac{4}{\xi} H + \mathcal{O}(H^2)$$

$$j_i^{MN} = -2 \frac{X_{i-1}^{[M} X_i^{N]}}{X_{i-1} \cdot X_i}$$

$$\dot{q}_i^{MN} = \{q_i^{MN}, H\} = -\frac{1}{2}\left(j_{i+1}^{MN} - j_i^{MN}\right), i = 0, \dots, J+1$$

$$\mathbb{L}_i = u \mathbb{I}_{4x4} + \frac{i}{2} q_i^{MN} \Sigma_{MN}, \mathbb{V}_i = -\frac{i}{4u} j_i^{MN} \Sigma_{MN}$$

$$\dot{\mathbb{L}}_i = \mathbb{L}_i \cdot \mathbb{V}_{i+1} - \mathbb{V}_i \cdot \mathbb{L}_i = \mathbb{V}_{i+1} \cdot \mathbb{L}_i - \mathbb{L}_i \cdot \mathbb{V}_i$$

$$\mathbb{T}(u) = \mathrm{tr} \mathbb{L}_{-J}(u) \cdot \mathbb{L}_{-1}(u) \cdot \mathbb{L}_0(u) \cdot \mathbb{L}_1(u) \cdots \mathbb{L}_J(u) \cdot \mathbb{L}_{J+1}(u) \cdot G^4 \cdot G^4,$$

$$\mathbb{L}_{-i}(u) = \boldsymbol{C^4} \cdot \mathbb{L}_i^t(-u) \cdot \boldsymbol{C^4}$$

$$C^{4ab} = C^{4}_{ab} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}_{ab}, G^{4}{}^{a}{}_{f} = \begin{pmatrix} e^{i\frac{\varphi}{2}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\varphi}{2}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\varphi}{2}} & 0 \\ 0 & 0 & 0 & e^{i\frac{\varphi}{2}} \end{pmatrix}^{a}_{f}$$

$$\xi\{(\mathbb{L}_n)^a{}_b(u), (\mathbb{L}_m)^c{}_d(v)\} = \frac{(\mathbb{L}_n)^a{}_d(u)(\mathbb{L}_n)^c{}_b(v) - (\mathbb{L}_n)^c{}_b(u)(\mathbb{L}_n)^a{}_d(v)}{u-v}\delta_{nm}$$

$$\xi\{\mathbb{L}_n(u), \mathbb{L}_m(v)\} = [r(u,v), \mathbb{L}_n(u) \otimes \mathbb{L}_m(v)]\delta_{nm}$$

$$\mathbb{K}(u) \equiv C \cdot \mathbb{L}_0(u)\,, \overline{\mathbb{K}}(u) \equiv G^{-1} \cdot \mathbb{L}_{J+1}(u) \cdot G \cdot C,$$

$$\xi\{\mathbb{K}_{ab}(u), \mathbb{K}_{cd}(v)\} = \frac{\mathbb{K}_{ad}(u)\mathbb{K}_{cb}(v) - \mathbb{K}_{ad}(v)\mathbb{K}_{cb}(u)}{u-v} - \frac{\mathbb{K}_{db}(u)\mathbb{K}_{ca}(v) - \mathbb{K}_{bd}(v)\mathbb{K}_{ac}(u)}{u+v}$$

$$\{\mathbb{T}(u), \mathbb{T}(v)\} = 0$$

$$\hat{\mathbb{L}}_i{}^a{}_b(u) = u\delta^a_b + \frac{i}{2}\hat{q}_i^{MN}\Sigma_{MN}{}^a{}_b$$

$$P_j^K \to \hat{P}_j^K = -i\partial_{X_{j,K}}$$

$$\hat{q}_j^{MN} = -\frac{i}{\xi}\left(X_j^N \frac{\partial}{\partial X_{j,M}} - X_j^M \frac{\partial}{\partial X_{j,N}}\right)$$

$$[\hat{q}_k^{MN}, \hat{q}_k^{KL}] = \frac{i}{\xi}\left(-\eta^{MK}\hat{q}_k^{NL} + \eta^{NK}\hat{q}_k^{ML} + \eta^{ML}\hat{q}_k^{NK} - \eta^{NL}\hat{q}_k^{MK}\right), k = 1, \dots, J$$

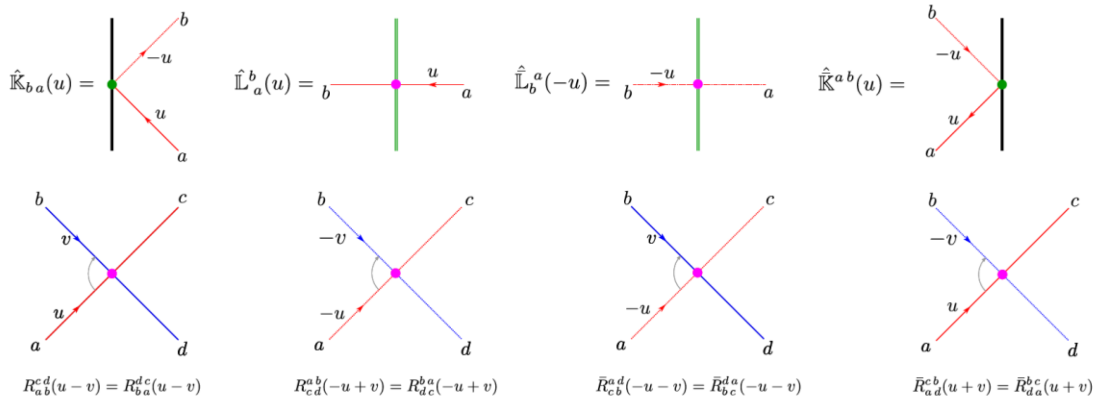
$$f(y_1, \dots, y_m) \to \frac{1}{X^{-1} + X^0} f\left(\frac{X^1}{X^{-1} + X^0}, \dots, \frac{X^4}{X^{-1} + X^0}\right)$$

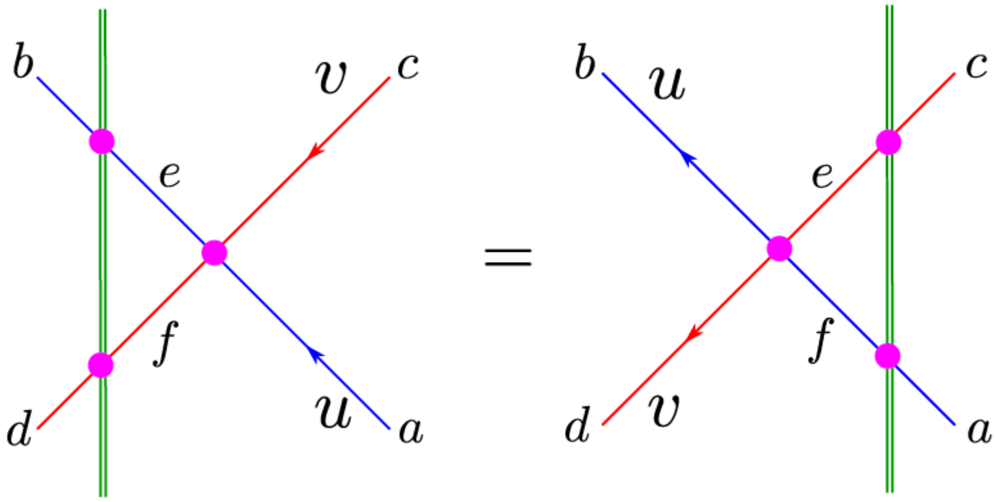
$$\hat{\mathbb{L}}_e^b(u)\hat{\mathbb{L}}_f^d(v)R_{ac}^{ef}(v-u) = R_{fe}^{bd}(v-u)\hat{\mathbb{L}}_c^e(v)\hat{\mathbb{L}}_a^f(u),$$

$$R_{ad}^{bc}(u) = \mathbb{I}_{(\text{aux}\times\text{aux})} + \frac{i}{\xi u}\mathbb{P} = \delta_a^b\delta_d^c + \frac{i}{\xi u}\delta_d^b\delta_a^c$$

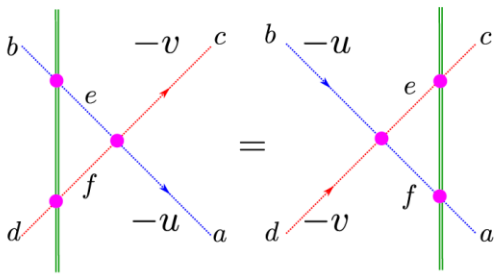
$$\hat{\hat{\mathbb{L}}}_{ib}{}^a(u) = -\hat{\mathbb{L}}_i{}^a{}_b(u),$$

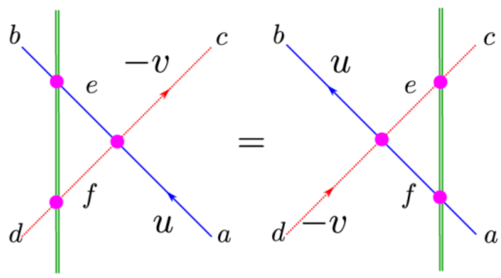
$$\bar{R}_{ad}^{bc}(u) = R_{ad}^{bc}(-u)$$





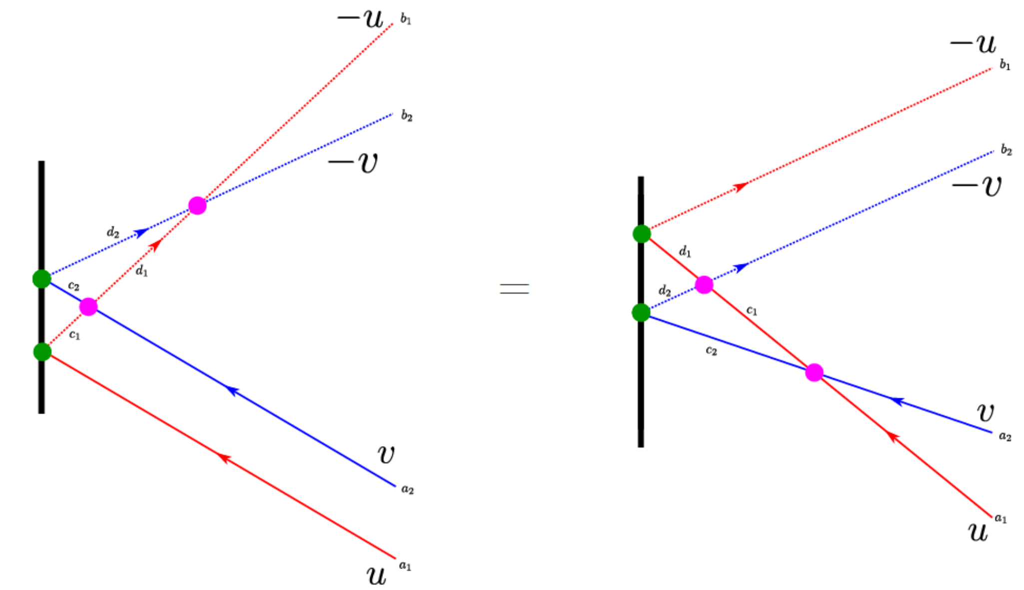
$$\hat{q}_0^{NM} \equiv -i\frac{2}{\xi}(Y_0^M\dot{Y}_0^N - Y_0^N\dot{Y}_0^M)\partial_t\,,\hat{q}_{J+1}^{NM} \equiv -i\frac{2}{\xi}\left(Y_{J+1}^M\dot{Y}_{J+1}^N - Y_{J+1}^N\dot{Y}_{J+1}^M\right)\partial_s,$$





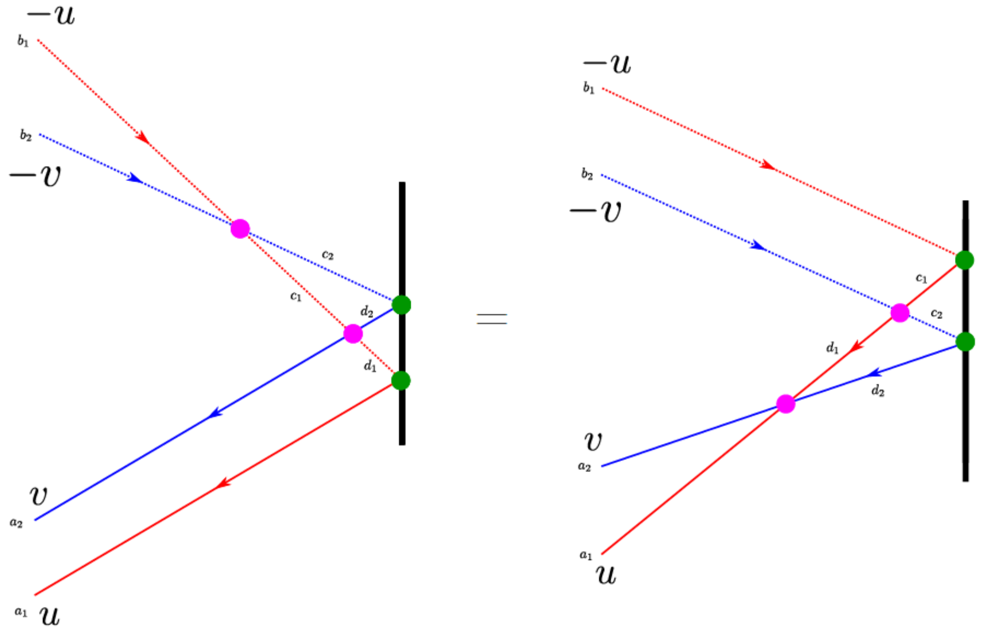
$$\hat{\mathbb{L}}_0{}^a{}_b(u) = u\delta_b^a + \frac{i}{2}\hat{q}_0^{MN}\Sigma_{MN}{}^a{}_b, \hat{\mathbb{L}}_{J+1}{}^a{}_b(u) = u\delta_b^a + \frac{i}{2}\hat{q}_{J+1}^{MN}\Sigma_{MN}{}^a{}_b.$$

$$\hat{\mathbb{K}}_{d_2c_2}(v)\bar{R}_{a_2d_1}^{c_2c_1}(u+v)\hat{\mathbb{K}}_{c_1a_1}(u)R_{b_1b_2}^{d_1d_2}(v-u) = R_{a_2a_1}^{c_2c_1}(v-u)\hat{\mathbb{K}}_{b_1d_1}(u)\bar{R}_{c_1b_2}^{d_1d_2}(u+v)\hat{\mathbb{K}}_{d_2c_2}(v).$$



$$\hat{\mathbb{K}}(u) = C \cdot \hat{\mathbb{L}}_0\left(u - \frac{i}{2\xi}\right),$$

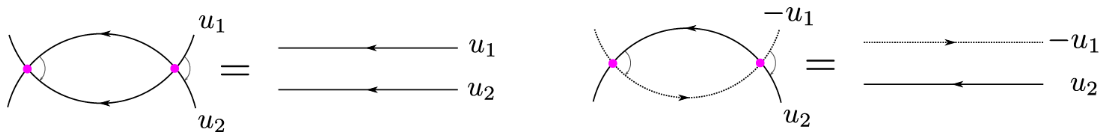
$$R^{b_1b_2}_{c_1c_2}(u-v)\hat{\bar{\mathbb{K}}}^{d_2c_2}(v)\bar{R}^{a_2c_1}_{d_2d_1}(-u-v)\hat{\bar{\mathbb{K}}}^{a_1d_1}(u) = \hat{\bar{\mathbb{K}}}^{c_1b_1}(u)\bar{R}^{d_1b_2}_{c_1c_2}(-u-v)\hat{\bar{\mathbb{K}}}^{d_2c_2}(v)R^{a_1a_2}_{d_1d_2}(u-v),$$



$$\hat{\bar{\mathbb{K}}}(u) = G^{-1} \cdot \mathbb{L}_{J+1}\left(u + \frac{i}{2\xi}\right) \cdot G \cdot C,$$

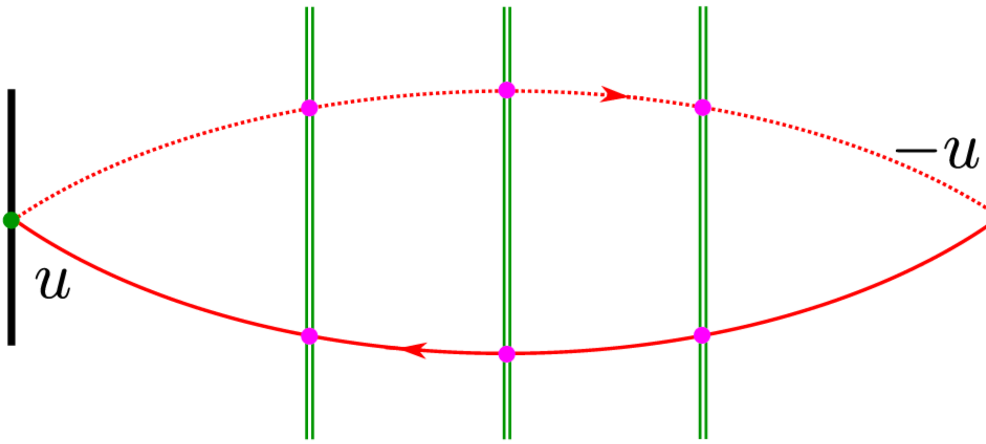
$$S(u) = a(u)R(u)\,, \bar{S}(u) = a(-u)\bar{R}(u),$$

$$S(u)S(-u) = I,$$



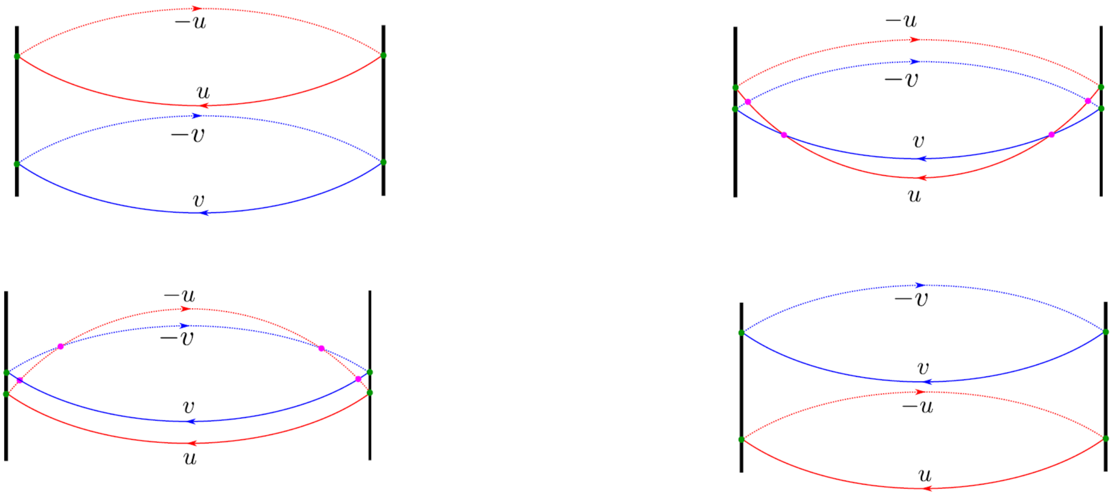
$$S_b^a{}_e^d(u)S_c^b{}_f^e(-u) = \bar{S}_b^a{}_e^d(-u)\bar{S}_c^b{}_c^e(u) = \delta_c^a\delta_f^d.$$

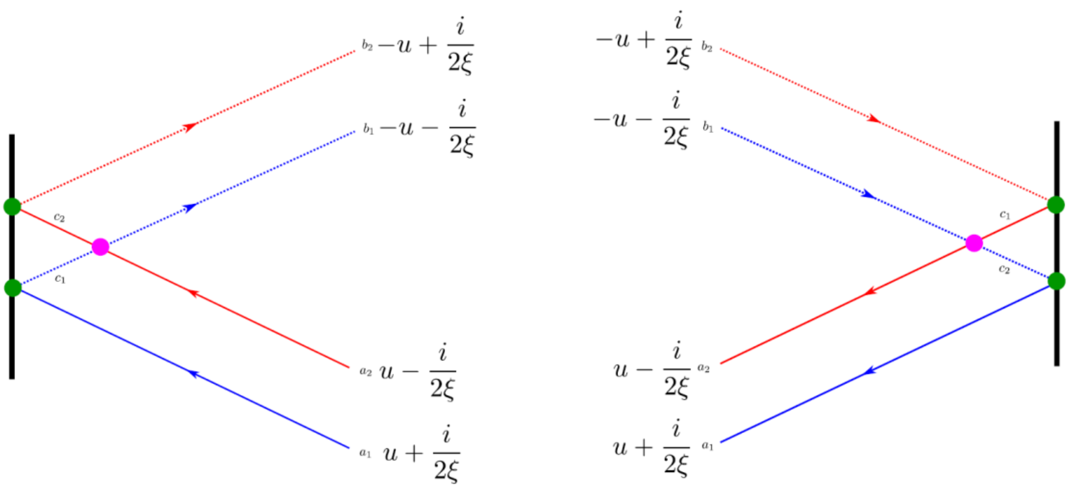
$$A(u) \equiv a(u)a(-u) = \frac{u^2\xi^2}{1-u^2\xi^2}$$



$$\hat{\mathbb{T}}^4(u) = \mathrm{tr}\left[\bar{\mathbb{L}}_J(-u)\cdots\bar{\mathbb{L}}_2(-u)\bar{\mathbb{L}}_1(-u)\mathbb{K}(u)\mathbb{L}_1(u)\ldots\mathbb{L}_{J-1}(u)\mathbb{L}_J(u)G\bar{\mathbb{K}}(u)G^t\right].$$

$$\left[\hat{\mathbb{T}}^4(u), \hat{\mathbb{T}}^4(v)\right] = 0.$$





(a) $\hat{\mathbb{K}}^6$ (b) $\hat{\hat{\mathbb{K}}}^6$

$$\hat{\mathbb{L}}_i^{6\ MN}(u) = \left(u^2 - \frac{1}{8}\mathrm{tr}\hat{q}_i^2\right)\eta^{MN} - u\hat{q}_i^{MN} + \left(\frac{1}{2}\hat{q}_i^{2MN} - \frac{i}{\xi}\hat{q}_i^{MN} + \frac{1}{4\xi^2}\eta^{MN}\right).$$

$$\hat{\hat{\mathbb{L}}}_i^6(u) \equiv \hat{\mathbb{L}}_i^6(-u).$$

$$\begin{aligned}\hat{\mathbb{K}}_{MN}^6(u) &= C_{MN}^6 u\left(u - \frac{i}{\xi}\right) + u\frac{2i}{\xi}(Y_N\dot{Y}_M - Y_M\dot{Y}_N)\partial_t \\ &\quad + \frac{2}{\xi^2}Y_N\hat{\partial}_t Y_M\hat{\partial}_t - \frac{2i}{\xi^3 u}Y_M\hat{\partial}_t Y_N\hat{\partial}_t\end{aligned}$$

$$\begin{aligned}\hat{\hat{\mathbb{K}}}_{MN}^6(u) &= C_{MN}^6 u\left(u + \frac{i}{\xi}\right) + u\frac{2i}{\xi}(Y_N\dot{Y}_M - Y_M\dot{Y}_N)\partial_s \\ &\quad + \frac{2}{\xi^2}Y_M\hat{\partial}_s Y_N\hat{\partial}_s + \frac{2i}{\xi^3 u}Y_N\hat{\partial}_s Y_M\hat{\partial}_s\end{aligned}$$

$$\hat{\mathbb{K}}^6(u) = C^6 u\left(u - \frac{i}{\xi}\right) - u\hat{q}_0 + \frac{1}{2}\hat{q}_0^2 - \frac{i}{2\xi u}(\hat{q}_0^2)^T.$$

$$\hat{\hat{\mathbb{K}}}^6(u) = (G^6)^{-1}\cdot\left(u\left(u + \frac{i}{\xi}\right) - u\hat{q}_{J+1} + \frac{1}{2}\left(\hat{q}_{J+1}^2\right)^T + \frac{i}{2\xi u}\hat{q}_{J+1}^2\right)\cdot G^6\cdot C^6$$

$$\mathbb{T}^6(0) = 4\lim_{u\to 0}u^2\xi^2\mathrm{tr}\left[\overline{\mathbb{L}}_J(0)\cdots\overline{\mathbb{L}}_2(0)\overline{\mathbb{L}}_1(0)\mathbb{K}(u)\mathbb{L}_1(u)\ldots\mathbb{L}_{J-1}(0)\mathbb{L}_J(0)G\overline{\mathbb{K}}(u)G^t\right]$$

$$\hat{\mathbb{L}}_i^{6\ MN}(0) = \frac{\hat{q}_i^{2MN}}{2} - \frac{i}{\xi}\hat{q}_i^{MN} - \frac{\eta^{MN}}{8}\mathrm{tr}\hat{q}_i^2 + \frac{\eta^{MN}}{4\xi^2} = \frac{:\hat{q}_i^2:^{MN}}{2} = \frac{1}{2\xi^2}X_i^M X_i^N\partial_{X_i^K}^2$$

$$u\xi\hat{\mathbb{K}}^{6\ MN}(u)\Big|_{u=0} = -\frac{i}{2}(\hat{q}_0^2)^{NM} = -\frac{2i}{\xi^2}Y_0^M\hat{\partial}_t Y_0^N\hat{\partial}_t$$

$$u\xi\left(G^6\hat{\hat{\mathbb{K}}}^6(u)(G^6)^{-1}\right)^{MN}\Bigg|_{u=0} = +\frac{i}{2}\left(\hat{q}_{J+1}^2\right)^{MN} = +\frac{2i}{\xi^2}Y_{J+1}^N\hat{\partial}_s Y_{J+1}^M\hat{\partial}_s$$

$$\begin{aligned}\mathbb{T}^6(0) &= 4\frac{4}{2^{2J}\xi^{4J+4}}\eta_{NM}X_J^M X_J\cdot X_{J-1}\ldots X_1\cdot Y_0\partial_t\prod_{i=1}^{J}\Box_i^{(6)} \\ &\quad\times Y_0\cdot X_1 X_1\cdot X_2\ldots X_J\cdot Y_{J+1}\partial_s Y_{J+1}^N\partial_s\partial_t\prod_{i=1}^{J}\Box_i^{(6)}\end{aligned}$$

$$\Box^{(6)} = \Box^{(4)} + \partial_{X_+}\partial_{X_-},$$

$$Y_0 \cdot X_1 X_1 \cdot X_2 \ldots X_J \cdot Y_{J+1} \partial_s \partial_t \prod_{i=1}^{J} \square_i^{(6)} =$$

$$\left(-\frac{1}{2}\right)^{J+1} \frac{\partial_s \partial_t}{|y_0'||y_{J+1}'|} \prod_{i=0}^{J} (y_i - y_{i+1})^2 \prod_{i=1}^{J} \square_i^{(4)} = \left(\frac{1}{2}\right)^{J+1} (4\hat{g}^2)^{J+1} \hat{B}^{-1}$$

$$X_1 \cdot X_2 = -\frac{1}{2}(x_1 - x_2)^2$$

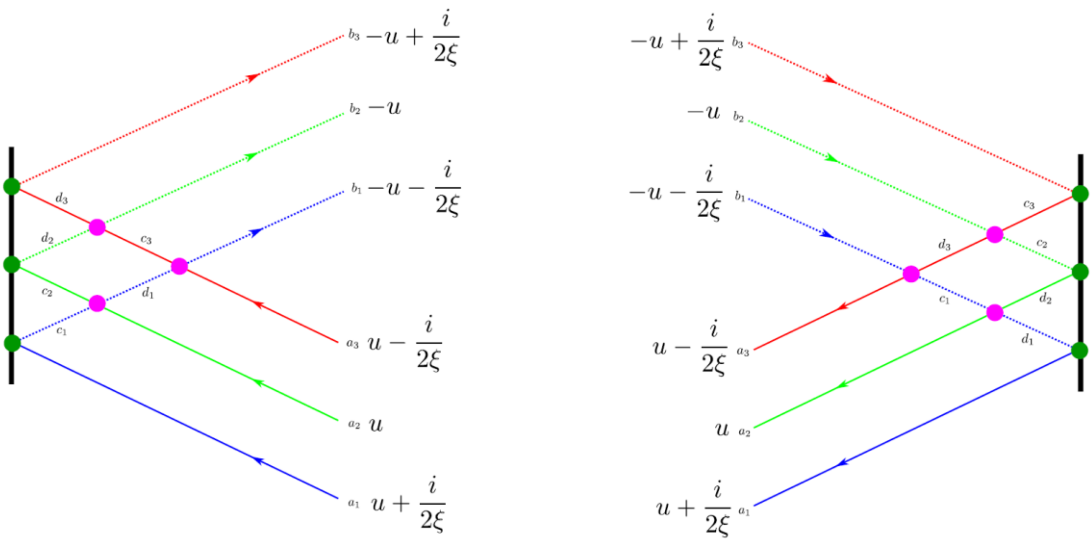
$$Y_0 \cdot X_1 = -\frac{e^{-t}}{2}(x_0 - x_1)^2$$

$$\mathbb{T}^{\mathbf{6}}(0) = 4\hat{B}^{-2}$$

$$\hat{\mathbb{L}}_{ka}^{\bar{4}}{}^{b}(u) = \left(u^2 - \frac{\mathrm{tr}\hat{q}_k^2}{8} + \frac{1}{\xi^2}\right) \hat{\bar{\mathbb{L}}}_{ka}^{4}{}^{b}(-u),$$

$$\hat{\bar{\mathbb{L}}}{}_{k}^{\bar{4}}{}_{b}(u) = \hat{\mathbb{L}}_{k}^{\bar{4}}{}_{b}{}^{a}(u)$$

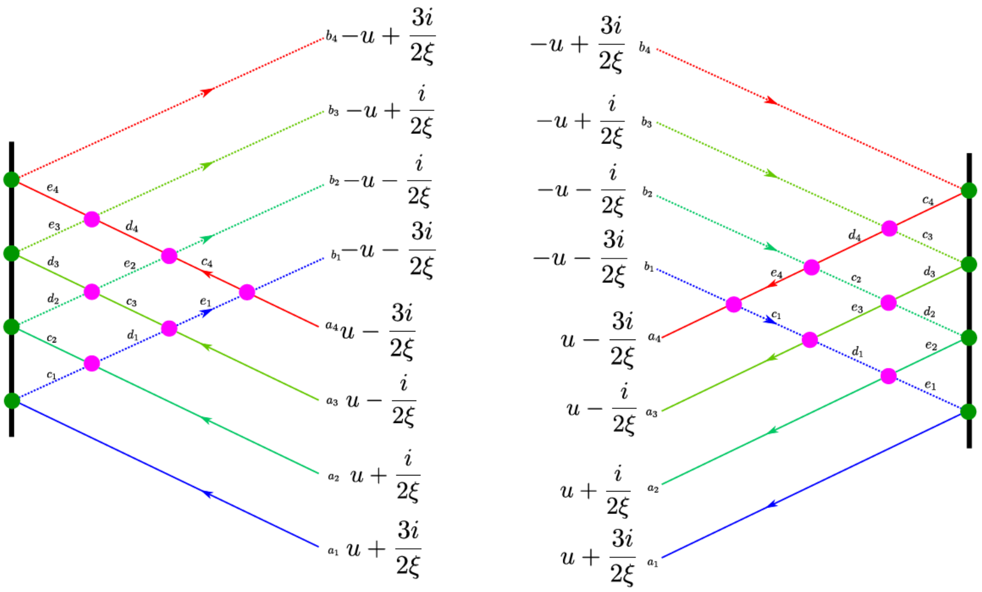
$$\hat{\mathbb{K}}^{\bar{4}ab}(u) = -\left(u^2 - \frac{iu}{\xi} + \frac{3}{4\xi^2}\right) \hat{\bar{K}}^{4ba}(-u), \hat{\bar{K}}_{ab}^{\bar{4}}(u) = -\left(u^2 + \frac{iu}{\xi} + \frac{3}{4\xi^2}\right) \bar{\mathbb{K}}_{ba}^{4}(-u)$$



(a) $\hat{\mathbb{K}}^{\bar{4}}$ (b) $\hat{\bar{\mathbb{K}}}^{\bar{4}}$

$$\hat{\mathbb{L}}_i^{\bar{\mathbf{1}}}(u) = \left(u^2 - \frac{\mathrm{tr}\hat{q}_i^2}{8} + \frac{5}{4\xi^2}\right)^2 + \frac{\mathrm{tr}\hat{q}_i^2}{8\xi^2} - \frac{1}{\xi^4}.$$

$$\mathbb{K}^{\bar{\mathbf{1}}}(u) = \left(u - \frac{2i}{\xi}\right)\left(u - \frac{i}{\xi}\right) u \left(u + \frac{i}{\xi}\right), \bar{\mathbb{K}}^{\bar{\mathbf{1}}}(u) = \left(u + \frac{2i}{\xi}\right)\left(u + \frac{i}{\xi}\right) u \left(u - \frac{i}{\xi}\right).$$



(a) $\mathbb{K}^{\bar{1}}$ (b) $\bar{\mathbb{K}}^{\bar{1}}$

$$\mathbb{T}^{4}(v) = \frac{1}{\xi^2}(4v^2\cos\varphi + \cos\varphi + 8\xi^2)$$

$$\mathbb{T}^{6}(v) = A(2v)\frac{v^2+1}{\xi^4 v^2}(v^4(2\cos(2\varphi)+4) + v^2(16\xi^2\cos\varphi - 4\Delta^2\sin^2\varphi) + 16\xi^4)$$

$$\mathbb{T}^{\bar{4}}(v) = A(2v)A(2v+i)A(2v-i)\frac{(4v^2+1)(4v^2+9)}{16\xi^6}(4v^2\cos\varphi + \cos\varphi + 8\xi^2)$$

$$\mathbb{T}^{\bar{1}}(v) = A^2(2v)A(2v+i)A(2v-i)A(2v+2i)A(2v-2i)\frac{v^2(v^2+1)^2(v^2+4)}{\xi^8}$$

$$\mathbb{T}^{1}(v) = 1$$

$$\mathbb{T}^{4}(v) \equiv \frac{P^{4}_{J+1}(v^2)}{\xi^{2J+2}}$$

$$\mathbb{T}^{6}(v) \equiv A(2v)\frac{v^2+1}{v^2}\frac{P^{6}_{2J+2}(v^2)}{\xi^{4J+4}}$$

$$\mathbb{T}^{\bar{4}}(v) = A(2v)A(2v+i)A(2v-i)\frac{\left(v^2+\frac{9}{4}\right)\left(v^2+\frac{1}{4}\right)^{2J+1} P^{\bar{4}}_{J+1}(v^2)}{\xi^{6J+6}}$$

$$\mathbb{T}^{\bar{1}}(v) = A^2(2v)A(2v+i)A(2v-i)A(2v+2i)A(2v-2i)\frac{(v^2+4)(v^2+1)^{2J+2}v^{4J+2}}{\xi^{8J+8}}$$

$$P^{4}_{J+1}(w) = \sum_{i=0}^{J+1} a_i w^{-i+J+1}$$

$$P^{6}_{2J+2}(w) = \sum_{i=0}^{2J+2} b_i w^{-i+2J+2}$$

$$P^{\bar{4}}_{J+1}(w) = \sum_{i=0}^{J+1} c_i w^{-i+J+1}$$

$$a_0 = c_0 = 4\cos\varphi, b_0 = 2\cos 2\varphi + 4$$

$$c_1 - a_1 = 8iS\Delta\sin\varphi$$
$$(a_1 + c_1)\cos\varphi - b_1 = 2(2S^2 + 2\Delta^2 + J)\sin^2\varphi + 2\cos^2\varphi$$

$$b_{2J+2} = \xi^{4J+4} = 16\hat{g}^{4J+4}$$

$$Q(v+2i)+\mathbb{T}^{\mathbf{4}}(v+i/2)Q(v+i)+\mathbb{T}^{\mathbf{6}}(v)Q(v)+\mathbb{T}^{\bar{\mathbf{4}}}(v-i/2)Q(v-i)+\mathbb{T}^{\bar{\mathbf{1}}}(v-i)Q(v-2i)=0$$

$$Q(v)=q(v)\frac{e^{\pi(J+1)v}\Gamma(-iv)\xi^{2i(J+1)v}\Gamma(iv+1)^{-2J-1}}{\Gamma\left(-iv-\frac{1}{2}\right)\Gamma(iv+2)}$$

$$\frac{P^{\mathbf{6}}_{2J+2}(v^2)}{v^{2J+3}}q(v)= -(v+i)^{2J+1}q(v+2i)-\frac{v+\frac{i}{2}}{v(v+i)}P^{4}_{J+1}\left(\left(v+\frac{i}{2}\right)^2\right)q(v+i)$$
$$-(v-i)^{2J+1}q(v-2i)-\frac{v-\frac{i}{2}}{v(v-i)}P^{\bar{4}}_{J+1}\left(\left(v-\frac{i}{2}\right)^2\right)q(v-i)$$

$$q(v)\left(\frac{2(8\hat{g}^2v^2\cos(\varphi)+8\hat{g}^4+v^4(\cos(2\varphi)+2))}{v^3}-\frac{4\Delta^2\sin^2(\varphi)}{v}\right)$$
$$+\frac{2(2v-i)q(v-i)(2\hat{g}^2+v(v-i)\cos(\varphi))}{v(v-i)}+\frac{2(2v+i)q(v+i)(2\hat{g}^2+v(v+i)\cos(\varphi))}{v(v+i)}$$
$$+(v-i)q(v-2i)+(v+i)q(v+2i)=0$$

$$q_1=e^{+\phi v}v^{+\Delta-S-J}\left(1+\frac{c_{1,1}}{v}+\cdots\right)$$
$$q_2=e^{-\phi v}v^{+\Delta+S-J}\left(1+\frac{c_{2,1}}{v}+\cdots\right)$$
$$q_3=e^{+\phi v}v^{-\Delta+S-J}\left(1+\frac{c_{3,1}}{v}+\cdots\right)$$
$$q_4=e^{-\phi v}v^{-\Delta-S-J}\left(1+\frac{c_{4,1}}{v}+\cdots\right)$$

$$q_i(v)\sim\frac{Q_i(v)}{v^{J+1/2}},$$

$$q_i^{\uparrow}(v)=\Omega_i^j(v)q_j^{\downarrow}(v),\Omega_i^j(v+i)=\Omega_i^j(v),$$

$$\Omega_i^j(v)=\frac{\epsilon^{jj_1j_2j_3}}{3!}\frac{\det_{n=0,\dots,3}\{q_i^{\uparrow}(v-in),q_{j_1}^{\downarrow}(v-in),q_{j_2}^{\downarrow}(v-in),q_{j_3}^{\downarrow}(v-in)\}}{\det_{n=0,\dots,3}\{q_1^{\downarrow}(v-in),q_2^{\downarrow}(v-in),q_3^{\downarrow}(v-in),q_4^{\downarrow}(v-in)\}}.$$

$$\Omega_i^j(v)=\frac{\sum_{n=0}^{2J+2}C_i^{(n)j}e^{2\pi nu}}{(1-e^{2\pi u})^{2J+2}},$$

$$\omega_{ik}=\Omega_i^j\Gamma_{jk}$$

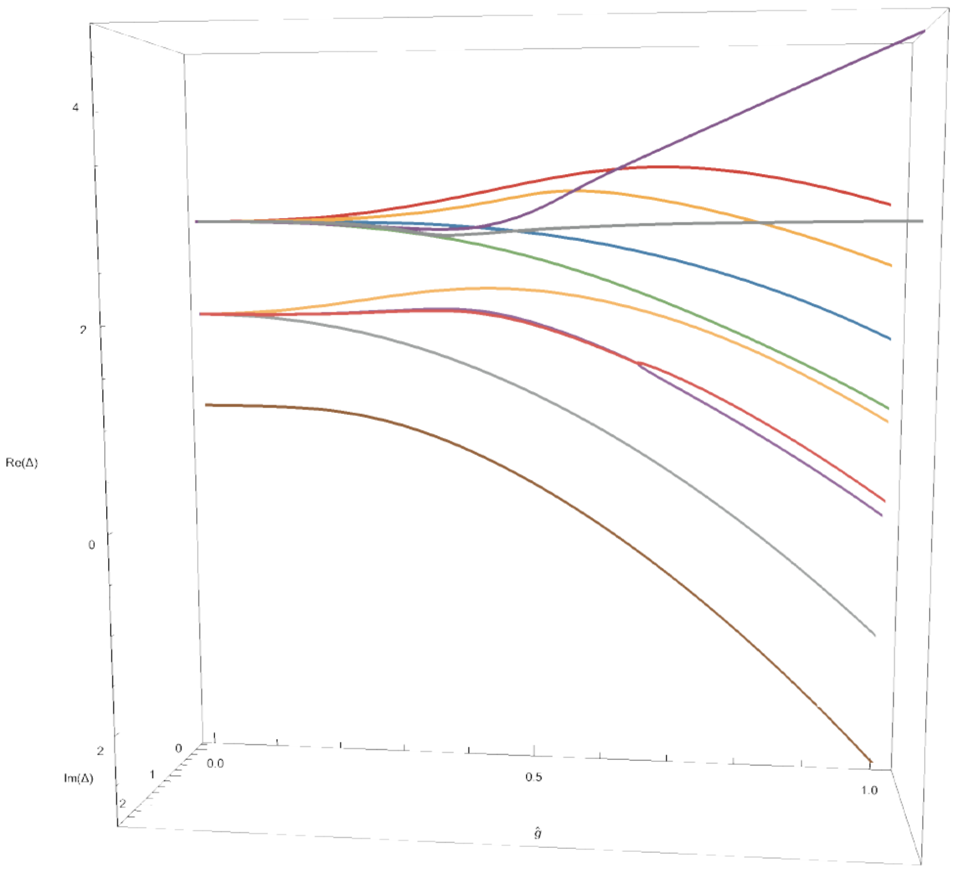
$$\Gamma_{jk}=\begin{pmatrix}0 & \gamma_1\sinh(2\pi v) & 0 & \gamma_3\\ \gamma_2\sinh(2\pi v) & 0 & \gamma_4 & 0\\ 0 & \gamma_5 & 0 & 0\\ \gamma_6 & 0 & 0 & 0\end{pmatrix}$$

$$\Omega_i^j\Gamma_{jk}=-\Omega_k^j\Gamma_{ji}$$

$$\Omega_{41}=\Omega_{32}=0$$

$$\Delta=J+\hat{g}^{2J+2}\frac{(-1)^J2^{4J+3}\pi^{2J+1}\csc(\varphi)B_{2J+1}\left(\frac{\varphi}{2\pi}\right)}{\Gamma(2J+2)}+\mathcal{O}(\hat{g}^{4J+4})$$

$$\Delta=2-124.08839542210\hat{g}^6+23271.513371517\hat{g}^{12}+\cdots$$



$$2+\hat{g}^{6}\frac{8}{45}\varphi(3\varphi^{4}-15\pi\varphi^{3}+20\pi^{2}\varphi^{2}-8\pi^{4})\csc\varphi=2-\hat{g}^{6}\frac{512\pi^{5}}{729\sqrt{3}}$$

$$\{\mathbb{T}(u),\mathbb{T}(v)\}=\{\mathbb{K}_{ab}(u),\mathbb{K}_{\alpha\beta}(v)\}\overline{\mathbb{K}}^{ba}(u)\overline{\mathbb{K}}^{\beta\alpha}(v)+\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\left\{\overline{\mathbb{K}}^{ba}(u),\overline{\mathbb{K}}^{\beta\alpha}(v)\right\}$$

$$\begin{aligned}\{\mathbb{T}(u),\mathbb{T}(v)\}=&-\frac{\overline{\mathbb{K}}^{ba}(u)\overline{\mathbb{K}}^{\beta\alpha}(v)}{\xi(u+v)}[\mathbb{K}_{\beta b}(u)\mathbb{K}_{\alpha a}(v)-\mathbb{K}_{b\beta}(v)\mathbb{K}_{a\alpha}(u)]\\&+\frac{\overline{\mathbb{K}}^{ba}(u)\overline{\mathbb{K}}^{\beta\alpha}(v)}{\xi(u-v)}[\mathbb{K}_{\alpha\beta}(u)\mathbb{K}_{ab}(v)-\mathbb{K}_{\alpha\beta}(v)\mathbb{K}_{ab}(u)]\\&-\frac{\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)}{\xi(u+v)}\left[\overline{\mathbb{K}}^{\alpha a}(u)\overline{\mathbb{K}}^{\beta b}(v)-\overline{\mathbb{K}}^{\alpha a}(v)\overline{\mathbb{K}}^{b\beta}(u)\right]\\&+\frac{\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)}{\xi(u-v)}\left[\overline{\mathbb{K}}^{b\alpha}(u)\overline{\mathbb{K}}^{\beta a}(v)-\overline{\mathbb{K}}^{b\alpha}(v)\overline{\mathbb{K}}^{\beta a}(u)\right]\end{aligned}$$

$$(\mathbb{L})^{a}{}_{b}(u)\equiv\left(\mathbb{L}_{1}(u)\cdot\mathbb{L}_{2}(u)\ldots\mathbb{L}_{J}(u)\right)^{a}{}_{b},(\overline{\mathbb{L}})_{a}{}^{b}(u)\equiv\left(\mathbb{L}_{-J}(u)\cdot\mathbb{L}_{-(J-1)}(u)\ldots\mathbb{L}_{-1}(u)\right)_{a}{}^{b}.$$

$$
\begin{aligned}
\{\mathbb{T}(u),\mathbb{T}(v)\} =& \{\mathbb{K}_{ab}(u),\mathbb{K}_{\alpha\beta}(v)\}(\bar{\mathbb{L}})_c{}^{a}(u)(\bar{\mathbb{L}})_\gamma{}^{\alpha}(v)\overline{\mathbb{K}}^{dc}(u)\overline{\mathbb{K}}^{\delta\gamma}(v)(\mathbb{L})^{b}{}_{d}(u)(\mathbb{L})^{\beta}{}_{\delta}(v) + \\
& \mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)(\bar{\mathbb{L}})_c{}^{a}(u)(\bar{\mathbb{L}})_\gamma{}^{\alpha}(v)\left\{\overline{\mathbb{K}}^{dc}(u),\overline{\mathbb{K}}^{\delta\gamma}(v)\right\}(\mathbb{L})^{b}{}_{d}(u)(\mathbb{L})^{\beta}{}_{\delta}(v) \\
& +(\mathbb{L}) = (\bar{\mathbb{L}})_c{}^{a}(u)(\bar{\mathbb{L}})_\gamma{}^{\alpha}(v)(\mathbb{L})^{b}{}_{d}(u)(\mathbb{L})^{\beta}{}_{\delta}(v) \\
& \Big\{-\frac{1}{\xi(u+v)}\Big[\mathbb{K}_{\beta b}(u)\mathbb{K}_{\alpha a}(v)\overline{\mathbb{K}}^{dc}(u)\overline{\mathbb{K}}^{\delta\gamma}(v) - \mathbb{K}_{b\beta}(v)\mathbb{K}_{a\alpha}(u)\overline{\mathbb{K}}^{dc}(u)\overline{\mathbb{K}}^{\delta\gamma}(v) \\
& +\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\overline{\mathbb{K}}^{\gamma c}(u)\overline{\mathbb{K}}^{\delta d}(v) - \mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\overline{\mathbb{K}}^{c\gamma}(v)\overline{\mathbb{K}}^{d\delta}(u)\Big] \\
& +\frac{1}{\xi(u-v)}\Big[\mathbb{K}_{\alpha\beta}(u)\mathbb{K}_{ab}(v)\overline{\mathbb{K}}^{dc}(u)\overline{\mathbb{K}}^{\delta\gamma}(v) - \mathbb{K}_{\alpha\beta}(v)\mathbb{K}_{ab}(u)\overline{\mathbb{K}}^{dc}(u)\overline{\mathbb{K}}^{\delta\gamma}(v) \\
& +\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\overline{\mathbb{K}}^{d\gamma}(u)\overline{\mathbb{K}}^{\delta c}(v) - \mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\overline{\mathbb{K}}^{d\gamma}(v)\overline{\mathbb{K}}^{\delta c}(u)\Big]\Big\} \\
& +(\mathbb{L})
\end{aligned}
$$

$$
\begin{aligned}
\{\mathbb{T}(u),\mathbb{T}(v)\} =& \mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\overline{\mathbb{K}}^{dc}(u)\overline{\mathbb{K}}^{\delta\gamma}(v) \\
& \Big\{-\frac{1}{\xi(u+v)}\Big[(\bar{\mathbb{L}})_c^{\beta}(u)(\bar{\mathbb{L}})_\gamma^{\alpha}(v)(\mathbb{L})_d^{b}(u)(\mathbb{L})_d^{a}(v) \\
& -(\bar{\mathbb{L}})_\delta^{a}(u)(\bar{\mathbb{L}})_\gamma^{\alpha}(v)(\mathbb{L})_d^{b}(u)(\mathbb{L})_c^{\beta}(v) \\
& +(\bar{\mathbb{L}})_c^{a}(u)(\bar{\mathbb{L}})_d^{\alpha}(v)(\mathbb{L})^{b}{}_{\gamma}(u)(\mathbb{L})^{\beta}{}_{\delta}(v) \\
& -(\bar{\mathbb{L}})_c^{a}(u)(\bar{\mathbb{L}})_\gamma^{b}(v)(\mathbb{L})_d^{\alpha}(u)(\mathbb{L})^{\beta}{}_{\delta}(v)\Big] \\
& +\frac{1}{\xi(u-v)}\Big[(\bar{\mathbb{L}})_c^{a}(u)(\bar{\mathbb{L}})_\gamma^{\alpha}(v)(\mathbb{L})_d^{\beta}(u)(\mathbb{L})^{b}{}_{\delta}(v) \\
& -(\bar{\mathbb{L}})_c^{\alpha}(u)(\bar{\mathbb{L}})_\gamma^{a}(v)(\mathbb{L})^{b}{}_{d}(u)(\mathbb{L})^{\beta}{}_{\delta}(v) \\
& +(\bar{\mathbb{L}})_\gamma^{a}(u)(\bar{\mathbb{L}})_c^{\alpha}(v)(\mathbb{L})^{b}{}_{d}(u)(\mathbb{L})^{\beta}{}_{\delta}(v) \\
& -(\bar{\mathbb{L}})_c^{a}(u)(\bar{\mathbb{L}})_\gamma^{\alpha}(v)(\mathbb{L})^{b}{}_{\delta}(u)(\mathbb{L})^{\beta}{}_{d}(v)\Big]\Big\} \\
& +(\mathbb{L}).
\end{aligned}
$$

$$
\begin{aligned}
\{\mathbb{T}(u),\mathbb{T}(v)\} =& (\mathbb{K}) \\
& +\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\overline{\mathbb{K}}^{dc}(u)\overline{\mathbb{K}}^{\delta\gamma}(v) \\
& \Big[\{(\bar{\mathbb{L}})_c^{a}(u),(\bar{\mathbb{L}})_\gamma^{\alpha}(v)\}(\mathbb{L})^{b}{}_{d}(u)(\mathbb{L})^{\beta}{}_{\delta}(v) \\
& +(\bar{\mathbb{L}})_\gamma^{\alpha}(v)\{(\bar{\mathbb{L}})_c^{a}(u),(\mathbb{L})^{\beta}{}_{\delta}(v)\}(\mathbb{L})^{b}{}_{d}(u) \\
& +(\bar{\mathbb{L}})_c^{a}(u)(\bar{\mathbb{L}})_\gamma^{\alpha}(v)\{(\mathbb{L})^{b}{}_{d}(u),(\mathbb{L})^{\beta}{}_{\delta}(v)\} \\
& +(\bar{\mathbb{L}})_c^{a}(u)\{(\mathbb{L})^{b}{}_{d}(u),(\bar{\mathbb{L}})_\gamma{}^{\alpha}(v)\}(\mathbb{L})^{\beta}{}_{\delta}(v)\Big].
\end{aligned}
$$

$$
\xi\{(\mathbb{L}_{-n})_a^{b}(u),(\mathbb{L}_{-m})_c^{d}(v)\} = \frac{(\mathbb{L}_{-n})_a^{d}(u)(\mathbb{L}_{-n})_c^{b}(v) - (\mathbb{L}_{-n})_c^{b}(u)(\mathbb{L}_{-n})_a^{d}(v)}{u-v}\delta_{nm}
$$

$$
\xi\{(\mathbb{L}_{-n})_a^{b}(u),(\mathbb{L}_{m})^{c}{}_{d}(v)\} = \frac{(\mathbb{L}_{-n})_a{}^{c}(u)(\mathbb{L}_{n})^{b}{}_{d}(v) - (\mathbb{L}_{-n})_d{}^{b}(u)(\mathbb{L}_{n})^{c}{}_{a}(v)}{u+v}\delta_{mn}
$$

$$
\begin{aligned}
\{\mathbb{T}(u),\mathbb{T}(v)\} =& (\mathbb{K}) \\
& +\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\overline{\mathbb{K}}^{dc}(u)\overline{\mathbb{K}}^{\delta\gamma}(v) \\
& \Big\{\frac{1}{\xi(u-v)}\Big[(\bar{\mathbb{L}})_c^{\alpha}(u)(\bar{\mathbb{L}})_\gamma^{a}(v)(\mathbb{L})_d^{b}(u)(\mathbb{L})_\delta^{\beta}(v) \\
& -(\bar{\mathbb{L}})_\gamma^{a}(u)(\bar{\mathbb{L}})_c^{\alpha}(v)(\mathbb{L})_d^{b}(u)(\mathbb{L})_\delta^{\beta}(v) \\
& +(\bar{\mathbb{L}})_c^{a}(u)(\bar{\mathbb{L}})_\gamma^{\alpha}(v)(\mathbb{L})_\delta^{b}(u)(\mathbb{L})_d^{\beta}{}_{d}(v) \\
& -(\bar{\mathbb{L}})_c^{a}(u)(\bar{\mathbb{L}})_\gamma^{\alpha}(v)(\mathbb{L})_d^{\beta}(u)(\mathbb{L})_\delta^{b}(v)\Big] + \\
& +\frac{1}{\xi(u+v)}\Big[(\bar{\mathbb{L}})_c^{\beta}(u)(\bar{\mathbb{L}})_\gamma^{\alpha}(v)(\mathbb{L})_d^{b}(u)(\mathbb{L})_\delta^{a}(v) \\
& -(\bar{\mathbb{L}})_\delta^{a}(u)(\bar{\mathbb{L}})_\gamma^{\alpha}(v)(\mathbb{L})_d^{b}(u)(\mathbb{L})_c^{\beta}(v) \\
& -(\bar{\mathbb{L}})_c^{a}(u)(\bar{\mathbb{L}})_\gamma^{b}(v)(\mathbb{L})_d^{\alpha}(u)(\mathbb{L})_\delta^{\beta}(v) \\
& +(\bar{\mathbb{L}})_c^{a}(u)(\bar{\mathbb{L}})_d^{\alpha}(v)(\mathbb{L})_\gamma^{b}(u)(\mathbb{L})_\delta^{\beta}(v)\Big]\Big\}.
\end{aligned}
$$

$$
\{\mathbb{T}(u),\mathbb{T}(v)\} = 0.
$$

$$\hat{\mathbb{T}}^4(-u) = \mathrm{Tr}\left(\hat{\hat{\mathbb{L}}}{}_J^4(u) \cdot \hat{\hat{\mathbb{L}}}{}_{J-1}^4(u) \ldots \hat{\hat{\mathbb{L}}}{}_1^4(u) \cdot \hat{\mathbb{K}}^4(-u) \cdot \hat{\mathbb{L}}_1^4(-u) \cdot \hat{\mathbb{L}}_2^4(-u) \ldots \hat{\mathbb{L}}_J^4(-u) \cdot G^4 \cdot \hat{\hat{\mathbb{K}}}{}^4(-u) \cdot G^4T\right)$$

$$\hat{\mathbb{T}}^4(-u) = \mathrm{Tr}\left(G^4 \cdot \hat{\hat{\mathbb{K}}}{}^{4T}(-u) \cdot G^{4T} \cdot \hat{\mathbb{L}}_J^{4T}(-u) \cdot \hat{\mathbb{L}}_{J-1}^{4T}(-u) \ldots \hat{\mathbb{L}}_1^{4T}(-u)\right.$$
$$\left. \cdot \hat{\mathbb{K}}^{4T}(-u) \cdot \hat{\hat{\mathbb{L}}}{}_1^{4T}(u) \cdot \hat{\hat{L}}{}_2^{4T}(u) \ldots \hat{\hat{L}}{}_J^{4T}(u)\right).$$

$$\hat{\mathbb{T}}^4(-u) = \mathrm{Tr}(\hat{\hat{\mathbb{L}}}{}_J^4(-u).\hat{\hat{\mathbb{L}}}{}_{J-1}^4(-u)\ldots\hat{\hat{\mathbb{L}}}{}_1^4(-u).\hat{\mathbb{K}}^{4T}(-u).\hat{\mathbb{L}}_1^4(u).\hat{\mathbb{L}}_2^4(u)\ldots\hat{\mathbb{L}}_J^4(u).G^4.\hat{\hat{\mathbb{K}}}{}^{4T}(-u).G^{4T})$$

$$\hat{\mathbb{T}}^4(-u) = \mathrm{Tr}\left(\hat{\hat{\mathbb{L}}}{}_J^4(-u) \cdot \hat{\hat{\mathbb{L}}}{}_{J-1}^4(-u) \ldots \hat{\hat{\mathbb{L}}}{}_1^4(-u) \cdot \hat{\mathbb{K}}^{4T}(-u) \cdot \bar{S}(-2u)\right.$$
$$\left. \cdot \hat{\mathbb{L}}_1^4(u) \cdot \hat{\mathbb{L}}_2^4(u) \ldots \hat{\mathbb{L}}_J^4(u) \cdot G^4 \cdot \bar{S}(2u) \cdot \hat{\hat{\mathbb{K}}}{}^{4T}(-u) \cdot G^{4T}\right)$$

$$\bar{\alpha}(2u)\bar{R}_{ab}^{cd}(2u)\hat{\mathbb{K}}_{cd}^4(-u) = \hat{\mathbb{K}}_{ba}^4(u)$$
$$\bar{\alpha}(-2u)\bar{R}_{dc}^{ab}(-2u)\left(\hat{\hat{\mathbb{K}}}{}^4\right)^{dc}(-u) = \left(\hat{\hat{\mathbb{K}}}{}^4\right)^{ba}(u)$$

$$\hat{\mathbb{K}}^{4T}(-u) \cdot \bar{S}(2u) = \hat{\mathbb{K}}^4(u),$$
$$\bar{S}(-2u) \cdot \hat{\hat{\mathbb{K}}}{}^{4T}(-u) = \hat{\hat{\mathbb{K}}}{}^4(u),$$

$$\hat{\mathbb{T}}^4(-u) = \left(\hat{\hat{\mathbb{L}}}{}_J^4(-u) \cdot \hat{\hat{\mathbb{L}}}{}_{J-1}^4(-u) \ldots \hat{\hat{\mathbb{L}}}{}_1^4(-u)\right)_a{}^{\gamma}\bar{S}^{\delta}{}_{\zeta\ \eta}(2u)\bar{S}^{\zeta}{}_{\gamma\ \beta}(-2u)\hat{\mathbb{K}}_{\alpha\delta}^4(-u)$$
$$\left(\hat{\mathbb{L}}_1^4(u) \cdot \hat{\mathbb{L}}_2^4(u) \ldots \hat{\mathbb{L}}_J^4(u)\right)^{\beta}{}_h\left(G^4 \cdot \hat{\hat{\mathbb{K}}}{}^{4\,T}(-u) \cdot G^4\right)^{ha}$$

$$\hat{\mathbb{T}}^4(-u) = \left(\hat{\hat{\mathbb{L}}}{}_J^4(-u) \cdot \hat{\hat{\mathbb{L}}}{}_{J-1}^4(-u) \ldots \hat{\hat{\mathbb{L}}}{}_1^4(-u)\right)_a^{\gamma}\bar{S}_{\gamma}^{\zeta\eta}{}_{\beta}(-2u)\hat{\mathbb{K}}_{\zeta\eta}^4(u)$$
$$\left(\hat{\mathbb{L}}_1^4(u) \cdot \hat{\mathbb{L}}_2^4(u) \ldots \hat{\mathbb{L}}_J^4(u)\right)_h^{\beta}\left(G^4 \cdot \hat{\hat{\mathbb{K}}}{}^{4T}(-u) \cdot G^{4T}\right)^{ha}.$$

$$\hat{\mathbb{T}}^4(-u) = \left(\hat{\hat{\mathbb{L}}}{}_J^4(-u) \cdot \hat{\hat{\mathbb{L}}}{}_{J-1}^4(-u) \ldots \hat{\hat{\mathbb{L}}}{}_2^4(-u)\right)_a{}^{\omega}\bar{S}_{\epsilon}^{\gamma\beta}(-2u)\hat{\mathbb{L}}_1^4{}_{\gamma\ \gamma}(u)\hat{\mathbb{L}}_{1\beta\ \beta}^4(-u)\hat{\mathbb{K}}_{\zeta\eta}^4(u)$$
$$\left(\hat{\mathbb{L}}_2^4(u) \ldots \hat{\mathbb{L}}_J^4(u)\right)^{\epsilon}{}_h\left(G^4 \cdot \hat{\hat{\mathbb{K}}}{}^{4T}(-u) \cdot G^{4T}\right)^{ha}.$$

$$\hat{\mathbb{T}}^4(-u) = \left(\hat{\hat{\mathbb{L}}}{}_J^4(-u) \cdot \hat{\hat{\mathbb{L}}}{}_{J-1}^4(-u) \ldots \hat{\hat{\mathbb{L}}}{}_1^4(-u) \cdot \hat{\mathbb{K}}^4(u) \cdot \hat{\mathbb{L}}_1^4(u) \cdot \hat{\mathbb{L}}_2^4(u) \ldots \hat{\mathbb{L}}_J^4(u)\right)_{\epsilon\omega}$$
$$\bar{S}_h^{\epsilon\omega}(-2u)\left(G^4 \cdot \hat{\hat{\mathbb{K}}}{}^{4\,T}(-u) \cdot G^4\right)^{ha}$$

$$\hat{\mathbb{T}}^4(-u) = \left(\hat{\hat{\mathbb{L}}}{}_J^4(-u) \cdot \hat{\hat{\mathbb{L}}}{}_{J-1}^4(-u) \ldots \hat{\hat{\mathbb{L}}}{}_1^4(-u) \cdot \hat{\mathbb{K}}^4(u) \cdot \hat{\mathbb{L}}_1^4(u) \cdot \hat{\mathbb{L}}_2^4(u) \ldots \hat{\mathbb{L}}_J^4(u)\right)_{\epsilon\omega}$$
$$\left(G^4 \cdot \hat{\hat{\mathbb{K}}}{}^4(u) \cdot G^4\right)^{\omega\epsilon} = \hat{\mathbb{T}}^4(u)$$

$$\hat{\mathbb{T}}^6(u) = \mathrm{Tr}\left(\hat{\mathbb{L}}_J^6(u) \ldots \hat{\mathbb{L}}_1^6(u) \cdot \hat{\mathbb{K}}^6(u) \cdot \hat{\mathbb{L}}_1^6(u) \ldots \hat{\mathbb{L}}_J^6(u) \cdot G^6 \cdot \hat{\hat{\mathbb{K}}}{}^6(u) \cdot G^{6T}\right).$$

$$\hat{\mathbb{T}}^6(-u) = \mathrm{Tr}\left(\hat{\mathbb{L}}_J^6(-u) \ldots \hat{\mathbb{L}}_1^6(-u) \cdot \hat{\mathbb{K}}^6(-u) \cdot \hat{\mathbb{L}}_1^6(-u) \ldots \hat{\mathbb{L}}_J^6(-u) \cdot G^6 \cdot \hat{\hat{\mathbb{K}}}{}^6(-u) \cdot G^6T\right)$$

$$\hat{\mathbb{T}}^6(-u) = \mathrm{Tr}\left(\hat{\mathbb{L}}_J^6(u) \ldots \hat{\mathbb{L}}_1^6(u) \cdot \hat{\mathbb{K}}^{6T}(-u) \cdot \hat{\mathbb{L}}_1^6(u) \ldots \hat{\mathbb{L}}_J^6(u) \cdot G^6 \cdot \hat{\hat{\mathbb{K}}}{}^{6T}(-u) \cdot G^{6T}\right)$$

$$\hat{\mathbb{L}}^6{}_E(u)\hat{\hat{\mathbb{L}}}{}^6{}_F{}^D(-v)\bar{S}^{6E}{}_{A\ C}(u+v) = \bar{S}^6{}_{F\ E}(u+v)\hat{\hat{\mathbb{L}}}{}^6{}_C{}^E(-v)\hat{\mathbb{L}}^6{}_A(u)$$

$$\bar{S}^{\mathbf{6}}{}_{B}^{A}{}_{D}^{C}(u) = c(u)\left(\delta_B^A\delta_D^C - \frac{i}{\xi u}\delta_D^A\delta_B^C + \frac{i}{\xi\left(u-\frac{2i}{\xi}\right)}\eta^{AC}\eta_{BD}\right)$$

$$c(u) = \frac{u\left(u-\frac{2i}{\xi}\right)}{\frac{2}{\xi^2}+u\left(u+\frac{i}{\xi}\right)}$$

$$\hat{\mathbb{K}}^{\mathbf{6}}T(-u)\cdot\bar{S}^{\mathbf{6}}(2u) = \hat{\mathbb{K}}^{\mathbf{6}}(u)$$
$$\bar{S}^{\mathbf{6}}(-2u)\cdot\hat{\bar{\mathbb{K}}}^{\mathbf{6}}T(-u) = \hat{\bar{\mathbb{K}}}^{\mathbf{6}}(u)$$

$$\hat{\mathbb{T}}^{\mathbf{6}}(-u) = \hat{\mathbb{T}}^{\mathbf{6}}(u).$$

$$\hat{\mathbb{T}}^{\bar{4}}(u) = \hat{\beta}(u)\mathrm{Tr}\left(\hat{\bar{\mathbb{L}}}_1^4(u)\ldots\hat{\bar{\mathbb{L}}}_J^4(u)\cdot(G^4)^{-T}\cdot\hat{\mathbb{K}}^4(-u)\cdot(G^4)^{-1}\cdot\hat{\mathbb{L}}_J^4(-u)\ldots\hat{\mathbb{L}}_1^4(-u)\cdot\hat{\bar{\mathbb{K}}}^4(-u)\right)$$

$$\mathbb{T}^{\bar{4}}(-u) = \mathbb{T}^{\bar{4}}(u)$$

$$\mathbb{K}^{\mathbf{1}}(u)\bar{\mathbb{K}}^{\mathbf{1}}(u) = \left(u^2+\frac{4}{\xi^2}\right)\left(u^2+\frac{1}{\xi^2}\right)^2 u^2$$

$$P^{\mathbf{4}} = P^{\bar{4}} = 4\cos\,\varphi v^2 - 8h + \cos\,\varphi$$
$$P^{\mathbf{6}} = (2\cos\,2\varphi+4)v^4 - (4\Delta^2\sin^2\,\varphi + 16h\cos\,\varphi)v^2 + 16h^2,$$

$$\hat{O}_{\pm}q \equiv q(u)(4\hat{g}^2 - 2u^2\cos\,(\phi) \pm 2\Delta u\sin\,(\phi)) + u^2q(u-i) + u^2q(u+i).$$

$$-\frac{1}{u^2}\hat{O}_+\frac{1}{u}\hat{O}_-q = -\frac{1}{u^2}\hat{O}_-\frac{1}{u}\hat{O}_+q = 0$$

$$P^4 = 4\cos\,\varphi v^4 + a_1v^2 + \frac{a_1}{4} + 8h - \frac{1}{4}\cos\,\varphi$$
$$P^6 = (2\cos\,2\varphi+4)v^8 + 2\left[\cos\,\varphi\frac{a_1+c_1}{2} + \cos\,2\varphi\Delta^2 - (\Delta^2+1)\right.$$
$$\left.-2S^2\sin^2\,\varphi\right]v^6 + b_2v^4 + b_3v^2 + 16h^2$$
$$P^{\bar{4}} = 4\cos\,\varphi v^4 + c_1v^2 + \frac{c_1}{4} + 8h - \frac{1}{4}\cos\,\varphi$$

$$P^4 = -4v^4 + (2\Delta^2 - 2\Delta + 2)v^2 + \frac{1}{4}(2\Delta^2 - 2\Delta + 3) - 8\hat{g}^4$$
$$P^6 = +6v^8 - (4(\Delta-1)\Delta + 6)v^6 + v^4((\Delta-1)^2\Delta^2 + 16\hat{g}^4) + b_3v^2 + 16\hat{g}^8$$
$$P^{\bar{4}} = -4v^4 + (2\Delta^2 - 2\Delta + 2)v^2 + \frac{1}{4}(2\Delta^2 - 2\Delta + 3) - 8\hat{g}^4$$

$$\hat{\mathbb{T}}_{\theta}^{\lambda}(u) = \mathrm{Tr}\left(\hat{\bar{\mathbb{L}}}_J^{\lambda}(-u-\theta_J)\ldots\hat{\bar{\mathbb{L}}}_1^{\lambda}(-u-\theta_1)\cdot\hat{\mathbb{K}}^{\lambda}(u)\cdot\hat{\mathbb{L}}_1^{\lambda}(u-\theta_1)\ldots\hat{\mathbb{L}}_J^{\lambda}(u-\theta_J)\cdot G^{\lambda}\cdot\hat{\bar{\mathbb{K}}}^{\lambda}(u)\cdot G^{\lambda t}\right)$$

$$\mathbb{T}_{\theta}^{\mathbf{1}}(v,\{\zeta_i\}) = 1$$

$$\mathbb{T}_{\theta}^{\mathbf{4}}(v,\{\zeta_i\}) \equiv \frac{P_{J+1}^{\mathbf{4}}(v^2,\{\zeta_i\})}{\xi^{2J+2}}$$

$$\mathbb{T}_{\theta}^{\mathbf{6}}(v,\{\zeta_i\}) \equiv A(2v)\frac{v^2+1}{v^2}\frac{P_{2J+2}^{\mathbf{6}}(v^2,\{\zeta_i\})}{\xi^{4J+4}}$$

$$\mathbb{T}_{\theta}^{\bar{4}}(v,\{\zeta_i\}) = A(2v)A(2v+i)A(2v-i)\frac{\left(v^2+\frac{9}{4}\right)\left(v^2+\frac{1}{4}\right)\prod_{i=1}^{J}\left(\zeta_i^2+\left(v^2-\zeta_i^2+\frac{1}{4}\right)^2\right)P_{J+1}^{\bar{4}}(v^2,\{\zeta_i\})}{\xi^{6J+6}}$$

$$\mathbb{T}_{\theta}^{\bar{\mathbf{1}}}(v,\{\zeta_i\}) = A^2(2v)A(2v+i)A(2v-i)A(2v+2i)A(2v-2i)$$
$$\frac{(v^2+4)(v^2+1)^2v^2\prod_{i=1}^{J}\left(\left(v^2-\zeta_i^2\right)^2\left(4\zeta_i^2+\left(v^2-\zeta_i^2+1\right)^2\right)\right)}{\xi^{8J+8}}$$

$$Q(v) \to \frac{\Gamma(-iv)\exp(\pi(J+1)v)q(v)\xi^{2iv(J+1)}}{\Gamma\left(-iv-\frac{1}{2}\right)\Gamma(iv+2)}\prod_{i=-J}^{J}\left(\Gamma(i(v+\zeta_i)+1)^{-1}\right),$$

$$\frac{P_{2J+2}^{\mathbf{6}}(v^2)}{v^2\prod_{i=-J}^{J}(v-\zeta_i)}q(v) = -\prod_{i=-J}^{J}(v+i-\zeta_i)q(v+2i) - \frac{v+\frac{i}{2}}{v(v+i)}P_{J+1}^{4}\left(\left(v+\frac{i}{2}\right)^2\right)q(v+i)$$
$$-\prod_{i=-J}^{J}(v-i-\zeta_i)q(v-2i) - \frac{v-\frac{i}{2}}{v(v-i)}P_{J+1}^{\bar{4}}\left(\left(v-\frac{i}{2}\right)^2\right)q(v-i)$$

CONCLUSIONES

En mérito a los resultados obtenidos, se concluye lo que sigue: 1) Las superpartículas, llamadas también partículas estrella o blancas, son aquellas cuyo centro de masa – energía es extremadamente denso, lo que le permite deformar el espacio – tiempo cuántico; 2) La deformación del espacio – tiempo cuántico, causada en supergravedad, puede ser por intervención gravitónica o no; de tal suerte que, es endógena, cuando la superpartícula, por su propia masa y energía, deforma el espacio – tiempo cuántico en tanto que, es exógena, cuando la superpartícula interactúa con el supergravitón o gravitino, lo que ocurre cuando el campo supergravitónico, permea el espacio – tiempo cuántico repercutido; 3) la deformación del espacio – tiempo cuántico en supergravedad, es drástica e intensa, en la medida en que, cuando la superpartícula interactúa, deforma el espacio – tiempo cuántico hasta formar un supercurvatura, en la que, el campo cuántico se vuelve perturbativo, y ésta supercurvatura deviene en la formación de un agujero negro cuántico, por la colisión o colapso inminente y previo de la partícula estrella o blanca, formándose consecuentemente, un agujero cuántico de gusano y finalmente, un agujero blanco cuántico, por la implosión de la singularidad. Téngase en cuenta, la definición concebida por este autor, respecto de la singularidad de un agujero negro cuántico. Véanse mis trabajos paralelos en este punto; 4) En supergravedad, el espacio – tiempo cuántico, se deforma en dimensiones altas, lo que da lugar a la creación de superespacios y por ende, supermembranas, entendidas éstas últimas, como regiones de dimensión disociada; 5) En supergravedad cuántica relativista, la supersimetría se vuelve esencial, incluso como contrapeso de la antisimetría; 6) En supergravedad, el entorno es eminentemente entrópico, por lo que, el principio de incertidumbre queda ratificado; 7) Las partículas supermasivas, son aquellas, cuya masa es extremadamente densa, con capacidad por tanto, de deformar el espacio – tiempo cuántico en términos racionales; 8) La colisión de una partícula supermasiva, produce agujeros negros cuánticos o en su defecto, su colapso en entornos cuánticos entrópicos; 9) En gravedad cuántica, es decir, la deformación del espacio – tiempo cuántico, al igual que en supergravedad, se produce por intervención gravitónica o sin intervención gravitónica, por lo que, ésta se tiene por exógena, cuando la partícula supermasiva interactúa con el gravitón en tanto que, se tiene por endógena, cuando la propia partícula supermasiva, a razón de su interacción, colisión o colapso, deforma el espacio – tiempo cuántico, en ambos casos, pudiendo provocar agujeros negros cuánticos; 9) La simetría se vuelve esencial en gravedad cuántica, como contrapeso de la simetría; y, 10) Las partículas blancas u oscuras, para efectos de determinar sus interacciones, matemáticamente es posible, a través de la identificación de sus osciladores y propagadores armónicos.

Algunas aclaraciones finales a tener en consideración y aplicar, a propósito de la Teoría Cuántica de Campos Relativistas o Curvos (TCCR) propuesta por este autor:

1. En todos los casos, este símbolo † será reemplazado por este símbolo†o por este símbolo †, equivaliendo lo mismo.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
†	†
	†

2. En todos los casos, este símbolo ‡, será reemplazado por este símbolo‡o por este símbolo ‡.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
‡	‡
	‡

3. En todos los casos, se añadirá y por ende, se calculará la magnitud§que equivale a un campo de Yang – Mills y por ende, a la teoría de Yang – Mills en sentido amplio, en relación a la Teoría Cuántica de Campos Relativistas propuesta por este autor.

4. Este símbolo • podrá usarse como exponente u operador, según sea el caso.

Las aclaraciones antes referidas aplican tanto a este trabajo como a todos los trabajos previos y posteriores publicados por este autor, según corresponda.

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