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AGUJEROS CUÁNTICOS: APROXIMACIONES TEÓRICAS RESPECTO DE LOS AGUJEROS NEGROS CUÁNTICOS, AGUJEROS BLANCOS CUÁNTICOS Y AGUJEROS CUÁNTICOS DE GUSANO EN RELACIÓN A CAMPOS CUÁNTICOS RELATIVISTAS O CURVOS Y EN CONDICIONES DE GRAVEDAD Y SUPERGRAVEDAD CUÁNTICAS. VOLUMEN I

QUANTUM HOLES: THEORETICAL APPROACHES TO QUANTUM BLACK HOLES, WHITE QUANTUM HOLES, AND QUANTUM WORMHOLES IN RELATION TO RELATIVISTIC OR CURVED QUANTUM FIELDS AND UNDER CONDITIONS OF QUANTUM GRAVITY AND SUPERGRAVITY. VOLUME I

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RESUMEN

Los agujeros negros cuánticos, son sin duda, un campo de estudio fascinante en la teoría cuántica de campos relativistas. Éstos, según la TCCR (Teoría Cuántica de Campos Relativistas), se forman por partículas supermasivas o en su defecto, por superpartículas, de tal suerte que, en escenarios de supergravedad o gravedad cuánticas, según sea el caso, su formación es inminente en la primera y potencial en la segunda. En este trabajo, abordaré más a detalle, la formación de agujeros negros cuánticos supermasivos, especialmente en escenarios de extrema gravitación e intentaremos descifrar la singularidad de los mismos, incluyendo la formación de agujeros cuánticos de gusano, que en sí, son disímiles a los agujeros negros cuánticos. Estos agujeros negros, de carácter subatómico, interactúan con campos cuánticos específicos, con la particularidad, de que devoran energía a escala microscópica, lo que explicaría, en forma exponencial, la formación de materia oscura.

Palabras Clave: materia oscura, agujero negro cuántico, relatividad cuántica, agujeros cuánticos de gusano, partícula oscura, partícula estrella.

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ABSTRACT

Quantum black holes are undoubtedly a fascinating field of study in the quantum theory of relativistic fields. These, according to the TCCR (Quantum Theory of Relativistic Fields), are formed by supermassive particles or, failing that, by superparticles, in such a way that, in scenarios of supergravity or quantum gravity, as the case may be, their formation is imminent in the former and potential in the latter. In this work, I will address in more detail the formation of supermassive quantum black holes, especially in extreme gravitation scenarios and we will try to decipher their singularity, including the formation of quantum wormholes, which in themselves, are dissimilar to quantum black holes. These black holes, of a subatomic nature, interact with specific quantum fields, with the particularity that they devour energy on a microscopic scale, which would explain, exponentially, the formation of dark matter.

Keywords: dark matter, quantum black hole, quantum relativity, wormholes, dark particle, star particle.

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INTRODUCCIÓN.

Los agujeros negros cuánticos, son aquellos fenómenos propios de la dualidad holográfica. Se forman principalmente por la deformación del espacio – tiempo cuántico, sea por una partícula supermasiva o en su defecto, por una superpartícula, aunque es inevitable en el segundo caso, sea por colapso o colisión. Estos agujeros negros cuánticos, a diferencia de los que ocurren a escala cosmológica, devoran energía para convertirla en materia oscura. En este trabajo se consolidará el modelo matemático de un agujero negro cuántico y de un agujero cuántico de gusano, éste último el mismo que se explica a través del puente de Einstein – Rosen, como ha quedado referido en trabajos anteriores, lo que además, explicaría la superposición y el entrelazamiento cuánticos. Se ha dicho que estos fenómenos microscópicos, ocurren por colapso o colisión de una partícula supermasiva o de una superpartícula, según sea el caso, sin embargo, los agujeros negros cuánticos también pueden colisionar, formando así, ondas cuánticas o denominadas también "ondas cuánticas gravitacionales", como se ha teorizado en trabajos anteriores. En este artículo, abordaré los fenómenos antes referidos, con especial énfasis, en la descripción matemática de los mismos, en un espacio de Hilbert – Einstein, tanto en supergravedad como en gravedad cuánticas, según corresponda, incluyendo la morfología propiamente de un agujero negro cuántico y un agujero cuántico de gusano.

RESULTADOS Y DISCUSIÓN.

El desarrollo matemático aquí contenido, complementa el sistema de ecuaciones desplegado en el volumen II de este manuscrito. Dicho esto, prosigo:

Modelo de campo cuántico relativista en supergravedad, en el que se forman agujeros negros cuánticos, agujeros blancos cuánticos y agujeros cuánticos de gusano.

$$\delta \mathrm{e}_{\mu}^{a} = \bar{\epsilon}\gamma^{a}\chi_{\mu}, \delta\chi_{\mu} = \left(D_{\mu} - \frac{A}{2}\gamma_{\mu}\right)\epsilon, \delta A = -\frac{1}{2}\bar{\epsilon}\gamma^{\mu\nu}\left(D_{\mu} - \frac{A}{2}\gamma_{\mu}\right)\chi_{\nu}$$

$$\mathrm{E} = \mathrm{e}\left[1 + \frac{1}{2}\bar{\theta}\gamma^{\mu}\chi_{\mu} + \frac{1}{2}\bar{\theta}\theta\left(-A + \frac{1}{4}\bar{\chi}_{\mu}\gamma^{\mu\nu}\chi_{\nu}\right)\right],$$

$$\mathcal{R} = -A + \bar{\theta}\gamma^{\mu\nu}\left(D_{\mu} - \frac{A}{2}\gamma_{\mu}\right)\chi_{\nu} + \frac{1}{2}\bar{\theta}\theta\left(\frac{R}{2} - A^{2} - \bar{\chi}^{\mu}\gamma^{\nu}D_{[\mu}\chi_{\nu]} - \frac{A}{4}\bar{\chi}^{\mu}\chi_{\mu}\right)$$

$$S_{\mathcal{N}=1}^{(h)} = \int_{\Sigma_h} \mathrm{d}^{2}x\, \mathrm{d}^{2}\theta \mathrm{E}\left(-\frac{\vartheta}{\pi}\mathcal{R} + 2\mathrm{i}\boldsymbol{\mu}\right)$$

$$\frac{\vartheta}{\pi}\int_{\Sigma_h} \mathrm{d}^2x\,\mathrm{d}^2\theta \mathrm{E}\mathcal{R} = \frac{\vartheta}{\pi}\int_{\Sigma_h} \mathrm{d}^2x \mathrm{e}\frac{R}{4} = \vartheta\chi_h$$

$$2\mathrm{i}\boldsymbol{\mu}\int_{\Sigma_h} \mathrm{d}^2x\,\mathrm{d}^2\theta \mathrm{E} = \boldsymbol{\mu}\int_{\Sigma_h} \mathrm{d}^2x \mathrm{e}\left(-\mathrm{i}A + \frac{\mathrm{i}}{4}\bar{\chi}_\mu\gamma^{\mu\nu}\chi_\nu\right)$$

$$\mathcal{Z}_{\text{grav}}^{\mathcal{N}=1} = \sum_{h=0}^{\infty} e^{\vartheta\chi_h}\int [\mathcal{D}\mathrm{e}_\mu^a][\mathcal{D}\chi_\mu][\mathcal{D}A]e^{\mu\int_{\Sigma_h}\mathrm{d}^2x\mathrm{e}\left(\mathrm{i}A-\frac{\mathrm{i}}{4}\bar{\chi}_\mu\gamma^{\mu\nu}\chi_\nu\right)} \times Z_{\text{SCFT}}^{(h)}[\mathrm{e}_\mu^a,\chi_\mu,A]$$

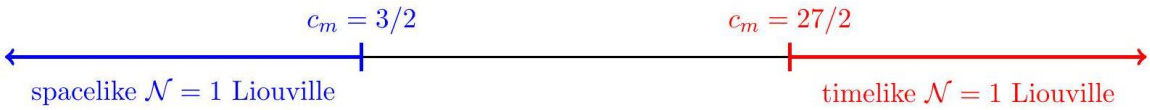
$$\mathrm{e}_\mu^a = e^{b\varphi}\tilde{\mathrm{e}}_\mu^a, \chi_\mu = e^{\frac{1}{2}b\varphi}\gamma_\mu\psi, A = \frac{\mathrm{i}}{b}e^{-b\varphi}F$$

$$Z_{\text{SCFT}}^{(0)}[\varphi,\psi,F] = \mathcal{C}e^{-S_{\text{s-anomaly}}[\varphi,\psi,F]},$$

$$S_{\text{s-anomaly}}[\varphi,\psi,F] = -\frac{c_m}{48\pi}\int_{S^2} \mathrm{d}^2x\tilde{\mathrm{e}}\left(\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\mathrm{i}}{2}\bar{\psi}\not{\partial}\psi - \frac{1}{2}F^2 + \tilde{R}\varphi\right)$$

$$S_{\text{L}}^{\mathcal{N}=1} = \frac{1}{4\pi}\int_{S^2} \mathrm{d}^2x\tilde{\mathrm{e}}\left(\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{\mathrm{i}}{2}\bar{\psi}\not{\partial}\psi - \frac{1}{2}F^2 + \frac{1}{2}Q\tilde{R}\varphi + \mu e^{b\varphi}F - \frac{\mathrm{i}}{2}\mu b e^{b\varphi}\bar{\psi}\psi\right)$$

$$\frac{\boldsymbol{\mu}}{b}\int \mathrm{d}^2x\tilde{\mathrm{e}}\left(e^{b\varphi}F - \frac{\mathrm{i}b}{2}e^{b\varphi}\bar{\psi}\psi\right)$$



$$c_{\text{L}}^{\mathcal{N}=1} + c_m + c_{\text{bc}} + c_{\beta\gamma} = 0 \Leftrightarrow Q = \sqrt{\frac{9}{2} - \frac{c_m}{3}}, b = \frac{1}{2}\left(\sqrt{\frac{9}{2} - \frac{c_m}{3}} - \sqrt{\frac{1}{2} - \frac{c_m}{3}}\right).$$

$$c_m = \frac{3}{2} - \frac{12}{p(p+2)}.$$

$$[\mathcal{J}_m,\mathcal{J}_n] = \mathrm{i}\epsilon_{mnr}\mathcal{J}_r, [\mathcal{J}_m,\mathcal{Q}^\alpha] = -\frac{1}{2}(\gamma_m)^\alpha{}_\beta\mathcal{Q}^\beta, \{\mathcal{Q}^\alpha,\mathcal{Q}^\beta\} = (\gamma^m)^{\alpha\beta}\mathcal{J}_m$$

$$\begin{aligned}
[J_m,J_n] &= \mathrm{i}\epsilon_{mnr}J_r, & [J_m,K_n] &= \mathrm{i}\epsilon_{nmr}K_r \\
[K_m,K_n] &= -\mathrm{i}\epsilon_{mnr}J_r, & \{Q^\alpha,Q^\beta\} &= (\gamma^m)^{\alpha\beta}J_m \\
\{S^\alpha,S^\beta\} &= -(\gamma^m)^{\alpha\beta}J_m, & \{Q^\alpha,S^\beta\} &= (\gamma^m)^{\alpha\beta}K_m \\
[J_m,Q^\alpha] &= -\frac{1}{2}(\gamma^m)^\alpha{}_\beta Q^\beta, & [K_m,Q^\alpha] &= -\frac{1}{2}(\gamma^m)^\alpha{}_\beta S^\beta \\
[J_m,S^\alpha] &= -\frac{1}{2}(\gamma^m)^\alpha{}_\beta S^\beta, & [K_m,S^\alpha] &= -\frac{1}{2}(\gamma^m)^\alpha{}_\beta Q^\beta
\end{aligned}$$

$$M=\begin{pmatrix} a & b & b\alpha-a\rho \\ c & d & d\alpha-c\rho \\ \alpha & \rho & 1-\alpha\rho \end{pmatrix}\equiv\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\mathrm{ber}(M)\equiv \mathrm{det}A\mathrm{det}(D-CA^{-1}B)^{-1}=1$$

$$z(Z')=\frac{az'+b}{cz'+d}+\theta'\frac{\alpha z'+\rho}{(cz'+d)^2}$$
$$\theta(Z')=\frac{\alpha z'+\rho}{cz'+d}+\frac{\theta'}{cz'+d}$$

$$\partial_{z'}z=\frac{1}{(cz'+d)^2}\left(1+\frac{\theta'}{(cz'+d)}(\alpha(d-cz')-2\rho c)\right)$$
$$\partial_{z'}\theta=\frac{1}{(cz'+d)^2}(\alpha d-\rho c)-\theta'\frac{c}{(cz'+d)^2}$$
$$\partial_{\theta'}z=\frac{\alpha z'+\rho}{(cz'+d)^2},\partial_{\theta'}\theta=\frac{1}{cz'+d}$$

$$\sigma\equiv \mathrm{ber}\begin{pmatrix} \partial_{z'}z & \partial_{z'}\theta \\ \partial_{\theta'}z & \partial_{\theta'}\theta \end{pmatrix}=\frac{1}{cz'+d}+\frac{\theta'}{(cz'+d)^2}(\alpha d+\rho c)$$

$$\Phi'[Z']=\Phi[z(Z')\mid\theta(Z')]+\frac{1}{\beta}\log\,\sigma^{-1},$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{grav},(0)}=e^{2\vartheta}\times\mathcal{A}\times\left(\frac{r}{\ell_{\mathrm{uv}}}\right)^{\frac{c_m+c_{\mathrm{bc}}+c_{\beta\gamma}}{3}}\times\int\frac{[\mathcal{D}\varphi][\mathcal{D}\psi][\mathcal{D}F]}{\mathrm{vol}_{OSp(1|2;\mathbb{C})}}e^{-\mathcal{S}^{\mathcal{N}=1}_{\mathrm{L}}}$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{L}}[\mu]\equiv\int\frac{[\mathcal{D}\varphi][\mathcal{D}\psi][\mathcal{D}F]}{\mathrm{vol}_{\mathrm{OSp}(1|2;\mathbb{C})}}e^{-\mathcal{S}^{\mathcal{N}=1}_{\mathrm{L}}}$$

$$\mathcal{S}^{\mathcal{N}=1}_{\mathrm{tL}}=\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\left(-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+\frac{\mathrm{i}}{2}\bar{\psi}\not{\partial}\psi+\frac{1}{2}F^2-\frac{1}{2}q\tilde{R}\varphi+\mathrm{i}\mu e^{\beta\varphi}F+\frac{1}{2}\mu\beta e^{\beta\varphi}\bar{\psi}\psi\right)$$

$$q=\sqrt{\frac{c_m}{3}-\frac{9}{2}},\beta=\frac{1}{2}\left(\sqrt{\frac{c_m}{3}-\frac{1}{2}}-\sqrt{\frac{c_m}{3}-\frac{9}{2}}\right)$$

$$\mathcal{S}^{\mathcal{N}=1}_{\mathrm{tL}}=\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\left[-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+\frac{\mathrm{i}}{2}\bar{\psi}\not{\partial}\psi+\frac{1}{2}F^2-\frac{1}{2}q\tilde{R}\varphi+\mathrm{i}\left(FW'(\varphi)-\frac{\mathrm{i}}{2}W''(\varphi)\bar{\psi}\psi\right)\right]$$

$$\delta\varphi=\bar{\epsilon}\psi,\delta\psi=\mathrm{i}\,\partial\varphi\epsilon+q\mathrm{i}\not{\partial}\epsilon-\mathrm{i}F\epsilon,\delta F=-\bar{\epsilon}\not{\partial}\psi$$

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\varphi=\mathcal{L}_\xi\varphi+\Delta q$$
$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\psi=\mathcal{L}_\xi\psi+\frac{1}{2}\Delta\psi$$
$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]F=\mathcal{L}_\xi F+\Delta F$$

$$\xi^\mu\equiv 2\mathrm{i}\bar{\epsilon}_2\gamma^\mu\epsilon_1,\Delta\equiv\frac{1}{2}\nabla_\mu\xi^\mu$$

$$\delta\varphi = \bar{\epsilon}\psi, \delta\psi = \mathrm{i}\,\partial\!\!\!/\varphi\epsilon - \frac{q}{r}\epsilon - \mathrm{i}F\epsilon, \delta F = -\bar{\epsilon}\partial\!\!\!/\psi$$

$$\mathcal{Z}_{\mathrm{tL}}^{\mathcal{N}} = 1[\mu] = \int \frac{[\mathcal{D}\varphi][\mathcal{D}\psi]}{\mathrm{vol}_{\mathrm{OSp}(1|2;\mathbb{C})}} e^{-\frac{1}{4\pi}\int_{S^2} \mathrm{d}^2x\tilde{\mathrm{e}}\left(-\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + \frac{\mathrm{i}}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{q}{r^2}\varphi + \frac{1}{2}\mu^2 e^{2\beta\varphi} + \frac{1}{2}\mu\beta e^{\beta\varphi}\bar{\psi}\psi\right)}$$

$$-\nabla^2\varphi = \mu^2\beta e^{2\beta\varphi} - \frac{q}{r^2} + \frac{1}{2}\mu\beta^2 e^{\beta\varphi}\bar{\psi}\psi, \partial\!\!\!/\psi = \mathrm{i}\mu\beta e^{\beta\varphi}\psi$$

$$\varphi_* = \frac{1}{2\beta}\log\frac{q}{r^2\mu^2\beta}, \psi_* = 0$$

$$\mathcal{Z}_{\mathrm{tL}}^{\mathcal{N}=1}[\mu] \approx \mathcal{Z}_{\mathrm{saddle}}^{\mathcal{N}=1}[\mu] = \left(\frac{q}{e\mu^2 r^2\beta}\right)^{\frac{q}{2\beta}}$$

$$\varphi = \varphi_* + \delta\varphi, \psi = \psi_* + \delta\psi$$

$$\mathcal{L}_{\mathrm{bos}}^{(2)}[\delta\varphi] = -\frac{1}{8\pi}\delta\varphi\left(-\nabla^2 - \frac{2q\beta}{r^2}\right)\delta\varphi, \mathcal{L}_{\mathrm{fer}}^{(2)}[\delta\psi] = \frac{\mathrm{i}}{8\pi}\delta\bar{\psi}\left(\partial\!\!\!/ - \mathrm{i}\frac{\sqrt{q\beta}}{r}\right)\delta\psi$$

$$\mathcal{Z}_{\mathrm{tL,pert}}^{\mathcal{N}=1} = \int [\mathcal{D}\delta\varphi][\mathcal{D}\delta\psi]e^{-\int_{S^2} \mathrm{d}^2x\,\mathrm{e}\left(\mathcal{L}_{\mathrm{bos}}^{(2)}[\delta\varphi] + \mathcal{L}_{\mathrm{fer}}^{(2)}[\delta\psi]\right)}$$

$$\varphi(\Omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\varphi_{lm}Y_{lm}(\Omega), -\nabla^2 Y_{lm}(\Omega) = \frac{1}{r^2}(l+1)Y_{lm}(\Omega)$$

$$\int_{S^2} \mathrm{d}\Omega Y_{lm}(\Omega)Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$$

$$\psi(\Omega) = \frac{1}{\sqrt{r}}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\alpha_{+lm}\psi_{+lm}^{(+)}(\Omega) + \alpha_{-lm}\psi_{-lm}^{(+)}(\Omega) + \beta_{+lm}\psi_{+lm}^{(-)}(\Omega) + \beta_{-lm}\psi_{-lm}^{(-)}(\Omega)\right)$$

$$\psi(\Omega) = \frac{1}{\sqrt{r}}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\alpha_{+lm}\psi_{+lm}^{(+)}(\Omega) + \alpha_{-lm}\psi_{-lm}^{(+)}(\Omega) - \mathrm{i}\alpha_{-lm}^*\psi_{+lm}^{(-)}(\Omega) - \mathrm{i}\alpha_{+lm}^*\psi_{-lm}^{(-)}(\Omega)\right)$$

$$\partial\!\!\!/\psi_{\pm lm}^{(s)}(\Omega) = \pm\frac{\mathrm{i}}{r}(l+1)\psi_{\pm,lm}^{(s)}(\Omega), s = \pm$$

$$\int_{S^2} \mathrm{d}\Omega\bar{\psi}_{\pm lm}^{(+)}(\Omega)\psi_{\pm l'm'}^{(-)}(\Omega) = -\mathrm{i}\delta_{l,l'}\delta_{m,m'}, \int_{S^2} \mathrm{d}\Omega\bar{\psi}_{\pm lm}^{(-)}(\Omega)\psi_{\pm l'm'}^{(+)}(\Omega) = +\mathrm{i}\delta_{l,l'}\delta_{m,m'}$$

$$[\mathcal{D}\psi] = \prod_{l=0}^{\infty}\prod_{m=0}^{l}\left(\frac{2\pi}{\Lambda_{\mathrm{uv}}r^2}\right)\prod_{\pm}\mathrm{d}\alpha_{\pm lm}\,\mathrm{d}\alpha_{\pm lm}^*, [\mathcal{D}\varphi] = \prod_{l=0}^{\infty}\prod_{m=-l}^{l}\left(\frac{\Lambda_{\mathrm{uv}}r^2}{\pi}\right)^{\frac{1}{2}}\mathrm{d}\varphi_{lm}$$

$$1 = \int [\mathcal{D}\psi]e^{-\frac{1}{2\sqrt{2\pi}}\Lambda_{\mathrm{uv}}^{1/2}\int_{S^2} \mathrm{d}^2x\tilde{\mathrm{e}}\bar{\psi}\psi}, 1 = \int [\mathcal{D}\varphi]e^{-\Lambda_{\mathrm{uv}}\int_{S^2} \mathrm{d}^2x\tilde{\mathrm{e}}\varphi(x)^2}$$

$$S^{(2)}_{\mathrm{tL,fer}}[\delta\psi]=\frac{1}{4\pi}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left((l+1-\sqrt{q\beta})\alpha_{+lm}\alpha^{*}_{-lm}-(l+1+\sqrt{q\beta})\alpha_{-lm}\alpha^{*}_{+lm}\right).$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{tL,fer,1-loop}}\equiv\frac{2\pi}{\Lambda_{\mathrm{uv}}r^2}\times\int\ [\mathcal{D}'\delta\psi]e^{-S^{(2)}_{\mathrm{tL,fer}}[\delta\psi]}=\frac{2\pi}{\Lambda_{\mathrm{uv}}r^2}\times\prod_{l=1}^{\infty}\left(\frac{(1+l)^2-q\beta}{8\pi\Lambda_{\mathrm{uv}}r^2}\right)^{l+1}$$

$$\sum_{l=1}^{\infty}(l+1)\log\frac{(1+l)^2-q\beta}{8\pi\Lambda_{\mathrm{uv}}r^2}$$
$$=-\int_0^{\infty}\frac{\mathrm{d}t}{2t}\left[\frac{2e^{-\frac{t}{2}}}{(1-e^{-t})}\left(\frac{2e^{-\frac{t}{2}+\mathrm{i}\nu t}}{(1-e^{-t})}-2e^{-\frac{t}{2}+\mathrm{i}\nu t}-e^{-\left(\frac{5}{2}+\mathrm{i}\nu\right)t}\frac{\left(1-2e^{t}+e^{2\mathrm{i}\nu t}\right)}{(1-e^{-t})}\right)\right]$$

$$\chi_{\Delta=3/2}(t)=\frac{e^{-\frac{3}{2}t}}{(1-e^{-t})}$$

$$-\frac{1}{2}\log\frac{\tilde{\lambda}}{8\pi r^2\Lambda_{\mathrm{uv}}}=\int_0^{\infty}\frac{\mathrm{d}\tau}{2\tau}e^{-\frac{\varepsilon^2}{4\tau}-\tilde{\lambda}\tau}=K_0(\sqrt{\tilde{\lambda}}\varepsilon)\approx-\frac{1}{2}\log\frac{\varepsilon^2e^{2\gamma_E\tilde{\lambda}}}{4}$$

$$\log\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{tL,fer,1\text{-}loop}}=-\frac{2}{\varepsilon^2}+\left(\frac{5}{6}-\beta^2\right)\log\varepsilon+\varrho$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{tL,\,bos\,,1-\,loop}}\equiv\pm\mathrm{i}\left(\frac{4\pi r^2\Lambda_{\mathrm{uv}}}{\beta q}\right)^{\frac{1}{2}}\left(\frac{r^2\Lambda_{\mathrm{uv}}}{\pi}\right)^{\frac{3}{2}}\times\int\ [\mathcal{D}'\delta\varphi]e^{-\frac{1}{4\pi}\int_{S^2}\ \mathrm{d}x^2\mathrm{e}\frac{1}{2}\delta\varphi\left(-\nabla^2-\frac{2}{r^2}q\beta\right)\delta\varphi}$$
$$=\pm\mathrm{i}\left(\frac{4\pi r^2\Lambda_{\mathrm{uv}}}{\beta q}\right)^{\frac{1}{2}}\left(\frac{r^2\Lambda_{\mathrm{uv}}}{\pi}\right)^{\frac{3}{2}}\times\prod_{l=2}^{\infty}\left(\frac{8\pi r^2\Lambda_{\mathrm{uv}}}{l(l+1)-2\beta q}\right)^{l+\frac{1}{2}}$$

$$-\frac{1}{2}\sum_{l=2}^{\infty}(2l+1)\log\left(\frac{l(l+1)-2\beta q}{8\pi\Lambda_{\mathrm{uv}}r^2}\right)=\int_0^{\infty}\frac{\mathrm{d}t}{2t}\left[\frac{1+e^{-t}}{(1-e^{-t})}\frac{2e^{-\frac{t}{2}+\mathrm{i}\nu t}}{(1-e^{-t})}\right.$$
$$\left.+e^{-\left(\frac{5}{2}-\mathrm{i}\nu\right)t}\left(\frac{3-2e^{t}+5e^{3t}}{1-e^{-t}}-e^{2t}\frac{(5e^{t}-3)\left(1-e^{-(3+2\mathrm{i}\nu)t}\right)}{(1-e^{-t})^2}\right)\right]$$

$$\chi_{\Delta=2}(t)=\frac{e^{-2t}}{(1-e^{-t})}$$

$$\log\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{bos\,,1\text{-}loop}}=\frac{2}{\varepsilon^2}-\left(\frac{7}{3}-2\beta^2\right)\log\varepsilon+\zeta^{\varsigma}_{\lambda}$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{tL}}\approx\pm\mathrm{i}\times\mathrm{const}\times\left(\frac{\mu}{\beta}\right)^{-\frac{1}{\beta^2}+1}\Lambda_{\mathrm{uv}}^{\frac{5}{4}-\beta^2}r_{\mathrm{tL}}^{c^{\mathcal{N}=1}/3}e^{-\frac{1}{2\beta^2}-\left(\frac{1}{\beta^2}-1\right)\log\beta^2}\times\beta^{-1}\times\left(1+\mathcal{O}(\beta^2)\right)$$

$$4\pi\mathcal{L}^{\mathcal{N}=1}_{\mathrm{tL,int}}[\delta\varphi,\delta\psi]=-\frac{2\mathrm{i}}{3r^2}q\beta^2\delta\varphi^3+\frac{1}{3r^2}q\beta^3\delta\varphi^4+\frac{\mathrm{i}}{2r}\beta\sqrt{q\beta}\delta\varphi\delta\bar{\psi}\delta\psi-\frac{1}{4r}\beta^2\sqrt{q\beta}\delta\varphi^2\delta\bar{\psi}\delta\psi$$

$$H(\Omega,\Omega') \equiv \frac{1}{\mathcal{Z}^{\mathcal{N}=1}_{\text{fer},1-\text{ loop}}} \frac{2\pi}{\Lambda_{\text{uv}} r^2} \int \; [\mathcal{D}'\delta\psi] e^{-S^{(2)}_{\text{pert}}[\delta\psi]} \delta\bar{\psi}(\Omega)\delta\psi(\Omega')$$

$$= \frac{4\pi}{r} \sum_{l\geq 1} \sum_{0\leq m\leq l} \frac{1}{(l+1)^2 - q\beta} \Big[\mathrm{i}(l+1+\sqrt{q\beta}) \left(\bar{\psi}^{(+)}_{+lm}(\Omega)\psi^{(-)}_{+lm}(\Omega') - \bar{\psi}^{(-)}_{+lm}(\Omega)\psi^{(+)}_{+lm}(\Omega') \right)$$

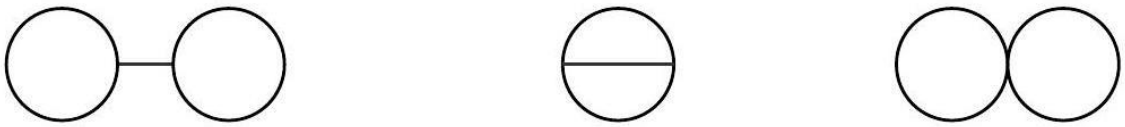
$$+\mathrm{i}(l+1-\sqrt{q\beta}) \left(\bar{\psi}^{(-)}_{-lm}(\Omega)\psi^{(+)}_{-lm}(\Omega') - \bar{\psi}^{(+)}_{-,lm}(\Omega)\psi^{(-)}_{-lm}(\Omega') \right) \Big],$$

$$H(\Omega_0,\Omega_0) = \frac{1}{4\pi}\int_{S^2} \mathrm{d}\Omega H(\Omega,\Omega) = +\frac{4\sqrt{q\beta}}{r} \sum_{l\geq 1} \frac{l+1}{(l+1)^2 - q\beta}$$

$$G(\Omega,\Omega') = 4\pi \sum_{l\neq 1} \sum_{-l\leq m\leq l} \frac{Y_{lm}(\Omega)Y_{lm}(\Omega')}{l(l+1) - 2q\beta}.$$

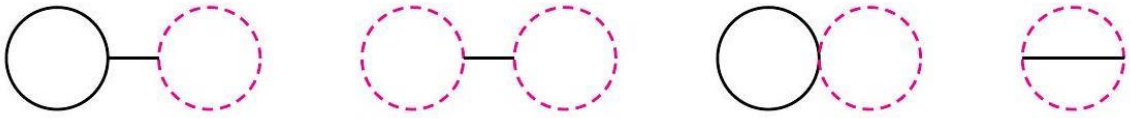
$$G(\Omega_0,\Omega_0) = \frac{1}{4\pi}\int_{S^2} \mathrm{d}\Omega G(\Omega,\Omega) = \sum_{l\neq 1} \frac{2l+1}{l(l+1) - 2q\beta}$$

$$\text{loops}_b = -\frac{\beta^2}{8\pi^2}\int_{S^2} \mathrm{d}\Omega\, \mathrm{d}\Omega' G(\Omega,\Omega)G(\Omega',\Omega')G(\Omega,\Omega') - \frac{\beta^2}{12\pi^2}\int_{S^2} \mathrm{d}\Omega\, \mathrm{d}\Omega' G(\Omega,\Omega')^3$$
$$-\frac{\beta^2}{4\pi}\int_{S^2} \mathrm{d}\Omega G(\Omega,\Omega)^2$$



$$\ominus \equiv -\frac{1}{6} \sum_{l_1,l_2,l_3\neq 1} \frac{(2l_1+1)(2l_2+1)(2l_3+1)}{(l_1(l_1+1)-2)(l_2(l_2+1)-2)(l_3(l_3+1)-2)} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2,$$

$$\text{loops}_f = \frac{r}{16\pi^2}\beta^2 \int_{S^2} \mathrm{d}\Omega\, \mathrm{d}\Omega' H(\Omega,\Omega)G(\Omega',\Omega')G(\Omega,\Omega') - \frac{r^2}{128\pi^2}\beta^2 \int_{S^2} \mathrm{d}\Omega\, \mathrm{d}\Omega' H(\Omega,\Omega)H(\Omega',\Omega')G(\Omega,\Omega')$$
$$+\frac{r}{16\pi}\beta^2 \int_{S^2} \mathrm{d}\Omega H(\Omega,\Omega)G(\Omega,\Omega) - \frac{r^2}{128\pi^2}\beta^2 \int_{S^2} \mathrm{d}\Omega\, \mathrm{d}\Omega' H(\Omega,\Omega')^2 G(\Omega,\Omega')$$



$$\frac{r}{16\pi^2}\int_{S^2} \mathrm{d}\Omega\, \mathrm{d}\Omega' H(\Omega,\Omega)G(\Omega',\Omega')G(\Omega,\Omega') = \frac{r}{(-2)} H(\Omega_0,\Omega_0)G(\Omega_0,\Omega_0) \approx -4\log^2(\Lambda_{\text{uv}} r^2) + \cdots,$$

$$-\frac{r^2}{128\pi^2}\int_{S^2} \mathrm{d}\Omega\, \mathrm{d}\Omega' H(\Omega,\Omega)F(\Omega',\Omega')G(\Omega,\Omega') = -\frac{r^2}{8}\frac{1}{(-2)} H(\Omega_0,\Omega_0)^2 \approx \log^2(\Lambda_{\text{uv}} r^2) + \cdots,$$

$$\frac{r}{16\pi}\int_{S^2} \mathrm{d}\Omega H(\Omega,\Omega)G(\Omega,\Omega) = \frac{r}{4} H(\Omega_0,\Omega_0)G(\Omega_0,\Omega_0) \approx 2\log^2(\Lambda_{\text{uv}} r^2) + \cdots,$$

$$-\frac{r^2}{128\pi^2}\beta^2 \int_{S^2} \mathrm{d}\Omega\, \mathrm{d}\Omega' H(\Omega,\Omega')^2 G(\Omega,\Omega') \approx \log^2(\Lambda_{\text{uv}} r^2) + \cdots$$

$$\mathcal{Z}_{\mathrm{tL}}^{\mathcal{N}=1} \approx \pm \mathrm{i}\ \mathrm{const} \times \left(\frac{\mu}{\beta}\right)^{-\frac{1}{\beta^2}+1} \Lambda_{\mathrm{uv}}^{\frac{5}{4}-\beta^2} r_{\mathrm{tL}}^{c_{\mathrm{tL}}^{\mathcal{N}}=1}/3 e^{-\frac{1}{2\beta^2}-\left(\frac{1}{\beta^2}-1\right)\log\,\beta^2} \times \beta^{-1} \times (1+\mathrm{loops}\beta^2+\cdots)$$

$$\mathcal{W}_\beta \equiv F e^{\beta\varphi} - \frac{\mathrm{i}}{2}\beta e^{\beta\varphi}\bar{\psi}\psi$$

$$\log\,\mathcal{Z}_{\mathrm{grav},(0)}^{\mathcal{N}=1} = 2\vartheta - \left(\frac{12}{\left(\sqrt{2c_m-27}-\sqrt{2c_m-3}\right)^2}-\frac{1}{2}\right)\log\frac{\mu^2}{\Lambda_{\mathrm{uv}}} + f_0(c_m)$$

$$\frac{1}{\pi}\int_{\Sigma_h} \mathrm{d}^2x \mathrm{e}\left(\vartheta\frac{R}{4}+\mathrm{i}\frac{v}{4}\epsilon^{\mu\nu}F_{\mu\nu}\right) = \vartheta\chi_h + \mathrm{i}vn_h$$

$$\mathcal{Z}_{\mathrm{grav}}^{\mathcal{N}=2} = \sum_{h=0}^{\infty}\sum_{n_h\in\mathbb{Z}} e^{\vartheta\chi_h+\mathrm{i}vn_h}\int [\mathcal{D}\mathrm{e}_\mu^a][\mathcal{D}\chi_\mu][\mathcal{D}A_\mu][\mathcal{D}B] e^{-\mathrm{i}\mu\int_{\Sigma_h}\mathrm{d}^2x\,\mathrm{d}^2\theta\,\mathrm{d}^2\tilde{\theta}(\mathrm{E}+\mathrm{h.c.})}$$

$$\times Z_{\mathrm{SCFT}}^{(h,n_h)}[\mathrm{e}_\mu^a,\chi_\mu,A_\mu,B]$$

$$\mathrm{e}_\mu^a = e^{\frac{1}{2}b(\varphi+\tilde{\varphi})}\tilde{\mathrm{e}}_\mu^a, A^\mu = -\frac{\mathrm{i}}{2}\epsilon^{\mu\nu}\partial_\nu(\varphi-\tilde{\varphi})$$

$$\chi_\mu = e^{\frac{1}{2}b\varphi}\gamma_\mu\psi, \chi_\mu^* = e^{\frac{1}{2}b\tilde{\varphi}}\gamma_\mu\tilde{\psi}, B = e^{-b\varphi}F, B^* = e^{-b\tilde{\varphi}}\tilde{F}$$

$$Z_{\mathrm{SCFT}}^{(0)}[\Phi,\tilde{\Phi}] = \mathcal{C}e^{-S_{\text{s-anomaly}}[\Phi,\tilde{\Phi}]}$$

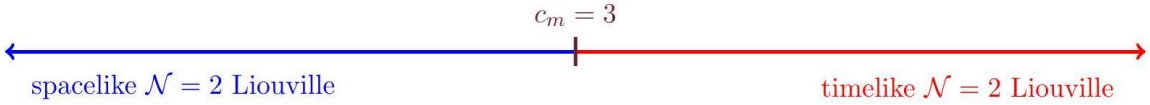
$$S_{\mathrm{L}}^{\mathcal{N}=2} = \frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\left(\partial_\mu\tilde{\varphi}\partial^\mu\varphi - \mathrm{i}\overline{\tilde{\psi}}\not{\partial}\psi - F\tilde{F} + \frac{Q\tilde{R}}{2}(\varphi+\tilde{\varphi})\right.$$
$$\left. +\mu e^{b\varphi}F - \frac{\mathrm{i}}{2}\mu b e^{b\varphi}\bar{\psi}\psi + \mu^* e^{b\tilde{\varphi}}\tilde{F} - \frac{\mathrm{i}}{2}\mu^* b e^{b\tilde{\varphi}}\overline{\tilde{\psi}}\tilde{\psi}\right)$$

$U(1)_V$: $\varphi \to \varphi + 2\mathrm{i}Q\rho, \tilde{\varphi}\to\tilde{\varphi}-2\mathrm{i}Q\rho, \psi\to e^{-\mathrm{i}\rho}\psi, \tilde{\psi}\to e^{\mathrm{i}\rho}\tilde{\psi}$
$F\to e^{-2\mathrm{i}\rho}F, \tilde{F}\to e^{2\mathrm{i}\rho}\tilde{F}$

$U(1)_A$: $\psi\to e^{\mathrm{i}\alpha\gamma_*}\psi, \tilde{\psi}\to e^{-\mathrm{i}\alpha\gamma_*}\tilde{\psi}.$

$$c_{\mathrm{L}}^{\mathcal{N}=2} + c_m + c_{\mathrm{bc}} + 2\times c_{\beta\gamma} + c_{U(1)_V} = 3 + 6Q^2 + c_m - 26 + 2\times 11 - 2 = 0.$$

$$Q = \frac{1}{b} = \sqrt{\frac{3-c_m}{6}}.$$



$$\psi\to e^{-\frac{\mathrm{i}}{2}b\varphi_2}\psi, \tilde{\psi}\to e^{\frac{\mathrm{i}}{2}b\varphi_2}\tilde{\psi}, F\to e^{-\mathrm{i}b\varphi_2}F, \tilde{F}\to e^{\mathrm{i}b\varphi_2}\tilde{F}$$

$$M=\begin{pmatrix} a & b & \alpha & \beta \\ c & d & \gamma & \delta \\ \zeta & \eta & e & f \\ \rho & \sigma & g & h \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix} \cdot \begin{pmatrix} \Omega_2 & 0 \\ 0 & \mathbb{S}_2 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Omega_2 & 0 \\ 0 & \mathbb{S}_2 \end{pmatrix}$$

$$\mathcal{Z}^{\mathcal{N}=2}_{\text{grav},(0)} = e^{2\vartheta} \times \mathcal{A} \times \left(\frac{r}{\ell_{\text{uv}}}\right)^{(c_m + c_{\mathfrak{bc}} + 2c_{\beta\gamma} + c_{U(1)})/3} \times \int \frac{[\mathcal{D}\Phi][\mathcal{D}\widetilde{\Phi}]}{\text{vol}_{OSp(2|2;\mathbb{C})}} e^{-\mathcal{S}^{\mathcal{N}=2}_{\text{L}}}$$

$$\mathcal{Z}^{\mathcal{N}=2}_{\text{L}}[\mu] \equiv \int \frac{[\mathcal{D}\varphi][\mathcal{D}\psi][\mathcal{D}F]}{\text{vol}_{\text{OSp}(2|2;\mathbb{C})}} e^{-\mathcal{S}^{\mathcal{N}=2}_{\text{L}}}$$

$$\mathcal{S}^{\mathcal{N}=2}_{\text{tL}} = \frac{1}{4\pi}\int_{S^2} \mathrm{d}^2x\tilde{\mathrm{e}}(-\partial_\mu\tilde{\varphi}\partial^\mu\varphi + \mathrm{i}\overline{\tilde{\psi}}\not{\partial}\psi + F\tilde{F} - \frac{q\tilde{R}}{2}(\varphi+\tilde{\varphi}) + \mathrm{i}\mu e^{\beta\varphi}F + \frac{1}{2}\mu\beta e^{\beta\varphi}\bar{\psi}\psi + \mathrm{i}\mu^* e^{\beta\tilde{\varphi}}\tilde{F} + \frac{1}{2}\mu^*\beta e^{\beta\tilde{\varphi}}\overline{\tilde{\psi}}\tilde{\psi}\Big)$$

$$\mathcal{S}^{\mathcal{N}=2}_{\text{tL}} = \frac{1}{4\pi}\int_{S^2} \mathrm{d}^2x\tilde{\mathrm{e}}[-\partial_\mu\tilde{\varphi}\partial^\mu\varphi + \mathrm{i}\overline{\tilde{\psi}}\not{\partial}\psi + F\tilde{F} - \frac{q\tilde{R}}{2}(\varphi+\tilde{\varphi}) + \mathrm{i}\left(FW'(\varphi) - \frac{\mathrm{i}}{2}W''(\varphi)\bar{\psi}\psi\right) + \mathrm{i}\left(\tilde{F}\widetilde{W}'(\tilde{\varphi}) - \frac{\mathrm{i}}{2}\widetilde{W}''(\tilde{\varphi})\overline{\tilde{\psi}}\tilde{\psi}\right)]$$

$$c^{\mathcal{N}=2}_{\text{tL}} = 3 - \frac{6}{\beta^2}$$

$$\begin{aligned} &\delta\varphi = \bar{\tilde{\epsilon}}\psi, && \delta\tilde{\varphi} = \bar{\epsilon}\tilde{\psi} \\ &\delta\psi = \mathrm{i}\,\partial\!\!\!/\varphi\epsilon + \mathrm{i}q\not{\partial}\epsilon - \mathrm{i}F\tilde{\epsilon}, && \delta\tilde{\psi} = \mathrm{i}\,\partial\!\!\!/\tilde{\varphi}\tilde{\epsilon} + \mathrm{i}q\not{\partial}\tilde{\epsilon} - \mathrm{i}\tilde{F}\epsilon \\ &\delta F = -\bar{\epsilon}\not{\partial}\psi, && \delta\tilde{F} = \bar{\tilde{\epsilon}}\not{\partial}\tilde{\psi} \end{aligned}$$

$$\tilde{\varphi} = \varphi^*, \tilde{\psi} = \gamma_*\psi^C, \tilde{F} = F^*$$

$$\begin{aligned} [\delta_\epsilon, \delta_{\tilde{\epsilon}}]\varphi &= \mathcal{L}_\xi\varphi + q\Delta + 2\mathrm{i}q\rho \\ [\delta_\epsilon, \delta_{\tilde{\epsilon}}]\psi &= \mathcal{L}_\xi\psi + \frac{1}{2}\Delta\psi - \mathrm{i}\rho\psi + \mathrm{i}\alpha\gamma_*\psi \\ [\delta_\epsilon, \delta_{\tilde{\epsilon}}]F &= \mathcal{L}_\xi F + \Delta F - 2\mathrm{i}\rho F \end{aligned}$$

$$\xi^\mu = \mathrm{i}\bar{\tilde{\epsilon}}\gamma^\mu\epsilon, \qquad \Delta = \frac{1}{2}\nabla_\mu\xi^\mu$$

$$\rho = -\frac{1}{4}\left(D_\mu\bar{\tilde{\epsilon}}\gamma^\mu\epsilon - \bar{\tilde{\epsilon}}\gamma^\mu D_\mu\epsilon\right), \qquad \alpha = \frac{1}{4}\left(D_\mu\bar{\tilde{\epsilon}}\gamma_*\gamma^\mu\epsilon - \bar{\tilde{\epsilon}}\gamma_*\gamma^\mu D_\mu\epsilon\right)$$

$$\begin{aligned} &\varphi \to \varphi + q\Delta, \quad \psi \to e^{\frac{\Delta}{2}}\psi, \quad F \to e^{\Delta}F \\ &\tilde{\varphi} \to \tilde{\varphi} - q\Delta, \quad \tilde{\psi} \to e^{-\frac{\Delta}{2}}\tilde{\psi}, \quad \tilde{F} \to e^{-\Delta}\tilde{F} \end{aligned}$$

$$0=\nabla^2\varphi-\frac{1}{\beta r^2}+\mathrm{i}\mu\beta e^{\beta\varphi}F+\frac{1}{2}\mu\beta^2\bar{\psi}\psi,0=\nabla^2\tilde{\varphi}-\frac{1}{\beta r^2}+\mathrm{i}\mu^*\beta e^{\beta\tilde{\varphi}}\tilde{F}+\frac{1}{2}\mu^*\beta^2\overline{\tilde{\psi}}\tilde{\psi}$$

$$0=-\mathrm{i}\not{\partial}\overline{\tilde{\psi}}+\frac{1}{2}\mu\beta e^{\beta\varphi}\bar{\psi},0=\mathrm{i}\not{\partial}\psi+\frac{1}{2}\mu^*\beta e^{\beta\tilde{\varphi}}\tilde{\psi}$$

$$0=\tilde{F}+\mathrm{i}\mu e^{\beta\varphi},0=F+\mathrm{i}\mu^* e^{\beta\tilde{\varphi}}$$

$$(\varphi+\tilde{\varphi})_*=\frac{1}{\beta}\log\frac{1}{|\mu|^2\beta^2r^2},(\varphi-\tilde{\varphi})_*=c$$

$$\mathcal{S}_{\mathrm{tL}}^{\mathcal{N}=2}=\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\sqrt{\tilde{g}}\left(-\partial_\mu\tilde{\varphi}\partial^\mu\varphi+\mathrm{i}\overline{\tilde{\psi}}\not{\partial}\psi+|\mu|^2e^{\beta(\varphi+\tilde{\varphi})}\right.$$
$$\left.-\frac{1}{\beta r^2}(\varphi+\tilde{\varphi})+\frac{\beta}{2}\left(\mu e^{\beta\varphi}\bar{\psi}\psi+\mu^*e^{\beta\tilde{\varphi}}\overline{\tilde{\psi}}\tilde{\psi}\right)\right)$$

$$\mathcal{Z}_{\mathrm{tL}}^{\mathcal{N}=2}[\mu]\approx\mathcal{Z}_{\mathrm{saddle}}^{\mathcal{N}=2}=\left(\frac{1}{e|\mu|^2\beta^2r^2}\right)^{\frac{1}{\beta^2}}$$

$$\varphi_1\equiv\frac{1}{2}(\varphi+\tilde{\varphi}),\varphi_2\equiv-\frac{\mathrm{i}}{2}(\varphi-\tilde{\varphi})$$

$$\varphi_1=\varphi_{1,*}+\delta\varphi_1,\varphi_2=\varphi_{2,*}+\delta\varphi_2,\psi=\psi_*+\delta\psi,\tilde{\psi}=\tilde{\psi}_*+\delta\tilde{\psi}$$

$$S_{\mathrm{pert}}^{(2)}=\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\left[-\delta\varphi_1\left(-\nabla^2-\frac{2}{r^2}\right)\delta\varphi_1-\delta\varphi_2(-\nabla^2)\delta\varphi_2\right.$$
$$\left.+\mathrm{i}\delta\overline{\tilde{\psi}}\not{\partial}\delta\psi+\frac{1}{2r}\frac{\mu}{|\mu|}\delta\bar{\psi}\delta\psi+\frac{1}{2r}\frac{\mu^*}{|\mu|}\delta\overline{\tilde{\psi}}\delta\tilde{\psi}\right]$$

$$1=\int[\mathcal{D}\varphi_1][\mathcal{D}\varphi_2]e^{-\Lambda_{\mathrm{uv}}\int_{S^2}\mathrm{d}^2x\,\mathrm{e}\,\varphi(x)\tilde{\varphi}(x)}$$

$$\mathcal{Z}_{\mathrm{tL,\,bos\,,1\text{-}loop}}^{\mathcal{N}=2}=\pm\mathrm{i}(2\pi r^2\Lambda_{\mathrm{uv}})^{\frac{1}{2}}\left(\frac{r^2\Lambda_{\mathrm{uv}}}{\pi}\right)^2\times\prod_{l=2}^{\infty}\left(\frac{4\pi r^2\Lambda_{\mathrm{uv}}}{l(l+1)-2}\right)^{l+\frac{1}{2}}\prod_{l=1}^{\infty}\left(\frac{4\pi r^2\Lambda_{\mathrm{uv}}}{l(l+1)}\right)^{l+\frac{1}{2}}$$

$$-\frac{1}{2}\sum_{l=2}^{\infty}(2l+1)\log\frac{l(l+1)-2}{4\pi r^2\Lambda_{\mathrm{uv}}}=\int_0^\infty\frac{\mathrm{d}t}{2t}\left[\frac{1+e^{-t}}{(1-e^{-t})}\left(\frac{2e^{-2t}}{(1-e^{-t})}+5e^{-t}-5e^{-2t}+3e^{-3t}\right)\right]$$

$$-\frac{1}{2}\sum_{l=1}^{\infty}(2l+1)\log\frac{l(l+1)}{4\pi r^2\Lambda_{\mathrm{uv}}}=\int_0^\infty\frac{\mathrm{d}t}{2t}\left[\frac{1+e^{-t}}{(1-e^{-t})}\left(\frac{2e^{-t}}{(1-e^{-t})}+e^{-t}\right)\right]$$

$$\chi_{\Delta=2}(t)=\frac{e^{-2t}}{(1-e^{-t})}\ \text{and}\ \chi_{\Delta=1}(t)=\frac{e^{-t}}{(1-e^{-t})}$$

$$\log\mathcal{Z}_{\mathrm{tL,bos,1\text{-}loop}}^{\mathcal{N}=2}=2\times\frac{2}{\varepsilon^2}-\frac{8}{3}\log\varepsilon+\aleph$$

$$\psi(\Omega) = \frac{1}{\sqrt{r}}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{\pm}\left(\alpha_{\pm lm}\psi^{(+)}_{\pm lm}(\Omega) + \beta_{\pm lm}\psi^{(-)}_{\pm lm}(\Omega)\right)$$

$$\tilde{\psi}(\Omega) = -\frac{\mathrm{i}}{\sqrt{r}}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{\pm}\left(\beta^{*}_{\mp lm}\psi^{(+)}_{\pm lm}(\Omega) + \alpha^{*}_{\mp lm}\psi^{(-)}_{\pm lm}(\Omega)\right)$$

$$[\mathcal{D}\psi] = \prod_{\substack{l\geq 0\\ 0\leq m\leq l}}\left(\frac{4\pi}{\Lambda_{\mathrm{uv}}r^2}\right)\prod_{\pm}\mathrm{d}\alpha_{\pm lm}\,\mathrm{d}\beta_{\pm lm}, [\mathcal{D}\tilde{\psi}] = \prod_{\substack{l\geq 0\\ 0\leq m\leq l}}\prod_{\pm}\left(\frac{4\pi}{\Lambda_{\mathrm{uv}}r^2}\right)\mathrm{d}\alpha^{*}_{\pm lm}\,\mathrm{d}\beta^{*}_{\pm lm}$$

$$\boldsymbol{x}^{T}_{lm} \equiv (\alpha_{+lm} \quad \alpha^{*}_{+lm} \quad \alpha_{-lm} \quad \alpha^{*}_{-lm} \quad \beta_{+lm} \quad \beta^{*}_{+lm} \quad \beta_{-lm} \quad \beta^{*}_{-lm}).$$

$$S^{(2)}_{\mathrm{pert}}[\delta\psi, \delta\tilde{\psi}] \equiv \frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\left(\mathrm{i}\delta\overline{\tilde{\psi}}\not{\mathrm{D}}\delta\psi + \frac{1}{r}\frac{\mu}{2|\mu|}\delta\bar{\psi}\delta\psi + \frac{1}{r}\frac{\mu^{*}}{2|\mu|}\delta\overline{\tilde{\psi}}\delta\tilde{\psi}\right) = \frac{1}{4\pi}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\boldsymbol{x}^{T}_{lm}B_{lm}\boldsymbol{x}_{lm}$$

B_{lm}

$$\equiv\begin{pmatrix}
0 & 0 & 0 & \frac{1}{2}(l+1) & -\mathrm{i}\frac{\mu}{2|\mu|} & 0 & 0 & 0\\
0 & 0 & \frac{1}{2}(l+1) & 0 & 0 & -\mathrm{i}\frac{\mu^{*}}{2|\mu|} & 0 & 0\\
0 & -\frac{1}{2}(l+1) & 0 & 0 & 0 & 0 & -\mathrm{i}\frac{\mu}{2|\mu|} & 0\\
-\frac{1}{2}(l+1) & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\frac{\mu^{*}}{2|\mu|}\\
\mathrm{i}\frac{\mu}{2|\mu|} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}(l+1)\\
0 & \mathrm{i}\frac{\mu^{*}}{2|\mu|} & 0 & 0 & 0 & 0 & -\frac{1}{2}(l+1) & 0\\
0 & 0 & \mathrm{i}\frac{\mu}{2|\mu|} & 0 & 0 & \frac{1}{2}(l+1) & 0 & 0\\
0 & 0 & 0 & \mathrm{i}\frac{\mu^{*}}{2|\mu|} & \frac{1}{2}(l+1) & 0 & 0 & 0
\end{pmatrix}$$

$$\mathcal{Z}^{\mathcal{N}=2}_{\mathrm{fer},1\text{-loop}} = \left(\frac{4\pi}{\Lambda_{\mathrm{uv}}r^2}\right)^{2}\prod_{l=1}^{\infty}\left(\frac{l(l+2)}{4\pi\Lambda_{\mathrm{uv}}r^2}\right)^{2(l+1)}$$

$$\sum_{l=1}^{\infty}2(l+1)\log\frac{l(l+2)}{4\pi\Lambda_{\mathrm{uv}}r^2} = -\int_0^{\infty}\frac{\mathrm{d}t}{2t}\left[\frac{2e^{-\frac{t}{2}}}{(1-e^{-t})}\left(4\chi_{\Delta=3/2}(t) + 4e^{-\frac{t}{2}} - 2e^{-\frac{3t}{2}} + 2e^{-\frac{5t}{2}}\right)\right]$$

$$\log\mathcal{Z}^{\mathcal{N}=2}_{\mathrm{fer,\ 1\text{-}loop}} = -\frac{4}{\varepsilon^2} + \frac{5}{3}\log\varepsilon + \lambda$$

$$\mathcal{Z}^{\mathcal{N}=2}_{\mathrm{tL}}[\mu] \approx \pm\mathrm{i}\times\mathrm{const}\times\left(\frac{\mu\mu^{*}}{\beta^2}\right)^{-\frac{1}{\beta^2}}\Lambda_{\mathrm{uv}}^{\frac{1}{2}}r^{c^{\mathcal{N}=2}_{\mathrm{tL}}/3}\times e^{-\frac{1}{\beta^2}-\frac{2}{\beta^2}\log\beta^2}\times\left(1+\mathcal{O}(\beta^2)\right)$$

$$S_{\mathrm{L,\,int}}^{\mathcal{N}=2}=\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\Big[-\frac{4\mathrm{i}}{3r^2}\beta\delta\varphi_1^3+\frac{2}{3r^2}\beta^2\delta\varphi_1^4+\frac{\beta}{2r}\frac{\mu}{|\mu|}\delta\bar{\psi}\delta\psi(\mathrm{i}\delta\varphi_1-\delta\varphi_2)$$
$$+\frac{\beta}{2r}\frac{\mu^*}{|\mu|}\delta\bar{\tilde{\psi}}\delta\tilde{\psi}(\mathrm{i}\delta\varphi_1+\delta\varphi_2)-\frac{1}{4r}\beta^2\frac{\mu}{|\mu|}\delta\bar{\psi}\delta\psi(\delta\varphi_1^2-\delta\varphi_2^2)-\frac{1}{4r}\beta^2\frac{\mu^*}{|\mu|}\delta\bar{\tilde{\psi}}\delta\tilde{\psi}(\delta\varphi_1^2-\delta\varphi_2^2)\Big]$$

$$G(\Omega;\Omega')=2\pi\sum_{l\neq1}\sum_{-l\leq m\leq l}\frac{Y_{lm}(\Omega)Y_{lm}(\Omega')}{l(l+1)-2}$$

$$G(\Omega_0,\Omega_0)=\frac{1}{4\pi}\int_{S^2}\mathrm{d}\Omega G(\Omega;\Omega)=\frac{1}{2}\sum_{l\neq1}\frac{2l+1}{l(l+1)-2}$$

$$\tilde{G}(\Omega;\Omega')=2\pi\sum_{l=1}^{\infty}\sum_{-l\leq m\leq l}\frac{Y_{lm}(\Omega)Y_{lm}(\Omega')}{l(l+1)}$$

$$H(\Omega;\Omega')\equiv\frac{1}{\mathcal{Z}_{\mathrm{fer},1-\mathrm{loop}}^{\mathcal{N}=2}}\left(\frac{4\pi}{\Lambda_{\mathrm{uv}}r^2}\right)^2\int[\mathcal{D}'\delta\psi]\left[\mathcal{D}'\delta\tilde{\psi}\right]e^{-S_{\mathrm{pert}}^{(2)}[\delta\psi,\delta\tilde{\psi}]}\delta\bar{\psi}(\Omega)\delta\psi(\Omega')$$
$$=\frac{4\pi\mathrm{i}}{r}\frac{\mu^*}{|\mu|}\sum_{l\geq1}\sum_{0\leq m\leq l}\frac{1}{l(l+2)}\Big(\bar{\psi}_{+lm}^{(+)}(\Omega)\psi_{+lm}^{(-)}(\Omega')+\bar{\psi}_{-lm}^{(+)}(\Omega)\psi_{-lm}^{(-)}(\Omega')$$
$$-\bar{\psi}_{+lm}^{(-)}(\Omega)\psi_{+lm}^{(+)}(\Omega')-\bar{\psi}_{-lm}^{(-)}(\Omega)\psi_{-lm}^{(+)}(\Omega')\Big)$$

$$\tilde{H}(\Omega;\Omega')\equiv\frac{1}{\mathcal{Z}_{\mathrm{fer\,},1\text{-loop}}^{\mathcal{N}=2}}\left(\frac{4\pi}{\Lambda_{\mathrm{uv}}r^2}\right)^2\int[\mathcal{D}'\delta\psi]\left[\mathcal{D}'\delta\tilde{\psi}\right]e^{-S_{\mathrm{pert}}^{(2)}[\delta\psi,\delta\tilde{\psi}]}\delta\bar{\tilde{\psi}}(\Omega)\delta\tilde{\psi}(\Omega')$$
$$=\frac{4\pi\mathrm{i}}{r}\frac{\mu}{|\mu|}\sum_{l\geq1}\sum_{0\leq m\leq l}\frac{1}{l(l+2)}\Big(\bar{\psi}_{+lm}^{(+)}(\Omega)\psi_{+lm}^{(-)}(\Omega')+\bar{\psi}_{-lm}^{(+)}(\Omega)\psi_{-lm}^{(-)}(\Omega')$$
$$-\bar{\psi}_{+lm}^{(-)}(\Omega)\psi_{+lm}^{(+)}(\Omega')-\bar{\psi}_{-lm}^{(-)}(\Omega)\psi_{-lm}^{(+)}(\Omega')\Big)$$

$$H(\Omega_0,\Omega_0)=\frac{1}{4\pi}\int_{S^2}\mathrm{d}\Omega H(\Omega,\Omega)=\frac{4}{r}\frac{\mu^*}{|\mu|}\sum_{l=1}^{\infty}\frac{l+1}{l(l+2)}$$
$$\tilde{H}(\Omega_0,\Omega_0)=\frac{1}{4\pi}\int_{S^2}\mathrm{d}\Omega\tilde{H}(\Omega,\Omega)=\frac{4}{r}\frac{\mu}{|\mu|}\sum_{l=1}^{\infty}\frac{l+1}{l(l+2)}$$

$$-\frac{1}{3\pi^2}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega'G(\Omega,\Omega')^3-\frac{1}{2\pi^2}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega'G(\Omega,\Omega)G(\Omega',\Omega')G(\Omega,\Omega')-\frac{1}{2\pi}\beta^2\int_{S^2}\mathrm{d}\Omega G(\Omega,\Omega)^2$$

$$-\frac{r}{16\pi}\frac{\mu}{|\mu|}\beta^2\int_{S^2}\mathrm{d}\Omega H(\Omega,\Omega)G(\Omega,\Omega)-\frac{r}{16\pi}\frac{\mu^*}{|\mu|}\beta^2\int_{S^2}\mathrm{d}\Omega\tilde{H}(\Omega,\Omega)G(\Omega,\Omega)$$
$$+\frac{r}{16\pi}\frac{\mu}{|\mu|}\beta^2\int_{S^2}\mathrm{d}\Omega H(\Omega,\Omega)\tilde{G}(\Omega,\Omega)+\frac{r}{16\pi}\frac{\mu^*}{|\mu|}\beta^2\int_{S^2}\mathrm{d}\Omega\tilde{H}(\Omega,\Omega)\tilde{G}(\Omega,\Omega)$$

$$+\frac{r}{8\pi^2}\frac{\mu}{|\mu|}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega' H(\Omega,\Omega)G(\Omega,\Omega')G(\Omega',\Omega')+\frac{r}{8\pi^2}\frac{\mu^*}{|\mu|}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega'\tilde{H}(\Omega,\Omega)G(\Omega,\Omega')G(\Omega',\Omega')$$
$$-\frac{r^2}{128\pi^2}\frac{\mu}{\mu^*}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega' H(\Omega,\Omega)G(\Omega,\Omega')F(\Omega',\Omega')-\frac{r^2}{128\pi^2}\frac{\mu^*}{\mu}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega'\tilde{H}(\Omega,\Omega)G(\Omega,\Omega')\tilde{H}(\Omega',\Omega')$$
$$-\frac{r^2}{64\pi^2}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega' H(\Omega,\Omega')\tilde{H}(\Omega',\Omega)G(\Omega,\Omega')$$

$$-\frac{r^2}{128\pi^2}\frac{\mu}{\mu^*}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega' H(\Omega,\Omega')^2G(\Omega,\Omega')-\frac{r^2}{128\pi^2}\frac{\mu^*}{\mu}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega'\tilde{H}(\Omega,\Omega')^2G(\Omega,\Omega')$$
$$+\frac{r^2}{128\pi^2}\frac{\mu}{\mu^*}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega' H(\Omega,\Omega')^2\tilde{G}(\Omega,\Omega')+\frac{r^2}{128\pi^2}\frac{\mu^*}{\mu}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega'\tilde{H}(\Omega,\Omega')^2\tilde{G}(\Omega,\Omega')$$

$$-\frac{r^2}{64\pi^2}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega' F(\Omega,\Omega')\tilde{H}(\Omega',\Omega)G(\Omega,\Omega')-\frac{r^2}{64\pi^2}\beta^2\int_{S^2}\mathrm{d}\Omega\,\mathrm{d}\Omega' H(\Omega,\Omega')\tilde{H}(\Omega',\Omega)\tilde{G}(\Omega,\Omega').$$

$$\mathcal{Z}_{\mathrm{tL}}^{\mathcal{N}=2}[\mu]\approx\pm\mathrm{i}\times\mathrm{const}\times\left(\frac{\mu\mu^*}{\beta^2}\right)^{-\frac{1}{\beta^2}}\Lambda_{\mathrm{uv}}^{\frac{1}{2}}r^{c_{\mathrm{tL}}^{\mathcal{N}=2}/3}\times e^{-\frac{1}{\beta^2}-\frac{2}{\beta^2}\log\beta^2}\times(1+\mathrm{loops}\beta^2)$$

$$\log\mathcal{Z}_{\mathrm{grav},(0)}^{\mathcal{N}=2}=2\vartheta-\left(\frac{c_m}{6}-\frac{3}{6}\right)\log\frac{|\mu|^2}{\Lambda_{\mathrm{uv}}}+f_0(c_m)$$

$$D_\mu\epsilon=\frac{\mathrm{i}}{2r}\gamma_\mu\epsilon,D_\mu\tilde{\epsilon}=\frac{\mathrm{i}}{2r}\gamma_\mu\tilde{\epsilon}$$

$$\begin{aligned}&\delta\varphi=\bar{\tilde{\epsilon}}\psi, && \delta\tilde{\varphi}=\bar{\epsilon}\tilde{\psi},\\ &\delta\psi=\mathrm{i}\,\partial\!\!\!/\varphi\epsilon-\frac{1}{\beta r}\epsilon-\mathrm{i}F\tilde{\epsilon}, && \delta\tilde{\psi}=\mathrm{i}\,\partial\!\!\!/\tilde{\varphi}\tilde{\epsilon}-\frac{1}{\beta r}\tilde{\epsilon}-\mathrm{i}\tilde{F}\epsilon,\\ &\delta F=-\bar{\epsilon}\partial\!\!\!/\psi, && \delta\tilde{F}=\tilde{\epsilon}\partial\!\!\!/\tilde{\psi}.\end{aligned}$$

$$\begin{aligned}[\delta_\epsilon,\delta_{\tilde{\epsilon}}]\varphi&=\mathcal{L}_\xi\varphi+\frac{2\mathrm{i}}{\beta}\rho\\ [\delta_\epsilon,\delta_{\tilde{\epsilon}}]\psi&=\mathcal{L}_\xi\psi-\mathrm{i}\rho\psi\\ [\delta_\epsilon,\delta_{\tilde{\epsilon}}]F&=\mathcal{L}_\xi F-2\mathrm{i}\rho F\end{aligned}$$

$$\int_{\mathcal{E}}[\mathcal{D}\Phi]e^{-S}\mathcal{O}=\mathrm{vol}(F)\int_{\mathcal{E}/F}[\widetilde{\mathcal{D}\Phi}]e^{-S}\mathcal{O}$$

$$Z(t)=\int_{\mathcal{E}}[\mathcal{D}\Phi]e^{-S-tQV}$$

$$Z'(t)=-\int_{\mathcal{E}}[\mathcal{D}\Phi]\mathcal{Q}\left(Ve^{-S-tQV}\right)$$

$$Q\equiv\epsilon_+^\alpha Q_\alpha,Q^\dagger\equiv(\epsilon_+^c)^\alpha\tilde{Q}_\alpha$$

$$\varphi^\dagger\equiv\tilde{\varphi},\psi^\dagger\equiv-\bar{\tilde{\psi}},F^\dagger=\tilde{F}$$

$$
\begin{aligned}
&Q\varphi = 0, && Q^\dagger\varphi = -\epsilon_+^\dagger\psi \\
&Q\psi = \left(\mathrm{i}\,\partial\varphi - \frac{q}{r}\right)\epsilon_+, && Q^\dagger\psi = -\mathrm{i}F\epsilon_+^c \\
&QF = -\overline{\epsilon_+}\not{\partial}\psi, && Q^\dagger F = 0 \\
&Q\varphi^\dagger = \psi^\dagger\epsilon_+, && Q^\dagger\varphi^\dagger = 0 \\
&Q\psi^\dagger = \mathrm{i}F^\dagger\overline{\epsilon_+}, && Q^\dagger\psi^\dagger = \epsilon_+^\dagger\left(-\mathrm{i}\,\partial\varphi^\dagger - \frac{q}{r}\right) \\
&QF^\dagger = 0, && Q^\dagger F^\dagger = D_\mu\psi^\dagger\gamma^\mu\epsilon_+^c
\end{aligned}
$$

$$\{Q, Q^\dagger\} = J + \frac{1}{2}R$$

$$
\begin{aligned}
\mathcal{L}_{\text{kin}} &= -\partial_\mu\varphi^\dagger\partial^\mu\varphi - \mathrm{i}\psi^\dagger\not{\partial}\psi + F^\dagger F - \frac{q}{r^2}\left(\varphi + \varphi^\dagger\right) - \frac{1}{\beta^2 r^2} \\
&= \frac{1}{s}\mathcal{Q}\left(-\mathrm{i}\epsilon_+^\dagger\,\partial\varphi^\dagger\psi - \frac{q}{r}\epsilon_+^\dagger\psi - \mathrm{i}F\psi^\dagger\epsilon_+^c - \frac{1}{r}\left(\varphi^\dagger - \varphi\right)\epsilon_+^\dagger\psi\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_W &\equiv FW'(\varphi) - \frac{\mathrm{i}}{2}W''(\varphi)\bar{\psi}\psi = \frac{\mathrm{i}}{s}\mathcal{Q}(W'(\varphi)\bar{\epsilon}_+\psi) \\
\mathcal{L}_{W^\dagger} &\equiv F^\dagger W'^\dagger\left(\varphi^\dagger\right) - \frac{\mathrm{i}}{2}W''^\dagger\left(\varphi^\dagger\right)\overline{\tilde{\psi}}\tilde{\psi} = -\frac{\mathrm{i}}{s}\mathcal{Q}\left(W'^\dagger\left(\varphi^\dagger\right)\psi^\dagger\epsilon_+^c\right)
\end{aligned}
$$

$$\delta F_W = -\nabla_\mu(\bar{\epsilon}\gamma^\mu\psi_W)$$

$$V_{\text{can}} = (\mathcal{Q}\psi)^\dagger\psi + \overline{\tilde{\psi}}(\mathcal{Q}\tilde{\psi})^\dagger$$

$$\mathcal{Q}V_{\text{can}}\,|_{\text{bos}} = 2\left[\partial_\mu\varphi^\dagger\partial^\mu\varphi + \frac{1}{\beta^2 r^2} + \mathrm{i}\frac{1}{r}v^\mu\varphi^\dagger\partial_\mu\varphi\right] - 2F^\dagger F - 2\mathrm{i}\left(\mathrm{i}Fw^\mu\partial_\mu\varphi^\dagger + \text{ c.c. }\right),$$

$$\epsilon_+ = e^{\mathrm{i}\frac{\phi}{2}}\begin{pmatrix}\cos\frac{\theta}{2} \\ \mathrm{i}\sin\frac{\vartheta}{2}\end{pmatrix}$$

$$
\begin{aligned}
0 &= \sin\frac{\theta}{2}\left(2\partial_-\varphi - \mathrm{i}Fe^{-\mathrm{i}\phi}\right) + \cos\frac{\theta}{2}\frac{1}{\beta r} \\
0 &= \cos\frac{\theta}{2}\left(2\partial_+\varphi - \mathrm{i}Fe^{-\mathrm{i}\phi}\right) - \sin\frac{\theta}{2}\frac{1}{\beta r} \\
0 &= \cos\frac{\theta}{2}\left(2\partial_-\varphi^\dagger + \mathrm{i}F^\dagger e^{\mathrm{i}\phi}\right) - \sin\frac{\theta}{2}\frac{1}{\beta r} \\
0 &= \sin\frac{\theta}{2}\left(2\partial_+\varphi^\dagger + \mathrm{i}F^\dagger e^{\mathrm{i}\phi}\right) + \cos\frac{\theta}{2}\frac{1}{\beta r}
\end{aligned}
$$

$$0 = F = \tilde{F}, 0 = \nabla^2\varphi - \frac{1}{\beta r^2} = \nabla^2\tilde{\varphi} - \frac{1}{\beta r^2}, 0 = \not{\partial}\psi = \not{\partial}\tilde{\psi}$$

$$\mathcal{L}_{\text{fermi}} = \mathrm{i}\tilde{\psi}_-(\partial_0 + \partial_1)\psi_- + \mathrm{i}\tilde{\psi}_+(\partial_0 - \partial_1)\psi_+ - \mathrm{i}\left(W''(\varphi)\psi_+\psi_- + \widetilde{W}''(\tilde{\varphi})\tilde{\psi}_-\tilde{\psi}_+\right)$$

$$\mathcal{C} = -\mathcal{C}^T, \gamma_a^T = -\mathcal{C}\gamma_a\mathcal{C}^{-1}$$

$$\bar{\lambda} = \lambda^T\mathcal{C},$$

$$P_L = \frac{1}{2}(\mathbb{I} + \gamma_*), P_R = \frac{1}{2}(\mathbb{I} - \gamma_*)$$

$$P_{L,R}\lambda_{L,R} = \lambda_{L,R}$$

$$\lambda^C \equiv \mathcal{C}^{-1}\lambda^*$$

$$\lambda = \gamma_*\lambda^C$$

$$(\bar{\chi}M\lambda)^C = -\overline{\chi^C}M^C\lambda^C$$

$$(\overline{\chi_m}\lambda_m)^C = \overline{\chi_m}\lambda_m, (\overline{\chi_m}\gamma_a\lambda_m)^C = \overline{\chi_m}\gamma_a\lambda_m, (\overline{\chi_m}\,\partial\chi_m)^C = \overline{\chi_m}\,\partial\chi_m$$

$$\bar{\lambda}\eta = \lambda^\alpha \mathcal{C}_{\alpha\beta}\eta^\beta$$

$$\bar{\lambda}\eta = \bar{\eta}\lambda, \bar{\lambda}\gamma_a\eta = -\bar{\eta}\gamma_a\lambda, \bar{\lambda}\gamma_{ab}\eta = -\bar{\eta}\gamma_{ab}\lambda$$

$$\delta^\alpha{}_\delta\delta^\gamma{}_\beta = \frac{1}{2}\delta^\alpha{}_\beta\delta^\gamma{}_\delta + \frac{1}{2}(\gamma^a)^\alpha{}_\beta(\gamma_a)^\gamma{}_\delta + \frac{1}{2}(\gamma_*)^\alpha{}_\beta(\gamma_*)^\gamma{}_\delta.$$

$$\gamma_a = \sigma_a, \gamma_* = \sigma_3$$

$$\lambda^1 = -i\lambda^{2*} \text{ and } \lambda^2 = -i\lambda^{1*}.$$

$$D_\mu\epsilon = \frac{1}{d}\gamma_\mu \not{D}_\epsilon$$

$$D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon$$

$$g_{\mu\nu}(x) \to \hat{g}_{\mu\nu}(x) = e^{2\sigma(x)}g_{\mu\nu}(x), \epsilon(x) \to \hat{\epsilon}(x)e^{\frac{1}{2}\sigma(x)}$$

$$D_\mu\epsilon = \kappa\gamma_\mu\epsilon$$

$$\kappa^2 = -\frac{R}{4d(d-1)}$$

$$\mathrm{d}s^2 = r^2(\,\mathrm{d}\theta^2 + \sin^2\,\theta\,\mathrm{d}\phi^2)$$

$$D_\mu\epsilon_\pm = \pm\frac{\mathrm{i}}{2r}\gamma_\mu\epsilon_\pm$$

$$\mathrm{e}^1 = r\,\mathrm{d}\theta, \mathrm{e}^2 = r\sin\,\theta\,\mathrm{d}\phi, \omega^{12} = -\cos\,\theta\mathrm{d}\phi$$

$$\epsilon_+ = C_1\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}\begin{pmatrix}\sin\frac{\theta}{2}\\ -\mathrm{i}\cos\frac{\theta}{2}\end{pmatrix} + C_2\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}\begin{pmatrix}\cos\frac{\theta}{2}\\ \mathrm{i}\sin\frac{\theta}{2}\end{pmatrix}$$

$$D_\mu = \partial_\mu + \mathrm{i}\hat{s}\cos\,\theta(\,\mathrm{d}\phi)_\mu$$

$$D^2 = \frac{1}{r^2}\left[\frac{1}{\sin\,\theta}\partial_\theta(\sin\,\theta\partial_\theta) + \frac{1}{\sin^2\,\theta}\left(\partial_\phi + \mathrm{i}\hat{s}\cos\,\theta\right)^2\right]$$

$$\mathbf{L} = \mathbf{r} \wedge \mathbf{p} - \hat{s}\frac{\mathbf{r}}{r},$$

$$L_{\pm} = \mathrm{e}^{\pm \mathrm{i}\phi}\left(\pm\partial_{\theta} + \mathrm{i}\cot\,\theta\partial_{\phi} - \hat{s}\frac{1}{\sin\,\theta}\right), L_z = -\mathrm{i}\partial_{\phi}$$

$$[L_z, L_{\pm}] = \pm L_{\pm}, [L_+, L_-] = 2L_z$$

$$L^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_z^2 = -r^2D^2 + \hat{s}^2$$

$$L^2\mathcal{Y}_{lm}(\Omega) = l(l+1)\mathcal{Y}_{lm}(\Omega), L_z\mathcal{Y}_{lm}(\Omega) = m\mathcal{Y}_{lm}(\Omega)$$

$$\mathcal{Y}_{lm}(\Omega) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\,\theta)\mathrm{e}^{\mathrm{i}m\phi},$$

$$\int_{S^2} \mathrm{d}\Omega(\mathcal{Y}_{lm}(\Omega))^*\mathcal{Y}_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$$

$$Y_{lm}(\Omega) \equiv \begin{cases} \frac{i}{\sqrt{2}}\left(\mathcal{Y}_{lm}(\Omega) - (-1)^m\mathcal{Y}_{l,-m}(\Omega)\right), & \text{if } m < 0 \\ \mathcal{Y}_{l0}(\Omega) \\ \frac{1}{\sqrt{2}}\left(\mathcal{Y}_{l,-m}(\Omega) + (-1)^m\mathcal{Y}_{lm}(\Omega)\right), & \text{if } m > 0 \end{cases}$$

$$\int_{S^2} \mathrm{d}\Omega Y_{lm}(\Omega)Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$$

$$\not{D}^2 = D^2 - \frac{R}{4} = D^2 - \frac{1}{2r^2} = -\frac{1}{r^2}L^2 - \frac{1}{4r^2}$$

$$L^2\psi_{\pm lm}^{(s)}(\Omega) = \left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right)\psi_{\pm lm}^{(s)}(\Omega), L_z\psi_{\pm lm}^{(s)}(\Omega) = s\left(m+\frac{1}{2}\right)\psi_{\pm lm}^{(s)}(\Omega)$$

$$\not{D}\psi_{\pm lm}^{(s)}(\Omega) = \pm\frac{\mathrm{i}}{r}(l+1)\psi_{\pm,lm}^{(s)}(\Omega), s = \pm$$

$$\psi_{\pm lm}^{(+)}(\Omega) = \frac{c_{lm}}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\left(m+\frac{1}{2}\right)\phi}\begin{pmatrix}\Phi_{lm}(\theta) \\ \pm\mathrm{i}\Psi_{lm}(\theta)\end{pmatrix}, \psi_{\pm lm}^{(-)}(\Omega) = \frac{c_{lm}}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\left(m+\frac{1}{2}\right)\phi}\begin{pmatrix}\pm\mathrm{i}\Psi_{lm}(\theta) \\ \Phi_{lm}(\theta)\end{pmatrix}$$

$$\Phi_{lm}(\theta) = \left(\cos\frac{\theta}{2}\right)^{m+1}\left(\sin\frac{\theta}{2}\right)^m P_{l-m}^{(m,m+1)}(\cos\,\theta)$$

$$\Psi_{lm}(\theta) = \left(\cos\frac{\theta}{2}\right)^m\left(\sin\frac{\theta}{2}\right)^{m+1} P_{l-m}^{(m+1,m)}(\cos\,\theta) = (-1)^{l-m}\Phi_{lm}(\pi-\theta)$$

$$c_{lm}^2 = \frac{(l+m+1)!\,(l-m)!}{2\pi(l!)^2}$$

$$\int_{S^2} \mathrm{d}\Omega\bar{\psi}_{\pm lm}^{(+)}(\Omega)\psi_{\pm l'm'}^{(-)}(\Omega) = -\mathrm{i}\delta_{l,l'}\delta_{m,m'}, \int_{S^2} \mathrm{d}\Omega\bar{\psi}_{\pm lm}^{(-)}(\Omega)\psi_{\pm l'm'}^{(+)}(\Omega) = +\mathrm{i}\delta_{l,l'}\delta_{m,m'}$$

$$\eta=\frac{1}{\sqrt{r}}\sum_{l\geq 0}\sum_{0\leq m\leq l}\left(\alpha_{+lm}\psi^{(+)}_{+lm}+\alpha_{-lm}\psi^{(+)}_{-lm}+\beta_{+lm}\psi^{(-)}_{+lm}+\beta_{-lm}\psi^{(-)}_{-lm}\right)$$
$$\tilde{\eta}=\frac{1}{\sqrt{r}}\sum_{l\geq 0}\sum_{0\leq m\leq l}\left(\tilde{\alpha}_{+lm}\psi^{(+)}_{+lm}+\tilde{\alpha}_{-lm}\psi^{(+)}_{-lm}+\tilde{\beta}_{+lm}\psi^{(-)}_{+lm}+\tilde{\beta}_{-lm}\psi^{(-)}_{-lm}\right)$$

$$\int_{S^2}\mathrm{d}\Omega\mathrm{i}\bar{\tilde{\eta}}\not{D}\eta=\frac{1}{r}\sum_{l\geq 0}\sum_{0\leq m\leq l}(l+1)\left[\tilde{\alpha}_{+lm}\beta_{+lm}-\tilde{\alpha}_{-lm}\beta_{-lm}-\tilde{\beta}_{+lm}\alpha_{+lm}+\tilde{\beta}_{-lm}\alpha_{-lm}\right]$$
$$\int_{S^2}\mathrm{d}\Omega\bar{\tilde{\eta}}\eta=-\frac{\mathrm{i}}{r}\sum_{l\geq 0}\sum_{0\leq m\leq l}\left[\tilde{\alpha}_{+lm}\beta_{+lm}+\tilde{\alpha}_{-lm}\beta_{-lm}-\tilde{\beta}_{+lm}\alpha_{+lm}-\tilde{\beta}_{-lm}\alpha_{-lm}\right]$$

$$\tilde{\alpha}_{\pm lm}=-\mathrm{i}\beta^{*}_{\mp lm},\tilde{\beta}_{\pm lm}=-\mathrm{i}\alpha^{*}_{\mp lm}$$

$$\int_{S^2}\mathrm{d}\Omega\mathrm{i}\bar{\tilde{\eta}}\not{D}\eta=\frac{1}{r}\sum_{l\geq 0}\sum_{0\leq m\leq l}(l+1)[-\alpha^{*}_{-lm}\alpha_{+lm}-\alpha_{-lm}\alpha^{*}_{+lm}+\beta^{*}_{-lm}\beta_{+lm}+\beta_{-lm}\beta^{*}_{+lm}]$$

$$\int_{S^2}\mathrm{d}\Omega\bar{\eta}\eta=-\frac{2\mathrm{i}}{r}\sum_{l\geq 0}\sum_{0\leq m\leq l}(\alpha_{+lm}\beta_{+lm}+\alpha_{-lm}\beta_{-lm})$$
$$\int_{S^2}\mathrm{d}\Omega\bar{\tilde{\eta}}\tilde{\eta}=-\frac{2\mathrm{i}}{r}\sum_{l\geq 0}\sum_{0\leq m\leq l}(\alpha^{*}_{+lm}\beta^{*}_{+lm}+\alpha^{*}_{-lm}\beta^{*}_{-lm})$$

$$\beta_{\pm lm}=-\mathrm{i}\alpha^{*}_{\mp lm}$$

$$\int_{S^2}\mathrm{d}\Omega\mathrm{i}\bar{\eta}\not{D}\eta=-\frac{2}{r}\sum_{l\geq 0}\sum_{0\leq m\leq l}(l+1)(\alpha^{*}_{-lm}\alpha_{+lm}+\alpha_{-lm}\alpha^{*}_{+lm})$$
$$\int_{S^2}\mathrm{d}\Omega\bar{\eta}\eta=\frac{2}{r}\sum_{l\geq 0}\sum_{0\leq m\leq l}(\alpha^{*}_{-lm}\alpha_{+lm}-\alpha_{-lm}\alpha^{*}_{+lm})$$

$$\mathcal{S}_{\mathrm{L}}^{\mathcal{N}=1}=\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\left(\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-\frac{\mathrm{i}}{2}\bar{\psi}\not{D}\psi-\frac{1}{2}F^2+\frac{1}{2}Q\tilde{R}\varphi+\mu e^{b\varphi}F-\frac{\mathrm{i}}{2}\mu be^{b\varphi}\bar{\psi}\psi\right),$$

$$\mathcal{Z}_{\mathrm{L}}^{\mathcal{N}=1}[\mu]=\int\frac{[\mathcal{D}\varphi][\mathcal{D}\psi]}{\mathrm{vol}_{OSp(1|2;\mathbb{C})}}e^{-\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\left(\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-\frac{\mathrm{i}}{2}\bar{\psi}\not{D}\psi+\frac{1}{2}Q\tilde{R}\varphi+\frac{1}{2}\mu^2e^{2b\varphi}-\frac{\mathrm{i}}{2}\mu be^{b\varphi}\bar{\psi}\psi\right)}$$

$$\varphi^{(n)}_{*}=\frac{1}{2b}\log\left(\frac{Q}{b\mu^2r^2}\right)+\frac{\pi\mathrm{i}}{2b}(2n+1),n\in\mathbb{Z}$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{L,\,saddle}}[\mu]\approx-\mathrm{i}\times\left(\frac{Q}{b\mu^2r^2}\right)^{-\frac{Q}{2b}}\times e^{-\frac{Q}{2b}}e^{-\frac{\mathrm{i}\pi}{2b^2}}\sum_{n\geq 0}(-1)^ne^{-\frac{\mathrm{i}\pi}{b^2}n}$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{L,\,bos.\,fluct.}}=\int[\mathcal{D}\delta\varphi]e^{-\frac{1}{8\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\delta\varphi\left(-\nabla^2-\frac{2Qb}{r^2}\right)\delta\varphi}$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{L,\,fer.\,fluct.}}=\int[\mathcal{D}\delta\psi]e^{-\frac{\mathrm{i}}{8\pi}\int_{S^2}\mathrm{d}^2x\,\tilde{\mathrm{e}}\,\delta\bar{\psi}\left(-\not{D}-\mathrm{i}(-1)^n\sqrt{\frac{Qb}{r}}\right)\delta\psi}$$

$$\mathcal{Z}_{\mathrm{L}}^{\mathcal{N}=1} \approx \text{ const } \times \frac{1}{\cos\dfrac{\pi}{2b^2}} \times (\mu b)^{\frac{1}{b^2}+1} \Lambda_{\mathrm{uv}}^{\frac{5}{4}+b^2} r^{c_{\mathrm{L}}^{\mathcal{N}=1}/3} e^{-\frac{1}{2b^2}+\frac{1}{b^2}\log\, b^2} \times b^{-1} \times \left(1+\mathcal{O}(b^2)\right)$$

$$\Delta_{\mathrm{NS}} = \frac{\alpha(Q-\alpha)}{2} = \frac{c_{\mathrm{L}}^{\mathcal{N}=1}}{24} - \frac{3}{48} + \frac{P^2}{2}, \Delta_{\mathrm{R}} = \Delta_{\mathrm{NS}} + \frac{1}{16},$$

$$\mathcal{W}_\alpha = \alpha^2 \bar{\psi}\psi e^{\alpha\varphi} - 2\pi \mathrm{i}\mu\alpha e^{(\alpha+b)\varphi}$$

$$C(b,b,b) \overset{b\to 0}{\approx} \mathrm{i}(\pi\mu b)^{\frac{1}{b^2}-2} e^{\frac{1}{b^2}\left(1+\log\, 2+\log\, b^2\right)} e^{-\frac{\mathrm{i}\pi}{2b^2}} \sum_{n=0}^{\infty} (-1)^n e^{-\frac{\mathrm{i}\pi}{b^2}n} b^{-\frac{9}{2}} \times \left(1 - \frac{11}{6}b^2 + \cdots\right)$$

$$S_{\mathrm{L}}^{\mathcal{N}=2} = \frac{1}{4\pi}\int_{S^2} \mathrm{d}^2x\tilde{\mathrm{e}}\left(\partial_\mu\tilde{\varphi}\partial^\mu\varphi - \mathrm{i}\bar{\tilde{\psi}}\not{\partial}\psi - F\tilde{F} + \frac{Q\tilde{R}}{2}(\varphi+\tilde{\varphi}) + \mu e^{b\varphi}F - \frac{\mathrm{i}}{2}\mu b e^{b\varphi}\bar{\psi}\psi + \mu^* e^{b\tilde{\varphi}}\tilde{F} - \frac{\mathrm{i}}{2}\mu^* b e^{b\tilde{\varphi}}\bar{\tilde{\psi}}\tilde{\psi}\right)$$

$$0 = -\nabla^2\varphi + \frac{1}{br^2} + \mu b e^{b\varphi}F - \frac{\mathrm{i}}{2}\mu b^2\bar{\psi}\psi, \quad 0 = -\nabla^2\tilde{\varphi} + \frac{1}{br^2} + \mu^* b e^{b\tilde{\varphi}}\tilde{F} - \frac{\mathrm{i}}{2}\mu^* b^2\bar{\tilde{\psi}}\tilde{\psi}$$
$$0 = \tilde{F} - \mu e^{b\varphi}, \qquad 0 = F - \mu^* e^{b\tilde{\varphi}}$$

$$(\varphi+\tilde{\varphi})_*^{(n)} = \frac{1}{b}\log\left(\frac{1}{b^2|\mu|^2r^2}\right) + \frac{\pi\mathrm{i}}{b}(2n+1), n \in \mathbb{Z}$$

$$\mathcal{Z}_{\mathrm{L,\,saddle}}^{\mathcal{N}=2}[\mu] = (|\mu^2|b^2r^2)^{\frac{1}{b^2}} e^{\frac{1}{b^2}-\frac{\mathrm{i}\pi}{b^2}} \sum_{n\geq 0} e^{-\frac{2\mathrm{i}\pi}{b^2}n} = -\frac{\mathrm{i}}{2}(e|\mu^2|b^2r^2)^{\frac{1}{b^2}} \times \frac{1}{\sin\left(\frac{\pi}{b^2}\right)}$$

$$\mathcal{L}_\xi\psi \equiv \xi^\mu D_\mu\psi + \frac{1}{4}\nabla_\mu\xi_\nu\gamma^{\mu\nu}\psi$$

$$\left\langle \mathcal{H}\left(\alpha_{\pm lm}, \alpha_{\pm l'm'}^*\right)\right\rangle \equiv \left(\mathcal{Z}_{\mathrm{fer,1-loop}}^{\mathcal{N}=1}\right)^{-1} \int [\mathcal{D}\delta\psi] e^{-S_{\mathrm{L,fer}}^{(2)}[\delta\psi]} \mathcal{H}\left(\alpha_{\pm lm}, \alpha_{\pm l'm'}^*\right)$$

$$\int \mathrm{d}\alpha_{+lm}\, \mathrm{d}\alpha_{+lm}^*\, \mathrm{d}\alpha_{-lm}\, \mathrm{d}\alpha_{-lm}\alpha_{+lm}\alpha_{+lm}^*\alpha_{-lm}\alpha_{-lm}^* = 1$$

$$\delta F = -\nabla_\mu(\bar{\epsilon}\gamma^\mu\psi) - \frac{r-2}{2}D_\mu\bar{\epsilon}\gamma^\mu\psi$$

$$L^2\psi_{\pm lm}^{(s)} = 1(1+1)\psi_{\pm lm}^{(s)}, L_z\psi_{\pm lm}^{(s)} = \mathrm{m}\psi_{\pm lm}^{(s)}, \not{D}\psi_{\pm lm}^{(s)} = \pm\mathrm{i}\frac{1+\frac{1}{2}}{r}\psi_{\pm lm}^{(s)}$$

$$C(b,b,b) = \frac{\mathrm{i}}{2}\left(\frac{\pi\mu}{2b}\gamma\left(\frac{1}{2}+\frac{b^2}{2}\right)b^{1-b^2}\right)^{\frac{1}{b^2}-2} \frac{\Upsilon_b\left(\frac{b}{2}\right)\Upsilon_b\left(\frac{1}{2b}\right)}{\Upsilon_b(b)} \times \frac{\gamma\left(\frac{1}{2}+\frac{b^2}{2}\right)^3\gamma\left(\frac{1}{2}-\frac{1}{2b^2}\right)}{\gamma\left(\frac{b^2}{2}-\frac{1}{2}\right)} b^{-\frac{1}{b^2}-2-2b^2},$$

$$n_H - n_V + 29n_T = 273$$

$$V = \frac{2\mathfrak{g}^2}{\kappa^4} U(q) e^{\varphi}$$

$$\mathcal{L}_6 = -\sqrt{-g}\left[\frac{1}{2\kappa^2} g^{MN}(R_{MN} + \partial_M\varphi\partial_N\varphi) + \frac{1}{4} e^{-\varphi} F_{MN}F^{MN} + \frac{1}{12} e^{-2\varphi} H_{MNP}H^{MNP} + \frac{2\mathfrak{g}^2}{\kappa^4} e^{\varphi}\right]$$

$$\Box_6\,\varphi + \frac{\kappa^2}{4} e^{-\varphi} F_{MN}F^{MN} + \frac{\kappa^2}{6} e^{-2\varphi} H_{MNP}H^{MNP} - \frac{2\mathfrak{g}^2}{\kappa^2} e^{\varphi} = 0$$

$$\nabla_M(e^{-\varphi}F^{MN}) + \kappa e^{-2\varphi} H^{PNQ}F_{PQ} = 0, \nabla_M(e^{-2\varphi}H^{MNP}) = 0$$

$$R_{MN} + \partial_M\varphi\partial_N\varphi + \kappa^2 e^{-\varphi} F_{MP}F_N^P + \frac{1}{2}(\Box_6\,\varphi) g_{MN} = 0$$

$$g_{MN} \to c g_{MN} \text{ and } e^{-\varphi} \to c e^{-\varphi} \text{ imply } \mathcal{L}_6 \to c^2 \mathcal{L}_6$$

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + g_{mn}(y)\mathrm{d}y^m\,\mathrm{d}y^n$$

$$e^{-\varphi} f = \pm 2\mathfrak{g}/\kappa^2$$

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + \rho^2(\,\mathrm{d}\phi^2 + \sin^2\,\phi\,\mathrm{d}\theta^2)$$

$$\frac{1}{\rho^2} = \kappa^2 f^2 e^{-\varphi}, \rho^2 e^{\varphi} = \left(\frac{\kappa}{2\mathfrak{g}}\right)^2$$

$$V = \frac{2\mathfrak{g}^2 e^{\varphi}}{\rho^2}\left(1 - \frac{\kappa^4}{4\mathfrak{g}^2 e^{\varphi}\rho^2}\right)^2$$

$$\mathrm{d}s^2 = \hat{g}_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N = W^2 g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + a^2\,\mathrm{d}\theta^2 + a^2 W^8\,\mathrm{d}\eta^2,$$

$$\hat{R}_{\mu\nu} + \frac{1}{2}(\Box_2\,\varphi)\hat{g}_{\mu\nu} = 0,$$

$$\int_{M_2} \mathrm{d}^2 y\sqrt{g_2}\hat{R}_4(x,y) = 2\oint_{\partial M_2} \mathrm{d}y n_m\sqrt{g_2}g^{mn}\partial_n\varphi$$

$$F_{\eta\theta} = Q a^2 e^{\varphi}$$

$$\varphi'' + \frac{\kappa^2}{2} Q^2 a^2 e^{\varphi} - \frac{2\mathfrak{g}^2}{\kappa^2} a^2 W^8 e^{\varphi} = 0$$

$$(\mu\nu): \quad \left(\ln\,W + \frac{\varphi}{2}\right)'' = 3\zeta H^2 a^2 W^6$$
$$(\theta\theta): \quad \left(\ln\,a + \frac{\varphi}{2}\right)'' = -\kappa^2 Q^2 a^2 e^{\varphi}$$

$$(\eta\eta): 6\zeta H^2 a^2 W^6 - \frac{4a'W'}{aW} - \frac{6(W')^2}{W^2} + \frac{1}{2}(\varphi')^2 + \frac{\kappa^2}{2} Q^2 a^2 e^{\varphi} - \frac{2\mathfrak{g}^2}{\kappa^2} a^2 W^8 e^{\varphi} = 0$$

$$\{\varphi, a, W, H\} \to \{\varphi + \varphi_0, a e^{-\varphi_0/2}, W, H e^{\varphi_0/2}\}$$

$$3\zeta H^2\int_{\eta_1}^{\eta_2}\mathrm{d}\eta a^2W^6=\left(\ln\,W+\frac{\varphi}{2}\right)'_{\eta=\eta_2}-\left(\ln\,W+\frac{\varphi}{2}\right)'_{\eta=\eta_1}$$

$$\frac{1}{\kappa_4^2}=\frac{1}{\kappa^2}\int\,\mathrm{d}^2x\sqrt{g_2}W^2=\frac{2\pi}{\kappa^2}\int_{-\infty}^{\infty}\mathrm{d}\eta a^2W^6$$

$$\zeta H^2=\frac{8}{3}\pi G_N\left[\left(\ln\,W+\frac{\varphi}{2}\right)'\right]_{\eta=-\infty}^{\eta=+\infty}$$

$$W^2e^{\varphi}=e^{-\lambda_3\eta},W^4=\left(\frac{\kappa^2Q\lambda_2}{2\mathrm{g}\lambda_1}\right)\frac{\cosh\left[\lambda_1(\eta-\eta_1)\right]}{\cosh\left[\lambda_2(\eta-\eta_2)\right]},F_{\eta\theta}=\left(\frac{Qa^2}{W^2}\right)e^{-\lambda_3\eta}$$

$$\text{and}\;\;\frac{1}{a^4}=\left(\frac{4\mathrm{g}\kappa^4Q^3}{\lambda_1^3\lambda_2}\right)e^{-2\lambda_3\eta}\cosh^3\left[\lambda_1(\eta-\eta_1)\right]\cosh\left[\lambda_2(\eta-\eta_2)\right]$$

$$\varphi''\simeq(\ln\,W)''\simeq(\ln\,a)''\simeq0$$

$$\varphi\simeq\mp\eta q_{\pm},W\simeq W_{\pm}e^{\mp\eta\omega_{\pm}}\;\text{and}\;\;a\simeq a_{\pm}e^{\mp\eta\alpha_{\pm}}$$

$$q_b^2=4\omega_b(2\alpha_b+3\omega_b)$$

$$2\alpha_b+6\omega_b>0,2\alpha_b+q_b>0\;\text{and}\;\;2\alpha_b+8\omega_b+q_b>0$$

$$\alpha_{\pm}=\frac{1}{4}(3\lambda_1+\lambda_2\mp2\lambda_3)\geq0,\omega_{\pm}=\frac{1}{4}(\lambda_2-\lambda_1)\geq0$$

$$q_{\pm}=\pm\lambda_3-\frac{1}{2}(\lambda_2-\lambda_1)$$

$$\omega_{\pm}+\frac{q_{\pm}}{2}=\pm\frac{\lambda_3}{2}$$

$$\zeta H^2=-\frac{8}{3}\pi G_N\sum_{b=\pm}\left(\frac{q_b}{2}+\omega_b\right)$$

$$S_{Ren}=S_{QED}+S_{\Phi}=-\int\,\mathrm{d}^4x\left\{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\Psi}[\not{\partial}+m]\Psi+\bar{\Phi}[\not{\partial}+M]\Phi\right\}$$

$$S_{NRen}=-\int\,\mathrm{d}^4x\left\{\frac{\tilde{c}_d}{2}(\bar{\Phi}\gamma^{\mu\nu}\Phi)F_{\mu\nu}+\tilde{c}_s(\bar{\Psi}\Psi)(\bar{\Phi}\Phi)+\tilde{c}_v(\bar{\Psi}\gamma^{\mu}\Psi)\left(\bar{\Phi}\gamma_{\mu}\Phi\right)+\cdots\right\}$$

$$S_{Nucl}=-\int_{\mathcal{P}}\mathrm{d}s\left\{\sqrt{-\dot{y}^2}M-Ze\dot{y}^{\mu}A_{\mu}+c_s\sqrt{-\dot{y}^2}(\bar{\Psi}\Psi)+ic_v\dot{y}^{\mu}\left(\bar{\Psi}\gamma_{\mu}\Psi\right)+\cdots\right\}$$

$$S_b=\int\,\mathrm{d}^Dx\int_W\mathrm{d}sL_b[y(s)]\delta^D[x-y(s)]=\int\,\mathrm{d}^Dx\frac{L_b}{\gamma}\delta^{D-1}[\mathbf{x}-\mathbf{y}(s)]$$

$$0=\int_{P_{\epsilon}}\mathrm{d}^{D-1}x\left\{-\partial_M\left[\sqrt{-g}f(\phi)F^{MN}\right]+\sqrt{-g}\nabla_M[f(\phi)F^{MN}]+\frac{1}{\gamma}\frac{\partial L_b}{\partial A_N}\delta^{D-1}[\mathbf{x}-\mathbf{y}(s)]\right\}$$

$$\simeq-\oint_{\partial P_{\epsilon}}\mathrm{d}^{D-2}xn_M\left[\sqrt{-g}fF^{MN}\right]+\frac{1}{\gamma}\left(\frac{\partial L_b}{\partial A_N}\right)_{\mathbf{x}=\mathbf{y}}$$

$$\oint_{C_\epsilon} \mathrm{d}\theta\left[n_N\sqrt{-g}f(\phi)F^{NM}\right] = -\left(\frac{\delta S_b}{\delta A_M}\right)_\epsilon$$

$$\mathcal{L} = -\frac{1}{2}\kappa^{-2}\sqrt{-g}\mathcal{G}_{AB}\partial^M\phi^A\partial_M\phi^B$$

$$\oint_{C_\epsilon} \mathrm{d}\theta\left[\frac{1}{\kappa^2}\sqrt{-g}\mathcal{G}_{AB}n_M\partial^M\phi^B\right] = -\left(\frac{\delta S_b}{\delta\phi^A}\right)_\epsilon$$

$$\lim_{r\to 0}\oint_{C_\epsilon} \mathrm{d}\theta\left[\frac{1}{2\kappa^2}\sqrt{-g}\left(K^{ij}-Kg^{ij}\right) - (\text{supercurvature})\right] = -\left(\frac{\delta S_b}{\delta g_{ij}}\right)_\epsilon$$

$$S_b = \int_W \mathrm{d}^4x\sqrt{-\gamma}T(\phi)$$

$$S_b = -\int \mathrm{d}^4x\sqrt{-\gamma}L_b(\varphi) = -\int \mathrm{d}^4x\sqrt{-g}W_b^4L_b(\varphi) = -\int \mathrm{d}^4x\sqrt{-g}T_b(\varphi)$$

$$\frac{2\pi}{\kappa^2}[aW^4\partial_r\varphi]_{r=\epsilon} = \frac{\partial}{\partial\varphi}[W_b^4L_b] \text{ and so } \lim_{\eta\to\pm\infty}[\mp\partial_\eta\varphi] = q_\pm = \frac{\kappa^2}{2\pi}\left(\frac{\partial T_\pm}{\partial\varphi}\right)$$

$$\mathrm{d}r := \mp aW^4\,\mathrm{d}\eta,$$

$$-\frac{2\pi}{\kappa^2}\left\{W^4\left[a\left(3\frac{\partial_r W}{W}+\frac{\partial_r a}{a}\right)-1\right]\right\}_{r=\epsilon} = W_b^4L_b(\varphi)$$

$$\lim_{\eta\to\pm\infty}\left\{\mp\left[3\left(\frac{\partial_\eta W}{W}\right)+\left(\frac{\partial_\eta a}{a}\right)\right]-W^4\right\} = 3\omega_\pm+\alpha_\pm-W_\pm^4 = -\frac{\kappa^2T_\pm}{2\pi}$$

$$a_b^2e^{-2\alpha_b|\eta|}\left(W_b^8\,\mathrm{d}\eta^2+\mathrm{d}\theta^2\right) \simeq \mathrm{d}r^2+\left(\frac{r\alpha_b}{W_b^4}\right)^2\mathrm{d}\theta^2$$

$$\delta_b := 2\pi\left(1-\frac{\alpha_b}{W_b^4}\right) = \frac{\kappa^2T_b}{W_b^4} = \kappa^2L_b$$

$$\frac{2\pi}{\kappa^2}[aW^4\partial_rW]_{r=\epsilon} = W_b^4\tilde{U}_b(\varphi) = U_b(\varphi) \text{ and so } \lim_{\eta\to\pm\infty}\left[\mp\left(\frac{\partial_\eta W}{W}\right)\right] = \omega_\pm = \frac{\kappa^2U_\pm}{2\pi}$$

$$\begin{aligned}0 &\simeq \left[-\frac{8a'W'}{aW}-\frac{12(W')^2}{W^2}+(\varphi')^2\right]_b = -8\alpha_b\omega_b-12\omega_b^2+q_b^2\\ &= -8\mathcal{U}_b\left(W_b^4-\mathcal{T}_b-3\mathcal{U}_b\right)-12\mathcal{U}_b^2+(\mathcal{T}_b{}')^2\\ &= W_b^8\left[-8\tilde{\mathcal{U}}_b\left(1-\mathcal{L}_b-3\tilde{\mathcal{U}}_b\right)-12\tilde{\mathcal{U}}_b^2+(\mathcal{L}_b{}')^2\right]\end{aligned}$$

$$\mathcal{U}_b = W_b^4\tilde{\mathcal{U}}_b = \frac{W_b^4}{3}\left[(1-\mathcal{L}_b)-\sqrt{(1-\mathcal{L}_b)^2-\frac{3}{4}(\mathcal{L}_{b'})^2}\right]$$

$$V_{\text{eff}}(\varphi_0) = -\sum_b \left(U_b + \frac{T_b'}{2}\right) = -\sum_b W_b^4 \left(\tilde{U}_b + \frac{L_b'}{2}\right),$$

$$\mathcal{L}_{10} = -\sqrt{-\hat{g}}\left[\frac{1}{2\kappa_{10}^2}\hat{g}^{MN}\left(\hat{R}_{MN} + \partial_M\hat{\phi}\partial_N\hat{\phi}\right) + \frac{1}{4}e^{-\hat{\phi}}\hat{F}_{MN}\hat{F}^{MN} + \frac{1}{12}e^{-2\varphi}\hat{H}_{MNP}\hat{H}^{MNP}\right]$$

$$y_1 + iy_2 = \cosh\ \rho e^{i\alpha};\ y_3 + iy_4 = \sinh\ \rho e^{i\beta}$$

$$\mathrm{d}\hat{s}_{10}^2 = (\cosh\ 2\rho)^{1/4}\left[e^{-\varphi/4}\ \mathrm{d}s_6^2 + e^{\varphi/4}\ \mathrm{d}z^2 + \frac{e^{\varphi/4}}{2\bar{g}^2}\left(\mathrm{d}\rho^2 + \frac{\cosh^2\ \rho}{\cosh\ 2\rho}(D\alpha)^2 + \frac{\sinh^2\ \rho}{\cosh\ 2\rho}(D\beta)^2\right)\right]$$

$$\hat{H}_{(3)} = H_{(3)} + \frac{\sinh\ 2\rho}{2\bar{g}(\cosh\ 2\rho)^2}\ \mathrm{d}\rho \wedge D\alpha \wedge D\beta + \frac{1}{2\bar{g}\cosh\ 2\rho}F_{(2)} \wedge (\cosh^2\ \rho D\alpha - \sinh^2\ \rho D\beta)$$

$$e^{\hat{\phi}} = (\cosh\ 2\rho)^{-1/2}e^{\varphi}$$

$$\mathcal{M}_{11} = \mathbb{R}^{1,4} \times Y_6;\ Y_6 \simeq B_4 \times T^2;\ T^2 = S_A^1 \times S_B^1$$

$$\mathcal{M}_{10} = \mathbb{R}^{1,4} \times \tilde{S}_B^1 \times B_4 \to \mathbb{R}^{1,5} \times B_4$$

$$S_{11} = -\frac{1}{2\kappa_{11}^2}\int\ \mathrm{d}^{11}x\left[\sqrt{-g}\left(R + \frac{1}{48}G_{MNPQ}G^{MNPQ}\right) + \frac{2}{3\cdot 4!^3}\epsilon^{M_1\cdots M_{11}}C_{M_1\cdots M_3}G_{M_4\cdots M_7}G_{M_8\cdots M_{11}}\right]$$

$$\mathrm{d}s^2 = \mathrm{d}s_5^2 + \mathrm{d}s_6^2$$

$$\int_{\gamma_\Lambda}\langle G_{(4)}\rangle = n_\Lambda;\ \langle G_{(4)}\rangle = n^\Lambda\omega_\Lambda$$

$$L^\Lambda = \frac{v^\Lambda}{\mathcal{V}^{1/3}}, \mathcal{N} \equiv \lambda_{\Lambda\Omega\Gamma}L^\Lambda L^\Omega L^\Gamma = 1$$

$$\lambda_{\Lambda\Omega\Gamma} = \int_{Y_6}\tilde{\omega}_\Lambda \wedge \tilde{\omega}_\Omega \wedge \tilde{\omega}_\Gamma$$

$$D_\mu\Phi = \partial_\mu\Phi + 2n_\Lambda A_\mu^\Lambda;\ V_{\text{flux}} = \frac{1}{8\mathcal{V}^2}G^{\Lambda\Sigma}n_\Lambda n_\Sigma$$

$$G_{\Lambda\Sigma} = \frac{1}{2\mathcal{V}^{1/3}}\left(\int_{Y_6}\tilde{w}_\Lambda \wedge w_\Sigma\right)$$

$$n_H = h^{2,1}(Y_6) + 1;\ n_T + 1 = h^{1,1}(B_4);\ \mathrm{rank}G = h^{1,1}(Y_6) - h^{1,1}(B_4) - 1$$

$$\langle F_{(2)}\rangle = n^\Lambda\tilde{\omega}_\Lambda;\ \int\ \mathrm{d}^8x\sqrt{-g}F^{MN}F_{MN} \Longrightarrow V_{\mathrm{D}7} \propto G^{\Lambda\Sigma}n_\Lambda n_\Sigma$$

$$-\frac{\mathcal{L}_6}{\sqrt{-g}} = \frac{1}{2\kappa^2}g^{MN}(R_{MN} + \partial_M\varphi\partial_N\varphi + h_{UV}D_Mq^UD_Nq^V) + \frac{1}{4}e^{-\varphi}F_{MN}F^{MN} + \frac{1}{12}e^{-2\varphi}H_{MNP}H^{MNP} + \frac{2\mathrm{g}^2U(q)}{\kappa^4}e^{\varphi}$$

$$h(\mathcal{V}) := h_{\mathcal{V}\mathcal{V}} = 4h_{\Phi\Phi} = \frac{1}{2\mathcal{V}^2};\ h_{\tau\tau^*} = \frac{1}{2\tau_2^2}$$

$$V(\varphi,\chi) = \frac{2\mathfrak{g}^2}{\kappa^4\mathcal{V}^2}e^{\varphi} = \frac{2\mathfrak{g}^2}{\kappa^4}e^{\varphi-2\chi} \text{ where } \mathcal{V} := e^{\chi}$$

$$R_{MN} = 4e^{-\varphi}\kappa^2 F_{MP}F_N^P + \partial_M\varphi\partial_N\varphi + \frac{1}{2}\partial_M\chi\partial_N\chi + \frac{1}{2}e^{-2\chi}D_M\Phi D_N\Phi + \frac{1}{2\tau_2^2}\partial_{(M}\tau\partial_{N)}\tau^* + \frac{1}{2}g_{MN}\,\Box\,\varphi$$

$$\nabla_R(e^{-\varphi}F^{LR}) = \frac{k}{32}e^{-2\chi}D^L\Phi$$

$$\Box\,\chi = -4e^{\varphi-2\chi}\tilde{V} - e^{-2\chi}D_M\Phi D^M\Phi, \Box\,\Phi = 2\partial_M\chi D^M\Phi,$$
$$\Box\,\varphi = \tilde{V}e^{\varphi-2\chi} - \kappa^2 e^{-\varphi}F^{MN}F_{MN} \text{ and } \Box\,\tau = -\frac{i}{\tau_2}\partial^M\tau\partial_M\tau,$$

$$\nabla_M(e^{-\varphi}F^{MN}) = 0$$

$$\Box\,\chi + 4\tilde{V}e^{\varphi-2\chi} = 0 \text{ and } \Box\,\varphi + \frac{1}{4}\kappa^2 e^{-\varphi}F^{MN}F_{MN} - \tilde{V}e^{\varphi-2\chi} = 0,$$

$$R_{MN} + \kappa^2 e^{-\varphi}F_{MP}F_N^P + \partial_M\varphi\partial_N\varphi + \frac{1}{2}\partial_M\chi\partial_N\chi + \frac{1}{2}g_{MN}\,\Box_6\,\varphi = 0$$

$$\mathrm{d}s^2 = \hat{g}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + a^2\,\mathrm{d}\theta^2 + a^2W^8\,\mathrm{d}\eta^2$$

$$F_{\eta\theta} = \hat{Q}a^2e^{\varphi}$$

$$\chi'' = -4\tilde{V}a^2W^8e^{\varphi-2\chi} \text{ and } \varphi'' = \tilde{V}a^2W^8e^{\varphi-2\chi} - \frac{1}{2}Q^2a^2e^{\varphi}$$

$$\left(\ln W + \frac{1}{2}\varphi\right)'' = 3\zeta H^2a^2W^6$$

$$\left(\ln a + \frac{1}{2}\varphi\right)'' = -Q^2a^2e^{\varphi}$$

$$\varphi'^2 + \frac{1}{2}\chi'^2 - 8(\ln W)'(\ln a)' - 12[(\ln W)']^2 + a^2e^{\varphi}\left(Q^2 - 2\tilde{V}W^8e^{-2\chi} + 12\zeta H^2W^6e^{-\varphi}\right) = 0$$

$$L := \left[\varphi'^2 + \frac{1}{2}\chi'^2 - 8(\ln W)'(\ln a)' - 12[(\ln W)']^2\right]N^{-1} - Na^2e^{\varphi}\left(Q^2 - 2\tilde{V}W^8e^{-2\chi} + 12\zeta H^2W^6e^{-\varphi}\right)$$

$$\varphi = \frac{1}{2}(\mathcal{X} - \mathcal{Y} - 2\mathcal{Z}), \ln W = \frac{1}{4}(\mathcal{Y} - \mathcal{X}), \ln a = \frac{1}{4}(3\mathcal{X} + \mathcal{Y} + 2\mathcal{Z})$$

$$X = \mathcal{X} + \ln Q, Y = \mathcal{Y} + \ln(2\tilde{V}) \text{ and } Z = \mathcal{Z} + \ln\left(\frac{6H^2}{\tilde{V}}\right)$$

$$L = \left[(X')^2 - (Y')^2 + (Z')^2 + \frac{1}{2}(\chi')^2\right]N^{-1} - N[e^{2X} - e^{2Y-2\chi} + \zeta e^{2Y+Z}]$$

$$X'' + e^{2X} = 0, \chi'' + 2e^{2Y-2\chi} = 0, Z'' + \frac{1}{2}\zeta e^{2Y+Z} = 0 \text{ and } Y'' + e^{2Y-2\chi} - \zeta e^{2Y+Z} = 0.$$

$$X = \ln\left[\frac{\lambda_1}{\cosh\left[\lambda_1(\eta - \eta_1)\right]}\right],$$

$$\chi = 4Z + 2Y + b\eta + c$$

$$A := \frac{1}{2}(2Y + Z) \text{ and } B := Y - \chi = -(Y + 4Z + b\eta + c)$$

$$B'' = \zeta e^{2A} + e^{2B} \text{ and } A'' = \frac{3}{4}\zeta e^{2A} - e^{2B}$$

$$B'' = e^{2B} \text{ (4D curve solutions)}$$

$$B = \ln\left[\frac{\lambda_q}{\sinh\left[\lambda_q(\eta - \eta_0)\right]}\right],$$

$$\mathrm{d}s^2 = e^{2\Gamma(r)} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2\Omega(r)}\mathrm{d}\theta^2 + \mathrm{d}r^2$$

$$J' = e^{\Omega+4\Gamma}\dot{J} \text{ and } J'' = e^{2\Omega+8\Gamma}[\ddot{J} + (\dot{\Omega} + 4\dot{\Gamma})\dot{J}],$$

$$\begin{aligned}
\ddot{\varphi} + (\dot{\Omega} + 4\dot{\Gamma})\dot{\varphi} &= \tilde{V}e^{\varphi-2\chi} - \frac{1}{2}Q^2 e^{\varphi-8\Gamma} \\
\ddot{\chi} + (\dot{\Omega} + 4\dot{\Gamma})\dot{\chi} &= -4\tilde{V}e^{\varphi-2\chi} \\
\ddot{\Gamma} + (\dot{\Omega} + 4\dot{\Gamma})\dot{\Gamma} &= 3\zeta H^2 e^{-2\Gamma} - \frac{1}{2}[\ddot{\varphi} + (\dot{\Omega} + 4\dot{\Gamma})\dot{\varphi}] \\
\ddot{\Omega} + (\dot{\Omega} + 4\dot{\Gamma})\dot{\Omega} &= -Q^2 e^{\varphi-8\Gamma} - \frac{1}{2}[\ddot{\varphi} + (\dot{\Omega} + 4\dot{\Gamma})\dot{\varphi}].
\end{aligned}$$

$$\ddot{\chi} + 3\ddot{\varphi} - 2\ddot{\Omega} = -(\dot{\Omega} + 4\dot{\Gamma})(\dot{\chi} + 3\dot{\varphi} - 2\dot{\Omega}),$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[e^{\Omega+4\Gamma}\left(\dot{\Gamma} + \frac{1}{2}\dot{\varphi}\right)\right] = 3\zeta H^2 e^{\Omega+2\Gamma} \text{ and } \frac{\mathrm{d}}{\mathrm{d}r}\left[e^{\Omega+4\Gamma}\left(\dot{\Omega} + \frac{1}{2}\dot{\varphi}\right)\right] = -Q^2 e^{\varphi+\Omega-4\Gamma}$$

$$\dot{\varphi}^2 + \frac{1}{2}\dot{\chi}^2 - 8\dot{\Omega}\dot{\Gamma} - 12\dot{\Gamma}^2 + 12\zeta H^2 e^{-2\Gamma} + Q^2 e^{\varphi-8\Gamma} - 2\tilde{V}e^{\varphi-2\chi} = 0$$

$$\varphi = \varphi_0 + q\ln\left(\frac{r}{r_0}\right), \chi = \chi_0 + s\ln\left(\frac{r}{r_0}\right), \Gamma = \Gamma_0 + w\ln\left(\frac{r}{r_0}\right) \text{ and } \Omega = \Omega_0 + \alpha\ln\left(\frac{r}{r_0}\right),$$

$$a(r) = a_0\left(\frac{r}{r_0}\right)^\alpha [1 + \mathcal{O}(r)], \text{ for } a(r) \propto e^\Omega$$

$$q > 2(s-1) \text{ and } q > 2(4w-1) \text{ and (if } H \neq 0) \; w < 1$$

$$\alpha + 4w = 1$$

$$q^2 + \frac{1}{2}s^2 - 8\alpha w - 12w^2 = q^2 + \frac{1}{2}s^2 - 8w + 20w^2 = 0$$

$$\frac{q}{r^2}(-1+\alpha+4w)=\tilde{V}e^{\varphi_0-2\chi_0}\left(\frac{r_0}{r}\right)^{2s-q}-\frac{1}{2}Q^2e^{\varphi_0-8\Gamma_0}\left(\frac{r_0}{r}\right)^{8w-q}$$

$$\frac{s}{r^2}(-1+\alpha+4w)=-4\tilde{V}e^{\varphi_0-2\chi_0}\left(\frac{r_0}{r}\right)^{2s-q}$$

$$\frac{w+\frac{1}{2}q}{r^2}(-1+\alpha+4w)=3\zeta H^2e^{-2\Gamma_0}\left(\frac{r_0}{r}\right)^{2w}$$

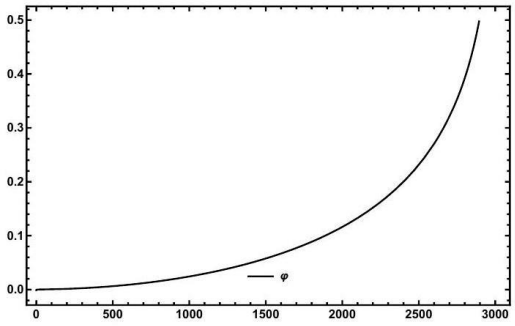
$$\frac{\alpha+\frac{1}{2}q}{r^2}(-1+\alpha+4w)=-Q^2e^{\varphi_0-8\Gamma_0}\left(\frac{r_0}{r}\right)^{8w-q}$$

$$(s+3q-2\alpha)(-1+\alpha+4w)\frac{1}{r^2}=0$$

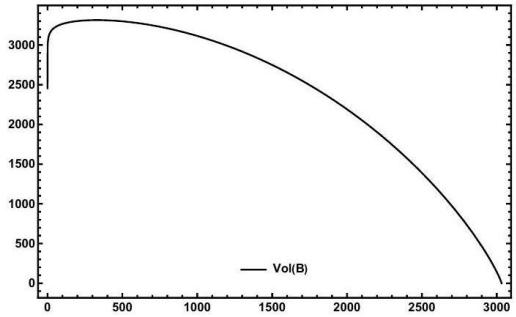
$$\frac{q^2+\frac{1}{2}s^2-8\alpha w-12w^2}{r^2}+12\zeta H^2e^{-2\Gamma_0}\left(\frac{r_0}{r}\right)^{2w}+Q^2e^{\varphi_0-8\Gamma_0}\left(\frac{r_0}{r}\right)^{8w-q}-2\tilde{V}e^{\varphi_0-2\chi_0}\left(\frac{r_0}{r}\right)^{2s-q}=0$$

$$q=2(s-1), q=2(4w-1) \text{ and (if } H\neq 0) w=1$$

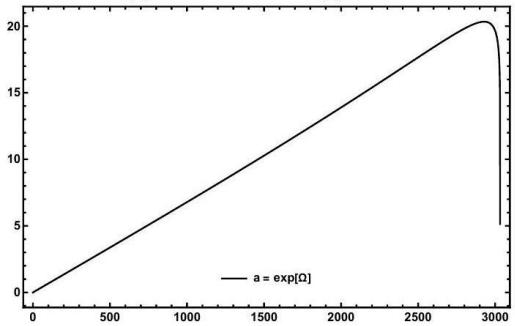
$$w=\frac{1}{9}, q=-\frac{2}{9}, s=\frac{8}{9}, \alpha=\frac{1}{9} \text{ and } \tilde{V}e^{\varphi_0-2\chi_0}r_0^2=\frac{8}{81}$$



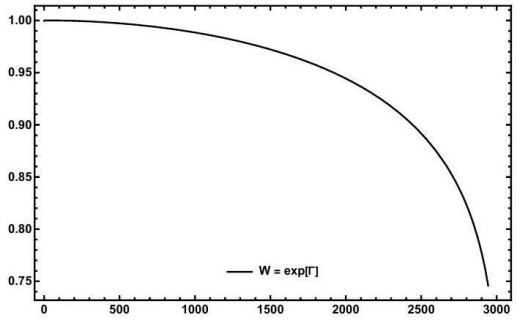
(a) The function $\varphi(r)$ vs r



(b) The function $\mathcal{V}=e^{\chi(r)}$ vs r.

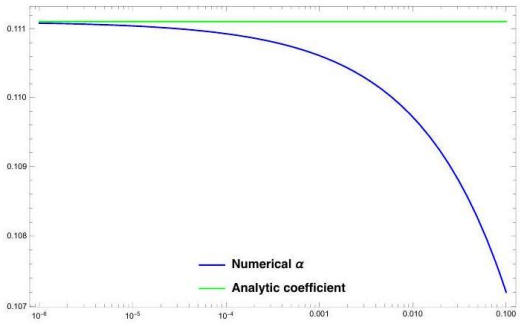


(c) The function $a(r)=e^{\Omega(r)}$ vs r.

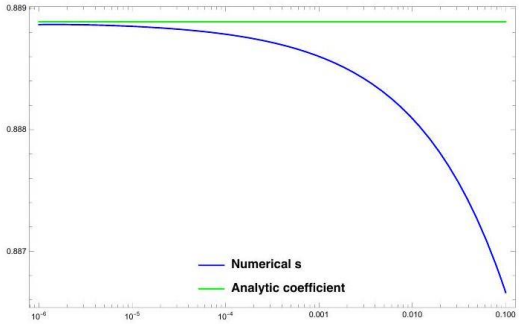


(d) The function $W(r)=e^{\Gamma(r)}$ vs r.

$$\alpha_0=0.9999\ q_0=0.0001, s_0\simeq 0.0200, w_0=0.000025$$



(a) $r\,\partial\Omega/\partial r$ *vs* $\alpha_e = \frac{1}{9}$



(b) $r\,\partial\chi/\partial r$ *vs* $s_e = \frac{8}{9}$

$$S_b = -\int \mathrm{d}^4x\sqrt{-g}W_b^4 L_b(\varphi,\chi) = -\int \mathrm{d}^4x\sqrt{-g}T_b(\varphi,\chi)$$

$$[a\partial_r\varphi]_{r=\epsilon} = \left(\frac{\partial\mathcal{L}_b}{\partial\varphi}\right)_{r=\epsilon} \text{ and } [a\partial_r\chi]_{r=\epsilon} = \left(\frac{\partial\mathcal{L}_b}{\partial\chi}\right)_{r=\epsilon}$$

$$1-\left[a\left(3\frac{\partial_r W}{W}+\frac{\partial_r a}{a}\right)\right]_{r=\epsilon} = (\mathcal{L}_b)_{r=\epsilon} \text{ and } [a\partial_r W]_{r=\epsilon} = (\tilde{\mathcal{U}}_b)_{r=\epsilon}$$

$$q_b\left(\frac{a_b}{r_b}\right)\left(\frac{\epsilon}{r_b}\right)^{\alpha_b-1} = \left(\frac{\partial\mathcal{L}_b}{\partial\varphi}\right)_{r=\epsilon} \text{ and } s_b\left(\frac{a_b}{r_b}\right)\left(\frac{\epsilon}{r_b}\right)^{\alpha_b-1} = \left(\frac{\partial\mathcal{L}_b}{\partial\chi}\right)_{r=\epsilon},$$

$$1-(3w_b+\alpha_b)\left(\frac{a_b}{r_b}\right)\left(\frac{\epsilon}{r_b}\right)^{\alpha_b-1} = (\mathcal{L}_b)_{r=\epsilon}$$

$$w_b\left(\frac{a_b}{r_b}\right)\left(\frac{\epsilon}{r_b}\right)^{\alpha_b-1} = (\tilde{\mathcal{U}}_b)_{r=\epsilon}$$

$$q_b\left(\frac{a_b}{r_b}\right) = \left(\frac{\partial\mathcal{L}_b}{\partial\varphi}\right)_{r=\epsilon}, s_b\left(\frac{a_b}{r_b}\right) = \left(\frac{\partial\mathcal{L}_b}{\partial\chi}\right)_{r=\epsilon}, w_b\left(\frac{a_b}{r_b}\right) = (\tilde{\mathcal{U}}_b)_{r=\epsilon}$$

$$1-(3w_b+\alpha_b)\left(\frac{a_b}{r_b}\right) = (\mathcal{L}_b)_{r=\epsilon}$$

$$0\simeq\left[\dot{\varphi}^2+\frac{1}{2}\dot{\chi}^2-8\dot{\Omega}\dot{\Gamma}-12\dot{\Gamma}^2\right]_b = q_b^2+\frac{1}{2}s_b^2-8\alpha_b w_b-12w_b^2$$
$$= W_b^4\left[-8\tilde{\mathcal{U}}_b(1-\mathcal{L}_b-3\tilde{\mathcal{U}}_b)-12\tilde{\mathcal{U}}_b^2+(\mathcal{L}_{b,\varphi})^2+\frac{1}{2}(\mathcal{L}_{b,\chi})^2\right]$$

$$\tilde{\mathcal{U}}_b = \frac{1}{3}\left[(1-\mathcal{L}_b)-\sqrt{(1-\mathcal{L}_b)^2-\frac{3}{4}(\mathcal{L}_{b,\varphi})^2-\frac{3}{8}(\mathcal{L}_{b,\chi})^2}\right] \simeq \frac{1}{8}(\mathcal{L}_{b,\varphi})^2+\frac{1}{16}(\mathcal{L}_{b,\chi})^2+\cdots$$

$$w_b\left(\frac{a_b}{r_b}\right) \simeq \frac{1}{8}\left(\frac{\partial\mathcal{L}_b}{\partial\varphi}\right)^2_{r=\epsilon}+\frac{1}{16}\left(\frac{\partial\mathcal{L}_b}{\partial\chi}\right)^2_{r=\epsilon}+\cdots$$

$$\frac{a_b}{r_b} \simeq 1-(\mathcal{L}_b)_{r=\epsilon}$$

$$\delta_b = 2\pi\left(1 - \frac{a_b}{r_b}\right) \simeq \kappa^2 L_b$$

$$\Omega_0 = \log\left(\frac{a_b}{\ell}\right) + \alpha\log\left(\frac{\ell}{r_b}\right) = \log\left(\frac{a_b}{r_b}\right) - (1-\alpha)\log\left(\frac{\ell}{r_b}\right)$$

$$3\zeta H^2 \int_0^{r_e} \mathrm{d}r a W^2 = W_e^4\left(w_e + \frac{1}{2}q_e\right)\left(\frac{a_e}{r_e}\right) - W_0^4\left(w_0 + \frac{1}{2}q_0\right)\left(\frac{a_0}{r_0}\right) = 0$$

$$S = \frac{A}{4G} + C\log\,\lambda + \cdots, (\lambda \to +\infty)$$

$$C = C_{\text{local}} + C_{\text{global}}$$

$$C_{\text{local}} \equiv \int d^d x\sqrt{g}a_4(x)$$

$$a_4(x) = -a_{\text{E}}E_4 + cW^2 + b_1R^2 + b_2RF_{\mu\nu}F^{\mu\nu}$$

Multiplet	a_{E}	c	b_1	b_2
Free scalar $(m\ell = -2)$	$\frac{1}{360}$	$\frac{1}{120}$	$\frac{1}{288}(\Delta(\Delta-3)+2)^2$	0
Free Dirac spinor $(m = 0)$	$-\frac{11}{360}$	$-\frac{1}{20}$	$\frac{1}{72}\left(\Delta - \frac{3}{2}\right)^2\left(\left(\Delta - \frac{3}{2}\right)^2 - 2\right)$	0
Free vector $(m = 0)$	$\frac{31}{180}$	$\frac{1}{10}$	0	0
Free gravitino $(m = 0)$	$-\frac{229}{720}$	$-\frac{77}{120}$	$-\frac{1}{9}$	0
Einstein-Maxwell	$\frac{53}{45}$	$\frac{137}{60}$	$-\frac{13}{36}$	0
$\mathcal{N} = 2$ gravitini	$-\frac{589}{360}$	$-\frac{137}{60}$	0	$\frac{13}{18}$
$\mathcal{N} = 2$ gravity multiplet	$-\frac{11}{24}$	0	$-\frac{13}{36}$	$\frac{13}{18}$

$$(4\pi)^2 a_4(x) = \frac{11}{24}E_4 - \frac{13}{36}R^2 + \frac{13}{18}RF_{\mu\nu}F^{\mu\nu}.$$

$$C_{\text{local}} = \frac{11}{6} - \frac{26}{3}\frac{a(\ell^2 - 4\ell - a^2)}{(\ell - a)(a^2 + 6a\ell + \ell^2)},$$

$$Z(\beta, \mu_\alpha) = \int D\Psi e^{-S_E(\Psi)}$$

$$S = \log\, Z + \beta M + \sum_\alpha \mu^\alpha q_\alpha$$

$$S = \frac{\text{Area}(q_\alpha)}{4G} + \cdots$$

$$Z(\beta, \mu_\alpha) \sim \frac{1}{\sqrt{\det Q}} e^{-S_E^{\text{classical}}}$$

$$\mathcal{Q} = \frac{\delta^2 S_E}{\delta \Psi^2}$$

$$\delta S = -\frac{1}{2}\log\ \det \mathcal{Q}$$

$$\kappa_n = \lambda^2 \kappa_n^{(0)}$$

$$S = \frac{A}{4G} + \sum_{i=1}^{k} C_i \log\ \ell_i + \cdots$$

$$C = \sum_{i=1}^{k} C_i$$

$$Z_{1-\text{ loop}}\,(\beta, \mu_\alpha) = Z_{\text{nz}} Z_{\text{zm}} e^{-S_E^{\text{class.}}}$$

$$S = \frac{A}{4G} + \left(C_{\text{local}} + C_{\text{global}}\right)\log\ \lambda + \cdots$$

$$\log\ Z_{\text{nz}} = -\frac{1}{2}{\sum_{n}}'\ \log\ \kappa_n$$

$$K(x,s) = \sum_n\ e^{-\kappa_n s} f_n^{\ell}(x) f_n^{\ell'}(x) G_{\ell\ell'}$$

$$\int_{\mathcal{M}} d^D x \sqrt{g} K(x,s) = \sum_n\ e^{-s\kappa_n} = {\sum_n}'\ e^{-s\kappa_n} + N_{\text{zm}}$$

$$\log\ \kappa - \log\ \kappa^{(0)} = -\lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{ds}{s}\left(e^{-s\kappa} - e^{-s\kappa^{(0)}}\right)$$

$$\log\ Z_{\text{nz}} - \log\ Z_{\text{nz}}^{(0)} = \frac{1}{2}\int_\epsilon^{\epsilon\lambda} \frac{ds}{s}\left(\int_{\mathcal{M}} d^D x \sqrt{g} K(x,s) - N_{\text{zm}}\right)$$

$$K(x,s) = \sum_{n \geqslant 0}\ s^{n - D/2} a_{2n}(x)$$

$$K(x,s) = s^{-2} a_0(x) + s^{-1} a_2(x) + s^0 a_4(x) + \mathcal{O}(s)$$

$$\log\ Z_{\text{nz}} = C_{\text{local}} \log\ \lambda + \cdots,$$

$$C_{\text{local}} \equiv \int\ d^4 x \sqrt{g} a_4(x)$$

$$\mathcal{Q}_m^n = (\Box) I_m^n + 2\left(\omega^\mu D_\mu\right)_m^n + P_m^n,$$

$$\mathcal{Q}_m^n = \left(\mathcal{D}^\mu \mathcal{D}_\mu\right) I_m^n + E_m^n,$$

$$E \equiv P - \omega^{\mu}\omega_{\mu} - \left(D^{\mu}\omega_{\mu}\right)$$

$$(4\pi)^2 a_4(x) = \mathrm{Tr}\left[\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{360}I\left(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu}\right)\right]$$

$$\Omega_{\mu\nu} = \left[D_{\mu} + \omega_{\mu}, D_{\nu} + \omega_{\nu}\right]$$

$$\mathcal{L} = \bar{\psi}\mathcal{D}\psi \text{ where } \mathcal{D} = \not{\partial} + L$$

$$\log\ \det\mathcal{D} = \frac{1}{2}\log\ \det\mathcal{D}^{\dagger}\mathcal{D}$$

$$\omega^{\mu} = \frac{1}{2}\left(\gamma^{\mu}L - L^{\dagger}\gamma^{\mu}\right), P = \mathcal{R} + (\not{\partial}L) - L^{\dagger}L$$

$$\mathcal{R} = -\frac{1}{4}g_{\mu\nu} + \frac{1}{2}\gamma^{\rho\sigma}R_{\mu\nu\rho\sigma}$$

$$C_{\text{global}} = C_{\text{ens}} + C_{\text{zm}}$$

$$\lambda^{\beta_{\Psi}}$$

$$C_{\text{zm}} = \sum_{\Psi}(\beta_{\Psi} - 1)n_{\Psi}^{0}$$

$$\beta_A = \frac{D}{2} - 1, \beta_{\psi} = D - 1, \beta_g = \frac{D}{2}$$

$$(4\pi)^2 a_4(x) = -a_{\text{E}}E_4 + cW^2 + b_1R^2 + b_2RF_{\mu\nu}F^{\mu\nu}$$

$$E_4 \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$
$$W^2 \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}E_4,$$

$$ds^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2, f(r) = 1 + \frac{r^2}{\ell^2} - \frac{2m}{r}$$

$$t \sim t + \beta, \beta = \frac{4\pi r_+}{1 + \frac{3r_+^2}{\ell^2}},$$

$$E_4 = \frac{24}{\ell^4} + \frac{48m^2}{r^6}, W^2 = \frac{48m^2}{r^6}, R^2 = \frac{144}{\ell^4}, RF_{\mu\nu}F^{\mu\nu} = 0$$

$$a_4^{\text{CT}} = \int_{\partial M} d^3y\sqrt{h}(c_1 + c_2\mathcal{R})$$

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}E_4 = 4, \qquad \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}W^2 = \frac{4(\ell^2+r_+^2)^2}{\ell^2(\ell^2+3r_+^2)}$$
$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R^2 = \frac{24r_+^2(\ell^2-r_+^2)}{\ell^2(\ell^2+3r_+^2)}, \quad \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}RF_{\mu\nu}F^{\mu\nu} = 0$$

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}E_4 = 2$$

$$C_{\text{local}} = \frac{4}{\ell^2(\ell^2+3r_+^2)}\left((c-a_{\text{E}})\ell^4 + (2c-3a_{\text{E}}+6b_1)\ell^2r_+^2 + (c-6b_1)r_+^4\right)$$

$$ds^2 = \left(1+\frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{\left(1+\frac{r^2}{\ell^2}\right)} + r^2d\Omega^2$$

$$E_4 = \frac{24}{\ell^4}, W^2 = 0, R^2 = \frac{144}{\ell^4}, F_{\mu\nu}F^{\mu\nu} = 0$$

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}E_4 = 0, \quad \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}W^2 = 0$$
$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R^2 = 0, \quad \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}RF_{\mu\nu}F^{\mu\nu} = 0$$

$$C_{\text{local}} = 0,$$

$$ds^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2, A = \frac{iq_e}{r}dt - q_m\cos\,\theta d\phi$$

$$f(r) = 1 + \frac{r^2}{\ell^2} - \frac{2m}{r} + \frac{q_e^2+q_m^2}{r^2},$$

$$T_H = \beta^{-1} = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+}\left(1+\frac{3r_+^2}{\ell^2} - \frac{(q_e^2+q_m^2)}{r_+^2}\right).$$

$$R^2 = \frac{144}{\ell^4}, \qquad E_4 = \frac{24}{\ell^4} + \frac{8(6m^2r^2 - 12m(q_e^2+q_m^2)r + 5(q_e^2+q_m^2)^2)}{r^8}$$
$$F_{\mu\nu}F^{\mu\nu} = -\frac{2(q_e^2-q_m^2)}{r^4}, \qquad W^2 = \frac{48\left(mr-(q_e^2+q_m^2)\right)^2}{r^8}$$

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}E_4 = 4$$
$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}W^2 = \frac{2}{5}\left(2 - \frac{14r_+^2}{\ell^2} + \frac{16\pi r_+}{\beta} + \frac{(\ell^4+\ell^2r_+^2+4r_+^4)}{\pi\ell^4r_+}\beta\right)$$
$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R^2 = \frac{12r_+(r_+^2+\ell^2)}{\pi\ell^4}\beta - \frac{24r_+^2}{\ell^2}$$
$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}RF_{\mu\nu}F^{\mu\nu} = \frac{6(3r_+^4+\ell^2r_+^2-2\ell^2q_m^2)}{\pi\ell^4r_+}\beta - \frac{24r_+^2}{\ell^2}$$

$$C_{\text{local}} = \frac{2}{5}\left(2(c-5a_{\text{E}}) - \frac{2r_+^2}{\ell^2}\left(7c+30(b_1+b_2)\right) + \frac{16\pi r_+}{\beta}c\right.$$
$$\left. + \frac{\beta}{\pi\ell^4 r_+}(c\ell^4 + (c+30b_1+15b_2)\ell^2 r_+^2 + (4c+30b_1+45b_2)r_+^4 - 30b_2\ell^2 q_m^2)\right)$$

$$r_+ = r_0 + \frac{2\pi\ell_2^2}{\beta} + O(\beta^{-2})$$

$$r_0^2 = \frac{1}{6}\ell\left(\sqrt{\ell^2+12q^2} - \ell\right), \ell_2^2 = \frac{r_0^2}{1+\frac{6r_0^2}{\ell^2}} = \frac{\ell^2}{6}\left(1 - \frac{\ell}{\sqrt{\ell^2+12q^2}}\right)$$

$$\int d^4x\sqrt{g}a_4(x) = C_1\beta + C_0 + O(\beta^{-1})$$

$$\lim_{\beta\to+\infty}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}E_4 = 4$$
$$\lim_{\beta\to+\infty}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}W^2 = -\frac{2(r_0^2-\ell_2^2)^2}{3r_0^2\ell_2^2}$$
$$\lim_{\beta\to+\infty}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R^2 = -\frac{2(r_0^2-\ell_2^2)^2}{r_0^2\ell_2^2}$$
$$\lim_{\beta\to+\infty}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}RF_{\mu\nu}F^{\mu\nu} = -\frac{(r_0^2-\ell_2^2)(r_0^4+r_0^2\ell_2^2-4q_m^2\ell_2^2)}{r_0^4\ell_2^2}$$

$$\lim_{\beta\to+\infty}C_{\text{local}} = -4a_{\text{E}} - \frac{r_0^2-\ell_2^2}{r_0^2\ell_2^2}\left(\left(\frac{2}{3}c+2b_1\right)(r_0^2-\ell_2^2) + b_2\left(r_0^2+\ell_2^2-\frac{4\ell_2^2q_m^2}{r_0^2}\right)\right)$$

$$r \to r_0 + \epsilon\tilde{r}, t \to \ell_2^2\frac{\tilde{t}}{\epsilon}$$

$$ds^2 = \ell_2^2\left(\tilde{r}^2d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2}\right) + r_0^2d\Omega_2^2, A = -\frac{i\ell_2^2q_e}{r_0^2}\tilde{r}d\tilde{t} + q_m\cos\theta d\phi$$

$$\frac{6}{\ell^2} = \frac{1}{\ell_2^2} - \frac{1}{r_0^2}, q_e^2 + q_m^2 = \frac{r_0^2(r_0^2+\ell_2^2)}{2\ell_2^2}$$

$$C_{\text{local}} = -4a_{\text{E}} - \frac{r_0^2-\ell_2^2}{r_0^2\ell_2^2}\left(\left(\frac{2}{3}c+2b_1\right)(r_0^2-\ell_2^2) + b_2\left(r_0^2+\ell_2^2-\frac{4\ell_2^2q_m^2}{r_0^2}\right)\right)$$

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + \frac{\rho^2dr^2}{\Delta_r} + \frac{\rho^2d\theta^2}{\Delta_\theta} + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\phi\right)^2$$

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{\ell^2}\right) - 2mr + q_e^2 + q_m^2, \Delta_\theta = 1 - \frac{a^2}{\ell^2}\cos^2\theta$$

$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$\Xi = 1 - \frac{a^2}{\ell^2}$$

$$A = -\frac{q_e r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right) - \frac{q_m \cos\theta}{\rho^2}\left(adt - \frac{r^2 + a^2}{\Xi}d\phi\right)$$

$$M = \frac{m}{\Xi^2}, J = \frac{am}{\Xi^2}, Q_e = \frac{q_e}{\Xi}, Q_m = \frac{q_m}{\Xi}$$

$$\beta = \frac{4\pi(r_+^2 + a^2)}{r_+\left(1 + \frac{a^2}{\ell^2} + 3\frac{r_+^2}{\ell^2} - \frac{a^2 + q_\ell^2 + q_m^2}{r_+^2}\right)}$$

$$C_{\text{local}} = -4a_{\text{E}} + (6A_1 + cW_1)\beta + (24A_2 + cW_2) + \frac{cW_3}{\beta}$$

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}E_4 = 2$$

$$A_1 = \frac{(2b_1 + b_2)(a^2 + \ell^2)r_+^3 + (2b_1 + 3b_2)r_+^5 + \left((2b_1 - b_2)a^2 - 2b_2 q_m^2\right)r_+\ell^2}{\pi\ell^2(\ell^2 - a^2)(a^2 + r_+^2)}$$

$$A_2 = -\frac{b_1 a^2 + (b_1 + b_2)r_+^2}{\ell^2 - a^2}$$

$$\begin{aligned}
W_1 =& \frac{1}{16\pi a^5 r_+^4 \ell^2(\ell^2 - a^2)(a^2 + r_+^2)}[3ar_+(a^8(\ell^2 - r_+^2)^2 + r_+^8(\ell^2 + 3r_+^2)^2) \\
& -4a^3 r_+^3\left(r_+^4(\ell^4 - 9r_+^4) + a^4(\ell^4 + 12\ell^2 r_+^2 + 3r_+^4) + 2a^5 r_+^5(\ell^4 - 14\ell^2 r_+^2 + 5r_+^4)\right. \\
& \left. -3(a^2 + r_+^2)\left(a^2(\ell^2 - r_+^2) - r_+^2(\ell^2 + 3r_+^2)\right)^2(r_+^4 - a^4)\arctan(a/r_+)\right] \\
W_2 =& \frac{a^2 + r_+^2}{2a^5 r_+^3(\ell^2 - a^2)}\left[4a^3\ell^2 r_+^3 + 3ar_+\left(a^4(\ell^2 - r_+^2) - r_+^4(\ell^2 + 3r_+^2)\right)\right. \\
& \left. -3\left(a^2(\ell^2 - r_+^2) - r_+^2(\ell^2 + 3r_+^2)\right)(r_+^4 - a^4)\arctan(a/r_+)\right] \\
W_3 =& \frac{\pi\ell^2(a^2 + r_+^2)}{a^5 r_+^2(\ell^2 - a^2)}[ar_+(3a^4 + 2a^2 r_+^2 + 3r_+^4) - 3(r_+^2 + a^2)(r_+^4 - a^4)\arctan(a/r_+)]
\end{aligned}$$

$$r_+ = r_0 + \frac{2\pi\ell_2^2}{\beta} + O(\beta^{-2})$$

$$\begin{aligned}
C_{\text{local}} = -4a_{\text{E}} + & \frac{1}{2ar_0^5(\ell^2 - a^2)(a^2 + r_0^2)(a^2 + \ell^2 + 6r_0^2)}[-3a^7 r_0(16b_1 r_0^4 + c(\ell^2 - r_0^2)^2) \\
& + a^5 r_0^3(c\ell^4 + 2(11c - 12b_2)\ell^2 r_0^2 - 3(13c - 8b_2 + 80b_1)r_0^4) \\
& + a^3 r_0^5(15c\ell^4 + 2(25c + 24b_2)\ell^2 r_0^2 - (49c + 336b_1 - 48b_2)r_0^4 - 48b_2\ell^2 q_m^2) \\
& + 3ar_0^7(c\ell^4 + 2(3c - 4b_2)\ell^2 r_0^2 - (7c + 48b_1 + 24b_2)r_0^4 + 16b_2\ell^2 q_m^2) \\
& - 3c(a^2 + r_0^2)\left(a^2(r_0^2 - \ell^2) + r_0^2(\ell^2 + 3r_0^2)\right)^2 \arctan(a/r_0)\Big].
\end{aligned}$$

$$r \to r_0 + \epsilon\tilde{r}, t \to \ell_2^2\frac{\tilde{t}}{\epsilon}, \phi \to \phi - \frac{ia\ell_2^2(\ell^2 - a^2)}{\ell^2(a^2 + r_0^2)}\frac{t}{\epsilon}$$

$$d\tilde{s}^2 = \frac{\ell_2^2(r_0^2 + a^2\cos^2\,\theta)}{a^2 + r_0^2}\left(\tilde{r}^2 d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2}\right) + \frac{\ell^2(r_0^2 + a^2\cos^2\,\theta)}{\ell^2 - a^2\cos^2\,\theta} d\theta^2$$
$$+ \frac{\ell^2(a^2 + r_0^2)^2(\ell^2 - a^2\cos^2\,\theta)\sin^2\,\theta}{(\ell^2 - a^2)^2(r_0^2 + a^2\cos^2\,\theta)}\left(d\phi - \frac{2\ell_2^2 a r_0(\ell^2 - a^2)}{\ell^2(a^2 + r_0^2)^2} i r dt\right)^2$$

$$\ell_2 = \ell\sqrt{\frac{a^2 + r_0^2}{a^2 + \ell^2 + 6r_0^2}}$$

$$\tilde{A} = \frac{1}{r_0^2 + a^2\cos^2\,\theta}\left[-\frac{i\ell_2^2}{a^2 + r_0^2}(q_e(r_0^2 - a^2\cos^2\,\theta) + 2q_m a r_0 \cos\,\theta)\tilde{r} d\tilde{t}\right.$$
$$\left. + \frac{\ell^2}{\ell^2 - a^2}(q_e a r_0 \sin^2\,\theta + q_m(a^2 + r_0^2)\cos\,\theta) d\tilde{\phi}\right]$$

$$r_0 = \sqrt{a\ell}, q_m = 0$$

$$M = \frac{\sqrt{a\ell}}{\left(1 - \frac{a}{\ell}\right)^2}, Q_e = \frac{\sqrt{a\ell}}{1 - \frac{a}{\ell}}, Q_m = 0, J = \frac{a\sqrt{a\ell}}{\left(1 - \frac{a}{\ell}\right)^2}$$

$$M = Q_e + \frac{J}{\ell}$$

$$C_{\text{local}} = -4a_{\text{E}} + \frac{3\ell_2^2}{2a\ell^2(\ell^2 - a^2)}[(9c - 8b_2)a\ell^2 - (9c + 48b_1 - 8b_2)a^2\ell - (c + 16b_1)a^3$$
$$\left. + c\ell^3 - \frac{c(a + \ell)^4}{\sqrt{al}}\arctan\left(\sqrt{a/\ell}\right)\right]$$

$$\ell_2 = \ell\sqrt{\frac{a(a + \ell)}{a^2 + 6a\ell + \ell^2}} \text{ (BPS case)}$$

$$S_{\text{EE}} = -\text{Tr}\rho_A \log\,\rho_A$$

$$\rho_A = \text{Tr}_{\bar{A}}|0\rangle\langle 0|$$

$$S_{\text{EE}} = \frac{\text{Area}}{4G} + (C_{\text{local}} + C_{\text{zm}})\log\,\ell + \cdots$$

$$ds^2 = \left(\frac{\rho^2}{\ell^2} - 1\right)dt^2 + \frac{d\rho^2}{\frac{\rho^2}{\ell^2} - 1} + \rho^2 ds_{H_2}^2, ds_{H_2}^2 = du^2 + \sinh^2\,u d\phi^2$$

$$\beta = 2\pi\ell$$

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}E_4 = 2 \qquad \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}W^2 = 0$$
$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R^2 = 12, \qquad \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}RF_{\mu\nu}F^{\mu\nu} = 0$$

$$C_{\text{local}} = -2a_{\text{E}} + 12b_1$$

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}E_4 = 1$$

$C_{\mathrm{zm}} = 0$ (full geometry)

$C_{\mathrm{zm}} = -3 + 8\delta_{\mathrm{BPS}}$ (near horizon geometry),

$$C_{\mathrm{ens}} = -K$$

Background spacetime	C_{zm}	C_{ens}	C_{global}
Schwarzschild	0	-3	-3
Reissner-Nordström	0	-3	-3
Kerr	0	-1	-1
Kerr-Newman	0	-1	-1
BPS Kerr-Newman	0	-1	-1
Reissner-Nordström near horizon	-3	-3	-6
Kerr-Newman near horizon	-3	-1	-4
BPS Kerr-Newman near horizon	5	-1	4
Thermal AdS_4	0	-3	-3
AdS_4-Rindler	0	-3	-3

$$S = -\frac{1}{2}\int d^4x\sqrt{g}((\partial\phi)^2 + m^2\phi^2)$$

$$P = E = m^2, \Omega = 0,$$

$$\Delta = \frac{1}{2}\left(3 + \sqrt{9 + 4m^2\ell^2}\right).$$

$$(4\pi)^2 a_4(x) = -\frac{1}{360}E_4 + \frac{1}{120}W^2 + \frac{1}{288}(\Delta(\Delta-3)-2)^2R^2.$$

$$C_{\text{local}} = -\frac{1}{90} - \frac{r_0^2}{20\ell^4}(24 + 5(\Delta+1)\Delta(\Delta-3)(\Delta-4))$$

$$S = \int d^{d+1}x\sqrt{g}\bar{\psi}(\gamma^\mu\nabla_\mu - m)\psi$$

$$\Delta = \frac{3}{2} + m\ell$$

$$(4\pi)^2 a_4(x) = \frac{11}{360}E_4 + \frac{1}{20}W^2 + \frac{1}{72}\left(\Delta - \frac{3}{2}\right)^2\left(\left(\Delta - \frac{3}{2}\right)^2 - 2\right)R^2$$

$$\mathcal{L} = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu}$$

$$\mathcal{L}_{\text{g.f.}} = \frac{1}{2}(\nabla_\mu a^\mu)^2,$$

$$\mathcal{L} + \mathcal{L}_{\text{g.f.}} = a^\mu \,\Box\, a_\mu - a^\mu R_{\mu\nu} a^\nu$$

$$(4\pi)^2 a_4(x) = -\frac{31}{180}E_4 + \frac{1}{10}W^2$$

$$\mathcal{L}_{3/2} = -\bar{\psi}_\mu \gamma^{\mu\rho\nu}\nabla_\rho \psi_\nu$$

$$\mathcal{L}_{\text{g.f}} = -\frac{1}{2}(\bar{\psi}_\mu\gamma^\mu)\gamma^\rho\nabla_\rho(\gamma^\nu\psi_\nu)$$

$$\mathcal{L}_{3/2} + \mathcal{L}_{\text{g.f}} = \bar{\chi}_\mu\gamma^\nu D_\nu\chi^\mu$$

$$(4\pi)^2 a_4(x) = \frac{229}{720}E_4 - \frac{77}{120}W^2 - \frac{1}{9}R^2$$

$$a_{\text{E}} = \frac{1}{720}(2n_S + 22n_F + 124n_V - 229n_\psi)$$

$$c = \frac{1}{120}(n_S + 6n_F + 12n_V - 77n_\psi)$$

$$b_1 = \frac{1}{72}(n_S - 8n_\psi)$$

$$C_{\text{local}} = \frac{1}{180\ell^2(\ell^2 + 3r_+^2)}\left[\ell^4(4n_S + 14n_F - 52n_V - 233n_\psi)\right.$$
$$\left.+3r_+^2\ell^2(22n_S + 2n_F - 76n_V - 239n_\psi) + 18r_+^4(-3n_S + 2n_F + 4n_V + n_\psi)\right]$$

$$\lim_{\ell\to+\infty} C_{\text{local}} = \frac{1}{180}(4n_S + 14n_F - 52n_V - 233n_\psi)$$

$$S_{\text{EE}} = \frac{\text{Area}}{4G} + C\log\,\beta + \cdots$$

$$C = \frac{29}{180} + \frac{1}{24}(\Delta+1)\Delta(\Delta-3)(\Delta-4)$$

$$S = \int d^4x\sqrt{g}\left(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}\right)$$

$$\delta g_{\mu\nu} = \sqrt{2}h_{\mu\nu}, \delta A_\mu = \frac{1}{2}a_\mu$$

$$S = -\int d^4x\sqrt{\det g}\left\{\left(D^\mu h_{\mu\rho} - \frac{1}{2}D_\rho h\right)\left(D^\nu h_\nu^\rho - \frac{1}{2}D^\rho h\right) + \frac{1}{2}\left(D^\mu a_\mu\right)\left(D^\nu a_\nu\right)\right\}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 2F_{\mu\rho}F_\nu^\rho - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$
$$D^\mu F_{\mu\nu} = 0$$

$$D_{[\mu}F_{\nu\rho]} = 0$$
$$R_{\mu[\nu\rho\sigma]} = 0$$

$$\phi_m I^{mn}\phi_n = h_{\mu\nu}\left(\frac{1}{2}g^{\mu\alpha}g^{\nu\beta} + \frac{1}{2}g^{\mu\beta}g^{\nu\alpha} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)h_{\alpha\beta} + a_\alpha g^{\alpha\beta}a_\beta$$
$$\phi_m E^{mn}\phi_n = h_{\mu\nu}\left(R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - g^{\mu\nu}R^{\alpha\beta} - g^{\alpha\beta}R^{\mu\nu} + \Lambda g^{\mu\nu}g^{\alpha\beta}\right)h_{\alpha\beta}$$
$$+a_\alpha\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - \Lambda g^{\alpha\beta}\right)a_\beta + \frac{\sqrt{2}}{2}h_{\mu\nu}(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu})a_\alpha$$
$$+\frac{\sqrt{2}}{2}a_\alpha(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu})h_{\mu\nu}$$
$$\phi_m(\Omega^{\rho\sigma})^{mn}\phi_n = h_{\mu\nu}\left\{\frac{1}{2}\left(g^{\nu\beta}R^{\mu\alpha\rho\sigma} + g^{\nu\alpha}R^{\mu\beta\rho\sigma} + g^{\mu\beta}R^{\nu\alpha\rho\sigma} + g^{\mu\alpha}R^{\nu\beta\rho\sigma}\right)\right.$$
$$\left.+[\omega^\rho, \omega^\sigma]^{\mu\nu\alpha\beta}\right\}h_{\alpha\beta} + a_\alpha\left\{R^{\alpha\beta\rho\sigma} + [\omega^\rho, \omega^\sigma]^{\alpha\beta}\right\}a_\beta$$
$$+h_{\mu\nu}\left(D^{[\rho}\omega^{\sigma]}\right)^{\mu\nu\alpha}a_\alpha + a_\alpha\left(D^{[\rho}\omega^{\sigma]}\right)^{\alpha\mu\nu}h_{\mu\nu}$$

$$\phi_m(\omega^\rho)^{mn}\phi_n = \frac{\sqrt{2}}{2}h_{\mu\nu}(g^{\alpha\mu}F^{\rho\nu} + g^{\alpha\nu}F^{\rho\mu} - g^{\mu\rho}F^{\alpha\nu} - g^{\nu\rho}F^{\alpha\mu} - g^{\mu\nu}F^{\rho\alpha})a_\alpha$$
$$-\frac{\sqrt{2}}{2}a_\alpha(g^{\alpha\mu}F^{\rho\nu} + g^{\alpha\nu}F^{\rho\mu} - g^{\mu\rho}F^{\alpha\nu} - g^{\nu\rho}F^{\alpha\mu} - g^{\mu\nu}F^{\rho\alpha})h_{\mu\nu}$$

$$(4\pi)^2 a_4^{\mathrm{EM}}(x) = -\frac{277}{180}E_4 + \frac{38}{15}W^2 + \frac{7}{18}R^2.$$

$$S_{\text{ghost},b} = \frac{1}{2}\int d^4x\sqrt{g}\left\{2b_\mu(g^{\mu\nu}\,\Box + R^{\mu\nu})c_\nu + 2b\,\Box\, c - 4bF^{\rho\nu}D_\rho c_\nu\right\}$$

$$\phi_n E_m^n \phi^m = b_\mu\left(R_\nu^\mu\right)b^\nu + c_\mu\left(R_\nu^\mu\right)c^\nu$$
$$\phi_n\left(\Omega_{\alpha\beta}\right)_m^n\phi^m = b_\mu\left(R_{\nu\alpha\beta}^\mu\right)b^\nu + c_\mu\left(R_{\nu\alpha\beta}^\mu\right)c^\nu - \frac{1}{2}\left(b_\mu - ic_\mu\right)\left(D^\mu F_{\alpha\beta}\right)(b + ic)$$
$$+\frac{1}{2}(b + ic)\left(D_\nu F_{\alpha\beta}\right)(b^\nu - ic^\nu)$$

$$a_4^{\text{ghosts}}(x) = \frac{13}{36}E_4 - \frac{1}{4}W^2 - \frac{3}{4}R^2,$$

$$(4\pi)^2 a_4^{\mathrm{B}}(x) = -\frac{53}{45}E_4 + \frac{137}{60}W^2 - \frac{13}{36}R^2.$$

$$a_{\mathrm{E}}=\frac{53}{45}, c=\frac{137}{60}, b_1=-\frac{13}{36}, b_2=0.$$

$$C_{\text{local}} =-\frac{212}{45}+\frac{\ell_2^2}{120a\ell^2(\ell^2-a^2)}(629a^3-579a^2\ell+3699a\ell^2 \\ +411\ell^3-411\frac{(a+\ell)^4}{\sqrt{a\ell}}\arctan\left(\sqrt{a/\ell}\right)\Big)$$

$$\mathcal{L}_f =\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho+\frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\psi_\sigma-\frac{1}{2\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$$

$$D_\mu\psi_\nu=\nabla_\mu\psi_\nu-\frac{i}{\ell}A_\mu\psi_\nu$$

$$\mathcal{L}_{\text{g.f.}} =-\frac{1}{4}(\bar{\psi}_\mu\gamma^\mu)\gamma^\nu D_\nu(\gamma^\rho\psi_\rho)$$

$$\psi_\mu=\sqrt{2}\left(\chi_\mu-\frac{1}{2}\gamma_\mu\gamma^\nu\chi_\nu\right)$$

$$\mathcal{L}_f=g^{\mu\nu}\bar{\chi}_\mu\gamma^\rho D_\rho\chi_\nu+\frac{i}{2}F^{\mu\nu}\bar{\chi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma-\frac{1}{\ell}\bar{\chi}_\mu\gamma^{\mu\nu}\chi_\nu$$

$$\chi^\mu=\chi_1^\mu+i\chi_2^\mu,$$

$$D_\mu\chi_A^\nu=\left(\delta_{AB}\nabla^\mu+\frac{1}{\ell}\varepsilon_{AB}A_\mu\right)\chi_B^\nu,$$

$$\mathcal{L}_f=\delta_{AB}g_{\mu\nu}\bar{\chi}_A^\mu\gamma^\rho D_\rho\chi_B^\nu-\frac{1}{2}\varepsilon_{AB}F^{\mu\nu}\bar{\chi}_A^\rho\gamma^\mu\gamma_{\rho\sigma}\gamma^\nu\chi_B^\sigma-\frac{1}{\ell}\delta_{AB}\bar{\chi}_A^\mu\gamma_{\mu\nu}\chi_B^\nu$$

$$\chi_1^\mu =\frac{1}{2}(\lambda_1^\mu+\gamma^5\lambda_2^\mu)$$
$$\chi_2^\mu =\frac{i}{2}(\lambda_1^\mu-\gamma^5\lambda_2^\mu)$$

$$\mathcal{L}_f=\mathcal{L}_1+\mathcal{L}_2,$$

$$\mathcal{L}_1 =g_{\mu\nu}\bar{\lambda}_1^\mu\gamma^\rho(\nabla_\rho+i\ell^{-1}A_\rho)\lambda_1^\nu-\frac{i}{2}F^{\mu\nu}\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma-\frac{1}{\ell}\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu$$
$$\mathcal{L}_2 =g_{\mu\nu}\bar{\lambda}_2^\mu\gamma^\rho(\nabla_\rho-i\ell^{-1}A_\rho)\lambda_2^\nu-\frac{i}{2}F^{\mu\nu}\bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma+\frac{1}{\ell}\bar{\lambda}_2^\mu\gamma_{\mu\nu}\lambda_2^\nu$$

$$(4\pi)^2a_4^{\text{F, gravitini}}(x)=\frac{139}{90}E_4-\frac{32}{15}W^2-\frac{2}{9}R^2+\frac{8}{9}RF_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{L}_{bc}=\delta_{AB}\bar{b}_A\gamma_\mu\delta_c\psi_B^\mu$$

$$\mathcal{L}_{bc}=\delta_{AB}\bar{b}_A\left(\gamma^\mu D_\mu+\frac{2}{\ell}\right)c_B$$

$$(4\pi)^2a_4^{\text{F, ghosts}}(x)=\frac{11}{120}E_4-\frac{3}{20}W^2+\frac{2}{9}R^2-\frac{1}{6}RF_{\mu\nu}F^{\mu\nu},$$

$$(4\pi)^2 a_4^{\mathrm{F}}(x) = \frac{589}{360}E_4 - \frac{137}{60}W^2 + \frac{13}{18}RF_{\mu\nu}F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = \frac{11}{24}E_4 - \frac{13}{36}R^2 + \frac{13}{18}RF_{\mu\nu}F^{\mu\nu}$$

$$C_{\text{local}} = \frac{11}{6} + \frac{26\left[r_+\beta\left(r_+^4 - \ell^2(a^2+q_m^2)\right) - \pi\ell^2(r_+^4 - a^4)\right]}{3\pi\ell^2(\ell^2-a^2)(a^2+r_+^2)}.$$

$$C_{\text{local}} = \frac{11}{6} + \frac{26\ell_2^2[(a^2(a+\ell)+r_0^2(3a-\ell))(a^2(a-\ell)+r_0^2(3a+\ell)) + 2\ell^2 q_m^2(r_0^2-a^2)]}{3\ell^2(\ell^2-a^2)(a^2+r_0^2)^2},$$

$$\ell_2 = \ell\sqrt{\frac{a^2+r_0^2}{a^2+\ell^2+6r_0^2}}$$

$$C_{\text{local}} = \frac{11}{6} - \frac{26}{3}\frac{a(\ell^2-4a\ell-a^2)}{(\ell-a)(a^2+6a\ell+\ell^2)}$$

$$\frac{1}{(4\pi)^2}\int d^2x\sqrt{g}W^2 = \frac{3\ell_2^2}{2a\ell^2(\ell^2-a^2)}\left((\ell-a)(\ell^2+10a\ell+a^2) - \frac{(a+\ell)^4}{\sqrt{a\ell}}\arctan\left(\sqrt{a/\ell}\right)\right)$$

$$(4\pi)^2 a_4(x) = \mathrm{Tr}\left[\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{360}\left(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu}\right)\right],$$

$$S = \int d^4x\sqrt{g}\left(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}\right)$$

$$\delta g_{\mu\nu} = \sqrt{2}h_{\mu\nu}, \delta A_\mu = \frac{1}{2}a_\mu$$

$$S = -\int d^4x\sqrt{\det g}\left\{\left(D^\mu h_{\mu\rho} - \frac{1}{2}D_\rho h\right)\left(D^\nu h_\nu^\rho - \frac{1}{2}D^\rho h\right) + \frac{1}{2}\left(D^\mu a_\mu\right)\left(D^\nu a_\nu\right)\right\}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 2F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$
$$D^\mu F_{\mu\nu} = 0$$

$$D_{[\mu}F_{\nu\rho]} = 0$$
$$R_{\mu[\nu\rho\sigma]} = 0$$

$$\begin{aligned}\phi_m Q^{mn}\phi_n =& G^{\mu\nu\alpha\beta}h_{\mu\nu}\,\Box\, h_{\alpha\beta} + g^{\alpha\beta}a_\alpha\,\Box\, a_\beta - a_\alpha R^{\alpha\beta}a_\beta \\ &+ h_{\mu\nu}\left\{R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - \frac{1}{2}\left(g^{\mu\alpha}R^{\nu\beta} + g^{\mu\beta}R^{\nu\alpha}\right.\right. \\ &\left. + g^{\nu\alpha}R^{\mu\beta} + g^{\nu\beta}R^{\mu\alpha}\right) - 2\left(F^{\mu\alpha}F^{\nu\beta} + F^{\mu\beta}F^{\nu\alpha}\right) \\ &\left. + \frac{1}{2}\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right)\left(F_{\theta\varphi}F^{\theta\varphi}\right) + 2G^{\alpha\beta\mu\nu}\Lambda\right\}h_{\alpha\beta} \\ &- h_{\mu\nu}\left\{\frac{1}{4}\left(D_\rho K^\rho\right)^{\mu\nu\alpha} - \frac{1}{2}(K^\rho)^{\mu\nu\alpha}D_\rho\right\}a_\alpha \\ &- a_\alpha\left\{\frac{1}{4}\left(D_\rho K^\rho\right)^{\mu\nu\alpha} + \frac{1}{2}(K^\rho)^{\mu\nu\alpha}D_\rho\right\}h_{\mu\nu}\end{aligned}$$

$$(K^{\rho})^{\mu\nu\alpha} = 2\sqrt{2}(g^{\alpha\mu}F^{\rho\nu} + g^{\alpha\nu}F^{\rho\mu} - g^{\mu\rho}F^{\alpha\nu} - g^{\nu\rho}F^{\alpha\mu} - g^{\mu\nu}F^{\rho\alpha})$$

$$G^{\mu\nu\alpha\beta} = \frac{1}{2}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)$$

$$h_{\mu\nu}\left(2g^{\mu\alpha}g^{\nu\beta}\Lambda - g^{\mu\nu}g^{\alpha\beta}\Lambda\right)h_{\alpha\beta} = h_{\mu\nu}\left(2G^{\alpha\beta\mu\nu}\Lambda\right)h_{\alpha\beta}$$

$$\begin{aligned}
\phi_m I^{mn}\phi_n &= h_{\mu\nu}G^{\mu\nu\alpha\beta}h_{\alpha\beta} + a_\alpha g^{\alpha\beta}a_\beta \\
\phi_m(\omega^{\rho})^{mn}\phi_n &= \frac{1}{4}h_{\mu\nu}(K^{\rho})^{\mu\nu\alpha}a_\alpha - \frac{1}{4}a_\alpha(K^{\rho})^{\mu\nu\alpha}h_{\mu\nu} \\
\phi_m P^{mn}\phi_n &= h_{\mu\nu}\Big\{R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - \frac{1}{2}\left(g^{\mu\alpha}R^{\nu\beta} + g^{\mu\beta}R^{\nu\alpha}\right. \\
&\quad \left. + g^{\nu\alpha}R^{\mu\beta} + g^{\nu\beta}R^{\mu\alpha}\right) - 2\left(F^{\mu\alpha}F^{\nu\beta} + F^{\mu\beta}F^{\nu\alpha}\right) \\
&\quad + \frac{1}{2}\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right)\left(F_{\theta\varphi}F^{\theta\varphi}\right) \\
&\quad + 2g^{\mu\alpha}g^{\nu\beta}\Lambda - g^{\mu\nu}g^{\alpha\beta}\Lambda\Big\}h_{\alpha\beta} \\
&\quad - a_\alpha R^{\alpha\beta}a_\beta + \frac{\sqrt{2}}{2}h_{\mu\nu}\{D^{\mu}F^{\alpha\nu} + D^{\nu}F^{\alpha\mu}\}a_\alpha \\
&\quad + \frac{\sqrt{2}}{2}a_\alpha\{D^{\mu}F^{\alpha\nu} + D^{\nu}F^{\alpha\mu}\}h_{\mu\nu}
\end{aligned}$$

$$\phi_m\Omega_{\mu\nu}^{mn}\phi_n = \phi_m\left[D_\mu + \omega_\mu, D_\nu + \omega_\nu\right]\phi_n = \phi_m\left\{\left[D_\mu, D_\nu\right]^{mn} + D_{[\mu}\omega_{\nu]}^{mn} + \left[\omega_\mu, \omega_\nu\right]^{mn}\right\}\phi_n$$

$$\left[D_\mu \cdot D_\nu\right], \left(D^{\rho}\omega_{\rho}\right)^{mn} \text{ and } (\omega^{\rho})^{mp}\left(\omega_{\rho}\right)_p{}^n$$

$$\begin{aligned}
\phi_m[D^{\rho}, D^{\sigma}]^{mn}\phi_n &= h_{\mu\nu}[D^{\rho}, D^{\sigma}]h^{\mu\nu} + a_\alpha[D^{\rho}, D^{\sigma}]a^{\alpha} \\
&= h_{\mu\nu}\{g^{\nu\beta}R^{\mu\alpha\rho\sigma} + g^{\mu\beta}R^{\nu\alpha\rho\sigma}\}h_{\alpha\beta} + a_\alpha R^{\alpha\beta\rho\sigma}a_\beta \\
&= \frac{1}{2}h_{\mu\nu}\{g^{\nu\beta}R^{\mu\alpha\rho\sigma} + g^{\nu\alpha}R^{\mu\beta\rho\sigma} + g^{\mu\beta}R^{\nu\alpha\rho\sigma} + g^{\mu\alpha}R^{\nu\beta\rho\sigma}\}h_{\alpha\beta} \\
&\quad + a_\alpha R^{\alpha\beta\rho\sigma}a_\beta \\
\phi_m\left(D^{\rho}\omega_{\rho}\right)^{mn}\phi_n &= -\frac{\sqrt{2}}{2}h_{\mu\nu}(D^{\nu}F^{\alpha\mu} + D^{\mu}F^{\alpha\nu})a_\alpha + \frac{\sqrt{2}}{2}a_\alpha(D^{\nu}F^{\alpha\mu} + D^{\mu}F^{\alpha\nu})h_{\mu\nu} \\
\phi_m(\omega^{\rho})^{mp}\left(\omega_{\rho}\right)_p{}^n\phi_n &= \frac{1}{16}h_{\mu\nu}(K^{\rho})^{\mu\nu\alpha}\left(-K_{\rho}\right)^{\delta\gamma\beta}g_{\alpha\beta}h_{\delta\gamma} \\
&\quad + \frac{1}{16}a_\alpha(K^{\rho})^{\mu\nu\alpha}\left(-K_{\rho}\right)^{\delta\gamma\beta}G_{\mu\nu\delta\gamma}a_\beta
\end{aligned}$$

$$\begin{aligned}
\phi_m(\omega^{\rho})^{mp}\left(\omega_{\rho}\right)_p^n\phi_n &= h_{\mu\nu}\left(-2F^{\mu\delta}F^{\nu\gamma} - 2F^{\mu\gamma}F^{\nu\delta} + 2F^{\gamma\rho}F^{\delta}{}_{\rho}g^{\mu\nu} - F^{\nu\rho}F^{\delta}{}_{\rho}g^{\mu\gamma}\right. \\
&\quad - F^{\nu\rho}F^{\gamma}{}_{\rho}g^{\mu\delta} - F^{\mu\rho}F^{\delta}{}_{\rho}g^{\nu\gamma} - F^{\mu\rho}F^{\gamma}{}_{\rho}g^{\nu\delta} + 2F^{\mu\rho}F^{\nu}{}_{\rho}g^{\gamma\delta} \\
&\quad \left. - \frac{1}{2}F_{\rho\sigma}F^{\rho\sigma}g^{\mu\nu}g^{\gamma\delta}\right)h_{\gamma\delta} + a_\alpha\left(-2F^{\alpha\gamma}F^{\beta}{}_{\gamma} - F_{\theta\phi}F^{\theta\phi}g^{\alpha\beta}\right)a_\beta
\end{aligned}$$

$$\begin{aligned}\phi_m(\omega^\rho)^{mp}\left(\omega_\rho\right)_p^n\phi_n =& h_{\mu\nu}\Big(-2F^{\mu\alpha}F^{\nu\beta}-2F^{\mu\beta}F^{\nu\alpha}-\frac{1}{2}F_{ad}F^{ad}g^{\mu\alpha}g^{\nu\beta}-\frac{1}{2}F_{ad}F^{ad}g^{\mu\beta}g^{\nu\alpha}\\&+\frac{1}{2}F_{ad}F^{ad}g^{\mu\nu}g^{\beta\alpha}+g^{\beta\alpha}R^{\mu\nu}-\frac{1}{2}g^{\nu\alpha}R^{\mu\beta}-\frac{1}{2}g^{\nu\beta}R^{\mu\alpha}-\frac{1}{2}g^{\mu\alpha}R^{\nu\beta}\\&-\frac{1}{2}g^{\mu\beta}R^{\nu\alpha}+g^{\mu\nu}R^{\beta\alpha}+\Lambda g^{\mu\alpha}g^{\nu\beta}+\Lambda g^{\mu\beta}g^{\nu\alpha}-2\Lambda g^{\mu\nu}g^{\beta\alpha}\Big)h_{\alpha\beta}\\&+\frac{1}{2}a_\alpha\left(-\frac{3}{2}F_{\theta\phi}F^{\theta\phi}g^{\alpha\beta}-R^{\alpha\beta}+\Lambda g^{\alpha\beta}\right)a_\beta\end{aligned}$$

$$\begin{aligned}\phi_m E^{mn}\phi_n =& h_{\mu\nu}\left(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+\Lambda g^{\mu\nu}g^{\alpha\beta}\right)h_{\alpha\beta}\\&+a_\alpha\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)a_\beta+\frac{\sqrt{2}}{2}h_{\mu\nu}(D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu})a_\alpha\\&+\frac{\sqrt{2}}{2}a_\alpha(D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu})h_{\mu\nu}\\\phi_m(\Omega^{\rho\sigma})^{mn}\phi_n =& h_{\mu\nu}\Big\{\frac{1}{2}\left(g^{\nu\beta}R^{\mu\alpha\rho\sigma}+g^{\nu\alpha}R^{\mu\beta\rho\sigma}+g^{\mu\beta}R^{\nu\alpha\rho\sigma}+g^{\mu\alpha}R^{\nu\beta\rho\sigma}\right)\\&+[\omega^\rho,\omega^\sigma]^{\mu\nu\alpha\beta}\Big\}h_{\alpha\beta}+a_\alpha\left\{R^{\alpha\beta\rho\sigma}+[\omega^\rho,\omega^\sigma]^{\alpha\beta}\right\}a_\beta\\&+h_{\mu\nu}\left(D^{[\rho}\omega^{\sigma]}\right)^{\mu\nu\alpha}a_\alpha+a_\alpha\left(D^{[\rho}\omega^{\sigma]}\right)^{\alpha\mu\nu}h_{\mu\nu}\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(RE) =& \mathrm{Tr}R\left[R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+\Lambda g^{\mu\nu}g^{\alpha\beta}\right]\\&+\mathrm{Tr}\left(\frac{3}{2}g^{\alpha\beta}RF_{\mu\nu}F^{\mu\nu}-R\Lambda g^{\alpha\beta}\right)\\=& R\left(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+g^{\mu\nu}g^{\alpha\beta}\Lambda\right)G_{\mu\nu\alpha\beta}\\&+R\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)g_{\alpha\beta}\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(RE) &= R\left(-2R+(D-2)R+\left(D-\frac{D^2}{2}\right)\Lambda\right)+R\left(\frac{3D}{2}F_{\mu\nu}F^{\mu\nu}-d\Lambda\right)\\&=-32\Lambda^2+6RF_{\mu\nu}F^{\mu\nu}\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(E^2) =& \mathrm{Tr}\left(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+g^{\mu\nu}g^{\alpha\beta}\Lambda\right)\\&\times\left(R^{\rho\tau\sigma\delta}+R^{\rho\delta\sigma\tau}-g^{\rho\sigma}R^{\tau\delta}-g^{\tau\delta}R^{\rho\sigma}+g^{\rho\sigma}g^{\tau\delta}\Lambda\right)\\&+\mathrm{Tr}\left(\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)\left(\frac{3}{2}g^{\tau\delta}F_{\theta\phi}F^{\theta\phi}-\Lambda g^{\tau\delta}\right)\right)\\&+\frac{1}{2}\mathrm{Tr}\left((D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu})(D^\rho F^{\beta\sigma}+D^\sigma F^{\beta\rho})\right)\\&+\frac{1}{2}\mathrm{Tr}\left((D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu})(D^\rho F^{\beta\sigma}+D^\sigma F^{\beta\rho})\right)\end{aligned}$$

$$\begin{aligned}&\left(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+g^{\mu\nu}g^{\alpha\beta}\Lambda\right)\\&\times\left(R^{\rho\tau\sigma\delta}+R^{\rho\delta\sigma\tau}-g^{\rho\sigma}R^{\tau\delta}-g^{\tau\delta}R^{\rho\sigma}+g^{\rho\sigma}g^{\tau\delta}\Lambda\right)G_{\mu\nu\rho\sigma}G_{\alpha\beta\tau\delta}\\&=16\Lambda^2-4R_{ab}R^{ab}+3R_{abcd}R^{abcd}\end{aligned}$$

$$\mathrm{Tr}\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)\left(\frac{3}{2}g^{\tau\delta}F_{\theta\phi}F^{\theta\phi}-\Lambda g^{\tau\delta}\right)=4\Lambda^2-12\Lambda F_{\alpha\beta}F^{\alpha\beta}+9\left(F_{\theta\phi}F^{\theta\phi}\right)^2,$$

$$\left(D_\rho F_{\mu\nu}\right)\left(D^\rho F^{\mu\nu}\right)= R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + \frac{1}{2}R^2 - \Lambda R - \frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma},$$

$$\left(D_\mu F_\rho{}^\nu\right)\left(D_\nu F^{\rho\mu}\right) = \frac{1}{2}\left(D_\rho F_{\mu\nu}\right)\left(D^\rho F^{\mu\nu}\right).$$

$$\begin{aligned} X \equiv & \frac{1}{2}\mathrm{Tr}\left(\left(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right)\left(D^\rho F^{\beta\sigma} + D^\sigma F^{\beta\rho}\right)\right) \\ & + \frac{1}{2}\mathrm{Tr}\left(\left(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right)\left(D^\rho F^{\beta\sigma} + D^\sigma F^{\beta\rho}\right)\right) \\ = & \left(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right)\left(D^\rho F^{\beta\sigma} + D^\sigma F^{\beta\rho}\right) g_{\alpha\beta}G_{\mu\nu\rho\sigma} \end{aligned}$$

$$X = 2\left(D_\rho F_{\alpha\sigma}\right)\left(D^\rho F^{\alpha\sigma}\right) + 2\left(D_\rho F_{\alpha\sigma}\right)\left(D^\sigma F^{\alpha\rho}\right).$$

$$\begin{aligned} X = & 2\left(-R^{\mu\nu}R_{\mu\nu} + \frac{1}{2}R^2 - \Lambda R - \frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha}\right) \\ & + \left(R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + \frac{1}{2}R^2 - \Lambda R - \frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha}\right) \\ = & 3\left(-R^{\mu\nu}R_{\mu\nu} + \frac{1}{2}R^2 - \Lambda R - \frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha}\right) \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}E^2 = & \left(16\Lambda^2 - 4R_{ab}R^{ab} + 3R_{abcd}R^{abcd}\right) + \left(4\Lambda^2 - 12\Lambda F_{\alpha\beta}F^{\alpha\beta} + 9\left(F_{\theta\phi}F^{\theta\phi}\right)^2\right) \\ & + 3\left(-R^{\mu\nu}R_{\mu\nu} + \frac{1}{2}R^2 - \Lambda R - \frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha}\right) \\ = & 32\Lambda^2 - 7R_{\mu\nu}R^{\mu\nu} + 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 18\Lambda F_{\rho\sigma}F^{\rho\sigma} + 3R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha} + 9\left(F_{\theta\phi}F^{\theta\phi}\right)^2. \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}(I) &= 14\,, \\ \mathrm{Tr}(RE) &= -32\Lambda^2 + 6RF_{\mu\nu}F^{\mu\nu} \\ \mathrm{Tr}(E^2) &= 32\Lambda^2 - 7R_{\mu\nu}R^{\mu\nu} + 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 18\Lambda F_{\rho\sigma}F^{\rho\sigma} \\ &\quad + 3R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + 9\left(F_{\theta\phi}F^{\theta\phi}\right)^2 \\ \mathrm{Tr}(\Omega^2) &= -224\Lambda^2 + 60\Lambda F_{\mu\nu}F^{\mu\nu} - 54F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + 56R_{\mu\nu}R^{\mu\nu} \\ &\quad - 18F^{\mu\nu}F^{\rho\sigma}R_{\mu\nu\rho\sigma} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \end{aligned}$$

$$(4\pi)^2 a_4^{\mathrm{EM}}(x) = -\frac{880}{180}\Lambda^2 + \frac{196}{180}R_{\mu\nu}R^{\mu\nu} + \frac{179}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

$$\mathcal{S}_{\mathrm{ghost},b} = \frac{1}{2}\int d^4x\sqrt{g}\left\{2b_\mu(g^{\mu\nu}\,\Box + R^{\mu\nu})c_\nu + 2b\,\Box\, c - 4bF^{\rho\nu}D_\rho c_\nu\right\}$$

$$\begin{aligned} \phi_n E_m^n \phi^m = & b_\mu\left(R_\nu^\mu\right)b^\nu + c_\mu\left(R_\nu^\mu\right)c^\nu \\ \phi_n\left(\Omega_{\alpha\beta}\right)_m^n\phi^m = & b_\mu\left(R_{\nu\alpha\beta}^\mu\right)b^\nu + c_\mu\left(R_{\nu\alpha\beta}^\mu\right)c^\nu - \frac{1}{2}\left(b_\mu - ic_\mu\right)\left(D^\mu F_{\alpha\beta}\right)(b + ic) \\ & + \frac{1}{2}(b + ic)\left(D_\nu F_{\alpha\beta}\right)\left(b^\nu - ic^\nu\right) \end{aligned}$$

$$a_4^{\text{ghosts, EM}}(x) = \frac{13}{36}E_4 - \frac{1}{4}W^2 - \frac{3}{4}R^2$$

$$(4\pi)^2 a_4^{\mathrm{B}}(x) = -\frac{53}{45}E_4 + \frac{137}{60}W^2 - \frac{13}{36}R^2$$

$$a_{\mathrm{E}}=\frac{53}{45}, c=\frac{137}{60}, b_1=-\frac{13}{36}, b_2=0$$

$$\phi_m E^{mn}\phi_n = h_{\mu\nu}\left(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+\Lambda g^{\mu\nu}g^{\alpha\beta}\right)h_{\alpha\beta}$$

$$\begin{aligned}\mathrm{Tr}(RE) &= -16\Lambda^2\\ \mathrm{Tr}(E^2) &= 16\Lambda^2-4R_{\mu\nu}R^{\mu\nu}+3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\\ \mathrm{Tr}(I) &= 10\end{aligned}$$

$$\begin{aligned}180(4\pi)^2 a_4^{\text{bulk},\Lambda} =& \frac{1}{2}\left[60(-16\Lambda^2)+180\left(16\Lambda^2-4R_{\mu\nu}R^{\mu\nu}+3R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\right)+30(-6)R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\right.\\ &\left.+10\left(5R^2+2R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}-2R_{\mu\nu}R^{\mu\nu}\right)\right]\\ =& -120\Lambda^2+190R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\end{aligned}$$

$$\phi_n\left(\Omega_{\alpha\beta}\right)_m^n\phi^m = b_\mu\left(R^\mu_{\nu\alpha\beta}\right)b^\nu + c_\mu\left(R^\mu_{\nu\alpha\beta}\right)c^\nu$$

$$\mathrm{Tr}(I)=\mathrm{Tr}(g^{\mu\nu})+\mathrm{Tr}(g^{\mu\nu})=8$$

$$\begin{aligned}-(4\pi)^2 a_4^{\text{ghost},\mathrm{B}}(x) =& \frac{1}{3}R^2+R_{\mu\nu}R^{\mu\nu}-\frac{1}{6}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\left(\frac{1}{9}R^2+\frac{2}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-\frac{2}{45}R_{\mu\nu}R^{\mu\nu}\right)\\ =& \frac{4}{9}R^2+\frac{43}{45}R_{\mu\nu}R^{\mu\nu}-\frac{43}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\\ =& \frac{11}{90}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-\frac{164}{15}\Lambda^2\end{aligned}$$

$$180(4\pi)^2 a_4^{\Lambda}(x) = 212R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-2088\Lambda^2$$

$$\mathcal{L}=\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho+\frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\psi_\sigma-\frac{1}{2\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$$

$$\psi_\mu=\psi_\mu^1+i\psi_\mu^2$$

$$\gamma^0=\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}, \gamma^1=\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}, \gamma^2=\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}, \gamma^3=\begin{pmatrix}0 & \sigma_3\\ \sigma_3 & 0\end{pmatrix}$$

$$\psi^*=\psi$$

$$\mathcal{L}_{\text{g.f.}}=-\frac{1}{4}\left(\bar{\psi}_\mu\gamma^\mu\right)\gamma^\nu D_\nu\left(\gamma^\rho\psi_\rho\right)$$

$$\psi_\mu=\sqrt{2}\left(\chi_\mu-\frac{1}{2}\gamma_\mu\gamma^\nu\chi_\nu\right)$$

$$\mathcal{L}_{\text{Fermi}}+\mathcal{L}_{\text{g.f.}}=\mathcal{L}_{\text{kin}}+\mathcal{L}_{\bar{\psi}F\psi}+\mathcal{L}_{\bar{\psi}\psi}$$

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= \frac{1}{2}g^{\mu\nu}\bar{\psi}_\mu\gamma^\rho\nabla_\rho\psi_\nu\\ \mathcal{L}_{\bar{\psi}F\psi} &= \frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\psi_\sigma\\ \mathcal{L}_{\bar{\psi}\psi} &= -\frac{1}{2\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\end{aligned}$$

$$\bar{\chi}_\mu \gamma^\nu D_\nu \chi^\mu = \frac{1}{2}\bar{\psi}_\mu \left(\gamma^{\mu\nu\rho} D_\nu - \frac{1}{2}\gamma^\mu \gamma^\nu \gamma^\rho D_\nu\right)\psi_\rho$$

$$\mathcal{L}_{\bar{\psi}\psi} = -\frac{1}{2\ell}\bar{\psi}_\mu \gamma^{\mu\nu}\psi_\nu$$
$$= -\frac{1}{\ell}\left(\bar{\chi}_\mu - \frac{1}{2}\bar{\chi}_\rho \gamma^\rho \gamma_\mu\right)\gamma^{\mu\nu}\left(\chi_\nu - \frac{1}{2}\gamma_\nu \gamma^\sigma \chi_\sigma\right)$$
$$= -\frac{1}{\ell}\left(\bar{\chi}_\mu \gamma^{\mu\nu}\chi_\nu - \frac{1}{2}\bar{\chi}_\rho \gamma^\rho \gamma_\mu \gamma^{\mu\nu}\chi_\nu - \frac{1}{2}\bar{\chi}_\mu \gamma^{\mu\nu}\gamma_\nu \gamma^\sigma \chi_\sigma + \frac{1}{4}\bar{\chi}_\rho \gamma^\rho \gamma_\mu \gamma^{\mu\nu}\gamma_\nu \gamma^\sigma \chi_\sigma\right)$$
$$= -\frac{1}{\ell}\left(\bar{\chi}_\mu \gamma^{\mu\nu}\chi_\nu - 3\bar{\chi}_\mu \gamma^\mu \gamma^\nu \chi_\nu + 3\bar{\chi}_\rho \gamma^\rho \gamma^\sigma \chi_\sigma\right)$$
$$= -\frac{1}{\ell}\bar{\chi}_\mu \gamma^{\mu\nu}\chi_\nu$$

$$\mathcal{L}_{\bar{\psi}F\psi} = \frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \psi_\sigma$$
$$= \frac{i}{2}F^{\mu\nu}\left(\bar{\chi}_\rho - \frac{1}{2}\bar{\chi}_\alpha \gamma^\alpha \gamma_\rho\right)\gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \left(\chi_\sigma - \frac{1}{2}\gamma_\sigma \gamma^\beta \chi_\beta\right)$$
$$= \frac{i}{2}F^{\mu\nu}\left(\bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \chi_\sigma - \frac{1}{2}\bar{\chi}_\alpha \gamma^\alpha \gamma_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \chi_\sigma - \frac{1}{2}\bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \gamma_\sigma \gamma^\beta \chi_\beta\right.$$
$$\left. + \frac{1}{4}\bar{\chi}_\alpha \gamma^\alpha \gamma_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \gamma_\sigma \gamma^\beta \chi_\beta\right)$$

$$\gamma^\mu \gamma_\nu \gamma_\mu = -2\gamma_\nu$$
$$\gamma_\rho \gamma_\mu \gamma^{\rho\sigma} = \gamma_\rho \gamma_\mu (\gamma^\rho \gamma^\sigma - g^{\rho\sigma}) = -\gamma_\mu \gamma^\sigma - 2\delta_\mu^\sigma$$
$$\gamma_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \gamma_\sigma = (-\gamma_\mu \gamma^\sigma - 2\delta_\mu^\sigma)\gamma_\nu \gamma_\sigma = 4\gamma_{\mu\nu}$$

$$-\frac{1}{2}\bar{\chi}_\alpha \gamma^\alpha \gamma_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \chi_\sigma = \frac{1}{2}\bar{\chi}_\alpha \gamma^\alpha (\gamma_\mu \gamma^\sigma + 2\delta_\mu^\sigma)\gamma_\nu \chi_\sigma$$
$$= \frac{1}{2}\bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \gamma^\sigma \gamma_\nu \chi_\sigma + \bar{\chi}_\alpha \gamma^\alpha \gamma_\nu \chi_\mu$$
$$= \frac{1}{2}\bar{\chi}_\alpha \gamma^\alpha \gamma_\mu (2\delta_\nu^\sigma - \gamma_\nu \gamma^\sigma)\chi_\sigma + \bar{\chi}_\alpha \gamma^\alpha \gamma_\nu \chi_\mu$$
$$= \bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \chi_\nu - \frac{1}{2}\bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \gamma_\nu \gamma^\sigma \chi_\sigma + \bar{\chi}_\alpha \gamma^\alpha \gamma_\nu \chi_\mu$$

$$\mathcal{L}_{\bar{\psi}F\psi} = \frac{i}{2}F^{\mu\nu}(\bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \chi_\sigma - \bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \gamma_\nu \gamma^\sigma \chi_\sigma + \bar{\chi}_\alpha \gamma^\alpha \gamma_{\mu\nu}\gamma^\beta \chi_\beta)$$
$$= \frac{i}{2}F^{\mu\nu}\bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \chi_\sigma$$

$$\mathcal{L}_f = g^{\mu\nu}\bar{\chi}_\mu \gamma^\rho D_\rho \chi_\nu + \frac{i}{2}F^{\mu\nu}\bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \chi_\sigma - \frac{1}{\ell}\bar{\chi}_\mu \gamma^{\mu\nu}\chi_\nu.$$

$$\chi_\mu = \chi_\mu^1 + i\chi_\mu^2,$$

$$\bar{\lambda}\gamma_{\mu_1 \dots \mu_r}\chi = t_r \bar{\chi}\gamma_{\mu_1 \dots \mu_r}\lambda,$$

$$t_r = \begin{cases} -1 & r = 1,2 \bmod 4 \\ 1 & r = 3,4 \bmod 4 \end{cases}.$$

$$\bar{\lambda}\gamma^{\mu_1}\gamma^{\mu_2} \dots \gamma^{\mu_p}\chi = (-1)^p \bar{\chi}\gamma^{\mu_p} \dots \gamma^{\mu_2}\gamma^{\mu_1}\lambda.$$

$$\bar{\chi}_\mu^1 \gamma^\rho D_\rho \chi_\nu^2 = -D_\rho \bar{\chi}_\nu^2 \gamma^\rho \chi_\mu^1 = \bar{\chi}_\nu^2 \gamma^\rho D_\rho \chi_\mu^1$$

$$\mathcal{L}_{\text{kin}} = \delta_{AB} g^{\mu\nu} \bar{\chi}_\mu^A \gamma^\rho D_\rho \chi_\nu^B$$

$$F^{\mu\nu} \bar{\psi}_\rho^1 \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \psi_\sigma^1 = F^{\mu\nu} \left(\gamma_\mu \psi_\rho^1\right)^\dagger \gamma^{\rho\sigma} (\gamma_\nu \psi_\sigma^1) = -F^{\mu\nu} (\gamma_\nu \psi_\sigma^1)^\dagger \gamma^{\rho\sigma} \left(\gamma_\mu \psi_\rho^1\right) = -F^{\mu\nu} \bar{\psi}_\rho^1 \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \psi_\sigma^1 = 0$$

$$\mathcal{L}_{\bar{\psi} F \psi} = -\frac{1}{2} \varepsilon_{AB} F^{\mu\nu} \bar{\chi}_\rho^A \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma^B$$

$$\bar{\chi}_\mu^1 \gamma^{\mu\nu} \chi_\nu^2 = -\bar{\chi}_\nu^2 \gamma^{\mu\nu} \chi_\mu^1 = \bar{\chi}_\mu^2 \gamma^{\mu\nu} \chi_\nu^1$$

$$\mathcal{L}_{\bar{\psi}\psi} = -\frac{1}{\ell} \delta_{AB} \bar{\chi}_\mu^A \gamma^{\mu\nu} \chi_\nu^B$$

$$\mathcal{L}_f + \mathcal{L}_{\text{g.f.}} = \delta_{AB} g^{\mu\nu} \bar{\chi}_\mu^A \gamma^\rho D_\rho \chi_\nu^B - \frac{1}{2} \varepsilon_{AB} F^{\mu\nu} \bar{\chi}_\rho^A \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma^B - \frac{1}{\ell} \delta_{AB} \bar{\chi}_\mu^A \gamma^{\mu\nu} \chi_\nu^B$$

$$(\gamma^\mu)^\dagger = (-1)^t A \gamma^\mu A^{-1}$$

$$(\gamma^\mu)^t = t_0 t_1 C \gamma^\mu C^{-1}$$

$$(\gamma^\mu)^* = -t_0 t_1 B \gamma^\mu B^{-1}$$

$$B = (CA^{-1})^t$$

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

$$(\gamma^5)^2 = 1, (\gamma^5)^\dagger = \gamma^5, (\gamma^5)^t = (\gamma^5)^* = -\gamma^5$$

$$\psi^* = B_+ \psi = \psi$$

$$B_- = (C_- A^{-1})^t = \gamma^5$$

$$\left(\lambda_A^\mu\right)^* = B_- \varepsilon_{AB} \lambda_B^\mu$$

$$\lambda_1^\mu = \chi_1^\mu - i\chi_\mu^2$$
$$\lambda_2^\mu = \gamma^5 \left(\chi_1^\mu + i\chi_2^\mu\right)$$

$$\chi_1^\mu = \frac{1}{2}\left(\lambda_1^\mu + \gamma^5 \lambda_2^\mu\right)$$
$$\chi_2^\mu = \frac{i}{2}\left(\lambda_1^\mu - \gamma^5 \lambda_2^\mu\right)$$

$$\bar{\chi}_1^\mu = \frac{1}{2}\left(\bar{\lambda}_1^\mu - \bar{\lambda}_2^\mu \gamma^5\right)$$
$$\bar{\chi}_2^\mu = -\frac{i}{2}\left(\bar{\lambda}_1^\mu + \bar{\lambda}_2^\mu \gamma^5\right)$$

$$\mathcal{L}_f + \mathcal{L}_{\text{g.f.}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\chi F \chi} + \mathcal{L}_{\chi\chi}$$

$$\mathcal{L}_{\text{kin}} = \delta_{AB} g_{\mu\nu} \bar{\chi}_A^\mu \gamma^\rho D_\rho \chi_B^\nu$$
$$\mathcal{L}_{\chi F \chi} = -\frac{1}{2}\varepsilon_{AB} F^{\mu\nu} \bar{\chi}_A^\rho \gamma_\mu \gamma_{\rho\sigma} \gamma_\nu \chi_B^\sigma$$
$$\mathcal{L}_{\chi\chi} = -\frac{1}{\ell}\delta_{AB} \bar{\chi}_A^\mu \gamma_{\mu\nu} \chi_B^\nu$$

$$g_{\mu\nu}\bar{\chi}_1^\mu \gamma^\rho \nabla_\rho \chi_1^\nu = \frac{1}{4} g_{\mu\nu}\left(\bar{\lambda}_1^\mu - \bar{\lambda}_2^\mu \gamma^5\right)\gamma^\rho \nabla_\rho \left(\lambda_1^\mu + \gamma^5 \lambda_2^\mu\right)$$
$$= \frac{1}{4} g_{\mu\nu}\left(\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu - \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu + \bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu - \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \lambda_1^\mu\right)$$
$$= \frac{1}{4} g_{\mu\nu}\left(\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu + \bar{\lambda}_2^\mu \gamma^\rho \nabla_\rho \lambda_2^\mu + \bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu - \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \lambda_1^\mu\right)$$

$$g_{\mu\nu}\bar{\chi}_2^\mu \gamma^\rho \nabla_\rho \chi_2^\nu = \frac{1}{4} g_{\mu\nu}\left(\bar{\lambda}_1^\mu + \bar{\lambda}_2^\mu \gamma^5\right)\gamma^\rho \nabla_\rho \left(\lambda_1^\mu - \gamma^5 \lambda_2^\mu\right)$$
$$= \frac{1}{4} g_{\mu\nu}\left(\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu - \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu - \bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu + \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \lambda_1^\mu\right)$$
$$= \frac{1}{4} g_{\mu\nu}\left(\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu + \bar{\lambda}_2^\mu \gamma^\rho \nabla_\rho \lambda_2^\mu - \bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu + \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \lambda_1^\mu\right)$$

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} g_{\mu\nu}\left(\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu + \bar{\lambda}_2^\mu \gamma^\rho \nabla_\rho \lambda_2^\mu\right)$$

$$D_\mu \chi_A^\nu = \left(\delta_{AB}\nabla_\mu + \frac{1}{\ell}\varepsilon_{AB} A_\mu\right)\chi_B^\nu,$$

$$D_\mu \chi_1^\nu = \nabla_\mu \chi_1^\nu + \frac{1}{\ell} A_\mu \chi_2^\nu$$
$$D_\mu \chi_2^\nu = \nabla_\mu \chi_2^\nu - \frac{1}{\ell} A_\mu \chi_1^\nu$$

$$D_\mu \lambda_1^\nu = D_\mu(\chi_1^\nu - i\chi_2^\nu)$$
$$= \nabla_\mu \chi_1^\nu + \frac{1}{\ell} A_\mu \chi_2^\nu - i\left(\nabla_\mu \chi_2^\nu - \frac{1}{\ell} A_\mu \chi_1^\nu\right)$$
$$= \nabla_\mu \lambda_1^\nu + \frac{i}{\ell} A_\mu \lambda_1^\nu$$

$$D_\mu \lambda_2^\nu = \gamma^5 D_\mu(\chi_1^\nu + i\chi_2^\nu)$$
$$= \gamma^5\left(\nabla_\mu \chi_1^\nu + \frac{1}{\ell} A_\mu \chi_2^\nu\right) + i\gamma^5\left(\nabla_\mu \chi_2^\nu - \frac{1}{\ell} A_\mu \chi_1^\nu\right)$$
$$= \nabla_\mu \lambda_2^\nu - \frac{i}{\ell} A_\mu \lambda_2^\nu$$

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} g_{\mu\nu}\left(\bar{\lambda}_1^\mu \gamma^\rho D_\rho \lambda_1^\mu + \bar{\lambda}_2^\mu \gamma^\rho D_\rho \lambda_2^\mu\right)$$

$$\mathcal{L}_{\chi\chi} = -\frac{1}{\ell}\delta_{AB}\bar{\chi}_A^{\mu}\gamma_{\mu\nu}\chi_B^{\nu}$$

$$\bar{\chi}_1^{\mu}\gamma_{\mu\nu}\chi_1^{\nu} = \frac{1}{4}\left(\bar{\lambda}_1^{\mu} - \bar{\lambda}_2^{\mu}\gamma^5\right)\gamma_{\mu\nu}\left(\lambda_1^{\nu} + \gamma^5\lambda_2^{\nu}\right)$$
$$= \frac{1}{4}\left(\bar{\lambda}_1^{\mu}\gamma_{\mu\nu}\lambda_1^{\nu} - \bar{\lambda}_2^{\mu}\gamma^5\gamma_{\mu\nu}\gamma^5\lambda_2^{\nu} + \bar{\lambda}_1^{\mu}\gamma_{\mu\nu}\gamma^5\lambda_2^{\nu} - \bar{\lambda}_2^{\mu}\gamma^5\gamma_{\mu\nu}\lambda_1^{\nu}\right)$$
$$= \frac{1}{4}\left(\bar{\lambda}_1^{\mu}\gamma_{\mu\nu}\lambda_1^{\nu} - \bar{\lambda}_2^{\mu}\gamma_{\mu\nu}\lambda_2^{\nu} + \bar{\lambda}_1^{\mu}\gamma_{\mu\nu}\gamma^5\lambda_2^{\nu} - \bar{\lambda}_2^{\mu}\gamma^5\gamma_{\mu\nu}\lambda_1^{\nu}\right)$$

$$\bar{\chi}_2^{\mu}\gamma_{\mu\nu}\chi_2^{\nu} = \frac{1}{4}\left(\bar{\lambda}_1^{\mu} + \bar{\lambda}_2^{\mu}\gamma^5\right)\gamma_{\mu\nu}\left(\lambda_1^{\nu} - \gamma^5\lambda_2^{\nu}\right)$$
$$= \frac{1}{4}\left(\bar{\lambda}_1^{\mu}\gamma_{\mu\nu}\lambda_1^{\nu} - \bar{\lambda}_2^{\mu}\gamma_{\mu\nu}\lambda_2^{\nu} - \bar{\lambda}_1^{\mu}\gamma_{\mu\nu}\gamma^5\lambda_2^{\nu} + \bar{\lambda}_2^{\mu}\gamma^5\gamma_{\mu\nu}\lambda_1^{\nu}\right)$$

$$\mathcal{L}_{\chi\chi} = -\frac{1}{2\ell}\left(\bar{\lambda}_1^{\mu}\gamma_{\mu\nu}\lambda_1^{\nu} - \bar{\lambda}_2^{\mu}\gamma_{\mu\nu}\lambda_2^{\nu}\right)$$

$$F^{\mu\nu}\bar{\chi}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\chi_2^{\sigma}$$
$$= \frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^{\rho} - \bar{\lambda}_2^{\rho}\gamma^5\right)\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\left(\lambda_1^{\sigma} - \gamma^5\lambda_2^{\sigma}\right)$$
$$= \frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma} + \bar{\lambda}_2^{\rho}\gamma^5\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\gamma^5\lambda_2^{\sigma} - \bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\gamma^5\lambda_2^{\sigma} - \bar{\lambda}_2^{\rho}\gamma^5\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma}\right)$$
$$= \frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma} + \bar{\lambda}_2^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_2^{\sigma} - \bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\gamma^5\lambda_2^{\sigma} - \bar{\lambda}_2^{\rho}\gamma^5\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma}\right)$$

$$F^{\mu\nu}\bar{\chi}_2^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\chi_1^{\sigma}$$
$$= -\frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^{\rho} + \bar{\lambda}_2^{\rho}\gamma^5\right)\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\left(\lambda_1^{\sigma} + \gamma^5\lambda_2^{\sigma}\right)$$
$$= -\frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma} + \bar{\lambda}_2^{\rho}\gamma^5\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\gamma^5\lambda_2^{\sigma} + \bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\gamma^5\lambda_2^{\sigma} + \bar{\lambda}_2^{\rho}\gamma^5\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma}\right)$$
$$= -\frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma} + \bar{\lambda}_2^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_2^{\sigma} + \bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\gamma^5\lambda_2^{\sigma} + \bar{\lambda}_2^{\rho}\gamma^5\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma}\right)$$

$$\mathcal{L}_{\chi F\chi} = -\frac{1}{2}\varepsilon_{AB}F^{\mu\nu}\bar{\chi}_A^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\chi_B^{\sigma} = -\frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma} + \bar{\lambda}_2^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_2^{\sigma}\right)$$

$$\mathcal{L}_f = \frac{1}{2}\delta_{AB}g_{\mu\nu}\bar{\lambda}_A^{\mu}\gamma^{\rho}D_{\rho}\lambda_A^{\nu} - \frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma} + \bar{\lambda}_2^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_2^{\sigma}\right) - \frac{1}{2\ell}\left(\bar{\lambda}_1^{\mu}\gamma_{\mu\nu}\lambda_1^{\nu} - \bar{\lambda}_2^{\mu}\gamma_{\mu\nu}\lambda_2^{\nu}\right)$$

$$\mathcal{L}_f = \mathcal{L}_1 + \mathcal{L}_2,$$

$$\mathcal{L}_1 = g_{\mu\nu}\bar{\lambda}_1^{\mu}\gamma^{\rho}\left(\nabla_{\rho} + i\ell^{-1}A_{\rho}\right)\lambda_1^{\nu} - \frac{i}{2}F^{\mu\nu}\bar{\lambda}_1^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_1^{\sigma} - \frac{1}{\ell}\bar{\lambda}_1^{\mu}\gamma_{\mu\nu}\lambda_1^{\nu}$$
$$\mathcal{L}_2 = g_{\mu\nu}\bar{\lambda}_2^{\mu}\gamma^{\rho}\left(\nabla_{\rho} - i\ell^{-1}A_{\rho}\right)\lambda_2^{\nu} - \frac{i}{2}F^{\mu\nu}\bar{\lambda}_2^{\rho}\gamma_{\mu}\gamma_{\rho\sigma}\gamma_{\nu}\lambda_2^{\sigma} + \frac{1}{\ell}\bar{\lambda}_2^{\mu}\gamma_{\mu\nu}\lambda_2^{\nu}$$

$$\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3, \hat{\gamma}^4$$

$$(\hat{\gamma}^{\mu})^{\dagger} = \hat{\gamma}^{\mu}$$

$$\bar{\lambda}_A^{\mu} = \left(\lambda_A^{\mu}\right)^{\dagger}$$

$$Z = \int D\bar{\psi}_\mu D\psi_\nu e^{-S[\bar{\psi}_\mu,\psi_\nu]}$$

$$1 = \int D\epsilon D\bar{\epsilon}\delta\left(\xi - \gamma^\mu\psi_\mu^{(\epsilon)}\right)\delta\left(\bar{\xi} - \gamma^\mu\bar{\psi}_\mu^{(\epsilon)}\right)\Delta_{\text{FP}}^{-1}$$

$$\Delta_{\text{FP}} = \det\left(\frac{\delta\left(\gamma^\mu\psi_\mu^{(\epsilon)}\right)}{\delta\epsilon}\right)\det\left(\frac{\delta\left(\gamma^\mu\bar{\psi}_\mu^{(\bar{\epsilon})}\right)}{\delta\bar{\epsilon}}\right)$$

$$\psi_\mu^{(\epsilon)} = \psi_\mu + \mathcal{D}_\mu\epsilon$$

$$1 = \frac{1}{\det\not{\mathcal{D}}}\int D\xi D\bar{\xi}\exp\left(-\bar{\xi}\not{\mathcal{D}}\xi\right)$$

$$Z = \frac{\Delta_{\text{FP}}^{-1}}{\det\not{\mathcal{D}}}\int D\bar{\psi}_\mu D\psi_\nu D\xi D\bar{\xi} D\epsilon D\bar{\epsilon}\exp\left(-\bar{\xi}\not{\mathcal{D}}\xi\right)\delta\left(\xi - \gamma^\mu\psi_\mu^{(\epsilon)}\right)\delta\left(\bar{\xi} - \gamma^\mu\bar{\psi}_\mu^{(\epsilon)}\right)e^{-S[\bar{\psi}_\mu,\psi_\nu]}$$
$$= \frac{\Delta_{\text{FP}}^{-1}}{\det\not{\mathcal{D}}}\int D\bar{\psi}_\mu D\psi_\nu D\epsilon D\bar{\epsilon}\exp\left(-\bar{\psi}_\nu\gamma^\nu\not{\mathcal{D}}\gamma^\mu\psi_\mu\right)e^{-S[\bar{\psi}_\mu,\psi_\nu]}$$

$$\psi_\mu \to \psi_\mu - \mathcal{D}_\mu\epsilon$$

$$\Delta_{\text{FP}}^{-1} = \int DbDc\exp\left(-b\gamma^\mu\mathcal{D}_\mu c\right)$$
$$\frac{1}{\det\not{\mathcal{D}}} = \int DdD\bar{d}\exp\left(-\bar{d}\gamma^\mu D_\mu d\right)$$

$$\mathcal{L}_{\text{ghost}} = \bar{b}_A\left(\gamma^\mu D_\mu + \frac{2}{\ell}\right)c_A + \bar{c}_A\left(\gamma^\mu D_\mu + \frac{2}{\ell}\right)b_A + \bar{e}_A\gamma^\mu D_\mu e_A$$

$$b_1 = \frac{1}{2}(\beta_1 + \gamma^5\beta_2), \qquad b_2 = \frac{i}{2}(\beta_1 - \gamma^5\beta_2)\ ,$$
$$c_1 = \frac{1}{2}(\eta_1 + \gamma^5\eta_2), \qquad c_2 = \frac{i}{2}(\eta_1 - \gamma^5\eta_2)\ ,$$
$$e_1 = \frac{1}{2}(\epsilon_1 + \gamma^5\epsilon_2), \qquad e_2 = \frac{i}{2}(\epsilon_1 - \gamma^5\epsilon_2)\ .$$

$$\bar{b}_1\gamma^\mu D_\mu c_1 = \frac{1}{4}(\bar{\beta}_1 - \bar{\beta}_2\gamma^5)\gamma^\mu D_\mu(\eta_1 + \gamma^5\eta_2)$$
$$= \frac{1}{4}(\bar{\beta}_1\gamma^\mu D_\mu\eta_1 - \bar{\beta}_2\gamma^5\gamma^\mu D_\mu\gamma^5\eta_2 + \bar{\beta}_1\gamma^\mu D_\mu\gamma^5\eta_2 - \bar{\beta}_2\gamma^5\gamma^\mu D_\mu\eta_1)$$
$$= \frac{1}{4}(\bar{\beta}_1\gamma^\mu D_\mu\eta_1 + \bar{\beta}_2\gamma^\mu D_\mu\eta_2 + \bar{\beta}_1\gamma^\mu D_\mu\gamma^5\eta_2 - \bar{\beta}_2\gamma^5\gamma^\mu D_\mu\eta_1)$$

$$\bar{b}_2\gamma^\mu D_\mu c_2 = \frac{1}{4}(\bar{\beta}_1 + \bar{\beta}_2\gamma^5)\gamma^\mu D_\mu(\eta_1 - \gamma^5\eta_2)$$
$$= \frac{1}{4}(\bar{\beta}_1\gamma^\mu D_\mu\eta_1 + \bar{\beta}_2\gamma^\mu D_\mu\eta_2 - \bar{\beta}_1\gamma^\mu D_\mu\gamma^5\eta_2 + \bar{\beta}_2\gamma^5\gamma^\mu D_\mu\eta_1)$$

$$\bar{b}_A\gamma^\mu D_\mu c_A = \frac{1}{2}(\bar{\beta}_1\gamma^\mu D_\mu\eta_1 + \bar{\beta}_2\gamma^\mu D_\mu\eta_2)$$

$$\bar{c}_A\gamma^\mu D_\mu b_A = \frac{1}{2}(\bar{\eta}_1\gamma^\mu D_\mu\beta_1 + \bar{\eta}_2\gamma^\mu D_\mu\beta_2)$$

$$\bar{e}_A\gamma^\mu D_\mu e_A = \frac{1}{2}(\bar{\epsilon}_1\gamma^\mu D_\mu\epsilon_1 + \bar{\epsilon}_2\gamma^\mu D_\mu\epsilon_2)$$

$$\bar{b}_1 c_1 = \frac{1}{4}(\bar{\beta}_1 - \bar{\beta}_2\gamma^5)(\eta_1 + \gamma^5\eta_2)$$
$$= \frac{1}{4}(\bar{\beta}_1\eta_1 - \bar{\beta}_2\eta_2 + \bar{\beta}_1\gamma^5\eta_2 - \bar{\beta}_2\gamma^5\eta_1)$$

$$\bar{b}_2 c_2 = \frac{1}{4}(\bar{\beta}_1 + \bar{\beta}_2\gamma^5)(\eta_1 - \gamma^5\eta_2)$$
$$= \frac{1}{4}(\bar{\beta}_1\eta_1 - \bar{\beta}_2\eta_2 - \bar{\beta}_1\gamma^5\eta_2 + \bar{\beta}_2\gamma^5\eta_1)$$

$$\bar{b}_A c_A = \frac{1}{2}(\bar{\beta}_1\eta_1 - \bar{\beta}_2\eta_2),$$

$$\bar{c}_A b_A = \frac{1}{2}(\bar{\eta}_1\beta_1 - \bar{\eta}_2\beta_2).$$

$$\mathcal{L}_{\text{ghosts}} = \bar{\beta}_1\left(\gamma^\mu D_\mu + \frac{2}{\ell}\right)\eta_1 + \bar{\eta}_1\left(\gamma^\mu D_\mu + \frac{2}{\ell}\right)\beta_1$$
$$+\bar{\beta}_2\left(\gamma^\mu D_\mu - \frac{2}{\ell}\right)\eta_2 + \bar{\eta}_2\left(\gamma^\mu D_\mu - \frac{2}{\ell}\right)\beta_2$$
$$+\bar{\epsilon}_1\gamma^\mu D_\mu\epsilon_1 + \bar{\epsilon}_2\gamma^\mu D_\mu\epsilon_2$$

$$\bar{\beta}_1 = \eta_1^\dagger, \bar{\beta}_2 = \eta_2^\dagger, \bar{\epsilon}_1 = \epsilon_1^\dagger$$

$$(4\pi)^2 a_4(x) = \frac{139}{90}E_4 - \frac{32}{15}W^2 - \frac{2}{9}R^2 + \frac{8}{9}RF_{\mu\nu}F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = \frac{11}{360}E_4 - \frac{1}{20}W^2 - \frac{1}{18}RF_{\mu\nu}F^{\mu\nu},$$

$$(4\pi)^2 a_4(x) = \frac{11}{360}E_4 - \frac{1}{20}W^2 + \frac{1}{9}R^2 - \frac{1}{18}RF_{\mu\nu}F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = \frac{589}{360}E_4 - \frac{137}{60}W^2 + \frac{13}{18}RF_{\mu\nu}F^{\mu\nu},$$

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{4}(\bar{\psi}_\mu\gamma^\mu)(\gamma^\nu D_\nu - m)(\gamma^\rho\psi_\rho)$$
$$= -\frac{1}{4}\delta_{AB}(\bar{\psi}_\mu^A\gamma^\mu)(\gamma^\nu D_\nu - m)(\gamma^\rho\psi_\rho^B)$$

$$\mathcal{L}_{\text{new}} = \frac{1}{4} m \bar{\psi}_\mu \gamma^\mu \gamma^\nu \psi_\nu$$
$$= \frac{1}{2} m \left(\bar{\chi}_\mu - \frac{1}{2} \bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \right) \gamma^\mu \gamma^\nu \left(\chi_\nu - \frac{1}{2} \gamma_\nu \gamma^\beta \chi_\beta \right)$$
$$= \frac{1}{2} m \delta_{AB} \bar{\chi}_\mu^A \gamma^\mu \gamma^\nu \chi_\nu^B$$

$$g_{\mu\nu} = \gamma^\mu \gamma^\nu - \gamma^{\mu\nu}$$

$$\mathcal{L}_f = \delta_{AB} g^{\mu\nu} \bar{\chi}_\mu^A \gamma^\rho D_\rho \chi_\nu^B - \frac{1}{2} \varepsilon_{AB} F^{\mu\nu} \bar{\chi}_\rho^A \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma^B + \frac{1}{\ell} \delta_{AB} \bar{\chi}_\mu^A g^{\mu\nu} \chi_\nu^B$$

$$(4\pi)^2 a_4(x) = \frac{139}{90} E_4 - \frac{32}{15} W^2 - \frac{1}{3} R^2 + \frac{8}{9} R F_{\mu\nu} F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = -\frac{11}{120} E_4 - \frac{3}{20} W^2 + \frac{1}{3} R^2 - \frac{1}{6} R F_{\mu\nu} F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = \frac{589}{360} E_4 - \frac{137}{60} W^2 + \frac{13}{18} R F_{\mu\nu} F^{\mu\nu}$$

$$S = S_{\text{classical}} + S_{1-\text{ loop}} + \cdots$$

$$S_{1-\text{loop}} = -\frac{1}{2} \int_\epsilon^{+\infty} \frac{ds}{s} \int d^4x \sqrt{g} K(x,s), K(x,s) = s^{-2} a_0(x) + s^{-1} a_2(x) + \cdots$$

$$S_{1-\text{ loop}} = \int d^4x \sqrt{g} \left(-\frac{1}{4\epsilon^2} a_0(x) - \frac{1}{2\epsilon} a_2(x) + \cdots \right)$$

$$(4\pi)^2 a_2(x) = d_1 R + d_2 F_{\mu\nu} F^{\mu\nu}$$

$$S_{1\text{-loop}} = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \left(-\frac{1}{4\epsilon^2} a_0 - \frac{1}{2\epsilon} \left(d_1 R + d_2 F_{\mu\nu} F^{\mu\nu} \right) + \cdots \right)$$

$$(4\pi)^2 a_0 = \text{Tr} 1$$
$$(4\pi)^2 a_2 = \text{Tr} \left(E + \frac{1}{6} R \right)$$

$$\frac{1}{32\pi^2} \int_{\mathcal{M}} d^d x \sqrt{g} E_4 + \frac{1}{32\pi^2} B = \chi$$

Multiplet	a_0	d_1	d_2
Free scalar	1	$\frac{1}{12}(2-\Delta(\Delta-3))$	0
Free Dirac fermion	-4	$-\frac{1}{12}(5+4\Delta(\Delta-3))$	0
Free vector	2	$-\frac{2}{3}$	0
Free gravitino	-2	$-\frac{1}{2}$	0
Einstein-Maxwell	4	$-\frac{10}{3}$	6
$\mathcal{N}=2$ gravitini	-4	7	-8
$\mathcal{N}=2$ gravity multiplet	0	$\frac{11}{3}$	-2

$$\lim_{r_c\to+\infty}\left[\frac{1}{32\pi^2}\int_{\mathcal{M}} d^dx\sqrt{g}E_4+\int_{\partial\mathcal{M}} d^3y\sqrt{h}(c_1+c_2\mathcal{R})\right]=\chi.$$

$$\chi=\frac{1}{32\pi^2}\int d^4x\sqrt{g}E_4+\frac{1}{32\pi^2}B$$

$$B\equiv-2\int \epsilon_{abcd}\theta^a_b\mathcal{R}^c_d+\frac{4}{3}\int \epsilon_{abcd}\theta^a_b\theta^c_e\theta^e_d$$

$$\frac{1}{32\pi^2}\int d^4x\sqrt{g}E_4=\int_{r_+}^{r_c} r^2dr\int d\Omega^2\int_0^\beta d\tau\left(\frac{24}{\ell^4}+\frac{48m^2}{r^6}\right)$$
$$=\frac{\beta}{\pi}\left[2m^2\left(\frac{1}{r_+^3}-\frac{1}{r_c^3}\right)+\frac{1}{\ell^4}(r_c^3-r_+^3)\right]$$

$$\theta_{01}=\frac{m\ell^2+r_c^3}{r_c^2\ell^2}d\tau$$

$$\theta_{12}=-\sqrt{\frac{r_c^3+r_c\ell^2-2m\ell^2}{r_c\ell^2}}d\theta$$

$$\theta_{12}=-\sqrt{\frac{r_c^3+r_c\ell^2-2m\ell^2}{r_c\ell^2}}\sin\theta d\phi$$

$$\mathcal{R}^a_b=\mathcal{R}^a{}_{bcd}e^c\wedge e^d=\mathcal{R}^a_{b\mu\nu}dx^\mu\wedge dx^\nu$$

$$\mathcal{R}^0_{2\tau\theta} = -\frac{m\ell^2 + r_c^3}{\ell^2 r_c^2}\sqrt{\frac{r_c^3 + r_c\ell^2 - 2m\ell^2}{r_c\ell^2}}$$

$$\mathcal{R}^0_{3\tau\phi} = -\frac{m\ell^2 + r_c^3}{\ell^2 r_c^2}\sqrt{\frac{r_c^3 + r_c\ell^2 - 2m\ell^2}{r_c\ell^2}}\sin\,\theta$$

$$\mathcal{R}^0_{3\theta\phi} = \frac{2m\ell^2 - r_c^3}{r_c\ell^2}$$

$$B\ = 8(4\pi\beta)\frac{(m\ell^2 + r_c^3)(2m\ell^2 - r_c^3)}{\ell^4 r_c^3}$$

$$= 4\pi\beta\left[-16m^2\frac{1}{r_c^3} + 8m\frac{1}{\ell^2} - \frac{8}{\ell^4}r_c^3\right].$$

$$4\pi\beta\left[16m^2\frac{1}{r_+^3} + 8m\frac{1}{\ell^2} - \frac{8}{\ell^4}r_+^3\right].$$

$$\chi = \frac{\beta}{\pi}\left[2m^2\frac{1}{r_+^3} + m\frac{1}{\ell^2} - \frac{1}{\ell^3}r_+^3\right] = 2,$$

$$ds^2 = \left(1 + \frac{r_c^2}{\ell^2} - \frac{2m}{r_c}\right)dt^2 + r_c^2 d\theta^2 + r_c^2\sin^2\,\theta d\phi^2$$

$$a^{\mathrm{CT}}\ = \int\ d^3y\sqrt{h}(c_1 + c_2\mathcal{R})$$

$$= \int_0^\beta dt\int_0^\pi \sin\,\theta d\theta\int_0^{2\pi} d\phi\sqrt{1 + \frac{r_c^2}{\ell^2} - \frac{2m}{r_c}}\,r_c^2(c_1 + c_2\mathcal{R})$$

$$= 4\pi\beta(c_1 r_c^2 + 2c_2)\sqrt{\frac{r_c^2}{\ell^2} - \frac{2m}{r_c} + 1}$$

$$= 4\pi\beta\left[\frac{c_1}{\ell}r_c^3 + \frac{(c_1\ell^2 + 4c_2)}{2\ell}r_c - c_1 m\ell\right] + \mathcal{O}(r_c^{-1})$$

$$c_1 = -\frac{8}{\ell^3}, c_2 = \frac{2}{\ell}$$

$$a_4(x) = \cdots + \frac{1}{30}(5\,\square\,E + \square\,R)$$

$$C_{\mathrm{local}}\ = \cdots + \frac{1}{(4\pi)^2}\frac{1}{30}\int\ d^3y\sqrt{h}n^\mu\nabla_\mu(5\mathrm{Tr}E + R\mathrm{Tr}1)$$

$$\mathrm{Tr}E = \alpha_1 R + \alpha_2 F_{\mu\nu}F^{\mu\nu},$$

$$C_{\mathrm{bdy}} = \int\ d^3y\sqrt{h}a_4^\partial(y),$$

$$a_4^\partial = B_1\mathrm{Tr}1 + B_2\mathrm{Tr}E,$$

$$B_1 = \frac{1}{360}[24K_{;bb} + 20RK + 4R_{anan}K - 12R_{anbn}K_{ab} + 4R_{abcb}K_{ac} + 480S^2K + 480S^3$$
$$+ \frac{1}{21}[(280\Pi_+ + 40\Pi_-)K^3 + (168\Pi_+ - 264\Pi_-)K_{ab}K_{ab}K_{cc}] + 120S_{;aa}$$
$$+ \frac{1}{21}(224\Pi_+ + 320\Pi_-)K_{ab}K_{bc}K_{ac} + 120SR + 144SK^2 + 48SK_{ab}K_{ab}]$$

$$B_2 = \frac{1}{3}(K + 6S)$$

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{24}{\ell^4} + \frac{8}{4(r^2 + a^2\cos^2\theta)^6}\Big[-24m^2(a^2\cos^2\theta + r^2)^3 + 192r^4\big((q_e^2 + q_m^2) - 2mr\big)^2$$
$$-192r^2\big((q_e^2 + q_m^2) - 3mr\big)\big((q_e^2 + q_m^2) - 2mr\big)(a^2\cos^2\theta + r^2)$$
$$+4\big((q_e^2 + q_m^2) - 6mr\big)(7(q_e^2 + q_m^2) - 18mr)(a^2\cos^2\theta + r^2)^2\Big]$$
$$= \frac{24}{\ell^4} + \tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{36}{\ell^4} + \frac{4(q_e^2 + q_m^2)^2}{(r^2 + a^2\cos^2\theta)^4}$$
$$= \frac{36}{\ell^4} + \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$$

$$R^2 = \frac{144}{\ell^4},$$

$$F_{\mu\nu}F^{\mu\nu} = -\frac{2(q_e^2 - q_m^2)(r^4 - 6a^2r^2\cos^2\theta + a^4\cos^4\theta) + 16q_eq_mra\cos\theta(r^2 - a^2\cos^2\theta)}{(r^2 + a^2\cos^2\theta)^4}.$$

$$E_4 = \frac{24}{\ell^4} + \frac{8}{(r^2 + a^2\cos^2\theta)^6}(6m^2(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta)$$
$$-12mr(q_e^2 + q_m^2)(r^4 - 10a^2r^2\cos^2\theta + 5a^4\cos^4\theta)$$
$$+(q_e^2 + q_m^2)^2(5r^4 - 38a^2r^2\cos^2\theta + 5a^4\cos^4\theta)\big)$$

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$
$$=-\frac{3(a^2-r_+^2)(a^2+r_+^2)^2\left(a^2(r_+^2-\ell^2)+r_+^2(\ell^2+3r_+^2)\right)\arctan\left(\frac{a}{r_+}\right)}{a^5\ell^2\Xi r_+^3}$$
$$+\left(a^2(\ell^2-r_+^2)-r_+^2(\ell^2+3r_+^2)\right)^2$$
$$\times\frac{\beta\left(3a^5r_+ + 2a^3r_+^3 + 3(a^2-r_+^2)(a^2+r_+^2)^2\arctan\left(\frac{a}{r_+}\right)+3ar_+^5\right)}{8\pi a^5\ell^4\Xi r_+^4(a^2+r_+^2)}$$
$$+\frac{2\pi(a^2+r_+^2)\left(3a^5r_+ + 2a^3r_+^3 + 3(a^2-r_+^2)(a^2+r_+^2)^2\arctan\left(\frac{a}{r_+}\right)+3ar_+^5\right)}{a^5\beta\Xi r_+^2}$$
$$+\frac{ar_+\left(a^6(3\ell^2-7r_+^2)+a^4(3\ell^2r_+^2-11r_+^4)+a^2r_+^4(\ell^2-9r_+^2)-3r_+^6(\ell^2+3r_+^2)\right)}{a^5\ell^2\Xi r_+^3}$$
$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R_{\mu\nu}R^{\mu\nu}$$
$$=\frac{96\beta r_+^4\left(\beta r_+(a^2+\ell^2+2r_+^2)-2\pi\ell^2(a^2+r_+^2)\right)-96\beta^2r_+^5(a^2+r_+^2)}{32\pi\beta\ell^4\Xi r_+^4}$$
$$+\left(a^2\left(\beta r_+^2-\ell^2(\beta+4\pi r_+)\right)+\ell^2r_+^2(\beta-4\pi r_+)+3\beta r_+^4\right)^2$$
$$\times\frac{\left(3a^5r_+ + 2a^3r_+^3 + 3(a-r_+)(a+r_+)(a^2+r_+^2)^2\arctan\left(\frac{a}{r_+}\right)+3ar_+^5\right)}{32\pi\beta\ell^4\Xi r_+^4\left(a^5(a^2+r_+^2)\right)}$$
$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R^2=\frac{12\left(\beta r_+(\ell^2+r_+^2)-2\pi\ell^2(a^2+r_+^2)\right)}{\pi\ell^4\Xi}$$
$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}=2r_+^2-\frac{\beta r_+(3r_+^4+(a^2+\ell^2)r_+^2-a^2\ell^2-2\ell^2q_m^2)}{2\pi\ell^2(a^2+r_+^2)}$$

Modelo de campo cuántico – relativista en el que se forman agujeros negros cuánticos en gravedad cuántica.

$$\psi_I^\alpha = ie^i(\gamma_i)^\alpha{}_\beta\chi_I^\beta$$

$$T^i=\mathcal{D}e^i\equiv de^i+\omega_j^i\wedge e^j=\tau\epsilon_{jk}^ie^j\wedge e^k$$

$$\mathcal{L}_{SUGRA}=\mathcal{L}_{CS}^{(G_+)}-\mathcal{L}_{CS}^{(G_-)}$$

$$\mathcal{A}_{(+)}=\frac{1}{2}\omega_{(+)}^{\iota J}\mathbb{J}_{\iota J}+A_{(+)}^{IJ}\mathbb{T}_{IJ}+\overline{\mathbb{Q}}^{\alpha|I}\psi_{\alpha|I}^{(+)}$$
$$\mathcal{A}_{(-)}=\frac{1}{2}\omega_{(-)}^{\hat{\iota}\hat{\jmath}}\mathbb{J}_{\hat{\iota}\hat{\jmath}}+A_{(-)}^{\dot{I}\dot{J}}\mathbb{T}_{\dot{I}\dot{J}}+\overline{\mathbb{Q}}^{\dot{\alpha}|\dot{I}}\psi_{\dot{\alpha}|\dot{I}}^{(-)}$$

$$\omega^{ij}=\frac{1}{2}\left(\omega_{(+)}^{\iota J}+\omega_{(-)}^{\hat{\iota}\hat{J}}\right)$$
$$E^k=\frac{\ell}{4}\left(\omega_{(+)\iota J}-\omega_{(-)\hat{\iota}\hat{J}}\right)\epsilon^{ijk}$$

$$\mathfrak{so}(1,2)_D\subset\mathfrak{so}(2,2)=\mathfrak{so}(1,2)_{(+)}\times\mathfrak{so}(1,2)_{(-)}$$

$$\mathcal{L}_{(3)}=\left(\mathcal{R}^{ij}-\frac{1}{3\ell^2}E^iE^j-\frac{1}{2\ell}\bar{\psi}_I\gamma^{ij}\psi_I\right)E^k\epsilon_{ijk}-\frac{1}{2\ell}AdA+2\bar{\psi}_I\left(\mathcal{D}\psi_I-\frac{1}{2\ell}\epsilon_{IJ}A\psi_J\right)$$

$$\begin{aligned}\mathcal{R}^{\iota J}_{(+)}&=\frac{\mathrm{i}}{\ell}\bar{\psi}_I\wedge\gamma_k\psi_I\epsilon^{\iota Jk}\\ \mathcal{D}_{(+)}\psi_I&=\frac{1}{2\ell}A\wedge\epsilon_{IJ}\psi_J\\ dA&=\epsilon_{IJ}\bar{\psi}_I\wedge\psi_J\\ \mathcal{R}^{\hat{\imath}\hat{\jmath}}_{(-)}&=0\end{aligned}$$

$$R^{\iota J}_{(+)}\equiv d\omega^{\iota J}_{(+)}+\omega^{\iota k}_{(+)}\wedge\omega^{J}_{(+)k},R^{\hat{\imath}\hat{\jmath}}_{(-)}\equiv d\omega^{\hat{\imath}\hat{\jmath}}_{(-)}+\omega^{\hat{\imath}\hat{k}}_{(-)}\wedge\omega^{\hat{\jmath}}_{(-)\hat{k}},A^{IJ}_{(+)}=\epsilon^{IJ}A$$

$$\psi^\alpha_I=\mathrm{i}(\gamma_i)^\alpha_\beta\chi^\beta_I e^i$$

$$\Gamma^{\hat{\mu}}\Psi_{\hat{\mu}I}=0\Rightarrow\gamma^\mu\psi_{\mu I}=-\gamma^r\psi_{rI}=3\mathrm{i}\chi_I\neq0$$

$$\psi^\alpha_I=\mathrm{i}(\gamma_i)^\alpha_{\beta'}\chi^{\beta'}_I e^i$$

$$\omega^{\alpha\beta}_{\pm}\equiv\frac{1}{2}\gamma^{\alpha\beta}_{ij}\left(\omega^{ij}\pm\frac{1}{\ell}E_k\epsilon^{ijk}\right)$$

$$\mathcal{R}^{\alpha\beta}_{+}=-i\psi^{\alpha I}\wedge\psi^\beta_I,(\mathcal{D}\psi)^\alpha_I=\frac{1}{2}A\wedge\epsilon_{IJ}\psi^\alpha_J,dA=-i\frac{1}{2}\epsilon^{IJ}\epsilon_{\alpha\beta}\psi^\alpha_I\wedge\psi^\beta_J$$
$$\mathcal{R}^{\alpha\beta}_{-}=0$$

$$\begin{aligned}\delta\omega^{\alpha\beta}_{-}&=0\\ \delta\omega^{\alpha\beta}_{+}&=-2i\epsilon^{(\alpha I}\psi^{\beta)}_I\\ \delta A&=-i\epsilon^{IJ}\epsilon_{\alpha\beta}\epsilon^\alpha_I\psi^\beta_J\\ \delta\psi^\alpha_I&=\mathcal{D}\epsilon^\alpha_I-\frac{1}{2}A\epsilon_{IJ}\epsilon^\alpha_J\equiv\nabla\epsilon^\alpha_I\end{aligned}$$

$$\begin{aligned}s\omega^{\alpha\beta}_{-}&=0\\ s\omega^{\alpha\beta}_{+}&=-2i\phi^{(\alpha I}\psi^{\beta)}_I\\ sA&=-i\epsilon^{IJ}\epsilon_{\alpha\beta}\phi^\alpha_I\psi^\beta_J\\ s\psi^\alpha_I&=\mathcal{D}\phi^\alpha_I-\frac{1}{2}A\epsilon_{IJ}\phi^\alpha_J\equiv\nabla\phi^\alpha_I\\ s\phi^\alpha_I&=0\end{aligned}$$

$$\mu^{(\alpha\beta)}_{+}=-\phi^\alpha_I\phi^\beta_I,\mu_{+[IJ]}=-\epsilon_{\alpha\beta}\phi^\alpha_I\phi^\beta_J=\epsilon_{IJ}\mu_+,$$

$$\begin{aligned}s^2\omega^{\alpha\beta}_{-}&=0\\ s^2\omega^{\alpha\beta}_{+}&=i\nabla\mu^{(\alpha\beta)}_{+}\\ s^2A&=i\nabla\mu_+\\ s^2\psi^\alpha_I&=\frac{i}{2}\mu^{+}_{IJ}\psi^\alpha_J+\frac{i}{2}\mu^{+|(\alpha\beta)}\epsilon_{\beta\gamma}\psi^\gamma_I\\ s^2\phi^\alpha_I&=0\end{aligned}$$

$$s\bar{\phi}_\alpha^I = \bar{\eta}_\alpha^I, s\bar{\eta}_\alpha^I = \frac{i}{2}\left(\mu_{IJ}^+\bar{\phi}_{\alpha J} + \mu^{+|(\delta\beta)}\epsilon_{\alpha\delta}\bar{\phi}_{\beta I}\right).$$

$$\mu_3^{\alpha\beta} = -\bar{\phi}_I^{(\alpha}\phi_I^{\beta)}, \mu_{3|IJ} = -\phi_{[I}^\alpha\bar{\phi}_{J]}^\beta\epsilon_{\alpha\beta}$$

$$\mu_-^{(\alpha\beta)} = -\bar{\phi}_I^\alpha\bar{\phi}_I^\beta, \mu_{-|[IJ]} = -\epsilon_{\alpha\beta}\bar{\phi}_I^\alpha\bar{\phi}_J^\beta = \epsilon_{IJ}\mu_-$$

$$\Phi_I^{\alpha|1} = \phi_I^\alpha, \Phi_I^{\alpha|2} = -\epsilon^{\alpha\beta}\bar{\phi}_{\beta I}$$

$$\Omega^{AB} = \epsilon_{\alpha\beta}d\Phi_I^{\alpha A}\wedge d\Phi_I^{\beta B}$$

$$\iota_{V^{(\alpha\beta)}}\Omega^{AB} = -d\mu^{AB|(\alpha\beta)}, \iota_{V_{IJ}}\Omega^{AB} = -d\mu_{IJ}^{AB}$$

$$\mu^{AB|(\alpha\beta)} = -\Phi_I^{(\alpha|A}\Phi_I^{\beta)B}, \mu_{IJ}^{AB} = -\epsilon_{\alpha\beta}\Phi_{[I}^{\alpha|A}\Phi_{J]}^{\beta|B} = \epsilon_{IJ}\mu^{AB}$$

$$\mu^{11|(\alpha\beta)} = \mu_+^{(\alpha\beta)};\ \mu^{22|(\alpha\beta)} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\mu_{-(\gamma\delta)};\ \mu^{12|(\alpha\beta)} = \mu_3^{(\alpha\beta)};$$
$$\mu^{11} = \mu_+;\ \mu^{22} = \mu_-;\ \mu^{12} = \mu_3$$

$$\mu_+^{(\alpha\beta)}\mu_{+|(\alpha\beta)} + 2\mu_+^2 = 0$$

$$\mathcal{L}^{(3)} = \mathcal{L}_+^{(3)} - \mathcal{L}_-^{(3)}$$

$$\mathcal{L}_+^{(3)} = \frac{1}{2}\left(\omega_+^{\alpha\beta}\wedge d\omega_{+,\alpha\beta} - \frac{1}{3}\omega_+^{\alpha\alpha'}\wedge\omega_{+,\alpha'\beta'}\wedge\omega_+^{\beta'\beta}\right) - 2i\epsilon_{\alpha\beta}\psi^{\alpha I}\nabla\psi_I^\beta - A\wedge dA$$
$$\nabla\psi_I^\alpha = \left(\delta_\beta^\alpha d + \frac{1}{2}\omega_\beta^\alpha\right)\psi_I^\beta - \frac{1}{2}\epsilon_{IJ}A\wedge\psi_J^\alpha$$
$$\mathcal{L}_-^{(3)} = \frac{1}{2}\left(\omega_-^{\alpha\beta}\wedge d\omega_{-,\alpha\beta} - \frac{1}{3}\epsilon_{\alpha\beta}\omega_-^{\alpha\alpha'}\wedge\omega_{-,\alpha'\beta'}\wedge\omega_-^{\beta'\beta}\right)$$

$$\begin{aligned}
\bar{s}\omega_-^{\alpha\beta} &= 0\\
\bar{s}\omega_+^{\alpha\beta} &= -2i\bar{\phi}_I^{(\alpha}\psi_I^{\beta)}\\
\bar{s}A &= -i\epsilon_{\alpha\beta}\epsilon_{IJ}\bar{\phi}_I^\alpha\psi_J^\beta\\
\bar{s}\psi_I^\alpha &= \mathcal{D}\bar{\phi}_I^\alpha - \frac{1}{2}A\epsilon_{IJ}\bar{\phi}_J^\alpha \equiv \nabla\bar{\phi}_I^\alpha\\
\bar{s}\bar{\phi}_\alpha^I &= 0
\end{aligned}$$

$$\bar{s}\phi_I^\alpha = -\bar{\eta}_I^\alpha, \bar{s}\bar{\eta}_I^\alpha = -\frac{i}{2}\left(\mu_-\epsilon_{IJ}\phi_J^\alpha + \mu_-^{(\alpha\beta)}\epsilon_{\beta\gamma}\phi_I^\gamma\right).$$

$$\frac{1}{2}(s\bar{s} + \bar{s}s) = -\frac{i}{2}\mu_3^{\alpha\beta}\delta_{(\alpha\beta)} - i\mu_3\delta$$

$$\frac{\delta S}{\delta g_{\mu\nu}} \equiv T_{\mu\nu} = s\Gamma_{\mu\nu}$$

$$\partial^\mu\Gamma_{\mu\nu} = \delta_\nu A_\mu^{\alpha\beta}\frac{\delta S}{\delta A_\mu^{\alpha\beta}} + \delta_\nu A_{\mu,[IJ]}\frac{\delta S}{\delta A_{\mu,[IJ]}} + \delta_\nu\psi_{\mu,I}^\alpha\frac{\delta S}{\delta\psi_{\mu,I}^\alpha} + \delta_\nu\bar{\eta}_\alpha^I\frac{\delta S}{\delta\bar{\eta}_\alpha^I} + \delta_\nu\phi_I^\alpha\frac{\delta S}{\delta\phi_I^\alpha} + \delta_\nu\bar{\phi}_\alpha^I\frac{\delta S}{\delta\bar{\phi}_\alpha^I}$$

$$\Lambda^{\alpha}_{I\alpha'\beta'} = i\gamma^{\mu}_{\alpha'\beta'}\psi^{\alpha}_{\mu,I} - \frac{1}{2}\epsilon_{\alpha'\beta'}\bar{\eta}^{\alpha}_{I}$$

$$\Lambda^{\alpha}_{I\alpha'\beta'} = i(\gamma^{\mu})^{\alpha}{}_{\alpha'}\chi_{\mu\beta',I} - 2\delta^{\alpha}{}_{\alpha'}\chi_{I\beta'}$$

$$\Lambda^{\alpha}_{I\alpha'\beta'} \longrightarrow \Lambda^{\alpha}_{I\alpha'\dot{\beta}'}$$

$$\epsilon^{\alpha'\dot{\alpha}'A} = i\left(\gamma^{i}\right)^{\alpha'\dot{\alpha}'}\epsilon^{A}_{i} + \epsilon^{\alpha'\dot{\alpha}'}\epsilon^{A},$$

$$\mathcal{L} = \mathcal{L}^{(3)} + \mathcal{L}^{g.f.}, \mathcal{L}^{g.f.} = s\left(\bar{\phi}^{I}_{\alpha}\mathcal{F}^{\alpha}_{I}(\omega_{+}, \psi^{\alpha}_{I}, A, \phi^{\alpha}_{I})\right), S = \int_{\mathcal{M}_3}\mathcal{L}$$

$$S_A = 2\int s(\bar{\phi}^{I}_{\alpha}\nabla\star\psi^{\alpha}_{I}) = -2\int (\nabla\phi^{\alpha}_{I}\wedge\star\nabla\bar{\phi}^{I}_{\alpha} + \psi^{\alpha}_{I}\wedge\star\nabla\bar{\eta}^{I}_{\alpha})$$
$$+2\int \left(-i\phi^{J(\alpha}\psi^{\beta)}_{J}\bar{\phi}^{I}_{\alpha}\star\psi_{\beta I} - \frac{1}{2}\epsilon^{KL}\phi^{\gamma}_{K}\psi^{\delta}_{L}\epsilon_{\gamma\delta}\bar{\phi}^{I}_{\alpha}\epsilon^{J}_{I}\star\psi^{\alpha}_{J}\right)$$

$$2\epsilon_{\alpha\beta}\nabla\psi^{\beta}_{I} + \star\nabla\bar{\eta}_{I\alpha} = 0, \nabla\star\psi^{\alpha}_{I} = 0$$

$$\epsilon^{\mu\nu\rho}\epsilon_{\alpha\beta}\nabla_{\nu}\psi^{\alpha}_{\rho,I} + \nabla^{\mu}\bar{\eta}_{I\alpha} = 0, \nabla^{\mu}\psi^{\alpha}_{\mu,I} = 0$$

$$2(\gamma_{\mu})_{\alpha\beta}\nabla^{\mu}(\gamma^{\nu}\psi_{\nu,I})^{\beta} + (\gamma_{\mu})^{\beta}_{\alpha}\nabla^{\mu}\bar{\eta}^{I}_{\beta} = \not{\nabla}\left(2\epsilon_{\alpha\beta}\left(\psi^{\beta}_{I}\right) + \bar{\eta}^{I}_{\alpha}\right) = 0$$

$$\bar{\eta}^{I}_{\alpha} = -2(\psi_{I})_{\alpha} + \sigma^{I}_{\alpha}$$

$$S_B = 2\int s(\beta\bar{\eta}^{I}_{\alpha}\epsilon^{\alpha\beta}\bar{\phi}_{\beta,I})V^{(3)}$$
$$= 2\int \beta\left(\bar{\phi}^{I}_{\alpha}\left(\mu_{+,IJ}(\phi)\epsilon^{\alpha\beta} + \mu^{\alpha\beta}_{+}(\phi)\delta_{IJ}\right)\bar{\phi}^{J}_{\beta} + \bar{\eta}^{I}_{\alpha}\epsilon^{\alpha\beta}\bar{\eta}_{\beta,I}\right)V^{(3)}$$
$$= 2\int \beta(\mu_{3,\alpha}{}^{\beta}\mu_{3,\gamma}{}^{\delta}\epsilon_{\alpha\gamma}\epsilon^{\beta\delta} + \mu^{I}_{3,K}\mu^{K}_{3,I} + \bar{\eta}^{I}_{\alpha}\epsilon^{\alpha\beta}\bar{\eta}_{\beta,I})V^{(3)}$$

$$S'_B = \int s\left(\bar{\eta}^{I}_{\alpha}V^{\alpha\beta}_{IJ}(\phi)\bar{\phi}^{J}_{\beta}\right)V^{(3)}$$
$$= \int \left(\bar{\phi}^{K}_{\alpha}\bar{\phi}^{J}_{\beta}\left(\mu^{I}_{+,K}V^{\alpha\beta}_{IJ} + \mu^{\alpha}_{+,\gamma}V^{\gamma\beta}_{IJ}\right) + \bar{\eta}^{I}_{\alpha}V^{\alpha\beta}_{IJ}\bar{\eta}^{J}_{\beta}\right)V^{(3)}$$

$$\mathcal{L} = 2\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho}\psi^{\alpha}_{\mu,I}\nabla_{\nu}\psi^{\beta}_{\rho,I} + \bar{\eta}^{I}_{\alpha}\left(\nabla^{\mu}\psi^{\alpha}_{\mu,I} - 2i\beta\left(\gamma^{i}\right)^{\alpha}{}_{\beta}\psi^{\beta}_{i,I}\right) + 2\beta\epsilon^{\alpha\beta}\bar{\eta}^{I}_{\alpha}\bar{\eta}_{\beta,I}$$

$$4\epsilon^{\mu\nu\rho}\nabla_{\nu}\psi^{\alpha}_{\rho,I} - 2\nabla^{\mu}\bar{\eta}^{\alpha}_{I} + 4i\beta\gamma^{\mu,\alpha}{}_{\beta}\bar{\eta}^{\beta}_{I} = 0$$
$$\nabla^{\mu}\psi^{\alpha}_{\mu,I} - 2i\beta\left(\gamma^{\mu}\psi_{\mu,I}\right)^{\alpha} + 2\beta\epsilon^{\alpha\beta}\bar{\eta}^{I}_{\beta} = 0$$

$$S_A = \int s\left(\psi^{\alpha}_{I}\wedge\star\nabla\bar{\phi}^{I}_{\alpha} + \psi^{\alpha}_{I}\wedge\star e^{a}\gamma^{\beta}_{a,\alpha}\bar{\phi}^{I}_{\beta}\right)$$
$$= \int \left[\nabla\phi^{\alpha}_{I}\wedge\star\nabla\bar{\phi}^{I}_{\alpha} + \psi^{\alpha}_{I}\wedge\star\nabla\bar{\eta}^{I}_{\alpha} + \psi^{\alpha}_{I}\wedge\star\left(\epsilon_{\alpha\beta}\psi^{(\beta}_{K}\eta^{KL}\phi^{\gamma)}_{L}\bar{\phi}^{I}_{\gamma} + \eta^{IJ}\psi^{\gamma}_{[J}\epsilon_{\gamma\beta}\phi^{\beta}_{K]}\bar{\phi}^{K}_{\alpha}\right)\right.$$
$$\left.+\nabla\phi^{\alpha}_{I}\wedge\star e^{i}\gamma^{\beta}_{i,\alpha}\bar{\phi}^{I}_{\beta} + \psi^{\alpha}_{I}\wedge\star e^{i}\gamma^{\beta}_{i,\alpha}\bar{\eta}^{I}_{\beta}\right]$$

$$\mathcal{L}^{g.f.} = s\bar{s}\Sigma$$

$$\mathcal{L}^{g.f.} = s\bar{s}\left(\psi_I^{\alpha}\epsilon_{\alpha\beta}\eta^{IJ} \wedge\star \psi_J^{\beta} + \bar{\eta}_{\alpha}^{I}\epsilon^{\alpha\beta}\bar{\eta}_{\beta}^{I} \star 1\right)$$

$$\mathcal{L}^{g.f.} = s\bar{s}\left(\psi_I^{\alpha}\epsilon_{\alpha\beta} \wedge\star \psi_I^{\beta} + \bar{\eta}_{\alpha}^{I}\epsilon^{\alpha\beta}\bar{\eta}_{\beta|I} \star 1 + \beta\psi_I^{\alpha} \wedge\star e^{i}\Gamma_{i\alpha}{}^{\beta}\bar{\eta}_{\beta}^{I}\right)$$

$$s\bar{s}\left(A^{(\alpha\beta)}\gamma_{i\alpha\beta} \wedge\star e^{i}\right)$$

$$s\left(\psi^{\alpha} \wedge \bar{\phi}^{\beta}(\gamma_i)_{\alpha\beta} \star e^{i}\right) = \nabla\phi^{\alpha} \wedge \bar{\phi}^{\beta}(\gamma_i)_{\alpha\beta} \star e^{i} - \psi^{\alpha} \wedge \bar{\eta}^{\beta}(\gamma_i)_{\alpha\beta} \star e^{i}$$

$$\delta_{\mathrm{Sp}(2)}\Phi_I^{\gamma A} = \frac{1}{2}\lambda_{(\alpha\beta)}\left(V^{(\alpha\beta)}\right)_I^{\gamma A} = \frac{1}{2}\lambda_{(\alpha\beta)}\epsilon^{\gamma(\alpha}\Phi_I^{\beta)A}\,,\delta_{\mathrm{SO}(2)}\Phi_I^{\alpha A} = \frac{1}{2}\lambda^{KL}(V_{KL})_I^{\alpha A} =$$

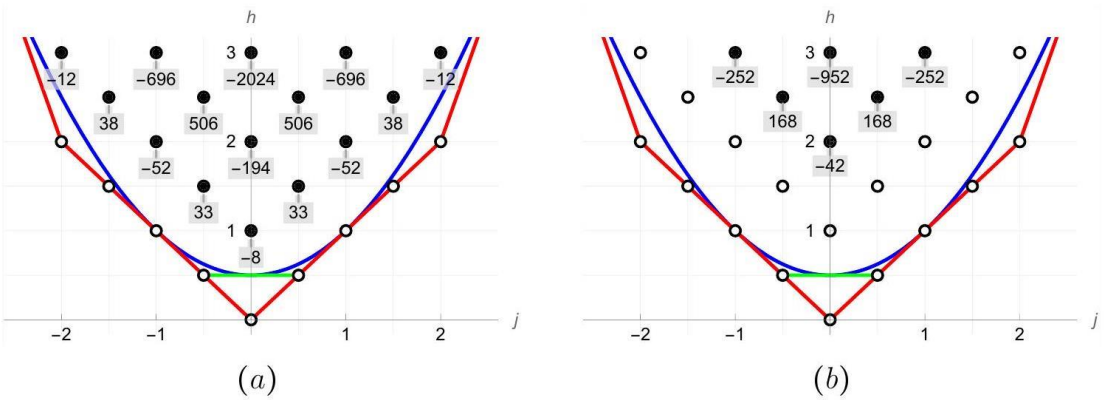
$$-\frac{1}{2}\lambda^{KL}\delta_{I[K}\Phi_{L]}^{\alpha A}$$

$$|\Psi\rangle = |\psi_1\rangle_{a_1}|\psi_2\rangle_{a_2}|\psi_3\rangle_{a_3}\cdots$$

$$|\Psi\rangle = \mathcal{O}_1|\phi_1\rangle_{a_1}\mathcal{O}_2|\phi_2\rangle_{a_2}\cdots$$

$$\mathcal{G}|\phi\rangle_a \underbrace{|0\rangle_1|0\rangle_1\cdots}_{N-a},$$

$$\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1|0\rangle_1\cdots$$



$$\mathcal{A}^{(\mathrm{T})}\left(\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1|0\rangle_1\cdots\right)$$
$$= \left(\mathcal{A}\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1|0\rangle_1\cdots\right) + \left(\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{A}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1|0\rangle_1\cdots\right) + \cdots$$
$$+\left(\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots\mathcal{A}|0\rangle_1|0\rangle_1\cdots\right) + \left(\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1\mathcal{A}|0\rangle_1\cdots\right) + \cdots$$

$$X^{\dot{A}A}(t,\sigma), \psi^{\alpha\dot{A}}(t,\sigma), \tilde{\psi}^{\dot{\alpha}\dot{A}}(t,\sigma)$$

$$\alpha \in \{+,-\}, \dot{\alpha} \in \{\dot{+},\dot{-}\}, A \in \{1,2\}, \dot{A} \in \{\dot{1},\dot{2}\}$$

$$SO(4)_E \cong SU(2)_L \times SU(2)_R$$

$$SO(4)_I \cong SU(2)_1 \times SU(2)_2$$

$$\partial X^{\dot{A}A}(t,\sigma + 2\pi R_\sigma) = \partial X^{\dot{A}A}(t,\sigma)\,, \psi^{\alpha\dot{A}}(t,\sigma + 2\pi R_\sigma) = -\psi^{\alpha\dot{A}}(t,\sigma)$$

$$\partial X^{\dot{A}A}(z)=\sum_{n\in\mathbb{Z}}\alpha_{-n}^{\dot{A}A}z^{n-1},\psi^{\alpha\dot{A}}(z)=\sum_{r\in\mathbb{Z}+\frac{1}{2}}\psi_{-r}^{\alpha\dot{A}}z^{r-\frac{1}{2}},\tilde{\psi}^{\dot{\alpha}\dot{A}}(\tilde{z})=\sum_{r\in\mathbb{Z}+\frac{1}{2}}\tilde{\psi}_{-r}^{\dot{\alpha}\dot{A}}\tilde{z}^{r-\frac{1}{2}}$$

$$\left\{L_n,J_n^{\pm,3},G_r^{\alpha A}\right\},\left\{\tilde{L}_n,\tilde{J}_n^{\dot{\pm},3},\tilde{G}_r^{\dot{\alpha}A}\right\}$$

$$\mathcal{H}\left(\mathrm{Sym}^N(\mathcal{M}_4)\right)\equiv\mathcal{H}_{\mathrm{seed}}^{\otimes N}/S_N=\bigoplus_{[g]}\mathcal{H}_{[g]}$$

$$[g]\equiv\{\tilde{g}\in S_N\mid\exists h\in S_N\text{ s.t. }h\tilde{g}h^{-1}=g\}$$

$$\sum_{a=1}^{N}an_a=N$$

$$\mathcal{H}_{\{n_a\}}=\bigotimes_{a=1}^{N}(\underbrace{\mathcal{H}_a\otimes\cdots\otimes\mathcal{H}_a}_{n_a})^{S_{n_a}}$$

$$N=\{\underbrace{1,\ldots,1}_{n_1},\underbrace{2,\ldots,2}_{n_2},\ldots\}$$

$$\left\{\partial X^{\dot{A}A(i)},\psi^{\alpha\dot{A}(i)},\tilde{\psi}^{\dot{\alpha}\dot{A}(i)}\right\}$$

$$L_n^{(\mathrm{T})}\equiv\sum_{i=1}^{N}L_n^{(i)}$$

$$\partial X^{\dot{A}A(i)}(t,\sigma+2\pi R_\sigma)=\partial X^{\dot{A}A(i+1)}(t,\sigma)\ \text{ for }\ i=1,\ldots,a-1,$$
$$\partial X^{\dot{A}A(a)}(t,\sigma+2\pi R_\sigma)=\partial X^{\dot{A}A(1)}(t,\sigma),$$

$$\mathcal{C}_{\{n_a\}}=\prod_{a=1}^{N}\mathcal{C}_{n_a}=\prod_{a=1}^{N}\left(S_{n_a}\ltimes\mathbb{Z}_a^{n_a}\right)$$

$$\begin{aligned}L_n(\eta)&=L_n+\eta J_n^3+\eta^2\frac{c}{24}\delta_{n,0}\\J_n^3(\eta)&=J_n^3+\eta\frac{c}{12}\delta_{n,0}\\J_n^{\pm}(\eta)&=J_{n\mp\eta}^{\pm}\\G_r^{\pm A}(\eta)&=G_{r\mp\frac{\eta}{2}}^{\pm A}\end{aligned}$$

$$\left\{L_{\pm1},L_0,G_{\pm\frac{1}{2}}^{\alpha A},J_0^a\right\}$$

$$\begin{aligned}L_n|\phi\rangle&=J_n^a|\phi\rangle=G_r^{\alpha A}|\phi\rangle=0\ \text{ for }n>0,r\geq\tfrac{1}{2},\\L_0|\phi\rangle&=h|\phi\rangle,J_0^3|\phi\rangle=j|\phi\rangle,\end{aligned}$$

$$h\geq|j|,h\geq J$$

$$G^{+A}_{-\frac{1}{2}}|\phi\rangle = 0$$

$$h = j = \frac{a-1}{2}, \frac{a}{2}, \frac{a+1}{2}.$$

$$|-\rangle_a : h = j = \frac{a-1}{2},$$
$$|\dot{A}\rangle_a : h = j = \frac{a}{2}$$
$$|+\rangle_a : h = j = \frac{a+1}{2},$$

$$|\dot{-}\rangle_a : \ \tilde{h} = \tilde{j} = \frac{a-1}{2},$$
$$|\dot{A}\rangle_a : \ \tilde{h} = \tilde{j} = \frac{a}{2},$$
$$|\dot{+}\rangle_a : \ \tilde{h} = \tilde{j} = \frac{a+1}{2}.$$

$\frac{1}{2}-$ BPS: | chiral primary $\rangle|\overline{\text{chiral primary}}\rangle$
$\frac{1}{4}-$ BPS: | anything $\rangle|\overline{\text{chiral primary}}\rangle$

$$(1)\ |\phi\rangle, \quad (4)\ G|\phi\rangle, \quad (6)\ G^2|\phi\rangle, \quad (4)\ G^3|\phi\rangle, \quad (1)\ G^4|\phi\rangle\ ,$$

$$(1)\ |\phi\rangle, \quad (2)\ G|\phi\rangle, \quad (1)\ G^2|\phi\rangle\ .$$

$$h(\eta) = h + \eta j + \frac{c}{24}\eta^2, j(\eta) = j + \frac{c}{12}\eta$$

$\frac{1}{2}-$ BPS: | R ground state $\rangle|\overline{\text{R ground state}}\rangle$,
$\frac{1}{4}-$ BPS: | anything $\rangle$ | $\overline{\text{R ground state}}\rangle$.

$$X^{\dot{A}A} \to -X^{\dot{A}A}, \psi^{\alpha\dot{A}} \to -\psi^{\alpha\dot{A}}, \tilde{\psi}^{\dot{\alpha}\dot{A}} \to -\tilde{\psi}^{\dot{\alpha}\dot{A}}$$

$$|-\dot{\pm}\rangle_a : h = j = \frac{a-1}{2}, \tilde{h} = \tilde{j} = \frac{a\pm1}{2}$$
$$|I\rangle_a : h = j = \tilde{h} = \tilde{j} = \frac{a}{2}$$
$$|+\dot{\pm}\rangle_a : h = j = \frac{a+1}{2}, \tilde{h} = \tilde{j} = \frac{a\pm1}{2}$$

$$\prod_{\psi} \ (|\psi\rangle)^{N_\psi}$$

$$\prod_{\psi,n,m} \ [(L_{-1})^n (J_0^-)^m |\psi\rangle]^{N_{\psi,n,m}}$$
$$\prod_{\psi,n,m} \left[(L_{-1})^n (J_0^-)^m G^{-A}_{-\frac{1}{2}}|\psi\rangle\right]^{N'_{\psi,n,m}}$$
$$\prod_{\psi,n,m} \left[(L_{-1})^n (J_0^-)^m \left(G^{-1}_{-\frac{1}{2}} G^{-2}_{-\frac{1}{2}} - \frac{1}{2h} L_{-1} J_0^-\right)|\psi\rangle\right]^{N''_{\psi,n,m}}$$

$$\mathcal{H}_N^{\text{graviton}} = \bigoplus\nolimits'_{\{j,\tilde{j}_i;d_i\}} \aleph \otimes_i \ (j_i, \tilde{j}_i; d_i)_s$$

$$Z_R^{\mathrm{CFT}}(q,y,\tilde{q},\tilde{y}) \equiv \mathrm{Tr}_{\mathcal{H}_R^{\mathrm{CFT}}}\left[(-1)^F q^{L_0-\frac{c}{24}} y^{2J_0^3} \tilde{q}^{\tilde{L}_0-\frac{c}{24}} \tilde{y}^{2\tilde{J}_0^3}\right]$$

$$q = e^{2\pi i\tau}, y = e^{2\pi i\nu}, \tilde{q} = e^{2\pi i\tilde{\tau}}, \tilde{y} = e^{2\pi i\tilde{\nu}}$$

$$\mathcal{E}_R^{\mathrm{CFT}}(q,y) \equiv Z_R^{\mathrm{CFT}}(q,y,1,1) = \mathrm{Tr}_{\mathcal{H}_R^{\mathrm{CFT}}}\left[(-1)^F q^{L_0-\frac{c}{24}} y^{2J_0^3}\right]$$

$$Z_{\mathrm{NS}}^{\mathrm{CFT}}(q,y,\tilde{q},\tilde{y}) \equiv (y\tilde{y})^{\frac{c}{6}}(q\tilde{q})^{\frac{c}{12}} Z_{\mathrm{R}}^{\mathrm{CFT}}\left(q, yq^{\frac{1}{2}}, \tilde{q}, \tilde{y}\tilde{q}^{\frac{1}{2}}\right) = \mathrm{Tr}_{\mathcal{H}(\mathrm{NS})}\left[(-1)^F q^{L_0} y^{2J_0^3} \tilde{q}^{\tilde{L}_0} \tilde{y}^{2\tilde{J}_0^3}\right]$$

$$\mathcal{E}_{\mathrm{NS}}^{\mathrm{CFT}}(q,y) \equiv Z_{\mathrm{NS}}^{\mathrm{CFT}}(q,y,1,1) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}^{\mathrm{CFT}}}\left[(-1)^F q^{L_0} y^{2J_0^3}\right]$$

$$\hat{\mathcal{E}}_{\mathrm{NS}}^{\mathrm{CFT}}(q,y) \equiv \frac{1}{2}\left(\tilde{y}\partial_{\tilde{y}}\right)^2 Z_{\mathrm{NS}}^{\mathrm{CFT}}(q,y,\tilde{q},\tilde{y})\Big|_{\tilde{q}=\tilde{y}=1} = \frac{1}{2}\mathrm{Tr}_{\mathcal{H}_{\mathrm{CFT}}^{\mathrm{NS}}}\left[(-1)^F q^{L_0} y^{2J_0^3}\left(2\tilde{J}_0^3\right)^2\right]$$

$$\begin{aligned}\hat{\mathcal{E}}_{\mathrm{NS}}^{\mathrm{CFT}}(q,y) &\supset \frac{1}{2}(-1)^{2j} q^h y^{2j}\left[(-1)^{2\tilde{j}}(2\tilde{j})^2 + 2(-1)^{2\tilde{j}+1}(2\tilde{j}+1)^2 + (-1)^{\tilde{j}+2}(2\tilde{j}+2)^2\right] \\ &= (-1)^{2j-2\tilde{j}} q^h y^{2j}\end{aligned}$$

$$\varepsilon_{R,K3}^{\mathrm{CFT}}(q,y) = 8\sum_{i=2}^4 \frac{\vartheta_i(\nu,\tau)}{\vartheta_i(0,\tau)} = \sum_{m,\ell} c_R^{K3}(m,\ell) q^m y^\ell$$

$$c_R^{K3}(m,\ell) = c(4m-\ell^2)$$

$$\mathcal{Z}_{R,K3}^{\mathrm{CFT}}(p,q,y) \equiv \sum_{N=1}^\infty p^N \mathcal{E}_R^{\mathrm{CFT}}(q,y;N) = \prod_{n,m,\ell}\left(1-p^n q^m y^\ell\right)^{-c_R^{K3}(nm,\ell)}$$

$$\mathcal{Z}_{\mathrm{NS},K3}^{\mathrm{CFT}}(p,q,y) \equiv \sum_{N=1}^\infty p^N \mathcal{E}_{\mathrm{NS}}^{\mathrm{CFT}}(q,y;N) = \prod_{n,m,\ell}\left(1-p^n q^m y^\ell\right)^{-c_{\mathrm{NS}}^{K3}(n,m,\ell)}$$

$$c_{\mathrm{NS}}^{K3}(n,m,\ell) = c(4nm-n^2-\ell^2)$$

$$\begin{aligned}z_{K3}^{\mathrm{graviton}}(p,q,y,\tilde{y}) &= \sum_{k=1}^\infty p^k\left[\phi_{\frac{k-1}{2}}^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2}}(\tilde{y}) + 20\phi_{\frac{k}{2}}^{(s)}(q,y)\tilde{\chi}_0(\tilde{y}) + \phi_{\frac{k+1}{2}}^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2}}(\tilde{y})\right] \\ &= \frac{p}{(1-q)(1-y^2)}\left[\frac{(1-\sqrt{q}y)^2\left(\tilde{\chi}_{\frac{1}{2}}(\tilde{y}) + 20\sqrt{q}y^{-1}\tilde{\chi}_0(\tilde{y}) + qy^{-2}\tilde{\chi}_{\frac{1}{2}}(\tilde{y})\right)}{1-p\sqrt{q}y^{-1}}\right] \\ &\quad - \frac{y^2\left(1-\sqrt{q}y^{-1}\right)^2\left(\tilde{\chi}_{\frac{1}{2}}(\tilde{y}) + 20\sqrt{q}y\tilde{\chi}_0(\tilde{y}) + qy^2\tilde{\chi}_{\frac{1}{2}}(\tilde{y})\right)}{1-p\sqrt{q}y}\Bigg] \\ &= \sum_{n,m,l} c_{\mathrm{graviton}}^{K3}(n,m,\ell,\tilde{\ell}) p^n q^m y^\ell \tilde{y}^{\tilde{\ell}}\end{aligned}$$

$$X_k^{K3}(q,y,\tilde{y}) \equiv \phi_{\frac{k-1}{2}}^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2},k}(\tilde{y}) + 20\phi_{\frac{k}{2}}^{(s)}(q,y)\tilde{\chi}_{0;k}(\tilde{y}) + \phi_{\frac{k+1}{2}}^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2},k}(\tilde{y})$$

$$\begin{aligned}\mathcal{Z}_{K3}^{\mathrm{graviton}}(p,q,y,\tilde{y}) &\equiv \sum_{N=1}^\infty p^N Z_{K3}^{\mathrm{graviton}}(q,y,\tilde{y};N) \\ &= \mathrm{PE}\left[z_{K3}^{\mathrm{graviton}}\right](p,q,y,\tilde{y}) \\ &\equiv \exp\left(\sum_{n=1}^\infty \frac{1}{n}\sum_{k=1}^\infty p^{nk} X_k^{K3}(q^n,y^n,\tilde{y}^n)\right) \\ &= \prod_{n,m,\ell,\tilde{\ell}}\left(1-p^n q^m y^\ell \tilde{y}^{\tilde{\ell}}\right)^{-c_{\mathrm{graviton}}^{K3}(n,m,\ell,\tilde{\ell})}\end{aligned}$$

$$\mathcal{E}_{K3}^{\text{graviton}}(q,y;N)=Z_{K3}^{\text{graviton}}(q,y,1;N)$$

$$z_{R,T^4}^{\text{CFT}}(q,y,\tilde{q},\tilde{y})=\left(\frac{\vartheta_1(v,\tau)}{\eta(q)^3}\right)^2\left(\frac{\vartheta_1(\tilde{v},\tilde{\tau})}{\eta(\tilde{q})^3}\right)^2=\sum_{m,\ell,\tilde{m},\tilde{\ell}}\ c_R^{T^4}(m,\ell,\tilde{m},\tilde{\ell})q^m y^\ell \tilde{q}^{\tilde{m}}\tilde{y}^{\tilde{\ell}},$$

$$Z_{R,T^4}^{\text{CFT}}(p,q,y,\tilde{q},\tilde{y})\equiv\sum_{N=1}^{\infty}\ p^N Z_{R,T^4}^{\text{CFT}}(q,y,\tilde{q},\tilde{y};N)=\prod_{n,m,\ell,\tilde{m},\tilde{\ell}}\left(1-p^n q^m y^\ell \tilde{q}^{\tilde{m}}\tilde{y}^{\tilde{\ell}}\right)^{-c_R^{T^4}(nm,\ell,n\tilde{m},\tilde{\ell})}$$

$$Z_{\text{NS},T^4}^{\text{CFT}}(q,y,\tilde{q},\tilde{y};N)=(y\tilde{y})^{\frac{c}{6}}(q\tilde{q})^{\frac{c}{12}}Z_{R,T^4}^{\text{CFT}}\left(q,yq^{\frac{1}{2}},\tilde{q},\tilde{y}\tilde{q}^{\frac{1}{2}};N\right),$$

$$z_{T^4}^{\text{graviton}}(p,q,y,\tilde{q},\tilde{y})=\sum_{k=1}^{\infty}\ p^k\left[\phi_{\frac{k-1}{2}}^{(s)}(q,y)-2\phi_{\frac{k}{2}}^{(s)}(q,y)+\phi_{\frac{k+1}{2}}^{(s)}(q,y)\right]\omega_k(\tilde{q},\tilde{y})$$
$$=\sum_{n,m,l,\tilde{m},\tilde{\ell}}\ c_{T^4}^{\text{graviton}}(n,m,l,\tilde{m},\tilde{\ell})p^n q^m y^l \tilde{q}^{\tilde{m}}\tilde{y}^{\tilde{\ell}}$$

$$\omega_k(\tilde{q}^n,\tilde{y}^n)\equiv\tilde{q}^{n\frac{k-1}{2}}\tilde{y}^{n(k-1)}\left(1-\tilde{q}^{\frac{n}{2}}\tilde{y}^n\right)^2.$$

$$X_k^{T^4}(q,y)\equiv\phi_{\frac{k-1}{2}}^{(s)}(q,y)-2\phi_{\frac{k}{2}}^{(s)}(q,y)+\phi_{\frac{k+1}{2}}^{(s)}(q,y),$$

$$Z_{T^4}^{\text{graviton}}(p,q,y,\tilde{q},\tilde{y})\equiv\sum_{N=1}^{\infty}\ p^N Z_{T^4}^{\text{graviton}}(q,y,\tilde{q},\tilde{y};N)$$
$$=\text{PE}\left[z_{T^4}^{\text{graviton}}\right](p,q,y,\tilde{q},\tilde{y})$$
$$=\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\ p^{nk}X_k^{T^4}(q^n,y^n)\omega_k(\tilde{q}^n,\tilde{y}^n)\right)$$
$$=\prod_{n,m,l,\tilde{m},\tilde{\ell}}\left(1-p^n q^m y^l \tilde{q}^{\tilde{m}}\tilde{y}^{\tilde{\ell}}\right)^{-c_{T^4}^{\text{graviton}}(n,m,l,\tilde{m},\tilde{\ell})}$$

$$h\leq\frac{N+1}{4}\ (K3)$$

$$h<\frac{N}{4}\ (T^4)$$

$$J_{-(n+1)}^+|-\rangle_1=\frac{1}{n!}L_{-1}^n|+\rangle_1, n\geq 0$$

$$\{|h\rangle,L_{-1}|h\rangle,L_{-1}^2|h\rangle,\dots\}=\{L_{-1}^n|h\rangle\}_{n=0}^{\infty}$$

$$z(q)=q^h(1+q+q^2+\cdots)=\frac{q^h}{1-q}=\chi_h^{\text{global}}(q)=\sum_m\ c(m)q^m$$

$$\prod_{n\geq 0}\ [(L_{-1})^n|h\rangle]^{N_n}$$

$$Z(q,p)=\sum_N\ Z_N(q)p^N$$

$$Z(q,p)=\prod_m\frac{1}{(1-q^m p)^{c(m)}}=\prod_{n\geq 0}\frac{1}{1-q^{h+n}p}$$

$$h^{(\text{T})}=2h\text{: }(|h\rangle)^2$$
$$h^{(\text{T})}=2h+1\text{: }(L_{-1}|h\rangle)|h\rangle$$
$$h^{(\text{T})}=2h+2\text{: }(L_{-1}^2|h\rangle)|h\rangle,(L_{-1}|h\rangle)^2$$
$$h^{(\text{T})}=2h+3\text{: }(L_{-1}^3|h\rangle)|h\rangle,(L_{-1}^2|h\rangle)(L_{-1}|h\rangle)$$
$$\dots$$

$$|\phi\rangle = a(L_{-1}^2|h\rangle)|h\rangle + 2b(L_1L_{-1}|h\rangle)(L_{-1}|h\rangle) + d(L_{-2}|h\rangle)|h\rangle$$

$$L_1^{(\mathrm{T})}|\phi\rangle = ((4h+1)a + 4hb + 3d)(L_{-1}|h\rangle)|h\rangle = 0$$
$$L_2^{(\mathrm{T})}|\phi\rangle = \left(6ha + \left(\frac{c}{2} + 4h\right)d\right)(|h\rangle)^2 = 0$$

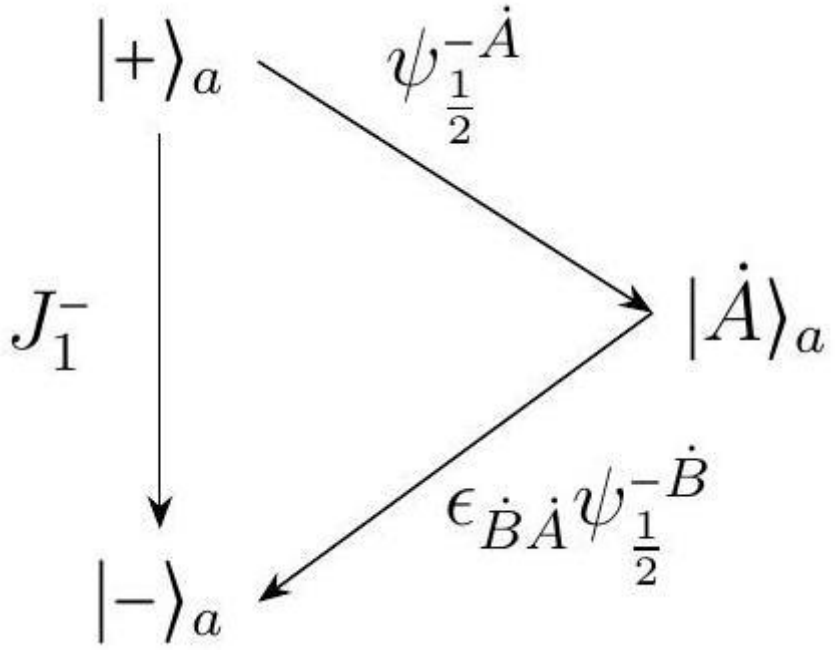
$$Z_N(q) = \frac{1}{1-q}\cdot(1-q)Z_N(q) = \sum_{h'}\ c_{h'}\chi_{h'}^{\text{global}}(q)$$

$$(1-q)Z_N(q) \equiv \sum_{h'}\ c_{h'}q^{h'}$$

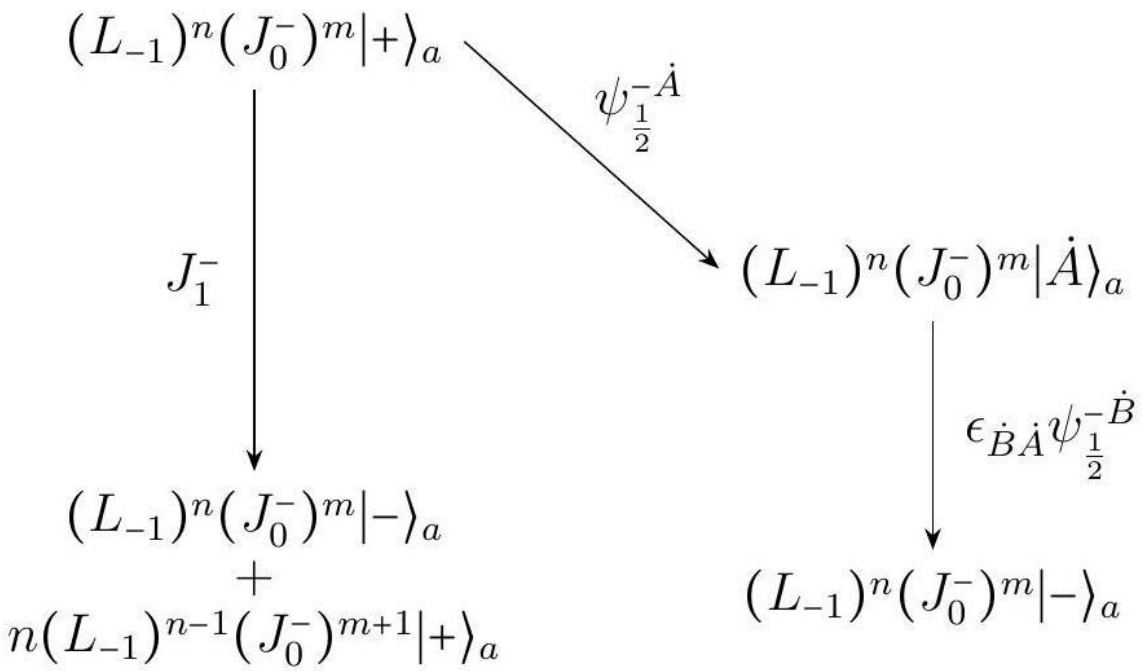
$$Z_k(q) = \sum_{h'}\ c_{h'}\chi_{h'}^{\text{global}}(q) \to \sum_{h'}\ c_{h'}\chi_{h'}^{\text{Vir}}(q), \chi_{h'}^{\text{Vir}}(q) = \frac{q^{h'}}{\prod_{n\geq 1}(1-q^n)}$$

$$Z_N(q) \to \frac{Z_N(q)}{\prod_{n\geq 2}(1-q^n)}$$

$$|-\rangle_a, \psi_{-\frac{1}{2}}^{+\dot{A}}|-\rangle_a = |\dot{A}\rangle_a, \psi_{-\frac{1}{2}}^{+\dot{1}}\psi_{-\frac{1}{2}}^{+\dot{2}}|-\rangle_a = J_{-1}^{+}|-\rangle_a = |+\rangle_a$$

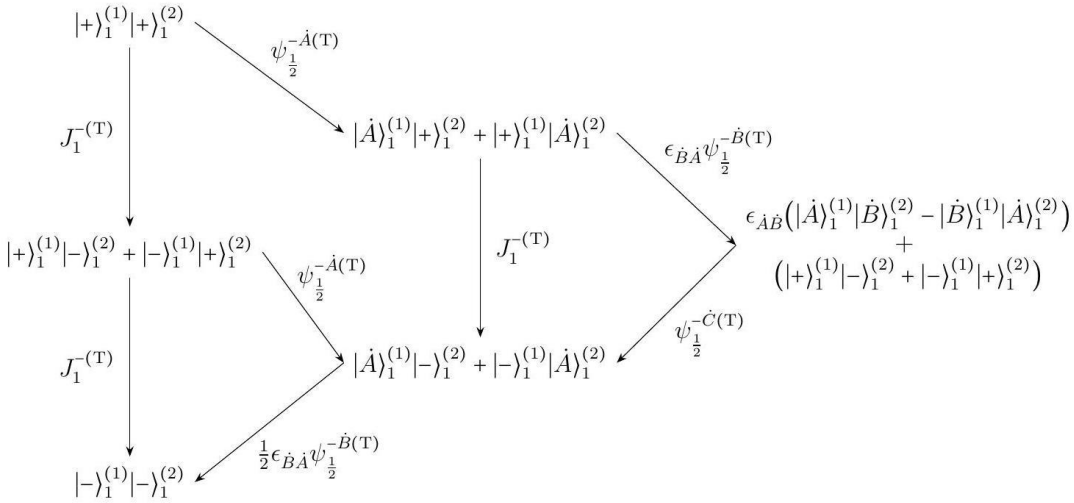


$$J_1^-\mathcal{O}|+\rangle_a = \mathcal{O}'|+\rangle_a + \mathcal{O}|-\rangle_a$$
$$\psi_{\frac{1}{2}}^{-\dot{A}}\mathcal{O}|+\rangle_a = \mathcal{O}'|+\rangle_a \pm \mathcal{O}|\dot{A}\rangle_a$$

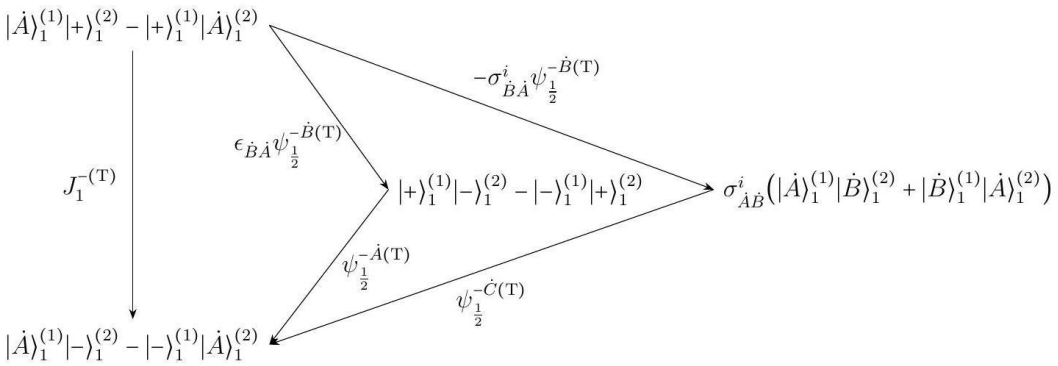


$$J_1^{-(\mathrm{T})}\left(\mathcal{O}_1^{(1)}|+\rangle_1^{(1)}\mathcal{O}_2^{(2)}|+\rangle_1^{(2)}\right)=\mathcal{O}_1^{(1)}|-\rangle_1^{(1)}\mathcal{O}_2^{(2)}|+\rangle_1^{(2)}+\mathcal{O}_1^{(1)}|+\rangle_1^{(1)}\mathcal{O}_2^{(2)}|-\rangle_1^{(2)}$$
$$+\mathcal{O}_1'^{(1)}|+\rangle_1^{(1)}\mathcal{O}_2^{(2)}|+\rangle_1^{(2)}+\mathcal{O}_1^{(1)}|+\rangle_1^{(1)}\mathcal{O}_2'^{(2)}|+\rangle_1^{(2)}$$

a)



b)



$$Z_{T^4}^{\text{graviton}}(q,y,\tilde{y};\{2\}) \equiv \prod_{m,l,\tilde{m},\tilde{l}} \left(1-p^2 q^m y^l \tilde{q}^{\tilde{m}} \tilde{y}^{\tilde{l}}\right)^{-c_{T^4}^{\text{graviton}}(2,m,l,\tilde{m},\tilde{l})}\Big|_{\tilde{p}^2}$$

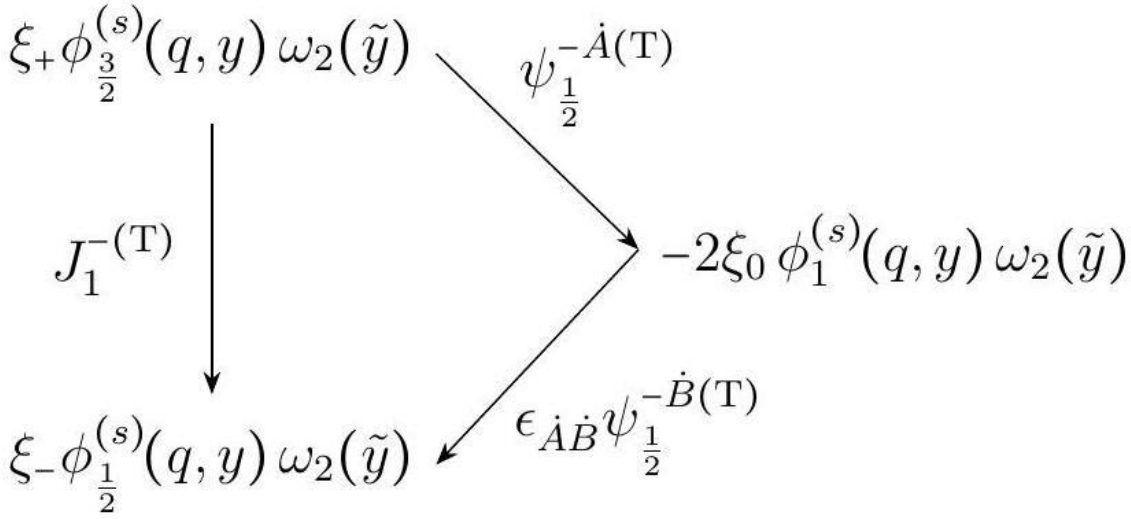
$$= X_2^{T^4}(q,y)\omega_2(\tilde{y})$$

$$\omega_k(\tilde{y}) \equiv \omega_k(1,\tilde{y}) = \tilde{\chi}_{\frac{1}{2};k}(\tilde{y}) - 2\tilde{\chi}_{0;k}(\tilde{y})$$

$$Z_{T^4}^{\text{graviton}}(q,y,\tilde{y},\xi;\{2\}) = \left(\xi_- \phi_{\frac{1}{2}}^{(s)}(q,y) - 2\xi_0 \phi_1^{(s)}(q,y) + \xi_+ \phi_{\frac{3}{2}}^{(s)}(q,y)\right)\omega_2(\tilde{y})$$

$$Z_{T^4}^{\text{red}}(q,y,\tilde{y},\xi;\{2\}) = \xi_- \phi_{\frac{1}{2}}^{(s)}(q,y)\omega_2(\tilde{y})$$

$$Z_{T^4}^{\text{gen}}(q,y,\tilde{y};\{2\}) = \mathbf{\Phi}_{\frac{1}{2};2}^{(s)}(q,y)\omega_2(\tilde{y})$$



$$Z_{T^4}^{\text{gen}}(q,y,\tilde{y};\{2\}) = \left[\phi_{\frac{1}{2}}^{(s)} - 2\phi_1^{(s)} + \phi_{\frac{3}{2}}^{(s)} - 2\phi_{0,1}^{(\ell)} - 12\phi_{0,2}^{(\ell)} + 5\phi_{\frac{1}{2}\frac{3}{2};2}^{(\ell)} - 4\phi_{1,2}^{(\ell)} + \cdots\right]\omega_2(\tilde{y})$$

$$= Z_{T^4}^{\text{graviton}}(q,y,\tilde{y};\{2\}) + \left[-2\phi_{0,1}^{(\ell)} - 12\phi_{0,2}^{(\ell)} + 5\phi_{\frac{1}{2}\frac{3}{2};2}^{(\ell)} - 4\phi_{1,2}^{(\ell)} + \cdots\right]\omega_2(\tilde{y})$$

$$\hat{\mathcal{E}}_{T^4}^{\text{gen}}(q,y;\{2\}) \equiv \frac{1}{2}\left(\tilde{y}\partial_{\tilde{y}}\right)^2 Z_{T^4}^{\text{gen}}(q,y,\tilde{y};\{2\})\Big|_{\tilde{y}=1} = \mathbf{\Phi}_{\frac{1}{2};2}^{(s)}(q,y).$$

$$Z_{T^4}^{\text{graviton}}(q,y,\tilde{y};\{1,1\}) \equiv \prod_{m,l,\tilde{m},\tilde{l}} \left(1-pq^m y^l \tilde{q}^{\tilde{m}} \tilde{y}^{\tilde{l}}\right)^{-c_{T^4}^{\text{graviton}}(1,m,l,\tilde{m},\tilde{l})}\Big|_{\substack{p^2 \\ \tilde{q}=1}}$$

$$= \frac{1}{2}\left(X_1^{T^4}(q,y)\omega_1(\bar{q},\bar{y})\right)^2 + \frac{1}{2}X_1^{T^4}(q^2,y^2)\omega_1(\bar{q}^2,\bar{y}^2)$$

$$Z^{\text{graviton}}_{T^4\{1,1\}} = \frac{1}{2}\xi_-^2\left[\left(\phi_0^{(s)}(q,y)\omega_1(\bar{y})\right)^2 + \phi_0^{(s)}(q^2,y^2)\omega_1(\bar{y}^2)\right] - 2\xi_-\xi_0\phi_0^{(s)}(q,y)\phi_{\frac{1}{2}}^{(s)}(q,y)(\omega_1(\bar{y}))^2$$
$$+\xi_-\xi_+\phi_0^{(s)}(q,y)\phi_1^{(s)}(q,y)(\omega_1(\bar{y}))^2 + \xi_0^2\left[2\left(\phi_{\frac{1}{2}}^{(s)}(q,y)\omega_1(\bar{y})\right)^2 - \phi_{\frac{1}{2}}^{(s)}(q^2,y^2)\omega_1(\bar{y}^2)\right]$$
$$-2\xi_0\xi_+\phi_{\frac{1}{2}}^{(s)}(q\cdot y)\phi_1^{(s)}(q,y)(\omega_1(\bar{y}))^2 + \frac{1}{2}\xi_+^2\left[\left(\phi_1^{(s)}(q,y)\omega_1(\bar{y})\right)^2 + \phi_1^{(s)}(q^2,y^2)\omega_1(\bar{y}^2)\right]$$

$$Z^{\text{graviton}}_{T^4\{1,1\}}\Big|_{\xi_-^2} = \frac{1}{2}\left[\left(\phi_0^{(s)}(q,y)\omega_1(\bar{y})\right)^2 + \phi_0^{(s)}(q^2,y^2)\omega_1(\bar{y}^2)\right]$$
$$= \phi_0^{(s)}(q,y)\frac{1}{2}[(\omega_1(\bar{y}))^2 + \omega_1(\bar{y}^2)]$$

$$\omega_{\text{sym}}(\tilde{y}) \equiv \frac{1}{2}[(\omega_1(\bar{y}))^2 + \omega_1(\bar{y}^2)] = \tilde{\chi}_{1;2}(\tilde{y}) - 2\tilde{\chi}_{\frac{1}{2};2}(\tilde{y}) + \tilde{\chi}_{0;2}(\tilde{y})$$

$$\omega_{\text{asym}}(\tilde{y}) \equiv \frac{1}{2}[(\omega_1(\bar{y}))^2 - \omega_1(\bar{y}^2)] = -2\tilde{\chi}_{\frac{1}{2};2}(\tilde{y}) + 4\tilde{\chi}_{0;2}(\tilde{y})$$

$$\phi_0^{(s)}\tilde{\chi}_{1;2}: |-\rangle_1^{(1)}|-\rangle_1^{(2)}\left(|\dot{\alpha}\rangle_1^{(1)}|\dot{\beta}\rangle_1^{(2)} + |\dot{\beta}\rangle_1^{(1)}|\dot{\alpha}\rangle_1^{(2)}\right)$$
$$-2\phi_0^{(s)}\tilde{\chi}_{\frac{1}{2};2}: |-\rangle_1^{(1)}|-\rangle_1^{(2)}\left(|\dot{\alpha}\rangle_1^{(1)}|\dot{C}\rangle_1^{(2)} + |\dot{C}\rangle_1^{(1)}|\dot{\alpha}\rangle_1^{(2)}\right)$$
$$\phi_0^{(s)}\tilde{\chi}_{0;2}: |-\rangle_1^{(1)}|-\rangle_1^{(2)}\left(|\dot{C}\rangle_1^{(1)}|\dot{D}\rangle_1^{(2)} - |\dot{D}\rangle_1^{(1)}|\dot{C}\rangle_1^{(2)}\right)$$

$$Z^{\text{graviton}}_{T^4\{1,1\}}\Big|_{\xi_-\xi_0} = -2\phi_0^{(s)}(q,y)\phi_{\frac{1}{2}}^{(s)}(q,y)(\omega_1(\bar{y}))^2$$
$$= -2\phi_{\frac{1}{2}}^{(s)}(q,y)\left(\tilde{\chi}_{1;2} - 4\tilde{\chi}_{\frac{1}{2};2} + 5\tilde{\chi}_{0;2}\right)$$

$$-2\phi_{\frac{1}{2}}^{(s)}(\tilde{\chi}_{1;2} + \tilde{\chi}_{0;2}): \left(|-\rangle_1^{(1)}|\dot{A}\rangle_1^{(2)} \pm |\dot{A}\rangle_1^{(1)}|-\rangle_1^{(2)}\right)\left(|\dot{\alpha}\rangle_1^{(1)}|\dot{\beta}\rangle_1^{(2)} \pm |\dot{\beta}\rangle_1^{(1)}|\dot{\alpha}\rangle_1^{(2)}\right),$$
$$8\phi_{\frac{1}{2}}^{(s)}\tilde{\chi}_{\frac{1}{2};2}: \left(|-\rangle_1^{(1)}|\dot{A}\rangle_1^{(2)} \pm |\dot{A}\rangle_1^{(1)}|-\rangle_1^{(2)}\right)\left(|\dot{\alpha}\rangle_1^{(1)}|\dot{C}\rangle_1^{(2)} \pm |\dot{C}\rangle_1^{(1)}|\dot{\alpha}\rangle_1^{(2)}\right),$$
$$-8\phi_{\frac{1}{2}}^{(s)}\tilde{\chi}_{0;2}: \left(|-\rangle_1^{(1)}|\dot{A}\rangle_1^{(2)} \pm |\dot{A}\rangle_1^{(1)}|-\rangle_1^{(2)}\right)\left(|\dot{C}\rangle_1^{(1)}|\dot{D}\rangle_1^{(2)} \mp |\dot{D}\rangle_1^{(1)}|\dot{C}\rangle_1^{(2)}\right).$$

$$-2\xi_-\xi_0\phi_{\frac{1}{2}}^{(s)}\tilde{\chi}_{1;2} \longrightarrow \xi_-^2\phi_0^{(s)}\tilde{\chi}_{1;2}$$
$$4\xi_-\xi_0\phi_{\frac{1}{2}}^{(s)}\tilde{\chi}_{\frac{1}{2};2} \longrightarrow -2\xi_-^2\phi_0^{(s)}\tilde{\chi}_{\frac{1}{2};2}$$
$$-2\xi_-\xi_0\phi_{\frac{1}{2}}^{(s)}\tilde{\chi}_{0;2} \longrightarrow \xi_-^2\phi_0^{(s)}\tilde{\chi}_{0;2}$$

$$Z^{\text{graviton}}_{T^4\{1,1\}}\Big|_{\xi_-\xi_+} = \phi_0^{(s)}(q,y)\phi_1^{(s)}(q,y)(\omega_1(\bar{y}))^2$$
$$= \phi_1^{(s)}(q,y)\left(\tilde{\chi}_{1;2} - 4\tilde{\chi}_{\frac{1}{2};2} + 5\tilde{\chi}_{0;2}\right)$$

$$\phi_1^{(s)}(\tilde{\chi}_{1;2} + \tilde{\chi}_{0;2}): \left(|-\rangle_1^{(1)}|+\rangle_1^{(2)} \pm |+\rangle_1^{(1)}|-\rangle_1^{(2)}\right)\left(|\dot{\alpha}\rangle_1^{(1)}|\dot{\beta}\rangle_1^{(2)} \pm |\dot{\beta}\rangle_1^{(1)}|\dot{\alpha}\rangle_1^{(2)}\right),$$
$$-4\phi_1^{(s)}\tilde{\chi}_{\frac{1}{2};2}: \left(|-\rangle_1^{(1)}|+\rangle_1^{(2)} \pm |+\rangle_1^{(1)}|-\rangle_1^{(2)}\right)\left(|\dot{\alpha}\rangle_1^{(1)}|\dot{C}\rangle_1^{(2)} \pm |\dot{C}\rangle_1^{(1)}|\dot{\alpha}\rangle_1^{(2)}\right),$$
$$4\phi_1^{(s)}\tilde{\chi}_{0;2}: \left(|-\rangle_1^{(1)}|+\rangle_1^{(2)} \pm |+\rangle_1^{(1)}|-\rangle_1^{(2)}\right)\left(|\dot{C}\rangle_1^{(1)}|\dot{D}\rangle_1^{(2)} \mp |\dot{D}\rangle_1^{(1)}|\dot{C}\rangle_1^{(2)}\right).$$

$$Z^{\text{graviton}}_{T^4\{1,1\}}\Big|_{\xi_0^2} = 2\left(\phi^{(s)}_{\frac{1}{2}}(q,y)\omega_1(\bar{y})\right)^2 - \phi^{(s)}_{\frac{1}{2}}(q^2,y^2)\omega_1(\bar{y}^2)$$
$$= \phi^{(s)}_1(q,y)\left[\omega_{\text{sym}}(\tilde{y}) + 3\omega_{\text{asym}}(\tilde{y})\right] + \sum_{\substack{h\geq 1\\ \text{odd}}} \phi^{(\ell)}_{0,h}(q,y)\left[3\omega_{\text{sym}}(\tilde{y}) + \omega_{\text{asym}}(\tilde{y})\right]$$
$$+ \sum_{\substack{h\geq 2\\ \text{even}}} \phi^{(\ell)}_{0,h}(q,y)\left[\omega_{\text{sym}}(\tilde{y}) + 3\omega_{\text{asym}}(\tilde{y})\right]$$

$$\phi^{(s)}_1\tilde{\chi}_{1;2}: \left(|\dot{A}\rangle^{(1)}_1|\dot{B}\rangle^{(2)}_1 - |\dot{B}\rangle^{(1)}_1|\dot{A}\rangle^{(2)}_1\right)\left(|\dot{\alpha}\rangle^{(1)}_1|\dot{\beta}\rangle^{(2)}_1 + |\dot{\beta}\rangle^{(1)}_1|\dot{\alpha}\rangle^{(2)}_1\right),$$
$$3\phi^{(s)}_1\tilde{\chi}_{0;2}: \left(|\dot{A}\rangle^{(1)}_1|\dot{B}\rangle^{(2)}_1 + |\dot{B}\rangle^{(1)}_1|\dot{A}\rangle^{(2)}_1\right)\left(|\dot{\alpha}\rangle^{(1)}_1|\dot{\beta}\rangle^{(2)}_1 - |\dot{\beta}\rangle^{(1)}_1|\dot{\alpha}\rangle^{(2)}_1\right),$$
$$-8\phi^{(s)}_1\tilde{\chi}_{\frac{1}{2};2}: \left(|\dot{A}\dot{\alpha}\rangle^{(1)}_1|\dot{B}\dot{C}\rangle^{(2)}_1 \pm |\dot{B}\dot{C}\rangle^{(1)}_1|\dot{A}\dot{\alpha}\rangle^{(2)}_1\right),$$
$$10\phi^{(s)}_1\tilde{\chi}_{0;2}: \left(|\dot{A}\rangle^{(1)}_1|\dot{B}\rangle^{(2)}_1 \pm |\dot{B}\rangle^{(1)}_1|\dot{A}\rangle^{(2)}_1\right)\left(|\dot{C}\rangle^{(1)}_1|\dot{D}\rangle^{(2)}_1 \pm |\dot{D}\rangle^{(1)}_1|\dot{C}\rangle^{(2)}_1\right),$$

$$Z^{\text{red}}_{T^4}(q,y,\tilde{y};\{1,1\}) = \phi^{(s)}_0(q,y)\omega_{\text{sym}}(\tilde{y}) - 2\phi^{(s)}_{\frac{1}{2}}(q,y)\omega_{\text{asym}}(\tilde{y})$$

$$Z^{\text{gen}}_{T^4}(q,y,\tilde{y};\{1,1\}) = \boldsymbol{\Phi}^{(s)}_{0;2}(q,y)\omega_{\text{sym}}(\tilde{y}) - 2\boldsymbol{\Phi}^{(s)}_{\frac{1}{2};2}(q,y)\omega_{\text{asym}}(\tilde{y})$$

$$Z^{\text{gen}}_{T^4\{1,1\}} \approx Z^{\text{graviton}}_{T^4\{1,1\}} + 6\phi^{(\ell)}_{0,1}\left(2\tilde{\chi}_{0;2} - \tilde{\chi}_{\frac{1}{2};2}\right) - 2\phi^{(\ell)}_{\frac{1}{2},\frac{3}{2}}\left(17\tilde{\chi}_{0;2} - 10\tilde{\chi}_{\frac{1}{2};2} + \tilde{\chi}_{1;2}\right)$$
$$+2\phi^{(\ell)}_{0,2}\left(47\tilde{\chi}_{0;2} - 31\tilde{\chi}_{\frac{1}{2};2} + 5\tilde{\chi}_{1;2}\right) + 2\phi^{(\ell)}_{1,2}\left(16\tilde{\chi}_{0;2} - 11\tilde{\chi}_{\frac{1}{2};2} + 2\tilde{\chi}_{1;2}\right) + \cdots$$

$$\hat{\mathcal{E}}^{\text{gen}}_{T^4}(q,y;\{1,1\}) \equiv \frac{1}{2}\left(\tilde{y}\partial_{\tilde{y}}\right)^2 Z^{\text{gen}}_{T^4}(q,y,\tilde{y};\{1,1\})\Big|_{\tilde{y}=1} = 2\boldsymbol{\Phi}^{(s)}_{0;2}(q,y) + 4\boldsymbol{\Phi}^{(s)}_{\frac{1}{2};2}(q,y)$$

$$\hat{\mathcal{E}}^{\text{gen}}_{T^4}(q,y;2) = 2\boldsymbol{\Phi}^{(s)}_{0;2}(q,y) + 5\boldsymbol{\Phi}^{(s)}_{\frac{1}{2};2}(q,y)$$

$$\hat{\mathcal{E}}^{\text{CFT}}_{\text{NS}}(q,y;2) = 2 + \sqrt{q}(y^{-1}+y) + q(-12 - 6y^{-2} - 6y^2) + q^{\frac{3}{2}}(y^{-3} + 39y^{-1} + 39y + y^3)$$
$$+q^2(-192 + 2y^{-4} - 56y^2 - 56y^{-2} + 2y^4) + \cdots$$
$$\hat{\mathcal{E}}^{\text{graviton}}_{T^4}(q,y;2) = 2 + \sqrt{q}(y^{-1}+y) + q(-4 - 6y^{-2} - 6y^2) + q^{\frac{3}{2}}(y^{-3} + 6y^{-1} + 6y + y^3)$$
$$+q^2(2 + 2y^{-4} - 4y^2 - 4y^{-2} + 2y^4) + \cdots$$
$$\hat{\mathcal{E}}^{\text{gen}}_{T^4}(q,y;2) = 2 + \sqrt{q}(y^{-1}+y) + q(-12 - 6y^{-2} - 6y^2) + q^{\frac{3}{2}}(y^{-3} + 39y^{-1} + 39y + y^3)$$
$$+q^2(-150 + 2y^{-4} - 56y^2 - 56y^{-2} + 2y^4) + \cdots$$

$$\hat{\mathcal{E}}^{\text{gen}}_{T^4}(q,y;2) - \hat{\mathcal{E}}^{\text{graviton}}_{T^4}(q,y;2) \approx -8q + 33q^{\frac{3}{2}}(y^{-1}+y) + q^2(-52y^{-2} - 152 - 52y^2).$$

$$\left(Z^{\text{gen}}_{T^4\{1,1\}} - Z^{\text{graviton}}_{T^4\{1,1\}}\right)\Big|_q = 3\phi^{(\ell)}_{0,1}(q,y)\omega_{\text{asym}}(\tilde{y})\Big|_q = 3\omega_{\text{asym}}(\tilde{y}) \overset{\text{MEG}}{\to} -6,$$
$$\left(Z^{\text{gen}}_{T^4\{2\}} - Z^{\text{graviton}}_{T^4\{2\}}\right)\Big|_q = -2\phi^{(\ell)}_{0,1}(q,y)\omega_2(\tilde{y})\Big|_q = -2\omega_2(\tilde{y}) \overset{\text{MEG}}{\to} -2,$$

$$\psi^{-\dot{A}}_{-\frac{1}{2}}|-\rangle_2 \otimes \begin{cases}|\dot{\alpha}\rangle_2\\ |\dot{B}\rangle_2\end{cases},$$

$$\sigma^{i}_{\dot{A}\dot{B}}\psi^{-\dot{A}(\mathrm{T})}_{-\frac{1}{2}}\left(|-\rangle^{(1)}_{1}|\dot{B}\rangle^{(2)}_{1}-|\dot{B}\rangle^{(1)}_{1}|-\rangle^{(2)}_{1}\right)\otimes\begin{cases}\left(|\dot{\alpha}\rangle^{(1)}_{1}|\dot{\beta}\rangle^{(2)}_{1}-|\dot{\beta}\rangle^{(1)}_{1}|\dot{\alpha}\rangle^{(2)}_{1}\right)\\ \left(|\dot{\alpha}\rangle^{(1)}_{1}|\dot{C}\rangle^{(2)}_{1}-|\dot{C}\rangle^{(1)}_{1}|\dot{\alpha}\rangle^{(2)}_{1}\right)\\ \left(|\dot{C}\rangle^{(1)}_{1}|\dot{D}\rangle^{(2)}_{1}+|\dot{D}\rangle^{(1)}_{1}|\dot{C}\rangle^{(2)}_{1}\right)\end{cases}$$

$$\psi^{+\dot{A}}_{-\frac{1}{2}}\psi^{-\dot{B}}_{-\frac{1}{2}}|-\rangle_{2}\otimes\begin{cases}|\dot{\alpha}\rangle_{2}\\ |\dot{B}\rangle_{2}\end{cases},J^{3}_{-1}|-\rangle_{2}\otimes\begin{cases}|\dot{\alpha}\rangle_{2}\\ |\dot{B}\rangle_{2}\end{cases},$$

$$G^{+\dot{A}(T)}_{-\frac{3}{2}}|-\rangle^{(1)}_{1}|-\rangle^{(2)}_{1}\otimes\begin{cases}\left(|\dot{\alpha}\rangle^{(1)}_{1}|\dot{\beta}\rangle^{(2)}_{1}+|\dot{\beta}\rangle^{(1)}_{1}|\dot{\alpha}\rangle^{(2)}_{1}\right)\\ \left(|\dot{\alpha}\rangle^{(1)}_{1}|\dot{C}\rangle^{(2)}_{1}+|\dot{C}\rangle^{(1)}_{1}|\dot{\alpha}\rangle^{(2)}_{1}\right),\\ \left(|\dot{C}\rangle^{(1)}_{1}|\dot{D}\rangle^{(2)}_{1}-|\dot{D}\rangle^{(1)}_{1}|\dot{C}\rangle^{(2)}_{1}\right)\end{cases}$$

$$\psi^{-\dot{A}(T)}_{-\frac{1}{2}}\psi^{+\dot{B}(T)}_{-\frac{1}{2}}\left(|-\rangle^{(1)}_{1}|\dot{C}\rangle^{(2)}_{1}-|\dot{C}\rangle^{(1)}_{1}|-\rangle^{(2)}_{1}\right)\otimes\begin{cases}\left(|\dot{\alpha}\rangle^{(1)}_{1}|\dot{\beta}\rangle^{(2)}_{1}-|\dot{\beta}\rangle^{(1)}_{1}|\dot{\alpha}\rangle^{(2)}_{1}\right)\\ \left(|\dot{\alpha}\rangle^{(1)}_{1}|\dot{C}\rangle^{(2)}_{1}-|\dot{C}\rangle^{(1)}_{1}|\dot{\alpha}\rangle^{(2)}_{1}\right)\\ \left(|\dot{C}\rangle^{(1)}_{1}|\dot{D}\rangle^{(2)}_{1}+|\dot{D}\rangle^{(1)}_{1}|\dot{C}\rangle^{(2)}_{1}\right)\end{cases}$$

$$\hat{\mathcal{E}}^{\mathrm{CFT}}_{\mathrm{NS}}(q,y;2)=2\boldsymbol{\Phi}^{(s)}_{0;2}+5\boldsymbol{\Phi}^{(s)}_{\frac{1}{2};2}-42\boldsymbol{\Phi}^{(\ell)}_{0,2;2}-70\boldsymbol{\Phi}^{(\ell)}_{0,3;2}-324\boldsymbol{\Phi}^{(\ell)}_{0,4;2}+\cdots$$

$$\hat{\mathcal{E}}^{\mathrm{CFT}}_{\mathrm{NS}}(q,y;\{1,1\})=2\boldsymbol{\Phi}^{(s)}_{0;2}+4\boldsymbol{\Phi}^{(s)}_{\frac{1}{2};2}+6\boldsymbol{\Phi}^{(\ell)}_{0,1;2}-14\boldsymbol{\Phi}^{(\ell)}_{0,2;2}+28\boldsymbol{\Phi}^{(\ell)}_{0,3;2}-42\boldsymbol{\Phi}^{(\ell)}_{0,4;2}+\cdots$$

$$\hat{\mathcal{E}}^{\mathrm{CFT}}_{\mathrm{NS}}(q,y;\{2\})=\boldsymbol{\Phi}^{(s)}_{\frac{1}{2};2}-6\boldsymbol{\Phi}^{(\ell)}_{0,1;2}-28\boldsymbol{\Phi}^{(\ell)}_{0,2;2}-98\boldsymbol{\Phi}^{(\ell)}_{0,3;2}-282\boldsymbol{\Phi}^{(\ell)}_{0,4;2}+\cdots$$

$$\left|+\dot{\alpha}\right\rangle_{a}$$

$$J^{-}_{1}\downarrow$$

$$\left|-\dot{\alpha}\right\rangle_{a}$$

$$Z_{K3}^{\text{graviton}}(q,y,\tilde{y};\{2\}) \equiv \prod_{m,\ell,\tilde{\ell}} \left(1-p^2 q^m y^\ell \tilde{y}^{\tilde{\ell}}\right)^{-c^{K3}_{\text{graviton}}(2,m,\ell,\tilde{\ell})}\Big|_{p^2} = X_2^{K3}(q,y,\tilde{y})$$

$$Z_{K3}^{\text{graviton}}(q,y,\tilde{y};\{1,1\}) \equiv \prod_{m,\ell,\tilde{\ell}} \left(1-pq^m y^\ell \tilde{y}^{\tilde{\ell}}\right)^{-c^{K3}_{\text{graviton}}(1,m,\ell,\tilde{\ell})}\Big|_{p^2}$$
$$= \frac{1}{2}(X_1^{K3}(q,y,\tilde{y}))^2 + \frac{1}{2}X_1^{K3}(q^2,y^2,\tilde{y}^2)$$

$$Z_{K3}^{\text{graviton}}(q,y,\tilde{y},\xi;\{2\}) = \xi_- \phi^{(s)}_{\frac{1}{2}}(q,y)\tilde{\chi}_{\frac{1}{2};2}(\tilde{y}) + 20\xi_0\phi_1^{(s)}(q,y)\tilde{\chi}_{0;2}(\tilde{y}) + \xi_+\phi^{(s)}_{\frac{3}{2}}(q,y)\tilde{\chi}_{\frac{1}{2};2}(\tilde{y})$$

$$Z_{K3}^{\text{gen}}(q,y,\tilde{y};\{2\}) = \Phi^{(s)}_{\frac{1}{2};2}(q,y)\tilde{\chi}_{\frac{1}{2};2}(\tilde{y}) + 20\Phi^{(s)}_{1;2}(q,y)\tilde{\chi}_{0;2}(\tilde{y})$$

$$Z_{K3\{1,1\}}^{\text{graviton}} \equiv Z_{K3}^{\text{graviton}}(q,y,\tilde{y},\xi;\{1,1\})$$

$$\begin{aligned}
Z_{K3\{1,1\}}^{\text{graviton}} =& \xi_-^2\frac{1}{2}\left[\left(\phi_0^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2};1}(\tilde{y})\right)^2 + \phi_0^{(s)}(q^2,y^2)\tilde{\chi}_{\frac{1}{2};1}(\tilde{y}^2)\right] + \xi_-\xi_+\phi_0^{(s)}(q,y)\phi_1^{(s)}(q,y)\left(\tilde{\chi}_{\frac{1}{2};1}(\tilde{y})\right)^2 \\
&+\xi_+^2\frac{1}{2}\left[\left(\phi_1^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2};1}(\tilde{y})\right)^2 + \phi_1^{(s)}(q^2,y^2)\tilde{\chi}_{\frac{1}{2};1}(\tilde{y}^2)\right] \\
&+20\xi_0\xi_-\phi^{(s)}_{\frac{1}{2}}(q,y)\phi_0^{(s)}(q,y)\tilde{\chi}_{0;1}(\tilde{y})\tilde{\chi}_{\frac{1}{2};1}(\tilde{y}) + 20\xi_0\xi_+\phi^{(s)}_{\frac{1}{2}}(q,y)\phi_1^{(s)}(q,y)\tilde{\chi}_{0;1}(\tilde{y})\tilde{\chi}_{\frac{1}{2};1}(\tilde{y}) \\
&+\xi_0^2\frac{1}{2}\left[\left(20\phi^{(s)}_{\frac{1}{2}}(q,y)\tilde{\chi}_{0;1}(\tilde{y})\right)^2 + 20\phi^{(s)}_{\frac{1}{2}}(q^2,y^2)\tilde{\chi}_{0;1}(\tilde{y}^2)\right]
\end{aligned}$$

$$\begin{aligned}
Z_{K3\{1,1\}}^{\text{graviton}}\Big|_{\xi_0} &= 20\left[\xi_-\phi^{(s)}_{\frac{1}{2}}(q,y)\phi_0^{(s)}(q,y) + \xi_+\phi^{(s)}_{\frac{1}{2}}(q,y)\phi_1^{(s)}(q,y)\right]\tilde{\chi}_{0;1}(\tilde{y})\tilde{\chi}_{\frac{1}{2};1}(\tilde{y}) \\
&= 20\left[\xi_-\phi^{(s)}_{\frac{1}{2}}(q,y) + \xi_+\left(\phi^{(s)}_{\frac{3}{2}}(q,y) + \sum_{h\geq 1}\phi^{(\ell)}_{\frac{1}{2},h+\frac{1}{2}}(q,y)\right)\right]\tilde{\chi}_{\frac{1}{2};2}(\tilde{y})
\end{aligned}$$

$$20\xi_0\xi_\pm\phi^{(s)}_{\frac{1}{2}}\tilde{\chi}_{\frac{1}{2};2}: |\pm\dot{\alpha}\rangle_1^{(1)}|I\rangle_1^{(2)} + |I\rangle_1^{(1)}|\pm\dot{\alpha}\rangle_1^{(2)}.$$

$$\begin{aligned}
Z_{K3\{1,1\}}^{\text{graviton}}\Big|_{(\xi_0)^0} \supset\ & \xi_-^2\phi_0^{(s)}(q,y)\tilde{\chi}_{1;2}(\tilde{y}) + \xi_-\xi_+\phi_1^{(s)}(q,y)\left[\tilde{\chi}_{1;2}(\tilde{y}) + \tilde{\chi}_{0;2}(\tilde{y})\right] \\
&+\xi_+^2\phi_2^{(s)}(q,y)\tilde{\chi}_{1;2}(\tilde{y})
\end{aligned}$$

$$\xi_-^2: |-\dot{\alpha}\rangle_1^{(1)}|-\dot{\beta}\rangle_1^{(2)} + |-\dot{\beta}\rangle_1^{(1)}|-\dot{\alpha}\rangle_1^{(2)}$$
$$\xi_-\xi_+: \left(|+\rangle_1^{(1)}|-\rangle_1^{(2)} \pm |-\rangle_1^{(1)}|+\rangle_1^{(2)}\right)\left(|\dot{\alpha}\rangle_1^{(1)}|\dot{\beta}\rangle_1^{(2)} \pm |\dot{\beta}\rangle_1^{(1)}|\dot{\alpha}\rangle_1^{(2)}\right)$$
$$\xi_+^2: |+\dot{\alpha}\rangle_1^{(1)}|+\dot{\beta}\rangle_1^{(2)} + |+\dot{\beta}\rangle_1^{(1)}|+\dot{\alpha}\rangle_1^{(2)}$$

$$|+\dot{\alpha}\rangle_1^{(1)}|+\dot{\beta}\rangle_1^{(2)} + |+\dot{\beta}\rangle_1^{(1)}|+\dot{\alpha}\rangle_1^{(2)}$$

$$\downarrow J_1^{-(\mathrm{T})}$$

$$\left(|+\rangle_1^{(1)}|-\rangle_1^{(2)} + |-\rangle_1^{(1)}|+\rangle_1^{(2)}\right)\left(|\dot{\alpha}\rangle_1^{(1)}|\dot{\beta}\rangle_1^{(2)} + |\dot{\beta}\rangle_1^{(1)}|\dot{\alpha}\rangle_1^{(2)}\right)$$

$$\downarrow J_1^{-(\mathrm{T})}$$

$$|-\dot{\alpha}\rangle_1^{(1)}|-\dot{\beta}\rangle_1^{(2)} + |-\dot{\beta}\rangle_1^{(1)}|-\dot{\alpha}\rangle_1^{(2)}$$

$$|+\dot{\alpha}\rangle_1^{(1)}|I\rangle_1^{(2)} + |I\rangle_1^{(1)}|+\dot{\alpha}\rangle_1^{(2)}$$

$$\downarrow J_1^{-(\mathrm{T})}$$

$$|-\dot{\alpha}\rangle_1^{(1)}|I\rangle_1^{(2)} + |I\rangle_1^{(1)}|-\dot{\alpha}\rangle_1^{(2)}$$

$$Z_{K3}^{\mathrm{red}}(q,y,\tilde{y},\xi;\{1,1\}) = \xi_-^2\phi_0^{(s)}(q,y)\tilde{\chi}_{1;2}(\tilde{y}) + \xi_-\xi_+\phi_1^{(s)}(q,y)\tilde{\chi}_{0;2}(\tilde{y}) + 20\xi_0\xi_-\phi_{\frac{1}{2}}^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2};2}(\tilde{y})$$
$$+\xi_0^2\left[210\phi_1^{(s)}(q,y) + 190\sum_{\substack{h\geq 1\\ \text{odd}}}\phi_{0,h}^{(\ell)}(q,y) + 210\sum_{\substack{h\geq 2\\ \text{even}}}\phi_{0,h}^{(\ell)}(q,y)\right]\tilde{\chi}_{0;2}(\tilde{y})$$

$$Z_{K3}^{\mathrm{gen}}(q,y,\tilde{y};\{1,1\}) = \Phi_{0;2}^{(s)}(q,y)\tilde{\chi}_{1;2}(\tilde{y}) + \Phi_{1;2}^{(s)}(q,y)\tilde{\chi}_{0;2}(\tilde{y}) + 20\Phi_{\frac{1}{2};2}^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2};2}(\tilde{y})$$
$$+\left[210\Phi_{1;2}^{(s)}(q,y) + 190\sum_{\substack{h\geq 1\\ \text{odd}}}\Phi_{0,h;2}^{(\ell)}(q,y) + 210\sum_{\substack{h\geq 2\\ \text{even}}}\Phi_{0,h;2}^{(\ell)}(q,y)\right]\tilde{\chi}_{0;2}(\tilde{y})$$

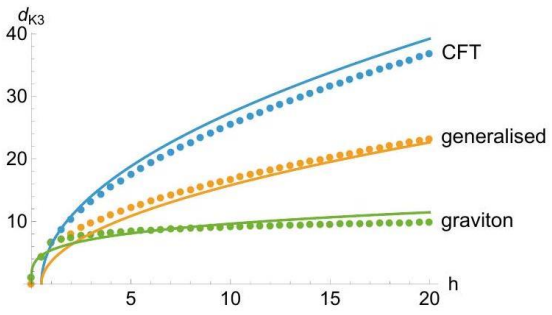
$$\mathcal{E}_{K3}^{\mathrm{gen}}(q,y;2) = 3\Phi_{0;2}^{(s)}(q,y) + 42\Phi_{\frac{1}{2};2}^{(s)}(q,y) + 231\Phi_{1;2}^{(s)}(q,y) + \sum_{h\geq 1} n_h^{(\ell)}\Phi_{0,h;2}^{(\ell)}(q,y)$$

$$\mathcal{E}_{K3}^{\mathrm{gen}}(q,y;2) - \mathcal{E}_{K3}^{\mathrm{graviton}}(q,y;2) = 2\phi_{\frac{1}{2},\frac{3}{2}}^{(\ell)} + 231\phi_{0,2}^{(\ell)} + 420\phi_{1,2}^{(\ell)} + \cdots$$

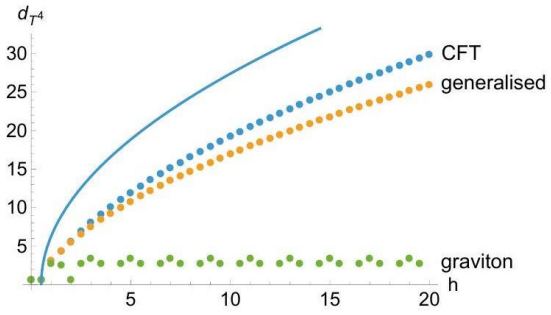
$$\mathcal{E}_{K3}^{\mathrm{gen}}(q,y;\{1,1\}) - \mathcal{E}_{K3}^{\mathrm{graviton}}(q,y;\{1,1\}) = 211\phi_{0,2}^{(\ell)} + 400\phi_{1,2}^{(\ell)} + \cdots$$
$$\mathcal{E}_{K3}^{\mathrm{gen}}(q,y;\{2\}) - \mathcal{E}_{K3}^{\mathrm{graviton}}(q,y;\{2\}) = 2\phi_{\frac{1}{2},\frac{3}{2}}^{(\ell)} + 20\phi_{0,2}^{(\ell)} + 20\phi_{1,2}^{(\ell)} + \cdots$$

$$J_{-1}^3|-\dot{\alpha}\rangle_2$$

$$\mathcal{E}_{\mathrm{NS}}^{\mathrm{CFT}}(q,y;2) \approx 3 + \sqrt{q}(42y^{-1} + 42y) + q(234y^{-2} + 360 + 234y^2)$$
$$+q^{\frac{3}{2}}(42y^{-3} - 3114y^{-1} - 3114y + 42y^3) + \cdots$$
$$\mathcal{E}_{K3}^{\mathrm{graviton}}(q,y;2) \approx 3 + \sqrt{q}(42y^{-1} + 42y) + q(234y^{-2} + 340 + 234y^2)$$
$$+q^{\frac{3}{2}}(42y^{-3} - 724y^{-1} - 724y + 42y^3) + \cdots$$
$$\mathcal{E}_{K3}^{\mathrm{gen}}(q,y;2) \approx 3 + \sqrt{q}(42y^{-1} + 42y) + q(234y^{-2} + 340 + 234y^2)$$
$$+q^{\frac{3}{2}}(42y^{-3} - 722y^{-1} - 722y + 42y^3) + \cdots$$



(a) $K3$



(b) T^4

$$T(z)=\frac{1}{2}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\partial X^{\dot{A}A}(z)\partial X^{\dot{B}B}(z)+\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}(z)\psi^{\beta\dot{B}}(z)$$
$$J^a(z)=\frac{1}{4}\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}(\sigma^{aT})^{\beta}{}_{\gamma}\psi^{\alpha\dot{A}}(z)\psi^{\gamma\dot{B}}(z)$$
$$G^{\alpha A}(z)=\epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}(z)\partial X^{\dot{B}A}(z)$$

$$(\sigma^1)^{\alpha}{}_{\beta}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},(\sigma^2)^{\alpha}{}_{\beta}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},(\sigma^3)^{\alpha}{}_{\beta}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\mathcal{O}(z)=\sum_n \mathcal{O}_n z^{-h-n}$$

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}m(m^2-1)\delta_{m+n,0},$$
$$[J_m^a,J_n^b]=\frac{c}{12}m\delta^{ab}\delta_{m+n,0}+i\epsilon_c^{ab}J_{m+n}^c,$$
$$\left\{G_r^{\alpha A},G_s^{\beta B}\right\}=\epsilon^{AB}\left[\epsilon^{\alpha\beta}\frac{c}{6}\left(\frac{1}{4}-r^2\right)\delta_{r+s,0}+(\sigma^{aT})^{\alpha}_{\gamma}\epsilon^{\beta\gamma}(r-s)J_{r+s}^a-\epsilon^{\alpha\beta}L_{r+s}\right],$$
$$[J_m^a,G_r^{\alpha A}]=\frac{1}{2}(\sigma^{aT})^{\alpha}_{\beta}G_{m+r}^{\beta A},$$
$$[L_m,G_r^{\alpha A}]=\left(\frac{m}{2}-r\right)G_{m+r}^{\alpha A},$$
$$[L_m,J_n^a]=-nJ_{m+n}^a.$$

$$\left[\alpha_n^{\dot{A}A},\alpha_m^{\dot{B}B}\right]=n\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\delta_{n+m,0}$$
$$\left\{\psi_r^{\alpha\dot{A}},\psi_s^{\beta\dot{B}}\right\}=-\epsilon^{\alpha\beta}\epsilon^{\dot{A}\dot{B}}\delta_{r+s,0}$$
$$\left[L_n,\alpha_m^{\dot{A}A}\right]=-m\alpha_{n+m}^{\dot{A}A}$$
$$\left[L_n,\psi_r^{\alpha\dot{A}}\right]=-\left(\frac{n}{2}+r\right)\psi_{n+r}^{\alpha\dot{A}}$$
$$\left[J_n^a,\alpha_m^{\dot{A}A}\right]=0$$
$$\left[J_n^a,\psi_r^{\alpha\dot{A}}\right]=\frac{1}{2}(\sigma^{aT})^{\alpha}{}_{\beta}\psi_{n+r}^{\beta\dot{A}}$$
$$\left[G_r^{\alpha A},\alpha_m^{\dot{B}B}\right]=-m\epsilon^{AB}\psi_{r+m}^{\alpha\dot{B}}$$
$$\left\{G_r^{\alpha A},\psi_s^{\beta\dot{A}}\right\}=\epsilon^{\alpha\beta}\alpha_{r+s}^{\dot{A}A}$$

$$J_n^{\pm}=J_n^1\pm iJ_n^2$$

$$[J_n^+,J_m^-]=n\frac{c}{6}\delta_{n+m,0}+2J_{n+m}^3$$
$$\left[J_n^3,J_m^{\pm}\right]=\pm J_{n+m}^{\pm}$$
$$\left[L_n,J_m^{\pm}\right]=-mJ_{n+m}^{\pm}$$
$$\left[J_n^{\pm},G_r^{\mp A}\right]=G_{n+r}^{\pm A}$$
$$\left[J_n^{\pm},\psi_r^{\mp\dot{A}}\right]=\psi_{n+r}^{\pm\dot{A}}$$

$$\vartheta(\nu,\tau)=\sum_{n=-\infty}^{\infty}\ e^{2\pi in\nu}e^{\pi in^2\tau}$$

$$\vartheta_{\alpha,\beta}(\nu,\tau)\equiv e^{\pi i\alpha^2\tau}e^{2\pi i\alpha(\nu+\beta)}\vartheta(\nu+\alpha\tau+\beta,\tau)$$

$$\vartheta_1(\nu,\tau)\equiv-\vartheta_{\frac{1}{2},\frac{1}{2}}(\nu,\tau)$$
$$\vartheta_2(\nu,\tau)\equiv\vartheta_{\frac{1}{2},0}(\nu,\tau)$$
$$\vartheta_3(\nu,\tau)\equiv\vartheta_{0,0}(\nu,\tau)$$
$$\vartheta_4(\nu,\tau)\equiv\vartheta_{0,\frac{1}{2}}(\nu,\tau)$$

$$\eta(q)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}\ (1-q^n)$$

$$\tilde{\chi}_{j;N}(\tilde{y})=\tilde{y}^N\frac{\tilde{y}^{2j+1}-\tilde{y}^{-2j-1}}{\tilde{y}-\tilde{y}^{-1}},$$

$$\Phi_{j;N}^{(s)}(q,y)\ =F(q,y)\varphi_{j;N}^{(s)}(q,y)$$
$$\Phi_{j,h;N}^{(\ell)}(q,y)\ =F(q,y)\varphi_{j,h;N}^{(\ell)}(q,y)$$

$$F(q,y)=\prod_{k=1}^{\infty}\frac{\left(1-yq^{k-1/2}\right)^2\left(1-y^{-1}q^{k-1/2}\right)^2}{\left(1-q^k\right)^2\left(1-y^2q^k\right)\left(1-y^{-2}q^{k-1}\right)}=\frac{-iq^{1/4}\vartheta_4(\nu,\tau)^2}{\vartheta_1(2\nu,\tau)\eta(q)^3}$$

$$\varphi_{j;N}^{(s)}(q,y)\ =q^j\sum_{k\in\mathbb{Z}}\left[\frac{y^{2(N+1)k+2j}}{\left(1-yq^{k+1/2}\right)^2}-\frac{y^{-2(N+1)k-2(j+1)}}{\left(1-y^{-1}q^{k+1/2}\right)^2}\right]q^{(N+1)k^2+(2j+1)k}$$
$$\varphi_{j,h;N}^{(\ell)}(q,y)\ =q^h\sum_{k\in\mathbb{Z}}\left(y^{2(N+1)k+2j}-y^{-2(N+1)k-2(j+1)}\right)q^{(N+1)k^2+(2j+1)k}$$

$$\mathbf{\Phi}_{j;N}^{(s)}(q,y)=\frac{q^{\frac{1}{8}}}{\eta(q)^3}\Phi_{0,0;1}^{(\ell)}(q,y)\Phi_{j;N-1}^{(s)}(q,y)$$

$$\mathbf{\Phi}_{j,h;N}^{(\ell)}(q,y)=\frac{q^{\frac{1}{8}}}{\eta(q)^3}\Phi_{0,0;1}^{(\ell)}(q,y)\Phi_{j,h;N-1}^{(\ell)}(q,y)$$

$$\phi_j^{(s)}(q,y)=\frac{q^j}{1-q}\frac{y^{2j}\left(y-2q^{\frac{1}{2}}+y^{-1}q\right)-y^{-2j}\left(y^{-1}-2q^{\frac{1}{2}}+yq\right)}{y-y^{-1}},$$

$$\phi_{j,h}^{(\ell)}(q,y)=\phi_{j,h}^{(s)}(q,y)-2\phi_{j+\frac{1}{2},h+\frac{1}{2}}^{(s)}(q,y)+\phi_{j+1,h+1}^{(s)}(q,y)$$
$$=\frac{q^h}{1-q}\frac{\left(y^{2j+1}-y^{-2j-1}\right)\left(y^{-1}-2q^{\frac{1}{2}}+qy\right)\left(y-2q^{\frac{1}{2}}+qy^{-1}\right)}{y-y^{-1}}$$

$$\phi_{j_1}^{(s)}(q,y)\times\phi_{j_2}^{(s)}(q,y)=\phi_{j_1+j_2}^{(s)}(q,y)+\sum_{h\geq j_1+j_2}\ \sum_{j=|j_1-j_2|}^{j_1+j_2-1}\ \phi_{j,h}^{(\ell)}(q,y).$$

$$\phi_j^{(s)}(q^2,y^2)=\phi_{2j}^{(s)}(q,y)+\sum_{h\geq 2j}\ \sum_{j'=0}^{2j-1}\ (-1)^{h+j'}\phi_{j',h}^{(\ell)}(q,y).$$

$$\tilde{\chi}_{j_1;N_1}(\bar{y})\times\tilde{\chi}_{j_2;N_2}(\bar{y})=\sum_{j=|j_1-j_2|}^{j_1+j_2}\ \tilde{\chi}_{j;N_1+N_2}(\bar{y}),$$

$$\tilde{\chi}_{j_1;N}(\bar{y}^2)=\sum_{j=0}^{2j_1}\ (-1)^{2j_1-j}\tilde{\chi}_{j;2N}(\bar{y})$$

$$+12q: \quad \sigma^{i}_{\dot{A}\dot{B}}|-\dot{\alpha}\rangle_1 \psi^{-\dot{A}}_{-\frac{1}{2}}|\dot{B}\dot{\beta}\rangle_1 \qquad (0;\mathbf{1},\mathbf{3})+2(1;\mathbf{1},\mathbf{3})+(2;\mathbf{1},\mathbf{3})$$

$$-192q: \quad \sigma^{i}_{\dot{A}\dot{B}}|-\dot{\alpha}\rangle_1 \alpha^{\dot{A}A}_{-\frac{1}{2}} \psi_0^{-\dot{B}}|r\rangle_1 \quad \left(\tfrac{1}{2};\mathbf{2},\mathbf{3}\right)_r+\left(\tfrac{3}{2};\mathbf{2},\mathbf{3}\right)_r$$

$$+4q: \quad \epsilon_{\dot{C}\dot{D}}\psi_0^{-\dot{C}}\psi_0^{-\dot{D}}|\dot{A}\dot{B}\rangle_2+f\psi^{-\dot{A}}_{-\frac{1}{2}}|-\dot{B}\rangle_2 \quad \left(\tfrac{1}{2};\mathbf{1},\mathbf{1}\right)+\left(\tfrac{1}{2};\mathbf{1},\mathbf{3}\right)$$

$$-12q: \quad \sigma^{i}_{\dot{A}\dot{B}}\alpha^{\dot{A}A}_{-\frac{1}{2}}\psi_0^{-\dot{B}}|-\dot{\alpha}\rangle_2 \qquad (0;\mathbf{2},\mathbf{3})+(1;\mathbf{2},\mathbf{3})$$

$$+160q: \quad \alpha^{\dot{A}A}_{-\frac{1}{4}}\alpha^{\dot{B}B}_{-\frac{1}{4}}J^{-}_{\frac{1}{2}}|r\rangle_2 \qquad (1,\mathbf{3},\mathbf{3})_r+\left(\tfrac{1}{2};\mathbf{1},\mathbf{1}\right)_r$$

$$+48q: \quad \sigma^{i}_{\dot{A}\dot{B}}\psi^{-\dot{A}}_{-\frac{1}{4}}\psi^{-\dot{B}}_{\frac{1}{4}}|r\rangle_2 \qquad \left(\tfrac{1}{2};\mathbf{1},\mathbf{3}\right)_r$$

$$(\tilde{\jmath};\mathbf{d_1},\mathbf{d_2}),\left(\tilde{\jmath}+\tfrac{1}{2};\mathbf{d_1}\otimes\mathbf{2},\mathbf{d_2}\right),(\tilde{\jmath}+1;\mathbf{d_1},\mathbf{d_2}).$$

$$(a\cdot\rho)R\wedge R+\sum_A\ (b_A\cdot\rho)F^A\wedge F^A$$

$$H-V=273-29T$$

$$y^2=x^3+fx+g$$

$$-12[K]=[\Delta]=\sum_A\ \nu_A[S_A]+[Y]$$

$$T=h^{1,1}(B)-1$$

$$r=h^{1,1}(\hat{X})-h^{1,1}(B)-1$$

$$H_{\text{neutral}}=h^{2,1}(\hat{X})+1$$

$$\chi(\hat{X})=2(h^{1,1}(X)-h^{2,1}(X))=2(r+T-H_{\text{neutral}}+3)$$

$$(K\cdot B)\wedge R\wedge R,(S_A\cdot B)\wedge F^A\wedge F^A$$

$$\begin{gathered}S^{(6)}=-\int\ \tfrac{1}{2}\left(j^\alpha\Omega_{\alpha\beta}a^\beta\right)\mathrm{tr}R\wedge* R+\tfrac{2}{\lambda_A}\left(j^\alpha\Omega_{\alpha\beta}b_A^\beta\right)\mathrm{tr}F^A\wedge* F^A\\+\tfrac{1}{4}\left(B^\alpha\Omega_{\alpha\beta}a^\beta\right)\wedge\mathrm{tr}R\wedge R+\tfrac{1}{\lambda_A}\left(B^\alpha\Omega_{\alpha\beta}b_A^\beta\right)\wedge\mathrm{tr}F^A\wedge F^A\\+\tfrac{1}{4}G_{\alpha\beta}H_3^\alpha\wedge* H_3^\beta+\tfrac{1}{2}G_{\alpha\beta}dj^\alpha\wedge* dj^\beta\end{gathered}$$

$$H_3^\alpha=dB_2^\alpha+\tfrac{1}{2}a^\alpha w_{\mathrm{CS}}(R)+2\tfrac{b_A^\alpha}{\lambda_A}w_{\mathrm{CS}}^A(F)$$

$$\begin{gathered}w_{\mathrm{CS}}(R)=\mathrm{tr}\left(\hat{\omega}\wedge d\hat{\omega}+\tfrac{3}{2}\hat{\omega}\wedge\hat{\omega}\wedge\hat{\omega}\right)\\w_{\mathrm{CS}}^B(F)=\mathrm{tr}\left(A^B\wedge dA^B+\tfrac{3}{2}A^B\wedge A^B\wedge A^B\right)\end{gathered}$$

$$\Omega_{\alpha\beta}*H_3^\beta=G_{\alpha\beta}H_3^\beta$$

$$S_{\mathrm{GS}}^{(6)}=-\tfrac{1}{2}\int\ \Omega_{\alpha\beta}B_2^\alpha\wedge X_4^\beta$$

$$X_4^\alpha=\tfrac{1}{2}a^\alpha\mathrm{tr}R\wedge R+\sum_A\ b_A^\alpha\left(\tfrac{2}{\lambda_A}\mathrm{tr}F^A\wedge F^A\right)$$

$$a^{\alpha}=K^{\alpha}, b_{A}^{\alpha}=C_{A}^{\alpha}$$

$$[K]=K^{\alpha}\omega_{\alpha},[S_{A}]=C_{A}^{\alpha}\omega_{\alpha}$$

$$\int_{M_6\times S_A} C_4\wedge\left(\mathrm{tr}\left(\hat{F}^{A}\right)^{2}-\frac{1}{48}\mathrm{tr}\hat{R}^{2}\right),$$

$$x\cdot y=x^{\alpha}\Omega_{\alpha\beta}y^{\beta}$$

$$\Gamma=H_{2}(B,\mathbb{Z})$$

$$\int_{M_6\times S_A}\mathrm{Tr}\left(\hat{F}^{A}\wedge *_{8}\hat{F}^{A}\right)=\int_{M_6}\mathrm{Tr}(F^{A}\wedge *F^{A})\cdot\frac{\int_{S_A}J_{\mathrm{b}}}{\mathcal{V}_{\mathrm{b}}^{1/2}}$$

$$j^{\alpha}=\frac{v_{\mathrm{b}}^{\alpha}}{(2\mathcal{V}_{\mathrm{b}})^{1/2}},j^{\alpha}\Omega_{\alpha\beta}j^{\beta}=1$$

$$G_{\alpha\beta}=-\Omega_{\alpha\beta}+\mathcal{O}(j^{2})$$

$$\omega_{f}=[\Sigma+(m/2)F],\omega_{b}=[F]$$

$$\Omega_{\alpha\beta}=\int_{B_2}\omega_{\alpha}\wedge\omega_{\beta},\Omega=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$-[K]=[2\Sigma+(2+m)F]=2\omega_{f}+2\omega_{b},a=(K^{\alpha})=(-2,-2)$$

$$[S]=p\omega_{f}+(q-pm/2)\omega_{b},b=(C^{\alpha})=(p,q-pm/2)$$

$$S_{\mathrm{GS}}^{(6)}=\frac{1}{2}\int\left(B_{f}+B_{b}\right)\wedge\mathrm{tr}R^{2}-\left(pB_{b}+(q-pm/2)B_{f}\right)\wedge\frac{2}{\lambda}\mathrm{tr}F^{2}$$

$$d\hat{H}=\frac{2}{\lambda}\mathrm{Tr}\hat{F}^{2}-\mathrm{tr}\hat{R}^{2},\hat{H}=d\hat{B}+\frac{2}{\lambda}w_{\mathrm{CS}}^{B}(F)-w_{\mathrm{CS}}(R)$$

$$S_{B}^{(10)}=\int-\frac{1}{4}e^{-2\phi}\hat{H}\wedge *\hat{H}-\frac{1}{4}e^{2\phi}\hat{H}_{7}\wedge *\hat{H}_{7}-\frac{1}{3}\hat{B}\wedge\hat{X}_{8}-\frac{1}{2}\hat{B}_{6}\wedge\hat{X}_{4}$$

$$\hat{X}_{8}=\frac{1}{24}\mathrm{Tr}\hat{F}^{4}-\frac{1}{7200}\left(\mathrm{Tr}\hat{F}^{2}\right)^{2}-\frac{1}{240}\mathrm{Tr}\hat{F}^{2}\mathrm{tr}\hat{R}^{2}+\frac{1}{8}\left(\mathrm{tr}\hat{R}^{4}\right)+\frac{1}{32}\left(\mathrm{tr}\hat{R}^{2}\right)^{2}$$

$$\hat{X}_{4}=\frac{2}{\lambda}\mathrm{tr}\hat{F}^{2}-\mathrm{tr}\hat{R}^{2}$$

$$S_{GS}^{(6)}(B_{1})=\frac{1}{2}\int B_{1}\wedge\left(\mathrm{tr}R^{2}-\frac{2}{\lambda}\mathrm{tr}F^{2}\right)$$

$$\frac{2}{\lambda}\int_{\mathrm{K3}}\mathrm{tr}\mathcal{F}^{2}=\int_{\mathrm{K3}}\mathrm{tr}\mathcal{R}^{2}=24$$

$$\mathrm{Tr}\hat{F}^{2}=30\mathrm{tr}\hat{F}^{2},\mathrm{Tr}\hat{F}^{4}=24\mathrm{tr}\hat{F}^{4}+3\left(\mathrm{tr}\hat{F}^{2}\right)^{2}$$

$$\hat{X}_{8}^{SO(32)}=\int_{\mathrm{K3}}\left(-\frac{1}{8}\mathrm{tr}F^{2}\mathrm{tr}\mathcal{R}^{2}-\frac{1}{8}\mathrm{tr}R^{2}\mathrm{tr}\mathcal{F}^{2}+\frac{1}{16}\mathrm{tr}R^{2}\mathrm{tr}\mathcal{R}^{2}\right).$$

$$S_{SO(32)}^{(6)}=\frac{1}{2}\int(B_{1}+B_{0})\wedge\mathrm{tr}R^{2}-(B_{1}-2B_{0})\wedge\mathrm{tr}F^{2}$$

$$a=(-2,-2),b=(1,-2)$$

$$\left(B_f,B_b\right)\leftrightarrow(B_0,B_1).$$

$$\mathrm{Tr}\hat{F}^2=\mathrm{tr}\hat{F}^2,\mathrm{Tr}\hat{F}^4=\frac{1}{100}\mathrm{tr}\hat{F}^4$$

$$\begin{aligned}\hat{X}_8^{E_8}=\int_{K3}\Big(&\frac{1}{3600}[2\mathrm{tr}F_1^2\mathrm{tr}\mathcal{F}_1^2+2\mathrm{tr}F_2^2\mathrm{tr}\mathcal{F}_2^2-\mathrm{tr}F_2^2\mathrm{tr}\mathcal{F}_1^2-\mathrm{tr}F_1^2\mathrm{tr}\mathcal{F}_2^2]\\&-\frac{1}{240}[\mathrm{tr}F_1^2\mathrm{tr}\mathcal{R}^2+\mathrm{tr}R^2\mathrm{tr}\mathcal{F}_1^2+\mathrm{tr}F_2^2\mathrm{tr}\mathcal{R}^2+\mathrm{tr}R^2\mathrm{tr}\mathcal{F}_2^2]\\&+\frac{1}{16}[\mathrm{tr}R^2\mathrm{tr}\mathcal{R}^2]\Big)\end{aligned}$$

$$S_{E_8}^{(6)}=\frac{1}{2}\int\ (B_1+B_0)\wedge\mathrm{tr}R^2-\frac{1}{30}\left(B_1-\frac{k}{2}B_0\right)\wedge\mathrm{tr}F_1^2-\frac{1}{30}\left(B_1+\frac{k}{2}B_0\right)\wedge\mathrm{tr}F_2^2$$

$$a=(-2,-2),b_1=(1,-k/2),b_2=(1,k/2)$$

$$J_{\mathrm{b}}=v_{\mathrm{b}}^{\alpha}\omega_{\alpha},C_4=B_2^{\alpha}\wedge\omega_{\alpha}$$

$$\begin{aligned}C_{\mathrm{cs}}&=h^{3,1}(\hat{X})-1\\C_{\mathrm{sa}}&=h^{1,1}(\mathcal{B}_3)+1\\C_{21}&=h^{2,1}(\hat{X})-h^{2,1}(\mathcal{B}_3)\end{aligned}$$

$$T_{\alpha}=\frac{1}{2}\kappa_{\alpha\beta\gamma}v_{\mathrm{b}}^{\beta}v_{\mathrm{b}}^{\gamma}+\frac{1}{4}d_{\alpha ab}(N+\bar{N})^a(N+\bar{N})^b+i\rho_{\alpha}$$

$$N^a\to N^a+i\Lambda^a,T_{\alpha}\to T_{\alpha}+i\Lambda_{\alpha}.$$

$$r_{\mathrm{v}}=h^{1,1}(\hat{X})-h^{1,1}(\mathcal{B}_3)-1$$

$$f_A=\frac{1}{2}C_A^{\alpha}T_{\alpha},[S_A]=C_A^{\alpha}\omega_{\alpha}$$

$$r_{21}=h^{2,1}(\mathcal{B}_3)$$

$$\tau_{\kappa\lambda}(z)=\frac{i}{2}\left(\int_B\ \beta^{\mu}\wedge\bar{\psi}^{\kappa}\right)^{-1}\int_B\ \alpha_{\lambda}\wedge\bar{\psi}^{\mu}$$

$$\Theta_{\Sigma\Lambda}=\int_{\hat{X}}\ \omega_{\Sigma}\wedge\omega_{\Lambda}\wedge G_4$$

$$DT_{\alpha}=dT_{\alpha}+i\Theta_{\alpha i_B}A^{i_B}$$

$$DT_{\alpha}=dT_{\alpha}+i\mathrm{Tr}\left(\underline{\Theta}_{\alpha B}A^B\right)$$

$$\Theta_{i_Aj_B}=\int_{\hat{X}}\ \omega_{i_A}\wedge\omega_{j_B}\wedge G_4$$

$$\frac{\chi(\hat{X})}{24}=\frac{1}{2}\int\ G_4\wedge G_4+N_3$$

$$h^{2,2}(X)=44+4h^{1,1}(X)+2h^{2,1}(X)-4h^{3,1}(X)$$

$$\begin{aligned}\chi(\hat{X})&=6\left(8+h^{1,1}(\hat{X})+h^{3,1}(\hat{X})-h^{2,1}(\hat{X})\right)\\&=6\left(9+C_{\mathrm{sa}}+C_{\mathrm{cs}}+r_{\mathrm{v}}-(C_{21}+r_{21})\right)\end{aligned}$$

$$\Delta C_{\mathrm{cs}}=3N_3,\Delta r_{\mathrm{v}}=N_3$$

$$S_{ax}^{(4)}=\frac{1}{8}\int\ \frac{1}{2}a^{\mathcal{A}}\rho_{\mathcal{A}}\mathrm{tr}R\wedge R+\frac{2}{\lambda_A}b_A^{\mathcal{A}}\rho_{\mathcal{A}}\mathrm{tr}F^A\wedge F^A$$

$$b_A^{\alpha}=C_A^{\alpha},[S_A]=C_A^{\alpha}\omega_{\alpha}$$

$$a^{\alpha}=K^{\alpha},c_1(\mathcal{B}_3)=-K^{\alpha}\omega_{\alpha}$$

$$S_{\mathrm{D7}}^{CS}(R,F)=-\frac{1}{2}\int_{M^{3,1}}\ C_{\hat{A}}^{\alpha}\rho_{\alpha}\left(\frac{1}{96}\mathrm{tr}(R\wedge R)N_{\mathrm{D7}}^{\hat{A}}+\frac{1}{2}\mathrm{tr}\left(F^{\hat{A}}\wedge F^{\hat{A}}\right)\right)+\cdots$$
$$S_{\mathrm{O7}}^{CS}(R)=+2\int_{M^{3,1}}\ \tilde{C}^{\alpha}\rho_{\alpha}\left(-\frac{1}{192}\mathrm{tr}(R\wedge R)\right)+\cdots$$

$$-12[K]=[\Delta]=2[S_{\mathrm{O7}}]+N_{\mathrm{D7}}^{\hat{A}}[S_{\hat{A}}]=\left(2\tilde{C}^{\alpha}+N_{\mathrm{D7}}^{\hat{A}}C_{\hat{A}}^{\alpha}\right)\omega_{\alpha}$$

$$T_0=-i\tau_{\mathrm{IIB}}=e^{-\phi}-iC_0$$

$$\frac{1}{2}\tilde{a}(z)\mathrm{tr}R\wedge R+\frac{2}{\lambda_A}\tilde{b}_A(z)\mathrm{tr}F^A\wedge F^A$$

$$H_3^{\alpha}=dB_2^{\alpha}+\frac{1}{2}a^{\alpha}w_{\mathrm{CS}}(R)+2\frac{b_A^{\alpha}}{\lambda_A}w_{\mathrm{CS}}^A(F)$$

$$\tilde{G}_{\alpha\beta}\propto\kappa_{\alpha\beta\gamma}L^{\gamma}+\cdots,$$

$$r=c_1(\mathcal{O}(1))=[\Sigma],t=c_1(\mathcal{L})$$

$$r(r+t)=0$$

$$c_1(\mathcal{B}_3)=c_1(\mathcal{B}_2)+(2r+t)$$
$$c_2(\mathcal{B}_3)=c_2(\mathcal{B}_2)+c_1(\mathcal{B}_2)(2r+t)$$

$$\omega_f=r+\frac{t}{2},\omega_i$$

$$\kappa_{ijk}=0,\kappa_{fij}=\kappa_{ij},\kappa_{ffi}=0,\kappa_{fff}=\frac{1}{4}t^it^j\kappa_{ij}$$

$$(a_F^{\alpha})\equiv(K^{\alpha})=\left(-2,K^i\right)$$

$$[S]=p\omega_f+\sum_i\ q_i\omega_i$$

$$t=k[H]$$

$$c_1\left(\tilde{\mathbb{F}}_k\right)=2[\Sigma]+(3+k)[H],c_2\left(\tilde{\mathbb{F}}_k\right)=6[\Sigma\cdot H]+(3+3k)[H]^2$$

$$\Sigma H^2=1,H^3=0,\Sigma^2H=-k,\Sigma^3=k^2,$$

$$\hat{B}=b_2+\chi^I\omega_I=b_2+\chi_f\omega_{\mathcal{B}}+\sum_i\ \chi^i\omega_i$$

$$c_I-\lambda_I=0$$

$$c_I = \int_{Z_3} \omega_I \wedge \mathrm{tr}\mathcal{R}^2 = \int_{Z_3} \omega_I \wedge c_2(Z_3), \lambda_I = \frac{2}{\lambda}\int_{Z_3} \omega_I \wedge \mathrm{Tr}\mathcal{F}^2$$

$$c_2(Z_3) = 11c_1(\mathcal{B}_2)^2 + c_2(\mathcal{B}_2) + 12c_1(\mathcal{B}_2) \wedge \omega_{\mathcal{B}},$$

$$c_i = -12\kappa_{ij}K^j$$
$$c_f = \int_{\mathcal{B}_2} c_2(Z_3) = \int_{\mathcal{B}_2} c_2(\mathcal{B}_2) - c_1(\mathcal{B}_2)^2 = -8 + 2h^{1,1}(\mathcal{B}_2)$$

$$\chi_0\left(\mathrm{tr}R^2 - \frac{2}{\lambda}\mathrm{tr}F^2\right)$$

$$S^{(4)}_{SO(32)} = \frac{1}{2}\int \left(\chi_0 + \frac{1}{24}\chi^I c_I\right)\mathrm{tr}R^2 - \left(\chi_0 - \frac{1}{12}\chi^I c_I\right)\mathrm{tr}F^2$$
$$= \frac{1}{2}\int \left(\chi_0 - \frac{1}{2}\chi^i\kappa_{ij}K^j + \frac{1}{24}\chi_f c_f\right)\mathrm{tr}R^2$$
$$-\left(\chi_0 + \chi^i\kappa_{ij}K^j - \frac{1}{12}\chi_f c_f\right)\mathrm{tr}F^2$$

$$\tilde{\chi}_0 = 8\chi_0, \tilde{\chi}_i = 8\kappa_{ij}\chi^j, \tilde{\chi}_f = 8\chi_f$$

$$a = \left(-2, K^i, -\frac{1}{12}c_f\right), b = \left(1, K^i, -\frac{1}{12}c_f\right)$$

$$\frac{1}{30}\mathrm{tr}\mathcal{F}_i^2 = \eta_i \wedge \omega_{\mathcal{B}} + \zeta_i$$

$$\mathrm{tr}\mathcal{F}_1^2 + \mathrm{tr}\mathcal{F}_2^2 = 30\mathrm{tr}\mathcal{R}^2$$

$$\eta_1 + \eta_2 = 12c_1(\mathcal{B}_2)$$
$$\zeta_1 + \zeta_2 = 11c_1(\mathcal{B}_2)^2 + c_2(\mathcal{B}_2) \equiv \mathcal{C}_2$$

$$\eta_1 = 6c_1(\mathcal{B}_2) + \tilde{t}, \eta_2 = 6c_1(\mathcal{B}_2) - \tilde{t}$$
$$\zeta_1 = \frac{1}{2}\mathcal{C}_2 + \Phi, \zeta_2 = \frac{1}{2}\mathcal{C}_2 - \Phi$$

$$S^{(4)}_{E_8} = \frac{1}{2}\int \left(\chi_0 + \frac{1}{24}\chi^I c_I\right)\mathrm{tr}R^2 - \left(\chi_0 - \frac{1}{12}\chi^I \tilde{c}_I\right)\mathrm{tr}F_1^2 - \left(\chi_0 + \frac{1}{12}\chi^I \tilde{c}_I\right)\mathrm{tr}F_2^2$$

$$\tilde{c}_I = \int_{Z_3} \omega_I \wedge (\tilde{t} \wedge \omega_{\mathcal{B}} + \Phi) = (-1)^i \int_{Z_3} \omega_I \wedge \left(\frac{1}{2}\mathrm{tr}\mathcal{R}^2 - \frac{1}{30}\mathrm{tr}\mathcal{F}_i\right)$$

$$\tilde{c}_i = \kappa_{ij}\tilde{t}^j, \tilde{c}_f = \int_{\mathcal{B}_2} \Phi - \tilde{t}^2$$

$$a = \left(-2, K^i, -\frac{1}{12}c_f\right), \quad b_1 = \left(1, -\frac{1}{2}\tilde{t}^i, -\frac{1}{12}\tilde{c}_f\right)$$
$$b_2 = \left(1, \frac{1}{2}\tilde{t}^i, \frac{1}{12}\tilde{c}_f\right)$$

$$\rho_0 \leftrightarrow \tilde{\chi}_f$$
$$\rho_b \leftrightarrow \tilde{\chi}_0$$
$$\rho_i \leftrightarrow \tilde{\chi}_i$$

$$t^i \leftrightarrow \tilde{t}^i,$$

$$\eta_1 = 8c_1(\mathcal{B}_2), \eta_2 = 4c_1(\mathcal{B}_2)$$

$$a \cdot a = 9 - T$$
$$273 = 29T + H$$

$$K \cdot K = 10 - h^{1,1}(B)$$

$$\chi_0(B) = \frac{1}{12}\int_B (c_1^2 + c_2)$$

$$\chi(B) = \int_B c_2 = \sum_i (-1)^i b^i = 2 + h^{1,1}$$

$$n_{U(1)} = h^{1,1}(X) - h^{1,1}(B) - 1,$$

$$\chi(X) = -60\int_B c_1(B)^2 = -60\Omega_{\alpha\beta}K^\alpha K^\beta = 2(T - H + 3),$$

$$30(9 - T) = H - T - 3 \Rightarrow 273 = H + 29T$$

$$\pi: B' \to B$$

$$c_1(B') = \pi^* c_1(B) - [E]$$

$$\chi(X') = \chi(X) - 60$$

$$\Delta T = 1, \Delta H_{\text{neutral}} = -29$$

$$(-12a - \nu_A b_A) \cdot b_A = 0$$
$$b_A \cdot b_B = 0, A \neq B$$
$$29T - 273 = V - H$$

$$\chi(\hat{X}) = -60\int_B c_1^2(B) - \sum_A r_{G_A} c_{G_A}(2 - 2g_A)$$

$$\chi(X) = 60T - 540 + 2\sum_A \left(r_{G_A} - \dim_{G_A}\right)$$

$$29T - 273 = \sum_A \dim_{G_A} - H$$

$$j \cdot (-a) > 0$$
$$j \cdot b > 0$$
$$j \cdot (-12a - \sum_A \nu_A b_A) > 0$$

$$-12K^\alpha - \sum_A \nu_A C_A^\alpha - Y^\alpha = 0$$

$$\chi(X) = 288 + 360\int c_1^3(\mathcal{B}_3)$$

$$39 - 60\kappa_{\alpha\beta\gamma}K^\alpha K^\beta K^\gamma = 39 - 60\langle\langle a, a, a\rangle\rangle = C_{\text{sa}} + C_{\text{cs}} - C_{21},$$

$$\langle\langle x, y, z\rangle\rangle \cong \kappa_{\alpha\beta\gamma}x^\alpha y^\beta z^\gamma$$

$$C_{\text{sa}} + C_{\text{cs}} - C_{21} \equiv 39(\text{mod}60)$$

$$\kappa = \kappa_{HHH} = 1$$

$$39 - 60\kappa a^3 = 3879 = 2 + C_{\text{cs}} - (C_{21} + r_{21}).$$

$$k = 2: 60\langle\langle -K, -K, -K\rangle\rangle = 3720$$

$$C_{\mathrm{sa}} + C_{\mathrm{cs}} - (C_{21} + r_{21}) = 3 + 3756 - (0 + 0) = 3759 = 39 - 60\langle\langle a, a, a\rangle\rangle.$$

$$C_{\mathrm{sa}} + C_{\mathrm{cs}} - C_{21} = 4359$$

$$\chi(X') = \chi(X) - 2880$$

$$\Delta C_{\mathrm{cs}} = -481, \Delta C_{\mathrm{sa}} = +1, \Delta C_{21} = \Delta r_{21} = 0$$

$$(x_1, x_2, x_3, x_4, x_5) \sim (\lambda x_1, \lambda\mu x_2, \lambda x_3, \mu x_4, \lambda x_5).$$

$$\chi(X') = \chi(X) - 1440\left(2 - 2g_{\mathcal{C}} + \int_{\mathcal{C}} c_1(\mathcal{B}_3)\right)$$

$$\Delta C_{\mathrm{cs}} = -481(1 - g_{\mathcal{C}}) - 240\int_{\mathcal{C}} c_1(\mathcal{B}_3), \Delta C_{\mathrm{sa}} = +1$$
$$\Delta r_{21} = g_{\mathcal{C}}, \Delta C_{21} = 0$$

$$39 - 60\kappa_{\alpha\beta\gamma}K^{\alpha}K^{\beta}K^{\gamma} = 39 - 60\langle\langle a, a, a\rangle\rangle = C_{\mathrm{sa}} + C_{\mathrm{cs}} - C_{21} - r_{21}$$

$$C_{\mathrm{sa}} + C_{\mathrm{cs}} - C_{21} - r_{21} \equiv 39(\mathrm{mod}60)$$

$$\kappa_{\alpha\beta\gamma}S_A^{\alpha}S_B^{\beta} = 0, \forall\gamma$$

$$\Theta_{i_A i_B} = \int_{\hat{X}} \omega_{i_A} \wedge \omega_{i_B} \wedge G_4 = 0$$

$$\chi(X) = 288 + 360\int_B c_1(B)^3 - \sum_A r_{G_A}c_{G_A}\left(c_{G_A} + 1\right)\int_{S_A} c_1(S_A)^2$$

$$39 - 60\langle\langle a, a, a\rangle\rangle$$
$$= C_{\mathrm{sa}} + C_{\mathrm{cs}} + r_v - (C_{21} + r_{21}) + \frac{1}{6}\sum_A r_{G_A}c_{G_A}\left(c_{G_A} + 1\right)\left\langle\langle a + b_A, a + b_A, b_A\rangle\right\rangle.$$

$$39 - 60\langle\langle a, a, a\rangle\rangle = 5199 = 3 + 5186 + 1 + \tfrac{1}{6}(1 \cdot 2 \cdot 3 \cdot 9),$$

$$\langle\langle -K, v, v\rangle\rangle = \kappa_{\alpha\beta\gamma}(-a^{\alpha})v^{\beta}v^{\gamma} > 0$$

$$\left\langle\langle S_A, v, v\rangle\right\rangle = \kappa_{\alpha\beta\gamma}b_A^{\alpha}v^{\beta}v^{\gamma} > 0$$

$$12\langle\langle -K, v, v\rangle\rangle \geq \sum_A \nu_A\left\langle\langle S_A, v, v\rangle\right\rangle > 0$$

$$48 \geq mN$$

$$\rho_{\alpha} \to \rho_{\alpha} + l_{\alpha}, l \in L$$

$$a, b_A \in L^*$$

$$C_{\mathrm{sa}} + C_{\mathrm{cs}} - C_{21} \equiv 39(\mathrm{mod}60)$$

$$c_1(B') = \pi^* c_1(B) - 2[E], c_2(B') = \pi^*(c_2(B))$$

$$E^2 = f, E \cdot f = -1, E \cdot \pi^* D = E \cdot \pi^* \tilde{C} = f \cdot \pi^* D = 0$$

$$c_1(B') = \pi^* c_1(B) - [E]$$
$$c_2(B') = \pi^*(c_2(B) + [\mathcal{C}]) - [E] \wedge \pi^* c_1(B)$$

$$E^2 = -\pi^*\mathcal{C} + \deg(N_B\mathcal{C})f, E \cdot f = -1$$
$$E \cdot \pi^* D = (C \cdot D)f, f \cdot \pi^* D = 0, E \cdot \pi^* \tilde{\mathcal{C}} = 0$$

$$\deg(N_B\mathcal{C}) = -\chi(\mathcal{C}) + \int_{\mathcal{C}} c_1(B)$$

$$I_8(R,F) = \frac{1}{2}\Omega_{\alpha\beta} X_4^\alpha X_4^\beta$$

$$X_4^\alpha = \frac{1}{2}a^\alpha \mathrm{tr}R^2 + \sum_i b_i^\alpha \left(\frac{2}{\lambda_i}\mathrm{tr}F_i^2\right)$$

$$H - V = 273 - 29T$$
$$0 = B_{\mathrm{adj}}^i - \sum_{\mathbf{R}} x_{\mathbf{R}}^i B_{\mathbf{R}}^i$$
$$a \cdot a = 9 - T$$
$$-a \cdot b_i = \frac{1}{6}\lambda_i\left(\sum_{\mathbf{R}} x_{\mathbf{R}}^i A_{\mathbf{R}}^i - A_{\mathrm{adj}}^i\right)$$
$$b_i \cdot b_i = \frac{1}{3}\lambda_i^2\left(\sum_{\mathbf{R}} x_{\mathbf{R}}^i C_{\mathbf{R}}^i - C_{\mathrm{adj}}^i\right)$$
$$b_i \cdot b_j = \lambda_i\lambda_j \sum_{\mathbf{RS}} x_{\mathbf{RS}}^{ij} A_{\mathbf{R}}^i A_{\mathbf{S}}^j$$

$$\mathrm{tr}_{\mathbf{R}}F^2 = A_{\mathbf{R}}\mathrm{tr}F^2, \mathrm{tr}_{\mathbf{R}}F^4 = B_{\mathbf{R}}\mathrm{tr}F^4 + C_{\mathbf{R}}(\mathrm{tr}F^2)^2$$

$$B \cdot X_4 = B^\alpha \Omega_{\alpha\beta}\left[\frac{1}{2}a^\beta \mathrm{tr}R^2 + \sum_i b_i^\beta \left(\frac{2}{\lambda_i}\mathrm{tr}F_i^2\right)\right]$$

$$r \to r' N^{1/3}, x^\mu \to \frac{x'^\mu}{N^{1/3}}$$

$$T_{\mu\nu} = \frac{2h_\phi(\beta)}{\beta\pi^2}\gamma^4(\theta)\left(4u_\mu u_\nu + g_{\mu\nu}\right)$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2$$
$$u^\mu \partial_\mu = \gamma\left(\partial_t - \omega_1\partial_{\phi_1} - \omega_2\partial_{\phi_2}\right)$$

$$\gamma(\theta) = \frac{1}{\sqrt{1 - \omega_1^2\sin^2\theta - \omega_2^2\cos^2\theta}}$$

$$Z_{\mathrm{gas}}(\beta, \omega_i) = \frac{4h_\phi(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$T_{\mu\nu} = \frac{2h_{\mathrm{YM}}(\beta)}{\beta\pi^2}\gamma^4(\theta)\left(4u_\mu u_\nu + g_{\mu\nu}\right)$$

$$ds^2 = -\Delta_{\tilde{\theta}}(1+r^2)\rho^2 dt^2 + \frac{f}{\rho^4}\left(\frac{\Delta_{\tilde{\theta}}dt}{\Xi_a\Xi_b} - \omega\right)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\tilde{\theta}^2}{\Delta_{\tilde{\theta}}} + \frac{r^2+a^2}{\Xi_a}\sin^2\tilde{\theta}d\phi_1^2 + \frac{r^2+b^2}{\Xi_b}\cos^2\tilde{\theta}d\phi_2^2$$

$$\Xi_a = 1 - a^2$$
$$\Xi_b = 1 - b^2$$
$$\Delta_{\tilde{\theta}} = 1 - a^2\cos^2\ \tilde{\theta} - b^2\sin^2\ \tilde{\theta}$$
$$\Delta_r = \frac{(r^2+a^2)(r^2+b^2)(1+r^2)}{r^2} - 2m$$
$$\rho^2 = r^2 + a^2\cos^2\ \tilde{\theta} + b^2\sin^2\ \tilde{\theta}$$
$$f = 2m\rho^2$$
$$\omega = a\sin^2\ \tilde{\theta}\,\frac{d\phi_1}{\Xi_a} + b\cos^2\ \tilde{\theta}\,\frac{d\phi_2}{\Xi_b}$$

$$a, b \in [-1,1]$$

$$(r_+^2 + a^2)(r_+^2 + b^2)(r_+^2 + 1) - 2mr_+^2 = 0$$

$$x^3 + x^2(a^2 + b^2 + 1) + x(a^2b^2 + a^2 + b^2 - 2m) + a^2b^2 = 0$$

$$2m > a^2 + b^2 + a^2b^2$$

$$m_{ext} = \frac{1}{2}\left(\frac{1}{6}\sqrt{(a^2 + b^2 + 1)^4 + 216a^2b^2(a^2 + b^2 + 1)}\cos\left(\frac{\alpha}{3}\right) + a^2(b^2 + 1) - \frac{1}{12}(a^2 + b^2 + 1)^2 + b^2\right)$$

$$\cos(\alpha) = \left(\frac{1}{(a^2+b^2+1)^4+216a^2b^2(a^2+b^2+1)}\right)^{3/2}$$
$$\times (5832a^4b^4 - (a^2 + b^2 + 1)^6 + 540a^2b^2(a^2 + b^2 + 1)^3)$$

$$x_{\text{ext}} = \frac{1}{6}(a^2 + b^2 + 1)\left(2\cos\left(\frac{1}{3}\xi\right) - 1\right)$$

$$\cos(\xi) = \left(\frac{54a^2b^2}{(a^2+b^2+1)^3} - 1\right)$$

$$x \approx \sqrt{2m}$$

$$ds^2 = -\Xi_a(1 + r^2)(r^2 + a^2)dt^2 + \frac{2m}{(r^2+a^2)\Xi_a^2}(dt - a\sigma_3)^2 + \frac{(r^2+a^2)dr^2}{\Delta_r} + \frac{(r^2+a^2)}{\Xi_a}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

$$\sigma_1 = \frac{1}{2}(\sin(\phi)d(2\tilde{\theta}) - \cos(\phi)\sin(2\tilde{\theta})d\psi)$$
$$\sigma_2 = \frac{1}{2}(\cos(\phi)d(2\tilde{\theta}) + \sin(\phi)\sin(2\tilde{\theta})d\psi)$$
$$\sigma_3 = \frac{1}{2}(d\phi - \cos(2\tilde{\theta})d\psi)$$

$$\phi = \phi_1 + \phi_2, \psi = \phi_1 - \phi_2$$

$$m_{\text{ext}} = \frac{1}{128}\left(\sqrt{8a^2 + 1} - 1\right)\left(\sqrt{8a^2 + 1} + 3\right)^3$$
$$(r_+)_{\text{ext}}^2 = \frac{1}{4}\left(\sqrt{1 + 8a^2} - 1\right)$$

$$y^2\sin^2\ \theta = \frac{(r^2+a^2)\sin^2\ \tilde{\theta}}{\Sigma_a}$$
$$y^2\cos^2\ \theta = \frac{(r^2+b^2)\cos^2\ \tilde{\theta}}{\Sigma_b}$$

$$ds^2 = -(1+y^2)dt^2 + \frac{dy^2}{1+y^2} + y^2(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2)$$
$$+\frac{2m}{y^6\Delta_\theta^3}dy^2 + \frac{2m}{y^2\Delta_\theta^3}dt^2$$
$$-\frac{4am\sin^2\theta}{y^2\Delta_\theta^3}dtd\phi - \frac{4bm\cos^2\theta}{y^2\Delta_\theta^3}dtd\psi$$
$$-\frac{2ma^2\sin^4\theta}{y^2\Delta_\theta^3}d\phi^2 + \frac{2mb^2\cos^4\theta}{y^2\Delta_\theta^3}d\psi^2$$
$$+\frac{2abm\sin^2\theta\cos^2\theta}{y^2\Delta_\theta^3}d\psi d\phi$$

$$ds^2 = -y^2dt^2 + \frac{dy^2}{y^2} + y^2(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2)$$

$$E = \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b)}{4\Xi_a^2\Xi_b^2}$$
$$J_a = \frac{2\pi am}{4\Xi_a^2\Xi_b}$$
$$J_b = \frac{2\pi bm}{4\Xi_a\Xi_b^2}$$
$$S = \frac{\pi^2(r_+^2+a^2)(r_+^2+b^2)}{2\Xi_a\Xi_b r_+}$$
$$T = \frac{r_+}{2\pi}(1+r_+^2)\left(\frac{1}{r_+^2+a^2} + \frac{1}{r_+^2+b^2}\right) - \frac{1}{2\pi r_+}$$
$$\omega_a = \frac{a(r_+^2+1)}{(r_+^2+a^2)}$$
$$\omega_b = \frac{b(r_+^2+1)}{(r_+^2+b^2)}$$
$$\Xi_a = 1 - a^2$$
$$\Xi_b = 1 - b^2$$

$$(r_+)_{ext} = \frac{\sqrt{ab}}{(1+b^2)^{\frac{1}{4}}}$$
$$m_{ext} = \frac{b^2}{2} + ab\sqrt{1+b^2}$$

$$E = -\frac{b^2(b^2-3\pi)}{8(b^2-1)^2}$$
$$J_a = \frac{ab^2\pi}{4(1-b^2)}$$
$$J_b = \frac{b^3\pi}{4(b^2-1)^2}$$
$$S = \frac{\sqrt{ab}b^2\pi^2}{2(1+b^2)^{\frac{1}{4}}(1-b^2)}$$

$$m \leq \frac{(1+a)^2(a+b^2)}{2}$$

$$\Delta m = a\left(\frac{1}{2} + b^2 - b\sqrt{1+b^2}\right),$$

$$m = \frac{a(1+a)^3}{2}.$$

$$T=\frac{1}{2\pi\sqrt{a}}$$
$$E=\frac{a(3+a^2)\pi}{8(1-a)^3}$$
$$J=J_a=J_b=\frac{\pi a^2}{(1-a)^3}$$
$$S=\frac{\pi^2 a^{3/2}}{2(a-1)^2}$$

$$E=-\frac{\pi+48\pi^5T^4}{8(1-4\pi^2T^2)^3},\frac{1}{2\pi}\le T<\infty$$

$$E\approx\frac{3}{32\pi T^2}$$

$$c=\left(\frac{dE}{dT}\right)\Big|_{\omega_a=\omega_b=1}=-\frac{3\pi^3T\left(4\pi^2T^2+1\right)^2}{(1-4\pi^2T^2)^4}$$

$$c\approx-\frac{3}{128\pi^2\left(T-\frac{1}{2\pi}\right)^4}$$

$$c\approx-\frac{3}{16\pi T^3}$$

$$\frac{dE_\omega(J)}{dJ}=\frac{3(a+1)^2}{2a(a+2)}$$

$$E_\omega(J)=\int_0^J\frac{dE_\omega(J)}{dJ}>2J$$

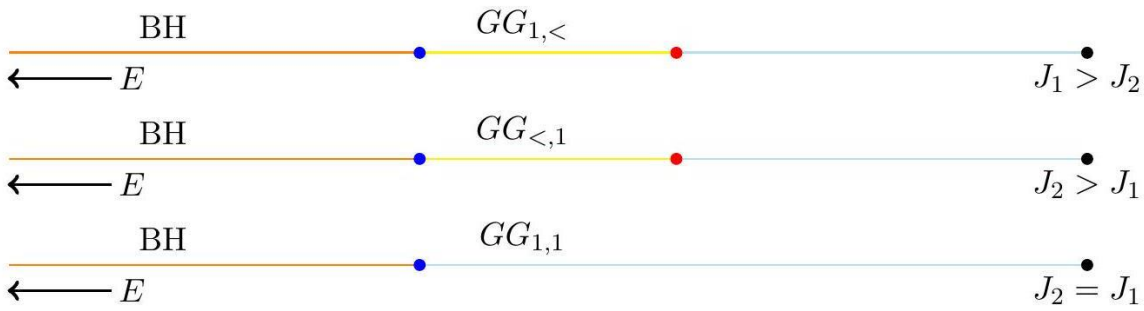
$$T^{tt}=\frac{m}{8\pi G_5}(4\gamma^6-\gamma^4)$$
$$T^{\phi_1\phi_1}=\frac{m}{8\pi G_5}\gamma^4\left(4\gamma^2a^2+\frac{1}{\sin^2(\theta)}\right)$$
$$T^{\phi_2\phi_2}=\frac{m}{8\pi G_5}\gamma^4\left(4\gamma^2b^2+\frac{1}{\cos^2(\theta)}\right)$$
$$T^{t\phi_1}=\frac{4m}{8\pi G_5}a\gamma^6,T^{t\phi_2}=\frac{4m}{8\pi G_5}b\gamma^6$$
$$T^{\phi_1\phi_2}=\frac{4m}{8\pi G_5}ab\gamma^6,T^{\theta\theta}=\frac{m}{8\pi G_5}\gamma^4$$

$$T^{\mu\nu}=\frac{N^2m}{4\pi^2}\gamma^4(\theta)\left(4u_{bh}^\mu u_{bh}^\nu+g^{\mu\nu}\right)$$
$$u_{bh}^\mu=\gamma\left(\partial_t-a\partial_{\phi_1}-b\partial_{\phi_2}\right),\gamma^{-2}=1-a^2\sin^2\theta-b^2\cos^2\theta$$

$$u_{bh}^\mu=\gamma\left(\partial_t-\omega_a\partial_{\phi_1}-\omega_b\partial_{\phi_2}\right),\gamma^{-2}=1-\omega_a^2\sin^2\theta-\omega_b^2\cos^2\theta$$

$$E_5=\int d\Omega_3T_0^0$$
$$=\frac{m}{8\pi G_5}\int_0^{2\pi}d\phi_1\int_0^{2\pi}d\phi_2\int_0^{\frac{\pi}{2}}d\theta\sin(\theta)\cos(\theta)(4\gamma^6-\gamma^4)$$
$$\equiv\int\rho(\theta)d\theta$$
$$=\frac{m\pi\left(2-2a^2+2-2b^2-\left(1-a^2\right)\left(1-b^2\right)\right)}{4G_5(1-a^2)^2(1-b^2)^2}$$

$$\rho(\theta)=\frac{m\pi}{2G_5}\frac{\sin(\theta)\cos(\theta)\left(3+a^2\sin^2(\theta)+b^2\cos^2(\theta)\right)}{(1-a^2\sin^2(\theta)-b^2\cos^2(\theta))^3}$$



$$(J_1, J_2, E) = \left(\alpha, \beta, E_{B_1}(\alpha, \beta)\right)$$

$$(\beta, \beta, E_\omega(\beta))$$

$$E_\omega(\beta) - E_{B_1}(\alpha, \beta) = \beta - \alpha$$

$$(\alpha, \beta, E_\omega(\beta - x) + (\alpha - (\beta - x)) + (\beta - (\beta - x))) = (\alpha, \beta, E_\omega(\beta - x) + \alpha - \beta + 2x)$$

$$\left(\alpha, \beta, E_{B_1}(\alpha, \beta) + E_\omega(\beta - x) - E_\omega(\beta) + 2x\right)$$

$$(\partial^2)^n \partial_{\mu_1} \dots \partial_{\mu_J} O$$

$$\mathrm{Tr} e^{-\beta H + \beta(\omega_1+\omega_2)J_L^z + \beta(\omega_1-\omega_2)J_R^z}$$

$$\ln Z(\beta, \omega_1, \omega_2) = -\sum_{n=0}^{\infty} \sum_{J=0}^{\infty} \sum_{m_L=-J/2}^{J/2} \sum_{m_R=-J/2}^{J/2} \ln\left(1 - e^{-\beta(\Delta+2n+J)+\beta\omega_1(m_L+m_R)+\beta\omega_2(m_L-m_R)}\right)$$

$$\ln Z(\beta, \omega_1, \omega_2) \approx \sum_{q=1}^{\infty} \frac{1}{q}\left[-\frac{e^{-q\beta(\Delta-6)}}{\left(e^{2\beta q}-1\right)^4} + \frac{e^{-q\beta(\Delta-2)}}{4\beta^2 q^2 (1-\omega_1)(1-\omega_2)\sinh^2(\beta q)}\right]$$

$$\approx \frac{h_\Delta(\beta)}{(1-\omega_1)(1-\omega_2)}$$

$$h_\Delta(\beta) = \sum_{q=1}^{\infty} \frac{e^{-q\beta(\Delta-2)}}{4\beta^2 q^3 \sinh^2(\beta q)}$$

$$\ln Z \approx \frac{4h_\Delta(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$J_L = \frac{J_1+J_2}{2}, J_R = \frac{J_1-J_2}{2}$$

$$\ln Z = -\sum_{n=0}^{\infty} \sum_{J=0}^{\infty} \sum_{|m_1|+|m_2|\le J} \ln\left(1 - e^{-\beta(\Delta+2n+J)+\beta\omega_1 m_1+\beta\omega_2 m_2}\right)$$

$$E = \frac{8h_\Delta(\beta)}{\beta}\left(\frac{1}{(1-\omega_2^2)^2(1-\omega_1^2)} + \frac{1}{(1-\omega_1^2)^2(1-\omega_2^2)}\right)$$

$$J_1 = \frac{1}{\beta}\frac{8h_\Delta(\beta)}{(1-\omega_1^2)^2(1-\omega_2^2)}$$

$$J_2 = \frac{1}{\beta}\frac{8h_\Delta(\beta)}{(1-\omega_2^2)^2(1-\omega_2^2)}$$

$$S = \frac{(\beta\partial_\beta - 1)4h_\Delta(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$d_n = \frac{(2n+4)(n+1)(n+2)(n+3)}{24}.$$

$$\ln Z \approx \sum_{n=0}^{\infty} d_n \frac{4h_\Delta(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$h_\Delta(\beta) = \sum_{q=1}^{\infty} \frac{e^{-q\beta(\Delta-2)}}{4\beta^2 q^3 \sinh^2(\beta q)}, \Delta = n+4,$$

$$\ln Z = \frac{1}{(1-\omega_1^2)(1-\omega_2^2)} \sum_{q=1}^{\infty} \frac{4\operatorname{csch}^6\left(\frac{\beta q}{2}\right)\operatorname{csch}(\beta q)}{128\beta^2 q^3}$$
$$= \frac{4h_\phi(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$\ln Z = \frac{4h_{YM}(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$h_{YM}(\beta) = h_{YM}^B(\beta) + h_{YM}^F(\beta)$$

$h_{YM}^B(\beta)$

$$= \sum_{n=1}^{\infty} \frac{(22\cosh(\beta n) + 17\cosh(2\beta n) + 6\cosh(3\beta n) + \cosh(4\beta n) + 18)\operatorname{csch}^7\left(\frac{\beta n}{2}\right)\operatorname{sech}\left(\frac{\beta n}{2}\right)}{128\beta^2 n^3}$$

$$h_{YM}^F(\beta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cosh(\beta n)(\cosh(\beta n)+\cosh(2\beta n)+2)\operatorname{csch}^7\left(\frac{\beta n}{2}\right)}{8\beta^2 n^3}$$

$$1-\omega_1 = \frac{\alpha_1}{N^{\frac{2}{3}}}$$
$$1-\omega_2 = \frac{\alpha_2}{N^{\frac{2}{3}}}$$

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta(H-\omega_1 J_1-\omega_2 J_2)}\right]$$

$$(\tau, \phi_1, \phi_2, \Omega_i) \sim (\tau+\beta, \phi_1 - i\beta\omega_1, \phi_2 - i\beta\omega_2, \Omega_i)$$

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{g_{\mu\nu}}{2}(\partial\Phi)^2$$

$$\langle T_{\mu\nu}\rangle = \lim_{x_1 \to x_2}\left[\partial_\mu^{x_1}\partial_\nu^{x_2}\tilde{G}(x_1-x_2) - \frac{g_{\mu\nu}}{2}\left(\partial^{\alpha,x_1}\partial_\alpha^{x_2}\tilde{G}(x_1-x_2)\right)\right]$$

$$\tilde{G}(x_1,x_2) = \sum_{q=-\infty}^{\infty} G\left(x_1, R^q(x_2)\right)$$

$$G(x_1,x_2) = \frac{\Gamma[4]}{4\pi^5}\left(\frac{1}{u+v}\right)^4$$

$$u = -2 + 2\left[\sqrt{(1+r^2)(1+r'^2)}\cosh(\tau^E - \tau'^E)\right.$$
$$\left. -rr'\left(\sin\theta\sin\theta'\cos(\phi_1-\phi_1') + \cos\theta\cos\theta'\cos(\phi_2-\phi_2')\right)\right]$$

$$u_q = -2 + 2[(1+r^2)\cosh(q\beta + \tau' - \tau)$$
$$-r^2\left(\sin^2\theta\cos(\phi_1 - \phi_1' - iq\beta\omega_1) + \cos^2\theta\cos(\phi_2 - \phi_2' - iq\beta\omega_2)\right)]$$

$$\tau' = \tau, \phi_1' = \phi_1, \phi_2' = \phi_2$$

$$r \sim \frac{1}{\sqrt{(1-\omega_1 \sin^2(\theta)-\omega_2 \cos^2(\theta))}}$$

$$r^2(1-\omega_1 \sin^2(\theta)-\omega_2 \cos^2(\theta)) = x^2$$

$$2(1-\omega_1 \sin^2(\theta)-\omega_2 \cos^2(\theta)) = 1-\omega_1^2 \sin^2(\theta)-\omega_2^2 \cos^2(\theta) = \frac{1}{\gamma^2(\theta)}$$

$$x^2 = \frac{r^2}{2\gamma^2(\theta)}$$

$$\begin{aligned} u_q &= 2(\cosh(\beta q)+\beta q r^2 \sinh(\beta q)(1-\omega_1 \sin^2(\theta)-\omega_2 \cos^2(\theta))-1) \\ &= 2(\cosh(\beta q)+\beta q x^2 \sinh(\beta q)-1) \end{aligned}$$

$$\begin{aligned} \partial_\tau u_q &= -\partial_{\tau'} u_q = -2\big((r^2+1)\sinh(\beta q)\big) \\ \partial_{\phi_1} u_q &= -\partial_{\phi_1'} u_q = 2ir^2 \sin^2(\theta)\sinh(\beta q \omega_1) \\ \partial_{\phi_2} u_q &= -\partial_{\phi_2'} u_q = 2ir^2 \sin^2(\theta)\sinh(\beta q \omega_2) \end{aligned}$$

$$w^\mu \partial_\mu = \big(\partial_t - \partial_{\phi_1} - \partial_{\phi_2}\big)$$

$$g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \partial_{\mu_1} \partial_{\mu_2} \langle \Phi(x_1)\Phi(x_2)\rangle_q \propto u^{\nu_1} u^{\nu_2}$$

$$\begin{aligned} T_{\mu_1\mu_2} &= -2\sum_{q=1}^{\infty} K^q_{\mu_1\mu_2} \\ K^q_{\mu_1\mu_2} &= \partial_{\mu_1}\partial_{\mu_2}\langle \Phi(x_1)\Phi(x_2)\rangle_q \\ &= \partial_u^2 G(u,v)\frac{\partial u_q}{\partial \mu_1}\frac{\partial u_q}{\partial \mu_2} + \partial_u G(u,v)\frac{\partial^2 u_q}{\partial \mu_1 \partial \mu_2} \end{aligned}$$

$$K^q_{\tau\tau} = \partial_u^2 G(u,v)\frac{\partial u_q}{\partial \tau}\frac{\partial u_q}{\partial \tau'}\Big|_{u=u_q}$$

$$\begin{aligned} K^q_{\tau\tau} &= -16\gamma^4(\theta)x^4 \sinh^2(q\beta)\partial_u^2 G(u,v)|_{v\to 0} \\ &= -\frac{4\gamma^4(\theta)x^4 \sinh^2(\beta q)\Gamma[6]}{\pi^5 u_q^6} \end{aligned}$$

$$\langle T_{00}\rangle = \sum_{q=1}^{\infty} \frac{8\gamma^4(\theta)x^4 \sinh^2(q\beta)\Gamma[6]}{\Gamma[2]\pi^5 u_q^6}$$

$$\begin{aligned} \langle T_{0\phi_1}\rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4 \sinh^2(q\beta)\sin^2\theta\Gamma[6]}{\pi^5 u_q^6} \\ \langle T_{0\phi_2}\rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4 \sinh^2(q\beta)\cos^2\theta\Gamma[6]}{\pi^5 u_q^6} \\ \langle T_{\phi_1\phi_1}\rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4 \sinh^2(q\beta)\sin^4\theta\Gamma[6]}{\pi^5 u_q^6} \\ \langle T_{\phi_2\phi_2}\rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4 \sinh^2(q\beta)\cos^4\theta\Gamma[6]}{\pi^5 u_q^6} \\ \langle T_{\phi_1\phi_2}\rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4 \sinh^2(q\beta)\sin^2\theta\cos^2\theta\Gamma[6]}{\pi^5 u_q^6} \\ \langle T_{\theta i}\rangle &= 0, i \in \{\tau_E, \theta, \phi_1, \phi_2\} \\ \langle T_{ri}\rangle &= 0 \\ \langle T_{\alpha\beta}\rangle &= 0 \\ \langle T_{\mu\alpha}\rangle &= 0, y_\alpha, y_\beta \in S^5, x_\mu \in AdS_5 \end{aligned}$$

$$\langle T_{\mu\nu}\rangle = 4\mathcal{F}(x)\gamma^4(\theta)\begin{pmatrix} 1 & 0 & 0 & \sin^2(\theta) & \cos^2(\theta) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sin^2(\theta) & 0 & 0 & \sin^4(\theta) & \sin^2(\theta)\cos^2(\theta) \\ \cos^2(\theta) & 0 & 0 & \sin^2(\theta)\cos^2(\theta) & \cos^4(\theta) \end{pmatrix}$$

$$\mathcal{F}(x) = \sum_{q=1}^{\infty} \frac{2x^4\Gamma[6]\sinh^2(\beta q)}{\pi^5 u_q^6}$$
$$= \sum_{q=1}^{\infty} \frac{x^4\Gamma[6]\sinh^2(\beta q)}{2^5\pi^5\left((\cosh(\beta q)+\beta q x^2\sinh(\beta q)-1)\right)^6}$$

$$T_{\mu\nu} = 4\mathcal{F}(x)\gamma^4(\theta)w_\mu w_\nu$$

$$\mathcal{F}(x) = x^4 D + \mathcal{O}(x^6)$$
$$D = \left(\frac{\Gamma[6]}{2^6\pi^5}\sum_{q=1}^{\infty} \frac{1}{2\sinh^2(\beta q(\cosh(\beta q)-1)^6}\right)$$

$$\mathcal{F}(x) = \frac{B}{x^8} + \mathcal{O}\left(\frac{1}{x^{10}}\right)$$
$$B = \frac{15\Gamma[6]}{4\pi^5\beta^6}\sum_{q=1}^{\infty} \frac{\operatorname{csch}^4(\beta q)}{q^6}$$

$$E = \int d^{10}x\sqrt{-g}g^{00}T_{00}$$
$$= \Omega_5 \int d\Omega_3 dr r \times 4\gamma^4(\theta)\mathcal{F}(x)$$
$$= \Omega_5 \int_{S_3} \gamma^6(\theta)\int_0^{\infty} dx x \times 8\mathcal{F}(x)$$

$$\int_{S_3} \gamma^6(\theta) = 4\pi^2\int_0^{\frac{\pi}{2}} \frac{d\theta \sin\theta\cos\theta}{(1-\omega_1^2\sin^2\theta-\omega_2^2\cos^2\theta)^3}$$
$$= \pi^2\left(\frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} + \frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2}\right)$$

$$\int_0^{\infty} dx x\mathcal{F}(x)$$
$$= \sum_{q=1}^{\infty} \int_0^{\infty} \frac{2\Gamma(6)\sinh^2(\beta q dx x^5}{\pi^5\left(2(\cosh(\beta q)+\beta q x^2\sinh(\beta q)-1)\right)^6}$$
$$= \sum_{q=1}^{\infty} \frac{\operatorname{csch}^6\left(\frac{\beta q}{2}\right)\operatorname{csch}(\beta q)}{128\pi^5\beta^3 q^3}$$
$$= \frac{h_\phi(\beta)}{\pi^5\beta}$$

$$E = \left(\frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} + \frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2}\right)\sum_{q=1}^{\infty} \frac{\operatorname{csch}^6\left(\frac{\beta q}{2}\right)\operatorname{csch}(\beta q)}{16\beta^3 q^3}$$
$$= \frac{8h_\phi(\beta)}{\beta}\left(\frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} + \frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2}\right)$$

$$ds^2 = d\theta^2 + \sin(\theta)^2 d\phi_1^2 + \cos(\theta)^2 d\phi_2^2$$

$$ds^2 = (\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2$$

$$ds^2_{AdS_5} = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2((\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2)$$

$$T = 4\mathcal{F}(x)\gamma^4(\theta)(dt-\sigma_3)(dt-\sigma_3)$$

$$x = \frac{r}{\sqrt{2}\gamma(\theta)}$$

$$t = t_0 + \frac{\tau}{\sqrt{2}\gamma(\theta_0)}$$
$$\phi = \phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)}$$
$$\psi = \psi_0 + \frac{\psi'}{\sqrt{2}\gamma(\theta_0)}$$
$$\theta = \theta_0 + \frac{\theta'}{\sqrt{2}\gamma(\theta_0)}$$

$$ds^2 = -x^2 d\tau^2 + \frac{dx^2}{x^2} + x^2((\sigma_1')^2 + (\sigma_2')^2 + (\sigma_3')^2) + d\Omega_5^2$$

$$\sigma_1' = \frac{1}{2}\left(\sin\left(\phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)}\right) d(2\theta') - \cos\left(\phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)}\right)\sin\left(2\theta_0 + \frac{2\theta'}{\sqrt{2}\gamma(\theta_0)}\right) d\psi'\right)$$
$$\sigma_2' = \frac{1}{2}\left(\cos\left(\phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)}\right) d(2\theta') + \sin\left(\phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)}\right)\sin\left(2\theta_0 + \frac{2\theta'}{\sqrt{2}\gamma(\theta_0)}\right) d\psi'\right)$$
$$\sigma_3' = \frac{1}{2}\left(d\phi' - \cos\left(2\theta_0 + \frac{2\theta'}{\sqrt{2}\gamma(\theta_0)}\right) d\psi'\right)$$

$$dw = \sigma_1', dy = \sigma_2', dz = \sigma_3'$$

$$ds^2 = \frac{dx^2}{x^2} + x^2(-d\tau^2 + dw^2 + dy^2 + dz^2)$$

$$T = \mathcal{F}(x) 2\gamma^2(\theta_0)(d\tau - dz)(d\tau - dz)$$

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} - 6 g_{\mu\nu} = 8\pi G_N^{(10)} T_{\mu\nu}$$

$$G^{(10)} = \frac{\pi^4 R_{AdS}^8}{2N^2}$$

$$G^{(5)} = \frac{G^{(10)}}{\Omega_5 R_{AdS}^5} = \frac{\pi^4 R_{AdS}^3}{2N^2\pi^3} = \frac{\pi R_{AdS}^3}{2N^2}$$

$$\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} - 6 g_{\mu\nu}\right) dx^\mu dx^\nu = 8\pi\left(\frac{\pi^4}{2N^2}\right)\mathcal{F}(x)\left(2\gamma^2(\theta_0)\right)(d\tau - dz)(d\tau - dz)$$
$$= \left(\frac{8\pi^5\gamma^2(\theta_0)}{N^2}\right)\mathcal{F}(x)(d\tau - dz)(d\tau - dz)$$

$$\delta ds^2 = \frac{8\pi^5\gamma^2(\theta_0)}{N^2} x^2 f(x)(d\tau - dz)(d\tau - dz)$$

$$\frac{1}{x^3}\partial_x(x^5\partial_x f) = -2\frac{x\mathcal{F}(x)}{x^3}$$

$$f(x) = 2\int_x^\infty dy\left(\frac{1}{y^5}\int_0^y dzz\mathcal{F}(z)\right)$$

$$ds_{AdS_5}^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2((\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2) + \frac{8\pi^2\gamma^2(\theta)}{N^2} r^2 f\left(\frac{r}{\gamma(\theta)}\right)(dt - \sigma_3)(dt - \sigma_3)$$

$$\int_0^y dzz\mathcal{F}(z) = \int_0^\infty dzz\mathcal{F}(z) - \int_y^\infty dzz\mathcal{F}(z)$$
$$= C - \frac{B}{6y^6} + \mathcal{O}(1/y^8)$$

$$C = \int_0^\infty dz z\mathcal{F}(z)$$
$$= \sum_{q=1}^\infty \frac{\operatorname{csch}^6\left(\frac{\beta q}{2}\right)\operatorname{csch}(\beta q)}{128\pi^5\beta^3 q^3}$$

$$C = \frac{h_\phi(\beta)}{\beta\pi^5}$$

$$f(x) = \frac{h_\phi(\beta)}{2\pi^5 x^4\beta} - \frac{B}{60x^{10}} + \mathcal{O}\left(\frac{1}{x^{12}}\right)$$

$$f(x) = \frac{1}{2}\int_x^\infty dy\frac{\mathcal{F}(y)}{y^3} + \frac{1}{2x^4}\int_0^x dyy\mathcal{F}(y)$$
$$= \frac{1}{2}\int_0^\infty dy\frac{\mathcal{F}(y)}{y^3} - \frac{1}{2}\int_0^x dy\frac{\mathcal{F}(y)}{y^3} + \frac{1}{2x^4}\int_0^x dyy\mathcal{F}(y)$$
$$= \frac{1}{2}\int_0^\infty dy\frac{\mathcal{F}(y)}{y^3} - \frac{D}{6}x^2 + \mathcal{O}(x^4)$$

$$ds^2_{AdS_5} = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2((\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2) + r^2(dt-\sigma_3)(dt-\sigma_3)\frac{16h_\phi(\beta)\gamma^6(\theta)}{N^2r^4\beta}$$

$$T_{\text{bdry}} = \left(\frac{4}{16\pi G_5}\right)\frac{16h_\phi(\beta)\gamma^6(\theta)}{\beta N^2}(dt-\sigma_3)(dt-\sigma_3)$$
$$= \frac{8h_\phi(\beta)\gamma^6(\theta)}{\pi^2\beta}(dt-\sigma_3)(dt-\sigma_3)$$

$$u^\mu\partial_\mu = \gamma(\theta)\left(\partial_t - \omega_1\partial_{\phi_1} - \omega_2\partial_{\phi_2}\right)$$

$$u^\mu\partial_\mu = \gamma(\theta)\left(\partial_t - \partial_{\phi_1} - \partial_{\phi_2} + \mathcal{O}\left(\frac{1}{\gamma(\theta)}\right)\right)$$

$$u_u dx^\mu = \gamma(\theta)(-dt + \sin^2\theta d\phi_1 + \cos^2\theta d\phi_2) = \gamma(\theta)(dt-\sigma_3)$$

$$\left(T_{\text{bdry}}\right)_{\mu\nu} = \frac{2h_\phi(\beta)\gamma^4(\theta)}{\beta\pi^2}\left(4u_\mu u_\nu + g_{\mu\nu}\right)$$

$$E = \frac{h_\phi(\beta)}{\beta\pi^5}\pi^3\int_{S^3} 8\gamma^6(\theta)$$
$$= \frac{8h_\phi(\beta)}{\beta}\left(\frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} + \frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2}\right)$$

$$\int_0^\infty dxx\mathcal{F}_{YM}(x) = \frac{h_{YM}(\beta)}{\pi^5\beta}$$

$$f_{YM}(x) = 2\int_x^\infty dy\left(\frac{1}{y^5}\int_0^y dzz\mathcal{F}_{YM}(z)\right)$$

$$f_{YM}(x) = \frac{h_{YM}(\beta)}{2\pi^5 x^4\beta} - \frac{B_{YM}}{60x^{10}} + \mathcal{O}\left(\frac{1}{x^{12}}\right)$$

$$\left(T^{YM}_{\text{bdry}}\right)_{\mu\nu} = \frac{2h_{YM}(\beta)\gamma^4(\theta)}{\beta\pi^2}\left(4u_\mu u_\nu + g_{\mu\nu}\right)$$

$$\omega J^{\frac{1}{D-2}} = g\left(\frac{M}{J^{\frac{D-3}{D-2}}}\right)$$

$$\omega \sim \frac{M}{J}$$

$$\Delta = a^4(r_1 - r_2)^2(r_1 - r_3)^2(r_2 - r_3)^2.$$

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$$

$$(r^2 + a^2)(r^2 + b^2)(r^2 + 1) - 2mr^2 = 0$$

$$\begin{aligned}\Delta = & -27a^4b^4 - 18a^2b^2(a^2 + b^2 + 1)(a^2(-b^2) - a^2 - b^2 + 2m) + 4(a^2(-b^2) - a^2 - b^2 + 2m)^3 \\ & + (a^2 + b^2 + 1)^2(a^2(-b^2) - a^2 - b^2 + 2m)^2 - 4a^2b^2(a^2 + b^2 + 1)^3\end{aligned}$$

$$\frac{1}{16}a^2b^2(a^6 + 3a^4(b^2 + 1) + 3a^2(b^4 - 7b^2 + 1) + (b^2 + 1)^3)^3 \geq 0$$

$$m_{\text{ext}} = \frac{1}{2}\left(\frac{1}{6}\sqrt{(a^2 + b^2 + 1)^4 + 216a^2b^2(a^2 + b^2 + 1)}\cos\left(\frac{\theta}{3}\right) + a^2(b^2 + 1) - \frac{1}{12}(a^2 + b^2 + 1)^2 + b^2\right)$$

$$\begin{aligned}\cos(\theta) = & \left(\frac{1}{(a^2 + b^2 + 1)^4 + 216a^2b^2(a^2 + b^2 + 1)}\right)^{3/2} \\ & \times (5832a^4b^4 - (a^2 + b^2 + 1)^6 + 540a^2b^2(a^2 + b^2 + 1)^3)\end{aligned}$$

$$\Delta_{a=b} = 4m(2a^2(-1 + a^2)^3 + m - 4a^2(5 + 2a^2)m + 8m^2)$$

$$m_{\text{ext}} = \frac{1}{128}\left(\sqrt{8a^2 + 1} - 1\right)\left(\sqrt{8a^2 + 1} + 3\right)^3$$

$$(r_+)^2_{\text{ext}} = \frac{1}{4}\left(\sqrt{1 + 8a^2} - 1\right)$$

$$\begin{aligned}\ln Z &= -\sum_{n=0}^{\infty}\sum_{J=0}^{\infty}\sum_{m_L,m_R=-\frac{J}{2}}^{\frac{J}{2}}\ln\left(1 - e^{-\beta(\Delta+2n+J)+\beta(\omega_1+\omega_2)m_L+\beta(\omega_1-\omega_2)m_R}\right) \\ &= -\sum_{n=0}^{\infty}\sum_{J=0}^{\infty}\sum_{a,b=0}^{J}\ln\left(1 - e^{-\beta\left(\Delta+2n+a(\omega_1+\omega_2)+b(\omega_1-\omega_2)\right)+\beta(1-\omega_1)J}\right) \\ &= \sum_{n=0}^{\infty}\sum_{J=0}^{\infty}\sum_{a,b=0}^{J}\sum_{q=1}^{\infty}\frac{e^{-q\beta\left(\Delta+2n+a(\omega_1+\omega_2)+b(\omega_1-\omega_2)\right)+q\beta(1-\omega_1)J}}{q}\end{aligned}$$

$$\begin{aligned}\ln Z = \sum_{n,J=0}^{\infty}\sum_{q=1}^{\infty}\frac{1}{q}\Bigg[& -\frac{\exp\left(\beta(J+1)q\omega_1 - \beta q(\Delta + J\omega_1 + J\omega_2 + J + 2n)\right)}{\left(e^{\beta q\omega_1} - e^{\beta q\omega_2}\right)\left(e^{\beta q(\omega_1+\omega_2)} - 1\right)} \\ & + \frac{\exp\left(\beta(J+1)q\omega_2 - \beta q(\Delta + J\omega_1 + J\omega_2 + J + 2n)\right)}{\left(e^{\beta q\omega_1} - e^{\beta q\omega_2}\right)\left(e^{\beta q(\omega_1+\omega_2)} - 1\right)} \\ & + \frac{\exp\left(-\beta q(\Delta + J\omega_1 + J\omega_2 + J + 2n) + \beta(J+1)q\omega_1 + \beta(J+1)q(\omega_1+\omega_2)\right)}{\left(e^{\beta q\omega_1} - e^{\beta q\omega_2}\right)\left(e^{\beta q(\omega_1+\omega_2)} - 1\right)} \\ & - \frac{\exp\left(-\beta q(\Delta + J\omega_1 + J\omega_2 + J + 2n) + \beta(J+1)q\omega_2 + \beta(J+1)q(\omega_1+\omega_2)\right)}{\left(e^{\beta q\omega_1} - e^{\beta q\omega_2}\right)\left(e^{\beta q(\omega_1+\omega_2)} - 1\right)}\Bigg]\end{aligned}$$

$$\begin{aligned}\ln Z = \sum_{q=1}^{\infty}\frac{1}{q}\Bigg[& -\frac{(\coth(\beta q) - 1)e^{\beta q(-\Delta+\omega_1+\omega_2+4)}}{2\left(e^{\beta q(\omega_1+1)} - 1\right)\left(e^{\beta q(\omega_2+1)} - 1\right)\left(e^{\beta q(\omega_1+\omega_2)} - 1\right)} \\ & + \frac{(\coth(\beta q) - 1)e^{\beta q(-\Delta+\omega_1+\omega_2+4)}}{2\left(e^{\beta q} - e^{\beta q\omega_1}\right)\left(e^{\beta q} - e^{\beta q\omega_2}\right)\left(e^{\beta q(\omega_1+\omega_2)} - 1\right)}\Bigg]\end{aligned}$$

$$\ln \mathcal{Z} = \sum_{q=1}^{\infty} \frac{1}{q}\left[-\frac{e^{\beta(\Delta-6)(-q)}}{\left(e^{2\beta q}-1\right)^4} + \frac{e^{\beta(\Delta-4)(-q)}}{\beta^2 q^2(\omega_1-1)(\omega_2-1)\left(e^{2\beta q}-1\right)^2} + O\left(\frac{1}{1-\omega_1}\right) + O\left(\frac{1}{1-\omega_2}\right)\right]$$
$$\approx \frac{e^{-q\beta(\Delta-2)}}{4\beta^2 q^3(1-\omega_1)(1-\omega_2)\sinh^2(\beta q)}$$

$$\Delta > j_1 + j_2 + 2 - \delta_{j_1,0} - \delta_{j_2,0} + \delta_{j_1,0}\delta_{j_2,0}$$

$$\mathrm{Tr}\left(s^H x^{j_1^z} y^{j_2^z}\right)$$

$$z_{B/F}^{\Delta j_1 j_2}(s,x,y) = \frac{s^\Delta \chi_{j_1}^L(x)\chi_{j_2}^R(y)}{\left(1-sx^{\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{\frac{1}{2}}y^{\frac{1}{2}}\right)}$$

$$z_{B/F}^{100}(s,x,y) = \frac{s\left(1-s^2\right)}{\left(1-sx^{\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{\frac{1}{2}}y^{\frac{1}{2}}\right)}$$

$$z_{B/F}^{j_1+1,j_1 0}(s,x,y) = \frac{s^{j_1+1}\left(\chi_{j_1}^L(x) - s\chi_{j_1-1/2}^L(x)\chi_{1/2}^R(y) + s^2\chi_{j_1-1}^L(x)\right)}{\left(1-sx^{\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{\frac{1}{2}}y^{\frac{1}{2}}\right)}$$

$$z_{B/F}^{j_1+j_2+2,j_1 j_2}(s,x,y) = \frac{s^{j_1+j_2+2}\left(\chi_{j_1}^L(x)\chi_{j_2}^R(y) - s\chi_{j_1-1/2}^L(x)\chi_{j_2-1/2}^R(y)\right)}{\left(1-sx^{\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{\frac{1}{2}}y^{\frac{1}{2}}\right)}$$

$$\mathrm{Tre}^{\mathrm{i}(\theta_1 \mathrm{R}_1 + \theta_2 \mathrm{R}_2 + \theta_3 \mathrm{R}_3)} = \mathrm{Tr}\left(\mathrm{p}^{\mathrm{R}_1}\mathrm{q}^{\mathrm{R}_2}\mathrm{r}^{\mathrm{R}_3}\right)$$

$$[1,0,0], [-1,1,0], [0,-1,1], \text{and } [0,0,-1]$$

$$v_1 = e^{i\theta_1}, v_2 = e^{i(\theta_2-\theta_1)}, v_3 = e^{i(\theta_3-\theta_2)}, v_4 = e^{-i\theta_3}.$$

$$\chi = \frac{\det\left[v_i^{l_j+n-j}\right]}{\det\left[v_i^{n-j}\right]}$$

$$\begin{aligned} l_1 &= r_1 + r_2 + r_3 \\ l_2 &= r_2 + r_3 \\ l_3 &= r_3 \\ l_4 &= 0 \end{aligned}$$

$$\chi_{(r_1,r_2,r_3)}(p,q,r) = \frac{\det\begin{pmatrix} v_1^{r_1+r_2+r_3+3} & v_2^{r_1+r_2+r_3+3} & v_3^{r_1+r_2+r_3+3} & v_4^{r_1+r_2+r_3+3} \\ v_1^{r_2+r_3+2} & v_2^{r_2+r_3+2} & v_3^{r_2+r_3+2} & v_4^{r_2+r_3+2} \\ v_1^{r_3+1} & v_2^{r_3+1} & v_3^{r_3+1} & v_4^{r_3+1} \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} v_1^3 & v_2^3 & v_3^3 & v_4^3 \\ v_1^2 & v_2^2 & v_3^2 & v_4^2 \\ v_1 & v_2 & v_3 & v_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}}$$

(j_1, j_2)	2Δ	Fields	$SU(4)$ Dynkin labels

(0,0)	2	$\varphi^{(1)}$	(0,1,0)
$\left(\frac{1}{2},0\right)$	3	$\lambda^{(1)}$	(0,0,1)
$\left(0,\frac{1}{2}\right)$	3	$\lambda^{(1)}$	(1,0,0)
(1,0)	4	$A_{\mu\nu}^{(1)}$	(0,0,0)
(0,1)	4	$\tilde{A}_{\mu\nu}^{(1)}$	(0,0,0)

$SO(4)$ Labels	2Δ	Fields	$SU(4)$ Dynkin Labels
$\left(\frac{1}{2},\frac{1}{2}\right)$	$2p+2=6$	$A_{\mu}^{(1)}$	$(1,p-2,1)=(1,0,1)$
$\left(1,\frac{1}{2}\right)$	$2p+3=7$	$\psi_{+\mu}^{(1)}$	$(1,p-2,0)=(1,0,0)$
$\left(\frac{1}{2},1\right)$	$2p+3=7$	$\psi_{-\mu}^{(1)}$	$(0,p-2,1)=(0,0,1)$
(1,1)	$2p+4=8$	$h_{\mu\nu}$	$(0,p-2,0)=(0,0,0)$

$$z_B = \mathcal{D}s\big(s\big(-s^2(s^2-1)xyr^4 + \big(s(s-q)(qs-1)(x+1)\sqrt{x}y^{3/2} + s^3(1-qs)xy^2 - (q(s-q)x^2s^3$$
$$+q(s-q)s^3 + (s^2+1)\big((q^2+1)s^3 + qs^2 - 2(q^2+1)s + q\big)x\big)y + s^3(1-qs)x$$
$$+s(s-q)(qs-1)(x+1)\sqrt{x}\sqrt{y}\big)r^2 - q^2s^2(s^2-1)xy\big)p^4 + (s^2(s-q)(qs-1)(r^4 + q(s^2+1)^2r^2$$
$$+q^2)(x+1)\sqrt{x}y^{3/2} + qs(q^3xs^3 + r^2(-sr^2+s^2+1)xs^3 + q^2(r^2(s^2+1)-s)xs^3$$
$$+qr^2\big((-2xs^4 - (x(x+3)+1)s^2 + r^2xs + x^2+1)s^2 + x\big)\big)y^2 + \big(s^2\big(2 - s(r^2+s)(s^2-1)\big)xq^4$$
$$+(r^2(s^2+1)-s)\big((x^2+x+1)s^4 + 2xs^2 + x\big)q^3 + s(s(-s^4+s^2+2)xr^4 - (xs^8 + (x(2x+3)+2)s^6$$
$$+(x(3x+7)+3)s^4 + 3xs^2 - (x-1)^2)r^2 + s^3 + sx(-s^4 + (x+1)s^2 + 2)\big)q^2$$
$$+r^2(-sr^2+s^2+1)\big((x^2+x+1)s^4 + 2xs^2 + x\big)q + r^4s^4 + r^4s^4x^2 + r^2s^2(-s^3 + s +$$
$$r^2(-s^4+s^2+2)\big)x\big)y + qs(q^3xs^3 + r^2(-sr^2+s^2+1)xs^3 + q^2(r^2(s^2+1)-s)xs^3$$
$$+qr^2\big((-2xs^4 - (x(x+3)+1)s^2 + r^2xs + x^2+1)s^2 + x\big)\big) + s^2(s-q)(qs-1)(r^4 + q(s^2+1)^2r^2$$
$$+q^2)(x+1)\sqrt{x}\sqrt{y}\big)p^2 - qr(r^2+q)\big(s^2(s^2+1)(q(s^3+s-q)-1)(x+1)\sqrt{x}y^{3/2}$$
$$-s^4(s-q)(qs-1)xy^2 + (q-s)(qs-1)\big((x^2+x+1)s^4 + 2xs^2 + x\big)y - s^4(s-q)(qs-1)x$$
$$+s^2(s^2+1)(q(s^3+s-q)-1)(x+1)\sqrt{x}\sqrt{y}\big)p - p^3r(r^2+q)(s^2(s^2+1)(q(s^3+s-q)$$
$$-1)(x+1)\sqrt{x}y^{3/2} - s^4(s-q)(qs-1)xy^2 + (q-s)(qs-1)\big((x^2+x+1)s^4 + 2xs^2 + x\big)y$$
$$-s^4(s-q)(qs-1)x + s^2(s^2+1)(q(s^3+s-q)-1)(x+1)\sqrt{x}\sqrt{y}\big) + q^2s(-s^2(s^2-1)xyr^4$$
$$+\big(s(s-q)(qs-1)(x+1)\sqrt{x}y^{3/2} + s^3(1-qs)xy^2 - \big(q(s-q)x^2s^3 + q(s-q)s^3 + (s^2+1)\big((q^2+1)s^3$$
$$+qs^2 - 2(q^2+1)s + q\big)x\big)y + s^3(1-qs)x + s(s-q)(qs-1)(x+1)\sqrt{x}\sqrt{y}\big)r^2 - q^2s^2(s^2-1)xy\big)\big)$$

$$z_F = -\mathcal{D}\big(s^{3/2}(s^2+1)\big(p^4rs^2\sqrt{x}\sqrt{y}(q+r^2)(\sqrt{x}(y+1)(qs-1) + x\sqrt{y}(s-q) + \sqrt{y}(s-q))$$
$$+p^3\big(y^{3/2}(q-s)\big(q^2s^2x(r^2s-1) - qr^2(s^2(x(s^2+x+4)+1)+x) + r^2sx(1-r^2s)\big)$$
$$+\sqrt{y}(q-s)(q^2s^2x(r^2s-1) - qr^2(s^2(xs^2+x+4)+1)+x) + r^2sx(1-r^2s)\big)$$
$$+(x+1)\sqrt{x}y(qs-1)\big(q^2s(s-r^2) + qr^2(s^4+4s^2+1) + r^2s^2(r^2-s)\big) + qr^2s^2(x+1)\sqrt{x}y^2(qs-1)$$
$$+qr^2s^2(x+1)\sqrt{x}(qs-1)\big) + p^2r(q+r^2)\big(-\big(y^{3/2}(qs-1)(q^2sx - q(s^2(x(s^2+x+4)+1)+x)$$
$$+s^3x)\big) - \sqrt{y}(qs-1)(q^2sx - q(s^2(x(s^2+x+4)+1)+x) + s^3x) + (x+1)\sqrt{x}y(q-s)(q^2s^3$$
$$-q(s^4+4s^2+1)+s) - qs^2\sqrt{x}(x+1)y^2(q-s) - qs^2(x+1)\sqrt{x}(q-s)\big) + pq\big(y^{3/2}(q-s)(q^2s^2x(r^2s -$$
$$-qr^2(s^2(x(s^2+x+4)+1)+x) + r^2sx(1-r^2s)\big) + \sqrt{y}(q-s)(q^2s^2x(r^2s-1)$$
$$-qr^2(s^2(x(s^2+x+4)+1)+x) + r^2sx(1-r^2s)\big) + (x+1)\sqrt{x}y(qs-1)(q^2s(s-r^2) + qr^2(s^4+4s^2$$
$$+1) + r^2s^2(r^2-s)\big) + qr^2s^2(x+1)\sqrt{x}y^2(qs-1) + qr^2s^2(x+1)\sqrt{x}(qs-1)\big)$$
$$+q^2rs^2\sqrt{x}\sqrt{y}(q+r^2)(\sqrt{x}(y+1)(qs-1) + x\sqrt{y}(s-q) + \sqrt{y}(s-q))\big)$$

$$\frac{1}{\mathcal{D}} = (q-s)(qs-1)(ps-r)(rs-p)(s\sqrt{x}-\sqrt{y})(s\sqrt{y}-\sqrt{x})(s-\sqrt{x}\sqrt{y})(s\sqrt{x}\sqrt{y}-1)(qs-pr)(q-prs)$$

$$i_{s.p} = \mathrm{Tr}_{s.p}\big[(-1)^F u^\delta t^{2(E+j_1)}\bar{y}^{2j_2}v^{R_2}w^{R_3}\big]$$

$$s \to ut^2, x \to u^{-2}t^2, y \to \bar{y}^2, p \to u^{-3/2}, q \to u^{-1}v, r \to u^{-1/2}w$$

$$i_{s.p} = -\frac{t^3}{\bar{y}\left(1-\frac{t^3}{\bar{y}}\right)} - \frac{t^3\bar{y}}{1-t^3\bar{y}} + \frac{t^2w}{v\left(1-\frac{t^2w}{v}\right)} + \frac{t^2v}{1-t^2v} + \frac{t^2}{w\left(1-\frac{t^2}{w}\right)}$$

$$z_B(s,x,y,1,1,1) = \mathcal{N}s\big(-s^3(s+1)x^2 - s^3(s+1) + s^2((s((s-5)s-2)-14)s^2+s-13)(x+1)\sqrt{x}y^{3/2}$$
$$+s^2((s((s-5)s-2)-14)s^2+s-13)(x+1)\sqrt{x}\sqrt{y} - y(s(s(s(s(s(s^2+s+4)+6)$$
$$+31)+5)+26)x + s((2(s-2)s+11)s^2+s+1) + ((2(s-2)s+11)s^3+s^2+s+1)x^2$$
$$-8x+1)+6x) - s((2(s-2)s+11)s^3+s^2+s+1)x$$
$$-sy^2(s(s((s+1)x^2+s(2(s-2)s+11)x+s+x+1)+x)+x)\big)$$
$$z_F(s,x,y,1,1,1) = -4\mathcal{N}s^{3/2}(s^2+1)(\sqrt{x}(y+1)+x\sqrt{y}+\sqrt{y})(s^2(x+1)y+s^2(x+1)$$
$$+(s(s((s-1)s+4)-1)+1)\sqrt{x}\sqrt{y})$$

$$\mathcal{N} = \frac{1}{(s-1)^5(s\sqrt{x}-\sqrt{y})(s\sqrt{y}-\sqrt{x})(s-\sqrt{x}\sqrt{y})(s\sqrt{x}\sqrt{y}-1)}$$

$$\ln Z_B = \sum_{n=1}^{\infty} \frac{1}{n} z_B(s^n, x^n, y^n, p^n, q^n, r^n)$$
$$\ln Z_F = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z_F(s^n, x^n, y^n, p^n, q^n, r^n)$$

$$\ln Z_B = \sum_{n=1}^{\infty} \frac{(22\cosh(\beta n)+17\cosh(2\beta n)+6\cosh(3\beta n)+\cosh(4\beta n)+18)\operatorname{csch}^7\left(\frac{\beta n}{2}\right)\operatorname{sech}\left(\frac{\beta n}{2}\right)}{32\beta^2 n^3(1-\omega_1^2)(1-\omega_2^2)}$$

$$\ln Z_F = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cosh(\beta n)(\cosh(\beta n)+\cosh(2\beta n)+2)\operatorname{csch}^7\left(\frac{\beta n}{2}\right)}{2\beta^2 n^3(1-\omega_1^2)(1-\omega_2^2)}$$

$$ds^2 = R_{AdS_5}^2\left(\frac{dz^2}{z^2} + \frac{dx_i^2}{z^2} + d\Omega_5^2\right)$$
$$= \frac{R_{AdS_5}^2}{z^2}\left(dx_i^2 + dz^2 + z^2 d\Omega_5^2\right)$$
$$= \frac{R_{AdS_5}^2}{z^2}\left(dx_i^2 + dy_j^2\right)$$

$$\nabla^2_{\text{conf, g}_{\mu\nu}} \equiv \left(g^{\mu\nu}\nabla_\mu\nabla_\mu - \frac{D-2}{4(D-1)}R\right)$$

$$\nabla^2_{x,\text{ conf, }} e^{2\chi}\, g_{\mu\nu}\left(e^{-\chi\frac{D-2}{2}}\phi\right) = e^{-\frac{(D+2)\chi}{2}}\nabla^2_{\text{conf, g}_{\mu\nu}}(\phi)$$

$$\nabla^2_{\text{conf, g}_{\mu\nu}} G(x,y) = \frac{1}{\sqrt{-g}}\delta^D(x-y)$$

$$\nabla^2_{\text{conf },e^2\chi}\, g_{\mu\nu}\left(e^{-\chi(x)\frac{D-2}{2}}G(x,y)\right) = \frac{e^{-\frac{(D+2)\chi(y)}{2}}\times e^{\chi(y)D}}{\sqrt{-ge^{2\chi(x)}}}\delta^D(x-y)$$
$$= \frac{e^{\frac{(D-2)\chi(y)}{2}}}{\sqrt{-ge^{2\chi(x)}}}\delta^D(x-y)$$

$$\nabla^2_{\text{conf },e^2\chi}\, g_{\mu\nu}\left(e^{-\chi(x)\frac{D-2}{2}}G(x,y)e^{-\chi(y)\frac{D-2}{2}}\right) = \frac{1}{\sqrt{-ge^{2\chi(x)}}}\delta^D(x-y)$$

$$G_{ge^{2\chi}}(x,y) = e^{-\chi(x)\frac{D-2}{2}}G_{g_{\mu\nu}}(x,y)e^{-\chi(y)\frac{D-2}{2}}$$

$$g_{\mu\nu} \to e^{2\chi}g_{\mu\nu} \Longrightarrow G(x,y) \to e^{-\chi(x)\frac{D-2}{2}}G(x,y)e^{-\chi(y)\frac{D-2}{2}}$$

$$\langle\phi(x)\phi(y)\rangle = z_x^4 G(x,y) z_y^4$$

$$G(x,x') = \frac{\Gamma[4]}{4\pi^5}\left(\frac{1}{\left(x_i - x_j'\right)^2}\right)^4$$

$$\langle \phi(\tilde{x}_1)\phi(\tilde{x}_2)\rangle = z_1^4 G(\tilde{x}_1,\tilde{x}_2) z_2^4$$
$$= \frac{\Gamma[4]}{4\pi^5} z_1^4 \left(\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}\right)^4 z_2^4$$
$$= \frac{\Gamma[4]}{2^6\pi^5}\left(\frac{1}{\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(y_1-y_2)^2}{2z_1z_2}}\right)^4$$
$$= \frac{\Gamma[4]}{2^6\pi^5}\left(\frac{1}{\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(z_1y_1'-z_2y_2')^2}{2z_1z_2}}\right)^4$$
$$= \frac{\Gamma[4]}{2^6\pi^5}\left(\frac{1}{\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(z_1-z_2)^2+2z_1z_2-2z_1z_2y_1'\cdot y_2'}{2z_1z_2}}\right)^4$$
$$= \frac{\Gamma[4]}{2^6\pi^5}\left(\frac{1}{\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(z_1-z_2)^2}{2z_1z_2}+1-y_1'\cdot y_2'}\right)^4$$
$$= \frac{\Gamma[4]}{4\pi^5}\left(\frac{1}{u+v}\right)^4$$

$$v = 2(1-y_1'\cdot y_2'), u = 2\left(\frac{(x_1-x_2)^2+(z_1-z_2)^2}{2z_1z_2}\right)$$

$$(y_1'-y_2')^2 = 2(1-y_1'\cdot y_2') = v$$

$$-X_{-1}^2 - X_0^2 + \sum_1^{d-1} X_i^2 = -1$$

$$X_i = \frac{x_i}{z}, i = 0,\ldots d-2$$
$$X_{-1} = \frac{1}{2}\left(\frac{(z^2+x^2)}{z}+\frac{1}{z}\right)$$
$$X_{d-1} = \frac{1}{2}\left(\frac{(z^2+x^2)}{z}-\frac{1}{z}\right)$$

$$(X_1-X_2)^2 = -2-2X_1\cdot X_2 = -2+2\left(1+\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(z_1-z_2)^2}{2z_1z_2}\right) = u$$

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2) + d\Omega_5^2$$

$$G^{(10)}\big((x,\theta),(x',\theta')\big) = \sum_{l=0}^{\infty}\sum_{\alpha=1}^{d_l} G_l^{(5)}(x,x')Y_l^{\alpha}(\theta)Y_l^{*\alpha}(\theta')$$

$$-\nabla_{10}^2 G^{(10)}\big((x,\theta),(x',\theta')\big) = \delta^{(10)}(x-x',\theta-\theta')$$
$$-\nabla_5^2 G_l^{(5)}(x,x') = \delta^{(5)}(x-x')$$

$$\int_{S^5} Y_l^{\alpha}(\theta)Y_l^{*\alpha}(\theta) = 1$$

$$T_{\mu\nu}^{(10)} = \sum_{l=0}^{\infty}\sum_{\alpha=1}^{d_l} (T_l)_{\mu\nu}^{(5)} Y_l^{\alpha}(\theta)Y_l^{*\alpha}(\theta)$$

$$\sum_\alpha \; Y_l^\alpha(\hat{n})Y_l^{*\alpha}(\hat{n}') = \frac{d_l}{\Omega_5 P^l(\hat{n}.\hat{n}'=1)} P^l(\hat{n}.\hat{n}')$$

$$T_{\mu\nu}^{(10)} = \sum_{l=0}^{\infty} \; d_l \frac{(T_l)_{\mu\nu}^{(5)}}{\Omega_5}$$

$$\int_{S^5} \; T_{\mu\nu}^{(10)} = \sum_{l=0}^{\infty} \; d_l (T_l)_{\mu\nu}^{(5)}$$

$$M^2 = \Delta(\Delta - 4) = m_{KK}^2 = l(l+4)$$

$$\begin{aligned} G_l^{(5)}(y,y') &= \frac{\Gamma(\Delta)}{2\pi^2\Gamma(\Delta-1)} \frac{1}{u^\Delta} \; {}_2F_1\left(\Delta, \Delta - \frac{3}{2}, 2\Delta - 3, -\frac{4}{u}\right) \\ &= \frac{\Gamma(l+4)}{2\pi^2\Gamma(l+3)} \frac{1}{u^{l+4}} \; {}_2F_1\left(l+4, l+5/2, 2l+5, -\frac{4}{u}\right) \end{aligned}$$

$$G^{(10)}(x,x') = \sum_{l=0}^{\infty} \; \sum_{\alpha=1}^{d_l} \; \frac{\Gamma(l+4)}{2\pi^2\Gamma(l+3)} \frac{1}{u^{l+4}} \; {}_2F_1\left(l+4, l+5/2, 2l+5, -\frac{4}{u}\right) Y_l^\alpha(\Omega) Y_l^{*\alpha}(\Omega')$$

$$\left\langle T_{\mu\nu}^{(10)} \right\rangle = -2 \sum_{q=1}^{\infty} \; \sum_{l=0}^{\infty} \; \sum_{\alpha=1}^{d_l} \; K_{l,\mu\nu}^q Y_l^\alpha(\Omega) Y_l^{*\alpha}(\Omega)$$

$$\begin{aligned} -K_{l,\tau\tau}^q &= -\frac{4x^4\gamma^4(\theta)\sinh^2(q\beta)}{2\pi^2} \frac{\Gamma[l+4](l+4)}{\Gamma[l+3](4+u_q)u_q^{l+6}} \left[5(u_q+2) \, {}_2F_1\left(l+\frac{5}{2}, l+5; 2l+5; -\frac{4}{u_q}\right)\right. \\ &\left. + (lu_q) \, {}_2F_1\left(l+\frac{5}{2}, l+4; 2l+5; -\frac{4}{u_q}\right)\right] \\ &= -\frac{4x^4\gamma^4(\theta)\sinh^2(q\beta)}{2\pi^2} \frac{\Gamma[l+5]}{\Gamma[l+3]u_q^{l+6}} \left[(l+5) \; {}_2F_1\left(l+\frac{5}{2}, l+6; 2l+5; -\frac{4}{u_q}\right)\right] \end{aligned}$$

$$\sum_{l=0}^{\infty} \; \sum_{\alpha=1}^{d_l} \; Y_l^\alpha(\Omega) Y_l^{*\alpha}(\Omega') = \sum_{l=0}^{\infty} \; \frac{(2l+4)\Gamma(2)}{4\Omega_5} C_l^{(2)}(\cos\,\Theta), \cos\,\Theta = 1 - \frac{v}{2}$$

$$\begin{aligned} \left\langle T_{00}^{(10)} \right\rangle &= \lim_{\Theta\to 0} \sum_{q=1}^{\infty} \; \sum_{l=0}^{\infty} \; \frac{4x^4\gamma^4(\theta)\sinh^2(q\beta)(2l+4)}{8\pi^5} \frac{\Gamma[(l+6)]}{\Gamma[l+3]u_q^{l+6}} \\ &\times \left[{}_2F_1\left(l+\frac{5}{2}, l+6; 2l+5; -\frac{4}{u_q}\right) C_l^{(2)}(\cos\,\Theta)\right] \\ &= \lim_{\Theta\to 0} \sum_{q=1}^{\infty} \; \sum_{l=0}^{\infty} \; \frac{x^4\gamma^4(\theta)\sinh^2(q\beta)}{\pi^5} \frac{\Gamma[(l+6)]}{\Gamma[l+2](u_q+2)^{l+6}} \\ &\times \left[{}_2F_1\left(\frac{l}{2}+3, \frac{l}{2}+\frac{7}{2}; l+3; \frac{4}{(u_q+2)^2}\right) C_l^{(2)}(\cos\,\Theta)\right] \end{aligned}$$

$$\sum_{l=0}^{\infty} \frac{\Gamma(l+\alpha)}{\Gamma(l+\beta)} \left(\frac{z}{2}\right)^l F\left(\frac{l}{2}+\frac{\alpha}{2}, \frac{l}{2}+\frac{\alpha}{2}+\frac{1}{2}; l+\beta+1; z^2\right) C_l^{(\beta)}(\cos\,\Theta) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \frac{1}{(1-z\cos\,\Theta)^\alpha}$$

$$\left\langle T_{00}^{(10)} \right\rangle = \lim_{\Theta\to 0} 2 \sum_{q=1}^{\infty} \; \frac{x^4\gamma^4(\theta)\sinh^2(q\beta)\Gamma[6]}{\Gamma[2]\pi^5(u_q+2)^6\left(1-\frac{2}{u_q+2}\cos\,\Theta\right)^6}$$

$$\left\langle T_{00}^{(10)} \right\rangle = \sum_{q=1}^{\infty} \; \frac{2x^4\gamma^4(\theta)\sinh^2(q\beta)\Gamma[6]}{\Gamma[2]\pi^5 u_q^6}$$

$$ds^2 = dt^2 + d\Omega_3^2$$
$$= \frac{dr^2}{r^2} + d\Omega_3^2$$
$$= \frac{1}{r^2}(dr^2 + r^2 d\Omega_3^2)$$

$$\vec{r} = e^t \hat{n}$$

$$e^{-t} G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') e^{-t'} = \frac{\mathcal{N}}{\left| e^t \hat{n}_1 - e^{t'} \hat{n}' \right|^2}$$
$$\Longrightarrow G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') = \frac{\mathcal{N} e^t e^{t'}}{\left| e^t \hat{n} - e^{t'} \hat{n}' \right|^2} = \frac{\mathcal{N}}{\left| e^{\frac{t-t'}{2}} \hat{n} - e^{\frac{t'-t}{2}} \hat{n}' \right|^2}$$

$$\mathcal{N} = \frac{1}{4\pi^2}$$

$$\hat{n} = (\sin\theta\cos\phi_1, \sin\theta\sin\phi_1, \cos\theta\cos\phi_2, \cos\theta\sin\phi_2)$$

$$G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') = \frac{\mathcal{N}}{e^{t-t'} + e^{t'-t} - 2\left(\cos\theta\cos\theta'\cos(\phi_1 - \phi_1') + \sin\theta\sin\theta'\cos(\phi_2 - \phi_2')\right)}$$

$$G_{S^3 \times S^1}(t, \hat{n}; t', \hat{n}') = G_{S^3 \times R}(t, \hat{n}; t', \hat{n}')$$
$$+ \sum_{q=-\infty; q\neq 0}^{\infty} \frac{\mathcal{N}}{\left(e^{t-t'+q\beta} + e^{t'-t-q\beta} - 2\left(\cos\theta\cos\theta'\cos(\phi_1 - \phi_1' - iq\beta\omega_1) + \sin\theta\sin\theta'\cos(\phi_2 - \phi_2' - iq\beta\omega_2)\right)\right)}$$

$$G_{S^3 \times S^1}(t, \hat{n}; t', \hat{n}') - G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') = \gamma^2(\theta) \sum_{q=1}^{\infty} \frac{2\mathcal{N}}{\beta q \sinh(\beta q)}$$

$$\lim_{\beta \to 0} G_{S^3 \times S^1}(t, \hat{n}; t', \hat{n}') - G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') = 2\sum_{q=1}^{\infty} \frac{\mathcal{N}}{q^2\beta^2\left(1 - \omega_2^2 \sin^2(\theta) - \omega_1^2 \cos^2(\theta)\right)}$$
$$= 2\sum_{q=1}^{\infty} \frac{\mathcal{N}\gamma^2(\theta)}{q^2\beta^2}$$

$$\partial_t \partial_{t'} G_{S^3 \times S^1}\left(t, \hat{n}, t', \hat{n'}\right)\Big|_{(t,\hat{n}) \to (t', \hat{n}')}$$
$$\approx \sum_{q=1}^{\infty} -\frac{8\gamma^6(\theta)\operatorname{csch}(\beta q)\left(2 - \beta q \coth(\beta q)\left(\frac{1}{2\gamma^2(\theta)}\right)\right)}{4\pi^2\beta^3 q^3}$$
$$\approx \sum_{q=1}^{\infty} -\frac{4\gamma^6(\theta)\operatorname{csch}(\beta q)}{\pi^2\beta^3 q^3}$$

$$\partial_{\phi_1}\partial_{\phi_1'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t',n')\to(t,n)} \approx \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{4\omega_1^2\cos^4(\theta)\mathrm{csch}(\beta q)}{\pi^2\beta^3q^3}$$

$$\partial_{\phi_2}\partial_{\phi_2'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})} \approx \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{4\omega_2^2\sin^4(\theta)\mathrm{csch}(\beta q)}{\pi^2\beta^3q^3}$$

$$\partial_{\phi_1}\partial_{\phi_2'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})} = \partial_{\phi_2}\partial_{\phi_1'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})}^{\infty}$$

$$\approx \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{4\omega_1\omega_2\sin^2(2\theta)\mathrm{csch}(\beta q)}{\pi^2(\beta q)^3}$$

$$\partial_t\partial_{\phi_2'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})} = \partial_{\phi_2}\partial_{t'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})}$$

$$\approx \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{i4\omega_2\sin^2(\theta)\mathrm{csch}(\beta q)}{\pi^2\beta^3q^3}$$

$$\partial_t\partial_{\phi_1'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})} = \partial_{\phi_1}\partial_{t'}G_{S^3\times S^1(t,\hat{n},t',\hat{n'})}\big|_{(t,\hat{n})\to(t',\hat{n'})}$$

$$\approx \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{i4\omega_1\cos^2(\theta)\mathrm{csch}(\beta q)}{\pi^2\beta^3q^3}$$

$$\partial_\theta\partial_{\phi_1'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})} = \partial_\theta\partial_{\phi_2'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})} \approx O\left(\frac{1}{\gamma^2(\theta)}\right)$$

$$\partial_\theta\partial_{\theta'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})} \approx \gamma^4(\theta)\sum_{q=1}^{\infty}\frac{\mathrm{csch}^2(\beta q)}{\pi^2\beta^2q^2}$$

$$\partial_t\partial_{\theta'}G_{S^3\times S^1}(t,\hat{n},t',\hat{n'})\big|_{(t,\hat{n})\to(t',\hat{n'})} \approx O\left(\frac{1}{\gamma^2(\theta)}\right)$$

$$S = \frac{1}{2}\int \sqrt{-g}d^4x\left(-g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \xi R\phi^2\right), \xi = \frac{1}{6}$$

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi + \xi G_{\mu\nu}\phi^2 + \xi\left(g_{\mu\nu}\nabla^\alpha\nabla_\alpha - \nabla_\mu\nabla_\nu\right)\phi^2$$

$$T_\mu^\mu = \left(1-\frac{D}{2}\right)\nabla_\alpha\phi\nabla^\alpha\phi + \xi R\left(1-\frac{D}{2}\right)\phi^2 + \xi(2(D-1)\nabla_\alpha\phi\nabla^\alpha\phi + 2(D-1)\phi\nabla^2\phi)$$
$$= \left(1-\frac{D}{2}\right)\phi(-\nabla^2 + \xi R)\phi = 0$$

$$\langle T_{00}\rangle = \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{4\mathrm{csch}(\beta q)}{\pi^2\beta^3q^3}$$

$$\langle T_{\phi_1\phi_1}\rangle = \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{4\omega_1^2\cos^4(\theta)\mathrm{csch}(\beta q)}{\pi^2\beta^3q^3}$$

$$\langle T_{\phi_2\phi_2}\rangle = \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{4\omega_2^2\sin^4(\theta)\mathrm{csch}(\beta q)}{\pi^2\beta^3q^3}$$

$$\langle T_{\theta\theta}\rangle = \gamma^4(\theta)\sum_{q=1}^{\infty}\frac{\mathrm{csch}^2(\beta q)}{\pi^2\beta^2q^2}$$

$$\langle T_{0\phi_1}\rangle = \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{4\omega_1\cos^2(\theta)\mathrm{csch}(\beta q)}{\pi^2\beta^3q^3}$$

$$\langle T_{0\phi_2}\rangle = \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{4\omega_2\sin^2(\theta)\mathrm{csch}(\beta q)}{\pi^2\beta^3q^3}$$

$$\langle T_{\phi_1\phi_2}\rangle = \gamma^6(\theta)\sum_{q=1}^{\infty}\frac{4\omega_1\omega_2\sin^2(2\theta)\mathrm{csch}(\beta q)}{\pi^2(\beta q)^3}$$

$$T_{\mu\nu} = \gamma^4(\theta)\left(4u_\mu u_\nu + g_{\mu\nu}\right)\sum_{q=1}^{\infty}\frac{1}{\pi^2\beta^3\sinh(q\beta)q^3}$$

$$\lim_{\beta\to 0}\langle T_{\mu\nu}\rangle = \frac{\pi^2}{90\beta^4}\gamma^4(\theta)\left(g_{\mu\nu} + 4u_\mu u_\nu\right), u_\mu = \gamma(-1,0,\omega_1\cos^2\theta, \omega_2\sin^2\theta).$$

$$T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + M^2A_\mu A_\nu - g_{\mu\nu}\left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{M^2}{2}A_\alpha A^\alpha\right)$$

$$\langle T_{\mu\nu}\rangle = \lim_{x_1\to x_2}\left[\langle F_{\mu\alpha}(x_1)F_\nu{}^\alpha(x_2)\rangle + M^2\langle A_\mu(x_1)A_\nu(x_2)\rangle - \frac{g_{\mu\nu}}{4}\left(\langle F_{\alpha\beta}(x_1)F^{\alpha\beta}(x_2)\rangle + \langle A_\alpha(x_1)A^\alpha(x_2)\rangle\right)\right]$$

$$= \lim_{x_1\to x_2}\left[\partial_\mu^{x_1}\partial_\nu^{x_2}\tilde{\Pi}_\alpha{}^\alpha(x_1,x_2) - \partial_\mu^{x_1}\partial^{x_2,\alpha}\tilde{\Pi}_{\alpha,\nu}(x_1,x_2) - \partial_\alpha^{x_1}\partial_\nu^{x_2}\tilde{\Pi}_\mu{}^\alpha(x_1,x_2) + \partial_\alpha^{x_1}\partial^{x_2,\alpha}\tilde{\Pi}_{\mu,\nu}(x_1,x_2)\right.$$

$$-\frac{g_{\mu\nu}}{4}\left(\partial_\alpha^{x_1}\partial^{x_2,\alpha}\tilde{\Pi}_\beta{}^\beta(x_1,x_2) + \partial_\beta^{x_1}\partial^{x_2,\beta}\tilde{\Pi}_\alpha{}^\alpha(x_1,x_2) - \partial_\alpha^{x_1}\partial^{x_2,\beta}\tilde{\Pi}_\beta{}^\alpha(x_1,x_2) - \partial_\beta^{x_1}\partial^{x_2,\alpha}\tilde{\Pi}_\alpha{}^\beta(x_1,x_2)\right)$$

$$\left. + M^2\tilde{\Pi}_{\mu,\nu}(x_1,x_2) - \frac{g_{\mu\nu}}{4}M^2\tilde{\Pi}_\alpha{}^\alpha(x_1,x_2)\right]$$

$$\tilde{\Pi}_{\mu,\nu}(x_1,x_2) = \sum_{q=-\infty}^{\infty}\Pi_{\mu,\nu}\left(x_1,R^q(x_2)\right)$$

$$\langle A_\mu(x_1)A_\nu(x_2)\rangle = \Pi_{\mu,\nu}(x_1,x_2) = -\frac{\partial^2 u}{\partial x_1^\mu\partial x_2^\nu}g_0(u) + \frac{\partial u}{\partial x_1^\mu}\frac{\partial u}{\partial x_2^\nu}g_1(u)$$

$$\begin{aligned} g_0(u) =& \frac{\mathcal{N}}{(u)^\Delta}\left[(4-\Delta)_2F_1\left(\Delta,\frac{1}{2}(-3+2\Delta);-3+2\Delta;-\frac{1}{u}\right)\right. \\ & \left. -\frac{2+u}{u}\,{}_2F_1\left(\Delta+1,\frac{1}{2}(-3+2\Delta);-3+2\Delta;-\frac{1}{u}\right)\right] \\ g_1(u) =& \frac{\mathcal{N}}{(u)^\Delta}\left[\frac{2(2+u)(4-\Delta)}{u(u+4)}\,{}_2F_1\left(\Delta,\frac{1}{2}(-3+2\Delta);-3+2\Delta;-\frac{1}{u}\right)\right. \\ & \left. -2\frac{4+(u+2)^2}{u^2(u+4)}\,{}_2F_1\left(\Delta+1,\frac{1}{2}(-3+2\Delta);-3+2\Delta;-\frac{1}{u}\right)\right] \end{aligned}$$

$$\mathcal{N} = \frac{\Gamma(\Delta+1)}{2\pi^2(3-\Delta)(\Delta-1)\Gamma(\Delta-1)}$$

$$\lim_{x_1\to x_2}\langle F_{0\lambda}(x_1)F_0^\lambda(x_2)\rangle = 4x^4\gamma^4(\theta)f^{(1)}(x)$$

$$\lim_{x_1\to x_2}\langle F_{0\lambda}(x_1)F_{\phi_1}^\lambda(x_2)\rangle = 4x^4\gamma^4(\theta)\sin^2(\theta)f^{(1)}(x)$$
$$\lim_{x_1\to x_2}\langle F_{0\lambda}(x_1)F_{\phi_2}^\lambda(x_2)\rangle = 4x^4\gamma^4(\theta)\cos^2(\theta)f^{(1)}(x)$$
$$\lim_{x_1\to x_2}\langle F_{\phi_1\lambda}(x_1)F_{\phi_1}^\lambda(x_2)\rangle = 4x^4\gamma^4(\theta)\sin^4(\theta)f^{(1)}(x)$$
$$\lim_{x_1\to x_2}\langle F_{\phi_2\lambda}(x_1)F_{\phi_2}^\lambda(x_2)\rangle = 4x^4\gamma^4(\theta)\cos^4(\theta)f^{(1)}(x)$$
$$\lim_{x_1\to x_2}\langle F_{\phi_1\lambda}(x_1)F_{\phi_2}^\lambda(x_2)\rangle = 4x^4\gamma^4(\theta)\sin^2(\theta)\cos^2(\theta)f^{(1)}(x)$$

$$f^{(1)}(x) = \sum_{q=1}^{\infty}\sinh^2(\beta q)\left(g_0''(u_q) + g_1'(u_q)\right)(3\cosh(\beta q) + \beta q x^2\sinh(\beta q)) + 2g_0'(u_q) + 2g_1(u_q)$$

$$g_{\mu\nu}\lim_{x_1\to x_2}\langle F_{\alpha\beta}(x_1)F^{\alpha\beta}(x_2)\rangle = O(\gamma^2(\theta))$$

$$\langle T_{\mu\nu}\rangle = 4\mathcal{F}_\Delta^{(1)}(x)\gamma^4(\theta)\begin{pmatrix} 1 & 0 & 0 & \sin^2(\theta) & \cos^2(\theta) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sin^2(\theta) & 0 & 0 & \sin^4(\theta) & \sin^2(\theta)\cos^2(\theta) \\ \cos^2(\theta) & 0 & 0 & \sin^2(\theta)\cos^2(\theta) & \cos^4(\theta) \end{pmatrix}$$

$$\mathcal{F}_{\Delta}^{(1)}(x)=\sum_{q=1}^{\infty}\left[\sinh^2(\beta q)\left(g_0''(u_q)+g_1'(u_q)\right)(3\cosh(\beta q)+\beta q x^2\sinh(\beta q))\right.$$
$$\left.+2g_0'(u_q)+2g_1(u_q)+M^2g_1(u_q)\right]$$
$$=\sum_{q=1}^{\infty}\left[\sinh^2(\beta q)\left(g_0''(u_q)+g_1'(u_q)\right)(3\cosh(\beta q)+\beta q x^2\sinh(\beta q))\right.$$
$$\left.+2g_0'(u_q)+2g_1(u_q)+(\Delta-1)(\Delta-3)g_1(u_q)\right]$$

$$T_{\mu\nu}=\mathcal{F}_{\Delta}^{(1)}(x)\gamma^4(\theta)4w_\mu w_\nu$$

$$y^2=\frac{r^2(1-a^2\cos^2\theta-b^2\sin^2\theta)+a^2\sin^2\theta+b^2\cos^2\theta-a^2b^2}{\Sigma_a\Sigma_b}$$

$$\gamma(x,\theta)\left(\partial_t-\omega_1\partial_{\phi_1}-\omega_2\partial_{\phi_2}\right),\gamma(x,\theta)=\frac{1}{\sqrt{1+\frac{r^2}{\gamma^2(\theta)}}}=\frac{1}{\sqrt{1+2x^2}}$$

$$\frac{\partial u}{\partial\tau^E}\frac{\partial u}{\partial\tau'^E}=-4r^4\sinh^2(q\beta)$$
$$=-16\gamma^4(\theta)x^4\sinh^2(q\beta)$$
$$\frac{\partial^2 u}{\partial\tau^E\partial\tau'^E}=-2r^2\cosh(q\beta)$$
$$=-4\gamma^2(\theta)x^2\cosh(q\beta)$$

$$\gamma=\frac{1}{\sqrt{1-\omega^2\sin^2\theta}}\approx\frac{\sqrt{2}}{\sqrt{\delta^2+\delta\theta^2}},1-\omega^2\approx\frac{\delta^2}{2}$$

$$TJ^{\frac{1}{D-2}}=f\left(\frac{M}{J^{\frac{D-3}{D-2}}}\right)$$

$$\sum\ y_i^2=z^2$$

$$\sum_{l,\alpha}Y_l^{\alpha}(\theta)Y_l^{*\alpha}(\theta')=\delta^{(5)}(\theta-\theta')$$

$${}_2F_1(a,b,2b,z)=\left(1-\frac{z}{2}\right)^{-a}\ {}_2F_1\left(a/2,a/2+1/2,b+1/2,\frac{z^2}{(2-z)^2}\right)$$

Formación de un agujero negro cuántico por colapso interior de una partícula oscura o blanca.

Modelo Unificado.

$$R^a_{bcd}=\Gamma^a_{bd,c}-\Gamma^a_{bc,d}+\Gamma^e_{bd}\Gamma^a_{ce}-\Gamma^e_{bc}\Gamma^a_{de},$$

$$\Gamma^a_{bd}=\frac{1}{2}g^{ae}\left(g_{be,d}+g_{ed,b}-g_{bd,e}\right)$$

$$R_{ab}=g^{cd}R_{acbd}$$

$$T_{(ab)}=\frac{1}{2}(T_{ab}+T_{ba}),T_{[ab]}=\frac{1}{2}(T_{ab}-T_{ba}).$$

$$\mathcal{A}=\frac{1}{2}\int\ d^4x\sqrt{-g}[R+2\mathcal{L}_m]$$

$$N^b_a\equiv h^b_a-e_ae^b=g^b_a+u_au^b-e_ae^b,N^a{}_a=2$$

$$\varepsilon_{ab} \equiv \varepsilon_{abc} e^c = \eta_{dabc} e^c u^d$$

$$\dot{X}^{a..b}{}_{c..d} \equiv u^e \nabla_e X^{a..b}{}_{c..d},$$
$$D_e X^{a..b}{}_{c..d} \equiv h^a{}_f h^p{}_{c...} h^b{}_g h^q{}_d h^r{}_e \nabla_r X^{f..g}{}_{p..q},$$
$$\hat{X}_{a..b}{}^{c..d} \equiv e^f D_f X_{a..b}{}^{c..d},$$
$$\delta_f X_{a..b}{}^{c..d} \equiv N_a{}^f \ldots N_b{}^g N_h{}^c \ldots N_i{}^d N_f{}^j D_j X_{f..g}{}^{i..j}$$

$$\{\phi, \xi, \Theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}\}$$

$$\{a^b, \mathcal{A}^a, \Omega^a, \Sigma^a, \mathcal{E}^a, \mathcal{H}^a\}$$

$$\{\zeta^{ab}, \Sigma^{ab}, \mathcal{E}^{ab}, \mathcal{H}^{ab}\}$$

$$\mathcal{A} = e_a \dot{u}^a, \mathcal{A}^a = N_{ab} \dot{e}^b,$$
$$\tilde{\theta} = \delta_a u^a, \bar{\theta} = a_b u^b, \tilde{\theta} + \bar{\theta} = D_a u^a = \Theta,$$
$$\Omega = \frac{1}{2} \varepsilon^{abc} D_{[a} u_{b]} e_a, \Omega^a = \frac{1}{2} \varepsilon^{abd} D_{[a} u_{b]} N_d^a,$$
$$\Sigma = \sigma^{ab} \left(e_a e_b - \frac{1}{2} N_{ab} \right), \Sigma_a = \sigma_{cd} e^c N_a^d,$$
$$\Sigma_{ab} = \left(N_{(a}^c N_{b)}^d - \frac{1}{2} N_{ab} N^{cd} \right) \sigma_{cd},$$
$$\sigma_{ab} = \left(n_{(a}^c h_{b)}^d - \frac{1}{3} h_{ab} h^{cd} \right) D_c u_d,$$
$$a_b = e^c D_c e_b = \hat{e}_b, \phi = \delta_a e^a,$$
$$\zeta_{ab} = \left(N_{(a}^c N_{b)}^d - \frac{1}{2} N_{ab} N^{cd} \right) \delta_c e_d,$$
$$\xi = \frac{1}{2} \varepsilon^{ab} \delta_a e_b,$$

$$\mathcal{E} = C^{ab}{}_{cd} u^c u^d \left(e_a e_b - \frac{1}{2} N_{ab} \right)$$
$$\mathcal{E}_a = C_{cdef} u^e u^f e^c N^d{}_a, \mathcal{E}_{ab} = C_{\{ab\}cd} u^c u^d$$
$$\mathcal{H} = \frac{1}{2} \varepsilon^a{}_{de} C^{deb}{}_c u^c \left(e_a e_b - \frac{1}{2} N_{ab} \right)$$
$$\mathcal{H}_a = \frac{1}{2} \varepsilon_{cfe} C^{fe}{}_{dh} u^h e^c N^d{}_a$$
$$\mathcal{H}_{ab} = \frac{1}{2} \varepsilon_{\{a}{}^{de} C_{b\}cde} u^c$$

$$\{\mu, p, Q, \Pi\},$$

$$\{Q^a, \Pi^a\},$$

$$\mu = T_{ab}u^a u^b,$$
$$p = \frac{1}{3}T_{ab}\left(e^a e^b + N^{ab}\right),$$
$$\Pi = \frac{1}{3}T_{ab}\left(2e^a e^b - N^{ab}\right),$$
$$Q = \frac{1}{2}T_{ab}e^a u^b,$$
$$Q_a = T_{cd}N^a{}_c u^d,$$
$$\Pi_a = T_{cd}N^a{}_c e^d,$$
$$\Pi_{ab} = T_{\{ab\}}.$$

$$p_r = p + \Pi$$
$$p_\perp = p - \frac{1}{2}\Pi$$

$$\hat{\phi} = -\frac{1}{2}\phi^2 - \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E},$$
$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right),$$
$$\hat{p} + \hat{\Pi} = -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - (\mu + p)\mathcal{A},$$
$$\hat{\mathcal{A}} = -(\mathcal{A} + \phi)\mathcal{A} + \frac{1}{2}(\mu + 3p),$$
$$\hat{K} = -\phi K,$$

$$0 = -\mathcal{A}\phi + \frac{1}{3}(\mu + 3p) - \mathcal{E} + \frac{1}{2}\Pi,$$
$$K = \frac{1}{3}\mu - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2.$$

$$K = K_0 e^{-\rho}$$

$$\rho = 2\ln(r/r_0),$$

$$X = \frac{\phi_{,\rho}}{\phi}, \quad Y = \frac{\mathcal{A}}{\phi}, \quad \mathcal{K} = \frac{K}{\phi^2}, \quad E = \frac{\mathcal{E}}{\phi^2}$$
$$\mathbb{M} = \frac{\mu}{\phi^2}, \quad \mathbb{P} = \frac{\Pi}{\phi^2}, \quad P = \frac{p}{\phi^2},$$

$$Y_{,\rho} = M + 3P - 2Y(X + Y + 1)$$
$$\mathcal{K}_{,\rho} = -\mathcal{K}(1 + 2X)$$
$$P_{,\rho} + \mathbb{P}_{,\rho} = -2Y(M + \mathbb{P}) - 2P(2X + Y)$$
$$-\mathbb{P}(4X + 3)$$

$$2\mathbb{M} + 2P + 2\mathbb{P} + 2X - 2Y + 1 = 0$$
$$1 - 4\mathcal{K} - 4P + 4Y - 4\mathbb{P} = 0$$
$$2\mathbb{M} + 6P + 3\mathbb{P} - 6Y - 6E = 0$$

$$\mathcal{A} = \frac{1}{2A\sqrt{B}}\frac{dA}{d\rho}, \phi = \frac{1}{\sqrt{B}}$$
$$K = \frac{1}{C_0}e^{-\rho} = K_0 e^{-\rho}$$
$$Y = \frac{1}{2}\frac{A_{,\rho}}{A}, \mathcal{K} = K_0 B(\rho) e^{-\rho}$$

$$\mathcal{A} = \frac{1}{2\sqrt{B}}\frac{dA}{dr}, \phi = \frac{2}{r\sqrt{B}}$$
$$K = \frac{\bar{K}_0}{r^2}$$
$$Y = \frac{1}{4}\frac{A_{,r}}{A}, \mathcal{K} = \bar{K}_0 B(r).$$

$$\left[g_{ab}^{\mathcal{S}}\right] = [N_{ab} + u_a u_b] = 0,$$
$$[\bar{K}_{cd}] = \left[\frac{1}{2}\phi N_{ab} - u_a u_b \mathcal{A}\right] = 0,$$

$$[u_a] = 0, [N_{ab}] = 0$$

$$[\mathcal{A}] = 0, [\phi] = 0$$

$$T_{ab}^{\mathcal{S}} = (N_{ab} + u_a u_b)[\bar{K}] - [\bar{K}_{cd}] =$$
$$= u_a u_b[\phi + 2\mathcal{A}] + N_{ab}\left[\frac{\phi}{2} + \mathcal{A}\right],$$

$$0 = [\mathcal{K}] = [P + \mathbb{P} - Y]$$

$$[P + \mathbb{P}] = 0$$

$$P_{,\rho} + \mathbb{P}_{,\rho} = P\left(\mathbb{M} - 2\mathbb{P} - 3\mathcal{K} + \frac{7}{4}\right)$$
$$+\mathbb{P}\left(\mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right)$$
$$+\left(\frac{1}{4} - \mathcal{K}\right)\mathbb{M} - P^2 - \mathbb{P}^2$$
$$\mathcal{K}_{,\rho} = -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathbb{M}\right)$$

$$\mathbb{P} = \frac{\mu_0^2 P_0 (1-h) e^{2\rho}\left(1 - \frac{\mu_0 e^{\rho}}{3}\right)^{h/2}}{18\left(1 - \frac{\mu_0 e^{\rho}}{3}\right)^2\left[\left(1 - \frac{\mu_0 e^{\rho}}{3}\right)^{h/2} + 3P_0\right]^2},$$

$$P = \frac{\mu_0 e^{\rho}}{\left[\left(1-\frac{\mu_0 e^{\rho}}{3}\right)^{h/2}+3P_0\right]^2}\left[\frac{3P_0^2+\left(1-\frac{\mu_0 e^{\rho}}{3}\right)^{h}}{4\left(\frac{\mu_0 e^{\rho}}{3}-1\right)}+\right.$$
$$\left.+\frac{2P_0[4(h+5)\mu_0 e^{\rho}-72]\left(1-\frac{\mu_0 e^{\rho}}{3}\right)^{h/2}}{(4\mu_0 e^{\rho}-12)^2}\right],$$

$$\mathcal{K} = -\frac{3K_0 e^{\rho/2}}{16\mu_0 e^{3\rho/2}-12K_0 e^{\rho/2}}$$
$$Y = \frac{\mu_0 e^{\rho}(3K_0-\mu_0 e^{\rho})^{h/2}}{2(\mu_0 e^{\rho}-3K_0)[(3K_0-\mu_0 e^{\rho})^{h/2}+9K_0P_0]}$$

$$ds^2 = -Adt^2+Bdr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$$
$$A = A_0\left[\left(1-\frac{\mu_0}{3}r^2\right)^{h/2}+3P_0\right]^{2/h}$$
$$B = \frac{3}{3-\mu_0 r^2}$$

$$p_r = -\frac{\mu_0\left[\left(1-\frac{\mu_0 r^2}{3}\right)^{h/2}+P_0\right]}{\left(1-\frac{\mu_0 r^2}{3}\right)^{h/2}+3P_0},$$

$$p_\perp = p_r - \frac{\mu_0^2 P_0 r^2(1-h)\left(1-\frac{\mu_0 r^2}{3}\right)^{\frac{h}{2}-1}}{3\left[\left(1-\frac{\mu_0 r^2}{3}\right)^{h/2}+3P_0\right]^2},$$

$$\mathbb{P} = \frac{1}{6}\mathbb{M}(1-4\mathcal{K}),$$

$$\mathcal{P}_{,\rho}+\mathcal{P}^2+\mathcal{P}\left(3\mathcal{K}-\mathbb{M}-\frac{7}{4}\right)=0$$

$$ds^2 = -Adt^2+Bdr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$$
$$A = \frac{A_0\left(e^{\mathcal{P}_0}r_0^2+r^2\right)^2}{\sqrt{3-\mu_0 r^2}}$$
$$B = \frac{3}{3-\mu_0 r^2}$$

$$p_r = \frac{3-\mu_0 r^2}{3(r^2+e^{\mathcal{P}_0}r_0^2)}$$
$$p_\perp = \frac{\mu_0^2 r^2}{4(3-\mu_0 r^2)}+p_r$$

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2\,\theta d\phi^2),$$

$$A = A_0\psi(12a^2 - 3a(4b-1)r^2 + b(4b-2)r^4)^{\frac{4b-5}{8b-4}}$$

$$\psi = \exp\left(\frac{\sqrt{3}(4b+1)\tan^{-1}\left(\frac{a(3-12b)+2b(4b-2)r^2}{a\sqrt{48b^2-24b-9}}\right)}{(2-4b)\sqrt{(4b-3)(4b+1)}}\right),$$

$$B = \frac{12(br^2-a)^2}{12a^2 - 3a(4b-1)r^2 + b(4b-2)r^4},$$

$$\mu = \frac{(4b+1)(9a^2 - 7abr^2 + 2b^2r^4)}{12(br^2-a)^3},$$

$$p_r = \frac{1}{a-br^2},$$

$$p_\perp = \frac{[4b^2r^4 + 3a(4a+r^2) - b(12ar^2+2r^4)]^{-1}}{48(br^2-a)^3} \times$$
$$\{9a^3(64a+19r^2)$$
$$-b^3(624a^2r^4 + 800ar^6 + 100r^8)$$
$$+ab^2r^2(432a^2 + 1608ar^2 + 356r^4)$$
$$-9a^2br^2(168a+47r^2)$$
$$+80b^4r^6(4a+2r^2) - 64b^5r^8\}.$$

$$P_{,\rho} + P^2 - P\left(\mathbb{M} - 3\mathcal{K} + \frac{7}{4}\right)$$
$$+\left(\frac{1}{4} - \mathcal{K}\right)\mathbb{M} =$$
$$\mathbb{P}_{,\rho} - \mathbb{P}^2 + \mathbb{P}\left(2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right),$$
$$\mathcal{K}_{,\rho} = -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathbb{M}\right).$$

$$\mathbb{P}_{,\rho} - \mathbb{P}^2 + \mathbb{P}\left(2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right) = 0$$

$$P = P_0 - \mathbb{P}, \mathbb{M} = \mathbb{M}_0 + \alpha\mathbb{M}_1$$

$$\mathbb{P} = \frac{1}{6}\big[P_0(4\alpha\mathbb{M}_1 + 4\mathbb{M}_0 - 12\mathcal{K} + 7) - 4P_{0,\rho}$$
$$-4P_0^2 - (4\mathcal{K}-1)(\alpha\mathbb{M}_1 + \mathbb{M}_0)\big].$$

$$\mathcal{K} = \frac{e^F}{\mathcal{K}_* - 2\int e^F d\rho}$$
$$F = \int \frac{1}{2}(4\alpha\mathbb{M}_1 + 4\mathbb{M}_0 + 1)d\rho$$

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2\ \theta d\phi^2)$$

$$A = A_0\left(\sqrt{3-\mu_0 r^2} + P_0\right)^2$$

$$B = \frac{3}{3-\mu_0 r^2}$$

$$p = -\frac{\mu_0\left(P_0 + 3\sqrt{3-\mu_0 r^2}\right)}{3\left(P_0 + \sqrt{3-\mu_0 r^2}\right)}$$

$$\mathbb{M}_1 = \frac{\mu_1 e^{\rho}}{K_0}\mathcal{K}$$

$$\mathcal{K} = \frac{3K_0}{4\left(3K_0 - 4e^{\rho}(\alpha\mu_1 + \mu_0)\right)}.$$

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2\ \theta d\phi^2),$$

$$A = \frac{A_0\sqrt{3-\mu_0 r^2}(3 - P_0^2 r_0^2 - \mu_0 r^2)}{\sqrt{3 - r^2(\alpha\mu_1 + \mu_0)}} \times$$

$$\exp\left(2\tanh^{-1}\left(\frac{\sqrt{3-\mu_0 r^2}}{P_0 r_0}\right)\right),$$

$$B = \frac{3}{3 - r^2(\alpha\mu_1 + \mu_0)},$$

$$p_r = \frac{\mu_0[3 - r^2(\alpha\mu_1 + \mu_0)]\left(3\sqrt{3-\mu_0 r^2} + P_0 r_0\right)}{3(\mu_0 r^2 - 3)\left(\sqrt{3-\mu_0 r^2} + P_0 r_0\right)}$$

$$p_\perp = -192(\mu_0 r^2 - 3)^2(r^2(\beta\mu_1 + \mu_0) - 3)p_r$$

$$-\frac{36\beta\mu_1 r^2(8\mu_0(r^2(\beta\mu_1 + \mu_0) - 3) - 12\beta\mu_1)}{48(\mu_0 r^2 - 3)^2\left(12 - 4r^2(\beta\mu_1 + \mu_0)\right)}$$

$$P = P_0 + P_1, Y = Y_0 + Y_1$$

$$A \to A_0(\rho)\exp\left(\int\ Y_1 d\rho\right)$$

$$B \to B_0(\rho)$$

$$C \to C_0(\rho)$$

$$P_{1,\rho} + P_1^2 + P_1\left(3\mathcal{K}_0 - \mathbb{M}_0 + 2P_0 + 2\mathbb{P}_0 - \frac{7}{4}\right) = 0,$$

$$Y_1 = P_1$$

$$P_1 = \frac{e^F}{P_* + \int\ e^F d\rho}$$

$$F = \int\ \left(3\mathcal{K}_0 - \mathbb{M}_0 + 2P_0 + 2\mathbb{P}_0 - \frac{7}{4}\right)d\rho$$

$$P = P_0 + P_1, \mathbb{M} = \mathbb{M}_0 + \mathbb{M}_1,$$
$$\mathcal{K} = \frac{1}{\mathcal{K}_0 + \mathcal{K}_1},$$

$$A \to A_0(\rho)$$
$$B^{-1} \to B_0(\rho) + \frac{e^{\rho}}{K_0}\mathcal{K}_1$$
$$C \to C_0(\rho)$$

$$\mathcal{K}_{1,\rho} = -\mathcal{K}_1\Phi + \Gamma$$
$$\Phi = \frac{12\mathcal{K}_0 - 4\mathbb{M}_0 - 1 + Y_0(8\mathcal{K}_0 - 8\mathbb{M}_0 - 2)}{2(1 + 2Y_0)}$$
$$\Gamma = \frac{8\mathcal{K}_0^2(Y_0 + 1) + \mathcal{K}_0(4\mathbb{M}_0 + 1)(2Y_0 + 1) - 4}{1 + 2Y_0}$$
$$P_1 = \frac{\mathcal{K}_0^2 + \mathcal{K}_1\mathcal{K}_0 - 1}{\mathcal{K}_0 + \mathcal{K}_1}$$
$$\mathbb{M}_1 = -\frac{[\mathcal{K}_0(\mathcal{K}_0 + \mathcal{K}_1) - 1](2Y_0 + 3)}{(\mathcal{K}_0 + \mathcal{K}_1)(2Y_0 + 1)}$$

$$\mathcal{K}_1 = e^{-F}\left(\mathcal{K}_* - \int e^{F}\Gamma d\rho\right)$$
$$F = \int \Phi d\rho$$

$$\mathbb{P} = \mathbb{P}_0 + \mathbb{P}_1, Y = Y_0 + Y_1$$

$$Y_1 = \mathbb{P}_1$$
$$\mathbb{P}_1 = \frac{e^{F}}{\mathbb{P}_* + \int e^{F} d\rho}$$
$$F = \int \left(3\mathcal{K}_0 - \mathbb{M}_0 + 2P_0 + 2\mathbb{P}_0 - \frac{1}{4}\right) d\rho$$

$$\mathbb{P} = \mathbb{P}_0 + \mathbb{P}_1, \mathbb{M} = \mathbb{M}_0 + \mathbb{M}_1$$
$$\mathcal{K} = \frac{1}{\mathcal{K}_0 + \mathcal{K}_1}$$

$$\mathcal{K}_{1,\rho} = \mathcal{K}_1\Phi + \Gamma$$
$$\Phi = \frac{Y_0(8\mathcal{K}_0 - 8\mathbb{M}_0 - 2) - 4\mathbb{M}_0 - 1}{2(1 + 2Y_0)}$$
$$\Gamma = \frac{\mathcal{K}_0^2(8Y_0 + 2) - \mathcal{K}_0(4\mathbb{M}_0 + 1)(2Y_0 + 1) + 2}{1 + 2Y_0}$$
$$\mathbb{P}_1 = \frac{\mathcal{K}_0^2 + \mathcal{K}_1\mathcal{K}_0 - 1}{\mathcal{K}_0 + \mathcal{K}_1}$$
$$\mathbb{M}_1 = -\frac{2[\mathcal{K}_0(\mathcal{K}_0 + \mathcal{K}_1) - 1]Y_0}{(\mathcal{K}_0 + \mathcal{K}_1)(2Y_0 + 1)}$$

$$\mathcal{K}_1 = e^{F}\left(\mathcal{K}_* - \int e^{-F}\Gamma d\rho\right)$$
$$F = \int \Phi d\rho$$

$$\mathbb{P} = \mathbb{P}_0 + \mathbb{P}_1, P = P_0 + P_1$$
$$\mathcal{K} = \frac{1}{\mathcal{K}_0 + \mathcal{K}_1}$$

$$\mathcal{K}_{1,\rho} = -\mathcal{K}_1\Phi + \Gamma$$
$$\Phi = \frac{1}{2}(4\mathbb{M}_0 + 1)$$
$$\Gamma = \frac{1}{2}[4\mathcal{K}_0^2 - 2\mathcal{K}_0(4\mathbb{M}_0 + 1) + 4]$$
$$\mathbb{P}_1 = \frac{(\mathcal{K}_0(\mathcal{K}_0 + \mathcal{K}_1) - 1)(2Y_0 + 3)}{3(\mathcal{K}_0 + \mathcal{K}_1)}$$
$$P_1 = \frac{2}{3}\left(\frac{1}{\mathcal{K}_0 + \mathcal{K}_1} - \mathcal{K}_0\right)Y_0$$

$$\mathcal{K}_1 = e^{-F}\left(\mathcal{K}_* - \int e^{F}\Gamma d\rho\right)$$
$$F = \int \Phi d\rho$$

$$\mathbb{P} = \mathbb{P}_0 + \mathbb{P}_1, P = P_0 + P_1,$$
$$\mathcal{K} = \mathcal{K}_1,$$

$$\mathcal{K}_{1,\rho} = -\mathcal{K}_1\left(-2\mathbb{M}_0 - \frac{1}{2}\right) - 2\mathcal{K}_1^2$$
$$\mathbb{P}_1 = (2\mathcal{K}_0 - 1)P_0 - \mathbb{P}_0 + Y_0 + \frac{1}{4}$$
$$\frac{1}{6}\mathcal{K}_1(-4P_0 - 4\mathbb{P}_0 - 5) + \frac{2}{3}\mathcal{K}_0(\mathbb{M}_0 + 3\mathbb{P}_0)$$
$$P_1 = \frac{1}{6}\mathcal{K}_1(4P_0 + 4\mathbb{P}_0 - 1)$$
$$-\frac{2}{3}\mathcal{K}_0(\mathbb{M}_0 + 3P_0 + 3\mathbb{P}_0)$$

$$\mathbb{M} = \frac{\mathcal{K}_\rho}{2\mathcal{K}} + \mathcal{K} - \frac{1}{4}$$
$$P = \frac{1}{3}\left(2Y_\rho + 2Y^2 + Y\right)$$
$$-\frac{2Y + 1}{6}\frac{\mathcal{K}_\rho}{\mathcal{K}} - \frac{1}{3}\mathcal{K} + \frac{1}{12}$$
$$\mathbb{P} = (2Y + 1)\mathcal{K}_\rho - 4\mathcal{K}^2$$
$$-\mathcal{K}\left[4Y_\rho + 4(Y - 1)Y - 1\right]$$

$$w_r = \frac{\mathcal{K}^2\left(-4\mathcal{K} - 4Y_{,\rho} + 4Y + 1\right) + (4Y+1)\mathcal{K}\mathcal{K}_{,\rho}}{4\mathcal{K}^3 - \mathcal{K}^2 + \mathcal{K}\left(\mathcal{K}_{,\rho} - 2\mathcal{K}_{,\rho\rho}\right) + 4\mathcal{K}_{,\rho}^2}$$

$$w_\perp = -\frac{1 + 16\mathcal{K}Y^2\left(\mathcal{K}_{,\rho} + \mathcal{K}\right) + \mathcal{K}\mathcal{K}_{,\rho}\left(1 - 12Y_{,\rho}\right)}{\left[4\mathcal{K}^3 - \mathcal{K}^2 + \mathcal{K}\left(\mathcal{K}_{,\rho} - 2\mathcal{K}_{,\rho\rho}\right) + 4\mathcal{K}_{,\rho}^2\right]}$$
$$+\frac{2\mathcal{K}^2\left(-8Y_{,\rho} + 8Y_{,\rho\rho} + 1\right) - 8\mathcal{K}^3}{\left[4\mathcal{K}^3 - \mathcal{K}^2 + \mathcal{K}\left(\mathcal{K}_{,\rho} - 2\mathcal{K}_{,\rho\rho}\right) + 4\mathcal{K}_{,\rho}^2\right]}$$
$$+\frac{Y\left(4\mathcal{K}\left(\mathcal{K}_{,\rho} + \mathcal{K}\left(-4\mathcal{K} + 8Y_{,\rho} + 1\right)\right) - 2\right)}{\left[4\mathcal{K}^3 - \mathcal{K}^2 + \mathcal{K}\left(\mathcal{K}_{,\rho} - 2\mathcal{K}_{,\rho\rho}\right) + 4\mathcal{K}_{,\rho}^2\right]}$$

$$A = (a + br^{2\rho})^\delta$$

$$B = \frac{3\mu_0}{3\mu_0 - r^2(\mu_0 - \mu_1 r^{2\alpha})^{\beta+1}\mathcal{F}}$$

$$\mathcal{F} = {}_2F_1\left(1, \beta + \frac{3}{2\alpha} + 1; 1 + \frac{3}{2\alpha}; \frac{r^{2\alpha\rho}\mu_1}{\mu_0}\right)$$

$$\mu = (\mu_0 - \mu_1 r^{2\alpha})^\beta$$

$$p_r = \frac{2b\delta\mu_0}{\mu_0(a + br^2)}$$
$$-\frac{[a + b(2\delta+1)r^2](\mu_0 - \mu_1 r^{2\alpha})^{\beta+1}}{3\mu_0(a + br^2)}\mathcal{F}$$

$$p_\perp = \frac{2[a^2 + ab(2-3\delta)r^2 + b^2(-2\delta^2 + \delta + 1)r^4]\mathcal{F}}{\mu_0 12(a + br^2)^2(\mu_0 - \mu_1 r^{2\alpha})^{-(\beta+1)}}$$
$$+\frac{12b\delta(2a + b\delta r^2) - 6(a + br^2)(a + b(\delta+1)r^2)}{12(a + br^2)^2(\mu_0 - \mu_1 r^{2\alpha})^{-\beta}}$$

$$w_r = \frac{(\mu_0 - \mu_1 r^{2\alpha})}{3\alpha\beta\mu_0\mu_1 r^{2\alpha}(a + br^2)^2} \times$$
$$\left\{\frac{3}{2}\mu_0(a + br^2)[a + b(2\delta+1)r^2]\right.$$
$$\frac{\mathcal{F}}{2}(\mu_0 - \mu_1 r^{2\alpha})[2b\delta r^2(a + 3br^2) + 3(a + br^2)^2]$$
$$\left.+6b^2\delta\mu_0 r^2(\mu_0 - \mu_1 r^{2\alpha})^{-\beta}\right\}$$

$$w_\perp = \frac{r^{-2\alpha}}{12\alpha\beta\mu_1(a + br^2)^3} \times$$
$$\left\{\frac{\mathcal{F}}{\mu_0}(\mu_0 - \mu_1 r^{2\alpha})^2[3a^3 - 3a^2b(\delta-3)r^2\right.$$
$$+ab^2(2(\delta-8)\delta + 9)r^4 + 3b^3(-2\delta^2 + \delta + 1)r^6]$$
$$+12b^2\delta r^2[b\delta r^2 - a(\delta-4)](\mu_0 - \mu_1 r^{2\alpha})^{1-\beta}$$
$$-3\mu_0(a + br^2)[a^2 + ab(2-5\delta)r^2$$
$$+b^2(-2\delta^2 + \delta + 1)r^4]$$
$$-3\mu_1 r^{2\alpha}(a + br^2)[ab\delta r^2(2\alpha\beta + 5) + 2b^2\delta^2 r^4$$
$$\left.+b^2\delta r^4(2\alpha\beta - 1) + (2\alpha\beta - 1)(a + br^2)^2]\right\}$$

$$A= (a + (c - br^2)^\gamma)^\delta$$

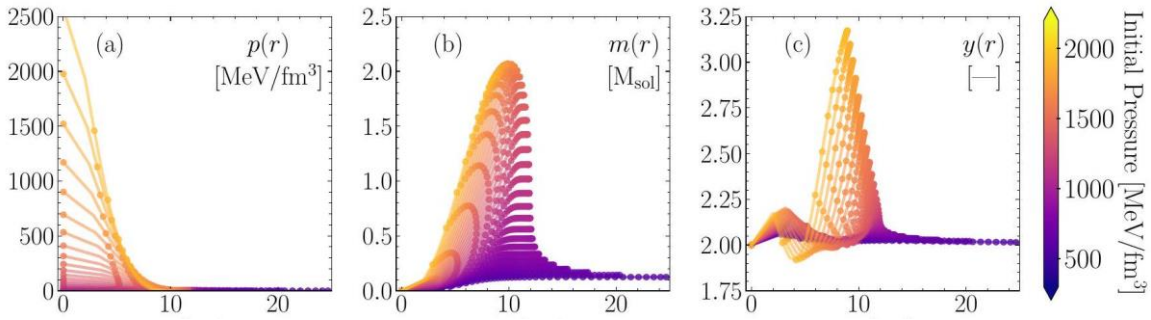
$$B = \frac{3\mu_0}{3\mu_0 - r^2(\mu_0 - \mu_1 r^{2\alpha})^{\beta+1}\mathcal{F}}$$

$$\mathcal{F} = {}_2F_1\left(1, \beta + \frac{3}{2\alpha} + 1; 1 + \frac{3}{2\alpha}; \frac{r^{2\alpha\rho}\mu_1}{\mu_0}\right)$$

$$\mu = (\mu_0 - \mu_1 r^{2\alpha})^\beta$$

$$p_r = \frac{1}{6(br^2 - c)(a + (c - br^2)^\gamma)} \times$$
$$\left\{12b\gamma\delta(c - br^2)^\gamma - 2\frac{\mathcal{F}}{\mu_0}[a(br^2 - c)\right.$$
$$+(c - br^2)^\gamma(br^2(2\gamma\delta + 1) - c)]\}$$

$$p_\perp = \frac{\mu_0^\beta(\mu_0 - \mu_1 r^{2\alpha})^{-\beta}}{12(br^2 - c)^2(a + (c - br^2)^\gamma)^2}$$
$$\{2\mathcal{F}(\mu_0 - \mu_1 r^{2\alpha})^\beta[a^2(br^2 - c)^2$$
$$-a(c - br^2)^\gamma[b^2r^4(4\gamma^2\delta - \gamma\delta - 2)$$
$$+bcr^2(4 - 3\gamma\delta) - 2c^2] + (c - br^2)^{2\gamma} \times$$
$$(b^2r^4(-2\gamma^2\delta^2 + \gamma\delta + 1) + bcr^2(3\gamma\delta - 2) + c^2)]$$
$$-3\mu_0^{-\beta}(\mu_0 - \mu_1 r^{2\alpha})^\beta\big[2a^2(br^2 - c)^2(\mu_0 - \mu_1 r^{2\alpha})^\beta$$
$$+2a(c - br^2)^\gamma\big(b^2r^4(\gamma\delta + 2)(\mu_0 - \mu_1 r^{2\alpha})^\beta$$
$$-4b^2\gamma^2\delta r^2 + bc\big(4\gamma\delta - r^2(\gamma\delta + 4)(\mu_0 - \mu_1 r^{2\alpha})^\beta\big)$$
$$+2c^2(\mu_0 - \mu_1 r^{2\alpha})^\beta\big) + (c - br^2)^{2\gamma} \times$$
$$\big(b^2r^2\big(2r^2(\gamma\delta + 1)(\mu_0 - \mu_1 r^{2\alpha})^\beta - 4\gamma^2\delta^2\big)$$
$$+2bc\big(4\gamma\delta - r^2(\gamma\delta + 2)(\mu_0 - \mu_1 r^{2\alpha})^\beta\big)$$
$$+2c^2(\mu_0 - \mu_1 r^{2\alpha})^\beta\big)]\}$$



$$\frac{dp}{dx} = -\frac{1}{2}\frac{(\varepsilon(x) + p(x))(m(x) + 3x^3p(x))}{x^2(1 - m(x)/x)}$$

$$\frac{dm}{dx} = 3x^2\varepsilon(x)$$

$$\frac{dy}{dx} = -\frac{y(x)^2}{x} - \frac{F(x)y(x)}{x} - \frac{Q(x)}{x}$$

$$F(x) = \left[1 - \frac{3}{2}(\varepsilon(x) - p(x))\right][1 - m(x)/x]^{-1},$$

$$Q(x) = \frac{\frac{3}{2}x^2}{\left(1 - \frac{m(x)}{x}\right)}\left[5\varepsilon(x) + 9p(x) + \frac{\varepsilon(x) + p(x)}{c_s^2(x)} - \frac{4}{x^2}\right] - \left(\frac{m(x)/x + 3x^2 p(x)}{\left(1 - \frac{m(x)}{x}\right)}\right)^2,$$

$$\frac{\partial y}{\partial r} = \mathcal{F}(y(\boldsymbol{x}, r)),$$

$$\frac{d\boldsymbol{y}}{dr} = \hat{\boldsymbol{A}}\boldsymbol{y},$$

	HF		SLM		Error (%)		Time (s)	
EOS	Max. Mass $[M_\odot]$	Radius [nm]	Max. Mass [$M_\odot$]	Radius [nm]	Max. Mass	Radius	HF	SLM
Tabular EOS								
SLy4	2.067	10.033	2.068	10.033	0.02	0.00	9.218	2.135×10^{-4}
APR	2.193	9.979	2.194	9.982	0.05	0.03	9.237	2.149×10^{-4}
FSU Garnet	2.054	10.998	2.068	11.000	0.66	0.02	9.226	2.319×10^{-4}
BL	2.083	10.194	2.085	10.204	0.10	0.11	9.246	2.099×10^{-4}

DS-CMF-5	2.023	11.896	2.026	11.647	0.17	2.09	9.228	2.215×10^{-4}
Quarkyonic EOS								
$\Lambda = 300.00, \kappa = 0.19$	2.947	14.763	2.917	14.666	1.00	0.66	9.286	3.822×10^{-4}
$\Lambda = 352.63, \kappa = 0.14$	2.653	12.512	2.675	12.561	0.83	0.39	9.299	4.051×10^{-4}
$\Lambda = 405.26, \kappa = 0.10$	2.441	11.199	2.441	11.063	0.40	1.21	9.234	4.361×10^{-4}
$\Lambda = 447.37, \kappa = 0.28$	2.294	10.462	2.294	10.463	0.01	0.01	9.292	4.261×10^{-4}
$\Lambda = 500.00, \kappa = 0.23$	2.226	10.180	2.221	10.177	0.22	0.03	9.156	3.898×10^{-4}

$$\frac{d\boldsymbol{y}^{\boldsymbol{\theta}}}{dr} = \hat{\boldsymbol{A}}(\boldsymbol{\theta})\boldsymbol{y}^{\boldsymbol{\theta}},$$

$$\frac{dp}{dr} = -\frac{G}{c^2}(\epsilon(r) + P(r))\frac{M(r) + 4\pi r^3 P(r)/c^2}{r(r - 2GM(r)/c^2)}$$

$$\frac{dm}{dr} = 4\pi r^2 \frac{\epsilon(r)}{c^2}$$

$$\frac{dy}{dr} = -\frac{y(r)^2}{r} - \frac{F(r)y(r)}{r} - \frac{Q(r)}{r}$$

$$F(r) = \frac{1 - 4\pi G r^2(\epsilon(r) - P(r))/c^4}{1 - 2\frac{GM}{rc^2}}$$

$$Q(r) = \frac{4\pi G r^2/c^4}{1 - \frac{2GM}{rc^2}}\left(5\epsilon(r) + 9p(r) + \frac{\epsilon(r) + p(r)}{c_s(r)^2}c^2 - \frac{6c^4}{4\pi r^2 G}\right) - 4\left(\frac{G(M(r)/(rc^2) + 4\pi r^2 p(r)/c^4)}{1 - 2GM/(rc^2)}\right)^2$$

$$y(r) = \frac{R\beta(R)}{H(R)} \text{ and } \beta = dH/dr^1$$

$$r = R_0 x, M = M_0 m(x), P = P_0 p(x), \epsilon = \varepsilon_0 \varepsilon(x)$$

$$\varepsilon_0 = P_0 \equiv \frac{1}{8\pi^2}\frac{(m_n c^2)^4}{(\hbar c)^3} \approx 1.285\text{GeV/fm}^3$$

$$\left[\frac{2GM_0}{c^2 R_0}\right] = \left[\frac{4\pi R_0^3 \varepsilon_0}{3M_0 c^2}\right] = 1.$$

$$R_0 = \sqrt{\frac{3\pi}{\alpha_G}}\lambda_n \approx 8.378 \text{ nm},$$

$$M_0 = \left(\frac{R_0}{R_s^{\odot}}\right)M_{\odot} \approx 2.837 M_{\odot},$$

$$\alpha_G = \frac{G m_n^2}{\hbar c} \approx 5.922 \times 10^{-39}$$

$$\lambda_n = \frac{\hbar c}{m_n c^2} \approx 0.210 \times 10^{-18} \text{ nm}$$

$$R_s^{\odot} = \frac{2GM_{\odot}}{c^2} \approx 2.953 \text{ nm}$$

$$\frac{dp}{dx} = -\frac{1}{2}\frac{(\varepsilon(x) + p(x))(m(x) + 3x^3 p(x))}{x^2(1 - m(x)/x)}$$

$$\frac{dm}{dx} = 3x^2\varepsilon(x)$$

$$\frac{dy}{dx} = -\frac{y(x)^2}{x} - \frac{F(x)y(x)}{x} - \frac{Q(x)}{x}$$

$$F(x) = \frac{1 - \frac{3}{2}(\varepsilon(x) - p(x))}{1 - \frac{m(x)}{x}},$$

$$Q(x)=\frac{\frac{3}{2}x^2}{\left(1-\frac{m(x)}{x}\right)}\left[5\varepsilon(x)+9p(x)+\frac{\varepsilon(x)+p(x)}{c_s^2(x)}-\frac{4}{x^2}\right]-\left(\frac{m(x)/x+3x^2p(x)}{\left(1-\frac{m(x)}{x}\right)}\right)^2$$

$$\begin{aligned}k_2=&\frac{8\beta^5}{5}(1-2\beta)^2[2+2\beta(y_R-1)-y_R]\\&\times\{2\beta\big(6-3y_R+3\beta(5y_R-8)\big)\\&+4\beta^3[13-11y_R+\beta(3y_R-2)+2\beta^2(1+y_R)]\\&+3(1-2\beta)^2[2-y_R+2\beta(y_R-1)]\log(1-2\beta)\}^{-1}.\end{aligned}$$

$$\frac{\partial y}{\partial r}=\mathcal{F}(y(\boldsymbol{x},r))$$

$$\frac{d\boldsymbol{y}}{dr}=\boldsymbol{F}(\boldsymbol{y}(r),r)\Big),$$

$$\frac{d\boldsymbol{y}}{dr}=\hat{\boldsymbol{A}}\boldsymbol{y}$$

$$\log\,\boldsymbol{y}_{r+\delta r}=\hat{\boldsymbol{A}}\log\,\boldsymbol{y}_r$$

$$X_j(y)=\{y,y^2,y^3,\dots,y^n\}$$

$$X_j(\boldsymbol{z})=\{\boldsymbol{z},\boldsymbol{z}^2\}$$

$$\begin{aligned}\boldsymbol{X}^-&=\begin{bmatrix}\boldsymbol{z}_0 & \boldsymbol{z}_1 & \dots & \boldsymbol{z}_{m-1} & \boldsymbol{z}_0^2 & \dots & \boldsymbol{z}_i\boldsymbol{z}_j & \dots & \boldsymbol{z}_{m-1}^2\end{bmatrix}\\\boldsymbol{X}^+&=\begin{bmatrix}\boldsymbol{z}_1 & \boldsymbol{z}_2 & \dots & \boldsymbol{z}_m & \boldsymbol{z}_1^2 & \dots & \boldsymbol{z}_i\boldsymbol{z}_j & \dots\boldsymbol{z}_m^2\end{bmatrix}\end{aligned}$$

$$s=\text{argmin}_j\frac{\sum_{i=1}^{i=j}\sigma_i}{\sum_{i=1}^{i=n}\sigma_i},\kappa$$

$$\boldsymbol{X}^-\approx\boldsymbol{U}_s\boldsymbol{\Sigma}_s\boldsymbol{V}_s^T$$

$$\boldsymbol{A}_s\equiv\boldsymbol{U}_s^T\boldsymbol{A}\boldsymbol{U}_s=\boldsymbol{U}_s^T\boldsymbol{X}^+\boldsymbol{V}_s\boldsymbol{\Sigma}_s^{-1}.$$

$$A_sW=\Lambda W.$$

$$\boldsymbol{y}(l)=\boldsymbol{\Phi}\boldsymbol{\Lambda}^{l/\Delta l}\boldsymbol{b}_0=\Sigma_i b_{0i}\phi_i(\lambda_i)^{l/\Delta l},$$

$$\frac{\partial y^\mu}{\partial r}=\mathcal{F}(y^\mu(\boldsymbol{x},r;\mu),r;\mu).$$

$$\frac{d\boldsymbol{y}^\mu}{dr}=\boldsymbol{F}(\boldsymbol{y}^\mu(r;\mu),r;\mu).$$

$$\frac{d\boldsymbol{y}^\theta}{dt}=\hat{\boldsymbol{A}}(\boldsymbol{\theta})\boldsymbol{y}^\theta,$$

$$\boldsymbol{A}_{js} = \boldsymbol{U}_{js}^{T}\boldsymbol{X}_{j}^{+}\boldsymbol{V}_{js}\boldsymbol{\Sigma}_{js}^{-1}$$

$$\boldsymbol{A}_{js}\boldsymbol{W}_{js} = \boldsymbol{\Lambda}_{js}\boldsymbol{W}_{js}.$$

$$u := \operatorname{argmin} E[w] : w \in \{u_1, \ldots, u_s\}$$

$$ds^2 = r^2(d\theta^2 + \sin^2\,\theta d\phi^2) + ..$$

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r) = 12\pi r^2 p(r)$$
$$\frac{dp(r)}{dr} = -\frac{\rho(r)+p(r)}{2}\frac{dv(r)}{dr} = -2p(r)\frac{dv(r)}{dr}$$
$$\frac{dv(r)}{dr} = \frac{N(r)}{1-2m(r)/r}$$
$$N(r) = (2/r^2)(m(r) + 4\pi r^3 p(r))$$

$$r \to r/L$$
$$m(r) \to m(r/L)/L$$
$$v(r) \to v(r/L)$$
$$p(r) \to p(r/L)L^2$$
$$\rho(r) \to \rho(r/L)L^2$$

$$t = \log r$$
$$dt = dr/r$$
$$\alpha(t) = m(r)/r$$
$$\delta(t) = 4\pi r^2 p(r)$$

$$\frac{d\alpha(t)}{dt} = 3\delta(t) - \alpha(t)$$
$$\frac{d\delta(t)}{dt} = -4\delta(t)\frac{\delta(t)+2\alpha(t)-1/2}{1-2\alpha(t)}$$

$$D(r) \equiv 1 - 2m(r)/r$$

$$r = f(r')$$
$$t = t'$$

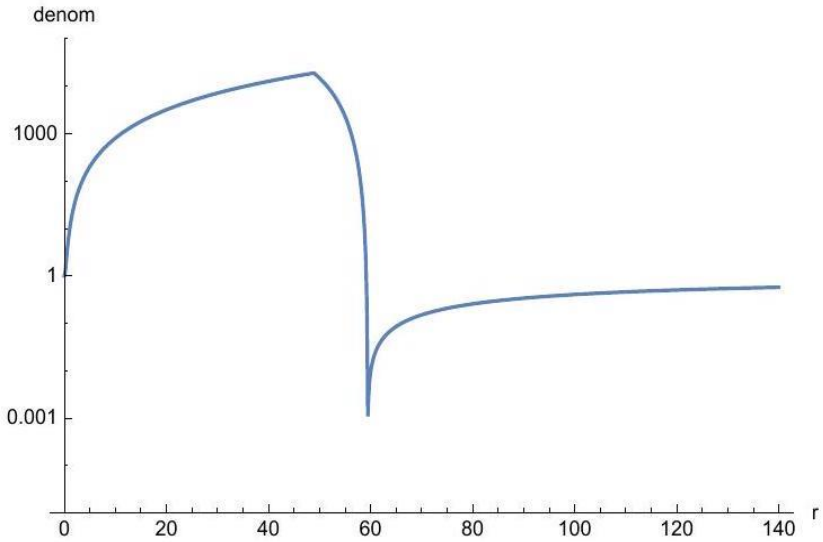
$$f' \equiv dr/dr' = df(r')/dr'$$

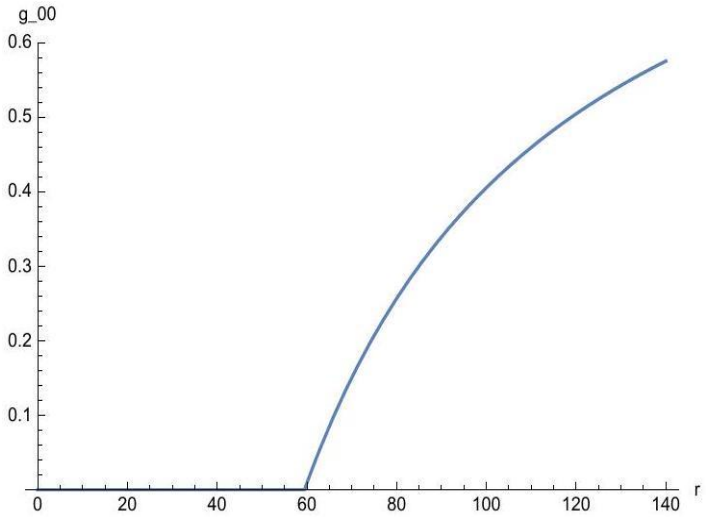
$$\frac{dm(r)}{dr'} = 12\pi r^2 f' p(r)$$
$$\frac{dp(r)}{dr'} = -2p(r)f'\frac{N(r)}{D(r)}$$
$$N(r) = (2/r^2)(m(r) + 4\pi r^3 p(r))$$
$$D(r) = D\big(f(r')\big) = 1 - 2m(r)/r$$

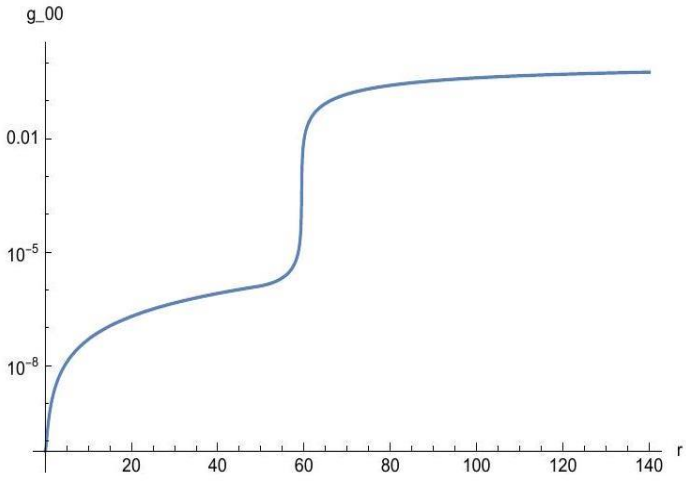
$$d\hat{D}(r')/dr'|_{r'=r'_{\min}} = f' dD/dr|_{r=r_{\min}} = 0$$

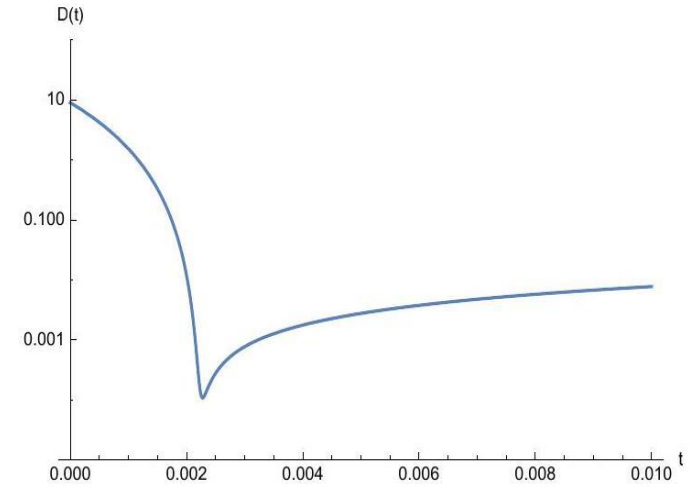
$$d^2\hat{D}(r')/(dr')^2|_{r'=r'_{\min}} = d^2D(r)/(dr)^2|_{r=r_{\min}}(f')^2\big|_{r'=r'_{\min}} + dD(r)/dr|_{r=r_{\min}} df'/dr'|_{r'=r'_{\min}}$$
$$= d^2D(r)/(dr)^2|_{r=r_{\min}}(f')^2\big|_{r'=r'_{\min}} > 0$$

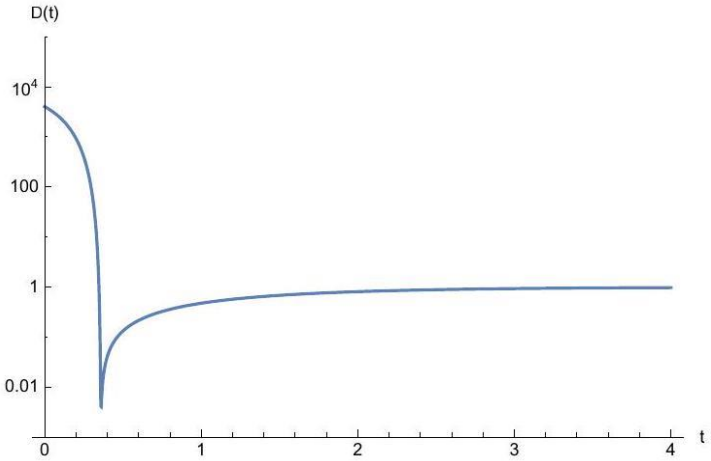
Figuras. Densidad de energía y masa de una partícula oscura y blanca para formar un agujero negro cuántico.

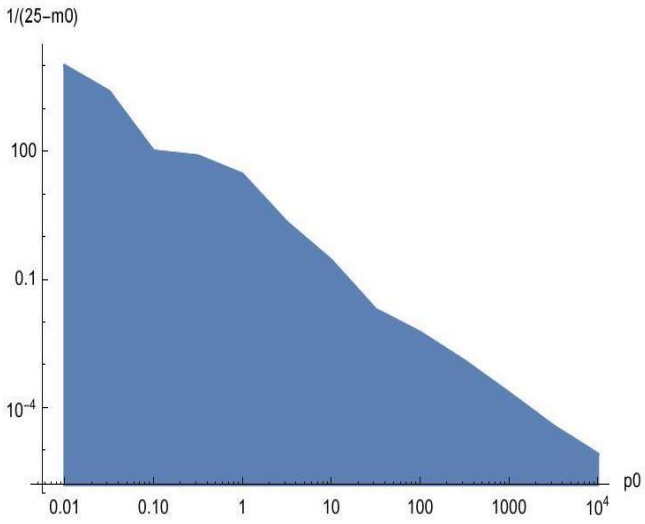


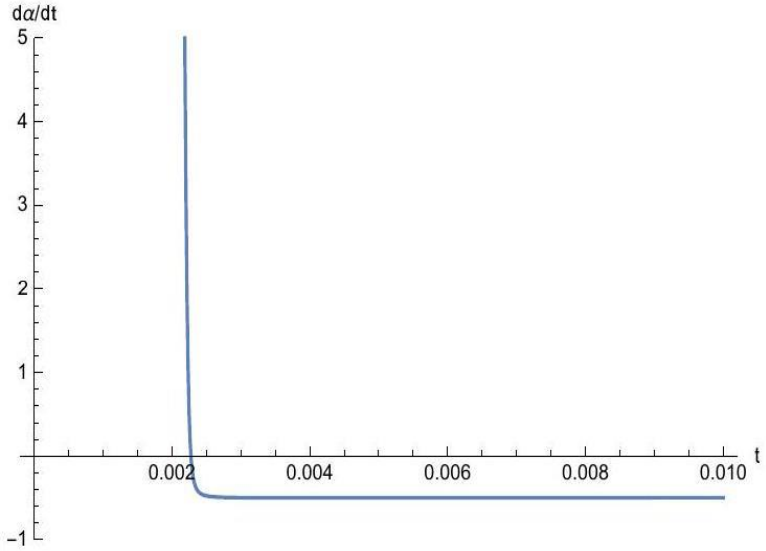





D(t)
10
0.100
0.001
0.000
0.002
0.004
0.006
0.008
0.010
t


D(t)
10^4
100
1
0.01
0
1
2
3
4
t


1/(25−m0)
100
0.1
10^−4
0.01
0.10
1
10
100
1000
10^4
p0



$\alpha(t) \simeq 1/2 - \theta_0 - \theta_2 t^2 + O(t^3)$

$\delta(t) \simeq 1/6 - (2/9)\text{ArcTan}\left((\theta_2/\theta_1)^{1/2}\text{t}\right)/(\theta_2\theta_1)^{1/2} + \text{O}(\theta_1, \theta_2, \text{t}^2)$

p_0	m_0	α_0	δ_0
0.01	24.9998999	. 499998	314.159
. 03162	24.9996004	. 499992	993.37
0.1	24.9899999	. 499800	3141.59
. 3162	24.9869469	. 499739	9933.71
1	24.9651	. 499302	31415.9
3.162	24.5310	. 49012	99337.1
10	21.366	. 42732	314159
31.62	-26.75	-. 535	993371
100	-153	-3.06	3141590
316.2	-806	-16.12	9933708
1000	-4,640	-92.8	31415900

3162	-27,600	-63.24	99337080
10000	-126,000	-2520	314159000

$$ds^2 = -L^2(x)dt^2 + h_{ij}(x)dx^i dx^j$$

$$T_{\mu\nu} = \rho n_\mu n_\nu + P\left(g_{\mu\nu} + n_\mu n_\nu\right)$$

$$\frac{\nabla_i P}{\rho + P} = -\frac{\nabla_i L}{L}$$

$$Tds = d\rho - \sum_a \mu_a dn_a$$

$$\mu_a = -T\frac{\partial s}{\partial n_a}$$

$$T = \left(\frac{\partial s}{\partial \rho}\right)^{-1}_{n_a}$$

$$\rho + P - Ts - \sum_a \mu_a n_a = 0$$

$$N_a = \int_C d^3x\sqrt{h}n_a$$

$$\Omega = S + \sum_a b_a N_a$$

$$\delta\Omega = \sum_a \int_C d^3x\sqrt{h}\left(-\frac{\mu_a}{T} + b_a\right)\delta n_a = 0$$

$$\omega(\rho, b_a) := s - \sum_a \frac{\partial s}{\partial n_a}n_a = s + \sum_a b_a n_a = \frac{\rho + P}{T}.$$

$$\Omega = \int_C d^3x\sqrt{h}\omega(\rho, b_a)$$

$$d\omega = \frac{d\rho}{T} + \sum_a n_a db_a$$

$$T^{-1} = (\partial\omega/\partial\rho)_{b_a}, n_a = (\partial\omega/\partial b_a)_\rho$$

$$dP = \omega dT + T\sum_a n_a db_a$$

$$\frac{dP}{dT} = \omega = \frac{P+\rho}{T}$$

$$\frac{\nabla_i T}{T} = -\frac{\nabla_i L}{L}$$

$$LT = T_\infty$$

$$n(T,b) = \frac{g}{\pi^2}\int_0^\infty \frac{dpp^2}{e^{-b+\epsilon_p/T} \pm 1}$$

$$P(T,b) = \frac{gT}{\pi^2}\int_0^\infty dpp^2 \log\left[1 \pm e^{b-\epsilon_p/T}\right]$$

$$\rho = T(\partial P/\partial T)_b - P$$

$$n(T,b) = -\frac{2}{\pi^2} f_1^{\pm}(b) T^3$$

$$P(T,b) = -\frac{2}{\pi^2} f_2^{\pm}(b) T^4$$

$$f_1^+ = -\mathrm{Li}_3\left(-e^b\right), f_1^- = \mathrm{Li}_3\left(e^b\right), f_2^+ = -\mathrm{Li}_4\left(-e^b\right), f_2^- = \mathrm{Li}_4\left(e^b\right); \mathrm{Li}_k(x) := \sum_{n=1} \frac{x^n}{n^k}$$

$$\omega(\rho,b) = \frac{8}{6^{3/4}\sqrt{\pi}} \rho^{3/4}\left[f_2^{\pm}(b)\right]^{1/4}$$

$$\epsilon_p \simeq m + \frac{p^2}{2m}$$

$$n(T,b) = \frac{1}{2}\left(\frac{2m}{\pi}\right)^{3/2} e^{b-\frac{m}{T}} T^{3/2}$$

$$P(T,b) = \frac{1}{2}\left(\frac{2m}{\pi}\right)^{3/2} e^{b-\frac{m}{T}} T^{5/2}$$

$$\rho = \left(m + \frac{3}{2}T\right) n$$

$$(\partial P/\partial T)_{b_a} = \omega \geq 0$$

$$\left(\frac{\partial^2 \rho}{\partial \omega^2}\right)_{b_a} \geq 0$$

$$\left(\frac{\partial^2 \rho}{\partial \omega^2}\right)_{b_a} = \left(\frac{\partial T}{\partial \rho}\right)_{b_a}$$

$$(\partial \rho/\partial T)_{b_a} \geq 0$$

$$2P \leq \omega T = T\left(\frac{\partial P}{\partial T}\right)_{b_a}$$

$$\rho = f(b_a)T, \omega = f(b_a)$$

$$\kappa := \left(\frac{\partial \log\, P}{\partial \log\, T}\right)_{b_a} \geq 2$$

$$\lim_{T\to\infty} P(T, b_a) = \lim_{T\to\infty} \rho(T, b_a) = \infty$$

$$\lim_{T\to 0} P(T, b_a) = \lim_{T\to 0} \rho(T, b_a) = 0.$$

$$\frac{d\rho}{dt} = \frac{\rho + f(\rho)}{f'(\rho)}$$

$$X_f := \frac{x+f(x)}{f'(x)} \frac{\partial}{\partial x}\, \boldsymbol{R}_*^+$$

$$f(x) = kx^a \text{ as } x \to \infty, k, a > 0$$

$$\frac{d\rho}{dt} = \frac{\rho^{2-a}}{ka}$$

$$\frac{1}{\rho^{1-a}} = \frac{1}{\rho_0^{1-a}} - \frac{1-a}{ak} \log\,(T/T_0)$$

$$T = T_0 \exp\left(\frac{ak}{(1-a)\rho_0^{1-a}}\right)$$

$$\frac{d\rho}{dt} = \left(k' a' \rho^{a'-2}\right)^{-1}$$

$$\rho^{a'-1} = \rho_0^{a'-1} + \frac{a'-1}{a'k'} \log\,(T/T_0).$$

$$T = T_0 \exp\left(-\frac{a'k'\rho_0' - 1}{a'-1}\right)$$

$$ds^2 = -L^2(r)dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2\,\theta d\phi^2)$$

$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r^2\left(1 - \frac{2m}{r}\right)}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{1}{L}\frac{dL}{dr} = -\frac{1}{\rho + P}\frac{dP}{dr}$$

$$\frac{d\log\, T}{dr} = -\frac{m + 4\pi r^3 P(T)}{r^2\left(1 - \frac{2m}{r}\right)}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho(T)$$

$$u: = \frac{2m}{r}$$
$$v: = 4\pi r^2 \rho$$
$$w: = 4\pi r^2 P \leq v$$
$$\xi: = -\log (r/r_B)$$
$$t: = \log (T/T_B).$$

$$\frac{dt}{d\xi} = \frac{\frac{1}{2}u + w}{1-u}$$
$$\frac{du}{d\xi} = u - 2v$$

$$u(0) = u_B = \tfrac{2M}{r_B},\ t(0) = 0$$

$$\lim_{T\to\infty} P/\rho = \lambda \leq 1$$

$$\frac{dt}{dx} = -\frac{\frac{1}{2} + w}{\epsilon}$$
$$\frac{d\epsilon}{dx} = 1 - 2v.$$

$$-\frac{dt}{dx} < \frac{1}{\epsilon} < \frac{1}{Cx}$$

$$-\log \frac{T(x)}{T(x_r)} < C^{-1} \log (x/x_r)$$

$$\frac{d\epsilon}{dx} = f(x)$$

$$\epsilon = F(x) = \int_0^x dx' f(x')$$

$$-\log \frac{T(x)}{T(x_r)} < \int_{x_r}^x \frac{dx'}{F(x')}$$

$$(\ P = \tfrac{1}{3}\rho \sim T^4\ )\ u_{\max} = 1 - \epsilon,\ u_B\sqrt{v(0)}$$

$$u_m < 1$$

$$m(r) < \tfrac{1}{2}r \text{ for all } r < r_B$$

$$\lim_{r\to 0} m(r) = 0 \text{ or } \lim_{r\to 0} m(r) < 0$$

$$\frac{dt}{dr} = f_1(r, m, t)\ \frac{dm}{dr} = f_2(r, m, t)$$

$$\xi = -\log (r_1/r_B)$$

$$du/d\xi < 0 \text{ for } u < 0, u(\xi) < 0 \text{ for all } \xi > \xi_1$$

$$\frac{dt}{d\xi}(\xi_1) = w(\xi_1) > 0, \xi > \xi_1$$

$$\frac{dt}{d\xi}(\xi_2) = 0$$

$$\frac{dt}{d\xi} > 0 \text{ for all } \xi > \xi_1, w > -\frac{1}{2}u \text{ for all } \xi > \xi_1.$$

$$v > -\frac{1}{2}u, u - 2v < 2u.$$

$$\frac{du}{d\xi} < 2u,$$

$$u < -ke^{2\xi}, k > 0.$$

$$r^{-1} \text{ as } r \to 0$$

$$\lim_{T\to\infty} \frac{P}{\rho} = \lambda \leq 1.$$

$$P = \lambda\rho \sim T^{1+\frac{1}{\lambda}}$$

$$\frac{d\log\, T}{dr} = \frac{m + 4\pi r^3 P}{2mr} = \frac{1}{2r} + \frac{\lambda}{2mr}\frac{dm}{dr} = \frac{1}{2r} + \frac{\lambda}{2r}\frac{d\log\, |m|}{dr}.$$

$$T|m|^{-\lambda/2} \sim \sqrt{r}$$

$$r^{-2}dm/dr$$

$$d|m|/dr = -Kr^2\left(r^{1/2}|m|^{\lambda/2}\right)^{1+\frac{1}{\lambda}} = -Kr^{\frac{5}{2}+\frac{1}{2\lambda}}|m|^{\frac{1}{2}+\frac{\lambda}{2}},$$

$$|m|^{(1-\lambda)/2} = c_1 - c_2 r^{\frac{7}{2}+\frac{1}{2\lambda}}, c_1, c_2 \geq 0.$$

$$|m| = Ce^{-\frac{Kr^5}{5}}$$

$$\frac{d^2t}{d\xi^2} = \frac{\left(\frac{1}{2} + w\right)}{(1-u)^2}\frac{du}{d\xi} - \frac{2w}{1-u} + \kappa\frac{w}{1-u}\frac{dt}{d\xi}$$

$$\frac{d^2t}{d\xi^2}(\xi_2) < 0$$

$$\bar{\xi}_2 > \xi_2, \frac{dt}{d\xi}(\bar{\xi}_2) < 0 \text{ for all } \xi \in (\xi_2, \bar{\xi}_2].$$

$$\frac{dy}{d\xi} > g(\xi)y,$$

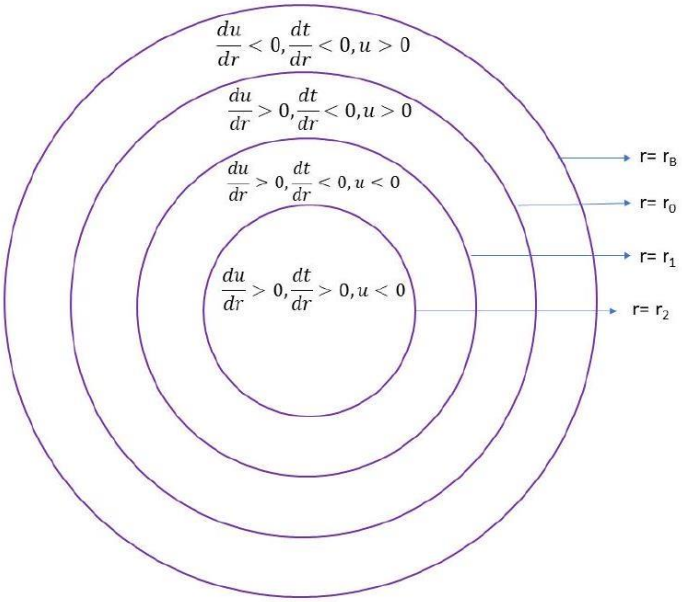
$$y = -\frac{dt}{d\xi}$$

$$y(\xi)/y(\bar{\xi}_2) = \exp\left[\int_{\bar{\xi}_2}^{\xi} d\xi' g(\xi')\right] > 0$$

$$\frac{dt}{d\xi}(\xi) < 0$$

$$\xi \to \infty, t(\xi) \sim -\frac{1}{2}\xi \text{ and } u(\xi) \sim e^{\xi}$$

$$0 < \frac{dt}{d\xi} + \frac{1}{2} = \frac{\frac{1}{2} + w}{1-u} < \frac{\frac{1}{2} + w(\xi_2)}{1-u} < \frac{\frac{1}{2} + w(\xi_2)}{-u} = \frac{1}{2}(1 + |u_2|^{-1})e^{-\xi}.$$

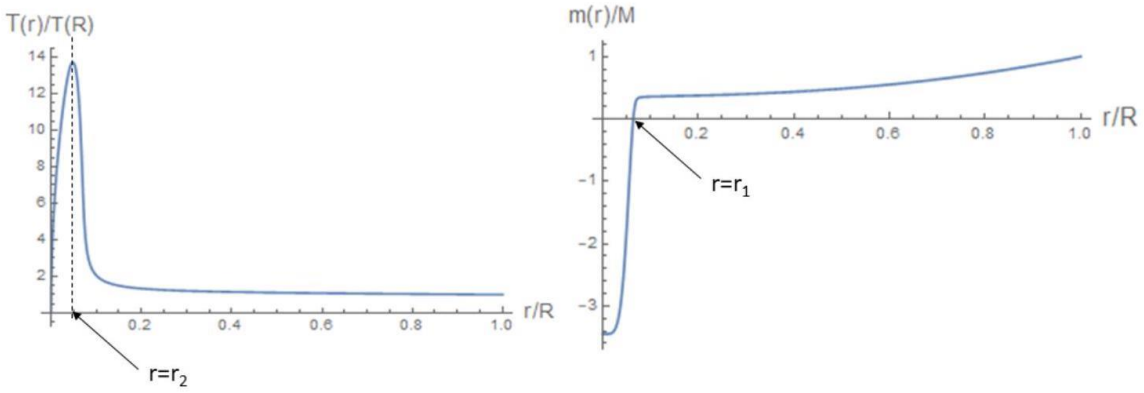


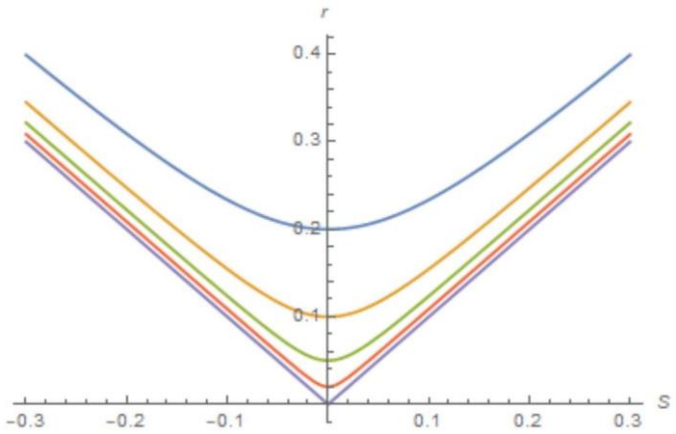
$$\frac{du}{dr}(r_B) > 0, r_0$$

$$u = 2m/r$$

$$r = r_2$$

Figuras. Termodinámica de una partícula oscura y blanca.





$$L(r) = \frac{\eta}{\sqrt{r}},$$

$$ds^2 = -\frac{\eta^2}{r}dt^2 + \frac{rdr^2}{2M_0} + r^2(d\theta^2 + \sin^2\ \theta d\phi^2)$$

$$ds^2 = -\frac{2M_0}{r}d\tilde{t}^2 + \frac{rdr^2}{2M_0} + r^2(d\theta^2 + \sin^2\ \theta d\phi^2)$$

$$\tilde{t} = \frac{\eta}{\sqrt{2M_0}}t$$

$$g_{rr} = \frac{r}{2M_0}\left[1 - \frac{r}{2M_0} + \frac{r^2}{4M_0^2} - \frac{r^3}{M_0^3} + O[(r/M_0)^4]\right]$$

$$r\frac{d\log\ T}{dr} = -\frac{-M_0 + kr^\delta + 4\pi r^3 P}{2M_0 + r - kr^\delta}.$$

$$r\frac{d\log\ T}{dr} = \frac{1}{2}\left(1 + \frac{r}{2M_0}\right)^{-1}$$

$$g_{tt} = \frac{\eta^2}{2M_0}\left(1 + \frac{2M_0}{r}\right) + O[(r/M_0)^4]$$

$$K := \frac{48M_0^2}{r^6}$$

$$\dot{r}^2 = \frac{2M_0\epsilon^2}{\eta^2} - \frac{2M_0\sigma}{r} - \frac{2M_0\ell^2}{r^3}$$

$$\dot{t}\eta^2 = \epsilon r$$

$$r_{\min} = (\eta\ell/\epsilon)^{2/3}$$

$$\dot{r} = \pm\sqrt{2M_0}\epsilon/\eta$$

$$r = \pm\left(\sqrt{2M_0}\epsilon/\eta\right)\lambda$$

$$t = t_0 + \frac{1}{2}\left(\sqrt{2M_0}\epsilon^2\eta^3\right)\lambda^2$$

$$S = \sqrt{2M_0}\epsilon\lambda/\eta$$

$$\delta\tau = \frac{2\eta}{\sqrt{2M_0}}r_0$$

$$a = \frac{d\log\ L}{dr}dr = -\frac{1}{2r}dr$$

$$\dot{t} = \frac{\sqrt{r}}{\eta}\sqrt{1 + \frac{r\dot{r}^2}{2M_0} + r^2\dot{\theta}^2 + r^2\sin^2\ \theta\dot{\phi}^2}$$

$$\dot{t} \simeq \frac{\sqrt{r}}{\eta}$$

$$\dot{t} = \frac{r|\dot{r}|}{\eta\sqrt{2M_0}}$$

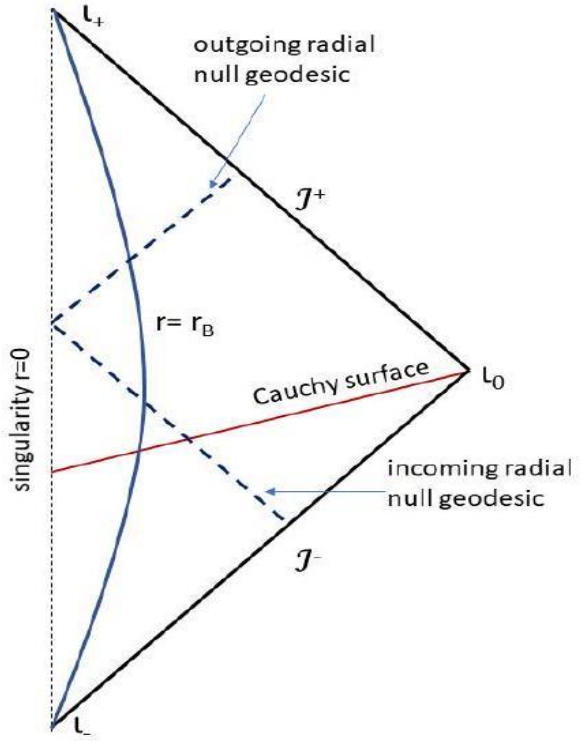
$$a^2 := a_\mu a^\mu \sim r^{-(4s+1)}$$

$$d\tilde{s}^2 = -d\tilde{t}^2 + dx^2 + (2x)^{3/2}(d\theta^2 + \sin^2\ \theta d\phi^2)$$

$$\tilde{t} = \eta^2/(8M_0^3)t \text{ and } x = \frac{r^2}{8M_0^2}$$

$$d\sigma^2 = dx^2 + (2x)^{3/2}(d\theta^2 + \sin^2\ \theta d\phi^2)$$

Figura. Diagrama Penrose aplicado a una partícula blanca y oscura.



$$n = \frac{8D}{m_f}\left[\theta^{3/2}F_{\frac{1}{2}}(\theta, b-\theta^{-1}) + \theta^{5/2}F_{\frac{3}{2}}(\theta, b-\theta^{-1})\right]$$

$$P = \frac{16D}{3}\left[\theta^{5/2}F_{\frac{3}{2}}(\theta, b-\theta^{-1}) + \frac{1}{2}\theta^{7/2}F_{\frac{5}{2}}(\theta, b-\theta^{-1})\right]$$

$$\rho = m_f n + 8D\left[\theta^{5/2}F_{\frac{3}{2}}(\theta, b-\theta^{-1}) + \theta^{7/2}F_{\frac{5}{2}}(\theta, b-\theta^{-1})\right]$$

$$D = \frac{m_f^4}{8\pi^2\hbar^3}.F_\alpha$$

$$F_\alpha(\theta, s) = \int_0^\infty dx \frac{x^\alpha\sqrt{2+\theta x}}{e^{x-s}+1}$$

$$n = \frac{D}{m_f}\nu(Y,b), P = D\sigma(Y,b), \rho = D\psi(Y,b)$$

$$F_\alpha(\theta, s) \simeq \int_0^{b-\theta^{-1}} x^\alpha\sqrt{2+\theta x}$$

$$\nu(Y,b) = \nu_0(Y) := \frac{8}{3}(Y^2-1)^{3/2}$$

$$\sigma(Y,b) = \sigma_0(Y) := \frac{1}{3}Y\sqrt{Y^2-1}(2Y^2-5) + \sinh^{-1}\sqrt{Y^2-1}$$

$$\psi(Y,b) = \psi_0(Y) := Y\sqrt{Y^2-1}(2Y^2-1) - \sinh^{-1}\sqrt{Y^2-1}$$

$$\frac{dz}{dx} = x^2\psi(Y,b)$$

$$\frac{dY}{dx} = -\frac{Y[z + x^3\sigma(Y,b)]}{x^2\left(1 - \frac{2z}{x}\right)}$$

$$x = x_B := \sqrt{4\pi D}r_B$$

$$z_B := \sqrt{4\pi D}M$$

$$T_\infty = \frac{m_f}{b}\sqrt{1 - \frac{2z_B}{x_B}}$$

$$z(0) - z(x_c) = \int_0^{x_c} dx x^2\psi(Y(x),b)$$

$$\psi(1,b) = c_1 b^{-3/2} + O\left(b^{-5/2}\right)$$

$$c_1 = 8\int_0^\infty dx\sqrt{2x}(e^x + 1)^{-1} \simeq 7.67$$

$$|z(0) - z(x_c)| < \frac{c_1 x_c^3}{3b^{3/2}}$$

$$N_0 = \frac{1}{\mu\sqrt{4\pi D}}\int_0^{x_c} dx \frac{x^2 v[Y(x),b]}{\sqrt{1 - \frac{2w(x)}{x}}}$$

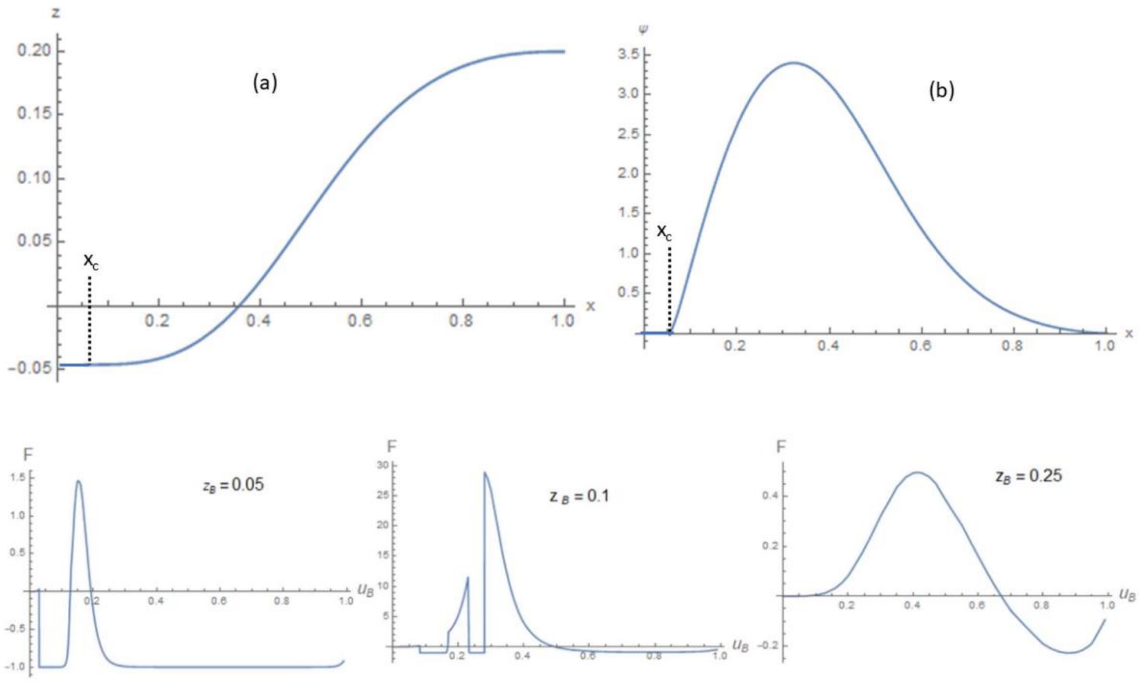
$$N_0 < \frac{c_1 x_c^3}{6\mu\sqrt{\pi D}b^{3/2}}$$

$M_0 = -\sqrt{4\pi D}z_0$, where $z_0 = -z(0) > 0$.

$$Y = \sqrt{\frac{1 + 2z_0/x_c}{1 + 2z_0/x}}$$

$$\eta = \frac{T_\infty b}{m_f(4\pi D)^{1/4}}\sqrt{\frac{2z_0}{1 + \frac{2z_0}{x_c}}}$$

Figuras. Masa y energía en dimensiones.



$$\rho = \rho_0 \left(\frac{P}{\rho_0}\right)^s + 3P$$

$$P = \rho_0 \left[\frac{(T/T_0)^{4(1-s)} - 1}{4}\right]^{\frac{1}{1-s}}$$

$$\rho = \frac{1}{4}\rho_0 \left[3(T/T_0)^{4(1-s)} + 1\right] \left[\frac{(T/T_0)^{4(1-s)} - 1}{4}\right]^{\frac{s}{1-s}}$$

$$P = Ce^{-m_f/T}T^{5/2}, \rho = Ce^{-m_f/T}T^{3/2}\left(m_f + \frac{3}{2}T\right)$$

$$ds^2 = -L^2(r,t)dt^2 + \frac{dr^2}{1 - \frac{2m(r,t)}{r}} + r^2(d\theta^2 + \sin^2\,\theta d\phi^2)$$

$$W\ddot{f} = (Sf')' + Qf$$

$$W(\xi) = \frac{1}{4\pi r_B^3} e^{3\xi} \frac{L(v+w)}{(1-u)^{3/2}} > 0$$

$$S(\xi) = \frac{1}{4\pi r_B^5} e^{5\xi} \frac{L^3 \Gamma_1 w}{\sqrt{1-u}} > 0$$

$$Q(\xi) = \frac{1}{4\pi r_B^5} e^{5\xi} \frac{L^3(v+w)}{\sqrt{1-u}} \left[(t')^2 + 4t' - \frac{2w}{1-u}\right]$$

$$f(r) = f_\omega(r)e^{-i\omega t}$$

$$(Sf_\omega')' + Qf_\omega + \omega^2 W f_\omega = 0$$

$$Lf := \frac{1}{W}(Sf')' + Qf$$

$$\omega_0^2 = \min_{f\in\mathcal{K}} R[f]$$

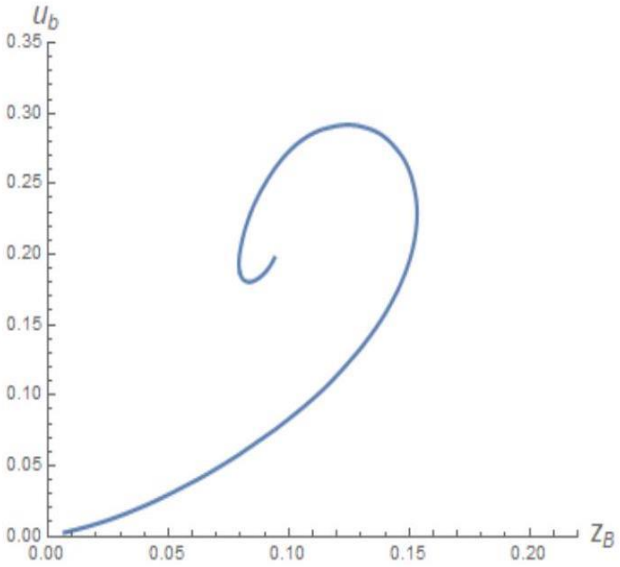
$$R[f] := \frac{\int_0^\infty d\xi (Sf'^2 - Qf^2)}{\int_0^\infty d\xi W f^2}$$

$$\int_0^{r_B} dr(S\zeta'^2 - Q\zeta^2) > 0$$

$$\frac{Q}{W} \to 8\pi \bar{L}\bar{P}$$

$$-\frac{7}{4}u^2 + 2u + w^2 + w - uw = \left(-\frac{7}{4}\delta^2 + 2\delta + 1\right)w^2 + 2(1-\delta)w$$

Figura. Curvatura previa a la formación de un agujero negro cuántico, provocada por la gravedad, sea por la partícula blanca y/o por la partícula oscura, según sea el caso.



$$F(z_B, u_B) = \exp\left[\frac{\omega_0^2(z_B, u_B)}{4\pi D^2}\right] - 1$$

$$R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}, \sum_M \nabla_M T_N^M = 0$$

$$ds^2 = -e^{2\lambda^0}dt^2 + \sum_i e^{2\lambda^i}\left(dx^i\right)^2 + e^{2\lambda}dr^2 + e^{2\sigma}d\Omega_n^2$$

$$\left(T_0^0, T_i^i, T_r^r, T_a^a\right) = (p_0, p_i, \Pi, p_a)$$

$$\Pi_r = \sum_\alpha (-\Pi + p_\alpha)\lambda_r^\alpha$$

$$\sum_{\alpha\beta} G_{\alpha\beta}\lambda_r^\alpha\lambda_r^\beta = 2\kappa^2\Pi e^{2\lambda} + n(n-1)e^{2\lambda-2\sigma}$$

$$\lambda_{rr}^\alpha + (\Lambda_r - \lambda_r)\lambda_r^\alpha = \kappa^2\sum_\beta G^{\alpha\beta}\left(\Pi + p_\beta\right)e^{2\lambda} + \delta^{\alpha a}(n-1)e^{2\lambda-2\sigma}$$

$$\alpha = (0, i, a)\,, \lambda^\alpha = \left(\lambda^0, \lambda^i, \lambda^a\right), p_\alpha = (p_0, p_i, p_a)$$

$$G_{\alpha\beta} = 1 - \delta_{\alpha\beta}\,, G^{\alpha\beta} = \frac{1}{D-2} - \delta^{\alpha\beta}\,, \Lambda = \sum_\alpha \lambda^\alpha$$

$$\sum_\beta G^{\alpha\beta}G_{\beta\gamma} = \delta_\gamma^\alpha\,, \sum_\beta G^{\alpha\beta} = \frac{1}{D-2}$$

$$\sum_\beta G^{\alpha\beta}\left(\Pi + p_\beta\right) = -p_\alpha + \frac{T}{D-2}$$

$$T_{MN} = \sum_I T_{MN(I)}\,;\ \sum_M \nabla_M T_{N(I)}^M = 0$$

$$\left(T_{0(I)}^0, T_{i(I)}^i, T_{r(I)}^r, T_{a(I)}^a\right) = (p_{0I}, p_{iI}, \Pi_I, p_{aI})$$

$$\Pi = \sum_I \Pi_I, p_\alpha = \sum_I p_{\alpha I}\,;\ (\Pi_I)_r = \sum_\alpha (-\Pi_I + p_{\alpha I})\lambda_r^\alpha$$

$$\Pi_I = \Pi_{I0}e^{\phi^I}, \phi^I = \sum_\alpha (c_{\alpha I} - 1)\lambda^\alpha$$

$$p_{0*} = p_{i*} = \Pi_*\,, p_* = \frac{n-2}{n}\Pi_*$$

$$T_* = \Pi_* + \sum_\beta p_{\beta*} = (n_c + n)\Pi_*$$

$$\sum_\beta G^{\alpha\beta}\left(\Pi_* + p_{\beta*}\right) = \Pi_* - p_{\alpha*} = \delta^{\alpha a}\frac{2}{n}\Pi_*$$

$$\tilde{\Pi} = \sum_{\tilde{I}} \Pi_{\tilde{I}}, \tilde{p}_\alpha = \sum_{\tilde{I}} p_{\alpha\tilde{I}}\,;\ (\Pi_{\tilde{I}})_r = \sum_\alpha (-\Pi_{\tilde{I}} + p_{\alpha\tilde{I}})\lambda_r^\alpha$$

$$e^\lambda dr = dl,$$

$$ds^2 = -e^{2\lambda^0}dt^2 + \sum_i e^{2\lambda^i}\left(dx^i\right)^2 + dl^2 + e^{2\sigma}d\Omega_n^2$$

$$\psi_l = e^{-\lambda}\psi_r, \psi_{ll} = e^{-2\lambda}(\psi_{rr} - \lambda_r\psi_r)$$

$$\sum_{\alpha\beta} G_{\alpha\beta}\lambda_l^\alpha\lambda_l^\beta = 2\kappa^2\tilde{\Pi}$$

$$\lambda_{ll}^\alpha + \Lambda_l\lambda_l^\alpha = \kappa^2\sum_\beta G^{\alpha\beta}\left(\tilde{\Pi} + \tilde{p}_\beta\right)$$

$$\tilde{\Pi} = \sum_{\tilde{I}} \Pi_{\tilde{I}}\,, \tilde{p}_\alpha = \sum_{\tilde{I}} p_{\alpha\tilde{I}}, \tilde{I} = *,1,2,\cdots,\mathcal{N}$$

$$(\Pi_{\tilde{I}})_l = \sum_\alpha (-\Pi_{\tilde{I}} + p_{\alpha\tilde{I}})\lambda_l^\alpha$$

$$dl^2 + e^{2\sigma}d\Omega_n^2 = e^{2\lambda}dr^2 + r^2 d\Omega_n^2.$$

$$e^{-\lambda} = \frac{dr}{dl} = e^\sigma\sigma_l.$$

$$Y_\alpha = \sum_\beta G_{\alpha\beta}\lambda_l^\beta = \Lambda_l - \lambda_l^\alpha \implies \lambda_l^\alpha = \sum_\beta G^{\alpha\beta}Y_\beta$$

$$\sum_\alpha Y_\alpha = (D-2)\Lambda_l, \sum_\alpha \lambda_l^\alpha Y_\alpha = 2\kappa^2\tilde{\Pi}.$$

$$\lambda_{ll}^\alpha + \sum_\beta \frac{\left(\lambda_l^\alpha - \lambda_l^\beta\right)Y_\beta}{D-2} = \kappa^2\sum_\beta G^{\alpha\beta}\left(-\tilde{\Pi} + \tilde{p}_\beta\right).$$

$$\tilde{\Pi}_l = \sum_\alpha \left(-\tilde{\Pi} + \tilde{p}_\alpha\right)\lambda_l^\alpha.$$

$$\alpha = (0,i,a)\,, m^\alpha = \left(m^0, m^i, m^a\right)$$

$$f^\alpha = f(m^\alpha)\,, g_\alpha = \frac{df^\alpha}{dm^\alpha}\,, X_\alpha = g_\alpha\sum_\beta G_{\alpha\beta}f^\beta$$

$$\sum_{\alpha\beta} G_{\alpha\beta}f^\alpha f^\beta = 2l_{qm}^2\kappa^2\tilde{\Pi}$$

$$(m^\alpha)_l + \sum_\beta \frac{\left(m^\alpha - m^\beta\right)X_\beta}{(D-2)l_{qm}} = l_{qm}\kappa^2\sum_\beta G^{\alpha\beta}\left(-\tilde{\Pi} + \tilde{p}_\beta\right)$$

$$l_{qm}\lambda_l^\alpha = \sum_\beta G^{\alpha\beta}X_\beta$$

$$\leftrightarrow X_\alpha = l_{qm}Y_\alpha = l_{qm}\sum_\beta G_{\alpha\beta}\lambda_l^\beta$$

$$l_{qm}(\Pi_{\tilde{I}})_l = \sum_{\alpha\beta} G^{\alpha\beta}(-\Pi_{\tilde{I}} + p_{\alpha\tilde{I}})X_\beta$$

$$l_{qm}\tilde{\Pi}_l = \sum_{\alpha\beta} G^{\alpha\beta}\left(-\tilde{\Pi} + \tilde{p}_\alpha\right)X_\beta$$

$$f^{\alpha} = cm^{\alpha} + c_0\,, g_{\alpha} = c\,, X_{\alpha} = c\sum_{\beta} G_{\alpha\beta} f^{\beta}\,, l_{qm}\lambda_l^{\alpha} = cf^{\alpha}$$

$$\begin{aligned}(m^0, f^0, g_0, X_0) &= (m, f, g, X)\\ (m^a, f^a, g_a, X_a) &= (\hat{m}, \hat{f}, \hat{g}, \hat{X})\end{aligned}$$

$$\sum_{\beta} G^{\alpha\beta} X_{\beta} = \hat{X} - X_{\alpha} + \frac{X}{n}$$

$$\sum_{\beta} G^{\alpha\beta}\left(-\tilde{\Pi} + \tilde{p}_{\beta}\right) = \tilde{p} - \tilde{p}_{\alpha} - \frac{\tilde{\Pi} + \tilde{\rho}}{n}$$

$$\begin{aligned}2nf\hat{f} + n(n-1)\hat{f}^2 &= 2l_{qm}^2\kappa^2\tilde{\Pi}\\ nl_{qm}\hat{m}_l + (\hat{m} - m)X &= -l_{qm}^2\kappa^2(\tilde{\Pi} + \tilde{\rho})\\ l_{qm}m_l + (m - \hat{m})\hat{X} &= l_{qm}^2\kappa^2\left(\tilde{p} + \tilde{\rho} - \frac{\tilde{\Pi} + \tilde{\rho}}{n}\right)\\ l_{qm}\lambda_l^0 &= \hat{g}f + (n-1)(\hat{g} - g)\hat{f}, l_{qm}\sigma_l = g\hat{f}\end{aligned}$$

$$(\Pi_I)_l = -(\Pi_I + \rho_I)\lambda_l^0 - n(\Pi_I - p_I)\sigma_l$$

$$2f\hat{f} + (n-1)\hat{f}^2 = l_{qm}^2\left(\frac{2\kappa^2}{n}\Pi + (n-1)e^{-2\sigma}\right)$$

$$\begin{aligned}&2l_{qm}\hat{m}_l + 2(\hat{m} - m)g\hat{f} + 2f\hat{f} + (n-1)\hat{f}^2\\ &= l_{qm}^2\left(-\frac{2\kappa^2}{n}\rho + (n-1)e^{-2\sigma}\right).\end{aligned}$$

$$\tilde{\Pi} \le f_{\max}^2\Pi_{qm}, \Pi_{qm} = \frac{(D-1)(D-2)}{2l_{qm}^2\kappa^2}$$

$$g_{\alpha} = \frac{df^{\alpha}}{dm^{\alpha}}$$

$$\Pi = p, \tilde{\Pi} = p + \Pi_*, \tilde{\rho} = \rho + \rho_*, \tilde{p} = p + p_*.$$

$$p + \Pi_* \le \Pi_{qm}$$

$$p_c < \Pi_{qm}$$

$$2\kappa^2\Pi_* \propto e^{-2\sigma}$$

$$e^{\sigma} > \mathcal{O}(1)l_{qm}$$

$$\rho \ll \Pi_{qm}$$

$$R_{\text{star}} \propto \rho^{-\frac{1}{2}}$$

$$M_{star} \propto \rho^{-\frac{n-1}{2}}, K_{star} \simeq \frac{4n}{(n+1)^2}\left(1-\frac{(n-1)^2\rho}{n(n+1)\Pi_{qm}}\right).$$

$$R_{star} \simeq \left(\kappa^2\Pi_{qm}\right)^{-\frac{1}{2}} \simeq l_{qm}, M_{star} \simeq l_{qm}^{n-1}, K_{star} \simeq \frac{2n-1}{n^2}.$$

$$\alpha = (0,a)\,, \lambda^{\alpha} = (\lambda^0, \sigma)\,, p_{\alpha} = (-\rho, p)$$

$$\Lambda = \lambda^0 + n\sigma, \sum_{\beta} G^{\alpha\beta}\left(\Pi + p_{\beta}\right) = -p_{\alpha} + \frac{T}{n}$$

$$\begin{aligned} 2n\lambda_r^0\sigma_r + n(n-1)(\sigma_r)^2 &= 2\kappa^2\Pi e^{2\lambda} + n(n-1)e^{2\lambda-2\sigma} \\ n\sigma_{rr} + n(\Lambda_r - \lambda_r)\sigma_r &= \kappa^2(\Pi-\rho)e^{2\lambda} + n(n-1)e^{2\lambda-2\sigma} \\ \lambda_{rr}^0 + (\Lambda_r - \lambda_r)\lambda_r^0 &= \kappa^2\left(\rho + \frac{T}{n}\right)e^{2\lambda} \end{aligned}$$

$$K(r) = \frac{M(r)}{r^{n-1}} = 1 - e^{-2\lambda(r)}$$

$$ds^2 = -e^{2\lambda^0}dt^2 + \frac{dr^2}{1-K} + r^2 d\Omega_n^2$$

$$\begin{aligned} 2nr\lambda_r^0 e^{-2\lambda} &= 2\kappa^2 r^2\Pi + n(n-1)K \\ 2nr\lambda_r e^{-2\lambda} &= nrK_r = 2\kappa^2 r^2\rho - n(n-1)K \end{aligned}$$

$$nM_r = 2\kappa^2 r^n\rho \;\longrightarrow\; M = \frac{2\kappa^2}{n}\int_0^r dr r^n\rho$$

$$\begin{aligned} \Pi_r &= -(\Pi+\rho)\lambda_r^0 - n(\Pi-p)\sigma_r \\ \lambda_r^0 &= \frac{2\kappa^2 r^{n+1}\Pi + n(n-1)M}{2nr(r^{n-1}-M)} \end{aligned}$$

$$M(r) = \frac{2\kappa^2\rho r^{n+1}}{n(n+1)}, K(r) = \frac{M(r)}{r^{n-1}} = \frac{2\kappa^2\rho r^2}{n(n+1)}.$$

$$p_r = -\kappa^2 r\frac{(p+\rho)((n+1)p + (n-1)\rho)}{n(n+1) - 2\kappa^2\rho r^2}$$

$$\frac{(n+1)p + (n-1)\rho}{(n-1)(p+\rho)} = \sqrt{\frac{1-K(r)}{1-K(R)}}$$

$$\frac{p}{\rho} = \frac{(n-1)(\sqrt{1-K(r)} - \sqrt{1-K(R)})}{(n+1)\sqrt{1-K(R)} - (n-1)\sqrt{1-K(r)}}.$$

$$e^{2\lambda^0} = \frac{\text{const}}{(\rho+p)^2} = \left(\frac{(n+1)\sqrt{1-K(R)} - (n-1)\sqrt{1-K(r)}}{2}\right)^2$$

$$e^{2\lambda^0} = e^{-2\lambda} = 1 - \frac{M(R)}{r^{n-1}}$$

$$K_{\text{star}} = \frac{M_{\text{star}}}{R_{\text{star}}^{n-1}} = \frac{2\kappa^2\rho R_{\text{star}}^2}{n(n+1)} = \frac{4p_c(np_c+(n-1)\rho)}{((n+1)p_c+(n-1)\rho)^2}$$

$$M_{star} \propto \rho^{-\frac{n-1}{2}}, K_{star} \simeq \frac{4n}{(n+1)^2}\left(1-\frac{(n-1)^2\rho}{n(n+1)p_c}\right)$$

$$R_{star} \simeq (\kappa^2 p_c)^{-\frac{1}{2}}, M_{star} \simeq (\kappa^2 p_c)^{-\frac{n-1}{2}}, K_{star} \simeq \frac{2n-1}{n^2}.$$

$$ds^2 = -dt^2 + \sum_i e^{2\lambda^i}\left(dx^i\right)^2$$

$$\sum_{ij} G_{ij}\lambda_t^i\lambda_t^j = 2\kappa^2\rho$$

$$\lambda_{tt}^i + \Lambda_t\lambda_t^i = \kappa^2\sum_j G^{ij}(\rho - p_j)$$

$$\rho_t + \sum_i (\rho + p_i)\lambda_t^i = 0$$

$$G_{ij} = 1 - \delta_{ij}, G^{ij} = \frac{1}{D-2} - \delta^{ij}, \Lambda = \sum_i \lambda^i$$

$$Y_i = \sum_j G_{ij}\lambda_t^j \Leftrightarrow \lambda_t^i = \sum_j G^{ij}Y_j$$

$$\lambda_{tt}^i + \sum_j \frac{(\lambda_t^i - \lambda_t^j)Y_j}{D-2} = -\kappa^2\sum_j G^{ij}(\rho + p_j)$$

$$\rho_t = -\sum_i (\rho + p_i)\lambda_t^i$$

$$f^i = f\left(m^i\right), g_i = \frac{df^i}{dm^i}, X_i = g_i\sum_j G_{ij}f^j$$

$$\sum_{ij} G_{ij}f^i f^j = 2\gamma^2\lambda_{qm}^2\kappa^2\rho$$

$$\left(m^i\right)_t + \sum_j \frac{(m^i - m^j)X_j}{(D-2)\gamma\lambda_{qm}} = -\gamma\lambda_{qm}\kappa^2\sum_j G^{ij}(\rho + p_j)$$

$$\gamma\lambda_{qm}\lambda_t^i = \sum_j G^{ij}X_j$$

$$\leftrightarrow X_i = \gamma\lambda_{qm}Y_i = \gamma\lambda_{qm}\sum_j G_{ij}\lambda_t^j$$

$$\lambda_{qm}^{D-2} = \mathcal{O}(1)\gamma\kappa^2$$

$$(\gamma\lambda_{qm})\rho_t = -\sum_{ij} G^{ij}(\rho + p_i)X_j$$

$$f^i = cm^i + c_0, g_i = c, (\gamma\lambda_{qm})\lambda_t^i = cf^i$$

Modelo matemático para describir el comportamiento fenomenológico y morfología de un Agujero Negro Cuántico y su conexión sustancial tanto con el Agujero Cuántico de Gusano como con el Agujero Blanco Cuántico (Modelo Unificado), todo esto, en escenarios de gravedad y supergravedad cuánticas respectivamente o teorizados como campos cuánticos relativistas. Incluye cálculos termodinámicos.

$$\begin{aligned}(16\pi G_5)e^{-1}\mathcal{L} =& R + \frac{i}{2}\bar{\psi}_\mu^a\gamma^{\mu\nu\rho}D_\nu\psi_{\rho a} - \frac{1}{8}G_{AB,CD}F_{\mu\nu}^{AB}F^{\mu\nu CD} - \frac{i}{12}\bar{\lambda}^{abc}\gamma^\mu D_\mu\lambda_{abc} - \frac{1}{6}P_{\mu abcd}P^{\mu abcd}\\ &-\frac{i}{3\sqrt{2}}P_{\mu abcd}\bar{\psi}_\mu^a\gamma^\rho\gamma^\mu\lambda^{bcd} - \frac{i}{8}F_{\mu\nu}^{ab}\left(\bar{\psi}_a^\rho\gamma_{[\rho}\gamma^{\mu\nu}\gamma_{\sigma]}\psi_b^\sigma + \frac{1}{\sqrt{2}}\bar{\psi}_\rho^c\gamma^{\mu\nu}\gamma^\rho\lambda_{abc}\right.\\ &\left.+\frac{1}{2}\bar{\lambda}_{acd}\gamma^{\mu\nu}\lambda_b^{cd}\right) + \frac{1}{24}e^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}(F_{\mu\nu})_b^a(F_{\rho\sigma})_c^b(A_\lambda)_a^c\end{aligned}$$

$$G_{AB,CD} = V_{ABab}\Omega^{ac}\Omega^{bd}V_{CDcd}$$

$$\begin{aligned}V_{ij}^{ab} &= \frac{1}{8}(\Gamma_{kl})^{ab}S_i^kS_j^l + \frac{1}{4}(\Gamma^{k\alpha})^{ab}\varphi_{ijk\alpha},\\ V_{i\alpha}^{ab} &= \frac{1}{4\sqrt{2}}(\Gamma^{kl})^{ab}\varphi_{ikl\alpha} + \frac{1}{2\sqrt{2}}(\Gamma_{k\beta})^{ab}S_i^k\Lambda_\alpha^\beta.\end{aligned}$$

$$G_{ij,kl} = \frac{1}{4}(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np})S_i^mS_j^nS_k^pS_l^q + \varphi_{ijm\alpha}\varphi_{klm\alpha} = \frac{1}{2}T_{ik}^{-1}T_{jl}^{-1} + \varphi_{ijm\alpha}\varphi_{klm\alpha}$$

$$W_{abcd} = \epsilon^{\alpha\beta}\delta^{IJ}V_{I\alpha ab}V_{J\beta cd}$$

$$\mathcal{T}_{ab} = \frac{15}{4}W_{acb}^c, \mathcal{A}_{abcd} = -3W_{a[bcd]|}$$

$$(16\pi G_5)e^{-1}\mathcal{L} = \mathcal{L}_{g+v} + \mathcal{L}_s + \mathcal{L}_f$$

$$\begin{aligned}\mathcal{L}_{g+v} =& R - \frac{1}{8}G_{ij.kl}F_{\mu\nu}^{ij}F^{\mu\nu kl} - \frac{1}{96}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}\epsilon_{ijklmn}\left(F_{\mu\nu}^{ij}F_{\rho\sigma}^{kl}A_\tau^{mn} + gF_{\mu\nu}^{ij}A_\rho^{kl}A_\sigma^{mp}A_\tau^{pn}\right.\\ &\left.+\frac{2}{5}g^2A_\mu^{ij}A_\nu^{kp}A_\rho^{pl}A_\sigma^{mr}A_\tau^{rn}\right) + \frac{1}{8g}e^{-1}\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma\tau}B_{\mu\nu}^{i\alpha}D_\rho B_{\sigma\tau}^{j\beta} - \frac{1}{4}G_{i\alpha j\beta}B_{\mu\nu i\alpha}B^{\mu\nu j\beta}\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{g+v} =& R\star 1 - \frac{1}{8}T_{ik}^{-1}T_{jl}^{-1}F^{ij}\wedge\star F^{kl} - \frac{1}{4}\varphi_{ijm\alpha}\varphi_{klm\alpha}F^{ij}\wedge\star F^{kl} - \frac{1}{48}\epsilon_{i_1\dots i_6}\left(F^{i_1i_2}\wedge F^{i_3i_4}\wedge A^{i_5i_6}\right.\\ &\left.-gF^{i_1i_2}\wedge A^{i_3i_4}\wedge A^{i_5j}\wedge A^{ji_6} + \frac{2}{5}g^2A^{i_1i_2}\wedge A^{i_3j}\wedge A^{ji_4}\wedge A^{i_5k}\wedge A^{ki_6}\right)\\ &+\frac{1}{8g}e^{-1}\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma\tau}B_{\mu\nu}^{i\alpha}D_\rho B_{\sigma\tau}^{j\beta} - \frac{1}{4}G_{i\alpha\ j\beta}B_{\mu\nu}^{i\alpha}B^{\mu\nu i\alpha}\end{aligned}$$

$$\mathcal{L}_s = -\frac{1}{6}P_{\mu abcd}P^{\mu abcd} + \frac{1}{6}g^2\left(\frac{64}{225}(\mathcal{T}_{ab})^2 - (\mathcal{A}_{abcd})^2\right)$$

$$\mathcal{L}_s = -\frac{1}{4}T_{ij}^{-1}T_{kl}^{-1}DT_{li} \wedge\star DT_{jk} - \frac{1}{2}D\Lambda_\alpha^\beta \wedge\star D\Lambda_\alpha^\beta - \frac{1}{2}D\varphi_{ijk\alpha} \wedge\star D\varphi_{ijk\alpha} - V \star 1$$

$$V = \frac{1}{2}g^2\left(2T_{ij}T^{ij} - \left(T_i^i\right)^2\right) - \frac{1}{4}g^2\varphi_{ijk\alpha}\varphi^{ijk\alpha}$$

$$\begin{aligned}\mathcal{L}_f =& \frac{i}{2}\bar{\psi}_\mu^a\gamma^{\mu\nu\rho}D_\nu\psi_{\rho a} - \frac{i}{12}\bar{\lambda}^{abc}\gamma^\mu D_\mu\lambda_{abc} - \frac{i}{3\sqrt{2}}P_{\mu abcd}\bar{\psi}_\mu^a\gamma^\rho\gamma^\mu\lambda^{bcd}\\ &-\frac{i}{8}F_{\mu\nu}^{ab}\left(\bar{\psi}_a^\rho\gamma_{[\rho}\gamma^{\mu\nu}\gamma_{\sigma]}\psi_b^\sigma + \frac{1}{\sqrt{2}}\bar{\psi}_\rho^c\gamma^{\mu\nu}\gamma^\rho\lambda_{abc} + \frac{1}{2}\bar{\lambda}_{acd}\gamma^{\mu\nu}\lambda_b^{cd}\right)\\ &+\frac{2i}{15}g\mathcal{T}_{ab}\bar{\psi}_\mu^a\gamma^{\mu\nu}\psi_\nu^b - \frac{i}{6\sqrt{2}}g\mathcal{A}_{dabc}\bar{\lambda}^{abc}\gamma^\mu\psi_\mu^d - ig\bar{\lambda}^{abc}\left(\frac{1}{2}\mathcal{A}_{bcde} - \frac{1}{45}\Omega_{bd}\mathcal{T}_{ce}\right)\lambda_a^{de}\end{aligned}$$

$$\begin{aligned}DT_{ij} &= dT_{ij} + g\left(A^{ik}T_{kj} + A^{jk}T_{ki}\right)\\ D\Lambda_\alpha^\beta &= d\Lambda_\alpha^\beta\\ D\varphi_{ijk\alpha} &= d\varphi_{ijk\alpha} + gA_{l[i}\varphi_{jk]l\alpha}\\ F^{ij} &= dA^{ij} + gA^{ik}\wedge A^{kj}\\ D_\mu B_{\rho\sigma}^{i\alpha} &= \partial_\mu B_{\rho\sigma}^{i\alpha} + gA_\mu^{ij}B_{\rho\sigma}^{j\alpha}\end{aligned}$$

$$D\left(T_{ik}^{-1}T_{jl}^{-1}\star F^{kl}\right) = -2gT_{k[i}^{-1}\star DT_{j]k} - \frac{1}{8}\epsilon_{ijk_1\ldots k_4}F^{k_1k_2}\wedge F^{k_3k_4},$$

$$D_{[\mu}B_{\nu\rho]}^{i\alpha} - \frac{1}{12}g\epsilon^{\alpha\beta}\epsilon_{\mu\nu\rho\sigma\tau}V_{i\beta}^{ab}V_{ab}^{j\gamma}B_{j\gamma}^{\sigma\tau} = 0$$

$$\begin{aligned}D\left(T_{ik}^{-1}\star DT_{kj}\right) =& -2g^2\left(2T_{ik}T_{jk} - T_{ij}T_{kk}\right)\star 1 + T_{ik}^{-1}T_{lm}^{-1}F^{lk}\wedge\star F^{mj}\\ &-\frac{1}{6}\delta_{ij}\left[-2g^2\left(T_{kl}T_{kl} - T_{kk}^2\right)\star 1 + T_{pk}^{-1}T_{lm}^{-1}F^{lk}\wedge\star F^{mp}\right].\end{aligned}$$

$$\begin{aligned}\Box\,\Lambda_\alpha^\beta &= 0\\ D_\mu D^\mu\varphi_{ijk\alpha} - 3g^2\varphi_{ijk\alpha} - \frac{1}{2}F_{\mu\nu}^{lm}F^{\mu\nu[ij}\varphi_{k]lm\alpha} &= 0\end{aligned}$$

$$\begin{aligned}&\gamma^{\mu\nu\rho}D_\nu\psi_{\rho a} - \frac{1}{3\sqrt{2}}P_{\nu abcd}\gamma^\nu\gamma^\mu\lambda^{bcd} - \frac{1}{4}F_{ab}^{\rho\sigma}\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]}\psi_\nu^b - \frac{1}{8\sqrt{2}}F_{\rho\sigma}^{bc}\gamma^{\rho\sigma}\gamma^\mu\lambda_{abc}\\ &+\frac{4}{15}g\mathcal{T}_{ab}\gamma^{\mu\nu}\psi_\nu^b - \frac{1}{6\sqrt{2}}g\mathcal{A}_{abcd}\gamma^\mu\lambda^{bcd} = 0\\ &\frac{1}{6}\gamma^\mu D_\mu\lambda_{abc} - \frac{1}{3\sqrt{2}}P_{\nu abcd}\gamma^\mu\gamma^\nu\psi_\mu^d - \frac{1}{8\sqrt{2}}\gamma^\rho\gamma_{\mu\nu}\psi_{\rho[a}F_{bc]|}^{\mu\nu}\\ &+\frac{1}{8}\gamma_{\mu\nu}\lambda_{[ab}^dF_{c]|d}^{\mu\nu} + \frac{1}{6\sqrt{2}}g\mathcal{A}_{dabc}\gamma^\mu\psi_\mu^d + g\left(\frac{1}{2}\mathcal{A}_{[ab}^{ef} + \frac{1}{45}\delta_{[a}^e\mathcal{T}_b^f\right)\lambda_{c]lef} = 0\end{aligned}$$

$$\mathcal{L}_{\text{EMAdS}} = \frac{1}{16\pi G_5}\left((R + 12g^2)\star 1 - \frac{3}{2}F\wedge\star F - F\wedge F\wedge A\right)$$

$$\begin{aligned}d\star F + F\wedge F &= 0\\ R_{\mu\nu} + 4g^2g_{\mu\nu} &= \frac{3}{2}F_{\mu\lambda}F_\nu^\lambda - \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}\end{aligned}$$

$$M = \frac{\pi}{4G_5}\frac{m(3+a^2g^2)+4qa^2g^2}{\Xi^3}$$
$$Q = \frac{\pi}{4G_5}\frac{q}{\Xi^2}$$
$$J = \frac{\pi}{4G_5}\frac{2ma+qa(1+a^2g^2)}{\Xi^3}$$

$$ds^2 = -\frac{[(1+g^2r^2)\rho^2dt+2qv]dt}{\Xi\rho^2}+\frac{2qv^2}{\Xi\rho^2}+\frac{f}{\Xi^2\rho^4}(dt-v)^2$$
$$+\frac{\rho^2dr^2}{\Delta_r}+\frac{\rho^2d\theta^2}{\Xi}+\frac{r^2+a^2}{\Xi}(\sin^2\theta d\phi^2+\cos^2\theta d\psi^2)$$
$$A = \frac{3q}{\Xi\rho^2}(dt-v)$$

$$\Delta_r = \frac{(r^2+a^2)^2(1+g^2r^2)+q^2+2a^2q}{r^2}-2m, \rho^2 = r^2+a^2$$
$$v = a(\sin^2\theta d\phi+\cos^2\theta d\psi), f = 2m\rho^2-q^2+2a^2g^2q\rho^2$$

$$S = \frac{\pi^2}{2G_5}\frac{(r_+^2+a^2)^2+a^2q}{\Xi^2r_+}$$
$$T = \frac{r_+^4\left(1+2g^2(r_+^2+a^2)\right)-(q+a^2)^2}{2\pi r_+((r_+^2+a^2)^2+a^2q)}$$
$$\Omega = a\frac{(r_+^2+a^2)(1+g^2r_+^2)+q}{(r_+^2+a^2)^2+a^2q}$$
$$\Phi = \frac{qr_+^2}{(r_+^2+a^2)^2+a^2q}$$

$$(1+g^2r_+^2)(r_+^2+a^2)^2+q^2+2a^2q-2mr_+^2 = 0$$

$$\Lambda_\alpha^\beta = \varphi_{ijk\alpha} = B_{\mu\nu}^{i\alpha} = \psi_\mu^a = \lambda_{abc} = 0$$

$$T_{ij} = \delta_{ij}, A_{\mu ij} = A_\mu\Omega_{ij}$$

$$\Omega^{ij} = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$$

$$\left(\left(\mathbf{3}_{-2}\oplus\overline{\mathbf{3}}_2\right)\otimes\left(\mathbf{3}_{-2}\oplus\overline{\mathbf{3}}_2\right)\right)_{\text{sym,tr}} = (\mathbf{3}_{-2}\otimes\mathbf{3}_{-2})_{\text{sym}}\oplus\left(\overline{\mathbf{3}}_2\otimes\overline{\mathbf{3}}_2\right)_{\text{sym}}\oplus(\overline{\mathbf{3}}\otimes\mathbf{3})_{\text{tr}} = \mathbf{6}_{-4}\oplus\overline{\mathbf{6}}_4\oplus\mathbf{8}_0.$$

$$T_{ij} = \delta_{ij}+t_{ij}, A^{ij} = A\Omega^{ij}+a^{ij}$$

$$DT_{ij} = \underbrace{dt_{ij}+gA\left(\Omega^{ik}t_{kj}+\Omega^{jk}t_{ki}\right)}_{Dt_{ij}}+\underline{g\left(a^{ik}t_{kj}+a^{jk}t_{ki}\right)}$$
$$F^{ij} = F\Omega^{ij}+\underbrace{da^{ij}+gA\wedge\left(\Omega^{ik}a^{kj}-a^{ik}\Omega^{kj}\right)}_{f^{ij}=Da^{ij}}+\underline{ga^{ik}\wedge a^{kj}}$$

$$D\star Dt_{ij}-4g^2t_{ij}\star 1+2\Omega^{il}t_{lm}\Omega^{mj}(F\wedge\star F)=-F\wedge\star\left(f^{kj}\Omega^{ki}+f^{ki}\Omega^{kj}\right),$$
$$D\star f^{ij}+F\wedge\star\left(\Omega^{ik}f^{kl}\Omega^{lj}\right)=D\left(\left(t_{ik}\Omega^{kj}+t_{jk}\Omega^{ki}\right)\star F\right).$$

$$t_{ij}=t_{ij}^{+}+t_{ij}^{-}$$

$$\Omega^{im}t_{mn}^{\pm}\Omega^{nj}=\pm t_{ij}^{\pm}$$

$$t_{ij}^{-}=\frac{1}{2}\begin{pmatrix}t_{11}+t_{22} & t_{13}+t_{24}\\ t_{33}+t_{44} & t_{35}+t_{26}\\ & t_{55}+t_{66}\end{pmatrix}\otimes I+\frac{1}{2}\begin{pmatrix}0t_{14}-t_{23} & t_{16}-t_{25}\\ 0 & t_{36}-t_{45}\\ 0 & 0\end{pmatrix}\otimes(i\sigma_2)$$
$$t_{ij}^{+}=\frac{1}{2}\begin{pmatrix}t_{11}-t_{22} & t_{13}-t_{24}\\ t_{33}-t_{44} & t_{35}-t_{26}\\ & t_{55}-t_{66}\end{pmatrix}\otimes\sigma_3+\frac{1}{2}\begin{pmatrix}2t_{12} & t_{14}+t_{23}\\ & t_{16}+t_{25}\\ & 2t_{34}\\ & t_{36}+t_{45}\end{pmatrix}\otimes\sigma_1.$$

$$a_{ij}=a_{ij}^{+}+a_{ij}^{-}$$

$$\Omega^{im}a_{mn}^{\pm}\Omega^{nj}=\pm a_{ij}^{\pm}.$$

$$a_{-}^{ij}=\frac{1}{2}\begin{pmatrix}0a^{13}+a^{24} & a^{15}+a^{26}\\ 0 & a^{35}+a^{46}\\ 0\end{pmatrix}\otimes I+\frac{1}{2}\begin{pmatrix}2a^{12} & a^{14}-a^{23}\\ & a^{16}-a^{25}\\ & 2a^{34}\\ & a^{36}-a^{45}\\ & \\ & 2a^{56}\end{pmatrix}\otimes(i\sigma_2)$$
$$a_{+}^{ij}=\frac{1}{2}\begin{pmatrix}0a^{13}-a^{24} & a^{15}-a^{26}\\ 0 & a^{35}-a^{46}\\ 0\end{pmatrix}\otimes\sigma_3+\frac{1}{2}\begin{pmatrix}0a^{14}+a^{23} & a^{16}+a^{25}\\ 0 & a^{36}+a^{45}\end{pmatrix}\otimes\sigma_1,$$

$$D\star Dt_{ij}^{+}-4g^2t_{ij}^{+}\star 1-2t_{ij}^{+}(F\wedge\star F)=0,$$

$$D\star f_{+}^{ij}+F\wedge f_{+}^{ij}=0$$

$$Dt_{ij}^{+}=dt_{ij}^{+}+2gA\Omega^{ik}t_{kj}^{+}$$
$$f_{+}^{ij}=Da_{+}^{ij}=da_{+}^{ij}+2g\Omega^{ik}A\wedge a_{+}^{kj}$$

$$D\star Dt_{ij}^{-}-4g^2t_{ij}^{-}\star 1+2t_{ij}^{-}(F\wedge\star F)=2F\wedge\star\left(f_{-}^{ki}\Omega^{kj}\right)$$
$$D\star\left(f_{-}^{ki}\Omega^{kj}\right)-F\wedge\left(f_{-}^{ki}\Omega^{kj}\right)=-2D\left(t_{ij}^{-}\star F\right)$$

$$Dt_{ij}^{-}=dt_{ij}^{-},f_{-}^{ij}=Da_{-}^{ij}=da_{-}^{ij}$$

$$D^{\mu}D_{\mu}\varphi_{ijk\alpha}-3g^2\varphi_{ijk\alpha}-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\Omega_{[ij}\varphi_{k]lm\alpha}\Omega^{lm}=0$$

$$\Omega_{[ij}\varphi_{k]lm\alpha}^{(\mathbf{1})}\Omega^{lm}=0$$
$$\Omega_{[ij}\varphi_{k]lm\alpha}^{(\mathbf{3})}\Omega^{lm}=0$$

$$+\frac{1}{3}F_{\mu\nu}F^{\mu\nu}\left(\varphi^{(\mathbf{6})}_{ijm\alpha}\right)^2,$$

$$D_\mu\varphi^{(\mathbf{1})}_{ijk\alpha} = \partial_\mu\varphi^{(\mathbf{1})}_{ijk\alpha} \pm 3gA_\mu\varphi^{(\mathbf{1})}_{ijk\alpha}$$
$$D_\mu\varphi^{(\mathbf{3})}_{ijk\alpha} = \partial_\mu\varphi^{(\mathbf{3})}_{ijk\alpha} \pm gA_\mu\varphi^{(\mathbf{3})}_{ijk\alpha}$$
$$D_\mu\varphi^{(\mathbf{6})}_{ijk\alpha} = \partial_\mu\varphi^{(\mathbf{6})}_{ijk\alpha} \pm gA_\mu\varphi^{(\mathbf{6})}_{ijk\alpha}$$

$$\Box\,\Lambda^\beta_\alpha = 0$$

$$\epsilon^{\mu\nu\rho\sigma\tau}D_\rho B^{i\pm}_{\sigma\tau} + iB^{i\pm}_{\mu\nu} = 0$$

$$B^{\mu\nu i\pm} = \frac{1}{2}\left(B^{\mu\nu i1} + \Omega^{ij}B^{\mu\nu j2}\right) \pm \frac{i}{2}\left(B^{\mu\nu i2} - \Omega^{ij}B^{\mu\nu j1}\right)$$

$$D_\rho B^{i\pm}_{\sigma\tau} = \partial_\rho B^{i\pm}_{\sigma\tau} \mp gA_\mu B^{j\pm}_{\sigma\tau}$$

$$\sigma_1 = -\sin\,\psi d\theta + \cos\,\psi\sin\,\theta d\phi$$
$$\sigma_2 = \cos\,\psi d\theta + \sin\,\psi\sin\,\theta d\phi$$
$$\sigma_3 = d\psi + \cos\,\theta d\phi$$

$$ds^2 = -\frac{r^2}{(r^2+a^2)^2(1-a^2)}\Delta_r e^{U_2-U_1}dt^2 + \frac{(r^2+a^2)dr^2}{\Delta_r} + e^{-U_1}(\sigma_1^2+\sigma_2^2) + e^{-U_2}(\sigma_3+\omega dt)^2,$$

$$e^{-U_1} = \frac{r^2+a^2}{4(1-a^2)}$$
$$e^{-U_2} = \frac{r^2+a^2}{4(1-a^2)}\left(1 + \frac{2(m+q)a^2}{(1-a^2)(r^2+a^2)^2} - \frac{a^2q^2}{(1-a^2)(r^2+a^2)^3}\right)$$
$$\omega = \frac{a}{2(1-a^2)}\left((r^2+a^2) - \frac{2m+q}{r^2+a^2} + \frac{q^2}{(r^2+a^2)^2}\right)e^{U_2} - 2a$$

$$A = \frac{q}{(1-a^2)(r^2+a^2)}\left(dt - \frac{1}{2}a\sigma_3\right).$$

$$q = r_+^2\sqrt{2r_+^2+1+2a^2} - a^2$$

$$2m = (r_+^2+a^2)(3r_+^2+2+a^2)$$

$$Q = \frac{\pi}{4G_5}\frac{1}{(1-a^2)^2}\left(r_+^2\sqrt{2r_+^2+1+2a^2} - a^2\right)$$
$$J = \frac{\pi}{4G_5}\frac{a}{(1-a^2)^3}\left(3r_+^4 + 2r_+^2(1+2a^2) + a^2 + r_+^2(1+a^2)\sqrt{2r_+^2+1+2a^2}\right)$$

$$\Phi = \frac{-a^2 + r_+^2\sqrt{2r_+^2 + 2a^2 + 1}}{r_+^2 + 2a^2 + a^2\sqrt{2r_+^2 + 2a^2 + 1}}$$

$$\Omega = a\frac{r_+^2 + a^2 + 1 + \sqrt{2r_+^2 + 1 + 2a^2}}{r_+^2 + 2a^2 + a^2\sqrt{2r_+^2 + 1 + 2a^2}}$$

$$S = \frac{\pi^2}{2G_5}\frac{r_+\left(r_+^2 + 2a^2 + a^2\sqrt{2r_+^2 + 2a^2 + 1}\right)}{(1-a^2)^2}$$

$$r_+^2 \geq \frac{1}{4}\left(\sqrt{1 + 8a^2} - 1\right)$$

$$\left(Q^3 + \frac{\pi}{4G_5}J^2\right) - \left(3Q + \frac{\pi}{4G_5}\right)\left(3Q^2 - \frac{\pi}{2G_5}J\right) = 0$$

$$M - 3Q - J = \frac{\pi}{4G_5}\frac{3-a}{(1-a)^2(1+a)^3}\left(\frac{1}{2}(r_+^2 + a^2)(3r_+^2 + 2 + a^2) - (1\right.$$

$$\left. + 2a)\left(r_+^2\sqrt{2r_+^2 + 1 + 2a^2} - a^2\right)\right) \geq 0$$

$$r_*^2 = a(2+a)$$

$$\frac{r^2}{(1-a^2)(r^2+a^2)^2}\Delta_r e^{U_2 - U_1} = e^{-U_2}\omega^2.$$

$$ds^2 = -A^2(r-r_+)^2dt^2 + \frac{B^2}{(r-r_+)^2}dr^2 + \frac{1}{4}C^2(\sigma_1^2 + \sigma_2^2)$$
$$+\frac{1}{4}D^2(\sigma_3 + E(r-r_+)dt)^2$$

$$\sigma_{3\text{as}} = \sigma_{3\text{hor}} - \omega|_{r=r_+}dt = \sigma_{3\text{hor}} + 2\Omega dt$$

$$A = \frac{2\sqrt{(r_+^2 + a^2)(3r_+^2 + 2a^2 + 1)}}{r_+^2 + 2a^2 + a^2\sqrt{2r_+^2 + 1 + 2a^2}}$$

$$B = \sqrt{\frac{r_+^2 + a^2}{4(3r_+^2 + 2a^2 + 1)}}$$

$$C = \sqrt{\frac{r_+^2 + a^2}{1-a^2}}$$

$$D = \frac{r_+}{(1-a^2)(r_+^2 + a^2)}\left(r_+^2 + 2a^2 + a^2\sqrt{2r_+^2 + 1 + 2a^2}\right)$$

$$E = \frac{4a(1-a^2)}{r_+}\frac{r_+^2 + (2r_+^2 + a^2)\sqrt{2r_+^2 + 1 + 2a^2}}{\left(r_+^2 + 2a^2 + a^2\sqrt{2r_+^2 + 1 + 2a^2}\right)^2}$$

$$\mathcal{A}_t|_{r=r_+} = 0$$

$$\mathcal{A} = -F(r - r_+)dt - G\sigma_3$$

$$\mathcal{F} = d\mathcal{A} = -Fdrdt - G\sigma_1\sigma_2$$

$$\mathcal{A} = \frac{q}{(1-a^2)}(1 - a\Omega)\left(\frac{1}{r^2+a^2} - \frac{1}{r_+^2+a^2}\right)dt - \frac{qa}{2(1-a^2)(r^2+a^2)}\sigma_3$$

$$\mathcal{A} = -\frac{2qr_+}{(r_+^2+a^2)\left(r_+^2+2a^2+a^2\sqrt{2r_+^2+1+2a^2}\right)}(r - r_+)dt - \frac{qa}{2(1-a^2)(r_+^2+a^2)}\sigma_3$$

$$F = \frac{2\left(r_+^2\sqrt{1+2a^2+2r_+^2} - a^2\right)r_+}{(r_+^2+a^2)\left(r_+^2+2a^2+a^2\sqrt{2r_+^2+1+2a^2}\right)}$$

$$G = \frac{\left(r_+^2\sqrt{1+2a^2+2r_+^2} - a^2\right)a}{2(1-a^2)(r^2+a^2)}$$

$$\mathcal{A} \wedge\star \mathcal{A} = \mathcal{A}_\mu\mathcal{A}^\mu \star 1 = \left(\frac{4G^2}{D^2} - \frac{(F-EG)^2}{A^2}\right)\star 1$$

$$= \frac{\left(a^2 - r_+^2\sqrt{1+2a^2+2r_+^2}\right)^2\left(-r_+^2 + a^2\left(1+3r_+^2+3a^2+2\sqrt{1+2a^2+2r_+^2}\right)\right)}{(r_+^2+a^2)(1+2a^2+3r_+^2)\left(r_+^2+a^2\left(2+\sqrt{1+2a^2+2r_+^2}\right)\right)^2}\star 1$$

$$\mathcal{F} \wedge\star \mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \star 1 = \left(\frac{16G^2}{C^4} - \frac{F^2}{A^2B^2}\right)\star 1$$

$$= \frac{4(a^2 - r_+^2)\left(a^2 - r_+^2\sqrt{1+2a^2+2r_+^2}\right)^2}{(r_+^2+a^2)^4}\star 1$$

$$E = \frac{16AB(ABD + C^2F)}{C^4DF}G$$

$$R_{\mu\nu} + 4g_{\mu\nu} - \left(\frac{3}{2}\mathcal{F}_{\mu\lambda}\mathcal{F}_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}\mathcal{F}_{\lambda\rho}\mathcal{F}^{\lambda\rho}\right) = 0$$

$$B = \frac{C}{2\sqrt{1+3C^2}}$$

$$E = \frac{1}{CD}\sqrt{\frac{2A^2D^2}{1+3C^2} + 8A^2C^2 - 6C^2F^2}$$

$$G = \frac{1}{2\sqrt{2}A}\sqrt{A^2(2C^2+2C^4-D^2) - C^2(1+3C^2)F^2}$$

$$ds_2^2 = -A^2(r - r_+)^2dt^2 + \frac{B^2}{(r-r_+)^2}dr^2$$

$$z = \frac{1}{r - r_+}, \tau = \frac{B}{A}t$$

$$ds_2^2 = \frac{B^2}{z^2}(-d\tau^2 + dz^2)$$

$$(\Box - m^2)\Phi = \left(\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\right) - m^2\right)\Phi$$
$$= \left(\frac{1}{B^2}\partial_r((r-r_+)^2\partial_r) + \frac{\omega^2}{A^2(r-r_+)^2} - m^2\right)\Phi = 0$$

$$\left(-\frac{\partial^2}{\partial z^2} + \frac{m^2B^2}{z^2}\right)\Phi = \frac{B^2}{A^2}\omega^2\Phi$$

$$m^2\ell_2^2 \geq -\frac{1}{4}$$

$$\Delta_0 = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2\ell_2^2}$$

$$\ell_2 = B = \sqrt{\frac{r_+^2 + a^2}{4(3r_+^2 + 2a^2 + 1)}},$$

$$\mathcal{L} = d\Phi_p \wedge\star\, d\Phi - m^2\Phi_p \wedge\star\, \Phi \;\leftrightarrow\; d\star d\Phi - (-1)^p m^2 \star \Phi_p = 0$$

$$d\star\Phi_p = 0\,, (m^2 \neq 0)$$

$$D\phi = d\phi + ie\mathcal{A}\phi$$

$$\mathcal{L} = -\frac{1}{2}d\phi \wedge d\phi^* - \frac{1}{2}(m^2 \star 1 + e^2\mathcal{A} \wedge\star \mathcal{A} - p\mathcal{F} \wedge\star \mathcal{F})\phi\phi^*$$

$$m_{\text{eff}}^2 = m^2 + e^2\mathcal{A}_\mu\mathcal{A}^\mu - \frac{1}{2}p\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$$

$$m_{t^+}^2 = -4 + 4\mathcal{A}_\mu\mathcal{A}^\mu - \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$$
$$= -4 + \frac{4\left(a^2\left(2\sqrt{2a^2+2r_+^2+1} + 3a^2 + 3r_+^2 + 1\right) - r_+^2\right)\left(a^2 - r_+^2\sqrt{2a^2+2r_+^2+1}\right)^2}{(a^2+r_+^2)(2a^2+3r_+^2+1)\left(a^2\left(\sqrt{2a^2+2r_+^2+1}+2\right)+r_+^2\right)^2}$$
$$+ \frac{8(r_+^2 - a^2)\left(a^2 - r_+^2\sqrt{2a^2+2r_+^2+1}\right)^2}{(a^2+r_+^2)^4}$$

$$m_{t^+}^2\ell_2^2 = \frac{1 - \sqrt{1+8a^2}}{2\sqrt{1+8a^2}} \in \left(-\frac{1}{3}, 0\right)$$
$$m_{t^+}^2\ell_2^2 = \frac{2 + 6a - 4a^2}{(1+5a)^2} \in \left(\frac{1}{9}, 2\right)$$
$$m_{t^+}^2\ell_2^2 = \frac{2 + 8r_+^2 + 7r_+^4}{(1+3r_+^2)^2} \in \left(\frac{7}{9}, 2\right)$$

$$\varphi^{(\mathbf{1})}: m_{\varphi(\mathbf{1})}^2 = -3 + 9\mathcal{A}_\mu\mathcal{A}^\mu$$

$$\varphi^{(3)}: m^2_{\varphi(3)} = -3 + \mathcal{A}_\mu \mathcal{A}^\mu$$

$$\varphi^{(6)}: m^2_{\varphi(6)} = -3 + \mathcal{A}_\mu \mathcal{A}^\mu - 2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$$

$$m^2 \ell_2^2 = \frac{3\left(1-\sqrt{1+8a^2}\right)}{8\sqrt{1+8a^2}} \in \left(-\frac{1}{4}, 0\right)$$

$$m^2_{\varphi(1)} \ell_2^2 = -\frac{6a(1-a)}{(1+5a)^2} \in \left(-\frac{1}{4}, 0\right)$$
$$m^2_{\varphi(3)} \ell_2^2 = -\frac{2a(1+3a)}{(1+5a)^2} \in \left(-\frac{2}{9}, 0\right)$$
$$m^2_{\varphi(6)} \ell_2^2 = \frac{2(2+9a-3a^2)}{(1+5a)^2} \in \left(\frac{4}{9}, 4\right)$$

$$m^2_{\varphi(1)} \ell_2^2 = \frac{-3r_+^2(4+9r_+^2)}{4(1+3r_+^2)^2} \in \left(-\frac{3}{4}, 0\right)$$
$$m^2_{\varphi(3)} \ell_2^2 = \frac{-r_+^2(4+11r_+^2)}{4(1+3r_+^2)^2} \in \left(-\frac{11}{36}, 0\right)$$
$$m^2_{\varphi(6)} \ell_2^2 = \frac{16+76r_+^2+85r_+^2}{4(1+3r_+^2)^2} \in \left(\frac{85}{36}, 4\right)$$

$$d \star dt_- + 4g^2 t_- \star 1 - t_-(\mathcal{F} \wedge \star\, \mathcal{F}) = 2\mathcal{F} \wedge \star\, f_-$$
$$d \star f_- - \mathcal{F} \wedge f_- = 2d(t_- \star \mathcal{F})$$

$$a_- = a_t^- dt + a_r^- dr + \sum_{i=1}^{3} a_i^- \sigma_i$$

$$\left[\frac{1}{A^2(r-r_+)^2}\partial_t^2 - \frac{1}{B^2}\partial_r((r-r_+)^2\partial_r) + \left(\frac{4}{D^2} - \frac{E^2}{A^2} - \frac{2F}{ABD}\right)\right] a_1^- = \frac{2E}{A^2(r-r_+)}\partial_t a_2^-$$
$$\left[\frac{1}{A^2(r-r_+)^2}\partial_t^2 - \frac{1}{B^2}\partial_r((r-r_+)^2\partial_r) + \left(\frac{4}{D^2} - \frac{E^2}{A^2} - \frac{2F}{ABD}\right)\right] a_2^- = -\frac{2E}{A^2(r-r_+)}\partial_t a_1^-$$

$$a_\perp^- = a_1^- + i a_2^-$$

$$\left[\frac{1}{A^2(r-r_+)^2}\partial_t^2 - \frac{1}{B^2}\partial_r((r-r_+)^2\partial_r) + \left(\frac{4}{D^2} - \frac{E^2}{A^2} - \frac{2F}{ABD}\right)\right] a_\perp^- = -\frac{2iE}{A^2(r-r_+)}\partial_t a_\perp^-$$

$$\left(\Box_2 - m^2_{\text{eff}}\right)\chi = 0$$

$$\Box_2 = -\frac{1}{A^2(r-r_+)^2}\partial_t^2 + \frac{1}{B^2}\partial_r((r-r_+)^2\partial_r)$$

$$m^2_{\perp -} = \frac{4}{D^2} - \frac{E^2}{A^2} - \frac{2F}{ABD}$$

$$0 = d\left(\partial_r a_t^- - \partial_t a_r^- - Er\partial_r a_3^- + 2Ft_- + \frac{8ABG}{DC^2} a_3^-\right)$$

$$\partial_r a_t^- - \partial_t a_r^- - Er\partial_r a_3^- + 2Ft_- + \frac{8ABG}{DC^2} a_3^- = 0$$

$$\square_2 \begin{pmatrix} a_3^- \\ t^- \end{pmatrix} - M_{\text{eff}}^2 \begin{pmatrix} a_3^- \\ t^- \end{pmatrix} = 0$$

$$M_{\text{eff}}^2 = \begin{pmatrix} \frac{4D^2}{C^4} - \frac{2D(F-EG)}{ABC^2} + \frac{16G^2}{C^4} & \frac{16DG}{C^4} + \frac{8FG}{ABC^2} \\ \frac{16DG}{C^4} + \frac{8FG}{ABC^2} & -4g^2 + \frac{16G^2}{C^4} + \frac{3F^2}{A^2B^2} \end{pmatrix}$$

$$m_{\perp -}^2 \ell_2^2 = -\frac{8(1-a^2)^2}{(8a^2+1)\left(3+\sqrt{8a^2+1}\right)^2} \in \left(-\frac{1}{2}, 0\right),$$
$$m_{\perp -}^2 \ell_2^2 = -\frac{6a(1-a)}{(1+5a)^2} \in \left(-\frac{1}{4}, 0\right),$$
$$m_{\perp -}^2 \ell_2^2 = \frac{1-\sqrt{1+2r_+^2}}{1+3r_+^2} \in \left(\sqrt{\frac{2}{3}} - 1{,}0\right).$$

$$\text{Kerr: } \{m_{3<}^2, m_{3>}^2\}\ell_2^2 = \left\{\frac{\sqrt{1+8a^2}-1}{2\sqrt{1+8a^2}}, \frac{2}{\sqrt{1+8a^2}}\right\} \in \left\{\left(-\frac{1}{3}, 0\right), \left(\frac{2}{3}, 2\right)\right\},$$
$$\text{BPS: } \{m_{3<}^2, m_{3>}^2\}\ell_2^2 = \left\{\frac{3+2a \pm \sqrt{9+20a+20a^2}}{2(1+5a)}\right\} \in \left\{\left(-\frac{1}{6}, 0\right), (1,3)\right\},$$
$$\text{RN: } \{m_{3<}^2, m_{3>}^2\}\ell_2^2 = \left\{\frac{4+5r_+^2+\sqrt{1+2r_+^2} \pm \sqrt{5+22r_+^2+25r_+^4+2(1+5r_+^2)\sqrt{1+2r_+^2}}}{2(1+3r_+^2)}\right\}$$
$$\in \left\{\left(-1+\sqrt{\frac{2}{3}}, 0\right), \left(\frac{5}{3}, 3\right)\right\}.$$

$$D \star f^+ + \mathcal{F} \wedge f^+ = 0$$

$$DB = i \star B$$

$$a_+ = a_t^+ dt + a_r^+ dr + \sum_{i=1}^{3} a_i^+ \sigma_i$$

$$a_\perp^+ = a_1^+ + i a_2^+$$

$$\left[\frac{1}{A^2(r-r_+)^2}\left(\partial_t + 2iE(r-r_+)\right)\partial_t - \frac{1}{B^2}\partial_r((r-r_+)^2\partial_r) + m_\perp^2\right] a_\perp^+ = 0,$$

$$m_{\perp +}^2 = \frac{4}{D^2} + \frac{2F}{ABD} - \frac{E^2}{A^2} + \frac{16G^2}{D^2} - \frac{4(F-EG)^2}{A^2}$$

$$m_{\perp+}^2\ell_2^2 = -\frac{8(1-a^2)^2}{(8a^2+1)\left(3+\sqrt{8a^2+1}\right)^2} \in \left(-\frac{1}{2},0\right)$$

$$m_{\perp+}^2\ell_2^2 = \frac{2-10a^2}{(1+5a)^2} \in \left(-\frac{2}{9},2\right)$$

$$m_{\perp+}^2\ell_2^2 = \frac{1+2r_+^2-2r_+^4+(1+3r_+^2)\sqrt{1+2r_+^2}}{(1+3r_+^2)^2} \in \left(-\frac{2}{9},2\right)$$

$$\partial_r a_t^+ - \partial_t a_r^+ - E(r-r_+)\partial_r a_3^+ + 2(F-EG)(r-r_+)ra_r^+ - \frac{8ABG}{C^2D}a_3^+ = 0$$

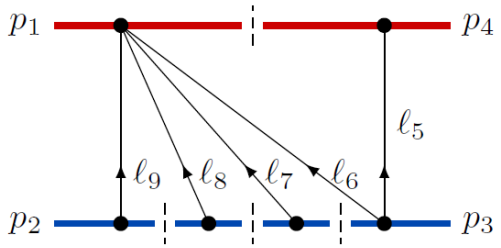
$$m_{3+}^2 = \frac{4D^2}{C^4} + \frac{2D(F-EG)}{ABC^2} - \frac{4(F-EG)^2}{A^2} + \frac{16G^2}{C^4}$$

$$m_{3+}^2\ell_2^2 = \frac{2}{\sqrt{1+8a^2}} \in \left(\frac{2}{3},2\right)$$

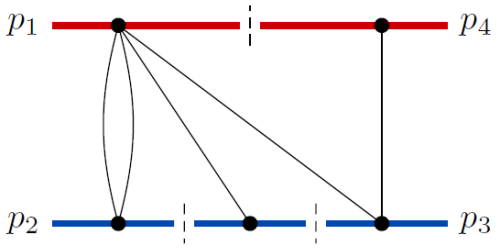
$$m_{3+}^2\ell_2^2 = \frac{4+24a+27a^2-a^3}{(2+a)(1+5a)^2} \in \left(\frac{1}{2},2\right)$$

$$m_{3+}^2\ell_2^2 = \frac{1-2r_+^4+\sqrt{1+2r_+^2}+r_+^2\left(2+3\sqrt{1+2r_+^2}\right)}{1+3r_+^2} \in \left(-\frac{2}{9},2\right)$$

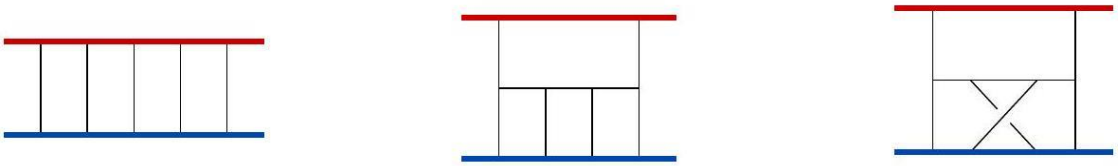
$$m_B^2 = 1 + \mathcal{A}^\mu\mathcal{A}_\mu = 1 + \frac{4G^2}{D^2} - \frac{(F-EG)^2}{A^2}$$



(a)



(b)



(a) (b) (c)

$$\mathcal{C}_{(a)} = \delta((p_2+\ell_9)^2-m_2^2)\delta((p_2+\ell_{89})^2-m_2^2)\delta((p_2+\ell_{789})^2-m_2^2)$$
$$\times\,\delta((p_4-\ell_5)^2-m_1^2)\prod_{i=5}^{9}\delta(\ell_i^2)M_0(p_1,-\ell_9,-\ell_8,-\ell_7,-\ell_6)M_0(p_4,-\ell_5)$$
$$\times\,M_0(p_2,\ell_9)M_0(p_2+\ell_9,\ell_8)M_0(p_2+\ell_{89},\ell_7)M_0(p_2+\ell_{879},\ell_6,\ell_5),$$

$$M_L = G^{L+1}\sum_i \int \prod_{j=1}^{L} \frac{\mathrm{d}^D \ell_j}{(2\pi)^D} \frac{N_i(p_e, \ell_j)}{\prod_{\alpha_i}\left(k_{\alpha_i}^2 - m_{\alpha_i}^2 + \mathrm{i}\varepsilon\right)},$$

$$k_1 = \begin{pmatrix} p_1 \\ 0 \\ m_1 \end{pmatrix}, k_2 = \begin{pmatrix} p_2 \\ m_2 \sin\,\phi \\ m_2 \cos\,\phi \end{pmatrix}, k_3 = \begin{pmatrix} p_3 \\ -m_2 \sin\,\phi \\ -m_2 \cos\,\phi \end{pmatrix}, k_4 = \begin{pmatrix} p_4 \\ 0 \\ -m_1 \end{pmatrix},$$

$$(k_1 + k_2)^2 \to s - \left|m_1 + m_2 e^{\mathrm{i}\phi}\right|^2, (k_1 + k_4)^2 \to t, (k_1 + k_3)^2 \to u - \left|m_1 + m_2 e^{\mathrm{i}\phi}\right|^2$$

$$s, u, m_1^2, m_2^2 \sim J^2 |t| \gg |t| = |q^2|$$

hard: $\ell \gg |q|$, soft: $\ell \sim |q|$,

potential: $\ell \sim (v, \mathbf{1})|q|$, radiation: $\ell \sim (v, \boldsymbol{v})|q|$.

$$p_1 = -\bar{m}_1 u_1 + \frac{q}{2}, p_2 = -\bar{m}_2 u_2 - \frac{q}{2}, p_3 = \bar{m}_2 u_2 - \frac{q}{2}, p_4 = \bar{m}_1 u_1 + \frac{q}{2},$$

$$u_1^2 = u_2^2 = 1, u_1 \cdot q = u_2 \cdot q = 0, \bar{m}_i^2 = m_i^2\left(1 - \frac{q^2}{4m_i^2}\right)$$

$$y = u_1 \cdot u_2$$

$$y = \sigma + \mathcal{O}(q^2), \text{ where } \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

$$\delta((p_2 + \ell_9)^2 - m_2^2) = \delta(2p_2 \cdot \ell_9) = \delta(-2\bar{m}_2 u_2 \cdot \ell_9 + q \cdot \ell_9)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \delta^{(n)}(-2\bar{m}_2 u_2 \cdot \ell_9)(q \cdot \ell_9)^n$$

$$\frac{1}{(p_2 + \ell_9)^2 - m_2^2 + \mathrm{i}\varepsilon} = \frac{1}{-2\bar{m}_2 u_2 \cdot \ell_9 + q \cdot \ell_9 + \mathrm{i}\varepsilon} = \sum_{n=0}^{\infty} \frac{(-q \cdot \ell_9)^n}{(-2\bar{m}_2 u_2 \cdot \ell_9 + \mathrm{i}\varepsilon)^{n+1}},$$

$$\frac{1}{x + \mathrm{i}\varepsilon} - \frac{1}{x - \mathrm{i}\varepsilon} = -2\pi\mathrm{i}\delta(x), \frac{1}{(x + \mathrm{i}\varepsilon)^{n+1}} - \frac{1}{(x - \mathrm{i}\varepsilon)^{n+1}} = -2\pi\mathrm{i}\frac{(-1)^n}{n!}\delta^{(n)}(x)$$

$$\delta(\omega_1 + \cdots \omega_n) \sum_{\text{perms } \omega_i} \frac{1}{\omega_1 + \mathrm{i}\varepsilon} \cdots \frac{1}{\omega_1 + \cdots + \omega_{n-1} + \mathrm{i}\varepsilon} = (-2\pi\mathrm{i})^n \delta(\omega_1) \cdots \delta(\omega_n),$$

$$\mathrm{i}\mathcal{M}(\boldsymbol{q}) = \int_J \left(e^{\mathrm{i}I_r(J)} - 1\right)$$

$$\tilde{I}_r(\boldsymbol{q}) = \int_J I_r(J) := \mu^{-2\epsilon} 4E|\boldsymbol{p}| \int \mathrm{d}^{D-2}\boldsymbol{b} e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{b}} I_r(J)$$

$$E = \sqrt{s} = \sqrt{2m_1 m_2 \sigma + m_1^2 + m_2^2}, |\boldsymbol{p}| = \frac{m_1 m_2 \sqrt{\sigma^2 - 1}}{E}$$

$$\int_J \frac{(\mathrm{i}I_r(J))^n}{n!} = \mathrm{i}\int_{\boldsymbol{\ell}} \frac{\tilde{I}_r(\boldsymbol{\ell}_1)\ldots\tilde{I}_r(\boldsymbol{\ell}_n)}{Z_1\ldots Z_{n-1}}$$

$$Z_j = -4E|\boldsymbol{p}|\left((\boldsymbol{\ell}_1+\boldsymbol{\ell}_2+\cdots+\boldsymbol{\ell}_j)\cdot\hat{\boldsymbol{z}}+\mathrm{i}\varepsilon\right)$$

$$\int_{\boldsymbol{\ell}} := \mu^{2(n-1)\epsilon}\int\left(\prod_{i=1}^{n}\frac{\mathrm{d}^{D-1}\boldsymbol{\ell}_i}{(2\pi)^{D-1}}\right)(2\pi)^{D-1}\delta^{(D-1)}\left(\sum_{j=1}^{n}\boldsymbol{\ell}_j-\boldsymbol{q}\right)$$

$$\mathcal{M}_1 = \tilde{I}_{r,1}, \mathcal{M}_2 = \tilde{I}_{r,2} + \int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,1}}{Z_1}$$

$$\mathcal{M} = \sum_{k=1}\mathcal{M}_k, \tilde{I}_r = \sum_{k=1}\tilde{I}_{r,k}$$

$$\mathcal{M}_k = \mathcal{O}(G^k), \tilde{I}_{r,k} = \mathcal{O}(G^k)$$

$$\mathcal{M}_5 = \tilde{I}_{r,5} + \mathcal{M}_5^{\mathrm{it}}$$

$$\mathcal{M}_5^{\mathrm{it.}} = \int_{\ell}\frac{\tilde{I}_{r,1}^5}{Z_1Z_2Z_3Z_4} + \int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,1}\tilde{I}_{r,1}\tilde{I}_{r,2}}{Z_1Z_2Z_3} + \int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,2}\tilde{I}_{r,2}}{Z_1Z_2} + \int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1Z_2} + \int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,4}}{Z_1} + \int_{\ell}\frac{\tilde{I}_{r,2}\tilde{I}_{r,3}}{Z_1}$$

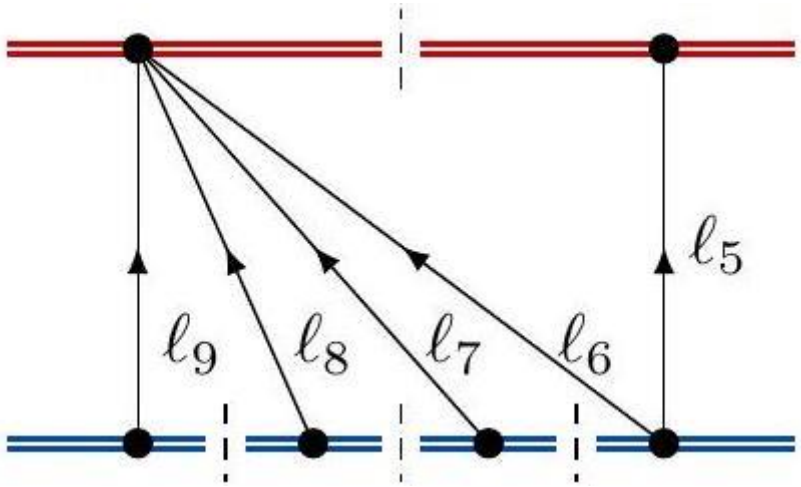
$$\tilde{I}_{r,1}\tilde{I}_{r,4} := \tilde{I}_{r,1}(\boldsymbol{\ell}_1)\tilde{I}_{r,4}(\boldsymbol{\ell}_2) + \tilde{I}_{r,4}(\boldsymbol{\ell}_1)\tilde{I}_{r,1}(\boldsymbol{\ell}_2),$$

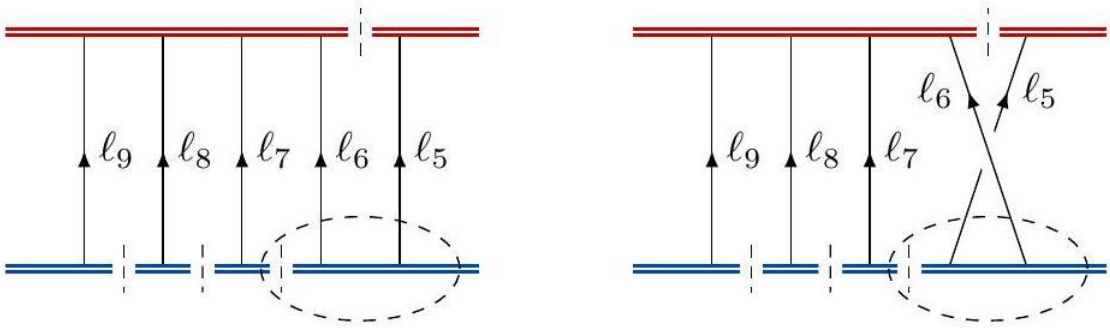
$$\chi = -\frac{\partial I_r(J)}{\partial J}$$

$$\chi = \sum_{k=1}\chi_k, \chi_k = \mathcal{O}(G^k)$$

$$\tilde{I}_{r,5} = m_1^2m_2^2(m_1^4+m_2^4)\tilde{I}_{r,5}^{0\mathrm{SF}} + m_1^3m_2^3(m_1^2+m_2^2)\tilde{I}_{r,5}^{1\mathrm{SF}} + m_1^4m_2^4\tilde{I}_{r,5}^{2\mathrm{SF}}$$
$$= M^8\nu^2\left[\tilde{I}_{r,5}^{0\mathrm{SF}} + \nu\left(\tilde{I}_{r,5}^{1\mathrm{SF}} - 4\tilde{I}_{r,5}^{0\mathrm{SF}}\right) + \nu^2\left(\tilde{I}_{r,5}^{2\mathrm{SF}} - 2\tilde{I}_{r,5}^{1\mathrm{SF}} + 2\tilde{I}_{r,5}^{0\mathrm{SF}}\right)\right]$$

$$M = m_1 + m_2, \text{ and } \nu = \frac{m_1m_2}{M^2},$$





$$\delta(\ell_{56}\cdot u_2)\frac{1}{2\ell_6\cdot u_2+\mathrm{i}\varepsilon}=-\delta(\ell_{56}\cdot u_2)\frac{1}{2\ell_5\cdot u_2-\mathrm{i}\varepsilon},$$

$$\delta^{(n)}(\ell_{56}\cdot u_2)\frac{1}{2\ell_6\cdot u_2+\mathrm{i}\varepsilon}\neq-\delta^{(n)}(\ell_{56}\cdot u_2)\frac{1}{2\ell_5\cdot u_2-\mathrm{i}\varepsilon},n\neq 0$$

$$\check{u}_2\cdot\partial_{\ell_5},\ \text{with}\ \check{u}_2^{\mu}=\frac{yu_1^{\mu}-u_2^{\mu}}{y^2-1}$$

$$\check{u}_2\cdot u_2=1,\check{u}_2\cdot u_1=0,\check{u}_2\cdot q=0$$

$$\partial_x\vec{I}=A(x,\epsilon)\vec{I},y=\frac{1+x^2}{2x},$$

$$\sqrt{y^2-1}=\frac{1-x^2}{2x}$$

$$\partial_x\vec{J}=\epsilon B(x)\vec{J},\vec{J}=T(\epsilon,x)\vec{I}.$$

$$\Psi(x):=\frac{K(1-x^2)}{\pi},$$

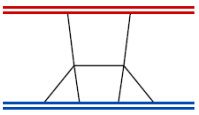
$$K(z):=\int_0^1\frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-zt^2}}.$$

$$B(x)=\sum_{i=1}^{10}b_ik_i(x)$$

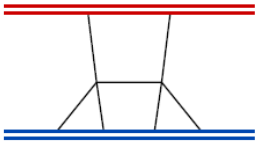
$$\mathbb{W}=\left\{\frac{1}{x},\frac{1}{x+1},\frac{1}{x-1},\frac{x}{x^2+1},\frac{1}{x(1-x^2)\Psi(x)^2},\frac{\Psi(x)^2}{x},x\Psi(x)^2\right.$$
$$\left.\frac{x\Psi(x)^2}{1-x^2},\frac{(1+x^2)\Psi(x)^2}{1-x^2},\frac{(1+x^2)(x^2-2x-1)(x^2+2x-1)\Psi(x)^4}{x(1-x^2)}\right\}$$

$$\Gamma_{i_1,\ldots i_n}(x)=\int_1^x\mathrm{d}x_nk_{i_n}(x_n)\cdots\int_1^{x_2}\mathrm{d}x_1k_{i_1}(x_1)$$

$$\begin{pmatrix} D_{1,1}(\epsilon,x) & 0 & 0 \\ D_{2,1}(\epsilon,x) & D_{2,2}(\epsilon,x) & 0 \\ \vdots & \vdots & \ddots \end{pmatrix} \xrightarrow{T} \begin{pmatrix} \epsilon D'_{1,1}(x) & 0 & 0 \\ D'_{2,1}(\epsilon,x) & \epsilon D'_{2,2}(x) & 0 \\ \vdots & \vdots & \ddots \end{pmatrix} \xrightarrow{T'} \begin{pmatrix} \epsilon D''_{1,1}(x) & 0 & 0 \\ \epsilon D''_{2,1}(x) & \epsilon D''_{2,2}(x) & 0 \\ \vdots & \vdots & \ddots \end{pmatrix}$$



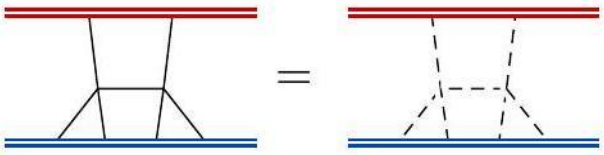
$$= \int \prod_{i=1}^{4} \frac{\mathrm{d}^D \ell_i}{(2\pi)^D} \frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 \left(\ell_{12} - \ell_4\right)^2 \ell_4^2 \left(\ell_{123} + \hat{q}\right)^2} \times \frac{1}{\left(\ell_4 + \hat{q}\right)^2 \left(-2u_1 \cdot \ell_4\right) \left(2u_2 \cdot \ell_1\right) \left(2u_2 \cdot \ell_{12}\right) \left(2u_2 \cdot \ell_{123}\right)}$$



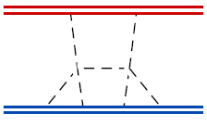
$$= c\, x \Psi^2 + \mathcal{O}(\epsilon)$$

$$u_1 = (y, 0, 0, v), u_2 = (1, 0, 0, 0), q = \left(0, \boldsymbol{q}_x, \boldsymbol{q}_y, 0\right), y = \frac{1+x^2}{2x}, v = \frac{1-x^2}{2x}.$$

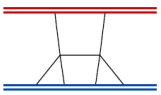
$$\frac{1}{\ell^2} = \frac{1}{\omega^2 - \ell^2} = -\frac{1}{\ell^2} \sum_{k=0}^{\infty} \frac{\omega^{2k}}{(\ell^2)^k}$$



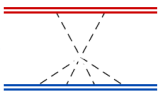
$$+ \mathcal{O}(v^2)\,.$$



$$= \int \prod_{i=1}^{4} \frac{\mathrm{d}\omega_i}{2\pi} \frac{\mathrm{d}^{D-1} \boldsymbol{\ell}_i}{(2\pi)^{D-1}} \frac{1}{\boldsymbol{\ell}_1^2 \boldsymbol{\ell}_2^2 \boldsymbol{\ell}_3^2 \left(\boldsymbol{\ell}_{12} - \boldsymbol{\ell}_4\right)^2 \boldsymbol{\ell}_4^2 \left(\boldsymbol{\ell}_{123} + \hat{\boldsymbol{q}}\right)^2} \times \frac{1}{\left(\boldsymbol{\ell}_4 + \hat{\boldsymbol{q}}\right)^2 \left(-2(\omega_4 - \hat{\boldsymbol{z}} \cdot \boldsymbol{\ell}_4)\right) 2\omega_1\, 2\omega_{12}\, 2\omega_{123}},$$

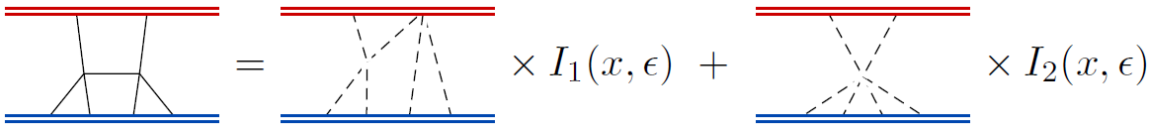


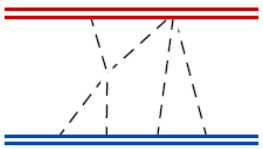
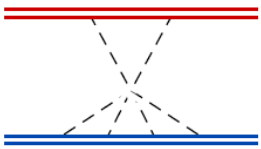
$$= \quad \times \left[\frac{1-10\epsilon}{2\epsilon} + \mathcal{O}(v^2)\right] +$$



$$\times \left[\frac{(4\epsilon-1)(6\epsilon-1)}{8\epsilon^2} + \mathcal{O}(v^2)\right].$$

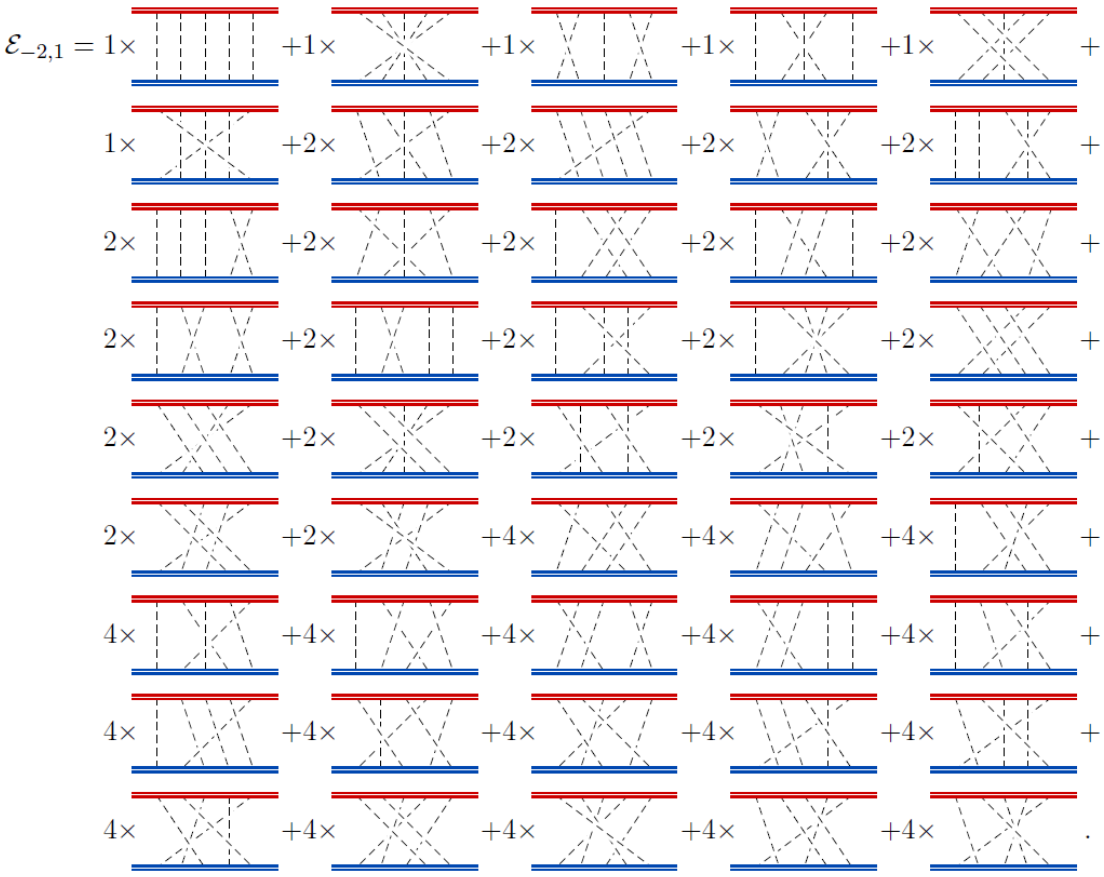
$$B(x)\vec{J}^{(0)} = \frac{1}{x-1}\left(b_3 - 2b_5 - \frac{1}{8}b_8 - \frac{1}{4}b_9 + \frac{1}{4}b_{10}\right)\vec{J}^{(0)} + \mathcal{O}((1-x)^0)$$
$$= \mathcal{O}((1-x)^0)\,,$$



$$\vec{J}^{(0)} = r_1(\epsilon)\vec{c}_1$$  $$+ r_2(\epsilon)\vec{c}_2$$  $$+ \cdots$$

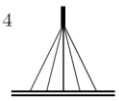
$$r_1(\epsilon) = \frac{1-10\epsilon}{2\epsilon}, r_2(\epsilon) = \frac{(4\epsilon-1)(6\epsilon-1)}{8\epsilon^2}$$

$$\mathcal{M}_5 = \left(\frac{|\boldsymbol{q}|^2}{\mu^2}\right)^{-4\epsilon}\left[\sum_{k=-2}^{2}|\boldsymbol{q}|^k\sum_{n=1}^{N_k}f_{k,n}(\sigma,\epsilon)\mathcal{E}_{k,n}(\epsilon) + \mathcal{O}(|\boldsymbol{q}|^3)\right]$$

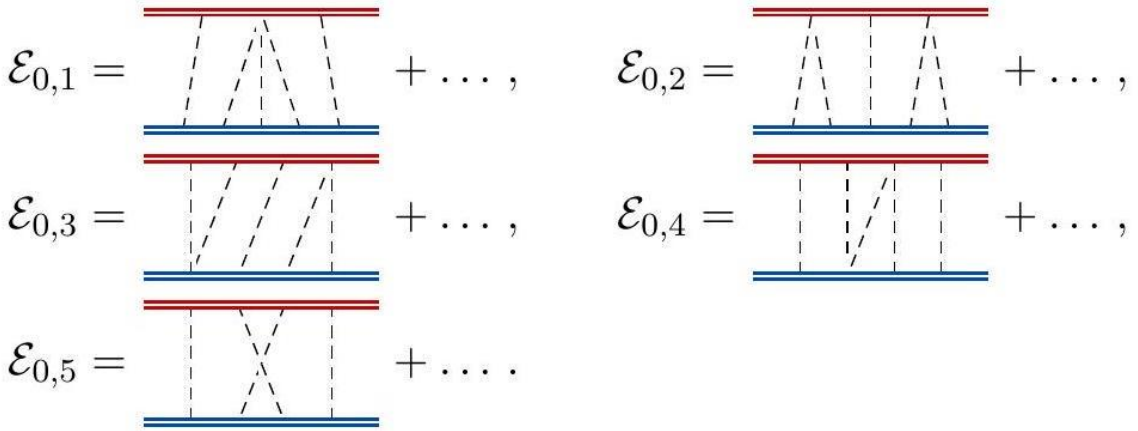


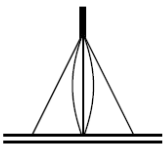
$$f_{-2,1} = \frac{\left(8\pi G 4 m_1^2 m_2^2 c_\phi^2\right)^5}{\left(m_1 m_2 \sqrt{\sigma^2 - 1}\right)^4} \propto \tilde{I}^5_{r,0}$$

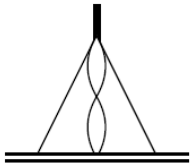
$$\frac{5!}{120}\mathcal{E}_{-2,1} = \left(\frac{\mathrm{i}}{2}\right)^4$$

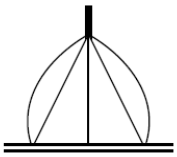


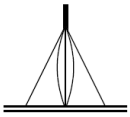
$$= \frac{1}{16}\int \prod_{i=1}^{4} \frac{\mathrm{d}^{3-2\epsilon}\boldsymbol{\ell}_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\boldsymbol{\ell}_1^2\boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^2\boldsymbol{\ell}_4^2(\boldsymbol{\ell}_{1234}+\hat{\boldsymbol{q}})^2} \frac{1}{(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_1)(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_{12})(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_{123})(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_{1234})}$$



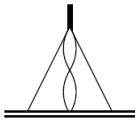
$$\frac{4!}{60}\mathcal{E}_{0,1} = \frac{4!}{144}\mathcal{E}_{0,4} = \left(\frac{\mathrm{i}}{2}\right)^4$$ 

$$\frac{4!}{72}\mathcal{E}_{0,5} = \left(\frac{\mathrm{i}}{2}\right)^4$$ 

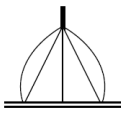
$$\frac{4!}{90}\mathcal{E}_{0,2} = \frac{4!}{72}\mathcal{E}_{0,3} = \left(\frac{\mathrm{i}}{2}\right)^4$$ 



$$= \left(\frac{1}{16\pi^2\epsilon} + \mathcal{O}(\epsilon^0)\right) \int \prod_{i=1}^{2} \frac{\mathrm{d}^{3-2\epsilon}\boldsymbol{\ell}_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\boldsymbol{\ell}_1^2(\boldsymbol{\ell}_2^2)^{2\epsilon}(\boldsymbol{\ell}_{12}+\hat{\boldsymbol{q}})^2} \frac{1}{(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_1)(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_{12})}$$



$$= \left(\frac{1}{16} + \mathcal{O}(\epsilon)\right) \int \prod_{i=1}^{2} \frac{\mathrm{d}^{3-2\epsilon}\boldsymbol{\ell}_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\boldsymbol{\ell}_1^2(\boldsymbol{\ell}_2^2)^{2\epsilon}(\boldsymbol{\ell}_{12}+\hat{\boldsymbol{q}})^2} \frac{1}{(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_1)(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_{12})}$$

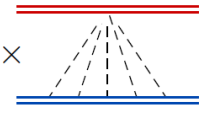


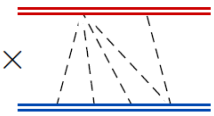
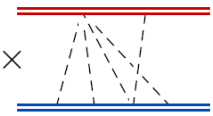
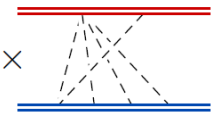
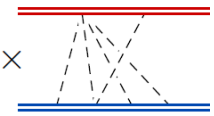
$$= \left(\frac{1}{16} + \mathcal{O}(\epsilon)\right) \int \prod_{i=1}^{2} \frac{\mathrm{d}^{3-2\epsilon}\boldsymbol{\ell}_i}{(2\pi)^{3-2\epsilon}} \frac{1}{(\boldsymbol{\ell}_1^2)^{1/2+2\epsilon}\boldsymbol{\ell}_2^2\left[(\boldsymbol{\ell}_{12}+\hat{\boldsymbol{q}})^2\right]^{1/2+2\epsilon}} \frac{1}{(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_1)(2\hat{\boldsymbol{z}}\cdot\boldsymbol{\ell}_{12})}$$

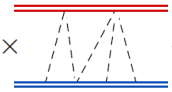
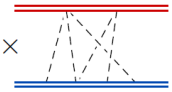
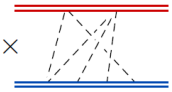
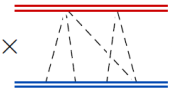
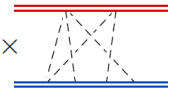
$$f_{0,1} = -(8\pi G)^5 \frac{\left(4m_1m_2c_\phi^2\right)^5(m_1^2+m_2^2)}{4m_1m_2(\sigma^2-1)^3} \propto \tilde{I}_{r,0}\tilde{I}_{r,2}^{0\mathrm{SF}}$$

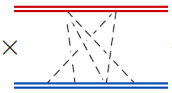
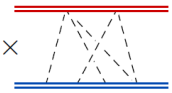
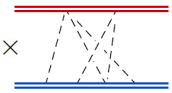
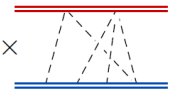
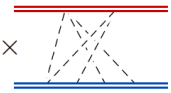
$$f_{0,4} = -(8\pi G)^5 \frac{(m_1m_2)^5\left(4c_\phi^2\right)^4}{(\sigma^2-1)^3}\left[\frac{2}{3}c_\phi^2\sigma - (\sigma^2-1)^{\frac{3}{2}}\mathrm{arccosh}(\sigma)\right] \propto \tilde{I}_{r,0}^2\tilde{I}_{r,2}^{1\mathrm{SF}}$$

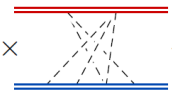
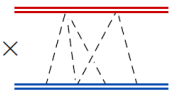
$$\tilde{I}_{r,2} = m_1^2m_2^2\left[(m_1^2+m_2^2)\tilde{I}_{r,2}^{0\mathrm{SF}} + m_1m_2\tilde{I}_{r,2}^{1\mathrm{SF}}\right]$$

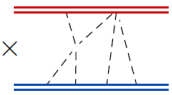
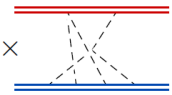
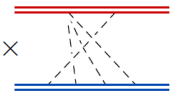
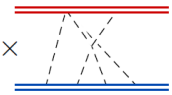
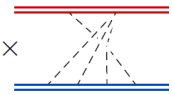
$$\mathcal{E}_{2,1} = 1\times$$ 

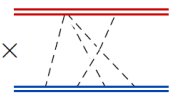
$\mathcal{E}_{2,2} = 1\times$  $+1\times$  $+1\times$  $+1\times$ 

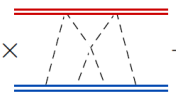
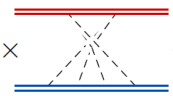
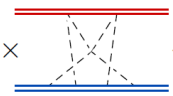
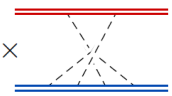
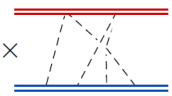
$\mathcal{E}_{2,3} = 1\times$  $+1\times$  $+1\times$  $+1\times$  $+1\times$  $+$

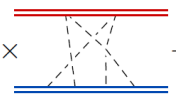
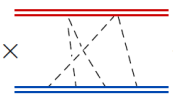
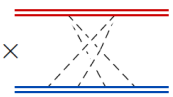
$1\times$  $+1\times$  $+1\times$  $+1\times$  $+1\times$  $+$

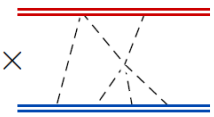
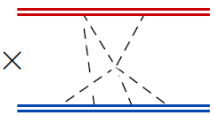
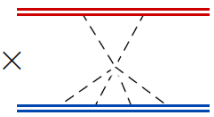
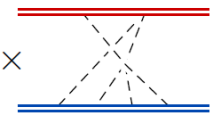
$1\times$  $+1\times$  .

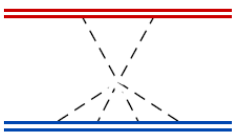
$\mathcal{E}_{2,4} = 1\times$  $+1\times$  $+1\times$  $+1\times$  $+1\times$  $+$

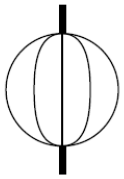
$1\times$  ,

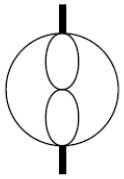
$\mathcal{E}_{2,5} = 1\times$  $+1\times$  $+1\times$  $+1\times$  $+2\times$  $+$

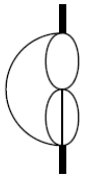
$2\times$  $+2\times$  $+2\times$  ,

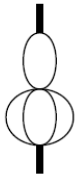
$\mathcal{E}_{2,6} = 1\times$  $+1\times$  $+1\times$  $+1\times$ 

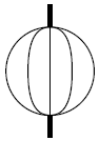
$\mathcal{E}_{2,7} =$ 

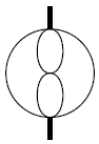
$$5! \times \mathcal{E}_{1,2} = \frac{4! \times 2!}{4} \times \mathcal{E}_{2,2} = \frac{4! \times 2!}{12} \times \mathcal{E}_{2,3} = \left(\frac{\mathrm{i}}{2}\right)^4$$  ,

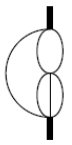
$$\frac{2! \times 4!}{6} \times \mathcal{E}_{2,4} = \frac{2! \times 4!}{12} \times \mathcal{E}_{2,5} = \left(\frac{\mathrm{i}}{2}\right)^4$$  ,

$$\frac{2! \times 4!}{4} \times \mathcal{E}_{2,6} = \left(\frac{\mathrm{i}}{2}\right)^4$$  ,

$$2! \times 4! \times \mathcal{E}_{2,7} = \left(\frac{\mathrm{i}}{2}\right)^4$$  .


$$= \int \prod_{i=1}^{4} \frac{\mathrm{d}^{3-2\epsilon}\boldsymbol{\ell}_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\boldsymbol{\ell}_1^2\boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^2\boldsymbol{\ell}_4^2(\boldsymbol{\ell}_{1234}+\hat{\boldsymbol{q}})^2} = \frac{1}{(16\pi^2)^4}\left[-\frac{16\pi^4}{3\epsilon}+\mathcal{O}(\epsilon^0)\right],$$


$$= \int \prod_{i=1}^{4} \frac{\mathrm{d}^{3-2\epsilon}\boldsymbol{\ell}_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\boldsymbol{\ell}_1^2\boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^2\boldsymbol{\ell}_4^2(\boldsymbol{\ell}_{123}+\hat{\boldsymbol{q}})^2\boldsymbol{\ell}_{234}^2} = \frac{1}{(16\pi^2)^4}\left[\frac{8\pi^6}{\epsilon}+\mathcal{O}(\epsilon^0)\right],$$


$$= \int \prod_{i=1}^{4} \frac{\mathrm{d}^{3-2\epsilon}\boldsymbol{\ell}_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\boldsymbol{\ell}_1^2\boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^2\boldsymbol{\ell}_4^2(\boldsymbol{\ell}_{123}+\hat{\boldsymbol{q}})^2\boldsymbol{\ell}_{1234}^2}$$

$$= \frac{1}{(16\pi^2)^4}\left[\frac{8\pi^4}{\epsilon^2}+\frac{32\pi^4}{\epsilon}(4-\gamma_{\mathrm{E}}+\ln(4\pi))+\mathcal{O}(\epsilon^0)\right],$$


$$= \int \prod_{i=1}^{4} \frac{\mathrm{d}^{3-2\epsilon}\boldsymbol{\ell}_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\boldsymbol{\ell}_1^2\boldsymbol{\ell}_2^2\boldsymbol{\ell}_3^2\boldsymbol{\ell}_4^2(\boldsymbol{\ell}_{123}+\hat{\boldsymbol{q}})^2(\boldsymbol{\ell}_4+\hat{\boldsymbol{q}})^2} = \mathcal{O}(\epsilon^0),$$

$$\tilde{I}_{r,5}^{0\mathrm{SF}} = G^5\pi c_\phi^4|\boldsymbol{q}|^2\left(\frac{|\boldsymbol{q}|^2}{\bar{\mu}^2}\right)^{-4\epsilon}\left[-\frac{4c_\phi^6}{5\epsilon(\sigma^2-1)^4}+\mathcal{O}(\epsilon^0)\right]$$

$$\tilde{I}_{r,5}^{1\mathrm{SF}} = G^5\pi c_\phi^4|\boldsymbol{q}|^2\left(\frac{|\boldsymbol{q}|^2}{\bar{\mu}^2}\right)^{-4\epsilon}\left[\frac{I_{r,5}^{1\mathrm{SF,div.}}}{\epsilon^2}+\frac{I_{r,5}^{1\mathrm{SF,fin.}}}{\epsilon}+\mathcal{O}(\epsilon^0)\right]$$

$$\tilde{I}_{r,5}^{1\mathrm{SF,\,div.}} = 16c_\phi^2\left\{\mathrm{F}_0\left[\frac{\sigma^2c_\phi^2}{(\sigma^2-1)^2}+\frac{4\sigma c_\phi}{\sigma^2-1}+4\right]+\frac{\sigma c_\phi^2}{(\sigma^2-1)^2}+\frac{2c_\phi}{\sigma^2-1}\right\}$$

$$\tilde{I}_{r,5}^{1\mathrm{SF,\,fin.}} = r_1 + r_2\,\mathrm{F}_0 + r_3\,\mathrm{F}_0^2 + r_4\,\mathrm{F}_1 + r_5\,\mathrm{F}_2$$

$$\mathrm{F}_0 = \frac{2x}{1-x^2}\ln(x)$$

$$\mathrm{F}_1 = \frac{2x}{1-x^2}\left[-\mathrm{Li}_2(1-x)-\mathrm{Li}_2(-x)-\ln(x)\ln(x+1)-\frac{1}{2}\zeta_2\right]$$

$$\mathrm{F}_2 = \frac{2x}{1-x^2}\left[-\mathrm{Li}_2(1-x)+\mathrm{Li}_2(-x)-\frac{1}{2}\ln^2(x)+\ln(x)\ln(x+1)+\frac{1}{2}\zeta_2\right]$$

$$r_1 = \frac{16c_\phi^3}{\sigma^2-1}\left[-\frac{\sigma(5\sigma^2-4)c_\phi^3}{5(\sigma^2-1)^3}-\frac{2c_\phi^2}{\sigma^2-1}+8\right]$$

$$r_2 = 32c_\phi^2\left[-\frac{\sigma^2c_\phi^4}{(\sigma^2-1)^3}-\frac{4\sigma c_\phi^3}{(\sigma^2-1)^2}-\frac{9(2\sigma^2-1)c_\phi^2}{2(\sigma^2-1)^2}-\frac{8\sigma c_\phi}{\sigma^2-1}+1\right]$$

$$r_3 = 16c_\phi\left[-\frac{\sigma^3c_\phi^5}{(\sigma^2-1)^3}-\frac{6\sigma^2c_\phi^4}{(\sigma^2-1)^2}+\frac{6\sigma(2-3\sigma^2)c_\phi^3}{(\sigma^2-1)^2}+\frac{(8-32\sigma^2)c_\phi^2}{\sigma^2-1}-\frac{92\sigma c_\phi}{3}-\frac{40}{3}(\sigma^2-1)\right]$$

$$r_4 = 32c_\phi\left[-\frac{\sigma c_\phi^3}{(\sigma^2-1)^2}-\frac{2c_\phi^2}{\sigma^2-1}-\frac{4\sigma c_\phi}{3(\sigma^2-1)}-\frac{8}{3}\right]$$

$$r_5 = 64c_\phi\left[\frac{\sigma^2c_\phi^3}{(\sigma^2-1)^2}+\frac{4\sigma c_\phi^2}{\sigma^2-1}+\frac{2(7\sigma^2-6)c_\phi}{3(\sigma^2-1)}+\frac{4\sigma}{3}\right]$$

$$16c_\phi^2\left[\mathrm{F}_0\left(\frac{\sigma^2c_\phi^2}{(\sigma^2-1)^2}+\frac{4\sigma c_\phi}{\sigma^2-1}+4\right)+\frac{\sigma c_\phi^2}{(\sigma^2-1)^2}+\frac{2c_\phi}{\sigma^2-1}\right]$$

$$\chi_5 = (m_1^4+m_2^4)\chi_5^{0\mathrm{SF}} + m_1m_2(m_1^2+m_2^2)\chi_5^{1\mathrm{SF}} + m_1^2m_2^2\chi_5^{2\mathrm{SF}}$$

$$|\boldsymbol{q}|^2\left(\frac{|\boldsymbol{q}|^2}{\bar{\mu}^2}\right)^{-4\epsilon} \mapsto \left(\mu^2\tilde{b}^2\right)^{5\epsilon}\frac{1}{b^5}\frac{\sqrt{2m_1m_2\sigma+m_1^2+m_2^2}}{\pi m_1^2m_2^2(\sigma^2-1)}\underbrace{\frac{2(e^{-9\epsilon\gamma_\mathrm{E}})\Gamma(3-5\epsilon)}{\Gamma(4\epsilon-1)}}_{=\left(-16\epsilon+184\epsilon^2+\mathcal{O}(\epsilon^3)\right)},$$

$$\chi_5^{0\mathrm{SF}} = \frac{G^5}{b^5}\left(\mu^2\tilde{b}^2\right)^{5\epsilon}c_\phi^4\frac{\sqrt{2m_1m_2\sigma+m_1^2+m_2^2}}{(\sigma^2-1)}\left[\frac{64c_\phi^6}{5(\sigma^2-1)^4}\right]+\mathcal{O}(\epsilon)$$

$$\chi_5^{1\mathrm{SF}} = \frac{G^5}{b^5}\left(\mu^2\tilde{b}^2\right)^{5\epsilon}c_\phi^4\frac{\sqrt{2m_1m_2\sigma+m_1^2+m_2^2}}{(\sigma^2-1)} \times\left[-\frac{16}{\epsilon}I_{r,5}^{1\mathrm{SF,\,div.}}+\left(184I_{r,5}^{1\mathrm{SF,\,div.}}-16I_{r,5}^{1\mathrm{SF,\,fin.}}\right)\right]+\mathcal{O}(\epsilon)$$

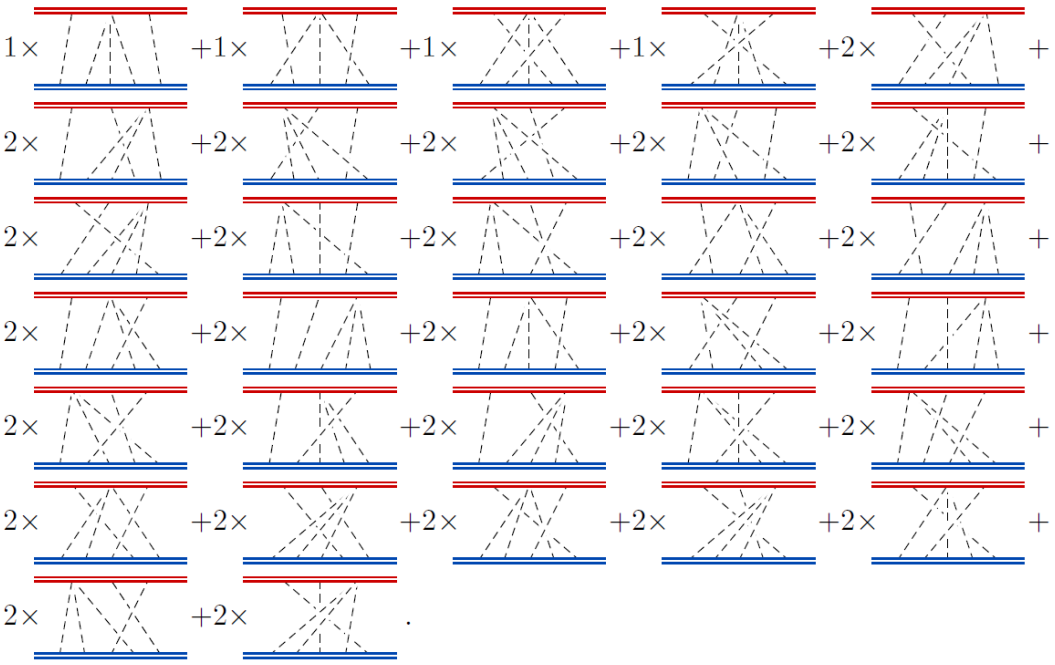
$$\chi^{0\mathrm{SF}} = 2\arctan\left(\frac{Gm_1m_2}{J}\frac{2c_\phi^2}{\sqrt{\sigma^2-1}}\right)$$

$$\chi = \pi - 2J\int_{r_{\min}}^{\infty}\frac{\mathrm{d}r}{r^2\sqrt{p_r^2(r)}}$$

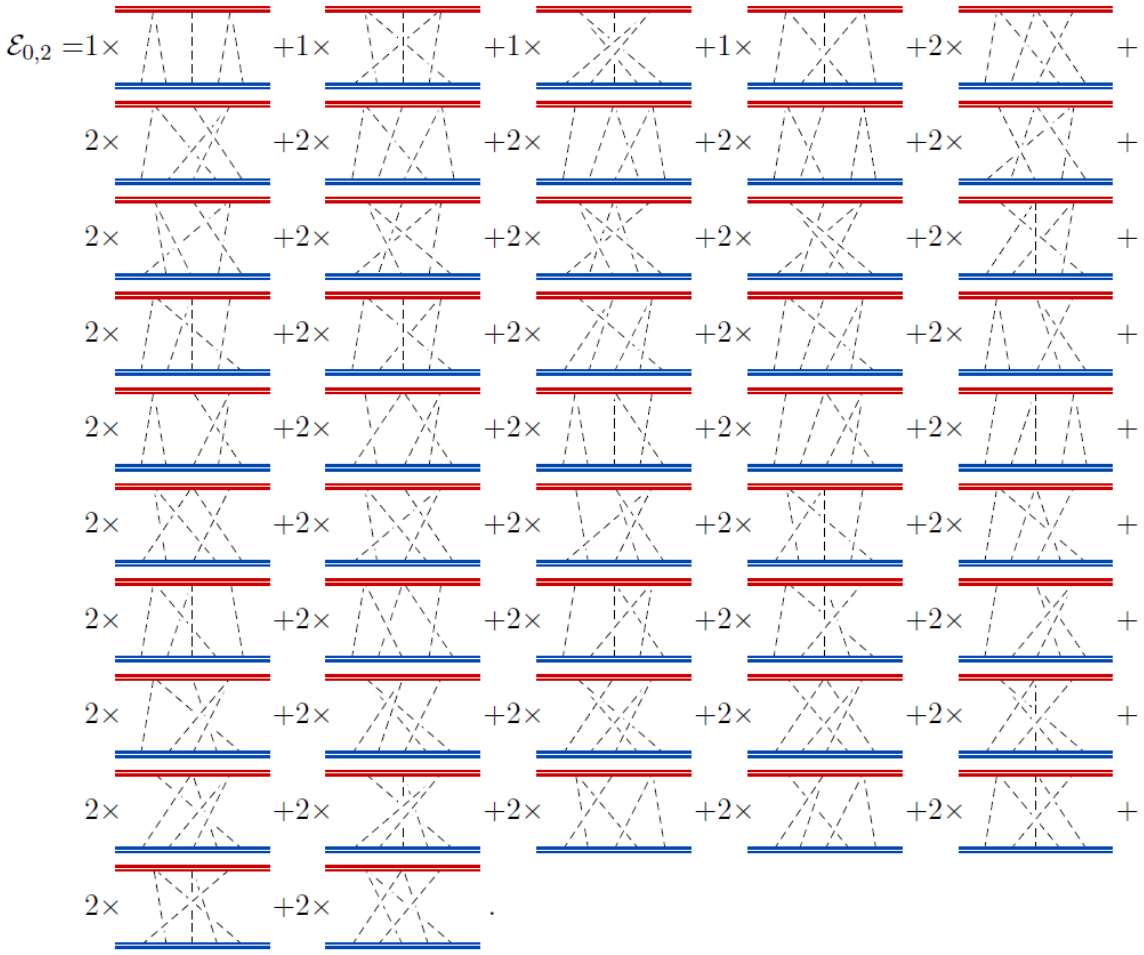
$$p_r^2(r) = \boldsymbol{p}^2 - \frac{J^2}{r^2} + \sum_{k=1} \left(\frac{G}{r}\right)^k (\mu r)^{2k\epsilon} P_k(\sigma)$$

$$\chi_5 = \frac{16\chi_1\chi_4\left[1+\epsilon\left(\frac{55}{6}-16\ln 2\right)\right]}{3\pi} - \frac{9\chi_1^5}{80} + \frac{3\chi_2\chi_1^3}{\pi} - \frac{3}{2}\chi_3\chi_1^2 - \frac{12\chi_2^2\chi_1}{\pi^2} + \frac{4\chi_2\chi_3}{\pi} + \mathcal{O}(v^{-2},\epsilon).$$

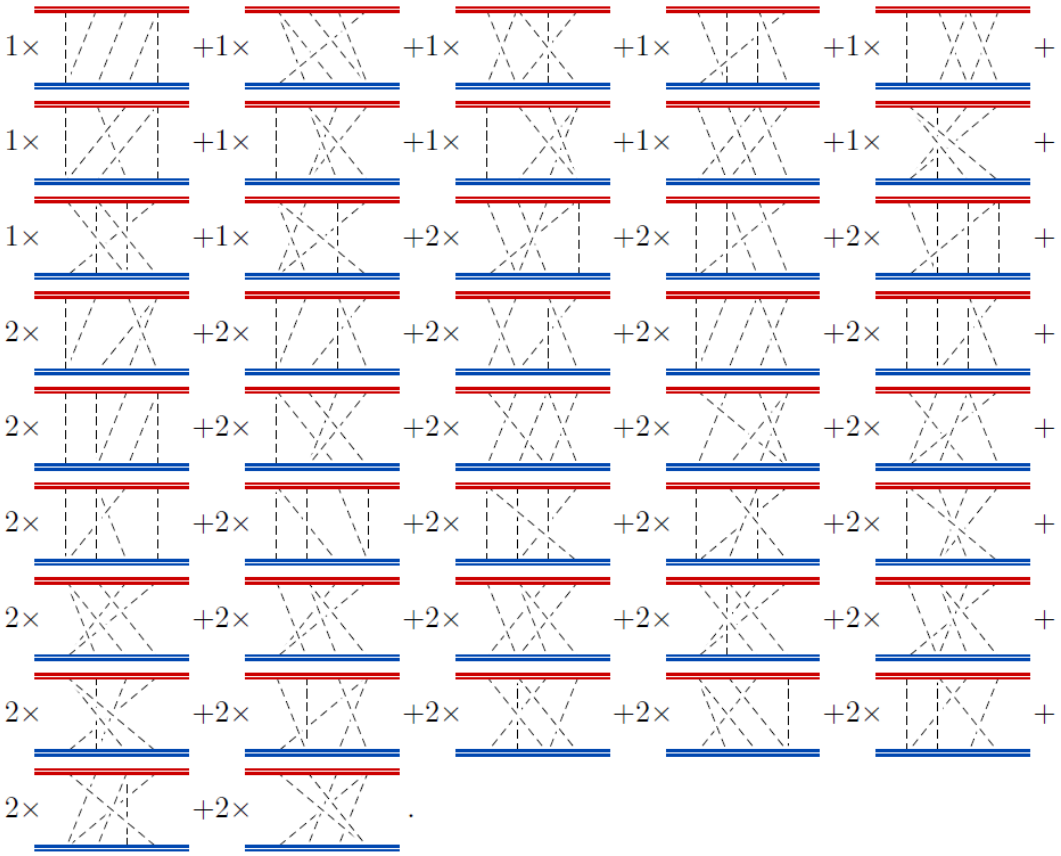
$\mathcal{E}_{0,1} =$



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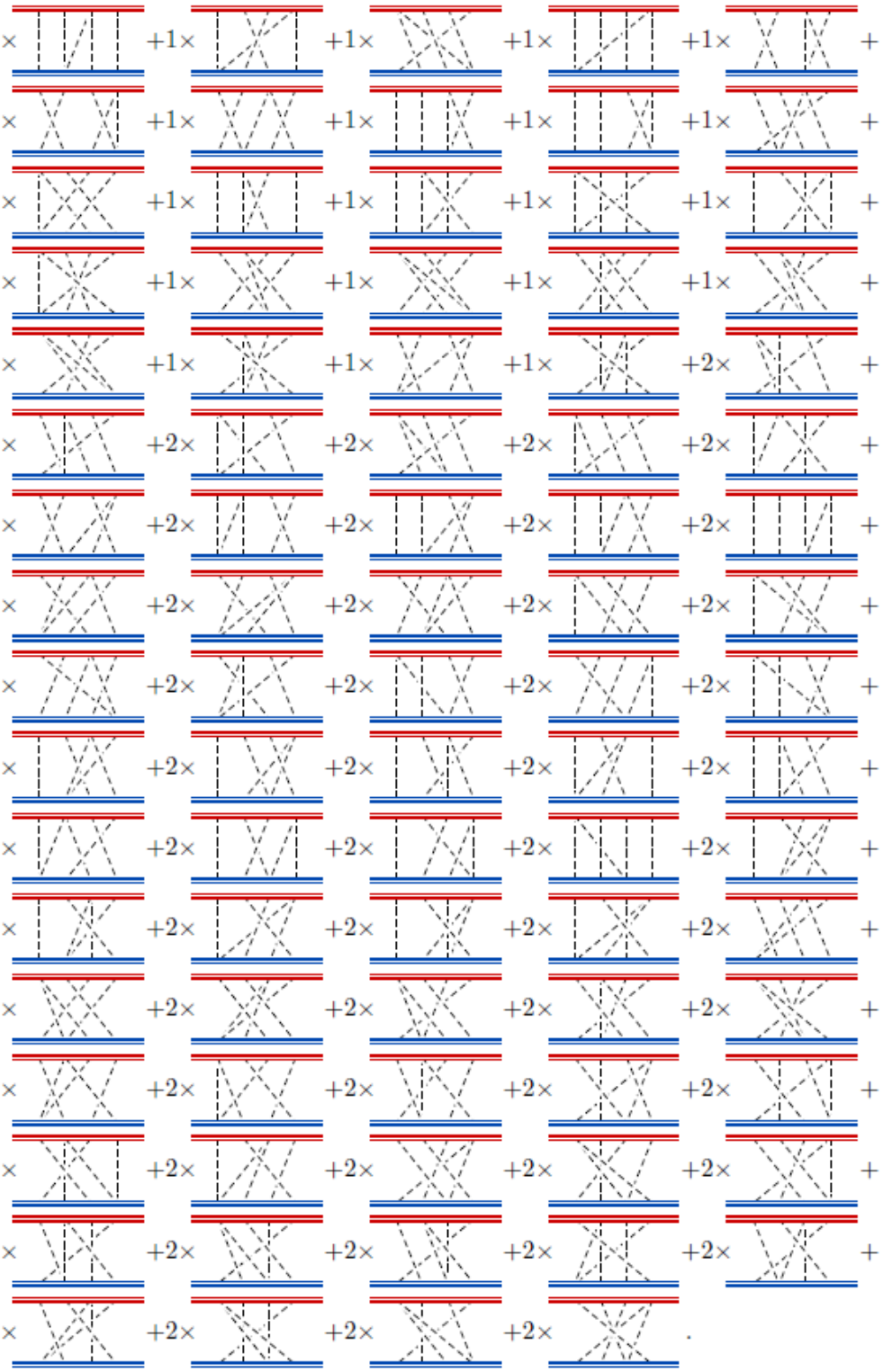


$\mathcal{E}_{0,3} =$



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$$\mathcal{E}_{0,4} = 1\times \quad +1\times \quad +1\times \quad +1\times \quad +1\times \quad +$$



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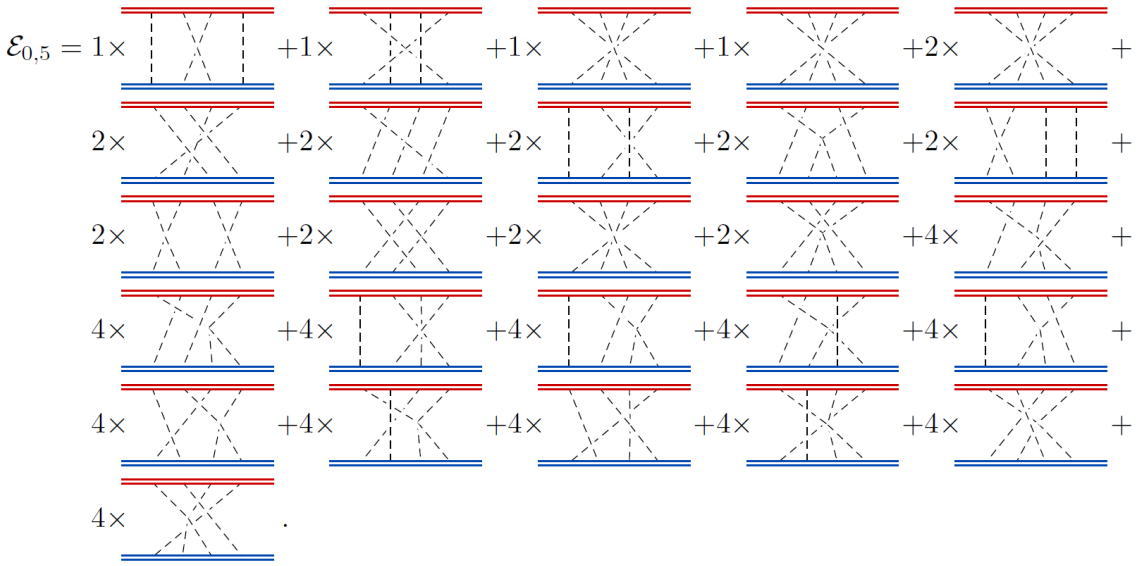
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$$I^{\sigma}(\varphi,\epsilon;n_{\pm})=\frac{i\pi}{8G_N^{(4)}}\frac{1}{\epsilon}\left(F(\varphi^I+\epsilon\mathfrak{n}^I)-\sigma F(\varphi^I-\epsilon\mathfrak{n}^I)\right)$$

$$\mathfrak{n}^R=\xi_I\mathfrak{n}^I=\frac{n_++\sigma n_-}{n_+n_-},\mathfrak{n}^m_\alpha=\zeta_{\alpha,I}\mathfrak{n}^I=0$$

$$S^{\sigma}(q,J,\varphi,\epsilon;n_{\pm})=\frac{i\pi}{2G_N^{(4)}}(\varphi^Iq_I+\epsilon J)-I^{\sigma}(\varphi,\epsilon;n_{\pm})+\lambda\left(\xi_I\varphi^I-\frac{n_+-\sigma n_-}{n_+n_-}\epsilon-2\right)+\mu^{\alpha}\zeta_{\alpha,I}\varphi^I$$

$$S^{\sigma}_{\text{on-shell}}\,(q,J;n_{\pm})=S^{\sigma}(q_I,J,\bar{\varphi}^I,\bar{\epsilon};n_{\pm}),\ \partial_{\varphi^I,\epsilon}S^{\sigma}\big|_{\bar{\varphi}^I,\bar{\epsilon}}=\partial_{\lambda,\mu^{\alpha}}S^{\sigma}\big|_{\bar{\varphi}^I,\bar{\epsilon}}=0$$

$$F=-2i\sqrt{X^0X^1X^2X^3}$$

$$\begin{aligned}e^{-1}\mathcal{L}=&\frac{1}{2}R-\sum_{i=1}^{3}\partial_\mu u_i\partial^\mu u_i-\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}e^{4\phi}D_\mu\sigma D^\mu\sigma-g^2\mathcal{V}\\&-\left[e^{2u_1+2u_2+2u_3}F^0_{\mu\nu}F^{0\mu\nu}+e^{2u_1-2u_2-2u_3}F^1_{\mu\nu}F^{1\mu\nu}\right.\\&\left.+e^{-2u_1+2u_2-2u_3}F^2_{\mu\nu}F^{2\mu\nu}+e^{-2u_1-2u_2+2u_3}F^3_{\mu\nu}F^{3\mu\nu}\right]\end{aligned}$$

$$D\sigma=d\sigma+g(e_0A^0-2A^1-2A^2-2A^3)$$

$$\mathcal{V}=\sum_{i=1}^{3}\left(\frac{\partial W}{\partial u_i}\right)^2+\left(\frac{\partial W}{\partial\phi}\right)^2-3W^2$$

$$W=\frac{1}{4}e^{-u_1-u_2-u_3}\left(8-e_0e^{2\phi}+2e^{2\phi}(e^{2u_1+2u_2}+e^{2u_2+2u_3}+e^{2u_3+2u_1})\right)$$

$$\begin{aligned}A^{\mathrm{R}}&=4\sqrt{2}A^0\\A^m&=\sqrt{2}(e_0A^0-2A^1-2A^2-2A^3)\\A^{\mathcal{B}_1}&=4\sqrt{2}(A^1-2A^2+A^3)\\A^{\mathcal{B}_2}&=4\sqrt{2}(A^1-A^3)\end{aligned}$$

$$\left[2\nabla_\mu - iB_\mu - gW\gamma_\mu + 4iH_{\mu\nu}\gamma^\nu\right]\epsilon = 0$$
$$\left[\partial_\mu u_i\gamma^\mu + g\partial_{u_i}W + i\partial_{u_i}H_{\mu\nu}\gamma^{\mu\nu}\right]\epsilon = 0$$
$$\left[\partial_\mu\phi\gamma^\mu + g\partial_\phi W + \frac{i}{2}\partial_\phi B_\mu\gamma^\mu\right]\epsilon = 0$$

$$H_{\mu\nu} = -\frac{1}{2\sqrt{2}}\left(e^{u_1+u_2+u_3}F_{\mu\nu}^{-0} + e^{u_1-u_2-u_3}F_{\mu\nu}^{-1} + e^{-u_1+u_2-u_3}F_{\mu\nu}^{-2} + e^{-u_1-u_2+u_3}F_{\mu\nu}^{-3}\right)$$
$$B_\mu = -4\sqrt{2}gA_\mu^0 + \frac{1}{2}e^{2\phi}D_\mu\sigma$$

$$e^{2u_i} = \sqrt{\frac{e_0}{6}}, e^{-2\phi} = \frac{e_0}{6}, L_{AdS_4} = \frac{1}{2}\left(\frac{e_0}{6}\right)^{3/4}$$

$$L_{AdS_4}^2 = -\frac{3}{g^2\mathcal{V}_*}$$

$$u_3 = u_2, A^3 = A^2$$

$$\mathcal{F}_{S^3} = \frac{\pi L_{AdS_4}^2}{2G_N^{(4)}}$$

$$\mathcal{F}_{S^3} = N^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{vol}(Y)}},$$

$$\mathcal{F}_{S^3}^{\mathrm{ABJM}} = \frac{\sqrt{2}\pi}{3}N^{3/2}, \qquad \mathrm{vol}(S^7) = \frac{\pi^4}{3}$$
$$\mathcal{F}_{S^3}^{Q^{1,1,1}/\mathbb{Z}_n} = \frac{4\pi}{3\sqrt{3}}n^{1/2}N^{3/2}, \qquad \mathrm{vol}(Q^{1,1,1}/\mathbb{Z}_n) = \frac{\pi^4}{8n}$$
$$\mathcal{F}_{S^3}^{M^{1,1,1}/\mathbb{Z}_n} = \frac{16\pi}{9\sqrt{3}}n^{1/2}N^{3/2}, \qquad \mathrm{vol}(M^{1,1,1}/\mathbb{Z}_n) = \frac{9\pi^4}{128n}$$

$$ds^2 = e^{2V}ds_{AdS_2}^2 + f^2dy^2 + h^2dz^2$$
$$A^I = a^I dz$$

$$B_\mu dx^\mu \equiv B_z dz$$

$$e^a = e^V\bar{e}^a, e^2 = fdy, e^3 = hdz$$

$$F_{23}^I = f^{-1}h^{-1}(a^I)'$$

$$-\frac{1}{\sqrt{2}}e^{2V}(e^{2u_1-2u_2-2u_3}F_{23}^1 - e^{-2u_1+2u_2-2u_3}F_{23}^2) = \mathcal{E}_{F_1}$$
$$-\frac{1}{\sqrt{2}}e^{2V}(e^{-2u_1+2u_2-2u_3}F_{23}^2 - e^{-2u_1-2u_2+2u_3}F_{23}^3) = \mathcal{E}_{F_2}$$
$$-\frac{1}{\sqrt{2}}e^{2V}(e^{-2u_1-2u_2+2u_3}F_{23}^3 - e^{2u_1-2u_2-2u_3}F_{23}^1) = \mathcal{E}_{F_3}$$

$$-\frac{1}{\sqrt{2}}e^{2V}\left(e^{2u_1+2u_2+2u_3}F_{23}^0+\frac{e_0}{2}e^{2u_1-2u_2-2u_3}F_{23}^1\right)=\mathcal{E}_{R_1},$$
$$-\frac{1}{\sqrt{2}}e^{2V}\left(e^{2u_1+2u_2+2u_3}F_{23}^0+\frac{e_0}{2}e^{-2u_1+2u_2-2u_3}F_{23}^2\right)=\mathcal{E}_{R_2},$$
$$-\frac{1}{\sqrt{2}}e^{2V}\left(e^{2u_1+2u_2+2u_3}F_{23}^0+\frac{e_0}{2}e^{-2u_1-2u_2+2u_3}F_{23}^3\right)=\mathcal{E}_{R_3},$$

$$(e^{2V-2u_1-2u_2+2u_3}F_{23}^3)'=\sqrt{2}ge^{2V}fh^{-1}e^{4\phi}D_z\sigma,$$

$$\gamma^m=\Gamma^m\otimes\sigma^3,\gamma^2=I_2\otimes\sigma^1,\gamma^3=I_2\otimes\sigma^2,$$

$$\epsilon=\psi\otimes\chi$$

$$D_m\psi=\frac{1}{2}\kappa\Gamma_m\psi$$

$$f^{-1}\xi'=2gW\cos\,\xi+\kappa e^{-V}$$
$$f^{-1}V'=gW\sin\,\xi$$
$$f^{-1}u_i'=-g\partial_{u_i}W\sin\,\xi$$
$$f^{-1}\phi'=-\frac{g\partial_\phi W}{\sin\,\xi}$$
$$f^{-1}\frac{h'}{h}=\frac{1}{\sin\,\xi}\left(\kappa e^{-V}\cos\,\xi+gW(1+\cos^2\,\xi)\right)$$

$$(s-B_z)\sin\,\xi=-2gWh\cos\,\xi-\kappa he^{-V}$$
$$g\partial_\phi W\cos\,\xi=\frac{1}{2}\partial_\phi B_z\sin\,\xi h^{-1}$$

$$\partial_{u_i}H_{23}=-\frac{1}{4}g\partial_{u_i}W\cos\,\xi$$
$$H_{23}=-\frac{1}{4}gW\cos\,\xi-\frac{1}{4}\kappa e^{-V}$$

$$he^{-V}=k\sin\,\xi$$

$$\xi'=-k^{-1}(s-B_z)(e^{-V}f),$$

$$(s-B_z)=-k[2gWe^V\cos\,\xi+\kappa]$$
$$g\partial_\phi W\cos\,\xi=\frac{1}{2}k^{-1}e^{-V}\partial_\phi B_z$$

$$\mathcal{E}_{R_1}=e^V[4ge^V\cos\,\xi+\kappa e^{u_1}\cosh\,(u_2+u_3)]-\frac{2-e_0}{4}\kappa e^Ve^{2u_1}e^{-u_1-u_2-u_3},$$
$$\mathcal{E}_{R_2}=e^V[4ge^V\cos\,\xi+\kappa e^{u_2}\cosh\,(u_3+u_1)]-\frac{2-e_0}{4}\kappa e^Ve^{2u_2}e^{-u_1-u_2-u_3},$$
$$\mathcal{E}_{R_3}=e^V[4ge^V\cos\,\xi+\kappa e^{u_3}\cosh\,(u_1+u_2)]-\frac{2-e_0}{4}\kappa e^Ve^{2u_3}e^{-u_1-u_2-u_3},$$

$$\mathcal{E}_{F_1}=\kappa e^Ve^{u_3}\sinh\,(u_1-u_2)$$
$$\mathcal{E}_{F_2}=\kappa e^Ve^{u_1}\sinh\,(u_2-u_3)$$
$$\mathcal{E}_{F_3}=\kappa e^Ve^{u_2}\sinh\,(u_3-u_1)$$

$$f = -e^V,$$

$$ds^2 = e^{2V}\left[ds^2_{AdS_2} + ds^2_\Sigma\right],$$

$$ds^2_\Sigma = dy^2 + k^2 \sin^2\,\xi dz^2.$$

$$\Delta z = 2\pi$$

$$h \le 0, \Leftrightarrow\ k\sin\,\xi \le 0$$

$$(k\sin\,\xi)'|_{N,S} = -\frac{(-1)^{l_{N,S}}}{n_{N,S}}, l_N = 0, l_S = 1$$

$$\cos\,\xi|_{N,S} = (-1)^{t_{N,S}}; \qquad (t_N, t_S) = (1,1)\ \text{or}\ (0,0)$$
$$(t_N, t_S) = (1,0)\ \text{or}\ (0,1)$$

$$(s - B_z)|_{N,S} = \frac{1}{n_{N,S}}(-1)^{l_{N,S}+t_{N,S}+1}$$

$$-gF^R = dB + d\left(\frac{1}{2}e^{2\phi}D\sigma\right)$$

$$\frac{1}{2\pi}\int_\Sigma gF^{\mathrm{R}} \equiv \frac{1}{2\pi}\int_\Sigma (-dB) = \frac{n_N(-1)^{t_S} + n_S(-1)^{t_N}}{n_N n_S}$$

$$\partial_\phi B_z\big|_{N,S} = \partial_\phi W\big|_{N,S} = 0$$

$$\phi|_N, \phi|_S \neq 0, \Rightarrow\ D_z\sigma|_N = D_z\sigma|_S = 0$$

$$\frac{1}{2\pi}\int_\Sigma \sqrt{2}g(e_0F^0 - 2F^1 - 2F^2 - 2F^3) = (D_z\sigma)|_{y_N}^{y_S} = 0$$

$$\left(e^{-2u_1} + e^{-2u_2} + e^{-2u_3} - \frac{e_0}{2}e^{-2u_1-2u_2-2u_3}\right)\Big|_{N,S} = 0, \Rightarrow\ W|_{N,S} = 2e^{-u_1-u_2-u_3}|_{N,S}$$

$$M_{(1)} \equiv -2ge^V e^{-u_1-u_2-u_3}, M_{(2)} \equiv -\frac{\kappa}{4}M_{(1)} + M^2_{(1)}\cos\,\xi$$

$$\mathcal{E}_{R_i} = -\frac{e_0}{2}\frac{\kappa}{4g}M_{(1)}e^{2u_i} + \frac{1}{g}M_{(2)}e^{2u_1+2u_2+2u_3}$$

$$M_{(1)} = \frac{1}{2}(-1)^{t_{N,S}}\kappa - \frac{1}{2kn_{N,S}}(-1)^{l_{N,S}},$$
$$M_{(2)} = \frac{1}{8}(-1)^{t_{N,S}} + \frac{1}{4k^2n^2_{N,S}}(-1)^{t_{N,S}} - \frac{3\kappa}{8kn_{N,S}}(-1)^{l_{N,S}}.$$

$$\mathcal{E}_{R_1}(u_{1N}, u_{2N}) = \mathcal{E}_{R_1}(u_{1S}, u_{2S}),$$
$$\mathcal{E}_{R_2}(u_{1N}, u_{2N}) = \mathcal{E}_{R_2}(u_{1S}, u_{2S}),$$
$$\mathcal{E}_{R_3}(u_{1N}, u_{2N}) = \mathcal{E}_{R_3}(u_{1S}, u_{2S}).$$

$$F^I_{yz} = (a^I)' = \left(\mathcal{J}^{(I)}\right)'$$

$$\mathcal{I}^{(0)} \equiv \frac{1}{\sqrt{2}} gke^{V} \cos\,\xi e^{-u_1-u_2-u_3}$$
$$\mathcal{I}^{(1)} \equiv \frac{1}{\sqrt{2}} gke^{V} \cos\,\xi e^{-u_1+u_2+u_3}$$
$$\mathcal{I}^{(2)} \equiv \frac{1}{\sqrt{2}} gke^{V} \cos\,\xi e^{u_1-u_2+u_3}$$
$$\mathcal{I}^{(3)} \equiv \frac{1}{\sqrt{2}} gke^{V} \cos\,\xi e^{u_1+u_2-u_3}$$

$$\frac{p_I}{n_N n_S} \equiv \frac{1}{2\pi}\int_{\Sigma} gF^I = g\mathcal{I}_I|_N^S$$

$$\mathcal{I}_{\mathrm{R}}|_{N,S} \equiv 4\sqrt{2}\mathcal{I}^{(0)}\big|_{N,S}$$
$$\mathcal{I}_m|_{N,S} \equiv \sqrt{2}\big(e_0\mathcal{I}^{(0)} - 2\mathcal{I}^{(1)} - 2\mathcal{I}^{(2)} - 2\mathcal{I}^{(3)}\big)\big|_{N,S}$$
$$\mathcal{I}_{\mathcal{B}_1}\big|_{N,S} \equiv 4\sqrt{2}\big(\mathcal{I}^{(1)} - 2\mathcal{I}^{(2)} + \mathcal{I}^{(3)}\big)\big|_{N,S}$$
$$\mathcal{I}_{\mathcal{B}_2}\big|_{N,S} \equiv 4\sqrt{2}\big(\mathcal{I}^{(1)} - 2\mathcal{I}^{(3)}\big)\big|_{N,S}$$

$$\begin{aligned} g\mathcal{I}_{\mathrm{R}}|_{N,S} &= 4\sqrt{2}g\mathcal{I}^{(0)}\big|_{N,S} \\ &= 4gke^{V}\cos\,\xi e^{-u_1-u_2-u_3}|_{N,S} \\ &= -2kM_{(1)}(-1)^{t_{N,S}+1}\big|_{N,S} \end{aligned}$$

$$\begin{aligned} g\mathcal{I}_m|_{N,S} &= \sqrt{2}g\big(e_0\mathcal{I}^{(0)} - 2\mathcal{I}^{(1)} - 2\mathcal{I}^{(2)} - 2\mathcal{I}^{(3)}\big)\big|_{N,S} \\ &= gke^{V}\cos\,\xi(e_0 e^{-u_1-u_2-u_3} - 2e^{-u_1+u_2+u_3} - 2e^{u_1-u_2+u_3} - 2e^{u_1+u_2-u_3})|_{N,S} \\ &= gke^{V}\cos\,\xi e^{u_1+u_2+u_3}(e_0 e^{-2u_1-2u_2-2u_3} - 2e^{-2u_1} - 2e^{-2u_2} - 2e^{-2u_3})|_{N,S} = 0 \end{aligned}$$

$$g\mathcal{I}_{\mathrm{R}}|_N^S = \frac{n_N(-1)^{t_S} + n_S(-1)^{t_N}}{n_N n_S}$$
$$g\mathcal{I}_m|_N^S = 0$$

$$\begin{aligned} \frac{p_{\mathcal{B}_1}}{n_N n_S} \equiv g\mathcal{I}_{\mathcal{B}_1}\big|_N^S &= 4\sqrt{2}g\big(\mathcal{I}^{(1)} - 2\mathcal{I}^{(2)} + \mathcal{I}^{(3)}\big)\big|_N^S \\ &= 4gke^{V}\cos\,\xi(e^{-u_1+u_2+u_3} - 2e^{u_1-u_2+u_3} + e^{u_1+u_2-u_3})|_N^S \\ &= 2kM_{(1)}(-1)^{t_{N,S}}(e^{2u_2+2u_3} - 2e^{2u_1+2u_3} + e^{2u_1+2u_2})\big|_N^S, \end{aligned}$$

$$\begin{aligned} \frac{p_{\mathcal{B}_2}}{n_N n_S} \equiv g\mathcal{I}_{\mathcal{B}_2}\big|_N^S &= 4\sqrt{2}g\big(\mathcal{I}^{(1)} - \mathcal{I}^{(3)}\big)\big|_N^S \\ &= 4gke^{V}\cos\,\xi(e^{-u_1+u_2+u_3} - e^{u_1+u_2-u_3})|_N^S \\ &= 2kM_{(1)}(-1)^{t_{N,S}}(e^{2u_2+2u_3} - e^{2u_1+2u_2})\big|_N^S, \end{aligned}$$

$$n_N = 4\,, \qquad n_S = 1\,, \qquad p_{\mathcal{B}_1} = 1\,, \qquad p_{\mathcal{B}_2} = 2\,,$$
$$g = 1\,, \qquad \kappa = +1\,, \qquad e_0 = 2\,,$$

$$\begin{aligned} e^{2u_{1N}} &\approx 0.174581, & e^{2u_{1S}} &\approx 0.0768503, \\ e^{2u_{2N}} &\approx 0.418164, & e^{2u_{2S}} &\approx 0.411234, \\ e^{2u_{3N}} &\approx 1.5639, & e^{2u_{3S}} &\approx 1.98408, \\ k &\approx -0.683914. \end{aligned}$$

$$\frac{1}{2G_N^{(4)}} = \left(\frac{6}{e_0}\right)^{3/2} \frac{16}{3\sqrt{3}} N^{3/2}$$

$$\left(G_N^{(2)}\right)^{-1} = \left(G_N^{(4)}\right)^{-1} \Delta z \int_{y_N}^{y_S} |fh| dy$$

$$fh = ke^V f \sin\,\xi = +\frac{k}{\kappa}(e^{2V}\cos\,\xi)'$$

$$S_{\mathrm{BH}} = \frac{1}{4G_N^{(2)}} = \left(\frac{6}{e_0}\right)^{3/2} \frac{16\pi}{3\sqrt{3}} N^{3/2} g^2 \left(-\frac{k}{\kappa}\right) [e^{2V}\cos\,\xi]_N^S$$

$$= -\left(\frac{6}{e_0}\right)^{3/2} \frac{16N^{3/2}k}{3\sqrt{3}\kappa} \left(\frac{1}{4}M_{(1)}^2\Big|_S e^{2u_{1S}+2u_{2S}+2u_{3S}}(-1)^{t_S} - \frac{1}{4}M_{(1)}^2\Big|_N e^{2u_{1N}+2u_{2N}+2u_{3N}}(-1)^{t_N}\right)$$

$$e^{2u_i} = \sqrt{\frac{e_0}{6}}, e^{-2\phi} = \frac{e_0}{6}, L_{AdS_4} = \frac{1}{2}\left(\frac{e_0}{6}\right)^{3/4}$$

$$ds^2 = \frac{1}{2g^2}\left[\frac{y^2}{4} ds_{AdS_2}^2 + \frac{y^2}{q(y)} dy^2 + \frac{q(y)}{4y^2} c_0^2 dz^2\right]$$

$$A^0 = A^1 = A^2 = A^3 = -\left[\frac{c_0\kappa}{2g}\left(1 - \frac{a}{y}\right) + \frac{s}{g}\right] dz$$

$$\sin\,\xi = -\frac{\sqrt{q(y)}}{y^2}, \cos\,\xi = \kappa\frac{2y-a}{y^2}.$$

$$q(y) = y^4 - 4y^2 + 4ay - a^2,$$

$$a = \frac{n_S^2 - n_N^2}{n_S^2 + n_N^2}$$

$$c_0 = \frac{\sqrt{n_S^2 + n_N^2}}{\sqrt{2} n_S n_N}$$

$$y_N = -1 + \sqrt{1+a}, y_S = 1 - \sqrt{1-a}.$$

$$S_{\mathrm{BH}} = \frac{\sqrt{2}\sqrt{n_S^2 + n_N^2} - (n_S + n_N)}{n_S n_N} \frac{\pi L_{AdS_4}^2}{4G_N^{(4)}}$$

$$= \frac{\sqrt{2}\sqrt{n_S^2 + n_N^2} - (n_S + n_N)}{n_S n_N} \frac{1}{2} \mathcal{F}_{S^3}^{Q^{1,1,1}},$$

$$n_N = 4, \quad n_S = 1, \quad p_{B_1} = 1$$
$$g = 1, \quad \kappa = +1, \quad e_0 = 2$$

$$e^{2u_{1N}} \approx 0.49043, \qquad e^{2u_{1S}} \approx 0.436592$$
$$e^{2u_{2N}} \approx 0.774299, \qquad e^{2u_{2S}} \approx 0.926937,$$
$$k \approx -0.725009.$$

$$\mathcal{B}(X^I, \epsilon) = \frac{i\pi}{8G_N^{(4)}} \frac{F(X^I)}{\epsilon},$$

$$\mathcal{F}_{\partial M_4}(\varphi, \epsilon) = \sum_{s=1}^{\chi_E(M_4)} \sigma_{(s)} \mathcal{B}\left(X^I_{(s)}, \epsilon_{(s)}\right),$$

$$\mathcal{F}_{S^3}(\varphi) = \mathcal{B}(X^I = 2\varphi^I, \epsilon = 1) = \frac{i\pi}{2G_N^{(4)}} F(\varphi^I)$$

$$S^\sigma(\varphi, \epsilon; \mathfrak{n}_+, \mathfrak{n}_-) = -\frac{i\pi}{8G_N^{(4)}\epsilon}\left(F(\varphi^I + \epsilon\mathfrak{n}^I) - \sigma F(\varphi^I - \epsilon\mathfrak{n}^I)\right)$$

$$\mathfrak{n}^R = \frac{1}{2\pi}\int_\Sigma dA^R = \frac{n_+ + \sigma n_-}{n_+ n_-}$$

$$\varphi^R - \frac{n_+ - \sigma n_-}{n_+ n_-}\epsilon = 2.$$

$$F = -2i\sqrt{X^0 X^1 X^2 X^3},$$

$$A^R = A^0 + A^1 + A^2 + A^3.$$

$$\mathcal{F}^{\text{ABJM}}_{S^3}(\varphi) = \frac{\pi}{G_N^{(4)}}\sqrt{\varphi^0\varphi^1\varphi^2\varphi^3}$$

$$\mathcal{F}^{\text{ABJM}}_{S^3} := \mathcal{F}^{\text{ABJM}}_{S^3}(\bar{\varphi}) = \frac{\pi}{4G_N^{(4)}} = \frac{\pi L^2_{AdS_4}}{2G_N^{(4)}},$$

$$\mathfrak{n} = \frac{n_+ + \sigma n_-}{4n_+ n_-},$$

$$\varphi - \frac{n_+ - \sigma n_-}{4n_+ n_-}\epsilon = \frac{1}{2}$$

$$S^-(\epsilon; n_+, n_-) = -\frac{\pi}{4G_N^{(4)}\epsilon}((\varphi + \epsilon\mathfrak{n})^2 + (\varphi - \epsilon\mathfrak{n})^2) = -\frac{\pi}{2G_N^{(4)}\epsilon}(\varphi^2 + \epsilon^2\mathfrak{n}^2)$$

$$\bar{\epsilon} = \pm \frac{\sqrt{2} n_+ n_-}{\sqrt{n_+^2 + n_-^2}},$$

$$S^-(\bar{\epsilon}; n_+, n_-) = \frac{\sqrt{2}\sqrt{n_+^2 + n_-^2} - (n_+ + n_-)}{2 n_+ n_-} \mathcal{F}_{S^3}^{\text{ABJM}}$$

$$F = -2i\sqrt{X^0 X^1 X^2 X^3}$$

$$A^R = A^0 + A^1 + A^2 + A^3, A^m = A^0 - A^1 - A^2 - A^3$$

$$F = -2i\sqrt{X^0 X^1 X^2 X^3} \;\Rightarrow\; F^{\text{eff}} = -2i\sqrt{(X^1 + X^2 + X^3) X^1 X^2 X^3}$$

$$\mathcal{F}_{S^3}^{\text{mABJM}}(\varphi) = \frac{\pi}{G_N^{(4)}} \sqrt{(\varphi^1 + \varphi^2 + \varphi^3)\varphi^1 \varphi^2 \varphi^3}$$

$$\mathcal{F}_{S^3}^{\text{mABJM}} := \mathcal{F}_{S^3}^{\text{mABJM}}(\bar{\varphi}) = \frac{\pi}{3\sqrt{3} G_N^{(4)}} = \frac{\pi L_{AdS_4}^2}{2 G_N^{(4)}}$$

$$\mathcal{F}_{S^3}^{\text{mABJM}} = \frac{4}{3\sqrt{3}} \mathcal{F}_{S^3}^{\text{ABJM}}$$

$$\mathfrak{n}^0 = \sum_{i=1}^{3} \mathfrak{n}^i = \frac{n_+ + \sigma n_-}{2 n_+ n_-},$$

$$\sum_{i=1}^{3} \varphi^i - \frac{n_+ - \sigma n_-}{2 n_+ n_-} \epsilon = 1$$

$$S^-(\varphi, \epsilon; \mathfrak{n}, n_+, n_-) = -\frac{i\pi}{8 G_N^{(4)} \epsilon} \left(F^{\text{eff}}\left(\varphi^i + \epsilon \mathfrak{n}^I\right) + F^{\text{eff}}\left(\varphi^i - \epsilon \mathfrak{n}^I\right) \right)$$

$$n_N = n_+, n_S = n_-, \frac{p_{F_1}}{n_+ n_-} = -\frac{1}{2}(\mathfrak{n}^1 - \mathfrak{n}^2), \frac{p_{F_2}}{n_+ n_-} = -\frac{1}{\sqrt{3}}(\mathfrak{n}^1 + \mathfrak{n}^2 - 2\mathfrak{n}^3)$$

$$\frac{\bar{\varphi}^1 + \mathfrak{n}^1 \bar{\epsilon}}{1 + \bar{\epsilon}/n_+} = e^{-\lambda_2 - \lambda_3}\big|_N,$$
$$\frac{\bar{\varphi}^1 - \mathfrak{n}^1 \bar{\epsilon}}{1 + \bar{\epsilon}/n_-} = e^{-\lambda_2 - \lambda_3}\big|_S,$$

$$F = -2i\sqrt{X^0 X^1 X^2 X^3}$$

$$A^R = 4\sqrt{2} A^0, A^m = \sqrt{2}(e_0 A^0 - 2A^1 - 2A^2 - 2A^3),$$

$$\xi_0 = 4\sqrt{2}, \xi_1 = \xi_2 = \xi_3 = 0$$

$$\zeta_0 = \sqrt{2} e_0, \zeta_1 = \zeta_2 = \zeta_3 = -2\sqrt{2}$$

$$L_{AdS_4}^2 = \sqrt{e_0^3}/4\sqrt{6^3}$$

$$F = -2i\sqrt{X^0X^1X^2X^3} \;\Rightarrow\; F^{\text{eff}} = -2i\sqrt{\frac{2}{e_0}(X^1+X^2+X^3)X^1X^2X^3}$$

$$\mathcal{F}_{S^3}^{Q^{1,1,1}}(\varphi) = \frac{\pi}{G_N^{(4)}}\sqrt{\frac{2}{e_0}(\varphi^1+\varphi^2+\varphi^3)\varphi^1\varphi^2\varphi^3}$$

$$\varphi^0 = 2/e_0\sum_{i=1}^{3}\varphi^i = 1/(2\sqrt{2})$$

$$\bar{\varphi}^i = e_0/(12\sqrt{2})$$

$$\mathcal{F}_{S^3}^{Q^{1,1,1}} := \mathcal{F}_{S^3}^{Q^{1,1,1}}(\bar{\varphi}) = \frac{\pi\sqrt{e_0^3}}{8\sqrt{6^3}G_N^{(4)}} = \frac{\pi L_{AdS_4}^2}{2G_N^{(4)}}$$

$$\mathfrak{n}^0 = \frac{1}{4\sqrt{2}}\frac{n_+ + \sigma n_-}{n_+n_-}, \sum_{i=1}^{3}\mathfrak{n}^i = \frac{e_0}{2}\frac{1}{4\sqrt{2}}\frac{n_+ + \sigma n_-}{n_+n_-}.$$

$$\varphi^0 = \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}\frac{n_+ - \sigma n_-}{n_+n_-}\epsilon, \sum_{i=1}^{3}\varphi^i = \frac{e_0}{2}\left(\frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}\frac{n_+ - \sigma n_-}{n_+n_-}\epsilon\right)$$

$$S^-(\varphi,\epsilon;\mathfrak{n},n_+,n_-) = -\frac{i\pi}{8G_N^{(4)}\epsilon}\left(F^{\text{eff}}\left(\varphi^i+\epsilon\mathfrak{n}^I\right) + F^{\text{eff}}\left(\varphi^i-\epsilon\mathfrak{n}^I\right)\right)$$

$$n_N = n_-, n_S = n_+, \frac{p_{\mathcal{B}_1}}{n_+n_-} = 4\sqrt{2}(\mathfrak{n}^1 - 2\mathfrak{n}^2 + \mathfrak{n}^3), \frac{p_{\mathcal{B}_2}}{n_+n_-} = 4\sqrt{2}(\mathfrak{n}^1 - \mathfrak{n}^3),$$

$$\frac{4\sqrt{2}}{e_0}\frac{\bar{\varphi}^1 + \mathfrak{n}^1\bar{\epsilon}}{1+\bar{\epsilon}/n_-} = e^{u_2+u_3}|_N$$
$$\frac{4\sqrt{2}}{e_0}\frac{\bar{\varphi}^1 - \mathfrak{n}^1\bar{\epsilon}}{1+\bar{\epsilon}/n_+} = e^{u_2+u_3}|_S$$

$$X^3 = X^2, \mathfrak{n}^3 = \mathfrak{n}^2, \varphi^3 = \varphi^2$$

$$\mathcal{M}_v \times \mathcal{M}_h = \left(\frac{SU(1,1)}{U(1)}\right)^3 \times \frac{SU(2,1)}{S(U(2)\times U(1))}$$

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4}e^{4\phi} & -\frac{1}{8}e^{4\phi}\tilde{\xi}_0 & \frac{1}{8}e^{4\phi}\xi^0 \\ 0 & -\frac{1}{8}e^{4\phi}\tilde{\xi}_0 & \frac{1}{4}e^{2\phi}\left(1+\frac{1}{4}e^{2\phi}\left(\tilde{\xi}_0\right)^2\right) & -\frac{1}{16}e^{4\phi}\xi^0\tilde{\xi}_0 \\ 0 & \frac{1}{8}e^{4\phi}\xi^0 & -\frac{1}{16}e^{4\phi}\xi^0\tilde{\xi}_0 & \frac{1}{4}e^{2\phi}\left(1+\frac{1}{4}e^{2\phi}(\xi^0)^2\right) \end{pmatrix}.$$

$$k^{\mathbb{R}} = \partial_\sigma, k^{U(1)} = -\tilde{\xi}_0\partial_{\xi^0} + \xi^0\partial_{\tilde{\xi}_0}$$

$$P^{\mathbb{R}}=\left(0,0,-\frac{1}{2}e^{2\phi}\right),P^{U(1)}=\left(\tilde{\xi}_0e^{\phi},-\xi^0e^{\phi},1-\frac{1}{4}\left((\xi^0)^2+\left(\tilde{\xi}_0\right)^2\right)e^{2\phi}\right)$$

$$P^{\mathbb{R}}=\left(0,0,-\frac{1}{2}e^{2\phi}\right),P^{U(1)}\left(\xi^0=\tilde{\xi}_0=0\right)=(0,0,1)$$

$$\omega^x=e^{\phi}\mathrm{d}\xi^0=0,\omega^y=e^{\phi}\mathrm{d}\tilde{\xi}_0=0,\omega^z=-\frac{1}{2}e^{2\phi}\left(\mathrm{d}\sigma+\xi^0\,\mathrm{d}\tilde{\xi}_0\right)=-\frac{1}{2}e^{2\phi}\,\mathrm{d}\sigma$$

$$\Omega^x:=\mathrm{d}\omega^x-\frac{1}{2}\epsilon^{xyz}\omega^y\wedge\omega^z$$

$$\Omega^z_{\sigma\phi}=-\Omega^z_{\phi\sigma}=\frac{1}{2}e^{2\phi}$$

$$X^0=1,X^1=z^2z^3,X^2=z^3z^1,X^3=z^1z^2$$

$$e^{-K}=8\mathrm{Re}[z^1]\mathrm{Re}[z^2]\mathrm{Re}[z^3]=8z^1z^2z^3$$

$$K_i=\partial_{z^i}K=-\left(2z^i\right)^{-1},g_{i\bar{\jmath}}=K_{ij}=\delta_{ij}\left(2z^i\right)^{-2}$$

$$\mathcal{N}=-i\mathrm{diag}\left(z^1z^2z^3,\frac{z^1}{z^2z^3},\frac{z^2}{z^3z^1},\frac{z^3}{z^1z^2}\right)$$

$$k_I=\left(\sqrt{2}\left(4k^{U(1)}+e_0k^{\mathbb{R}}\right),-2\sqrt{2}k^{\mathbb{R}},-2\sqrt{2}k^{\mathbb{R}},-2\sqrt{2}k^{\mathbb{R}}\right)$$

$$P_I^z=\left(\sqrt{2}\left(4-\frac{1}{2}e^{2\phi}e_0\right),e^{2\phi},e^{2\phi},e^{2\phi}\right)$$

$$e^{-1}\mathcal{L}=\frac{1}{2}R+\sum_i\frac{\left(\partial z^i\right)^2}{4(z^i)^2}+(\partial\phi)^2+\frac{1}{4}e^{4\phi}(D\sigma)^2+\mathrm{Im}\mathcal{N}_{IJ}F^I_{\mu\nu}F^{I\mu\nu}-g^2\mathcal{V}$$

$$D_\mu\sigma=\partial_\mu\sigma+gA^m_\mu$$

$$A^m=\sqrt{2}(e_0A^0-2A^1-2A^2-2A^3),$$

$$A^R=4\sqrt{2}A^0$$

$$\begin{aligned}\delta\psi_{\mu A}=&\nabla_\mu\varepsilon_A+\frac{i}{2}\left(4\sqrt{2}gA^0_\mu-\frac{1}{2}e^{2\phi}\nabla_\mu\sigma\right)\sigma^3_A{}^B\varepsilon_B-\frac{g}{2}e^{K/2}P^3_IX^I\gamma_\mu\sigma^3{}_{AB}\varepsilon^B\\&+2ie^{K/2}X^I\mathrm{Im}\mathcal{N}_{IJ}F^{-J}_{\mu\nu}\gamma^\nu\epsilon_{AB}\varepsilon^B\\\delta\lambda^{iA}=&i\partial_\mu z^i\gamma^\mu\varepsilon^A+ige^{K/2}g^{ij}\left(\partial_j+K_j\right)X^IP^3_I\sigma^{3AB}\varepsilon_B\\&-e^{K/2}g^{ij}\left(\partial_j+K_j\right)X^I\mathrm{Im}\mathcal{N}_{IJ}F^{-J}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B\\\delta\zeta_\alpha=&U_{u\alpha A}\left(i\nabla_\mu q^u\gamma^\mu\varepsilon^A+2ge^{K/2}k^u_IX^I\epsilon^{AB}\varepsilon_B\right)\end{aligned}$$

$$W\equiv e^{K/2}P^3_IX^I=\frac{1}{4\sqrt{z^1z^2z^3}}\left(8-e_0e^{2\phi}+2e^{2\phi}(z^1z^2+z^3z^1+z^2z^3)\right)$$

$$H_{\mu\nu}\equiv e^{K/2}X^I\mathrm{Im}\mathcal{N}_{IJ}F^{-J}_{\mu\nu}=-\frac{1}{2\sqrt{2z^1z^2z^3}}\left(z^1z^2z^3F^{-0}_{\mu\nu}+z^1F^{-1}_{\mu\nu}+z^2F^{-2}_{\mu\nu}+z^3F^{-3}_{\mu\nu}\right)$$

$$z^i = ib^i + v^i$$

$$v^i = e^{2u_i}.$$

$$\begin{aligned} e^{-1}\mathcal{L} =& \frac{1}{2}R - \sum_{i=1}^{3} \partial_\mu u_i \partial^\mu u_i - \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{4\phi} D_\mu \sigma D^\mu \sigma - g^2 \mathcal{V} \\ & - \left[e^{2u_1+2u_2+2u_3} F^0_{\mu\nu} F^{0\mu\nu} + e^{2u_1-2u_2-2u_3} F^1_{\mu\nu} F^{1\mu\nu} \right. \\ & \left. + e^{-2u_1+2u_2-2u_3} F^2_{\mu\nu} F^{2\mu\nu} + e^{-2u_1-2u_2+2u_3} F^3_{\mu\nu} F^{3\mu\nu} \right] \end{aligned}$$

$$\begin{aligned} \mathcal{V} =& -8e^{2\phi}(e^{-2u_1} + e^{-2u_2} + e^{-2u_3}) + e^{4\phi}(e^{-2u_1+2u_2+2u_3} + e^{2u_1-2u_2+2u_3} + e^{2u_1+2u_2-2u_3}) \\ & + \frac{1}{4} e_0^2 e^{4\phi} e^{-2u_1-2u_2-2u_3}, \end{aligned}$$

$$\mathcal{V} = \sum_{i=1}^{3} \left(\frac{\partial W}{\partial u_i}\right)^2 + \left(\frac{\partial W}{\partial \phi}\right)^2 - 3W^2$$

$$W = \frac{1}{4} e^{-u_1-u_2-u_3} \left(8 - e_0 e^{2\phi} + 2e^{2\phi}(e^{2u_1+2u_2} + e^{2u_2+2u_3} + e^{2u_3+2u_1})\right)$$

$$\begin{aligned} A^{\mathrm{R}} &= 4\sqrt{2}A^0 \\ A^m &= \sqrt{2}(e_0 A^0 - 2A^1 - 2A^2 - 2A^3) \\ A^{\mathcal{B}_1} &= 4\sqrt{2}(A^1 - 2A^2 + A^3) \\ A^{\mathcal{B}_2} &= 4\sqrt{2}(A^1 - A^3) \end{aligned}$$

$$\begin{aligned} \delta\psi_{\mu A} &\sim 2\nabla_\mu \varepsilon_A - iB_\mu \sigma^3{}_A{}^B \varepsilon_B - gW\gamma_\mu \sigma^3{}_{AB} \varepsilon^B + 4iH_{\mu\nu}\gamma^\nu \epsilon_{AB}\varepsilon^B \\ \delta\lambda^{iA} &\sim \partial_\mu u_i \gamma^\mu \varepsilon^A + g\partial_{u_i} W \sigma^{3AB} \varepsilon_B - i\partial_{u_i} H_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B \\ U^{\alpha A}_\phi \delta\zeta_\alpha &\sim U^{\alpha A}_\sigma \delta\zeta_\alpha \sim \partial_\mu \phi \gamma^\mu \varepsilon^A + \frac{i}{2}\partial_\phi B_\mu \gamma^\mu \sigma^3{}_B{}^A \varepsilon^B + g\partial_\phi W \sigma^{3AB}\varepsilon_B \end{aligned}$$

$$\begin{aligned} H_{\mu\nu} &= -\frac{1}{2\sqrt{2}}\left(e^{u_1+u_2+u_3} F^{-0}_{\mu\nu} + e^{u_1-u_2-u_3} F^{-1}_{\mu\nu} + e^{-u_1+u_2-u_3} F^{-2}_{\mu\nu} + e^{-u_1-u_2+u_3} F^{-3}_{\mu\nu}\right) \\ B_\mu &= -4\sqrt{2} g A^0_\mu + \frac{1}{2} e^{2\phi} D_\mu \sigma \end{aligned}$$

$$F^{-I} = \frac{1}{2}(F^I - i * F^I)$$

$$F^{-0}=\frac{1}{\sqrt{2}}e^{-u_1-u_2-u_3}(\bar{F}^{-12}+\bar{F}^{-34}+\bar{F}^{-56}+\bar{F}^{-78})$$
$$F^{-1}=\frac{1}{\sqrt{2}}e^{-u_1+u_2+u_3}(\bar{F}^{-12}-\bar{F}^{-34}-\bar{F}^{-56}+\bar{F}^{-78})$$
$$F^{-2}=\frac{1}{\sqrt{2}}e^{u_1-u_2+u_3}(-\bar{F}^{-12}+\bar{F}^{-34}-\bar{F}^{-56}+\bar{F}^{-78})$$
$$F^{-3}=\frac{1}{\sqrt{2}}e^{u_1+u_2-u_3}(-\bar{F}^{-12}-\bar{F}^{-34}+\bar{F}^{-56}+\bar{F}^{-78})$$

$$\bar{F}_{\mu\nu}^{-12}=-4\partial_{u_1}H_{\mu\nu},\bar{F}_{\mu\nu}^{-34}=-4\partial_{u_2}H_{\mu\nu},\bar{F}_{\mu\nu}^{-56}=-4\partial_{u_3}H_{\mu\nu},\bar{F}_{\mu\nu}^{-78}=4H_{\mu\nu}.$$

$$\epsilon=\varepsilon_1+\varepsilon^2.$$

$$\left[2\nabla_\mu-iB_\mu-gW\gamma_\mu+4iH_{\mu\nu}\gamma^\nu\right]\epsilon=0$$
$$\left[\partial_\mu u_i\gamma^\mu+g\partial_{u_i}W+i\partial_{u_i}H_{\mu\nu}\gamma^{\mu\nu}\right]\epsilon=0$$
$$\left[\partial_\mu\phi\gamma^\mu+g\partial_\phi W+\frac{i}{2}\partial_\phi B_\mu\gamma^\mu\right]\epsilon=0$$

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+g^2\mathcal{V}g_{\mu\nu}-2\left(T_{\mu\nu}^{\phi}+T_{\mu\nu}^{u_1}+T_{\mu\nu}^{u_2}+T_{\mu\nu}^{u_3}\right)-\frac{1}{2}e^{4\phi}T_{\mu\nu}^{\sigma}$$
$$-e^{2(u_1+u_2+u_3)}T_{\mu\nu}^{A^0}-e^{2(u_1-u_2-u_3)}T_{\mu\nu}^{A^1}-e^{-2(u_1-u_2+u_3)}T_{\mu\nu}^{A^2}-e^{-2(u_1+u_2-u_3)}T_{\mu\nu}^{A^3}=0,$$

$$T_{\mu\nu}^{X}=\partial_\mu X\partial_\nu X-\frac{1}{2}g_{\mu\nu}\partial_\rho X\partial^\rho X,$$
$$T_{\mu\nu}^{A^I}=g^{\rho\sigma}F_{\mu\rho}^{I}F_{\nu\sigma}^{I}-\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}^{I}F^{I\rho\sigma},$$

$$\partial_\nu\left(\sqrt{-g}e^{2(u_1+u_2+u_3)}F^{0\mu\nu}\right)+\frac{e_0}{2}\sqrt{2}\sqrt{-g}e^{4\phi}g^{\mu\nu}D_\nu\sigma=0,$$
$$\partial_\nu\left(\sqrt{-g}e^{2(u_1-u_2-u_3)}F^{1\mu\nu}\right)-\sqrt{2}\sqrt{-g}e^{4\phi}g^{\mu\nu}D_\nu\sigma=0,$$
$$\partial_\nu\left(\sqrt{-g}e^{-2(u_1-u_2+u_3)}F^{2\mu\nu}\right)-\sqrt{2}\sqrt{-g}e^{4\phi}g^{\mu\nu}D_\nu\sigma=0,$$
$$\partial_\nu\left(\sqrt{-g}e^{-2(u_1+u_2-u_3)}F^{3\mu\nu}\right)-\sqrt{2}\sqrt{-g}e^{4\phi}g^{\mu\nu}D_\nu\sigma=0.$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu u_1\right)-\frac{g^2}{2}\frac{\partial\mathcal{V}}{\partial u_1}$$
$$+e^{2(u_1+u_2+u_3)}F_{\mu\nu}^{0}F^{0\mu\nu}+e^{2(u_1-u_2-u_3)}F_{\mu\nu}^{1}F^{1\mu\nu}$$
$$-e^{-2(u_1-u_2+u_3)}F_{\mu\nu}^{2}F^{2\mu\nu}-e^{-2(u_1+u_2-u_3)}F_{\mu\nu}^{3}F^{3\mu\nu}=0$$
$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu u_2\right)-\frac{g^2}{2}\frac{\partial\mathcal{V}}{\partial u_2}$$
$$+e^{2(u_1+u_2+u_3)}F_{\mu\nu}^{0}F^{0\mu\nu}-e^{2(u_1-u_2-u_3)}F_{\mu\nu}^{1}F^{1\mu\nu}$$
$$+e^{-2(u_1-u_2+u_3)}F_{\mu\nu}^{2}F^{2\mu\nu}-e^{-2(u_1+u_2-u_3)}F_{\mu\nu}^{3}F^{3\mu\nu}=0$$
$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu u_3\right)-\frac{g^2}{2}\frac{\partial\mathcal{V}}{\partial u_3}$$
$$+e^{2(u_1+u_2+u_3)}F_{\mu\nu}^{0}F^{0\mu\nu}-e^{2(u_1-u_2-u_3)}F_{\mu\nu}^{1}F^{1\mu\nu}$$
$$-e^{-2(u_1-u_2+u_3)}F_{\mu\nu}^{2}F^{2\mu\nu}+e^{-2(u_1+u_2-u_3)}F_{\mu\nu}^{3}F^{3\mu\nu}=0$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)-\frac{g^2}{2}\frac{\partial\mathcal{V}}{\partial\phi}-\frac{1}{2}e^{4\phi}D_\mu\sigma D^\mu\sigma=0$$

$$e^{2u_i}=\sqrt{\frac{e_0}{6}},e^{-2\phi}=\frac{e_0}{6},L_{AdS_4}=\frac{1}{2}\left(\frac{e_0}{6}\right)^{3/4}$$
$$A\equiv A^0=A^1=A^2=A^3$$

$$e^{-1}\mathcal{L}=\frac{1}{2}R+12g^2-4F_{\mu\nu}F^{\mu\nu}$$

$$ds^2=e^{2V}ds^2_{AdS_2}+f^2dy^2+h^2dz^2$$
$$A^I=a^Idz\text{(B.1)}$$

$$\gamma^m=\Gamma^m\otimes\sigma^3,\gamma^2=1_2\otimes\sigma^1,\gamma^3=1_2\otimes\sigma^2$$

$$\epsilon=\psi\otimes\chi$$

$$D_m\psi=\frac{1}{2}\kappa\Gamma_m\psi$$

$$[-i(\kappa e^{-V}+4H_{23})\gamma^{23}+V'f^{-1}\gamma^2]\epsilon=gW\epsilon.$$

$$[i\cos\xi\gamma^{23}+\sin\xi\gamma^2]\epsilon=\epsilon$$

$$-\kappa e^{-V}-4H_{23}=gW\cos\xi,V'f^{-1}=gW\cos\xi$$

$$\epsilon=e^{i\frac{\xi}{2}\gamma^3}\eta,\gamma^2\eta=i\gamma^3\eta$$

$$\xi=0\pi,\gamma^{23}\epsilon=\pm i\epsilon$$

$$\left[\partial_y\eta-\frac{1}{2}V'\eta+\frac{i}{2}\left(\partial_y\xi+8fH_{23}+\kappa fe^{-V}\right)\gamma^3\right]\eta=0$$

$$[2\partial_z-iB_z+if^{-1}h'\cos\xi-4iH_{23}h\sin\xi$$
$$+(f^{-1}h'\sin\xi-gWh+4H_{23}h\cos\xi)\gamma^3]\eta=0$$

$$\eta=e^{V/2}e^{isz}\eta_0$$

$$\partial_y\xi+8fH_{23}+\kappa fe^{-V}=0$$
$$(s-B_z)+f^{-1}h'\cos\xi-4H_{23}h\sin\xi=0$$
$$f^{-1}h'\sin\xi-gWh+4H_{23}h\cos\xi=0$$

$$f^{-1}h'=gWh\sin\xi-(s-B_z)\cos\xi$$
$$hH_{23}=\frac{1}{4}gWh\cos\xi+\frac{1}{4}(s-B_z)\sin\xi$$

$$(s-B_z)\sin\xi=-2gWh\cos\xi-\kappa he^{-V},$$

$$H_{23}=-\frac{1}{4}gW\cos\xi-\frac{1}{4}\kappa e^{-V}$$
$$f^{-1}\partial_y\xi=2gW\cos\xi+\kappa e^{-V}$$

$$f^{-1}\frac{h'}{h}\sin\,\xi = \kappa e^{-V}\cos\,\xi + gW(1+\cos^2\,\xi).$$

$$f^{-1}u_i' + g\partial_{u_i}W\sin\,\xi = 0,$$

$$g\partial_{u_i}W\cos\,\xi + 4\partial_{u_i}H_{23} = 0.$$

$$f^{-1}\phi'\sin\,\xi + g\partial_\phi W = 0$$

$$g\partial_\phi W\cos\,\xi - \frac{1}{2}\partial_\phi B_z\sin\,\xi h^{-1} = 0$$

$$f^{-1}\xi' = 2gW\cos\,\xi + \kappa e^{-V}$$
$$f^{-1}V' = gW\sin\,\xi$$
$$f^{-1}u_i' = -g\partial_{u_i}W\sin\,\xi$$
$$f^{-1}\phi' = -\frac{g\partial_\phi W}{\sin\,\xi}$$

$$f^{-1}\frac{h'}{h}\sin\,\xi = \kappa e^{-V}\cos\,\xi + gW(1+\cos^2\,\xi)$$

$$(s-B_z)\sin\,\xi = -2gWh\cos\,\xi - \kappa h e^{-V}$$
$$g\partial_\phi W\cos\,\xi = \frac{1}{2}\partial_\phi B_z\sin\,\xi h^{-1}$$

$$\partial_{u_i}H_{23} = -\frac{1}{4}g\partial_{u_i}W\cos\,\xi$$
$$H_{23} = -\frac{1}{4}gW\cos\,\xi - \frac{1}{4}\kappa e^{-V}$$

$$\partial_y W = -gf\sin\,\xi\left[\sum_{i=1}^{3}\left(\partial_{u_i}W\right)^2 + \frac{1}{\sin^2\,\xi}\left(\partial_\phi W\right)^2\right]$$

$$he^{-V} = k\sin\,\xi$$

$$f^{-1}\xi' = -k^{-1}(s-B_z)e^{-V}$$
$$f^{-1}V' = gW\sin\,\xi$$
$$f^{-1}u_i' = -g\partial_{u_i}W\sin\,\xi$$
$$f^{-1}\phi' = -\frac{g\partial_\phi W}{\sin\,\xi}$$

$$(s-B_z) = -k(2gWe^V\cos\,\xi + \kappa)$$
$$g\partial_\phi W\cos\,\xi = \frac{1}{2}k^{-1}e^{-V}\partial_\phi B_z$$

$$\partial_\phi B_z = e^{2\phi}D_z\sigma.$$

$$D_z\sigma = \frac{2gke^V\partial_\phi W\cos\,\xi}{e^{2\phi}}$$

$$F_{yz}^I = (a^I)' = \left(\mathcal{J}^{(I)}\right)'$$

$$\mathcal{J}^{(0)} \equiv \frac{1}{\sqrt{2}} gke^{V} \cos\,\xi e^{-u_1-u_2-u_3}$$
$$\mathcal{J}^{(1)} \equiv \frac{1}{\sqrt{2}} gke^{V} \cos\,\xi e^{-u_1+u_2+u_3}$$
$$\mathcal{J}^{(2)} \equiv \frac{1}{\sqrt{2}} gke^{V} \cos\,\xi e^{u_1-u_2+u_3}$$
$$\mathcal{J}^{(3)} \equiv \frac{1}{\sqrt{2}} gke^{V} \cos\,\xi e^{u_1+u_2-u_3}$$

$$h \to -h, z \to -z,$$

$$U_{u\alpha A} U_v^{\alpha B} = \frac{1}{2} h_{uv} \delta_A{}^B + \frac{i}{2} \Omega_{uv}^{x} \sigma^3{}_A{}^B$$

$$\delta\lambda^{iA} \sim \frac{1}{4(z^i)^2} \partial_\mu z^i \gamma^\mu \varepsilon^A + g\partial_{z^i} W \sigma^{3AB} \varepsilon_B - i\partial_{z^i} H_{\mu\nu} \gamma^{\mu\nu} \epsilon^{AB} \varepsilon_B$$

$$S = \frac{1}{16\pi G_N} \int \Big[R \star 1 + 2g^2 \left(8X_1X_2 + \frac{4(X_1+X_2)}{X_1^2 X_2^2} - \frac{1}{X_1^4 X_2^4} \right) \star 1 - \frac{1}{2} \sum_{I=1}^{2} d\varphi_I \wedge \star\, d\varphi_I$$
$$- \frac{1}{2} \sum_{I=1}^{2} \frac{1}{X_I^2} F_{(2)}^I \wedge \star\, F_{(2)}^I - \frac{1}{2} X_1^2 X_2^2 F_{(4)} \wedge \star\, F_{(4)}$$
$$+ g F_{(4)} \wedge A_{(3)} + F_{(2)}^1 \wedge F_{(2)}^2 \wedge A_{(3)} \Big]$$

$$X_1 = e^{-\frac{1}{\sqrt{10}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}, X_2 = e^{-\frac{1}{\sqrt{10}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}, F_{(2)}^I = dA_{(1)}^I, F_{(4)} = dA_{(3)}$$

$$0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left[R + 2g^2 \left(8X_1X_2 + \frac{4(X_1+X_2)}{X_1^2 X_2^2} - \frac{1}{X_1^4 X_2^4} \right) \right]$$
$$- \sum_{I=1}^{2} \left[\frac{1}{2} \partial_\mu \varphi_I \partial_\nu \varphi_I - \frac{1}{4} g_{\mu\nu} \partial^\rho \varphi_I \partial_\rho \varphi_I \right] - \sum_{I=1}^{2} \frac{1}{X_I^2} \left[\frac{1}{2} F_{\mu\rho}^I F_\nu^{I\rho} - \frac{1}{8} g_{\mu\nu} F_{\rho\sigma}^I F^{I\rho\sigma} \right]$$
$$- X_1^2 X_2^2 \left[\frac{1}{12} F_{\mu\rho\sigma\lambda} F_\nu^{\rho\sigma\lambda} - \frac{1}{96} g_{\mu\nu} F_{\rho\sigma\lambda\delta} F^{\rho\sigma\lambda\delta} \right].$$

$$\Box\, \varphi_1 = \frac{1}{2\sqrt{10}} \sum_{I=1}^{2} \frac{1}{X_I^2} F_{\mu\nu}^I F^{I\mu\nu} - \frac{1}{12\sqrt{10}} X_1^2 X_2^2 F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma}$$
$$+ \frac{8g^2}{\sqrt{10}} \left(4X_1X_2 - \frac{3(X_1+X_2)}{X_1^2 X_2^2} + \frac{2}{X_1^4 X_2^4} \right)$$
$$\Box\, \varphi_2 = \frac{1}{2\sqrt{2}} \left[\frac{1}{X_1^2} F_{\mu\nu}^1 F^{1\mu\nu} - \frac{1}{X_2^2} F_{\mu\nu}^2 F^{2\mu\nu} \right] + 4\sqrt{2} g^2 \frac{X_1 - X_2}{X_1^2 X_2^2}$$

$$d\left(X_1^{-2} \star F_{(2)}^1\right) = F_{(2)}^2 \wedge F_{(4)}$$
$$d\left(X_2^{-2} \star F_{(2)}^2\right) = F_{(2)}^1 \wedge F_{(4)}$$
$$d\left(X_1^2 X_2^2 \star F_{(4)}\right) = 2gF_{(4)} + F_{(2)}^1 \wedge F_{(2)}^2$$

$$X_1^2 X_2^2 \star F_{(4)} = 2gA_{(3)} + \frac{1}{2}\left(A^1_{(1)} \wedge F^2_{(2)} + A^2_{(1)} \wedge F^1_{(2)}\right) - dA_{(2)}$$

$$\begin{aligned} ds^2 =& (H_1 H_2)^{\frac{1}{5}} \Bigg(-\frac{(1+g^2r^2)(1-g^2y^2)(1-g^2z^2)}{\Xi_1\Xi_2\Xi_3} dt^2 + \frac{(r^2+y^2)(r^2+z^2)}{U} dr^2 \\ &+ \frac{(r^2+y^2)(y^2-z^2)y^2}{(1-g^2y^2)(a_1^2-y^2)(a_2^2-y^2)(a_3^2-y^2)} dy^2 \\ &+ \frac{(r^2+z^2)(z^2-y^2)z^2}{(1-g^2z^2)(a_1^2-z^2)(a_2^2-z^2)(a_3^2-z^2)} dz^2 \\ &+ \frac{(a_1^2+r^2)(a_1^2-y^2)(a_1^2-z^2)}{\Xi_1(a_1^2-a_2^2)(a_1^2-a_3^2)} d\phi_1^2 + \frac{(a_2^2+r^2)(a_2^2-y^2)(a_2^2-z^2)}{\Xi_2(a_2^2-a_1^2)(a_2^2-a_3^2)} d\phi_2^2 \\ &+ \frac{(a_3^2+r^2)(a_3^2-y^2)(a_3^2-z^2)}{\Xi_3(a_3^2-a_1^2)(a_3^2-a_2^2)} d\phi_3^2 + \frac{1-\frac{1}{H_1}}{1-(s_2/s_1)^2} K_1^2 + \frac{1-\frac{1}{H_2}}{1-(s_1/s_2)^2} K_2^2 \Bigg) \end{aligned}$$

$$s_I = \sinh\,\delta_I, c_I = \cosh\,\delta_I$$

$$\Xi_i = 1 - a_i^2 g^2$$

$$H_I(r,y,z) = 1 + \frac{2ms_I^2}{(r^2+y^2)(r^2+z^2)}$$

$$\begin{aligned} U(r) =& \frac{(1+g^2r^2)\prod_{i=1}^3 \left(r^2+a_i^2\right)}{r^2} - 2m + mg^2(s_1^2+s_2^2)\left(2r^2 + \sum_{i=1}^3 a_i^2\right) + \frac{4m^2g^2s_1^2s_2^2}{r^2} \\ &- \frac{2mg(s_1^2+s_2^2)a_1a_2a_3}{r^2} + \frac{2mg(c_1-c_2)^2}{r^2}(a_1+a_2a_3g)(a_2+a_3a_1g)(a_3+a_1a_2g) \\ &- \frac{1}{2}mg^2(c_1-c_2)^2(-2(a_1^2+a_2^2+a_3^2) - 8a_1a_2a_3g \\ &-(a_1+a_2+a_3)(a_2+a_3-a_1)(a_3+a_1-a_2)(a_1+a_2-a_3)g^2) \end{aligned}$$

$$K_1 = \frac{c_1+c_2}{2s_1}\mathcal{A}[y^2,z^2,0] + \frac{c_1-c_2}{2s_1}\mathcal{Y}$$

$$K_2 = \frac{c_1+c_2}{2s_2}\mathcal{A}[y^2,z^2,0] - \frac{c_1-c_2}{2s_2}\mathcal{Y}$$

$$\mathcal{A}[y^2,z^2,0] = \frac{(1-g^2y^2)(1-g^2z^2)}{\Xi_1\Xi_2\Xi_3}dt - \sum_{i=1}^3 \frac{a_i\left(a_i^2-y^2\right)\left(a_i^2-z^2\right)}{\Xi_i\left(a_i^2-a_j^2\right)\left(a_i^2-a_k^2\right)} d\phi_i$$

$$\begin{aligned} \mathcal{Y} =& \frac{(1-g^2y^2)(1-g^2z^2)(1-(a_1^2+a_2^2+a_3^2)g^2 - 2a_1a_2a_3g^3)}{\Xi_1\Xi_2\Xi_3} dt \\ &+ \sum_{i=1}^3 \frac{a_i\left(a_i^2-y^2\right)\left(a_i^2-z^2\right)\left(1-\left(a_i^2-a_j^2-a_k^2\right)g^2 + \frac{2a_ja_k}{a_i}g\right)}{\Xi_i\left(a_i^2-a_j^2\right)\left(a_i^2-a_k^2\right)} d\phi_i \end{aligned}$$

$$A^I_{(1)} = \left(1 - \frac{1}{H_I}\right)K_I + \alpha_I dt$$

$$
\begin{aligned}
A_{(3)} =& 2ms_1s_2g^4a_1a_2a_3\left[\mathcal{A}[y^2,z^2,0]-\mathcal{A}[y^2,z^2,g^{-2}]\right]\\
&\wedge\left[\frac{dz\wedge(\mathcal{A}[y^2,0,0]-\mathcal{A}[y^2,0,g^{-2}])}{(r^2+y^2)z}+\frac{dy\wedge(\mathcal{A}[z^2,0,0]-\mathcal{A}[z^2,0,g^{-2}])}{(r^2+z^2)y}\right]\\
&+2ms_1s_2g^3\mathcal{A}[y^2,z^2,0]\\
&\wedge\left[\frac{zdz\wedge(\mathcal{A}[y^2,0,0]-\mathcal{A}[y^2,0,g^{-2}])}{(r^2+y^2)}+\frac{ydy\wedge(\mathcal{A}[z^2,0,0]-\mathcal{A}[z^2,0,g^{-2}])}{(r^2+z^2)}\right]
\end{aligned}
$$

$$
\mathcal{A}[v_1,v_2,v_3]=\left[\prod_{i=1}^{3}\frac{1-g^2v_i}{\Xi_i}\right]dt-\sum_{i=1}^{3}\left[\frac{(a_i^2-v_1)(a_i^2-v_2)(a_i^2-v_3)}{a_i\Xi_i\prod_{j=1(\neq i)}^{3}(a_i^2-a_j^2)}\right]d\phi_i
$$

$$
X_I=\frac{(H_1H_2)^{\frac{2}{5}}}{H_I}
$$

$$
\begin{aligned}
A_{(2)} =& \left(\frac{1}{H_1}+\frac{1}{H_2}\right)\frac{ms_1s_2(a_1+a_2a_3g)}{(r^2+y^2)(r^2+z^2)}\left(\frac{(1-g^2y^2)(1-g^2z^2)\mu_1^2}{\Xi_1(1-a_2^2g^2)(1-a_3^2g^2)}dt\wedge d\phi_1\right.\\
&\left.+\frac{g(a_3^2-a_2^2)\mu_2^2\mu_3^2}{\Xi_2\Xi_3}d\phi_2\wedge d\phi_3\right)\\
&+(\text{ cyclic-permutations })\\
&-\frac{1}{2}dt\wedge\left(\alpha_1A_{(1)}^2+\alpha_2A_{(1)}^1\right)
\end{aligned}
$$

$$
\mu_i^2=\frac{(a_i^2-y^2)(a_i^2-z^2)}{\prod_{j=1(\neq i)}^{3}(a_i^2-a_j^2)},
$$

$$
(1-g^2a_i^2)\tilde{r}^2\tilde{\mu}_i^2=(r^2+a_i^2)\mu_i^2
$$

$$
ds^2\sim\frac{d\tilde{r}^2}{g^2\tilde{r}^2}+\tilde{r}^2\left(-g^2dt^2+\sum_i\left(d\tilde{\mu}_i^2+\tilde{\mu}_i^2d\phi_i^2\right)\right)+\cdots
$$

$$
0\le a_1\le z=\sqrt{a_1^2\cos^2\theta_2+a_2^2\sin^2\theta_2}\le a_2\le y=\sqrt{a_2^2\cos^2\theta_1+a_3^2\sin^2\theta_1}\le a_3\le g^{-1},
$$

$$
\begin{aligned}
S=&\frac{1}{4G_N}\int_{S_5}d^5x\sqrt{h}\Big|_{r=r_+}=\frac{1}{4G_N}\int_{a_1}^{a_2}dz\int_{a_2}^{a_3}dy\int_0^{2\pi}d\phi_1d\phi_2d\phi_3\sqrt{h}\Big|_{r=r_+}\\
=&\frac{\pi^3}{4G_N\Xi_1\Xi_2\Xi_3}\frac{\sqrt{\mathcal{S}(r_+)}}{r_+}
\end{aligned}
$$

$$
h_{ij}=g_{\mu\nu}\frac{\partial x^\mu}{\partial x^i}\frac{dx^\nu}{\partial x^j}\Big|_{t,r=\aleph}
$$

$$
\begin{aligned}
h=&\frac{y^2z^2(y^2-z^2)^2}{\Xi_1^2\Xi_2^2\Xi_3^2(a_1^2-a_2^2)^2(a_2^2-a_3^2)^2(a_3^2-a_1^2)^2}\\
&\times\left[\frac{\Xi_1\Xi_2\Xi_3r^2\left((r^2+y^2)(r^2+z^2)-r^2(r^2+g^2y^2z^2)\right)}{(1-g^2y^2)(1-g^2z^2)}U(r)+\frac{\mathcal{S}(r)}{r^2}\right.\\
&\left.-\left((a_1^2+a_2^2+a_3^2+a_1^2a_2^2a_3^2g^4)r^4+(1-g^2r^2)\left((a_1^2a_2^2+a_2^2a_3^2+a_3^2a_1^2)r^2+a_1^2a_2^2a_3^2\right)\right)U(r)\right]
\end{aligned}
$$

$$\mathcal{S}(r) \equiv \prod_{I=1}^{2} \left((r^2+a_1^2)(r^2+a_2^2)(r^2+a_3^2) + 2ms_I^2(r^2 - a_1a_2a_3g)\right)$$
$$+2mg(c_1-c_2)^2(r^2+a_1^2)(r^2+a_2^2)(r^2+a_3^2)(a_1+a_2a_3g)(a_2+a_3a_1g)(a_3+a_1a_2g)$$

$$J_1 = -\frac{1}{16\pi G_N}\int_{S^5} \star\, dK_1$$
$$= \frac{\pi^2 m}{16G_N\Xi_1\prod_{j=1}^{3}\Xi_j}\left[4a_1c_1c_2 + 4g(1-c_1c_2)(a_2+a_1a_3g)(a_3+a_1a_2g)\right.$$
$$\left.+(c_1-c_2)^2\left(2a_2a_3g + a_1\left(1+2\Xi_1 - \sum_{j=1}^{3}\Xi_j\right)\right)\left(1+2a_1a_2a_3g^3 - \sum_{j=1}^{3}\Xi_j\right)\right]$$

$$K_i^{\mu}\partial_{\mu} = -\partial_{\phi_i} \;\Leftrightarrow\; K_i = -g_{\mu\phi_i}dx^{\mu}$$

$$Q_1 = -\frac{1}{16\pi G_N}\int_{S^5}\left(X_1^{-2}\star F_{(2)}^1 - F_{(2)}^1\wedge A_{(3)}\right)$$
$$= \frac{\pi^2 m s_1}{4G_N\Xi_1\Xi_2\Xi_3}\left[2c_1 - (c_1-c_2)g^2(a_1^2+a_2^2+a_3^2+2a_1a_2a_3g)\right]$$

$$ds^2|_{\rho^2\ll r_+} = (H_1H_2)^{\frac{1}{5}}\frac{(r_+^2+y^2)(y^2-z^2)y^2}{(1-g^2y^2)(a_1^2-y^2)(a_2^2-y^2)(a_3^2-y^2)}dy^2$$
$$+(H_1H_2)^{\frac{1}{5}}\frac{(r_+^2+z^2)(z^2-y^2)z^2}{(1-g^2z^2)(a_1^2-z^2)(a_2^2-z^2)(a_3^2-z^2)}dz^2$$
$$+(H_1H_2)^{\frac{1}{5}}(r_+^2+y^2)(r_+^2+z^2)\left[\frac{r_+^4}{\mathcal{S}(r_+)}U'(r_+)\rho^2d\tau^2 + \frac{4d\rho^2}{U'(r_+)}\right]$$
$$+\sum_{i,j=1}^{3} g_{ij}(r_+,y,z)(d\phi_i + \mathrm{i}\Omega_i d\tau)\left(d\phi_j + \mathrm{i}\Omega_j d\tau\right)$$

$$(\tau,\phi_i)\sim(\tau+\beta,\phi_i-\mathrm{i}\Omega_i\beta) \text{ where } \beta = 2\pi\frac{2\sqrt{\mathcal{S}(r_+)}}{r_+^2U'(r_+)}.$$

$$T = \beta^{-1} = \frac{r_+^2U'(r_+)}{4\pi\sqrt{\mathcal{S}(r_+)}}$$

$$\ell = \partial_t + \sum_{i=1}^{3}\Omega_i\partial_{\phi_i}$$

$$w_1(r)=\frac{1}{\mathcal{S}(r)}\left[\frac{1}{2}\left(\prod_{i=1}^{3}\left(r^2+a_i^2\right)+2ms_1^2(r^2-a_1a_2a_3g)\right)\right.$$
$$\times\left(a_1(1+g^2r^2)(r^2+a_2^2)(r^2+a_3^2)+2ms_2^2(a_1g^2r^2-a_2a_3g)\right)$$
$$+\frac{1}{2}\left(\prod_{i=1}^{3}\left(r^2+a_i^2\right)+2ms_2^2(r^2-a_1a_2a_3g)\right)$$
$$\times\left(a_1(1+g^2r^2)(r^2+a_2^2)(r^2+a_3^2)+2ms_1^2(a_1g^2r^2-a_2a_3g)\right)$$
$$-m(c_1-c_2)^2(r^2+a_2^2)(r^2+a_3^2)\{2g(gr^2-a_1)(a_1+a_2a_3g)(a_2+a_3a_1g)(a_3+a_1a_2g)$$
$$+(1+a_1g)^2(a_1+a_2a_3g)(1-a_1g-a_2g-a_3g)(1-a_1g+a_2g+a_3g)r^2\}$$

$$\Phi_I=\ell^\mu A^I_{(1)\mu}\big|_{r=r_+}-\ell^\mu A^I_{(1)\mu}\big|_{r\to\infty}$$

$$\Phi_1=\frac{2mr_+^2s_1c_1\left(\prod_i\left(r_+^2+a_i^2\right)+2ms_2^2(r_+^2-a_1a_2a_3g)\right)}{\mathcal{S}(r_+)}$$
$$-\frac{mgr_+^2s_1(c_1-c_2)}{\mathcal{S}(r_+)}\left\{g(a_1^2+a_2^2+a_3^2+2a_1a_2a_3g)\left(\prod_i\left(r_+^2+a_i^2\right)+2ms_2^2(r_+^2-a_1a_2a_3g)\right)\right.$$
$$\left.+4ms_2^2(a_1+a_2a_3g)(a_2+a_3a_1g)(a_3+a_1a_2g)\right\}$$

$$E=\frac{1}{8(D-3)\pi g^3}\int_{\mathcal{B}}d\mathcal{B}_a\widetilde{\Omega}^{-(D-3)}\bar{n}^c\bar{n}^d\bar{C}^a{}_{cbd}\xi^b$$

$$\widetilde{\Omega}=\frac{1}{gr}$$

$$g_{rr}=\frac{1}{g^2r^2},g^{rr}=g^2r^2,\bar{C}^a_{cbd}=\mathcal{O}(r^{-8})$$
$$\bar{n}^r=\bar{g}^{ra}\bar{n}_a=\bar{g}^{rr}\left(-\frac{1}{gr^2}\right)=\widetilde{\Omega}^{-2}g^{rr}\left(-\frac{1}{gr^2}\right)=-\widetilde{\Omega}^{-2}g$$

$$\mathrm{vol}_6=\frac{yz|y^2-z^2|dt\wedge dy\wedge dz\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3}{g^5(a_1^2-a_2^2)(a_3^2-a_1^2)(a_2^2-a_3^2)\Xi_1\Xi_2\Xi_3}+\mathcal{O}(r^{-1})$$

$$d\mathcal{B}_t=\frac{yz|y^2-z^2|dy\wedge dz\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3}{g^5(a_1^2-a_2^2)(a_3^2-a_1^2)(a_2^2-a_3^2)\Xi_1\Xi_2\Xi_3}$$

$$E=\frac{1}{32\pi g}\int_{\mathcal{B}}d\mathcal{B}_t\widetilde{\Omega}^{-8}\bar{C}^t{}_{rtr}$$

$$E=\frac{m\pi^2}{8G_N\Xi_1\Xi_2\Xi_3}\left[\sum_{i=1}^{3}\frac{2}{\Xi_i}-1+\frac{5(s_1^2+s_2^2)}{2}+\frac{s_1^2+s_2^2}{2}\sum_{i=1}^{3}\left(\frac{2\left(1+a_i^2g^2-\Sigma_2-2\Pi_1\right)}{\Xi_i}-\Xi_i\right)\right]$$
$$+\frac{m\pi^2(c_1-c_2)^2}{32G_N\Xi_1^2\Xi_2^2\Xi_3^2}[-10\Sigma_2-16\Pi_1+11\Sigma_4+13\Pi_{22}+32\Pi_1\Sigma_2-3(\Sigma_6+5\Pi_{42}+4\Pi_1^2)$$
$$-16\Pi_1\Sigma_2^2+\Pi_{62}+3\Pi_{44}-5\Pi_1^2\Sigma_2+8\Pi_1(2\Pi_2+\Pi_{42})+\Pi_1^2(\Sigma_4+3\Pi_{22})]$$

$$\Sigma_n \equiv (a_1 g)^n + (a_2 g)^n + (a_3 g)^n$$
$$\Pi_n \equiv (a_1 g)^n (a_2 g)^n (a_3 g)^n$$
$$\Pi_{nm} \equiv (a_1 g)^n ((a_2 g)^m + (a_3 g)^m) + (a_2 g)^n ((a_3 g)^m + (a_1 g)^m) + (a_3 g)^n ((a_1 g)^m + (a_2 g)^m)$$

$$dE = TdS + \sum_{i=1}^{3} \Omega_i dJ_i + \sum_{I=1}^{2} \Phi_I dQ_I$$

$$\begin{aligned}
I =& -S + \beta \left[E - \sum_{i=1}^{3} \Omega_i J_i - \sum_{I=1}^{2} \Phi_I Q_I \right] \\
=& \frac{\pi^2 \beta}{16 G_N \Xi_1 \Xi_2 \Xi_3 r_+^2} \Bigg[(1 - g^2 r_+^2) \prod_{i=1}^{3} \left(r_+^2 + a_i^2 \right) - 2m(s_1^2 + s_2^2)(g^2 r_+^4 + a_1 a_2 a_3 g) \\
& - \frac{2m^2 s_1^2 s_2^2}{g^4 \mathcal{S}(r_+)} \left(\prod_{i=1}^{3} \left(r_+^2 + a_i^2 \right) + 2m s_1^2 (r_+^2 - a_1 a_2 a_3 g) \right) \\
& \times \left(g^6 r_+^6 + (\Sigma_2 + 2\Pi_1) g^4 r_+^4 - \left(2\Pi_1 + \frac{1}{2}\Pi_{22} - 2g^4 m s_2^2 \right) g^2 r_+^2 + \Pi_1 (2g^4 m s_2^2 - \Pi_1) \right) \\
& - \frac{2m^2 s_1^2 s_2^2}{g^4 \mathcal{S}(r_+)} \left(\Pi_{i=1}^{3} \left(r_+^2 + a_i^2 \right) + 2m s_2^2 (r_+^2 - a_1 a_2 a_3 g) \right) \\
& \times \left(g^6 r_+^6 + (\Sigma_2 + 2\Pi_1) g^4 r_+^4 - \left(2\Pi_1 + \frac{1}{2}\Pi_{22} - 2g^4 m s_1^2 \right) g^2 r_+^2 + \Pi_1 (2g^4 m s_1^2 - \Pi_1) \right) \\
& + \frac{m(c_1 - c_2)^2 (a_1 + a_2 a_3 g)(a_2 + a_3 a_1 g)(a_3 + a_1 a_2 g)}{g^{11} \mathcal{S}(r_+)} \\
& \times \Bigg(-2\Xi_1 \Xi_2 \Xi_3 \left(g^6 r_+^6 + \Sigma_2 g^4 r_+^4 + \left(\frac{1}{2}\Pi_{22} + 2m g^4 (s_1^2 + s_2^2) \right) g^2 r_+^2 + \Pi_2 \right. \\
& \left. - 2g^4 m (1 + s_1^2 + s_2^2) - \frac{1}{2}(c_1 - c_2)^2 g^4 m (-2\Sigma_2 - 8\Pi_1 + \Sigma_4 - \Pi_{22}) + (s_1^2 + s_2^2) g^4 m \Sigma_2 \right) \\
& + g^4 m \left(4 + 8 g^4 m s_1^2 s_2^2 + (2 - \Sigma_2 - 2\Pi_1) \left(2(s_1^2 + s_2^2) - (c_1 - c_2)^2 (\Sigma_2 + 2\Pi_1) \right) \right) \\
& \times \frac{g^8 r_+^2 \Pi_{i=1}^{3} \left(r_+^2 + a_i^2 \right) - \Xi_1 \Xi_2 \Xi_3}{1 + g^2 r_+^2} - 8 g^8 m^2 s_1^2 s_2^2 (2 g^6 r_+^6 + \Sigma_2 g^4 r_+^4 - \Pi_2) \Bigg) \Bigg].
\end{aligned}$$

$$(a_1 + a_2 + a_3) g = \frac{2}{1 - e^{\delta_1 + \delta_2}}$$

$$E + g \sum_{i=1}^{3} J_i - \sum_{I=1}^{2} Q_I = 0$$

$$\beta \left[g - \sum_{i=1}^{3} \Omega_i - 2g \sum_{I=1}^{2} \Phi_I \right] = \pm 2\pi \mathrm{i}$$

$$I = -S + \beta \left[E - \sum_{i=1}^{3} \Omega_i J_i - \sum_{I=1}^{2} \Phi_I Q_I \right] = \frac{\pi^2}{8 G_N g} \frac{\varphi_1^2 \varphi_2^2}{\omega_1 \omega_2 \omega_3}$$

$$\varphi_I \equiv \beta(\Phi_I - 1), \omega_i \equiv \beta(\Omega_i + g)$$

$$T = 0 \;\Leftrightarrow\; U'(r_+) = 0$$

$$m^\star = \frac{M_1}{g^4 M_2},$$

$$M_1 = 2(2\Sigma_1 - 2\Sigma_2 - \Pi_{11} + 2\Pi_{21} + 2\Pi_1)(2\Pi_1 + \Pi_{21}) \prod_{i=1}^{3} (1 - a_i g),$$

$$\begin{aligned} M_2 = {} & 2(1-\Sigma_1)^3 \left(2 - (\Sigma_2 + 2\Pi_{11})(s_1^2 + s_2^2)\right) - 8(s_1^2 + s_2^2)(1-\Sigma_1)^2(2\Pi_1 + \Pi_{21}) \\ & + (c_1 - c_2)^2 (1-\Sigma_1)^2 (-2\Sigma_2 + 2\Sigma_1\Sigma_2 - 8\Pi_1 + \Sigma_4 - \Pi_{22} + 8\Sigma_1\Pi_1 \\ & - \Sigma_1^2(\Sigma_3 - \Pi_{21} + 2\Pi_1)\Big), \end{aligned}$$

$$r_+^\star = \sqrt{\frac{a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3 g}{1 - (a_1 + a_2 + a_3) g}}$$

$$\Omega_i^\star = -g, \Phi_I^\star = 1$$

$$\begin{aligned} & \frac{\frac{2}{g^3} Q_1^\star Q_2^\star (Q_1^\star + Q_2^\star) - \frac{\pi^2}{G_N g^5} (J_1^\star J_2^\star + J_2^\star J_3^\star + J_3^\star J_1^\star)}{\frac{8}{g}(Q_1^\star + Q_2^\star) - \frac{\pi^2}{G_N g^5}} \\ & = \left(-\frac{\pi^2}{16 G_N g^5} (J_1^\star + J_2^\star + J_3^\star) + \frac{1}{8g^2}((Q_1^\star)^2 + (Q_2^\star)^2) + \frac{1}{2g^2} Q_1^\star Q_2^\star \right) \\ & \times \left(1 - \sqrt{1 - \frac{\pi^2}{\left(-\frac{\pi^2}{16 G_N g^5} (J_1^\star + J_2^\star + J_3^\star) + \frac{1}{8g^2}((Q_1^\star)^2 + (Q_2^\star)^2) + \frac{1}{2g^2} Q_1^\star Q_2^\star \right)^2}} \right) \end{aligned}$$

$$S^\star = 2\pi \sqrt{\frac{\frac{2}{g^3} Q_1^\star Q_2^\star (Q_1^\star + Q_2^\star) - \frac{\pi^2}{G_N g^5} (J_1^\star J_2^\star + J_2^\star J_3^\star + J_3^\star J_1^\star)}{\frac{8}{g}(Q_1^\star + Q_2^\star) - \frac{\pi^2}{G_N g^5}}}$$

$$-\log Z_{S^1 \times S^5} = \frac{N^3}{24} \frac{\Delta_1^2 \Delta_2^2}{\hat{\omega}_1 \hat{\omega}_2 \hat{\omega}_3},$$

$$\Delta_1 + \Delta_2 - \hat{\omega}_1 - \hat{\omega}_2 - \hat{\omega}_3 = \pm 2\pi \mathrm{i}$$

$$N^3 = \frac{3\pi^2}{16 G_N g^5}$$

$$\hat{\omega}_i = \beta(\Omega_i + g) = \omega_i,$$

$$\Delta_I = -2g\beta(\Phi_I - 1) = -2g\varphi_I$$

$$-\log Z_{S^1 \times S^5} = \frac{N^3}{24} \frac{\Delta_1^2 \Delta_2^2}{\hat{\omega}_1 \hat{\omega}_2 \hat{\omega}_3} = \frac{\pi^2}{8 G_N g} \frac{\varphi_1^2 \varphi_2^2}{\omega_1 \omega_2 \omega_3}.$$

$$J_i \sim N^3, \frac{Q_I}{g} \sim N^3.$$

$$N^3 = \frac{3\pi^2}{64 G_N g^5}$$

$$-\log Z_{S^1 \times S^5} = \frac{N^3}{6} \frac{\Delta_1^2 \Delta_2^2}{\hat{\omega}_1 \hat{\omega}_2 \hat{\omega}_3}$$

$$X_{(p)} = \frac{1}{p!} X_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots dx^{\mu_p} \ (0 \le p \le d)$$

$$\star \left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\right) = \frac{\sqrt{|g|}}{(d-p)!} \varepsilon^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_{d-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{d-p}}$$

$$\varepsilon_{\mu_1 \cdots \mu_d} = \begin{cases} +1 & \text{if } \mu_1 \cdots \mu_d \text{ even permutation} \\ -1 & \text{if } \mu_1 \cdots \mu_d \text{ odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \star \left(\star X_{(p)}\right) &= (-1)^{s+p(d-p)} X_{(p)} \\ X_{(p)} \wedge \star Y_{(p)} &= Y_{(p)} \wedge \star X_{(p)} \\ &= \frac{1}{p!} X_{\mu_1 \cdots \mu_p} Y^{\mu_1 \cdots \mu_p} \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^d \end{aligned}$$

$$\begin{aligned} ds^2 =& H^{\frac{2}{5}} \Bigg(-\frac{(1+g^2r^2)(1-g^2y^2)(1-g^2z^2)}{(1-a_1^2g^2)(1-a_2^2g^2)(1-a_3^2g^2)} dt^2 + \frac{(r^2+y^2)(r^2+z^2)}{R} dr^2 \\ &+ \frac{(r^2+y^2)(y^2-z^2)y^2}{(1-g^2y^2)(a_1^2-y^2)(a_2^2-y^2)(a_3^2-y^2)} dy^2 \\ &+ \frac{(r^2+z^2)(z^2-y^2)z^2}{(1-g^2z^2)(a_1^2-z^2)(a_2^2-z^2)(a_3^2-z^2)} dz^2 + \frac{(a_1^2+r^2)(a_1^2-y^2)(a_1^2-z^2)}{(a_1^2-a_2^2)(a_1^2-a_3^2)(1-a_1^2g^2)} d\phi_1^2 \\ &+ \frac{(a_2^2+r^2)(a_2^2-y^2)(a_2^2-z^2)}{(a_2^2-a_1^2)(a_2^2-a_3^2)(1-a_2^2g^2)} d\phi_2^2 + \frac{(a_3^2+r^2)(a_3^2-y^2)(a_3^2-z^2)}{(a_3^2-a_1^2)(a_3^2-a_2^2)(1-a_3^2g^2)} d\phi_3^2 \\ &+ \frac{1-\frac{1}{H}}{H} \mathcal{A}[y^2, z^2, 0](C_0 dt + C_1 d\phi_1 + C_2 d\phi_2 + C_3 d\phi_3) \Bigg) \end{aligned}$$

$$A^I_{(1)} = \frac{2msc}{H(r^2+y^2)(r^2+z^2)} \mathcal{A}[y^2, z^2, 0] = \left(1 - \frac{1}{H}\right) \frac{c}{s} \mathcal{A}[y^2, z^2, 0]$$

$$A_{(3)} = 2ms^2g^4a_1a_2a_3\left[\mathcal{A}[y^2,z^2,0]-\mathcal{A}[y^2,z^2,g^{-2}]\right]$$
$$\wedge\left[\frac{dz\wedge(\mathcal{A}[y^2,0,0]-\mathcal{A}[y^2,0,g^{-2}])}{(r^2+y^2)z}+\frac{dy\wedge(\mathcal{A}[z^2,0,0]-\mathcal{A}[z^2,0,g^{-2}])}{(r^2+z^2)y}\right]$$
$$+2ms^2g^3\mathcal{A}[y^2,z^2,0]$$
$$\wedge\left[\frac{zdz\wedge(\mathcal{A}[y^2,0,0]-\mathcal{A}[y^2,0,g^{-2}])}{(r^2+y^2)}+\frac{ydy\wedge(\mathcal{A}[z^2,0,0]-\mathcal{A}[z^2,0,g^{-2}])}{(r^2+z^2)}\right]$$

$$A_{(2)} = \frac{2ms^2(a_1+a_2a_3g)}{H(r^2+y^2)(r^2+z^2)}\left(\frac{(1-g^2y^2)(1-g^2z^2)\mu_1^2}{(1-a_1^2g^2)(1-a_2^2g^2)(1-a_3^2g^2)}dt\wedge d\phi_1\right.$$
$$\left.+\frac{g(a_3^2-a_2^2)\mu_2^2\mu_3^2}{(1-a_2^2g^2)(1-a_3^2g^2)}d\phi_2\wedge d\phi_3\right)$$
$$+(\text{ cyclic-permutations })$$

$$s = \sinh\,\delta, c = \cosh\,\delta$$

$$R(r) = \frac{1+g^2r^2}{r^2}(r^2+a_1^2)(r^2+a_2^2)(r^2+a_3^2)-2m+g^2r^2\left(r^2+\frac{2ms^2}{r^2}\right)^2-g^2r^6$$
$$+2ms^2g^2(a_1^2+a_2^2+a_3^2)-\frac{4ms^2ga_1a_2a_3}{r^2}$$

$$C_0 = \frac{(1-g^2y^2)(1-g^2z^2)(c^2/s^2+(1-(a_1^2+a_2^2+a_3^2)g^2-2a_1a_2a_3g^3)H)}{(1-a_1^2g^2)(1-a_2^2g^2)(1-a_3^2g^2)}$$

$$C_i = -\frac{a_i(a_i^2-y^2)(a_i^2-z^2)\left(c^2/s^2-\left(1-(a_i^2-a_j^2-a_k^2)g^2+\frac{2a_ja_k}{a_i}g\right)H\right)}{(a_i^2-a_j^2)(a_i^2-a_k^2)(1-a_i^2g^2)}$$

$$\delta=\delta_1=\delta_2$$

$$(t,r,y=a\sin\,\theta_1,\theta_2,\phi_1,\phi_2,\phi_3)=(t,r,y=a\cos\,\theta,\psi,\xi,\zeta,\phi)^{\blacksquare}$$
$$=(t,r,y,\vartheta,\phi_2,\phi_3,\phi)^{\#}$$

$$ds^2 = (H_1H_2)^{\frac{1}{5}}\left(\frac{r^2+y^2}{\Delta_r}dr^2+\frac{r^2+y^2}{\Delta_y}dy^2+\frac{r^2y^2}{a^2}(d\theta_2^2+\sin^2\,\theta_2d\phi_1^2+\cos^2\,\theta_2d\phi_2^2)\right.$$
$$+\frac{1}{H_1H_2(1-a^2g^2)^2(r^2+y^2)}\left[-(V_y^2\Delta_r-\mathcal{V}_1\mathcal{V}_2\Delta_y)dt^2+(\mathcal{W}_1\mathcal{W}_2\Delta_y-\tilde{V}_y^2\Delta_r)a^2d\phi_3^2\right.$$
$$\left.\left.-\frac{4m'c_1'c_2'\tilde{c}_1'\tilde{c}_2'\Delta_y}{ar^2}dtd\phi_3\right]\right)$$
$$=(H_1H_2)^{\frac{1}{5}}\left(-\frac{(1+g^2r^2)(1-g^2y^2)}{1-a^2g^2}dt^2+\frac{r^2+y^2}{\Delta_r}dr^2+\frac{r^2+y^2}{1-g^2y^2}d\theta_1^2\right.$$
$$+r^2\sin^2\,\theta_1(d\theta_2^2+\sin^2\,\theta_2d\phi_1^2+\cos^2\,\theta_2d\phi_2^2)+\frac{r^2+a^2}{1-a^2g^2}\cos^2\,\theta_1d\phi_3^2$$
$$\left.+\frac{2m's_1'k_1^2}{r^2(r^2+y^2)H_1(1-a^2g^2)^2(s_1'^2-s_2'^2)}+\frac{2m's_2^2k_2^2}{r^2(r^2+y^2)H_2(1-a^2g^2)^2(s_2'^2-s_1'^2)}\right)$$

$$A_{\#}^1 = \frac{2m's_1'}{H_1(r^2+y^2)r^2(1-a^2g^2)}k_1$$
$$A_{\blacksquare}^2 = \frac{2m's_2'}{H_2(r^2+y^2)r^2(1-a^2g^2)}k_2$$

$$A_{\#} = -\frac{2m's_1's_2'a}{r^2+y^2}\sin^4\theta_1 \mathrm{vol}_{S^3} - \frac{2m's_1's_2'ag}{(1-a^2g^2)r^2}\sin\theta_1\cos\theta_1 dt\wedge d\theta_1\wedge d\phi_3,$$

$$\mathrm{vol}_{S^3} = \sin\theta_2\cos\theta_2 d\theta_2\wedge d\phi_1\wedge d\phi_2$$

$$A_{\triangle} = \left(\frac{1}{H_1}+\frac{1}{H_2}\right)\frac{m's_1's_2'(1-g^2y^2)a}{(r^2+y^2)r^2(1-a^2g^2)}\cos^2\theta_1 dt\wedge d\phi_3$$

$$s_I' = \sinh\delta_I', c_I' = \cosh\delta_I'$$

$$\tilde{c}_I' = \sqrt{1+a^2g^2s_I'^2}$$

$$H_I(r,y) = 1+\frac{2m's_I'^2}{(r^2+y^2)r^2}$$

$$\Delta_r(r) = \left(r^2+a^2-\frac{2m'}{r^2}\right)\left(1+g^2r^2-\frac{2m'g^2s_1'^2s_2'^2}{r^2}\right)+2m'g^2c_1'^2c_2'^2$$

$$\Delta_y(y) = (1-g^2y^2)(a^2-y^2)$$

$$\mathcal{V}_I(r) = 1+g^2r^2+\frac{2m's_I'^2g^2}{r^2}$$

$$V_y(y) = 1-g^2y^2$$

$$\mathcal{W}_I(r) = 1+\frac{r^2}{a^2}+\frac{2m's_I'^2}{r^2a^2}$$

$$\tilde{V}_y(y) = 1-\frac{y^2}{a^2}$$

$$k_1 = c_1'\tilde{c}_2'(1-g^2y^2)dt - c_2'\tilde{c}_1'a\cos^2\theta_1 d\phi_3$$

$$k_2 = c_2'\tilde{c}_1'(1-g^2y^2)dt - c_1'\tilde{c}_2'a\cos^2\theta_1 d\phi_3$$

$$\frac{c_1-\frac{1}{2}a^2g^2(c_1-c_2)}{s_1} = \frac{c_1'\tilde{c}_2'}{s_1'}, \frac{c_2+\frac{1}{2}a^2g^2(c_1-c_2)}{s_2} = \frac{c_2'\tilde{c}_1'}{s_2'}, ms_I^2 = m's_I'^2$$

$$\begin{aligned} ds^2 =& (H_1H_2)^{\frac{1}{5}}\left(-\frac{(1+g^2r^2)(1-g^2y^2)}{1-a^2g^2}dt^2+\frac{r^2+y^2}{\Delta_r}dr^2+\frac{r^2+y^2}{1-g^2y^2}d\theta_1^2\right.\\ &+r^2\sin^2\theta_1(d\theta_2^2+\sin^2\theta_2 d\phi_1^2+\cos^2\theta_2 d\phi_2^2)+\frac{r^2+a^2}{1-a^2g^2}\cos^2\theta_1 d\phi_3^2\\ &\left.+\frac{1-\frac{1}{H_1}}{1-(s_2/s_1)^2}K_1^2+\frac{1-\frac{1}{H_2}}{1-(s_1/s_2)^2}K_2^2\right) \end{aligned}$$

$$A_{(1)}^I = \left(1-\frac{1}{H_I}\right)K_I$$

$$A_{(2)} = \left(\frac{1}{H_1}+\frac{1}{H_2}\right)\frac{ms_1s_2(1-g^2y^2)a}{(r^2+y^2)r^2(1-a^2g^2)}\cos^2\theta_1 dt\wedge d\phi_3$$

$$A_{(3)} = -\frac{2ms_1s_2a}{r^2+y^2}\sin^4\theta_1\mathrm{vol}_{S^3} - \frac{2ms_1s_2ag}{(1-a^2g^2)r^2}\sin\theta_1\cos\theta_1 dt\wedge d\theta_1\wedge d\phi_3$$

$$K_1=\frac{\left(c_1-\frac{1}{2}a^2g^2(c_1-c_2)\right)(1-g^2y^2)dt-\left(c_2+\frac{1}{2}a^2g^2(c_1-c_2)\right)a\cos^2\ \theta_1 d\phi_3}{s_1(1-a^2g^2)}$$

$$K_2=\frac{\left(c_2+\frac{1}{2}a^2g^2(c_1-c_2)\right)(1-g^2y^2)dt-\left(c_1-\frac{1}{2}a^2g^2(c_1-c_2)\right)a\cos^2\ \theta_1 d\phi_3}{s_2(1-a^2g^2)}$$

$$\lim_{a_2\to 0}\lim_{a_1\to 0}\lim_{a_3\to a}.$$

$$T=T^{\triangle}=\frac{1}{2}T^{\odot},\quad \Omega_i=(0,0,\Omega)^{\boxdot},\qquad \Phi_I=\Phi_I^{\oplus}$$
$$S=\frac{1}{G_N}S^{\triangle},\qquad J_i=\frac{1}{G_N}(0,0,J)^{\circledcirc},\quad Q_I=\frac{1}{G_N}Q_I^{\boxplus}$$
$$E=\frac{1}{G_N}E^{\otimes},$$

$$t\to(1-ag)t,\ \text{ii})\ \tau\to\tau-gt,$$

$$\triangle=[1,13,30]|_{g\to -g}$$

$$g\to -g.$$

$$(t,r,\psi,\xi,\varphi_1,\varphi_2,\varphi_3)=\left(t,\sqrt{\rho^2-a^2},\tau,\xi,\varphi_1,\varphi_2,\varphi_3\right)^{\boxtimes}$$

$$\sigma_1=\cos\ \varphi_3 d\varphi_1+\sin\ \varphi_1\sin\ \varphi_3 d\varphi_2$$
$$\sigma_2=\sin\ \varphi_3 d\varphi_1-\sin\ \varphi_1\cos\ \varphi_3 d\varphi_2$$
$$\sigma_3=d\varphi_3+\cos\ \varphi_1 d\varphi_2$$

$$\begin{aligned}ds^2=&(H_1H_2)^{\frac{1}{5}}\left(-\frac{W}{f_1}dt^2+\frac{r^2(r^2+a^2)^2}{W}dr^2+\frac{r^2+a^2}{1-a^2g^2}ds^2_{\mathbb{CP}^2}\right.\\&\left.+\frac{f_1}{(r^2+a^2)^2H_1H_2(1-a^2g^2)^2}\left(\sigma+gdt-\frac{2f_2}{f_1}(1+ag)dt\right)^2\right)\\=&(H_1H_2)^{\frac{1}{5}}\left(-\frac{1+g^2r^2}{1-a^2g^2}dt^2+\frac{r^2(r^2+a^2)^2}{W}dr^2+\frac{r^2+a^2}{1-a^2g^2}\left(\sigma^2+ds^2_{\mathbb{CP}^2}\right)\right.\\&+\frac{2ms_1^2}{(s_1^2-s_2^2)(r^2+a^2)^2H_1(1-a^2g^2)^2}(\alpha_1(1+ag)dt-a\alpha_2(\sigma+gdt))^2\\&\left.+\frac{2ms_2^2}{(s_2^2-s_1^2)(r^2+a^2)^2H_2(1-a^2g^2)^2}(\alpha_2(1+ag)dt-a\alpha_1(\sigma+gdt))^2\right)\end{aligned}$$

$$\sigma=d\psi+\frac{1}{2}\sin^2\ \xi\sigma_3$$
$$ds^2_{\mathbb{CP}^2}=d\xi^2+\frac{1}{4}\sin^2\ \xi(\sigma_1^2+\sigma_2^2)+\frac{1}{4}\sin^2\ \xi\cos^2\ \xi\sigma_3^2$$

$$A^1_{(1)}=\frac{2ms_1}{(1-a^2g^2)H_1(r^2+a^2)^2}(\alpha_1(1+ag)dt-a\alpha_2(\sigma+gdt))$$
$$A^2_{(1)}=\frac{2ms_2}{(1-a^2g^2)H_2(r^2+a^2)^2}(\alpha_2(1+ag)dt-a\alpha_1(\sigma+gdt))$$

$$A_{(3)} = \frac{\text{mas}_1 s_2}{(1-a^2g^2)(1+ag)(r^2+a^2)}(\sigma + gdt) \wedge d\sigma$$
$$A_{(2)} = \left(\frac{1}{H_1}+\frac{1}{H_2}\right)\frac{\text{mas}_1 s_2}{(1+ag)(r^2+a^2)^2} dt \wedge \sigma$$

$$s_I = \sinh\ \delta_I, c_I = \cosh\ \delta_I$$
$$\alpha_1 = c_1 - \frac{1}{2}(1-(1-ag)^2)(c_1-c_2)$$
$$\alpha_2 = c_2 + \frac{1}{2}(1-(1-ag)^2)(c_1-c_2)$$
$$H_I(r) = 1 + \frac{2ms_I^2}{(r^2+a^2)^2}$$
$$f_1(r) = (1-a^2g^2)H_1H_2(r^2+a^2)^3 - \frac{4(1-ag)^2m^2a^2s_1^2s_2^2}{(r^2+a^2)^2} + \frac{1}{2}ma^2\left(4(1-ag)^2 + 2(1-(1-ag)^4)c_1c_2 + (1-(1-ag)^2)^2(c_1^2+c_2^2)\right)$$
$$f_2(r) = \frac{1}{2}g(1-ag)H_1H_2(r^2+a^2)^3 + \frac{1}{4}ma\left(2(1+(1-ag)^4)c_1c_2 + (1-(1-ag)^4)(c_1^2+c_2^2)\right)$$
$$W(r) = g^2H_1H_2(r^2+a^2)^4 + (1-a^2g^2)(r^2+a^2)^3 + \frac{1}{2}ma^2(4(1-ag)^2 + 2(1-(1-ag)^4)c_1c_2 + (1-(1-ag)^2)^2(c_1^2+c_2^2)) - \frac{1}{2}m(r^2+a^2)(4(1-a^2g^2) + 2a^2g^2(6-8ag+3a^2g^2)c_1c_2 - a^2g^2(2-ag)(2-3ag)(c_1^2+c_2^2))$$

$$(\psi, \xi, \varphi_1, \varphi_2, \varphi_3) = (-\phi_3, \theta_1, 2\theta_2, \phi_1-\phi_2, 2\phi_3-\phi_1-\phi_2)$$

$$\sigma = -\cos^2\ \theta_1 d\phi_3 - \sin^2\ \theta_1(\sin^2\ \theta_2 d\phi_1 + \cos^2\ \theta_2 d\phi_2)$$
$$ds^2_{\mathbb{CP}^2} = d\theta_1^2 + \sin^2\ \theta_1 d\theta_2^2 + \sin^2\ \theta_1 \sin^2\ \theta_2 \cos^2\ \theta_2 (d\phi_1 - d\phi_2)^2 + \sin^2\ \theta_1 \cos^2\ \theta_1 (d\phi_3 - \sin^2\ \theta_2 d\phi_1 - \cos^2\ \theta_2 d\phi_2)^2$$

$$ds^2 = (H_1H_2)^{\frac{1}{5}}\left(-\frac{1+g^2r^2}{1-a^2g^2}dt^2 + \frac{r^2(r^2+a^2)^2}{W}dr^2 + \frac{1-\frac{1}{H_1}}{1-(s_2/s_1)^2}K_1^2 + \frac{1-\frac{1}{H_2}}{1-(s_1/s_2)^2}K_2^2 + \frac{r^2+a^2}{1-a^2g^2}(d\theta_1^2 + \sin^2\ \theta_1 d\theta_2^2 + \sin^2\ \theta_1 \sin^2\ \theta_2 d\phi_1^2 + \sin^2\ \theta_1 \cos^2\ \theta_2 d\phi_2^2 + \cos^2\ \theta_1 d\phi_3^2)\right)$$

$$A^I_{(1)} = \left(1-\frac{1}{H_I}\right)K_I$$
$$A_{(3)} = \frac{mas_1s_2}{(1-a^2g^2)(1+ag)(r^2+a^2)}(\sigma + gdt) \wedge d\sigma$$
$$A_{(2)} = \left(\frac{1}{H_1}+\frac{1}{H_2}\right)\frac{mas_1s_2}{(1+ag)(r^2+a^2)^2} dt \wedge \sigma$$

$$K_1 = \frac{1}{s_1(1-a^2g^2)}(\alpha_1(1+ag)dt - a\alpha_2(\sigma+gdt))$$
$$K_2 = \frac{1}{s_2(1-a^2g^2)}(\alpha_2(1+ag)dt - a\alpha_1(\sigma+gdt))$$

$$\lim_{a_3\to -a}\lim_{a_2\to -a}\lim_{a_1\to -a}$$

$$T = \left[gT^{\oplus}\right]_{g\to -g'} \qquad \Omega_i = \left[g\Omega^{\boxdot}\right]_{g\to -g'}, \Phi_I = \left[\frac{1}{2}\Phi^{\dagger}\right]_{g\to -g}$$
$$S = \left[S^{\boxtimes}\right]_{g\to -g'} \qquad J_i = \left[J^{\boxplus}\right]_{g\to -g'}, Q_I = [2gQ_I^{\star}]_{g\to -g}$$
$$E = \left[gE^{\boxminus}\right]_{g\to -g'}$$

$$\omega_i = \frac{\Omega_i + g}{T} = \frac{[g\Omega^{\circ}]_{g\to -g} + g}{[gT^{\circledast}]_{g\to -g}} = \frac{\Omega^{\vee}-1}{T^{\triangle}}\Big|_{g\to -g} = -\zeta^{\ominus}\big|_{g\to -g}$$
$$\varphi_I = \frac{\Phi_I - 1}{T} = \frac{\left[\frac{1}{2}\Phi_I^{\odot}\right]_{g\to -g} - 1}{[gT^{\oslash}]_{g\to -g}} = \frac{\Phi_I^{\circledcirc} - 2}{2gT^{[13]}}\Big|_{g\to -g} = -\frac{1}{2g}\xi_I^{\wedge}\Big|_{g\to -g}$$

$$\begin{aligned} ds^2 =& ds_5^2 + (H_1(r,y,z)H_2(r,y,z))^{\frac{1}{5}}\frac{(r^2+y^2)(y^2-z^2)y^2}{(1-g^2y^2)(a_1^2-y^2)(a_2^2-y^2)(a_3^2-y^2)}dy^2 \\ &+(H_1(r,y,z)H_2(r,y,z))^{\frac{1}{5}}\frac{(r^2+z^2)(z^2-y^2)z^2}{(1-g^2z^2)(a_1^2-z^2)(a_2^2-z^2)(a_3^2-z^2)}dz^2 \end{aligned}$$
$$\begin{aligned} ds_5^2 =& -g_{00}(r,y,z)dt^2 + \sum_{i,j=1}^{3} g_{ij}(r,y,z)(d\phi_i - W_i(r,y,z)dt)\big(d\phi_j - W_j(r,y,z)dt\big) \\ &+(H_1(r,y,z)H_2(r,y,z))^{\frac{1}{5}}\frac{(r^2+y^2)(r^2+z^2)}{U}dr^2 \end{aligned}$$

$$g_{00}(r,y,z) = \frac{(H_1(r,y,z)H_2(r,y,z))^{\frac{1}{5}}r^4(r^2+y^2)(r^2+z^2)U(r)}{\mathcal{S}(r)}\left[1+\frac{Q(r,y,z)}{P(r,y,z)}U(r)\right]$$
$$W_i(r,y,z) = \frac{\frac{1+g^2r^2}{r^2}(1-g^2y^2)(1-g^2z^2)\mathcal{S}(r)}{P(r,y,z)}\left[w_i(r) - a_ir^2\big(r^2+a_j^2\big)\big(r^2+a_k^2\big)\frac{U(r)}{\mathcal{S}(r)}\right]$$

$$\begin{aligned} Q(r,y,z) =& (1+g^2r^2)\big[a_1^2a_2^2a_3^2\big(1-g^2(r^2+y^2+z^2)+g^4(r^2+y^2)(r^2+z^2)\big) \\ &+(a_1^2a_2^2+a_2^2a_3^2+a_3^2a_1^2)r^2\big(1-g^2(r^2+y^2+z^2)\big) \\ &+(a_1^2+a_2^2+a_3^2)r^2(r^2+g^2y^2z^2)+r^2\big(r^2(r^2+g^2y^2z^2)-(r^2+y^2)(r^2+z^2)\big)\big] \end{aligned}$$
$$\begin{aligned} P(r,y,z) =& \frac{1+g^2r^2}{r^2}(1-g^2y^2)(1-g^2z^2)\mathcal{S}(r) \\ &+\left[r^2(r^2+y^2)(r^2+z^2)\Xi_1\Xi_2\Xi_3 - (1-g^2y^2)(1-g^2z^2)\prod_{i=1}^{3}\left(r^2+a_i^2\right)\right]U(r) \end{aligned}$$

$$g_{00}(r,y,z) = \frac{\big(H_1(r_+,y,z)H_2(r_+,y,z)\big)^{\frac{1}{5}}r_+^4(r_+^2+y^2)(r_+^2+z^2)}{\mathcal{S}(r_+)}U'(r_+)\rho^2 + \mathcal{O}(\rho^4)$$
$$W_i(r_+,y,z) = \Omega_i + \mathcal{O}(\rho^2)$$

$$r = r_+ + \rho^2 \ (\rho^2 \ll r_+)$$

$$ds_5^2\big|_{\rho^2 \ll r_+}$$
$$= (H_1(r_+, y, z)H_2(r_+, y, z))^{\frac{1}{5}}(r_+^2 + y^2)(r_+^2 + z^2)\Big[- \frac{r_+^4}{\mathcal{S}(r_+)}U'(r_+)\rho^2 dt^2 + \frac{4d\rho^2}{U'(r_+)}\Big]$$
$$+ \sum_{i,j=1}^{3} g_{ij}(r_+, y, z)(d\phi_i - \Omega_i dt)(d\phi_j - \Omega_j dt)\,,$$

$$I_{11} = \frac{1}{16\pi G_{11}}\int \Big(\star_{11} R - \frac{1}{2}\mathcal{F} \wedge \star_{11} \mathcal{F} + \frac{1}{6}\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A}\Big),$$

$$\mathrm{d}s_4^2 = \frac{1}{\mu^2}\big(e^{-2\lambda_1}[\,\mathrm{d}\nu_1^2 + \nu_1^2(D\phi_1)^2] + e^{-2\lambda_2}[\,\mathrm{d}\nu_2^2 + \nu_2^2(D\phi_2)^2] + e^{4\lambda_1+4\lambda_2}\,\mathrm{d}\nu_3^2\big)$$

$$\nu_1 = \sin\,\alpha\cos\,\beta, \nu_2 = \sin\,\alpha\sin\,\beta, \nu_3 = \cos\,\alpha;\ \nu_1^2 + \nu_2^3 + \nu_3^2 = 1$$

$$\begin{aligned}
\mathrm{d}s_{11}^2 =& \Delta^{1/3}\,\mathrm{d}s_7^2 + \Delta^{-2/3}\,\mathrm{d}s_4^2\\
\mathcal{F} =& -e^{-4\lambda_1-4\lambda_2}\nu_3 \star_7 S + \frac{1}{\mu}S \wedge \mathrm{d}\nu_3\\
&+\frac{1}{\Delta\mu^2}\big[-e^{2\lambda_1}\nu_1^2 D\phi_1 \wedge F_2 \wedge \mathrm{d}\nu_3 - e^{2\lambda_2}\nu_2^2 D\phi_2 \wedge F_1 \wedge \mathrm{d}\nu_3\\
&+e^{-4\lambda_1-4\lambda_2}\nu_1\nu_3 D\phi_1 \wedge F_2 \wedge \mathrm{d}\nu_1 + e^{-4\lambda_1-4\lambda_2}\nu_2\nu_3 D\phi_2 \wedge F_1 \wedge \mathrm{d}\nu_2\big]\\
&+\frac{U\nu_1\nu_2}{\Delta^2\mu^3}\big[\nu_1 D\phi_1 \wedge D\phi_2 \wedge \mathrm{d}\nu_2 \wedge \mathrm{d}\nu_3 + \nu_2 D\phi_2 \wedge D\phi_1 \wedge \mathrm{d}\nu_1 \wedge \mathrm{d}\nu_3\\
&+\nu_3 D\phi_1 \wedge D\phi_2 \wedge \mathrm{d}\nu_1 \wedge \mathrm{d}\nu_2\big]\\
&+\frac{\nu_1\nu_2}{\Delta^2\mu^3}\big[2e^{2\lambda_1+2\lambda_2}\nu_1\nu_2 D\phi_1 \wedge D\phi_2 \wedge \mathrm{d}(\lambda_1 - \lambda_2) \wedge \mathrm{d}\nu_3\\
&+2e^{-2\lambda_1-4\lambda_2}\nu_1\nu_3 D\phi_1 \wedge D\phi_2 \wedge \mathrm{d}(3\lambda_1 - 2\lambda_2) \wedge \mathrm{d}\nu_2\\
&+2e^{-4\lambda_1-2\lambda_2}\nu_2\nu_3 D\phi_1 \wedge D\phi_2 \wedge \mathrm{d}(3\lambda_1 + 2\lambda_2) \wedge \mathrm{d}\nu_1\big]
\end{aligned}$$

$$\begin{aligned}
\Delta =& e^{2\lambda_1}\nu_1^2 + e^{2\lambda_2}\nu_2^2 + e^{-4\lambda_1-4\lambda_2}\nu_3^2\\
U =& -\big(2e^{2\lambda_1+2\lambda_2} + e^{-2\lambda_1-4\lambda_2}\big)\nu_1^2 - \big(2e^{2\lambda_1+2\lambda_2} + e^{-4\lambda_1-2\lambda_2}\big)\nu_2^2\\
&+\big(e^{-8\lambda_1-8\lambda_2} - 2e^{-2\lambda_1-4\lambda_2} - 2e^{-4\lambda_1-2\lambda_2}\big)\nu_3^2
\end{aligned}$$

$$\begin{aligned}
I_7 = \frac{1}{16\pi G_7}\int\ & [\star_7(R - V) - 5\,\mathrm{d}\lambda_+ \wedge \star_7\ \mathrm{d}\lambda_+ - \mathrm{d}\lambda_- \wedge \star_7\ \mathrm{d}\lambda_- - \frac{1}{2}e^{-4\lambda_1}F_1 \wedge \star_7 F_1\\
&-\frac{1}{2}e^{-4\lambda_2}F_2 \wedge \star_7 F_2 - \frac{1}{2}e^{-4\lambda_+}S \wedge \star_7 S + \frac{1}{2\mu}S \wedge \mathrm{d}S\\
&-\frac{1}{\mu}F_1 \wedge F_2 \wedge S + \frac{1}{4\mu}(A_1 \wedge F_1 \wedge F_2 \wedge F_2 + A_2 \wedge F_2 \wedge F_1 \wedge F_1)\Big]
\end{aligned}$$

$$V = \frac{\mu^2}{2}\big(e^{-8\lambda_1-8\lambda_2} - 4e^{-2\lambda_1-4\lambda_2} - 4e^{-4\lambda_1-2\lambda_2} - 8e^{2\lambda_1+2\lambda_2}\big)$$

$$\mathrm{d}S = \mu e^{-4\lambda_+} S + F_1 \wedge F_2$$

$$\mathrm{d}\left(e^{-4\lambda_{1,2}\star_7 F_{1,2}}\right) = \frac{1}{\mu} F_{1,2} \wedge F_{2,1} \wedge F_{2,1} - \frac{1}{\mu} F_{2,1} \wedge \mathrm{d}S$$

$$\mathrm{d}\star_7\ \mathrm{d}\left(3\lambda_{1,2} + 2\lambda_{2,1}\right) = -\frac{1}{2} e^{-4\lambda_{1,2}} F_{1,2} \wedge\star_7 F_{1,2} - \frac{1}{2} e^{-4\lambda_+} S \wedge\star_7 S + \frac{1}{4} \star_7 \frac{\partial V}{\partial \lambda_{1,2}}$$

$$R_{MN} = 5\partial_M \lambda_+ \partial_N \lambda_- + \partial_M \lambda_- \partial_N \lambda_+ + \frac{1}{2} e^{-4\lambda_1} F_{1MN}^2 + \frac{1}{2} e^{-4\lambda_2} F_{2MN}^2$$
$$+ \frac{1}{4} e^{-4\lambda_+} S_{MN}^2 + \frac{1}{5} g_{MN} \left[V - \frac{1}{4} e^{-4\lambda_1} F_1^2 - \frac{1}{4} e^{-4\lambda_2} F_2^2 - \frac{1}{6} e^{-4\lambda_+} S^2\right]$$

$$\delta\tilde{\psi}_M = [\partial_M + \frac{1}{4}\omega_{M\hat{A}\hat{B}}\tilde{\gamma}^{\hat{A}\hat{B}} \otimes 1_4 + \frac{\mu}{2} 1_8 \otimes (A_{1M}\Gamma^{12} + A_{2M}\Gamma^{34})$$
$$+ \left(\frac{\mu}{4} e^{-4\lambda_+}\tilde{\gamma}_M + \frac{1}{2}\tilde{\gamma}_M\tilde{\gamma}^N\partial_N\lambda_+\right) \otimes 1_4$$
$$+ \frac{1}{4}\left(e^{-2\lambda_1} F_{1MN}\tilde{\gamma}^N \otimes \Gamma^{12} + e^{-2\lambda_2} F_{2MN}\tilde{\gamma}^N \otimes \Gamma^{34}\right)$$
$$- \frac{1}{8} e^{-2\lambda_+} S_{MNP}\tilde{\gamma}^{NP} \otimes \Gamma^5\Big]\tilde{\epsilon} \otimes \mathcal{E}$$

$$\delta\tilde{\zeta}_{1,2} = \Big[\frac{\mu}{4}\left(e^{2\lambda_{1,2}} - e^{-4\lambda_+}\right) - \frac{1}{4}\tilde{\gamma}^M\partial_M\left(3\lambda_{1,2} + 2\lambda_{2,1}\right) \otimes 1_4$$
$$- \frac{1}{16} e^{-2\lambda_{1,2}F_{1,2MN}\tilde{\gamma}^{MN}} \otimes \Gamma^{12} + \frac{1}{48} e^{-2\lambda_+} S_{MNP}\tilde{\gamma}^{MNP} \otimes \Gamma^5\Big]\tilde{\epsilon} \otimes \mathcal{E}.$$

$$\mathrm{d}s_2^2 = (\mathrm{d}x_1^2 + \mathrm{d}x_2^2)/x_2^2$$

$$\Lambda_1 = \frac{1}{10}\ln\left(\frac{1 + 7z + 7z^2 + 33z^3 - (1 + 4z + 19z^2)\sqrt{1 + 3z^2}}{4z(1 - z)^2}\right)$$

$$\Lambda_2 = \Lambda_1 - \frac{1}{2}\ln\left(\frac{1 + z}{2z + \sqrt{1 + 3z^2}}\right)$$

$$f_0 = 4\Lambda_+$$

$$g_0 = \frac{1}{2}\ln\left(\frac{1}{8} e^{2\Lambda_+}\left[(1 - z)e^{2\Lambda_1} + (1 + z)e^{2\Lambda_2}\right]\right)$$

$$\mathrm{d}s_7^2 = e^{2f_0}\,\mathrm{d}s_5^2 + e^{2g_0} L^2\,\mathrm{d}s_2^2$$

$$S = c e^{f_0 + 2g_0 + 2\Lambda_+} L^2 \star_7 (F \wedge \mathrm{d}\omega)$$

$$A_{1,2} = a(1 \pm z)L\omega + b e^{f_0 + 2\Lambda_{1,2}} A$$

$$I_5 = \frac{1}{16\pi G_5}\int \left[\star_5\left(R + \frac{12}{L^2}\right) - \frac{\chi^2}{2} F \wedge\star_5 F + \frac{\chi^3}{3\sqrt{3}} A \wedge F \wedge F\right]$$

$$G_7 = e^{3f_0 + 2g_0} L^2 4\pi(\mathrm{g} - 1)G_5$$

$$\mathrm{d}\star_5 F = \frac{\chi}{\sqrt{3}} F \wedge F$$

$$R_{\mu\nu} = -\frac{4}{L^2} g_{\mu\nu} + \frac{\chi^2}{2} F_\mu{}^\alpha F_{\nu\alpha} - \frac{\chi^2}{12} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

$$\delta\psi_\mu = \left[\partial_\mu + \frac{1}{4}\omega_{\mu\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}\hat{\beta}} - \frac{\mathrm{i}\chi}{8\sqrt{3}}\left(\gamma_{\mu\alpha\beta}F^{\alpha\beta} - 4F_{\mu\alpha}\gamma^\alpha\right) - \frac{1}{2L}\left(\gamma_\mu + \mathrm{i}\sqrt{3}\chi A_\mu\right)\right]\epsilon$$

$$\tilde{\gamma}_{\mu}=-\chi_7\gamma_{\mu}\otimes\sigma_1,\tilde{\gamma}_{\hat{x}_1}=\chi_7 1_4\otimes\sigma_2,\tilde{\gamma}_{\hat{x}_2}=\chi_7 1_4\otimes\sigma_3,\tilde{\epsilon}=\epsilon\otimes\begin{pmatrix}1\\ \chi_7\end{pmatrix},$$

$$\left(\tilde{\gamma}_{\hat{x}_1\hat{x}_2}\otimes\Gamma^{12}\right)(\tilde{\epsilon}\otimes\mathcal{E})=\chi_7\tilde{\epsilon}\otimes\mathcal{E},(1_8\otimes\Gamma^5)(\tilde{\epsilon}\otimes\mathcal{E})=\tilde{\epsilon}\otimes\mathcal{E}$$

$$a=-\frac{\chi_7}{4},c=b=\frac{\chi}{\sqrt{3}},\partial_{x_1}(\tilde{\epsilon}\otimes\mathcal{E})=\partial_{x_2}(\tilde{\epsilon}\otimes\mathcal{E})=0$$

$$-1<ag<1,-1<bg<1,0\leq q\leq\frac{m}{1+ag+bg}$$

$$\begin{aligned}\mathrm{d}s^2_{\mathrm{CCLP}}=&-\frac{\Delta_\eta[(1+g^2r^2)\rho^2\,\mathrm{d}t+2qv]\mathrm{d}t}{\Xi_a\Xi_b\rho^2}+\frac{2qv\omega}{\rho^2}+\frac{f}{\rho^4}\left(\frac{\Delta_\eta\mathrm{d}t}{\Xi_a\Xi_b}-\omega\right)^2\\&+\frac{\rho^2\,\mathrm{d}r^2}{\Delta_r}+\frac{\rho^2\,\mathrm{d}\eta^2}{\Delta_\eta}+\frac{r^2+a^2}{\Xi_a}\sin^2\,\eta\,\mathrm{d}\xi_1^2+\frac{r^2+b^2}{\Xi_b}\cos^2\,\eta\,\mathrm{d}\xi_2^2\\A_{\mathrm{CCLP}}=&\frac{\sqrt{3}q}{\chi\rho^2}\left(\frac{\Delta_\eta\mathrm{d}t}{\Xi_a\Xi_b}-\omega\right)+\alpha\mathrm{d}t\end{aligned}$$

$$v=b\sin^2\,\eta\,\mathrm{d}\xi_1+a\cos^2\,\eta\,\mathrm{d}\xi_2,\omega=\frac{a\sin^2\,\eta}{\Xi_a}\,\mathrm{d}\xi_1+\frac{b\cos^2\,\eta}{\Xi_b}\,\mathrm{d}\xi_2$$

$$\begin{aligned}\Delta_r(r)=&\frac{(r^2+a^2)(r^2+b^2)(1+g^2r^2)+q^2+2abq}{r^2}-2m\\\Delta_\eta(\eta)=&1-a^2g^2\cos^2\,\eta-b^2g^2\sin^2\,\eta\\\rho^2(r,\eta)=&r^2+a^2\cos^2\,\eta+b^2\sin^2\,\eta\\f(r,\eta)=&2(m+abqg^2)\rho^2(r,\eta)-q^2\end{aligned}$$

$$\Xi_a=1-a^2g^2,\Xi_b=1-b^2g^2$$

$$-\infty<t<\infty,r_+<r<\infty,0<\eta<\frac{\pi}{2},\xi_1\sim\xi_1+2\pi,\xi_2\sim\xi_2+2\pi$$

$$m=\frac{(r_+^2+a^2)(r_+^2+b^2)(1+g^2r_+^2)+q^2+2abq}{2r_+^2}$$

$$\mathrm{d}s^2_{\mathrm{CCLP},\mathfrak{E}}=\mathrm{d}s^2_{\mathrm{CCLP}}\big|_{t\to-\mathrm{i}\tau},A_{\mathrm{CCLP},\mathfrak{E}}=A_{\mathrm{CCLP}}|_{t\to-\mathrm{i}\tau}.$$

$$\begin{aligned}\mathrm{d}s^2_{\mathrm{CCLP},\mathcal{B}}=&\lim_{r\to\infty}g^{-2}r^{-2}\,\mathrm{d}s^2_{\mathrm{CCLP}}\\=&-\frac{\Delta_\eta}{\Xi_a\Xi_b}\,\mathrm{d}t^2+\frac{1}{g^2\Delta_\eta}\mathrm{d}\eta^2+\frac{\sin^2\,\eta}{g^2\Xi_a}\,\mathrm{d}\xi_1^2+\frac{\cos^2\,\eta}{g^2\Xi_b}\,\mathrm{d}\xi_2^2\end{aligned}$$

$$\beta=\frac{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]}{r_+^4[1+g^2(2r_+^2+a^2+b^2)]-(ab+q)^2}$$

$$\Omega_1=\frac{a(r_+^2+b^2)(1+g^2r_+^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq},\Omega_2=\frac{b(r_+^2+a^2)(1+g^2r_+^2)+aq}{(r_+^2+a^2)(r_+^2+b^2)+abq}$$

$$\Phi=\frac{\sqrt{3}qr_+^2}{\chi[(r_+^2+a^2)(r_+^2+b^2)+abq]}$$

$$\mathrm{d}s_{\mathcal{H}}^2=\mathrm{d}s_{\mathrm{CCLP}}^2\big|_{\mathrm{d}t\to 0,\,\mathrm{d}r\to 0,r\to r_+}$$

$$S=\frac{\pi^2[(r_+^2+a^2)(r_+^2+b^2)+abq]}{2\Xi_a\Xi_b r_+ G_5}$$

$$E=\left(E_{\mathrm{AdS}}-\tilde{\zeta}'\right)+\frac{m\pi(2\Xi_a+2\Xi_b-\Xi_a\Xi_b)+2abq^2\pi(\Xi_a+\Xi_b)}{4\Xi_a^2\Xi_b^2G_5}$$

$$J_1=\frac{\pi[2am+bq(1+a^2g^2)]}{4\Xi_a^2\Xi_bG_5},J_2=\frac{\pi[2bm+aq(1+b^2g^2)]}{4\Xi_a\Xi_b^2G_5}$$

$$Q=\frac{\chi\sqrt{3}\pi q}{4\Xi_a\Xi_bG_5}$$

$$E_{\mathrm{AdS}}=\frac{3\pi}{32g^2G_5}\left(1+\frac{(\Xi_a-\Xi_b)^2}{9\Xi_a\Xi_b}\right),\tilde{\zeta}'=\frac{9g\pi\beta\left(\Xi_a^2-\Xi_a\Xi_b+\Xi_b^2\right)}{\Xi_a\Xi_bG_5}\zeta'$$

$$\mathcal{I}_{\mathrm{bs}}=\frac{\pi\beta}{4\Xi_a\Xi_bG_5}\left[m-g^2(a^2+r_+^2)(b^2+r_+^2)-\frac{q^2r_+^2}{(a^2+r_+^2)(b^2+r_+^2)+abq}\right]$$

$$\mathcal{I}=\beta\left(E_{\mathrm{AdS}}-\tilde{\zeta}'\right)+\mathcal{I}_{\mathrm{bs}}$$

$$\mathcal{I}=\beta E-S-\beta\Omega_1J_1-\beta\Omega_2J_2-\beta\Phi Q.$$

$$\mathrm{d}E=\frac{1}{\beta}\,\mathrm{d}S+\Omega_1\,\mathrm{d}J_1+\Omega_2\,\mathrm{d}J_2+\Phi\mathrm{d}Q$$

$$\mathrm{d}E=\delta_\sigma E+\frac{1}{\beta}\,\mathrm{d}S+\Omega_1\,\mathrm{d}J_1+\Omega_2\,\mathrm{d}J_2+\Phi\mathrm{d}Q$$

$$\delta_\sigma\mathcal{I}=\beta\mathrm{d}\left(E_{\mathrm{AdS}}-\tilde{\zeta}'\right)=\beta\delta_\sigma E$$

$$\mathrm{d}\left(E-\left(E_{\mathrm{AdS}}-\tilde{\zeta}'\right)\right)=\frac{1}{\beta}\,\mathrm{d}S+\Omega_1\,\mathrm{d}J_1+\Omega_2\,\mathrm{d}J_2+\Phi\mathrm{d}Q$$

$$\mathcal{I}(\beta,\Omega_1,\Omega_2,\Phi)=-\log\,\mathcal{Z}(\beta,\Omega_1,\Omega_2,\Phi)$$

$$P_i=\frac{1}{\mathcal{Z}}e^{-\beta\left(E_i-\Omega_1J_{1,i}-\Omega_2J_{2,i}-\Phi Q_i\right)},\mathcal{Z}=\sum_i e^{-\beta\left(E_i-\Omega_1J_{1,i}-\Omega_2J_{2,i}-\Phi Q_i\right)}$$

$$E=\sum_i P_i E_i = \frac{\partial \mathcal{I}}{\partial \beta} + \Omega_1 J_1 + \Omega_2 J_2 + \Phi Q$$

$$J_1 = \sum_i P_i J_{1,i} = -\frac{1}{\beta}\frac{\partial \mathcal{I}}{\partial \Omega_1}, J_2 = \sum_i P_i J_{2,i} = -\frac{1}{\beta}\frac{\partial \mathcal{I}}{\partial \Omega_2}$$

$$Q = \sum_i P_i Q_i = -\frac{1}{\beta}\frac{\partial \mathcal{I}}{\partial \Phi}$$

$$\frac{\partial \mathcal{I}}{\partial(a,b,r_+,q)} = \frac{\partial \mathcal{I}}{\partial \beta}\frac{\partial \beta}{\partial(a,b,r_+,q)} + \frac{\partial \mathcal{I}}{\partial \Omega_1}\frac{\partial \Omega_1}{\partial(a,b,r_+,q)} + \frac{\partial \mathcal{I}}{\partial \Omega_2}\frac{\partial \Omega_2}{\partial(a,b,r_+,q)} + \frac{\partial \mathcal{I}}{\partial \Phi}\frac{\partial \Phi}{\partial(a,b,r_+,q)}$$

$$\left(\frac{\partial \mathcal{I}}{\partial \beta}, \frac{\partial \mathcal{I}}{\partial \Omega_1}, \frac{\partial \mathcal{I}}{\partial \Omega_2}, \frac{\partial \mathcal{I}}{\partial \Phi}\right)$$

$$S(E, J_1, J_2, Q) = \beta E - \beta \Omega_1 J_1 - \beta \Omega_2 J_2 - \beta \Phi Q - \mathcal{I}(\beta, \Omega_1, \Omega_2, \Phi)$$

$$\beta = \frac{\partial S}{\partial E}, \Omega_1 = -\frac{1}{\beta}\frac{\partial S}{\partial J_1}, \Omega_2 = -\frac{1}{\beta}\frac{\partial S}{\partial J_2}, \Phi = -\frac{1}{\beta}\frac{\partial S}{\partial Q}.$$

$$\frac{\partial S}{\partial(a,b,r_+,q)} = \frac{\partial S}{\partial E}\frac{\partial E}{\partial(a,b,r_+,q)} + \frac{\partial S}{\partial J_1}\frac{\partial J_1}{\partial(a,b,r_+,q)} + \frac{\partial S}{\partial J_2}\frac{\partial J_2}{\partial(a,b,r_+,q)} + \frac{\partial S}{\partial Q}\frac{\partial Q}{\partial(a,b,r_+,q)}$$

$$\left(\frac{\partial S}{\partial E}, \frac{\partial S}{\partial J_1}, \frac{\partial S}{\partial J_2}, \frac{\partial S}{\partial Q}\right)$$

$$q = \frac{m}{1+ag+bg}$$

$$\tilde{m} = \frac{mg}{(a+b)(1+ag)(1+bg)(1+ag+bg)} - 1$$

$$r_*^2 = \frac{1}{g}(a+b+abg),$$

$$q = -ab + r_+^2(1+ag+bg) \pm \sqrt{-r_+^2(r_*^2 - r_+^2)}$$

$$\beta=-\frac{2\pi(a-\mathrm{i}r_+)(b-\mathrm{i}r_+)(gr_*^2+\mathrm{i}r_+)}{g(r_*^2-r_+^2)[2r_+(1+ag+bg)+\mathrm{i}g(r_*^2-3r_+^2)]}$$

$$\Omega_1=\frac{g(r_*^2+\mathrm{i}ar_+)(1-\mathrm{i}gr_+)}{(gr_*^2+\mathrm{i}r_+)(a-\mathrm{i}r_+)},\Omega_2=\Omega_1|_{a\leftrightarrow b}$$

$$\Phi=\frac{\sqrt{3}\mathrm{i}r_+(1-\mathrm{i}gr_+)}{\chi(gr_*^2+\mathrm{i}r_+)},$$

$$\tilde{E}=\frac{\pi[3-g^2(a^2(1+bg)+b^2(1+ag)-ab)][-ab+r_+^2(1+ag+bg)+\mathrm{i}gr_+(r_*^2-r_+^2)]}{4\Xi_a\Xi_b(1-ag)(1-bg)G_5}$$

$$J_1=\frac{\pi[2a+b(1+ag)][-ab+r_+^2(1+ag+bg)+\mathrm{i}gr_+(r_*^2-r_+^2)]}{4\Xi_a\Xi_b(1-ag)G_5},J_2=J_1\Big|_{a\leftrightarrow b},$$

$$Q=\frac{\chi\sqrt{3}\pi[-ab+r_+^2(1+ag+bg)+\mathrm{i}gr_+(r_*^2-r_+^2)]}{4\Xi_a\Xi_bG_5},$$

$$S=\frac{\pi^2[r_+(r_+^2+a^2(1+bg)+b^2(1+ag)+ab)+\mathrm{i}abg(r_*^2-r_+^2)]}{2\Xi_a\Xi_bG_5}$$

$$\tilde{\jmath}=\frac{\pi^2(a-\mathrm{i}r_+)^2(b-\mathrm{i}r_+)^2(1+g^2r_*^2)}{2\Xi_a\Xi_b[2r_+(1+ag+bg)+\mathrm{i}g(r_*-3r_+^2)]G_5},$$

$$E-gJ_1-gJ_2-\frac{\sqrt{3}}{\chi}Q=E_{\mathrm{AdS}}-\tilde{\zeta}'$$

$$\beta\left(g+\Omega_1+\Omega_2-g\chi\sqrt{3}\Phi\right)=2\pi\mathrm{i}$$

$$\omega_1=\beta(\Omega_1-g),\omega_2=\beta(\Omega_2-g),\varphi=\beta\left(\Phi-\frac{\sqrt{3}}{\chi}\right)$$

$$\mathcal{J}=\frac{\pi\chi^3}{12\sqrt{3}G_5}\frac{\varphi^3}{\omega_1\omega_2}$$

$$\mathcal{J}=-S-\omega_1J_1-\omega_2J_2-\varphi Q;\ \omega_1+\omega_2-g\chi\sqrt{3}\varphi=2\pi\mathrm{i}$$

$$\omega_1=-\frac{\pi(1-ag)(b-\mathrm{i}r_*)}{r_*[1+g(a+b-\mathrm{i}r_*)]},\omega_2=\omega_1|_{a\leftrightarrow b},\varphi=\frac{\sqrt{3}\pi(a-\mathrm{i}r_*)(b-\mathrm{i}r_*)}{r_*[1+g(a+b-\mathrm{i}r_*)]},$$

$$\boldsymbol{E}=\frac{\pi(a+b)[3-g^2(a^2(1+bg)+b^2(1+ag)-ab)]}{4g(1-ag)^2(1-bg)^2G_5},\boldsymbol{Q}=\frac{\chi\sqrt{3}\pi(a+b)}{4(1-ag)(1-bg)G_5},$$

$$\boldsymbol{J}_1=\frac{\pi(a+b)[2a+b(1+ag)]}{4g(1-ag)^2(1-bg)G_5},\boldsymbol{J}_2=\boldsymbol{J}_1|_{a\leftrightarrow b},\boldsymbol{S}=\frac{\pi^2(a+b)r_*}{2g(1-ag)(1-bg)G_5},$$

$$\mathcal{J}=\mu\frac{\varphi^3}{\omega_1\omega_2}$$

$$\mathcal{J}=-S-\omega_1J_1-\omega_2J_2-\varphi Q;\ \omega_1+\omega_2-\nu\varphi=2\pi\mathrm{i}$$

$$\boldsymbol{S}(\boldsymbol{J}_1,\boldsymbol{J}_2,\boldsymbol{Q})=-\mathcal{J}(\omega_1,\omega_2,\varphi)-\omega_1\boldsymbol{J}_1-\omega_2\boldsymbol{J}_2-\varphi\boldsymbol{Q}-\lambda(\omega_1+\omega_2-\nu\varphi-2\pi\mathrm{i})$$

$$-\frac{\partial \mathcal{I}}{\partial \boldsymbol{\omega}_1} = \boldsymbol{J}_1 + \lambda\,, \quad -\frac{\partial \mathcal{I}}{\partial \boldsymbol{\omega}_2} = \boldsymbol{J}_2 + \lambda\,, \quad -\frac{\partial \mathcal{I}}{\partial \boldsymbol{\varphi}} = \boldsymbol{Q} - \nu\lambda$$

$$\Downarrow$$

$$\mu\frac{\varphi^3}{\omega_1^2\omega_2} = \boldsymbol{J}_1 + \lambda\,, \quad \mu\frac{\varphi^3}{\omega_1\omega_2^2} = \boldsymbol{J}_2 + \lambda\,, \quad -\mu\frac{3\varphi^2}{\omega_1\omega_2^2} = \boldsymbol{Q} - \nu\lambda$$

$$\boldsymbol{S} = 2\pi\mathrm{i}\lambda$$

$$0 = \frac{1}{\nu^3}\left[\left(\frac{\partial \mathcal{J}}{\partial \varphi}\right)^3 - 27\mu\frac{\partial \mathcal{J}}{\partial \boldsymbol{\omega}_1}\frac{\partial \mathcal{J}}{\partial \boldsymbol{\omega}_2}\right] \Longrightarrow 0 = p_0 + p_1\lambda + p_2\lambda^2 + \lambda^3,$$

$$p_0 = -\frac{1}{\nu^3}\boldsymbol{Q}^3 - \frac{27\mu}{\nu^3}\boldsymbol{J}_1\boldsymbol{J}_2, p_1 = \frac{3}{\nu^2}\boldsymbol{Q}^2 - \frac{27\mu}{\nu^3}(\boldsymbol{J}_1 + \boldsymbol{J}_2), p_2 = -\frac{3}{\nu}\boldsymbol{Q} - \frac{27\mu}{\nu^3}.$$

$$\left(\frac{2}{g\chi\sqrt{3}}\boldsymbol{Q}\right)^3 + \frac{2\pi}{g^3G_5}\boldsymbol{J}_1\boldsymbol{J}_2 = \left(\frac{2\sqrt{3}}{g\chi}\boldsymbol{Q} + \frac{\pi}{2g^3G_5}\right)\left[\left(\frac{2}{g\chi}\boldsymbol{Q}\right)^2 - \frac{\pi}{g^3G_5}(\boldsymbol{J}_1 + \boldsymbol{J}_2)\right].$$

$$\boldsymbol{S} = \frac{\pi}{g}\sqrt{\frac{4}{\chi^2}\boldsymbol{Q}^2 - \frac{\pi}{gG_5}(\boldsymbol{J}_1 + \boldsymbol{J}_2)}.$$

$$\log \mathcal{J}_{\mathrm{QFT}} = \frac{\varphi_R^3}{48\omega_1\omega_2}k_{RRR} - \frac{\varphi_R(\omega_1^2 + \omega_2^2 - 4\pi^2)}{48\omega_1\omega_2}k_R + \log|G|,$$

$$\varphi_R = (\omega_1 + \omega_2 + 2\pi\mathrm{i}n_0)$$

$$k_{RRR} = \frac{2(\mathfrak{g}-1)}{27z^2}\left[9z^2 - 1 + (3z^2+1)^{\frac{3}{2}}\right]N^3, \qquad k_{RRF} = \frac{(\mathfrak{g}-1)}{9}zN$$

$$k_{RFF} = -\frac{(\mathfrak{g}-1)}{3}\sqrt{3z^2+1}N^3, \qquad k_{FFF} = (\mathfrak{g}-1)zN^3$$

$$k_R = \frac{\mathfrak{g}-1}{3}\left[4 - \sqrt{3z^2+1}\right]N, \qquad k_F = (\mathfrak{g}-1)zN$$

$$a = \frac{1}{32}(9k_{RRR} - 3k_R) = (\mathfrak{g}-1)\frac{\left(9z^2 - 1 + (1+3z^2)^{\frac{3}{2}}\right)}{48z^2}N^3$$

$$c = \frac{1}{32}(9k_{RRR} - 5k_R) = (\mathfrak{g}-1)\frac{\left(9z^2 - 1 + (1+3z^2)^{\frac{3}{2}}\right)}{48z^2}N^3$$

$$k_{RRR} = \frac{4\pi L^3}{9G_5}$$

$$\omega_{1,2} \leftrightarrow \omega_{1,2}, \varphi_R \leftrightarrow \frac{\chi\sqrt{3}}{L}\varphi$$

$$\log \mathcal{J}_{\mathrm{QFT}} = \frac{k_{RRR}\varphi_R^3 + k_{RRF}\varphi_R^2\varphi_F + k_{RFF}\varphi_R\varphi_F^2 + k_{FFF}\varphi_F^3}{48\omega_1\omega_2},$$

$$\omega_{1,2} \leftrightarrow \omega_{1,2}, \varphi_R \leftrightarrow \nu\varphi_R, \varphi_F \leftrightarrow \rho\varphi_F,$$

$$\mathcal{J}=\frac{\mu_1\varphi_R^3+\mu_2\varphi_R^2\varphi_F+\mu_3\varphi_R\varphi_F^2+\mu_4\varphi_F^3}{\omega_1\omega_2},$$

$$\mu_1=\frac{k_{RRR}\nu^3}{48},\mu_2=\frac{k_{RRF}\nu^2\rho}{48},\mu_3=\frac{k_{RFF}\nu\rho^2}{48},\mu_4=\frac{k_{FFF}\rho^3}{48},$$

$$\mathcal{J}=\frac{\pi}{24\omega_1\omega_2}\kappa_{IJK}\varphi^I\varphi^J\varphi^K$$

$$\kappa_{111}=\frac{24\mu_1}{\pi},\kappa_{112}=\frac{8\mu_2}{\pi},\kappa_{122}=\frac{8\mu_3}{\pi},\kappa_{222}=\frac{24\mu_4}{\pi},$$

$$\boldsymbol{S}=-I-\varphi^I\boldsymbol{Q}_I-\omega_1\boldsymbol{J}_1-\omega_2\boldsymbol{J}_2-\lambda(\omega_1+\omega_2-\nu\varphi_R-2\pi\mathrm{i}),$$

$$-\frac{\partial\mathcal{J}}{\partial\omega_1}=\boldsymbol{J}_1+\lambda,-\frac{\partial\mathcal{J}}{\partial\omega_2}=\boldsymbol{J}_2+\lambda,-\frac{\partial\mathcal{J}}{\partial\varphi_R}=\boldsymbol{Q}_R-\nu\lambda,-\frac{\partial\mathcal{J}}{\partial\varphi_F}=\boldsymbol{Q}_F.$$

$$0=\frac{1}{6}\kappa^{IJK}\frac{\partial\mathcal{J}}{\partial\varphi^I}\frac{\partial\mathcal{J}}{\partial\varphi^J}\frac{\partial\mathcal{J}}{\partial\varphi^K}-\frac{\pi}{4}\frac{\partial\mathcal{J}}{\partial\omega_1}\frac{\partial\mathcal{J}}{\partial\omega_2},$$

$$\kappa_{IJK}\kappa_{J'(LM}\kappa_{PQ)K'}g^{JJ'}g^{KK'}=\frac{4}{3}g_{I(L}\kappa_{MPQ)}$$

$$g_{IJ}=6\left(\frac{\pi}{2\mu_1}\right)^{-2/3}\delta_{IJ},g^{IJ}=\frac{1}{6}\left(\frac{\pi}{2\mu_1}\right)^{2/3}\delta^{IJ}$$

$$\kappa^{111}=\frac{\pi}{18\mu_1},\kappa^{112}=\frac{\pi\mu_2}{54\mu_1^2},\kappa^{122}=\frac{\pi\mu_3}{54\mu_1^2},\kappa^{222}=\frac{\pi\mu_4}{18\mu_1^2}$$

$$0=p_0+p_1\lambda+p_2\lambda^2+\lambda^3$$
$$p_0=-\frac{1}{\nu^3}\boldsymbol{Q}_R^3-\frac{27\mu_1}{\nu^3}\boldsymbol{J}_1\boldsymbol{J}_2-\frac{\mu_2}{\mu_1\nu^3}\boldsymbol{Q}_R^2\boldsymbol{Q}_F-\frac{\mu_4}{\mu_1\nu^3}\boldsymbol{Q}_F^3$$
$$p_1=-\frac{27\mu_1}{\nu^3}(\boldsymbol{J}_1+\boldsymbol{J}_2)+\frac{3}{\nu^2}\boldsymbol{Q}_R^2+\frac{2\mu_2}{\mu_1\nu^2}\boldsymbol{Q}_R\boldsymbol{Q}_F+\frac{\mu_3}{\mu_1\nu^2}\boldsymbol{Q}_F^2$$
$$p_2=-\frac{27\mu_1}{\nu^3}-\frac{3}{\nu}\boldsymbol{Q}_R-\frac{\mu_2}{\mu_1\nu}\boldsymbol{Q}_F$$

$$p_0=p_1p_2,\boldsymbol{S}=2\pi\sqrt{p_1}$$

$$\mu_1=\frac{k_{RRR}\nu^3}{48}=\frac{1}{48}\frac{4\pi L^3}{9G_5}\left(\frac{\chi\sqrt{3}}{L}\right)^3=\frac{\pi\chi^3}{12\sqrt{3}G_5}=\mu$$

$$\mu_3=-\frac{3}{2}\mu_1,\mu_4=\frac{1}{\sqrt{2}}\mu_1$$

$$k_{RFF}\rho^2=-\frac{3}{2}k_{RRR}\nu^2,k_{FFF}\rho^3=\frac{1}{\sqrt{2}}k_{RRR}\nu^3$$

$$\varepsilon_{(7)}=\star_7 1,\varepsilon_{(5)}=\star_5 1,\varepsilon_{(2)}=\star_2 1=\mathrm{d}\omega$$

$$\varepsilon_{(7)}=e^{5f_0+2g_0}L^2\varepsilon_{(5)}\wedge\mathrm{d}\omega$$

$$S = ce^{2f_0+2\Lambda_+} \star_5 F$$
$$\star_7 S = -ce^{f_0+2g_0+2\Lambda_+} L^2 F \wedge \mathrm{d}\omega$$
$$F_{1,2} = a(1 \pm z)L\,\mathrm{d}\omega + be^{f_0+2\Lambda_{1,2}} F$$
$$\star_7 F_{1,2} = ae^{5f_0-2g_0}(1 \pm z)L^{-1}\varepsilon_{(5)} + be^{2f_0+2g_0+2\Lambda_{1,2}} L^2 \star_5 F \wedge \mathrm{d}\omega$$

$$R_{(7)} = e^{-2f_0} R_{(5)} + e^{-2g_0} L^{-2} R_{(2)}, R_{(2)} = -2$$

$$\star_7 R = -2e^{5f_0}\varepsilon_{(5)} \wedge \mathrm{d}\omega + L_0^2 \star_5 R \wedge \mathrm{d}\omega$$
$$-\star_7 V = -e^{5f_0+2g_0} L^2 V \varepsilon_{(5)} \wedge \mathrm{d}\omega$$
$$-\frac{1}{2} e^{-4\Lambda_{1,2}} F_{1,2} \wedge \star_7 F_{1,2} = -\frac{a^2}{2} e^{5f_0-2g_0-4\Lambda_{1,2}} (1 \pm z)^2 \varepsilon_{(5)} \wedge \mathrm{d}\omega - \frac{b^2}{2} L_0^2 F \wedge \star_5 F \wedge \mathrm{d}\omega,$$
$$-\frac{1}{2} e^{-4\Lambda_+} S \wedge \star_7 S = \frac{c^2}{2} L_0^2 F \wedge \star_5 F \wedge \mathrm{d}\omega,$$
$$\frac{L}{4} S \wedge \mathrm{d}S = 0,$$
$$-\frac{L}{2} F_1 \wedge F_2 \wedge S = -4abcL_0^2 F \wedge \star_5 F \wedge \mathrm{d}\omega,$$
$$\frac{L}{8}[A_1 \wedge F_1 \wedge F_2 \wedge F_2 + (1 \leftrightarrow 2)] = 4ab^3 L_0^2 A \wedge F \wedge F \wedge \mathrm{d}\omega.$$

$$-\frac{1}{2L^2} e^{2f_0-2g_0-2\Lambda_+} \left(2e^{2g_0+4\Lambda_+}(2 + e^{2g_0} L^2 V) + a^2\left(e^{4\Lambda_1}(1-z)^2 + e^{4\Lambda_2}(1+z)^2\right)\right)$$

$$a = \pm\frac{1}{4}$$

$$I_7 = \frac{L_0^2}{16\pi G_7} \int \left[\star_5 \left(R + \frac{12}{L^2}\right) - \frac{1}{2}(2b^2 + 2bc - c^2) F \wedge \star_5 F + b^3 A \wedge F \wedge F\right] \wedge \mathrm{d}\omega$$

$$\Sigma_{\mathfrak{g}}: \int \mathrm{d}\omega = 4\pi(\mathfrak{g} - 1)$$

$$g_{MN} = e^{2f_0} g_{\mu\nu} \oplus e^{2g_0} L^{-2} g_{mn}$$

$$e_M{}^{\hat{M}} = e^{f_0} e_\mu^{\hat{\mu}} \oplus e^{g_0} L e_m^{\hat{m}}$$

$$S_{MNR} = ce^{2f_0+2\Lambda_+} \frac{1}{2!} F^{\alpha\beta} \varepsilon_{\alpha\beta\mu\nu\rho}$$
$$F_{1,2MN} = be^{f_0+2\Lambda_{1,2}} F_{\mu\nu} \oplus a(1 \pm z)L(\,\mathrm{d}\omega)_{mn}$$
$$A_{1,2M} = be^{f_0+2\Lambda_{1,2}} A_\mu \oplus a(1 \pm z)L\omega_m$$

$$\tilde{\gamma}^{MN} = e^{-2f_0}\tilde{\gamma}^{\mu\nu} \oplus e^{-2g_0} L^{-2}\tilde{\gamma}^{mn}, \tilde{\gamma}^M = e^{-f_0}\tilde{\gamma}^\mu \oplus e^{-g_0} L^{-1}\tilde{\gamma}^m$$

$$\omega_{M\hat{A}\hat{B}} = \omega_{\mu\hat{a}\hat{\beta}} \oplus \omega_{m\hat{a}\hat{b}}$$

$$\omega_{x_1\hat{x}_1\hat{x}_2} = -x_2^{-1}$$

$$\tilde{\gamma}^{MNR} = \chi_7 \frac{1}{4!} \varepsilon^{MNRABCD} \tilde{\gamma}_{ABCD}, \tilde{\gamma}^{MN} = \chi_7 \frac{1}{5!} \varepsilon^{MNABCDE} \tilde{\gamma}_{ABCDE}$$

$$\tilde{\gamma}^{\mu\nu\rho} = \chi_7 \frac{1}{2!} e^{-3f_0} \varepsilon^{\mu\nu\rho\alpha\beta} \tilde{\gamma}_{\alpha\beta} \tilde{\gamma}_{\hat{x}_1\hat{x}_2}, \tilde{\gamma}^{\mu\nu} = \chi_7 \frac{1}{3!} e^{-2f_0} \varepsilon^{\mu\nu\alpha\beta\gamma} \tilde{\gamma}_{\alpha\beta\gamma} \tilde{\gamma}_{\hat{x}_1\hat{x}_2}$$

$$\tilde{\gamma}^m (\,\mathrm{d}\omega)_{mn} = -\tilde{\gamma}_n \tilde{\gamma}_{x_1 x_2}$$

$$\delta\tilde{\zeta}_1 = \frac{L^{-1}}{2}\Big[\left(e^{2\Lambda_1} - e^{-4\Lambda_+}\right) - \frac{a}{4}(1+z)e^{-2g_0-2\Lambda_1}\tilde{\gamma}_{\hat{x}_1\hat{x}_2} \otimes \Gamma^{12}\Big]\tilde{\epsilon} \otimes \mathcal{E}$$
$$-\frac{e^{-f_0}}{16}\tilde{\gamma}^{\mu\nu}F_{\mu\nu}\big[61_8 \otimes \Gamma^{12} + \chi_7 c\tilde{\gamma}_{\hat{x}_1\hat{x}_2} \otimes \Gamma^5\big]\tilde{\epsilon} \otimes \mathcal{E}$$

$$\left(\tilde{\gamma}_{\widehat{x_1}\hat{x}_2} \otimes 1_4\right)(\tilde{\epsilon} \otimes E) = p_2(1_8 \otimes \Gamma^{12})(\tilde{\epsilon} \otimes E), (1_8 \otimes \Gamma^5)(\tilde{\epsilon} \otimes E) = p_5\tilde{\epsilon} \otimes E$$

$$p_2 = s_a, p_5 = -\chi_7 s_a;\ a = \frac{s_a}{4}, c = b = \frac{\chi}{\sqrt{3}},$$

$$(1_8 \otimes \Gamma^5)(\tilde{\epsilon} \otimes \mathcal{E}) = p_5\tilde{\epsilon} \otimes \mathcal{E} \Longrightarrow (1_8 \otimes \Gamma^{34})(\tilde{\epsilon} \otimes \mathcal{E}) = p_5(1_8 \otimes \Gamma^{12})(\tilde{\epsilon} \otimes \mathcal{E})$$

$$\delta\tilde{\psi}_m \overset{p_5=1}{=} [\partial_m - \frac{1}{2}x_2^{-1}\delta_{mx_1}\left(\tilde{\gamma}_{\hat{x}_1\hat{x}_2} \otimes 1_4 - 4a1_8 \otimes \Gamma^{12}\right)$$
$$+\frac{1}{2}e^{-f_0+g_0}\tilde{\gamma}_m\left(1_8 \otimes 1_4 + 4a\tilde{\gamma}_{\hat{x}_1\hat{x}_2} \otimes \Gamma^{12}\right)\Big]\tilde{\epsilon} \otimes \mathcal{E}$$

$$p_2 = -\chi_7, p_5 = 1;\ a = -\frac{\chi_7}{4}, c = b = \frac{\chi}{\sqrt{3}};\ \partial_m(\tilde{\epsilon} \otimes \mathcal{E}) = 0$$

$$\delta\tilde{\psi}_\mu = \Big(\Big[\partial_\mu + \frac{1}{4}\omega_{\mu\hat{\alpha}\hat{\beta}}\tilde{\gamma}^{\hat{\alpha}\hat{\beta}} + \chi_7\frac{\chi}{8\sqrt{3}}\left(\tilde{\gamma}_{\mu\alpha\beta}F^{\alpha\beta} - 4F_{\mu\alpha}\tilde{\gamma}^\alpha\right)\tilde{\gamma}_{\hat{x}_1\hat{x}_2}$$
$$+\frac{1}{2L}\left(\tilde{\gamma}_\mu - \chi_7\sqrt{3}\chi A_\mu\tilde{\gamma}_{\hat{x}_1\hat{x}_2}\right)\Big]\tilde{\epsilon}\Big) \otimes E$$

$$\tilde{\gamma}_\mu = \gamma_\mu \otimes g_1, \tilde{\gamma}_{\hat{x}_1\hat{x}_2} = 1_4 \otimes g_2, \tilde{\epsilon} = \epsilon \otimes \eta$$

$$\Big[\partial_\mu + \frac{1}{4}\omega_{\mu\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}\hat{\beta}} \otimes g_1^2 + \chi_7\frac{\chi}{8\sqrt{3}}\left(\gamma_{\mu\alpha\beta}F^{\alpha\beta} \otimes g_1^3 g_2 - 4F_{\mu\alpha}\gamma^\alpha \otimes g_1 g_2\right)$$
$$-\frac{1}{2L}\left(-\gamma_\mu \otimes g_1 + \chi_7\sqrt{3}\chi A_\mu 1_4 \otimes g_2\right)\Big]\epsilon \otimes \eta$$

$$g_1\eta = -\eta, g_2\eta = \mathrm{i}\chi_7\eta$$

$$\Gamma^{12}\mathcal{E} = -\mathrm{i}\mathfrak{E}, \Gamma^5\mathcal{E} = \mathcal{E}: \Longrightarrow \mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = 0$$

$$g_1 = -\chi_7\sigma_1, g_2 = \mathrm{i}\sigma_1$$

$$\varepsilon_{01234} = \sqrt{-g}, \varepsilon^{01234} = \frac{s}{\sqrt{-g}},$$

$$I = \frac{1}{16\pi G_5}\int \mathrm{d}^5x\sqrt{-g}\left(R + \frac{12}{L^2} - \frac{\chi^2}{4}F^{\mu\nu}F_{\mu\nu} + \frac{s\chi^3}{12\sqrt{3}}\varepsilon^{\nu\alpha\beta\gamma\delta}A_\nu F_{\alpha\beta}F_{\gamma\delta}\right)$$

$$t \to -\mathrm{i}\tau$$

$$T_{0ij\ldots} \to \mathrm{i}T_{0ij\ldots}, T^{0ij\ldots} \to -\mathrm{i}T^{0ij\ldots},$$

$$\mathrm{d}^5x \to -\mathrm{i}\mathrm{d}^5x, \sqrt{-g} \to \sqrt{g}, R \to R, F^{\mu\nu}F_{\mu\nu} \to F^{\mu\nu}F_{\mu\nu}$$

$$\varepsilon_{0ijkl} \to (-\mathrm{i}) \cdot \mathrm{i}\varepsilon_{0ijkl} = \varepsilon_{0ijkl}, \varepsilon^{0ijkl} \to \mathrm{i} \cdot (-\mathrm{i}) \cdot s\varepsilon^{0ijkl} = s\varepsilon^{0ijkl},$$

$$\varepsilon_{01234} = \sqrt{g}, \varepsilon^{01234} = \frac{1}{\sqrt{g}}$$

$$\begin{aligned}\varepsilon^{\nu\alpha\beta\gamma\delta}A_\nu F_{\alpha\beta}F_{\gamma\delta} &= \left(\varepsilon^{0ijkl}A_0F_{ij}F_{kl} - \varepsilon^{0ijkl}A_iF_{0j}F_{kl} + \cdots\right)\\ &\to \mathrm{i}s\left(\varepsilon^{0ijkl}A_0F_{ij}F_{kl} - \varepsilon^{0ijkl}A_iF_{0j}F_{kl} + \cdots\right)\\ &= \mathrm{i}s\varepsilon^{\nu\alpha\beta\gamma\delta}A_\nu F_{\alpha\beta}F_{\gamma\delta}\end{aligned}$$

$$I_E = -\frac{1}{16\pi G_5}\int \mathrm{d}^5x\sqrt{g}\left(R + \frac{12}{L^2} - \frac{\chi^2}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{i}\chi^3}{12\sqrt{3}}\varepsilon^{\nu\alpha\beta\gamma\delta}A_\nu F_{\alpha\beta}F_{\gamma\delta}\right)$$

$$\begin{aligned}R_{\mu\nu} + \frac{4}{L^2}g_{\mu\nu} - \frac{\chi^2}{2}F_\mu^\alpha F_{\nu\alpha} + \frac{\chi^2}{12}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} &= 0\\ \nabla_\mu F^{\mu\nu} + \frac{\mathrm{i}\chi}{4\sqrt{3}}\varepsilon^{\nu\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} &= 0\end{aligned}$$

$$r = r_+ + \varrho^2,$$

$$\begin{aligned}\mathrm{d}s^2_{\mathrm{CCLP},\mathcal{H}} = \tilde{g}_{\varrho\varrho}\left(\mathrm{d}\varrho^2 - \left(\frac{2\pi}{\beta^{\mathcal{H}}}\right)^2\varrho^2\,\mathrm{d}t^2\right) + \tilde{g}_{\eta\eta}\mathrm{d}\eta^2 + 2\tilde{g}_{12}\left(\mathrm{d}\xi_1 - \Omega_1^{\mathcal{H}}\mathrm{d}t\right)\left(\mathrm{d}\xi_2 - \Omega_2^{\mathcal{H}}\mathrm{d}t\right)\\ + \tilde{g}_{11}\left(\mathrm{d}\xi_1 - \Omega_1^{\mathcal{H}}\mathrm{d}t\right)^2 + \tilde{g}_{22}\left(\mathrm{d}\xi_2 - \Omega_2^{\mathcal{H}}\mathrm{d}t\right)^2\end{aligned}$$

$$\begin{aligned}\beta^{\mathcal{H}} &= \frac{2\pi r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{r_+^4[1 + g^2(2r_+^2 + a^2 + b^2)] - (ab + q)^2}\\ \Omega_1^{\mathcal{H}} &= \frac{a(r_+^2 + b^2)(1 + g^2r_+^2) + bq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}, \Omega_2^{\mathcal{H}} = \Omega_1^{\mathcal{H}}\big|_{a\leftrightarrow b},\end{aligned}$$

$$\begin{aligned}\tilde{g}_{\varrho\varrho} &= \frac{4\rho_+^2}{\Delta'_{r,+}}, \tilde{g}_{\eta\eta} = \frac{\rho_+^2}{\Delta_\eta},\\ \frac{\tilde{g}_{11}}{\mathcal{S}_\eta^2} &= -\frac{(\Delta_\eta - \Xi_a)(a^2f_+ + 2abq\Xi_a\rho_+^2) - (a^2 + r_+^2)\Xi_a(\Xi_a - \Xi_b)\rho_+^4}{\Xi_a^2(\Xi_a - \Xi_b)\rho_+^4}\\ &\quad - \frac{(\Delta_\eta - \Xi_a)[a^2f'_+\rho_+^2 - 2a(af_+ + bq\Xi_a\rho_+^2)(\rho_+^2)'] - 2r_+\Xi_a(\Xi_a - \Xi_b)\rho_+^6}{\Xi_a^2(\Xi_a - \Xi_b)\rho_+^6}\varrho^2,\\ \frac{\tilde{g}_{12}}{\mathcal{S}_\eta^2\mathcal{C}_\eta^2} &= \frac{abf_+ + (b^2\Xi_a + a^2\Xi_b)q\rho_+^2}{\Xi_a\Xi_b\rho_+^4}\\ &\quad + \frac{ab[f'_+\rho_+^2 - 2f_+(\rho_+^2)'] - (b^2\Xi_a + a^2\Xi_b)\rho_+^2(\rho_+^2)'}{\Xi_a\Xi_b\rho_+^6}\varrho^2,\\ \frac{\tilde{g}_{22}}{\mathcal{C}_\eta^2} &= \frac{(\Delta_\eta - \Xi_b)(b^2f_+ + 2abq\Xi_b\rho_+^2) + (b^2 + r_+^2)\Xi_b(\Xi_a - \Xi_b)\rho_+^4}{\Xi_b^2(\Xi_a - \Xi_b)\rho_+^4}\\ &\quad + \frac{(\Delta_\eta - \Xi_b)[b^2f'_+\rho_+^2 - 2b(bf_+ + aq\Xi_b\rho_+^2)(\rho_+^2)'] + 2r_+\Xi_b(\Xi_a - \Xi_b)\rho_+^6}{\Xi_b^2(\Xi_a - \Xi_b)\rho_+^6}\varrho^2,\end{aligned}$$

$$f_+ = f(r_+,\eta), \rho_+^2 = \rho^2(r_+,\eta), S_\eta = \sin\,\eta, C_\eta = \cos\,\eta$$
$$f_+' = \frac{\partial f}{\partial r}(r_+,\eta), (\rho_+^2)' = \frac{\partial \rho^2}{\partial r}(r_+,\eta), \Delta_{r,+}' = \frac{\partial \Delta_r}{\partial r}(r_+)$$

$$r = \hat{r}\left(1 + \frac{1}{\hat{r}^2}f_1(\hat{\eta}) + \frac{1}{\hat{r}^4}f_2(\hat{\eta})\right), \eta = \hat{\eta} + \frac{1}{\hat{r}^4}h_1(\hat{\eta}) + \frac{1}{\hat{r}^6}h_2(\hat{\eta})$$

$$\begin{matrix}\hat{g}_{tt} & \hat{g}_{t1} & \hat{g}_{t2} & \hat{g}_{\hat{r}\hat{r}} & \hat{g}_{\hat{r}\hat{\eta}} & \hat{g}_{\hat{\eta}\hat{\eta}} & \hat{g}_{11} & \hat{g}_{12} \\ O(\hat{r}^{-2}) & O(\hat{r}^{-2}) & O(\hat{r}^{-2}) & O(\hat{r}^{-6}) & O(\hat{r}^{-5}) & O(\hat{r}^{-2}) & O(\hat{r}^{-2}) & O(\hat{r}^{-2})\end{matrix} O(\hat{r}^{-2}).$$

$$f_1(\hat{\eta}) = -\frac{1}{4g^2}\left[2 - \Xi_a - \Xi_b + \Delta_\eta(\hat{\eta})\right]$$
$$f_2(\hat{\eta}) = \frac{1}{8g^4}[1 - 2g^2m - \Xi_a\sin^2\,\hat{\eta} - \Xi_b\cos^2\,\hat{\eta} - \cos^2\,\hat{\eta}\sin^2\,\hat{\eta}(\Xi_a - \Xi_b)^2]$$
$$h_1(\hat{\eta}) = -\frac{\Xi_a - \Xi_b}{8g^4}\cos\,\hat{\eta}\sin^2\,\hat{\eta}\Delta_\eta(\hat{\eta})$$
$$h_2(\hat{\eta}) = -\frac{\Xi_a - \Xi_b}{16g^6}\cos\,\hat{\eta}\sin^2\,\hat{\eta}\left[2\Xi_a^2\cos^2\,\hat{\eta} - \Xi_a\Xi_b + 2\Xi_b^2\sin^2\,\hat{\eta} - 3\cos^2\,\hat{\eta}\sin^2\,\hat{\eta}(\Xi_a - \Xi_b)^2\right]$$

$$\mathrm{d}s^2_{\mathrm{CCLP},\mathcal{A}} = \frac{1}{g^2\hat{r}^2}\,\mathrm{d}\hat{r}^2 + \hat{g}_{tt}\,\mathrm{d}t^2 + 2\hat{g}_{t1}\,\mathrm{d}t\,\mathrm{d}\xi_1 + 2\hat{g}_{t2}\,\mathrm{d}t\,\mathrm{d}\xi_2 + \hat{g}_{\hat{\eta}\eta}\mathrm{d}\hat{\eta}^2$$
$$+\hat{g}_{11}\,\mathrm{d}\xi_1^2 + 2\hat{g}_{12}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 + \hat{g}_{22}\,\mathrm{d}\xi_2^2$$

$$\hat{g}_{tt} = -\frac{g^2\hat{\Delta}_\eta\hat{r}^2}{\Xi_a\Xi_b} - \frac{\hat{\Delta}_\eta\left(\Xi_a + \Xi_b - \hat{\Delta}_\eta\right)}{2\Xi_a\Xi_b}$$
$$+\frac{2\hat{\Delta}_\eta^2(abg^2q + m) - \frac{\Xi_a\Xi_b\hat{\Delta}_\eta\left(-2\hat{\Delta}_\eta(\Xi_a + \Xi_b) + (\Xi_a + \Xi_b)^2 + \hat{\Delta}_\eta^2 + 8g^2m\right)}{16g^2}}{\Xi_a^2\Xi_b^2\hat{r}^2}$$
$$\frac{\hat{g}_{t1}}{\hat{S}_\eta^2} = -\frac{\hat{\Delta}_\eta(a^2bg^2q + 2am + bq)}{\Xi_a^2\Xi_b\hat{r}^2}$$
$$\frac{\hat{g}_{t2}}{\hat{C}_\eta^2} = -\frac{\hat{\Delta}_\eta(ab^2g^2q + aq + 2bm)}{\Xi_a\Xi_b^2\hat{r}^2},$$
$$\hat{g}_{\hat{\eta}\eta} = \frac{\hat{r}^2}{\hat{\Delta}_\eta} + \frac{\Xi_a + \Xi_b - 3\hat{\Delta}_\eta}{2g^2\hat{\Delta}_\eta} + \frac{-6\hat{\Delta}_\eta(\Xi_a + \Xi_b) + (\Xi_a + \Xi_b)^2 + 9\hat{\Delta}_\eta^2 + 8g^2m}{16g^4\hat{\Delta}_\eta\hat{r}^2}$$
$$\frac{\hat{g}_{11}}{\hat{S}_\eta^2} = \frac{\hat{r}^2}{\Xi_a} - \frac{\Xi_a - \Xi_b + \hat{\Delta}_\eta}{2g^2\Xi_a}$$
$$-\frac{\frac{8g^2\left(4a^2g^2m\hat{\Delta}_\eta + 4abg^2q\left(\hat{\Delta}_\eta - \Xi_a\right) + m\Xi_a(3\Xi_a + \Xi_b - 4)\right)}{\Xi_a - \Xi_b} - \Xi_a\left(\Xi_a - \Xi_b + \hat{\Delta}_\eta\right)^2}{16g^4\Xi_a^2\hat{r}^2}$$
$$\frac{\hat{g}_{12}}{\hat{S}_\eta^2\hat{C}_\eta^2} = \frac{a^2q + 2abm + b^2q}{\Xi_a\Xi_b\hat{r}^2},$$
$$\frac{\hat{g}_{22}}{\hat{C}_\eta^2} = \frac{\hat{r}^2}{\Xi_b} + \frac{\Xi_a - \Xi_b - \hat{\Delta}_\eta}{2g^2\Xi_b}$$
$$+\frac{\frac{8g^2\left(4b^2g^2m\hat{\Delta}_\eta + 4abg^2q\left(\hat{\Delta}_\eta - \Xi_b\right) + m\Xi_b(\Xi_a + 3\Xi_b - 4)\right)}{\Xi_a - \Xi_b} + \Xi_b\left(-\Xi_a + \Xi_b + \hat{\Delta}_\eta\right)^2}{16g^4\Xi_b^2\hat{r}^2}$$

$$\widehat{\Delta}_\eta \equiv \Delta_\eta(\hat{\eta}), \hat{S}_\eta = \sin\,\hat{\eta}, \hat{C}_\eta = \cos\,\hat{\eta}$$

$$\Omega_i = \Omega_i^{\mathcal{H}} - \Omega_i^{\mathcal{B}}, \Omega_i^{\mathcal{H}} = \frac{\mathrm{d}\xi_i}{\mathrm{d}t}\Big|_{r\to r_+}, \Omega_i^{\mathcal{B}} = \frac{\mathrm{d}\xi_i}{\mathrm{d}t}\Big|_{r\to\infty}, i = 1{,}2$$

$$\Omega_1 = \frac{a(r_+^2+b^2)(1+g^2r_+^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq}, \Omega_2 = \frac{b(r_+^2+a^2)(1+g^2r_+^2)+aq}{(r_+^2+a^2)(r_+^2+b^2)+abq}$$

$$\mathrm{d}s^2_{\mathrm{CCLP},\mathcal{H},\mathcal{E}} = \tilde{g}_{\varrho\varrho}\left(\mathrm{d}\varrho^2 + \left(\frac{2\pi}{\beta^{\mathcal{H}}}\right)^2 \varrho^2\,\mathrm{d}\tau^2\right) + \tilde{g}_{\eta\eta}\mathrm{d}\eta^2 + 2\tilde{g}_{12}(\,\mathrm{d}\xi_1 + \mathrm{i}\Omega_1\,\mathrm{d}t)(\mathrm{d}\xi_2 + \mathrm{i}\Omega_2\,\mathrm{d}t)$$
$$+\tilde{g}_{11}(\,\mathrm{d}\xi_1 + \mathrm{i}\Omega_1\,\mathrm{d}t)^2 + \tilde{g}_{22}(\,\mathrm{d}\xi_2 + \mathrm{i}\Omega_2\,\mathrm{d}t)^2$$

$$\tau \sim \tau + \beta^{\mathcal{H}}.$$

$$(\tau, \xi_1, \xi_2) \sim \left(\tau + \beta^{\mathcal{H}}, \xi_1 - \mathrm{i}\Omega_1\beta^{\mathcal{H}}, \xi_2 - \mathrm{i}\Omega_2\beta^{\mathcal{H}}\right).$$

$$\beta = \frac{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]}{r_+^4[1+g^2(2r_+^2+a^2+b^2)]-(ab+q)^2}$$

$$V = V^\mu\partial_\mu = \frac{\partial}{\partial t} + \Omega_1\frac{\partial}{\partial\xi_1} + \Omega_2\frac{\partial}{\partial\xi_2};\ \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$$

$$V_\mu V^\mu\big|_{r=r_+} = 0$$

$$\kappa_{\mathcal{H}} = \sqrt{-\frac{1}{2}\nabla^\mu V^\nu \nabla_\mu V_\nu}\Bigg|_{r\to r_+}$$

$$\Phi = V^\mu A_\mu\big|_{r=r_+} - V^\mu A_\mu\big|_{r\to\infty}$$

$$V^\mu A_\mu\big|_{r=r_+} = \frac{\sqrt{3}qr_+^2}{\chi[(r_+^2+a^2)(r_+^2+b^2)+abq]} + \alpha,\ V^\mu A_\mu\big|_{r\to\infty} = \alpha$$

$$\Phi = \frac{\sqrt{3}qr_+^2}{\chi[(r_+^2+a^2)(r_+^2+b^2)+abq]}$$

$$\alpha = -\Phi$$

$$A^\mu A_\mu\big|_{r=r_+} = 0,\ V^\mu A_\mu\big|_{r=r_+} = 0$$

$$K_{(t)} = \frac{\partial}{\partial t}, K_{(\xi_1)} = \frac{\partial}{\partial\xi_1}, K_{(\xi_2)} = \frac{\partial}{\partial\xi_2}.$$

$$\widetilde{K}_\odot = K^\mu_\odot g_{\mu\nu}\mathrm{d}x^\nu$$

$$J_i = \lim_{r\to\infty}\frac{1}{16\pi G_5}\int_{S^3} \star_5\ \mathrm{d}\widetilde{K}_{(\xi_i)}, i = 1{,}2$$

$$J_1=\frac{\pi[2am+bq(1+a^2g^2)]}{4\Xi_a^2\Xi_bG_5},J_2=\frac{\pi[2bm+aq(1+b^2g^2)]}{4\Xi_a\Xi_b^2G_5}.$$

$$0=\sigma\left[\mathrm{d}\star_5F-\frac{1}{\sqrt{3}}F\wedge F\right]=\sigma\left[\mathrm{d}\star_5F-\frac{1}{\sqrt{3}}\,\mathrm{d}(F\wedge A)\right]=\mathrm{d}\left[\sigma\left(\star_5F-\frac{1}{\sqrt{3}}F\wedge A\right)\right]$$

$$Q=\lim_{r\to\infty}\frac{\sigma}{16\pi G_5}\int_{S^3}\left(\star_5F-\frac{1}{\sqrt{3}}F\wedge A\right)=-\frac{\sigma\sqrt{3}\pi q}{4\chi\Xi_a\Xi_bG_5}$$

$$Q=\frac{\chi\sqrt{3}\pi q}{4\Xi_a\Xi_bG_5}$$

$$\begin{aligned}I_{\mathrm{GH}}&=\frac{1}{8\pi G_5}\int\ \mathrm{d}^4x\sqrt{-\gamma}K\\ I_{\mathrm{ct}}&=\frac{1}{8\pi G_5}\int\ \mathrm{d}^4x\sqrt{-\gamma}\left(-3g-\frac{1}{4g}\mathcal{R}\right)\\ I_{\mathrm{fct}}&=\frac{1}{8\pi G_5}\int\ \mathrm{d}^4x\sqrt{-\gamma}\left(\zeta'\mathcal{R}^2-\zeta''F_{ij}F^{ij}\right)\end{aligned}$$

$$K_{ij}=\frac{g\hat{r}}{2}\frac{\partial}{\partial\hat{r}}\gamma_{ij},K=\gamma^{ij}K_{ij}$$

$$I_{\mathrm{an}}=\frac{1}{8\pi G_5}\int\ \mathrm{d}^4x\sqrt{-\gamma}\frac{1}{16g^3}\left(\mathcal{R}_{ij}\mathcal{R}^{ij}-\frac{1}{3}\mathcal{R}^2-4F_{ij}F^{ij}\right)\log\,e^{-2\hat{R}_0}$$

$$\mathcal{I}_{\mathrm{hr}}=I_{\mathcal{E}}+I_{\mathrm{GH}}^{\mathcal{E}}+I_{\mathrm{ct}}^{\mathcal{E}}+I_{\mathrm{fct}}^{\mathcal{E}},$$

$$\begin{aligned}T_{ij}&=-\frac{2}{\sqrt{-\gamma}}\frac{\delta}{\delta\gamma^{ij}}(\mathcal{L}_{\mathrm{GH}}+\mathcal{L}_{\mathrm{ct}}+\mathcal{L}_{\mathrm{fct}})\\ &=\frac{1}{8\pi G_5}\left[K_{ij}-\gamma_{ij}K+3g\gamma_{ij}-\frac{1}{2g}\mathcal{R}_{ij}+\frac{1}{4g}\mathcal{R}\gamma_{ij}\right.\\ &\quad\left.+4\zeta'\left(\mathcal{R}_{ij}\mathcal{R}-\frac{1}{4}\gamma_{ij}\mathcal{R}^2-\nabla_i\nabla_j\mathcal{R}+\gamma_{ij}\nabla_k\nabla^k\mathcal{R}\right)\right]\end{aligned}$$

$$\mathcal{Q}[K]=\int_0^{2\pi}\mathrm{d}\xi_1\int_0^{2\pi}\mathrm{d}\xi_2\int_0^{\pi/2}\mathrm{d}\hat{\eta}\sqrt{-\gamma}T_j^iK^j$$

$$J_1=\mathcal{Q}[\partial_{\xi_1}]=\frac{\pi\left(2am+bq(1+a^2g^2)\right)}{4\Xi_a^2\Xi_bG_5},J_2=\mathcal{Q}[\partial_{\xi_2}]=J_1|_{b\leftrightarrow a}$$

$$E=\mathcal{Q}[\partial_t]=\left(E_{\mathrm{AdS}}-\tilde{\zeta}'\right)+\frac{m\pi(2\Xi_a+2\Xi_b-\Xi_a\Xi_b)+2abq^2\pi(\Xi_a+\Xi_b)}{4\Xi_a^2\Xi_b^2G_5},$$

$$\epsilon=\exp\left\{\frac{\mathrm{i}}{2}[g(1+\sqrt{3}\chi\alpha)t+\xi_1+\xi_2]\right\}\frac{1}{\rho}[\tilde{m}\rho^2-(1+\tilde{m})(r^2-r_*^2)]^{1/2}\epsilon_0$$

$$\frac{\mathrm{i}}{2}(\gamma^{12}-\gamma^{34})\epsilon_0=\epsilon_0$$

$$
\begin{aligned}
e^0 =& -\frac{g^2\Delta_\eta}{\Xi_a\Xi_b}\left[\frac{(a+b)(1+\tilde{m})(1+g^2r_*^2)}{g^3\rho^2}-(g^{-2}+r_*^2)-\frac{\rho^2}{1+\tilde{m}}\left(1-\frac{\tilde{m}\rho}{h}\right)\right]\mathrm{d}t\\
&+\frac{g\sin^2\eta}{\Xi_a}\left[\frac{a(a+b)(1+\tilde{m})(1+g^2r_*^2)}{g^2\rho^2}-(a^2+r_*^2)-\frac{\rho^2}{1+\tilde{m}}\left(1-\frac{\tilde{m}\rho}{h}\right)\right]\mathrm{d}\xi_1\\
&+\frac{g\cos^2\eta}{\Xi_b}\left[\frac{b(a+b)(1+\tilde{m})(1+g^2r_*^2)}{g^2\rho^2}-(b^2+r_*^2)-\frac{\rho^2}{1+\tilde{m}}\left(1-\frac{\tilde{m}\rho}{h}\right)\right]\mathrm{d}\xi_1\\
e^1 =& \sqrt{\frac{\rho^2}{\Delta_r}}\,\mathrm{d}r\\
e^2 =& \frac{r\sqrt{\Delta_r\rho^2}}{h}\left(-\frac{g\Delta_\eta}{\Xi_a\Xi_b}\,\mathrm{d}t+\frac{\sin^2\eta}{\Xi_a}\,\mathrm{d}\xi_2+\frac{\cos^2\eta}{\Xi_b}\,\mathrm{d}\xi_1\right)\\
e^3 =& -\sqrt{\frac{\rho^2}{\Delta_\eta}}\mathrm{d}\eta\\
e^4 =& \frac{\sin\eta\cos\eta\sqrt{\Delta_\eta\rho^2}}{h}\left\{-\frac{g(\Xi_a-\Xi_b)}{\Xi_a\Xi_b}(g^{-2}+(1+\tilde{m})r_*^2-r^2)\mathrm{d}t\right.\\
&\left.+\frac{1}{\Xi_a}(a^2+(1+\tilde{m})r_*^2-r^2)\mathrm{d}\xi_1+\frac{1}{\Xi_b}(b^2+(1+\tilde{m})r_*^2-r^2)\mathrm{d}\xi_2\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_V\epsilon &= V^\mu\left(\partial_\mu\epsilon+\omega_{\mu ab}\gamma^{ab}\epsilon\right)-\frac{1}{4}\nabla_\mu V_\nu\gamma^{\mu\nu}\epsilon\\
&=\frac{\mathrm{i}}{2}\left(g+\Omega_1+\Omega_2+g\chi\sqrt{3}\alpha+2V^\mu A_\mu\big|_{r=r_+}\right)\epsilon
\end{aligned}
$$

$$
\mathcal{L}_V\epsilon=\frac{\pi}{\beta}\epsilon\Longrightarrow e^{t\mathcal{L}_V}\epsilon\Big|_{t=t_0}^{t=t_0-\mathrm{i}\beta}=e^{-\mathrm{i}\beta\mathcal{L}_V}\epsilon=-\epsilon
$$

$$
\omega=\frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\,\mathrm{d}x^{\mu_p}
$$

$$
A=A_\mu\mathrm{d}x^\mu, F=\frac{1}{2!}F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu, F=\mathrm{d}A=\partial_{[\mu}A_{\nu]}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu
$$

$$
T_{[\mu_1\cdots\mu_p]}=\frac{1}{p!}\sum_{\sigma\in S_p}\mathrm{sign}(\sigma)T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(p)}}
$$

$$
F_{\mu\nu}=2!\,\partial_{[\mu}A_{\nu]}=\partial_\mu A_\nu-\partial_\nu A_\mu
$$

$$
\star\,\omega=\frac{1}{p!\,(D-p)!}\omega_{\mu_1\cdots\mu_p}\varepsilon^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_{D-p}}\,\mathrm{d}x^{\nu_1}\wedge\cdots\wedge\,\mathrm{d}x^{\nu_{D-p}}
$$

$$
\varepsilon_{\mu_1\cdots\mu_D}=\sqrt{|g|}\eta_{\mu_1\cdots\mu_D}, \varepsilon^{\mu_1\cdots\mu_D}=\frac{\mathrm{sgn}(g)}{\sqrt{|g|}}\eta^{\mu_1\cdots\mu_D}
$$

$$
R^\rho_{\mu\nu\ \sigma}=\partial_\mu\Gamma^\rho_{\nu\sigma}-\partial_\nu\Gamma^\rho_{\mu\sigma}+\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}-\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma},
$$

$$
R_{\mu\nu}=R^\rho_{\mu\nu\rho}=R^\rho_{\mu\rho\nu}, R=R^\mu_\mu
$$

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \partial_\mu\chi\partial^\mu\chi - \frac{1}{4}\sinh^2(2\chi)D_\mu\psi D^\mu\psi - \sum_{i=1}^{3}\partial_\mu\lambda_i\partial^\mu\lambda_i - g^2\mathcal{P}$$
$$-\frac{1}{4}\left[e^{-2(\lambda_1+\lambda_2+\lambda_3)}F^0_{\mu\nu}F^{0\mu\nu} + e^{-2(\lambda_1-\lambda_2-\lambda_3)}F^1_{\mu\nu}F^{1\mu\nu} + e^{2(\lambda_1-\lambda_2+\lambda_3)}F^2_{\mu\nu}F^{2\mu\nu} + e^{2(\lambda_1+\lambda_2-\lambda_3)}F^3_{\mu\nu}F^{3\mu\nu}\right]$$

$$D\psi \equiv d\psi - g(A^0 - A^1 - A^2 - A^3).$$

$$\mathcal{P} = \frac{1}{2}\left(\frac{\partial W}{\partial\chi}\right)^2 + \frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial W}{\partial\lambda_i}\right)^2 - \frac{3}{2}W^2$$

$$W = \frac{2e^{2(\lambda_1+\lambda_2+\lambda_3)}\sinh^2\chi - \cosh^2\chi\left(e^{2(\lambda_1+\lambda_2+\lambda_2)} + e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3}\right)}{2e^{\lambda_1+\lambda_2+\lambda_3}}.$$

$$A_R = -g(A^0 + A^1 + A^2 + A^3)$$
$$A_m = g(A^0 - A^1 - A^2 - A^3)$$
$$A_{F_1} = \frac{1}{2}g(A^1 - A^2)$$
$$A_{F_2} = \frac{1}{\sqrt{3}}g(A^1 + A^2 - 2A^3)$$

$$\left[2\nabla_\mu - iB_\mu - \frac{g}{\sqrt{2}}W\gamma_\mu - i\frac{1}{2\sqrt{2}}H_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\right]\epsilon = 0$$

$$H_{\mu\nu} \equiv \bar{F}^{78}_{\mu\nu} = \frac{1}{2}\left(e^{-\lambda_1-\lambda_2-\lambda_3}F^0_{\mu\nu} + e^{-\lambda_1+\lambda_2+\lambda_3}F^1_{\mu\nu} + e^{\lambda_1-\lambda_2+\lambda_3}F^2_{\mu\nu} + e^{\lambda_1+\lambda_2-\lambda_3}F^3_{\mu\nu}\right)$$
$$B_\mu \equiv -g\left(A^0_\mu + A^1_\mu + A^2_\mu + A^3_\mu\right) - \frac{1}{2}(\cosh(2\chi) - 1)D_\mu\psi$$

$$\left[\gamma^\mu\partial_\mu\lambda_1 + \frac{g}{\sqrt{2}}\partial_{\lambda_1}W + i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}^{12}_{\mu\nu}\right]\epsilon = 0$$
$$\left[\gamma^\mu\partial_\mu\lambda_2 + \frac{g}{\sqrt{2}}\partial_{\lambda_2}W + i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}^{34}_{\mu\nu}\right]\epsilon = 0$$
$$\left[\gamma^\mu\partial_\mu\lambda_3 + \frac{g}{\sqrt{2}}\partial_{\lambda_3}W + i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}^{56}_{\mu\nu}\right]\epsilon = 0$$
$$\left[\gamma^\mu\partial_\mu\chi + \frac{g}{\sqrt{2}}\partial_\chi W + i\frac{1}{2}\partial_\chi B_\mu\gamma^\mu\right]\epsilon = 0$$

$$\tanh\chi = \frac{1}{\sqrt{3}}, \psi = \frac{\pi}{2}, \tanh\lambda_i = 2 - \sqrt{3}, \varphi_i = \pi$$

$$L^2_{AdS_4} = -\frac{3}{g^2\mathcal{P}_*} = \begin{cases}\frac{1}{2g^2}, & SO(8)\\ \frac{2}{3\sqrt{3}g^2}, & SU(3)\times U(1)_R\end{cases}$$

$$\mathcal{F}_{S^3} = \frac{\pi L^2_{AdS_4}}{2G^{(4)}_N}$$

$$\mathcal{F}_{S^3}^{\text{ABJM}} = \frac{\sqrt{2}\pi}{3} N^{3/2}$$
$$\mathcal{F}_{S^3}^{\text{mABJM}} = \frac{4\sqrt{2}\pi}{9\sqrt{3}} N^{3/2}$$

$$\frac{\mathcal{F}_{S^3}}{L_{AdS_4}^2} = \frac{2\sqrt{2}\pi}{3} g^2 N^{3/2}$$

$$ds^2 = e^{2V} ds_{AdS_2}^2 + f^2 dy^2 + h^2 dz^2$$
$$A^\alpha = a^\alpha dz$$

$$B_\mu dx^\mu \equiv B_z dz$$

$$e^a = e^V \bar{e}^a, e^2 = f dy, e^3 = h dz$$

$$F_{23}^\alpha = f^{-1} h^{-1} (a^\alpha)'.$$

$$e^{2V}\left(e^{-2\lambda_1+2\lambda_2+2\lambda_3} F_{23}^1 - e^{2\lambda_1-2\lambda_2+2\lambda_3} F_{23}^2\right) = \mathcal{E}_{F_1}$$
$$e^{2V}\left(e^{2\lambda_1-2\lambda_2+2\lambda_3} F_{23}^2 - e^{2\lambda_1+2\lambda_2-2\lambda_3} F_{23}^3\right) = \mathcal{E}_{F_2}$$
$$e^{2V}\left(e^{2\lambda_1+2\lambda_2-2\lambda_3} F_{23}^3 - e^{-2\lambda_1+2\lambda_2+2\lambda_3} F_{23}^1\right) = \mathcal{E}_{F_3}$$

$$e^{2V}\left(e^{-2\lambda_1-2\lambda_2-2\lambda_3} F_{23}^0 + e^{-2\lambda_1+2\lambda_2+2\lambda_3} F_{23}^1\right) = \mathcal{E}_{R_1}$$
$$e^{2V}\left(e^{-2\lambda_1-2\lambda_2-2\lambda_3} F_{23}^0 + e^{2\lambda_1-2\lambda_2+2\lambda_3} F_{23}^2\right) = \mathcal{E}_{R_2}$$
$$e^{2V}\left(e^{-2\lambda_1-2\lambda_2-2\lambda_3} F_{23}^0 + e^{2\lambda_1+2\lambda_2-2\lambda_3} F_{23}^3\right) = \mathcal{E}_{R_3}$$

$$\left(e^{2V+2\lambda_1+2\lambda_2-2\lambda_3} F_{23}^3\right)' = -e^{2V} f h^{-1} \frac{1}{2} \sinh^2(2\chi) D_z \psi,$$

$$\gamma^m = \Gamma^m \otimes \sigma^3, \gamma^2 =]\,]_2 \otimes \sigma^1, \gamma^3 =]\,]_2 \otimes \sigma^2$$

$$\epsilon = \psi \otimes \chi$$

$$D_m \psi = \frac{1}{2} \kappa \Gamma_m \psi$$

$$\chi = e^{V/2} e^{isz/2} \begin{pmatrix} \sin\frac{\xi}{2} \\ \cos\frac{\xi}{2} \end{pmatrix}$$

$$f^{-1}\xi' = \sqrt{2} g W \cos\xi + \kappa e^{-V}$$
$$f^{-1}V' = \frac{g}{\sqrt{2}} W \sin\xi$$
$$f^{-1}\lambda_i' = -\frac{g}{\sqrt{2}} \partial_{\lambda_i} W \sin\xi$$
$$f^{-1}\chi' = -\frac{g}{\sqrt{2}} \frac{\partial_\chi W}{\sin\xi}$$
$$f^{-1}\frac{h'}{h} = \frac{1}{\sin\xi}\left(\kappa e^{-V}\cos\xi + \frac{gW}{\sqrt{2}}(1+\cos^2\xi)\right)$$

$$(s-B_z)\sin\,\xi=-\sqrt{2}gWh\cos\,\xi-\kappa he^{-V}$$
$$\sqrt{2}g\partial_\chi W\cos\,\xi=\partial_\chi B_z\sin\,\xi h^{-1}$$

$$\bar{F}_{23}^{12}=-g\partial_{\lambda_1}W\cos\,\xi$$
$$\bar{F}_{23}^{34}=-g\partial_{\lambda_2}W\cos\,\xi$$
$$\bar{F}_{23}^{56}=-g\partial_{\lambda_3}W\cos\,\xi$$
$$H_{23}=-gW\cos\,\xi-\sqrt{2}\kappa e^{-V}$$

$$he^{-V}=k\sin\,\xi$$

$$\xi'=-k^{-1}(s-B_z)(e^{-V}f)$$

$$(s-B_z)=-k\left[\sqrt{2}gWe^{V}\cos\,\xi+\kappa\right]$$
$$\sqrt{2}g\partial_\chi W\cos\,\xi=k^{-1}e^{-V}\partial_\chi B_z$$

$$\mathcal{E}_{R_1}=e^{V}\left[2ge^{V}\cos\,\xi-\sqrt{2}\kappa e^{-\lambda_1}\cosh\,(\lambda_2+\lambda_3)\right],$$
$$\mathcal{E}_{R_2}=e^{V}\left[2ge^{V}\cos\,\xi-\sqrt{2}\kappa e^{-\lambda_2}\cosh\,(\lambda_3+\lambda_1)\right],$$
$$\mathcal{E}_{R_3}=e^{V}\left[2ge^{V}\cos\,\xi-\sqrt{2}\kappa e^{-\lambda_3}\cosh\,(\lambda_1+\lambda_2)\right],$$

$$\mathcal{E}_{F_1}=\sqrt{2}\kappa e^{V}e^{\lambda_3}\sinh\,(\lambda_1-\lambda_2)$$
$$\mathcal{E}_{F_2}=\sqrt{2}\kappa e^{V}e^{\lambda_1}\sinh\,(\lambda_2-\lambda_3)$$
$$\mathcal{E}_{F_3}=\sqrt{2}\kappa e^{V}e^{\lambda_2}\sinh\,(\lambda_3-\lambda_1)$$

$$f=e^{V}$$

$$ds^2=e^{2V}\left[ds^2_{AdS_2}+ds^2_{\Sigma}\right]$$

$$ds^2_{\Sigma}=dy^2+k^2\sin^2\,\xi dz^2$$

$$\Delta z=2\pi$$

$$\cos\,\xi_{N,S}=(-1)^{t_{N,S}}2\pi\left(1-\frac{1}{n_{N,S}}\right)|(k\sin\,\xi)'|_{N,S}=\frac{1}{n_{N,S}}$$

$$h\geq 0,\Leftrightarrow\ k\sin\,\xi\geq 0$$

$$(k\sin\,\xi)'|_{N,S}=\frac{(-1)^{l_{N,S}}}{n_{N,S}},l_N=0,l_S=1$$

$$\cos\,\xi|_{N,S}=(-1)^{t_{N,S}};\qquad (t_N,t_S)=(1,1)\quad\text{or }(0,0)$$
$$(t_N,t_S)=(1,0)\quad\text{or }(0,1)$$

$$(s-B_z)|_{N,S}=\frac{1}{n_{N,S}}(-1)^{l_{N,S}+t_{N,S}+1}$$

$$F^R=dB+d\left(\frac{1}{2}(\cosh\,(2\chi)-1)D\psi\right)$$

$$\frac{1}{2\pi}\int_{\bar{\Sigma}} F^R \equiv \frac{1}{2\pi}\int_{\bar{\Sigma}} -g(F^0+F^1+F^2+F^3) = \frac{n_N(-1)^{t_S+1}+n_S(-1)^{t_N+1}}{n_N n_S}$$

$$\partial_z B = -\frac{1}{2}\sinh(2\chi) D_z\psi$$

$$\partial_\chi B_z\big|_{N,S} = \partial_\chi W\big|_{N,S} = 0.$$

$$\chi|_N, \chi|_S \neq 0, \Rightarrow D_z\psi|_N = D_z\psi|_S = 0$$

$$\frac{1}{2\pi}\int_{\Sigma} g(F^0 - F^1 - F^2 - F^3) = (D_z\psi)|_{y_N}^{y_S} = 0$$

$$\left(e^{2\lambda_1}+e^{2\lambda_2}+e^{2\lambda_3}-e^{2\lambda_1+2\lambda_2+2\lambda_3}\right)\big|_{N,S} = 0, \Rightarrow W|_{N,S} = -e^{\lambda_1+\lambda_2+\lambda_3}\big|_{N,S}.$$

$$M_{(1)} \equiv g e^{\lambda_1+\lambda_2+\lambda_3} e^V, M_{(2)} \equiv -\frac{\kappa}{\sqrt{2}} M_{(1)} + 2M_{(1)}^2 \cos\,\xi$$

$$\mathcal{E}_{R_i} = -\frac{\kappa}{\sqrt{2}g} M_{(1)} e^{-2\lambda_i} + \frac{1}{g} M_{(2)} e^{-2\lambda_1-2\lambda_2-2\lambda_3},$$

$$M_{(1)|_{N,S}} = \frac{1}{\sqrt{2}}(-1)^{t_{N,S}}\kappa - \frac{1}{\sqrt{2}kn_{N,S}}(-1)^{l_{N,S}},$$
$$M_{(2)|_{N,S}} = \frac{1}{2}(-1)^{t_{N,S}} + \frac{1}{k^2 n_{N,S}^2}(-1)^{t_{N,S}} - \frac{3\kappa}{2kn_{N,S}}(-1)^{l_{N,S}}.$$

$$\mathcal{E}_{R_1}(\lambda_{1N},\lambda_{2N}) = \mathcal{E}_{R_1}(\lambda_{1S},\lambda_{2S}),$$
$$\mathcal{E}_{R_2}(\lambda_{1N},\lambda_{2N}) = \mathcal{E}_{R_2}(\lambda_{1S},\lambda_{2S}),$$
$$\mathcal{E}_{R_3}(\lambda_{1N},\lambda_{2N}) = \mathcal{E}_{R_3}(\lambda_{1S},\lambda_{2S}).$$

$$F_{yz}^{\alpha} = (a^{\alpha})' = \left(\mathcal{J}^{(\alpha)}\right)'$$

$$\mathcal{J}^{(0)} \equiv \frac{1}{\sqrt{2}} k e^V \cos\,\xi e^{\lambda_1+\lambda_2+\lambda_3}$$
$$\mathcal{J}^{(1)} \equiv \frac{1}{\sqrt{2}} k e^V \cos\,\xi e^{\lambda_1-\lambda_2-\lambda_3}$$
$$\mathcal{J}^{(2)} \equiv \frac{1}{\sqrt{2}} k e^V \cos\,\xi e^{-\lambda_1+\lambda_2-\lambda_3}$$
$$\mathcal{J}^{(3)} \equiv \frac{1}{\sqrt{2}} k e^V \cos\,\xi e^{-\lambda_1-\lambda_2+\lambda_3}$$

$$\frac{p_\alpha}{n_N n_S} \equiv \frac{1}{2\pi}\int_{\Sigma} g F^{\alpha} = g\mathcal{J}^{\alpha}|_N^S$$

$$\mathcal{J}_0|_{N,S} \equiv -\left(\mathcal{J}^{(0)}+\mathcal{J}^{(1)}+\mathcal{J}^{(2)}+\mathcal{J}^{(3)}\right)\big|_{N,S}$$
$$\mathcal{J}_{\Delta_1}\big|_{N,S} \equiv \frac{1}{2}\left(\mathcal{J}^{(1)}-\mathcal{J}^{(2)}\right)\Big|_{N,S}$$
$$\mathcal{J}_{\Delta_2}\big|_{N,S} \equiv \frac{1}{\sqrt{3}}\left(\mathcal{J}^{(1)}+\mathcal{J}^{(2)}-2\mathcal{J}^{(3)}\right)\Big|_{N,S}$$

$$g\mathcal{I}_0|_{N,S}= -g\left(\mathcal{I}^{(0)}+\mathcal{I}^{(1)}+\mathcal{I}^{(2)}+\mathcal{I}^{(3)}\right)\Big|_{N,S}$$
$$= -\frac{1}{\sqrt{2}}gke^{V}\cos\,\xi\left(e^{\lambda_1+\lambda_2+\lambda_3}+e^{\lambda_1-\lambda_2-\lambda_3}+e^{-\lambda_1+\lambda_2-\lambda_3}+e^{-\lambda_1-\lambda_2+\lambda_3}\right)\Big|_{N,S}$$
$$= -\frac{1}{\sqrt{2}}gke^{V}\cos\,\xi e^{-\lambda_1-\lambda_2-\lambda_3}\left(e^{2\lambda_1+2\lambda_2+2\lambda_3}+e^{2\lambda_1}+e^{2\lambda_2}+e^{2\lambda_3}\right)\Big|_{N,S}$$
$$= -\frac{1}{\sqrt{2}}gke^{V}\cos\,\xi 2e^{\lambda_1+\lambda_2+\lambda_3}\Big|_{N,S}$$
$$= \sqrt{2}kM_{(1)}\big|_{N,S}(-1)^{t_{N,S}+1}$$

$$g\left(\mathcal{I}^{(0)}-\mathcal{I}^{(1)}-\mathcal{I}^{(2)}-\mathcal{I}^{(3)}\right)\Big|_{N,S}$$
$$=\frac{1}{\sqrt{2}}gke^{V}\cos\,\xi\left(e^{\lambda_1+\lambda_2+\lambda_3}-e^{\lambda_1-\lambda_2-\lambda_3}-e^{-\lambda_1+\lambda_2-\lambda_3}-e^{-\lambda_1-\lambda_2+\lambda_3}\right)\Big|_{N,S}$$
$$=\frac{1}{\sqrt{2}}gke^{V}\cos\,\xi e^{-\lambda_1-\lambda_2-\lambda_3}\left(-e^{2\lambda_1+2\lambda_2+2\lambda_3}+e^{2\lambda_1}+e^{2\lambda_2}+e^{2\lambda_3}\right)\Big|_{N,S}=0,$$

$$g\mathcal{I}_0|_N^S = -g\left(\mathcal{I}^{(0)}+\mathcal{I}^{(1)}+\mathcal{I}^{(2)}+\mathcal{I}^{(3)}\right)\Big|_N^S=\frac{n_N(-1)^{t_S+1}+n_S(-1)^{t_N+1}}{n_Nn_S}$$
$$g\left(\mathcal{I}^{(0)}-\mathcal{I}^{(1)}-\mathcal{I}^{(2)}-\mathcal{I}^{(3)}\right)\Big|_N^S=0$$

$$\frac{p_{F_1}}{n_Nn_S}\equiv g\mathcal{I}_{\Delta_1}\big|_N^S=\frac{1}{2}g\left(\mathcal{I}^{(1)}-\mathcal{I}^{(2)}\right)\Big|_N^S$$
$$=\frac{1}{2\sqrt{2}}gke^{V}\cos\,\xi\left(e^{\lambda_1-\lambda_2-\lambda_3}-e^{-\lambda_1+\lambda_2-\lambda_3}\right)\Big|_N^S$$
$$=\frac{1}{2\sqrt{2}}kM_{(1)}(-1)^{t_{N,S}}\left(e^{-2\lambda_2-2\lambda_3}-e^{-2\lambda_3-2\lambda_1}\right)\Big|_N^S$$
$$\frac{p_{F_2}}{n_Nn_S}\equiv g\mathcal{I}_{\Delta_2}\big|_N^S=\frac{1}{\sqrt{3}}g\left(\mathcal{I}^{(1)}+\mathcal{I}^{(2)}-2\mathcal{I}^{(3)}\right)\Big|_N^S$$
$$=\frac{1}{\sqrt{6}}gke^{V}\cos\,\xi\left(e^{\lambda_1-\lambda_2-\lambda_3}+e^{-\lambda_1+\lambda_2-\lambda_3}-2e^{-\lambda_1-\lambda_2+\lambda_3}\right)\Big|_N^S$$
$$=\frac{1}{\sqrt{6}}kM_{(1)}(-1)^{t_{N,S}}\left(e^{-2\lambda_2-2\lambda_3}+e^{-2\lambda_3-2\lambda_1}-2e^{-2\lambda_1-2\lambda_2}\right)\Big|_N^S$$

$$n_N=8,n_S=1,p_{F_1}=1,p_{F_2}=2$$
$$g=1,\kappa=-1,$$

$$e^{-2\lambda_{1N}}\approx 0.0452746,\qquad e^{-2\lambda_{1S}}\approx 0.012202,$$
$$e^{-2\lambda_{2N}}\approx 0.110402,\qquad e^{-2\lambda_{2S}}\approx 0.101903,$$
$$e^{-2\lambda_{3N}}\approx 6.39148,\qquad e^{-2\lambda_{3S}}\approx 8.75297,$$
$$k\approx 0.631939.$$

$$\frac{1}{2G_N^{(4)}}=\frac{2\sqrt{2}}{3}g^2N^{3/2}.$$

$$\left(G_N^{(2)}\right)^{-1} = \left(G_N^{(4)}\right)^{-1} \Delta z \int_{y_N}^{y_S} |fh| dy$$

$$fh = ke^V f \sin\,\xi = -\frac{k}{\kappa}(e^{2V}\cos\,\xi)'.$$

$$S_{\mathrm{BH}} = \frac{1}{4G_N^{(2)}} = \frac{2\sqrt{2}\pi}{3} N^{3/2} g^2 \left(-\frac{k}{\kappa}\right) [e^{2V}\cos\,\xi]_N^S$$
$$= -\frac{2\sqrt{2}N^{3/2}k}{3\kappa}\left(M_{(1)}^2\big|_S e^{-2(\lambda_{1S}+\lambda_{2S}+\lambda_{3S})}(-1)^{t_S} - M_{(1)}^2\big|_N e^{-2(\lambda_{1N}+\lambda_{2N}+\lambda_{3N})}(-1)^{t_N}\right),$$

$$\tanh\,\chi = \frac{1}{\sqrt{3}}, \psi = \frac{\pi}{2}, \tanh\,\lambda_i = 2-\sqrt{3}, \varphi_i = \pi$$

$$ds^2 = \frac{2}{3\sqrt{3}g^2}\left[\frac{y^2}{4} ds_{AdS_2}^2 + \frac{y^2}{q(y)} dy^2 + \frac{q(y)}{4y^2} c_0^2 dz^2\right]$$
$$\frac{1}{3}A^0 = A^1 = A^2 = A^3 = -\left[\frac{2c_0\kappa}{3g}\left(1-\frac{a}{y}\right) + \frac{s}{g}\right] dz$$

$$\sin\,\xi = -\frac{\sqrt{q(y)}}{y^2}, \cos\,\xi = \kappa\frac{2y-a}{y^2}$$

$$L_{AdS_4}^2 = \frac{2}{3\sqrt{3}g^2}$$

$$q(y) = y^4 - 4y^2 + 4ay - a^2$$

$$a = \frac{n_S^2 - n_N^2}{n_S^2 + n_N^2}$$
$$c_0 = \frac{\sqrt{n_S^2 + n_N^2}}{\sqrt{2}n_S n_N}$$

$$y_N = -1 + \sqrt{1+a}, y_S = 1 - \sqrt{1-a}$$

$$S_{\mathrm{BH}} = \frac{\sqrt{2}\sqrt{n_S^2 + n_N^2} - (n_S + n_N)}{n_S n_N} \frac{\pi L_{AdS_4}^2}{4G_N^{(4)}}$$
$$= \frac{\sqrt{2}\sqrt{n_S^2 + n_N^2} - (n_S + n_N)}{n_S n_N} \frac{1}{2} \mathcal{F}_{S^3}^{\mathrm{mABJM}},$$

$$\mathcal{F}_{S^3}^{\mathrm{mABJM}} = \frac{4\sqrt{2}\pi}{9\sqrt{3}} N^{3/2}$$

$$\frac{1}{2}(T_{12} - T_{34}), \frac{1}{\sqrt{3}}(T_{12} + T_{34} - 2T_{56}),$$

$$A^{12}=\frac{1}{2}(A^0+A^1-A^2-A^3),\quad A^{34}=\frac{1}{2}(A^0-A^1+A^2-A^3),$$
$$A^{56}=\frac{1}{2}(A^0-A^1-A^2+A^3),\quad A^{78}=\frac{1}{2}(A^0+A^1+A^2+A^3).$$

$$\mathcal{V}\equiv\begin{pmatrix}u_{ij}^{IJ} & v_{ijKL}\\ v^{klIJ} & u^{kl}KL\end{pmatrix}=\exp\begin{pmatrix}0 & \phi_{ijkl}\\ \phi^{ijkl} & 0\end{pmatrix}\in E_{7(7)}/SU(8),$$

$$\phi_{ijkl}=\frac{1}{24}\epsilon_{ijklmnpq}\phi^{mnpq},\phi^{ijkl}=\left(\phi_{ijkl}\right)^*$$

$$\phi_{1278}=-\frac{1}{2}\lambda_1e^{i\varphi_1},\phi_{3478}=-\frac{1}{2}\lambda_2e^{i\varphi_2},\phi_{5678}=-\frac{1}{2}\lambda_3e^{i\varphi_3}$$
$$\phi_{1234}=-\frac{1}{2}\lambda_3e^{-i\varphi_3},\phi_{1256}=-\frac{1}{2}\lambda_2e^{-i\varphi_2},\phi_{3456}=-\frac{1}{2}\lambda_1e^{-i\varphi_1}$$
$$\phi_{1357}=-\phi_{1467}=-\phi_{2367}=-\phi_{2457}=\frac{1}{4}(\chi_1\cos\,\psi_1+i\chi_2\sin\,\psi_2)$$
$$\phi_{1367}=\phi_{1457}=\phi_{2357}=-\phi_{2467}=-\frac{1}{4}(\chi_1\sin\,\psi_1-i\chi_2\cos\,\psi_2)$$
$$\phi_{1368}=\phi_{1458}=\phi_{2358}=-\phi_{2468}=-\frac{1}{4}(\chi_1\cos\,\psi_1-i\chi_2\sin\,\psi_2)$$
$$\phi_{1358}=-\phi_{1468}=-\phi_{2368}=-\phi_{2458}=-\frac{1}{4}(\chi_1\sin\,\psi_1+i\chi_2\cos\,\psi_2)$$

$$\mathcal{M}_V\times\mathcal{M}_H=\left[\frac{SU(1,1)}{U(1)}\right]^3\times\frac{SU(2,1)}{SU(2)\times U(1)}.$$

$$\chi\equiv\chi_2,\psi\equiv\psi_2,\chi_1=0,\psi_1=\frac{\pi}{2},\varphi_1=\varphi_2=\varphi_3=\pi$$

$$e^{-1}\mathcal{L}=\frac{1}{2}R-\partial_\mu\chi\partial^\mu\chi-\frac{1}{4}\sinh^2(2\chi)D_\mu\psi D^\mu\psi-\sum_{i=1}^{3}\partial_\mu\lambda_i\partial^\mu\lambda_i-g^2\mathcal{P}$$
$$-\frac{1}{4}\left[e^{-2(\lambda_1+\lambda_2+\lambda_3)}F^0_{\mu\nu}F^{0\mu\nu}+e^{-2(\lambda_1-\lambda_2-\lambda_3)}F^1_{\mu\nu}F^{1\mu\nu}+e^{2(\lambda_1-\lambda_2+\lambda_3)}F^2_{\mu\nu}F^{2\mu\nu}+e^{2(\lambda_1+\lambda_2-\lambda_3)}F^3_{\mu\nu}F^{3\mu\nu}\right]$$

$$D\psi\equiv d\psi-g(A^0-A^1-A^2-A^3)$$

$$\mathcal{P}=\frac{1}{2}\left(\frac{\partial W}{\partial\chi}\right)^2+\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial W}{\partial\lambda_i}\right)^2-\frac{3}{2}W^2$$

$$W=\frac{2e^{2(\lambda_1+\lambda_2+\lambda_3)}\sinh^2\chi-\cosh^2\chi\left(e^{2(\lambda_1+\lambda_2+\lambda_2)}+e^{2\lambda_1}+e^{2\lambda_2}+e^{2\lambda_3}\right)}{2e^{\lambda_1+\lambda_2+\lambda_3}}.$$

$$A_R=-g(A^0+A^1+A^2+A^3)$$
$$A_m=g(A^0-A^1-A^2-A^3)$$
$$A_{F_1}=\frac{1}{2}g(A^1-A^2)$$
$$A_{F_2}=\frac{1}{\sqrt{3}}g(A^1+A^2-2A^3)$$

$$\delta\chi^{ijk} = -\mathcal{A}_{\mu}{}^{ijkl}\gamma^{\mu}\epsilon_{l} + \frac{3}{2}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{-[ij}\epsilon^{k]} - 2gA_{2l}{}^{ijk}\epsilon^{l}$$

$$\delta\psi_{\mu}{}^{i} = 2D_{\mu}\epsilon^{i} + \frac{\sqrt{2}}{4}\bar{F}_{\nu\rho}^{-ij}\gamma^{\nu\rho}\gamma_{\mu}\epsilon_{j} + \sqrt{2}gA_{1}{}^{ij}\gamma_{\mu}\epsilon_{j}$$

$$F^{-IJ} = \left(u_{ij}^{IJ} + v_{ijIJ}\right)\bar{F}^{-ij}, F^{-IJ} = \frac{1}{2}(F^{IJ} - i * F^{IJ})$$

$$\epsilon^{1} = \cdots = \epsilon^{6} = 0$$

$$A_{1}{}^{77}\epsilon_{7} = -\frac{1}{2}W\epsilon_{7}, A_{1}{}^{88}\epsilon_{8} = -\frac{1}{2}W\epsilon_{8}$$

$$B_{\mu} \equiv B_{\mu}{}^{7}{}_{8} = -B_{\mu}{}^{8}{}_{7},$$
$$H_{\mu\nu} \equiv \bar{F}_{\mu\nu}^{78},$$

$$H_{\mu\nu} \equiv \bar{F}_{\mu\nu}^{78} = \frac{1}{2}\left(e^{-\lambda_1-\lambda_2-\lambda_3}F_{\mu\nu}^{0} + e^{-\lambda_1+\lambda_2+\lambda_3}F_{\mu\nu}^{1} + e^{\lambda_1-\lambda_2+\lambda_3}F_{\mu\nu}^{2} + e^{\lambda_1+\lambda_2-\lambda_3}F_{\mu\nu}^{3}\right)$$
$$B_{\mu} \equiv -g\left(A_{\mu}^{0} + A_{\mu}^{1} + A_{\mu}^{2} + A_{\mu}^{3}\right) - \frac{1}{2}(\cosh(2\chi) - 1)D_{\mu}\psi$$

$$\epsilon = (\epsilon_{7} + \epsilon^{7}) + i(\epsilon_{8} + \epsilon^{8})$$

$$\left[2\nabla_{\mu} - iB_{\mu} - \frac{g}{\sqrt{2}}W\gamma_{\mu} - i\frac{1}{2\sqrt{2}}H_{\nu\rho}\gamma^{\nu\rho}\gamma_{\mu}\right]\epsilon = 0$$

$$\left[\gamma^{\mu}\partial_{\mu}\lambda_{1} + \frac{g}{\sqrt{2}}\partial_{\lambda_1}W + i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{12}\right]\epsilon = 0,$$
$$\left[\gamma^{\mu}\partial_{\mu}\lambda_{2} + \frac{g}{\sqrt{2}}\partial_{\lambda_2}W + i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{34}\right]\epsilon = 0,$$
$$\left[\gamma^{\mu}\partial_{\mu}\lambda_{3} + \frac{g}{\sqrt{2}}\partial_{\lambda_3}W + i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{56}\right]\epsilon = 0,$$

$$\left[\gamma^{\mu}\partial_{\mu}\chi + \frac{g}{\sqrt{2}}\partial_{\chi}W + i\frac{1}{2}\partial_{\chi}B_{\mu}\gamma^{\mu}\right]\epsilon = 0$$

$$F^{0} = \frac{1}{2}e^{\lambda_1+\lambda_2+\lambda_3}(\bar{F}^{12} + \bar{F}^{34} + \bar{F}^{56} + \bar{F}^{78})$$
$$F^{1} = \frac{1}{2}e^{\lambda_1-\lambda_2-\lambda_3}(\bar{F}^{12} - \bar{F}^{34} - \bar{F}^{56} + \bar{F}^{78})$$
$$F^{2} = -\frac{1}{2}e^{-\lambda_1+\lambda_2-\lambda_3}(\bar{F}^{12} - \bar{F}^{34} + \bar{F}^{56} - \bar{F}^{78})$$
$$F^{3} = -\frac{1}{2}e^{-\lambda_1-\lambda_2+\lambda_3}(\bar{F}^{12} + \bar{F}^{34} - \bar{F}^{56} - \bar{F}^{78})$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + g^{2}\mathcal{P}g_{\mu\nu} - 2\left(T_{\mu\nu}^{\chi} + T_{\mu\nu}^{\lambda_1} + T_{\mu\nu}^{\lambda_2} + T_{\mu\nu}^{\lambda_3}\right) - \frac{1}{2}\sinh^{2}(2\chi)T_{\mu\nu}^{\psi}$$
$$-e^{-2(\lambda_1+\lambda_2+\lambda_3)}T_{\mu\nu}^{A^0} - e^{-2(\lambda_1-\lambda_2-\lambda_3)}T_{\mu\nu}^{A^1} - e^{2(\lambda_1-\lambda_2+\lambda_3)}T_{\mu\nu}^{A^2} - e^{2(\lambda_1+\lambda_2-\lambda_3)}T_{\mu\nu}^{A^3} = 0$$

$$T_{\mu\nu}^{X} = \partial_\mu X \partial_\nu X - \frac{1}{2} g_{\mu\nu} \partial_\rho X \partial^\rho X,$$

$$T_{\mu\nu}^{A^\alpha} = g^{\rho\sigma} F_{\mu\rho}^{\alpha} F_{\nu\sigma}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}^{\alpha} F^{\alpha\rho\sigma},$$

$$\partial_\nu\left(\sqrt{-g} e^{-2(\lambda_1+\lambda_2+\lambda_3)} F^{0\mu\nu}\right) - \frac{1}{2}\sqrt{-g}\sinh^2(2\chi) g^{\mu\nu} D_\nu \psi = 0$$

$$\partial_\nu\left(\sqrt{-g} e^{-2(\lambda_1-\lambda_2-\lambda_3)} F^{1\mu\nu}\right) + \frac{1}{2}\sqrt{-g}\sinh^2(2\chi) g^{\mu\nu} D_\nu \psi = 0$$

$$\partial_\nu\left(\sqrt{-g} e^{2(\lambda_1-\lambda_2+\lambda_3)} F^{2\mu\nu}\right) + \frac{1}{2}\sqrt{-g}\sinh^2(2\chi) g^{\mu\nu} D_\nu \psi = 0$$

$$\partial_\nu\left(\sqrt{-g} e^{2(\lambda_1+\lambda_2-\lambda_3)} F^{3\mu\nu}\right) + \frac{1}{2}\sqrt{-g}\sinh^2(2\chi) g^{\mu\nu} D_\nu \psi = 0$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g} g^{\mu\nu} \partial_\nu \lambda_1\right) - \frac{g^2}{2}\frac{\partial \mathcal{P}}{\partial \lambda_1}$$
$$+\frac{1}{4} e^{-2(\lambda_1+\lambda_2+\lambda_3)} F_{\mu\nu}^{0} F^{0\mu\nu} + \frac{1}{4} e^{-2(\lambda_1-\lambda_2-\lambda_3)} F_{\mu\nu}^{1} F^{1\mu\nu}$$
$$-\frac{1}{4} e^{2(\lambda_1-\lambda_2+\lambda_3)} F_{\mu\nu}^{2} F^{2\mu\nu} - \frac{1}{4} e^{2(\lambda_1+\lambda_2-\lambda_3)} F_{\mu\nu}^{3} F^{3\mu\nu} = 0$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g} g^{\mu\nu} \partial_\nu \lambda_2\right) - \frac{g^2}{2}\frac{\partial \mathcal{P}}{\partial \lambda_2}$$
$$+\frac{1}{4} e^{-2(\lambda_1+\lambda_2+\lambda_3)} F_{\mu\nu}^{0} F^{0\mu\nu} - \frac{1}{4} e^{-2(\lambda_1-\lambda_2-\lambda_3)} F_{\mu\nu}^{1} F^{1\mu\nu}$$
$$+\frac{1}{4} e^{2(\lambda_1-\lambda_2+\lambda_3)} F_{\mu\nu}^{2} F^{2\mu\nu} - \frac{1}{4} e^{2(\lambda_1+\lambda_2-\lambda_3)} F_{\mu\nu}^{3} F^{3\mu\nu} = 0$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g} g^{\mu\nu} \partial_\nu \lambda_3\right) - \frac{g^2}{2}\frac{\partial \mathcal{P}}{\partial \lambda_3}$$
$$+\frac{1}{4} e^{-2(\lambda_1+\lambda_2+\lambda_3)} F_{\mu\nu}^{0} F^{0\mu\nu} - \frac{1}{4} e^{-2(\lambda_1-\lambda_2-\lambda_3)} F_{\mu\nu}^{1} F^{1\mu\nu}$$
$$-\frac{1}{4} e^{2(\lambda_1-\lambda_2+\lambda_3)} F_{\mu\nu}^{2} F^{2\mu\nu} + \frac{1}{4} e^{2(\lambda_1+\lambda_2-\lambda_3)} F_{\mu\nu}^{3} F^{3\mu\nu} = 0$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g} g^{\mu\nu} \partial_\nu \chi\right) - \frac{g^2}{2}\frac{\partial \mathcal{P}}{\partial \chi} + \frac{1}{8}\sinh(4\chi) D_\mu \psi D^\mu \psi = 0$$

$$\chi = 0, \psi = \frac{\pi}{2}$$

$$\mathcal{P} = -2[\cosh(2\lambda_1) + \cosh(2\lambda_2) + \cosh(2\lambda_3)]$$

$$\lambda_i = 0, \varphi_i = \pi, A \equiv A^0 = A^1 = A^2 = A^3$$

$$e^{-1}\mathcal{L} = \frac{1}{2}R - 6g^2 - F_{\mu\nu}F^{\mu\nu}$$

$$\tanh\chi = \frac{1}{\sqrt{3}}, \psi = \frac{\pi}{2}, \tanh\lambda_i = 2 - \sqrt{3}, \varphi_i = \pi$$
$$\frac{1}{3}A \equiv \frac{1}{3}A^0 = A^1 = A^2 = A^3$$

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{9\sqrt{3}}{2}g^2 - \frac{1}{12\sqrt{3}}F_{\mu\nu}F^{\mu\nu}$$

$$ds^2 = e^{2V}ds^2_{AdS_2} + f^2dy^2 + h^2dz^2$$
$$A^\alpha = a^\alpha dz$$

$$\gamma^m = \Gamma^m \otimes \sigma^3, \gamma^2 =]\,]_2 \otimes \sigma^1, \gamma^3 =]\,]_2 \otimes \sigma^2$$

$$\epsilon = \psi \otimes \chi,$$

$$D_m\psi = \frac{1}{2}\kappa\Gamma_m\psi$$

$$\left[-i\left(\sqrt{2}\kappa e^{-V} + H_{23}\right)\gamma^{23} + \sqrt{2}V'f^{-1}\gamma^2\right]\epsilon = gW\epsilon.$$

$$[i\cos\,\xi\gamma^{23} + \sin\,\xi\gamma^2]\epsilon = +\epsilon$$

$$-\sqrt{2}\kappa e^{-V} - H_{23} = gW\cos\,\xi, \sqrt{2}V'f^{-1} = gW\sin\,\xi$$

$$\epsilon = e^{i\frac{\xi}{2}\gamma^3}\eta, \gamma^2\eta = +i\gamma^3\eta$$

$$\xi = 0, \pi, \gamma^{23}\epsilon = \pm i\epsilon$$

$$\left[\partial_y - \frac{1}{2}V' + \frac{i}{2}\left(\partial_y\xi + \sqrt{2}fH_{23} + \kappa fe^{-V}\right)\gamma^3\right]\eta = 0$$

$$\left[2\partial_z - iB_z + if^{-1}h'\cos\,\xi - i\frac{1}{\sqrt{2}}H_{23}h\sin\,\xi + \left(f^{-1}h'\sin\,\xi - \frac{1}{\sqrt{2}}gWh + \frac{1}{\sqrt{2}}H_{23}h\cos\,\xi\right)\gamma^3\right]\eta = 0$$

$$\eta = e^{V/2}e^{isz}\eta_0$$

$$\partial_y\xi + \sqrt{2}fH_{23} + \kappa fe^{-V} = 0$$
$$(s - B_z) + f^{-1}h'\cos\,\xi - \frac{1}{\sqrt{2}}H_{23}h\sin\,\xi = 0$$
$$f^{-1}h'\sin\,\xi - \frac{gWh}{\sqrt{2}} + \frac{1}{\sqrt{2}}H_{23}h\cos\,\xi = 0$$

$$f^{-1}h' = \frac{gWh}{\sqrt{2}}\sin\,\xi - (s - B_z)\cos\,\xi$$
$$hH_{23} = gWh\cos\,\xi + \sqrt{2}(s - B_z)\sin\,\xi$$

$$(s - B_z)\sin\,\xi = -\sqrt{2}gWh\cos\,\xi - \kappa he^{-V}$$

$$H_{23} = -gW\cos\,\xi - \sqrt{2}\kappa e^{-V}$$
$$f^{-1}\partial_y\xi = \sqrt{2}gW\cos\,\xi + \kappa e^{-V}$$

$$f^{-1}\frac{h'}{h}\sin\,\xi = \kappa e^{-V}\cos\,\xi + \frac{gW}{\sqrt{2}}(1 + \cos^2\,\xi)$$

$$f^{-1}\lambda_i' + \frac{g}{\sqrt{2}}\partial_{\lambda_i} W \sin\,\xi = 0$$

$$g\partial_{\lambda_1} W\cos\,\xi + \bar{F}_{23}^{12} = 0$$
$$g\partial_{\lambda_2} W\cos\,\xi + \bar{F}_{23}^{34} = 0$$
$$g\partial_{\lambda_3} W\cos\,\xi + \bar{F}_{23}^{56} = 0$$

$$f^{-1}\chi' \sin\,\xi + \frac{g}{\sqrt{2}}\partial_\chi W = 0$$
$$\sqrt{2}g\partial_\chi W\cos\,\xi - \partial_\chi B_z \sin\,\xi h^{-1} = 0$$

$$f^{-1}\xi' = \sqrt{2}gW\cos\,\xi + \kappa e^{-V}$$
$$f^{-1}V' = \frac{g}{\sqrt{2}} W\sin\,\xi$$
$$f^{-1}\lambda_i' = -\frac{g}{\sqrt{2}}\partial_{\lambda_i} W\sin\,\xi$$
$$f^{-1}\chi' = -\frac{g}{\sqrt{2}}\frac{\partial_\chi W}{\sin\,\xi}$$
$$f^{-1}\frac{h'}{h}\sin\,\xi = \kappa e^{-V}\cos\,\xi + \frac{gW}{\sqrt{2}}(1+\cos^2\,\xi)$$

$$(s - B_z)\sin\,\xi = -\sqrt{2}gWh\cos\,\xi - \kappa h e^{-V}$$
$$\sqrt{2}g\partial_\chi W\cos\,\xi = \partial_\chi B_z \sin\,\xi h^{-1}$$

$$\bar{F}_{23}^{12} = -g\partial_{\lambda_1} W\cos\,\xi$$
$$\bar{F}_{23}^{34} = -g\partial_{\lambda_2} W\cos\,\xi$$
$$\bar{F}_{23}^{56} = -g\partial_{\lambda_3} W\cos\,\xi$$
$$H_{23} = -gW\cos\,\xi - \sqrt{2}\kappa e^{-V}$$

$$\partial_y W = -\frac{g}{\sqrt{2}} f\sin\,\xi\left[\sum_{i=1}^{3}\left(\partial_{\lambda_i} W\right)^2 + \frac{1}{\sin^2\,\xi}\left(\partial_\chi W\right)^2\right]$$

$$he^{-V} = k\sin\,\xi$$

$$f^{-1}\xi' = -k^{-1}(s - B_z)e^{-V}$$
$$f^{-1}V' = \frac{g}{\sqrt{2}} W\sin\,\xi$$
$$f^{-1}\lambda_i' = -\frac{g}{\sqrt{2}}\partial_{\lambda_i} W\sin\,\xi$$
$$f^{-1}\chi' = -\frac{g}{\sqrt{2}}\frac{\partial_\chi W}{\sin\,\xi}$$

$$(s - B_z) = -k\left(\sqrt{2}gWe^{V}\cos\,\xi + \kappa\right)$$
$$\sqrt{2}g\partial_\chi W\cos\,\xi = k^{-1}e^{-V}\partial_\chi B_z$$

$$\partial_\chi B_z = -\sinh(2\chi) D_z\psi$$

$$D_z\psi = -\frac{\sqrt{2}gke^V\partial_\chi W\cos\,\xi}{\sinh\,(2\chi)}$$

$$F_{yz}^{\alpha} = (a^{\alpha})' = \left(\mathcal{J}^{(\alpha)}\right)'$$

$$\mathcal{J}^{(0)} \equiv \frac{1}{\sqrt{2}}ke^V\cos\,\xi e^{\lambda_1+\lambda_2+\lambda_3}$$
$$\mathcal{J}^{(1)} \equiv \frac{1}{\sqrt{2}}ke^V\cos\,\xi e^{\lambda_1-\lambda_2-\lambda_3}$$
$$\mathcal{J}^{(2)} \equiv \frac{1}{\sqrt{2}}ke^V\cos\,\xi e^{-\lambda_1+\lambda_2-\lambda_3}$$
$$\mathcal{J}^{(3)} \equiv \frac{1}{\sqrt{2}}ke^V\cos\,\xi e^{-\lambda_1-\lambda_2+\lambda_3}$$

$$C(\Delta) = \sum_{c\geq 1} K_c(\Delta)\int \overbrace{\frac{d\phi^0}{(\phi^0)^{3/2}}}^{\substack{\text{integration measure}\\ \text{on localization locus}}} \underbrace{(\phi^0)^5}_{\substack{\text{Bulk modes}\\ \text{one-loop determinant}}} \overbrace{\frac{1}{c\phi^0}}^{\substack{\text{Induced}\\ \text{one-loop determinant}}} \underbrace{\exp\left[-\frac{\pi\Delta\phi^0}{4c}-\frac{\pi}{c\phi^0}\right]}_{\substack{\text{Supergravity action}\\ \text{along locus}}},$$

$$\Delta(\mathcal{Q}) = C^{abcd}\mathcal{Q}_a\mathcal{Q}_b\mathcal{Q}_c\mathcal{Q}_d$$

$$S_{\mathrm{BH}}(\mathcal{Q}) = \pi\sqrt{\Delta(\mathcal{Q})}$$

$$\Delta = 4n - \ell^2$$

$$\varphi_{-2,1}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}$$

$$\varphi_{-2,1}(\tau,z) = \sum_{n,\ell\in\mathbb{Z}} c_{-2,1}(n,\ell)q^n\zeta^\ell, q = \mathrm{e}^{2\pi\mathrm{i}\tau}, \zeta = \mathrm{e}^{2\pi\mathrm{i}z}$$

$$c_{-2,1}(n,\ell) = C(\Delta), \Delta = 4n - \ell^2$$

$$C(\Delta) = 2\pi\left(\frac{\pi}{2}\right)^{7/2}\sum_{c=1}^{\infty} c^{-9/2}K_c(\Delta)\tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right)$$

$$\tilde{I}_\rho(z) = \left(\frac{z}{2}\right)^{-\rho} I_\rho(z) = \frac{1}{2\pi\mathrm{i}}\int_{\epsilon-\mathrm{i}\infty}^{\epsilon+\mathrm{i}\infty}\frac{d\sigma}{\sigma^{\rho+1}}\exp\left(\sigma+\frac{z^2}{4\sigma}\right)$$

$$K_c(\Delta) := \mathrm{e}^{5\pi\mathrm{i}/4}\sum_{\substack{-c\leq d<0\\(d,c)=1}} \mathrm{e}^{2\pi\mathrm{i}\frac{d}{c}(\Delta/4)}M\left(\gamma_{c,d}\right)_{\nu 1}^{-1}\mathrm{e}^{2\pi\mathrm{i}\frac{a}{c}(-1/4)}, \nu = \Delta\mathrm{mod}2$$

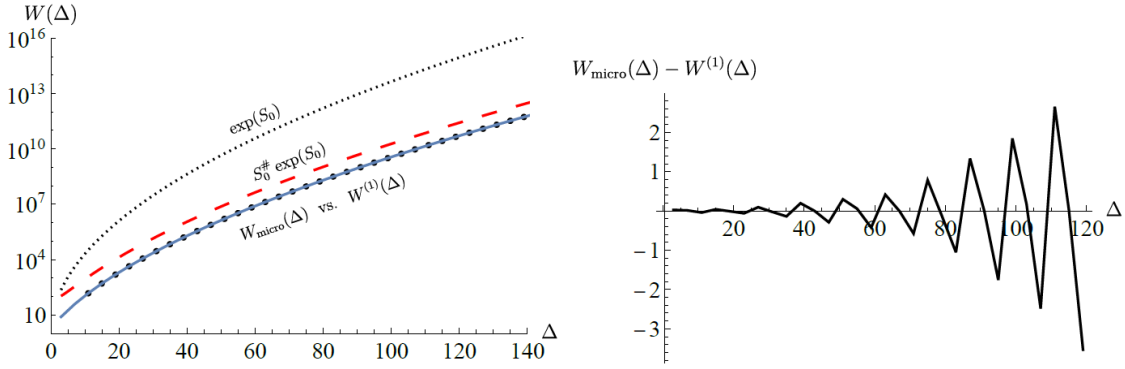
$$M^{-1}\left(\gamma_{c,d}\right)_{\nu 1} = \mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\frac{1}{\sqrt{6c}}\exp\left(-\frac{\mathrm{i}\pi}{6}\Phi(\gamma)\right)\times$$
$$\sum_{\varepsilon=\pm 1}\sum_{n=0}^{c-1}\varepsilon\exp\left(\frac{\mathrm{i}\pi}{6c}(d(\nu+1)^2 - 4(\nu+1)(3n+\varepsilon) + 4a(3n+\varepsilon)^2)\right)$$

$$W_{\text{micro}}(\Delta) = (-1)^{\Delta+1} C(\Delta)$$

$$\log W_{\text{micro}}(\Delta) \sim \pi\sqrt{\Delta} + \cdots, \Delta \to \infty$$

$$\log W_{\text{micro}}(\Delta) = \pi\sqrt{\Delta} - 2\log \Delta + O(1/\sqrt{\Delta})$$

$$W_{\text{micro}}(\Delta) = \underbrace{\frac{\pi^{9/2}}{8} \tilde{I}_{7/2}(\pi\sqrt{\Delta})}_{\equiv W^{(1)}(\Delta)} + O\left(e^{\sqrt{\Delta}/2}\right)$$



$$\frac{1}{2\pi}\left(\frac{2}{\pi}\right)^{7/2} W_{\text{micro}}(\Delta) = \sum_{c=1}^{\infty} A_c(\Delta), A_c(\Delta) = c^{-9/2} K_c(\Delta) \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right)$$

$$W(q,p) = \left\langle \exp\left(-iq_\Lambda \oint A^\Lambda\right)\right\rangle^{\text{reg}}_{\text{AdS}_2\times S^2 \text{ with T}\to 0}$$
$$= \int [DgD\Psi DADΦ] e^{-S^{\text{bulk}}_{\text{suggra}} - S^{\text{bdry}}_{\text{sugra}}}$$

$$S^{\text{bdry}}_{\text{sugra}} \supset \int_{\partial(\text{AdS}_2\times S^2)} d^3x\sqrt{h} A^\mu F_{\mu\nu} n^\nu$$

$$S_{\text{sugra}} = \int \mathrm{d}^4x \left(\mathcal{L}^{\text{phys}}_{\text{sugra}} - Q_{\text{eq}}(b_\alpha F^\alpha)\right)$$

$$S_{\text{sugra}} = S(0) \to S(t) = S(0) + \lambda Q_{\text{eq}} \mathcal{V}$$

$$\mathcal{V} = \int d^4x \sqrt{\mathring{g}} \sum_{\psi} \bar{\psi} Q_{\text{eq}} \psi$$

$$ds^2 = d\rho^2 + \sinh^2 \rho d\tau^2 + d\theta^2 + \sin^2 \theta d\psi^2$$

$$(\tau, \psi) \sim (\tau + 2\pi, \psi) \sim (\tau, \psi + 2\pi)$$

$$F^\Lambda_{\rho\tau} = -ie^\Lambda_\star \sinh \rho, F^\Lambda_{\theta\psi} = p^\Lambda \sin \theta$$

$$X^{\Lambda}=\frac{1}{2}\left(e_{\star}^{\Lambda}+\mathrm{i}p^{\Lambda}\right)+\frac{\phi^{\Lambda}-e_{\star}^{\Lambda}}{2\cosh\,\rho},\bar{X}^{\Lambda}=\frac{1}{2}\left(e_{\star}^{\Lambda}-\mathrm{i}p^{\Lambda}\right)+\frac{\phi^{\Lambda}-e_{\star}^{\Lambda}}{2\cosh\,\rho}.$$

$$Y_1^{\Lambda,1}=-Y_2^{\Lambda,2}=\frac{\phi^{\Lambda}-e_{\star}^{\Lambda}}{\cosh^2\,\rho}$$

$$\ell^2=\mathrm{e}^{-\mathcal{K}}\equiv-\mathrm{i}\left(X^{\Lambda}\bar{F}_{\Lambda}-\bar{X}^{\Lambda}F_{\Lambda}\right)\Big|_{\rho=0}$$

$$(\tau,\phi)\sim\left(\tau+\frac{2\pi}{c},\psi+\frac{2\pi}{c}\right)\sim(\tau,\psi+2\pi).$$

$$A^0_{(c,d)}=A^0+\frac{d}{c}d\tau,A^0=-ie_{\star}^0(\cosh\,\rho-1)d\tau$$

$$F(X^I)=C_{IJK}\frac{X^IX^JX^K}{X^0}+\cdots$$

$$W^{(1)}(q,p)=\int_{\mathcal{M}_Q}[\,\mathrm{d}^{n_v+1}\phi]\exp\left(-\pi q_{\Lambda}\phi^{\Lambda}+4\pi\mathrm{Im}F\left(\frac{\phi^{\Lambda}+\mathrm{i}p^{\Lambda}}{2}\right)\right)Z_{1-\mathrm{loop}}^{Q_{\mathrm{eq}}\mathcal{V}}\left(\phi^{\Lambda}\right)$$

$$W^{(\mathrm{c})}(q,p)=\int_{\mathcal{M}_Q}[\,\mathrm{d}^{n_v+1}\phi]_c\exp\left(-\frac{\pi q_{\Lambda}\phi^{\Lambda}}{c}+\frac{4\pi}{c}\mathrm{Im}F\left(\frac{\phi^{\Lambda}+\mathrm{i}p^{\Lambda}}{2}\right)\right)Z_{1-\mathrm{loop},c}^{Q_{\mathrm{eq}}\mathcal{V}}\left(\phi^{\Lambda}\right)Z_c^{\mathrm{top}}(q,p)$$

$$F^{(0)}(X)=C_{IJK}\frac{X^IX^JX^K}{X^0},C_{IJK}\equiv\frac{1}{6}\int_{T^6}\alpha_I\wedge\alpha_J\wedge\alpha_K$$

$$\Delta=4C_{IJK}p^Ip^Jp^K\left(q_0+\frac{1}{12}q_IC^{IJ}q_J\right)$$

$$\int\,[D\Phi^m]\mathrm{e}^{-\int\sqrt{g}G_{mn}(\Phi)\delta\Phi^m\delta\Phi^n}$$

$$[\mathrm{d}^{16}\phi]_c\equiv\,\mathrm{d}^{16}\phi\sqrt{2\det\left(\frac{1}{2c}\mathrm{Im}F_{\Lambda\Sigma}\right)},$$

$$\mathrm{d}^{16}\phi=\prod_{\Lambda=0}^{16}\,\mathrm{d}\phi^{\Lambda}$$

$$\mathrm{Im}\left(F_{IJ}\right)=\frac{6C_{IJ}}{\phi^0}$$

$$\mathrm{Im}(F_{0I})=\mathrm{Im}(F_{I0})=-\frac{6C_{IJ}\phi^J}{(\phi^0)^2}$$

$$\mathrm{Im}(F_{00})=-\frac{2C_{IJ}p^Ip^J}{(\phi^0)^3}+\frac{6C_{IJ}\phi^I\phi^J}{(\phi^0)^3}$$

$$[\mathrm{d}^{16}\phi]_c = \mathrm{d}^{16}\phi \frac{1}{c^8}\sqrt{\frac{-2C_{IJK}p^I p^J p^K \det\left(3C_{IJ}\right)}{(\phi^0)^{18}}}$$

$$Z_c^{\mathrm{top}} = \sqrt{c}K_c(\Delta).$$

$$\mathrm{e}^{-\mathcal{K}(\phi^\Lambda)} = \frac{-2C_{IJK}p^I p^J p^K}{\phi^0}$$

$$Z_{\mathrm{index}} = \sqrt{\frac{\det_\Psi(H/\ell)}{\det_\Phi(H/\ell)}}$$

$$\ell^2 = e^{-\mathcal{K}(\phi^\Lambda + \mathrm{i}p^\Lambda)}$$

$$\mathrm{ind}(D_{10})(t) := \mathrm{Tr}_{\mathrm{Ker}\ D_{10}}\mathrm{e}^{\frac{tH}{\ell}} - \mathrm{Tr}_{\mathrm{Coker}\ D_{10}}\mathrm{e}^{\frac{tH}{\ell}}$$

$$\mathrm{ind}(D_{10})(t) = \sum_n \alpha(n)\mathrm{e}^{\mathrm{i}\lambda_n t}$$

$$Z_{\mathrm{index}} = \prod_n (\lambda_n)^{-\frac{1}{2}\alpha(n)},$$

$$\mathrm{ind}(D_{10})(t) = \sum_{\{x|\tilde{x}=x\}} \frac{\mathrm{Tr}_\Phi e^{\frac{tH}{\ell}} - \mathrm{Tr}_\Psi e^{\frac{tH}{\ell}}}{\det(1-\partial\tilde{x}/\partial x)}$$

$$ds^2 = d\rho^2 + \sinh^2\,\rho d\tau^2 + d\theta^2 + \sin^2\,\theta d\psi^2 = \frac{4dwd\bar{w}}{(1-w\bar{w})^2} + \frac{4dzd\bar{z}}{(1+z\bar{z})^2}$$

$$H \equiv L_0 - J_0 = \left(\partial_\tau - \partial_\psi\right)$$

$$(w,z) \to \left(e^{\mathrm{i}t/\ell}w, e^{-\mathrm{i}t/\ell}z\right)$$

$$\det(1-\partial\tilde{x}/\partial x) = (1-q)^2(1-q^{-1})^2$$

$$NP\colon \varepsilon^i_{+\alpha} = 0, \varepsilon^i_{-\dot{\alpha}} = \left(\sigma_3 \exp\left[\mathrm{i}\frac{(\tau+\psi)}{2}\sigma_3\right]\right)^i \dot{\alpha}, H = L_0 - J_0 = 2J_0^{(+)}$$
$$SP\colon \varepsilon^i_{+\alpha} = \left(-\mathrm{i}\sigma_3 \exp\left[\mathrm{i}\frac{(\tau+\psi)}{2}\sigma_3\right]\right)^i_\alpha, \varepsilon^i_{-\dot{\alpha}} = 0, H = L_0 - J_0 = 2J_0^{(-)}$$

$$NP\colon \mathrm{Tr}_{(m,n)}e^{\frac{tH}{\ell}} = n\left(q^{-|m-1|} + q^{-|m-1|+2} \dots + q^{|m-1|-2} + q^{|m-1|}\right)$$
$$SP\colon \mathrm{Tr}_{(m,n)}e^{\frac{tH}{\ell}} = m\left(q^{-|n-1|} + q^{-|n-1|+2} \dots + q^{|n-1|-2} + q^{|n-1|}\right)$$

$$\log\, Z_{\mathrm{index}} = \frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\mathrm{ind}(D_{10})(t)$$

$$\frac{1}{4}\mathrm{ind}(D_{10})(t) = \cdots + \frac{a_{-2}}{t^2} + a_0 + a_2 t^2 + \cdots$$

$$\{Q^i, \bar{Q}_j\} = (\gamma^a P_a + \mathrm{i}Z)\delta^i_j$$

$$\left[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)\right] = \delta_{\mathrm{cgct}}(\xi) + \delta_M(\varepsilon) + \delta_K(\Lambda_K) + \delta_S(\eta) + \delta_{\mathrm{gauge}}$$

$$\varepsilon^{ab} = \mathrm{i}\frac{1}{2}\bar{\varepsilon}_{2i+}\varepsilon^i_{1+}T^{ab+} + \mathrm{i}\frac{1}{2}\bar{\varepsilon}_{2i-}\varepsilon^i_{1-}T^{ab-}$$

$$\{Q^i, \bar{S}_j\} = -\delta^i_j\left(\gamma^{ab}M_{ab} + Dil - \gamma_5 A\right) + 2V^i_j$$

$$(\tau, \psi) \sim \left(\tau + \frac{2\pi}{c}, \psi + \frac{2\pi}{c}\right) \sim (\tau, \psi + 2\pi).$$

$$\frac{1}{|\mathbb{Z}_c|}\sum_{\gamma\in\mathbb{Z}_c}\delta^4(\tilde{x}(\gamma, t) - x)$$

$$\tilde{x}(\gamma, t) = \left(\gamma \cdot \mathrm{e}^{\frac{tH}{\ell}}\right)x = \mathrm{e}^{\left(\frac{t}{\ell}+\frac{2\pi}{c}\right)H}x$$

$$\mathrm{ind}_c(D_{10})(t) = \frac{1}{c}\sum_{\gamma\in\mathbb{Z}_c}\sum_{\{x|\tilde{x}(\gamma,t)=x\}}\frac{\mathrm{Tr}_\Phi\left(\gamma \cdot \mathrm{e}^{\frac{tH}{\ell}}\right) - \mathrm{Tr}_\Psi\left(\gamma \cdot \mathrm{e}^{\frac{tH}{\ell}}\right)}{\det(1 - \partial\tilde{x}/\partial x)},$$

$$q = \mathrm{e}^{\frac{t}{\ell}+\frac{2\pi}{c}}$$

$$\lambda_{(2,2)} \equiv \lambda_\mu \equiv \bar{\varepsilon}_i\gamma_\mu\lambda^i, \lambda_{(1,1)} \equiv \lambda \equiv \bar{\varepsilon}_i\gamma_5\lambda^i, \lambda_{(1,3)} \equiv \lambda^{ij} \equiv -2\varepsilon^{(i}C\lambda^{j)},$$

$$Q_{\mathrm{eq}}A_\mu = \lambda_\mu + \partial_\mu c.$$

$$Q_{\mathrm{eq}}\mathrm{Im}X = \lambda,$$

$$Q_{\mathrm{eq}}\lambda^{ij} \supset Y^{ij},$$

$$Q_{\mathrm{eq}}c \supset i\left(\bar{\epsilon}_i\epsilon^i\right)\mathrm{Re}X,$$

$$Q_{\mathrm{eq}}b = B,$$

$$\mathrm{Tr}^{NP}_\Phi\left(\mathrm{e}^{\frac{tH}{\ell}}\right) = \mathrm{Tr}^{SP}_\Phi\left(\mathrm{e}^{\frac{tH}{\ell}}\right) = \mathrm{Tr}_{(2,2)}\left(\mathrm{e}^{\frac{tH}{\ell}}\right) + \mathrm{Tr}_{(1,1)}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)$$
$$\mathrm{Tr}^{NP}_\Psi\left(\mathrm{e}^{\frac{tH}{\ell}}\right) = \mathrm{Tr}^{SP}_\Psi\left(\mathrm{e}^{\frac{tH}{\ell}}\right) = \mathrm{Tr}_{(1,3)}\left(\mathrm{e}^{\frac{tH}{\ell}}\right) + 2\mathrm{Tr}_{(1,1)}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)$$

$$\mathrm{ind}^{\mathcal{N}=2}_{\mathrm{vect.}}(D_{10})(t) = \frac{4(q + q^{-1}) - 8}{(1-q)^2(1-q^{-1})^2} = \frac{4q}{(1-q)^2} \text{ with } q = e^{\frac{it}{\ell}}$$

$$\mathrm{ind}^{\mathcal{N}=2\ \mathrm{s.c.}}_{\mathrm{vect.},c}(D_{10})(t)\big|_{\mathrm{NP/SP}} = \frac{2cq^c}{(1-q^c)^2}\Big|_{\mathrm{NP/SP}}, \text{ with } q = e^{\frac{it}{\ell}},$$

$$\frac{2cq^c}{(1-q^c)^2} = \sum_{n\geq 1} 2cn e^{-it\frac{n}{\ell}c}$$

$$\left\langle e^{\partial_s(\sum_{n\geq 1} 2cn(nc/\ell)^{-s})} \right|_{s\to 0}$$

$$\log\left(Z^{\text{vec}}_{1-\text{loop}}\right) = -\frac{i\pi}{12}c + \frac{c}{6}\log\, c + 2c\zeta'(-1) - \frac{c}{12}\log\, \ell^2.$$

$$\begin{aligned}
\delta e_\mu{}^a &= \bar{\epsilon}^i \gamma^a \psi_{\mu i} + \text{ h.c.}\,,\\
\delta\psi_\mu{}^i &\supset -b_\mu \epsilon^i - 2V_\mu{}^i{}_j \epsilon^j - \frac{1}{2}\gamma^{ab} T_{ab}{}^{ij}\gamma_\mu \epsilon_j - \gamma_\mu \eta^i,\\
\delta b_\mu &\supset -\frac{1}{2}\bar{\psi}_\mu{}^i \eta_i + \text{ h.c.}\,,\\
\delta V_\mu{}^i{}_j &\supset \bar{\epsilon}^k \gamma_\mu \chi^i{}_{kj} - \bar{\psi}_\mu{}^i \eta_j + \frac{1}{4}\delta^i{}_j \bar{\psi}_\mu{}^k \eta_k,\\
\delta\Lambda_i &\supset E_{ij}\epsilon^j + \frac{1}{2}\varepsilon_{ijkl} T_{bc}{}^{kl}\gamma^{bc}\epsilon^j,\\
\delta E_{ij} &\supset -2\bar{\epsilon}^k \chi^{mn}_{(i}\varepsilon_{j)kmn} + 2\bar{\eta}_{(i}\Lambda_{j)},\\
\delta T_{ab}{}^{ij} &\supset -2\bar{\epsilon}^{[i}\left(\gamma_{[a}\phi_{b]}{}^{j]} + \frac{1}{2}\gamma\cdot T^{j]k}\gamma_{[a}\psi_{b]k}\right) + \frac{1}{2}\bar{\epsilon}^k \gamma_{ab}\chi^{ij}{}_k - \frac{1}{4}\varepsilon^{ijkl}\bar{\eta}_k \gamma_{ab}\Lambda_l,\\
\delta\chi^{ij}{}_k &\supset -\frac{1}{2}\not{D} T^{ij}_{ab}\epsilon_k + D^{ij}{}_{kl}\epsilon^l + \frac{1}{2}T^{ij}_{ab}\gamma^{ab}\eta_k + \frac{2}{3}\delta^{[i}_k T_{ab}{}^{j]l}\gamma^{ab}\eta_l - \frac{1}{2}\varepsilon^{ijlm}E_{kl}\eta_m,\\
\delta D^{ij}{}_{kl} &\supset \varepsilon^{ijmn}\bar{\epsilon}^p T^{ab}{}_{kl}\left[2T_{abmp}\lambda_m + T_{abmn}\Lambda_p\right],
\end{aligned}$$

$$\begin{aligned}
e^a_\mu &= e_{(3,3,1)\oplus(1,3,1)\oplus(3,1,1)\oplus(1,1,1)} \in \Phi\\
\psi^a_\mu &= \psi_{(2,2,1)\oplus(2,2,1)} = \psi_{(3,3,1)\oplus(1,3,1)\oplus(3,1,1)\oplus(1,1,1)} \notin \Psi
\end{aligned}$$

$$\psi_{(2,4,1)} \in \Psi, \psi_{(2,2,1)\otimes(1,1,1)} \in \Psi$$

$$V_{(2,2,1)} \notin \Phi, V_{(2,4,1)} \notin \Phi.$$

$$\psi_{(2,2,1)'} \in \Psi, (c_S)_{(2,2,1)} \notin \Phi$$

$$\begin{aligned}
\text{ind}^{\mathcal{N}=2\text{ s.c.}}_{\text{vect.}}(D_{10})(t)\big|_{\text{NP/SP}} &= \left.\frac{2(q+q^{-1})-4}{(1-q)^2(1-q^{-1})^2}\right|_{\text{NP/SP}},\\
\text{ind}^{\mathcal{N}=2\text{ s.c.}}_{\text{hyper}}(D_{10})(t)\big|_{\text{NP/SP}} &= -\left.\frac{2(q+q^{-1})-4}{(1-q)^2(1-q^{-1})^2}\right|_{\text{NP/SP}},\\
\text{ind}^{\mathcal{N}=2\text{ s.c.}}_{\text{Weyl}}(D_{10})(t)\big|_{\text{NP/SP}} &= \left.\frac{2(q^2+q^{-2})-6(q+q^{-1})+8}{(1-q)^2(1-q^{-1})^2}\right|_{\text{NP/SP}},\\
\text{ind}^{\mathcal{N}=2}_{\text{spin }3/2}\text{ s.c. }(D_{10})(t)|_{\text{NP/SP}} &= \left.\frac{-2(q^2+q^{-2})+4(q+q^{-1})-4}{(1-q)^2(1-q^{-1})^2}\right|_{\text{NP/SP}},\\
\text{ind}^{\mathcal{N}=2\text{ s.c.}}_{\text{chiral}}(D_{10})(t)\big|_{\text{NP/SP}} &= 0|_{\text{NP}},
\end{aligned}$$

$$\mathrm{ind}^{\mathcal{N}=2\ \text{s.P.}}_{\text{vect.},\,c}(D_{10})(t)\big|_{\text{NP}} = \frac{2cq^c}{(1-q^c)^2}\Big|_{\text{NP}},$$

$$\mathrm{ind}^{\mathcal{N}=2\ \text{s.P.}}_{\text{hyper.}\,c}(D_{10})(t)\big|_{\text{NP}} = \frac{-2cq^c}{(1-q^c)^2}\Big|_{\text{NP}},$$

$$\mathrm{ind}^{\mathcal{N}=2\ \text{s.P.}}_{\text{Graviton},\,c}(D_{10})(t)\big|_{\text{NP}} = \frac{2q^{2c}+2(c-2)q^c+2}{(1-q^c)^2}\Big|_{\text{NP}},$$

$$\mathrm{ind}^{\mathcal{N}=2\ \text{s.P.}}_{\text{Gravitino},\,c}(D_{10})(t)\big|_{\text{NP}} = -\frac{2q^{2c}+2(c-2)q^c+2}{(1-q^c)^2}\Big|_{\text{NP}},$$

$$\log\left(Z^{\mathcal{N}=2\ \text{vector}}_{\text{index}}\right) = -\frac{\mathrm{i}\pi}{12}c + \frac{c}{6}\log\, c + 2c\zeta'(-1) - \frac{c}{12}\log\,\ell^2,$$

$$\log\left(Z^{\mathcal{N}=2\ \text{hyper}}_{\text{index}}\right) = \frac{\mathrm{i}\pi}{12}c - \frac{c}{6}\log\, c - 2c\zeta'(-1) + \frac{c}{12}\log\,\ell^2,$$

$$\log\left(Z^{\mathcal{N}=2\ \text{graviton}}_{\text{index}}\right) = -\frac{\mathrm{i}\pi}{12}c + \frac{c}{6}\log\, c + 2c\zeta'(-1) + \left(1-\frac{c}{12}\right)\log\,\ell^2$$

$$\log\left(Z^{\mathcal{N}=2\ \text{gravitino}}_{\text{index}}\right) = \frac{\mathrm{i}\pi}{12}c - \frac{c}{6}\log\, c - 2c\zeta'(-1) - \left(1-\frac{c}{12}\right)\log\,\ell^2.$$

$$Z^{\mathcal{N}=4\ \text{vector}}_{\text{index}} = 1$$

$$\log\left(Z^{\mathcal{N}=4\ \text{graviton}}_{\text{index}}\right) = -\log\,\ell^2.$$

$$\log\left(Z^{\mathcal{N}=8\ \text{graviton}}_{\text{index}}\right) = -5\log\,\ell^2.$$

$$Z^{Q_{\text{eq}}\mathcal{V}}_{\text{1-loop}} \equiv Z^{Q_{\text{eq}}\mathcal{V}}_{\text{zero-modes}}\, Z^{Q_{\text{eq}}\mathcal{V}}_{\text{non-zero}}$$

$$Q^2_{\text{eq}}\, A^{\text{bdry}}_{\mu} = HA^{\text{bdry}}_{\mu} = n^{\nu}F^{\text{bdry}}_{\nu\mu} = 0$$

$$Q^2_{\text{eq}}\Phi^{\text{bdry}} = H\Phi^{\text{bdry}} = 0$$

$$Q_{\text{eq}}\,\mathcal{V}\big|_{A^{\text{bdry}}_{\mu}} = 0$$

$$Q_{\text{eq}}\mathcal{V}\big|_{\Phi^{\text{bdry}}} = 0$$

$$Q_{\text{eq}}\mathcal{V}\big|_{Q_{\text{eq}}\Phi^{\text{bdry}}} = -\left(Q_{\text{eq}}\Phi^{\text{bdry}}\right)H\left(Q_{\text{eq}}\Phi^{\text{bdry}}\right) = -Q_{\text{eq}}\left[\Phi^{\text{bdry}}H\left(Q_{\text{eq}}\Phi^{\text{bdry}}\right)\right]$$

$$Q_{\text{eq}}\mathcal{V}\big|_{\Psi^{\text{bdry}}} = Q_{\text{eq}}\mathcal{V}\big|_{Q_{\text{eq}}\,\Phi^{\text{pre-bdry}}} = 0$$

$$Q_{\text{eq}}\mathcal{V}\big|_{Q_{\text{eq}}\Psi^{\text{bdry}}} = -\left(Q_{\text{eq}}\Psi^{\text{bdry}}\right)1\left(Q_{\text{eq}}\Psi^{\text{bdry}}\right),\ Q_{\text{eq}}\mathcal{V}\big|_{\Phi^{\text{pre-bdry}}} = \Phi^{\text{pre-bdry}}\,H^2\Phi^{\text{pre-bdry}}$$

$$Z^{Q_{\text{eq}}\mathcal{V}}_{\text{non-zero}} = \sqrt{\det{}^{*_f}K_f/\det{}^{*_b}K_b}$$

$$Z^{Q_{\text{eq}}\mathcal{V}}_{\text{non-zero}} = \sqrt{\frac{\det^{*f} K_f}{\det^{*b} K_b}} = \sqrt{\frac{\det_{Q_{\text{eq}}\Phi^{\text{bdry}}}(H)}{\det_{\Phi^{\text{pre-bdy}}}(H^2)}\frac{\det' K_f}{\det' K_b}} = \sqrt{\frac{\det_{\Phi^{\text{bdry}}}(H)}{\det_{\Phi^{\text{pre-bdy}}}(H^2)}\frac{\det'_{\Psi} H}{\det'_{\Phi} H}}$$

$$\begin{aligned}\log Z^{Q_{\text{eq}}\mathcal{V}}_{\text{non-zero}} =& -\frac{1}{2}\int \frac{dt}{t}\left(\text{Tr}_{\Phi^{\text{bdry}}} e^{\frac{tH}{\ell}} - 2\text{Tr}_{\Phi^{\text{pre-bdry}}} e^{\frac{tH}{\ell}} + \text{Tr}_{\Psi'} e^{\frac{tH}{\ell}} - \text{Tr}_{\Phi'} e^{\frac{tH}{\ell}}\right)\\ =& \int \frac{dt}{t}\left[-\text{Tr}_{\Phi^{\text{bdry}}} e^{\frac{tH}{\ell}} + \frac{1}{2}\left(\text{Tr}_{\Psi^{\text{bdry}}} e^{\frac{tH}{\ell}} + \text{Tr}_{Q_{\text{eq}}\Phi^{\text{pre-bdry}}} e^{\frac{tH}{\ell}}\right)\right]\\ & -\frac{1}{2}\int \frac{dt}{t}\left(\text{Tr}_{\Psi} e^{\frac{tH}{\ell}} - \text{Tr}_{\Phi} e^{\frac{tH}{\ell}}\right)\end{aligned}$$

$$\text{ind}^{\mathcal{N}=8\text{ s.P.}}_{\text{Weyl}}(D_{10})(t) = \text{Tr}_{\Phi} e^{\frac{tH}{\ell}} - \text{Tr}_{\Psi} e^{\frac{tH}{\ell}}$$

$$n^{\text{bos}}_{\text{bdry}} \equiv \text{Tr}_{\Phi^{\text{bdry}}} e^{\tilde{t}H}, n^{\text{ferm}}_{\text{bdry}} \equiv \text{Tr}_{\Psi^{\text{bdry}}} e^{\tilde{t}H}, \text{ with } \tilde{t} = \frac{t}{\ell}$$

$$n^{\text{bos}}_{\text{bdry}} \equiv n^{1-\text{ form}}_{\text{bdry}} + n^{\text{grav}}_{\text{bdry}}, n^{\text{grav}}_{\text{bdry}} = -6, n^{1-\text{ form}}_{\text{bdry}} = -n_v, n^{\text{ferm}}_{\text{bdry}} = -8, \text{ for } c = 1.$$

$$n^{\text{grav}}_{\text{bdry},c} = -2, n^{\text{1-form}}_{\text{bdry},c} = -n_v, n^{\text{ferm}}_{\text{bdry},c} = -4, \text{ for } c > 1.$$

$$\log Z^{Q_{\text{eq}}\mathcal{V}}_{\text{non-zero},c} = \left(-n^{\text{bos}}_{\text{bdry},c} + \frac{1}{2} n^{\text{ferm}}_{\text{bdry},c}\right)\log(\ell) + \log Z^{Q_{\text{eq}}\mathcal{V}}_{\text{index},c}$$

$$\log Z^{Q_{\text{eq}}\mathcal{V}}_{\text{index},c} = \frac{1}{2}\int_{\epsilon}^{\infty} \frac{dt}{t} \text{ind}^{\mathcal{N}=8\text{ s.P.}}_{\text{Weyl},c}(D_{10})(t)$$

$$-n^{\text{bos}}_{\text{bdry},c} + \frac{1}{2} n^{\text{ferm}}_{\text{bdry},c} = 2 + \frac{n_v}{2}$$

$$-n^{\text{bos}}_{\text{bdry},c} + \frac{1}{2} n^{\text{ferm}}_{\text{bdry},c} = \frac{n_v}{2}$$

$$Z^{\text{bdy.}}_{\text{zero-modes},c} = \int_{\frac{\text{Large-diffeomorphisms}}{\text{Super-isometries}}} \left[d(A_\mu)^{\text{Large gauge transf.}}\right]\left[dh_{\mu\nu}^{\text{Large super-diffs.}}\right]\left[d\psi_\mu^{\text{Large super-diffs.}}\right],$$

$$\int \left[d(A_\mu)\right] e^{-\int d^4x\sqrt{g} g^{\mu\nu} A_\mu A_\nu} = 1$$

$$Z^{\text{bdy.}}_{\text{zero-modes},c=1} = (\ell)^{c_{\text{zero-modes}}} \int_{\frac{\text{Diff}(S^{1|4})}{\text{PSU}(1,1|2)}} \mathcal{D}_{large\ diffs} \int_{[\text{Loop } U(1)]^{nv}} \mathfrak{D}_{large\ gauge}$$

$$Z^{\text{bdy.}}_{\text{zero-modes},c} = (\ell)^{c_{\text{zero-modes},c}} \int_{\frac{\text{Diff}(S^{1|4})}{H_{\text{orb.}}}} \mathcal{D}_{large\ diffs} \int_{[\text{Loop } U(1)]^{nv}} \mathfrak{D}_{large\ gauge}$$

$$c_{\text{zero-modes},c} = \beta^{\text{1-form}} n^{\text{1-form}}_{\text{bdry},c} + \beta^{\text{grav}} n^{\text{grav}}_{\text{bdry},c} - \frac{1}{2}\beta^{\text{ferm}} n^{\text{ferm}}_{\text{bdry},c}.$$

$$\beta^{\text{1-form}} = 1, \beta^{\text{grav}} = 2, \beta^{\text{ferm}} = 3$$

$$c_{\text{non-index}} = \left(\beta^{1-\text{form}} - 1\right) n_{\text{bdry},c}^{1\text{-form}} + \left(\beta^{\text{grav}} - 1\right) n_{\text{bdry},c}^{\text{grav}} - \frac{1}{2}\left(\beta^{\text{ferm}} - 1\right) n_{\text{bdry},c}^{\text{ferm}}$$

$$\text{Vol}\left[\frac{\text{Diff}\left(S^{1|4}\right)}{H_{\text{Disk}}}\right] = \int_{\frac{\text{Diff}(S^{1|4})}{H_{\text{Disk}}}} \mathbb{D}_{zero\ modes}\ \text{Vol}\left[\frac{\text{Diff}\left(S^{1|4}\right)}{H_{\text{Orb}}}\right] = \int_{\frac{\text{Diff}(\ S^{1|4}}{H_{\text{Orb}}}} \mathbb{D}_{zero\ modes}$$

$$\text{Vol}[\text{Loop}(U(1))] = \int_{\text{Loop}(U(1))} \mathfrak{D}_{large\ gauge}$$

$$S_{\text{sugra, bdy}} \supset \int_{\partial(\text{AdS}_2 \times S^2)} d^3x\sqrt{h} A^\mu F_{\mu\nu} n^\nu$$

$$I_{\text{Particle on } U(1)} = -\frac{T}{E_{U(1)}} \int_0^{2\pi} d\tau (\partial_\tau \Lambda(\tau))^2$$

$$\Lambda(\tau + 2\pi) = \Lambda(\tau) + \alpha_{U(1)}$$

$$Z^{\text{Particle on } U(1)}(\beta, \tilde{q} = 0) = \int_0^{2\pi} d\alpha_{U(1)} Z\left(\beta, \alpha_{U(1)}\right)$$
$$= \int_0^{2\pi} d\alpha_{U(1)} \int_{\frac{\text{Loop}(U(1))}{U(1)}} D\Lambda e^{-I_{\text{Particle on } U(1)}[\Lambda, \alpha_{U(1)}]}$$

$$\text{Vol}[\text{Loop}(U(1))] = \lim_{T \to 0} \left[Z^{\text{Particle on } U(1)}(\beta, \tilde{q} = 0)\right] = 1.$$

$$I_{\mathcal{N}=4 \text{ Schw.}} = -\frac{T}{E_{\text{gap}}} \int_0^{2\pi} d\tau [\text{Sch}(f, \tau) + \text{Tr}(g^{-1}\partial_\tau g)^2 + (\text{ white particles})]$$

$$E_{\text{gap}} \sim \frac{1}{\ell_{\text{P1}} \Delta^{3/2}}$$

$$\text{Vol}\left[\frac{\text{Diff}\left(S^{1|4}\right)}{H_{\text{Disk}}}\right] = \lim_{T \to 0} \int_{\frac{\text{Dif}(S^{1|4})}{\text{PSU}(1,1|2)}} DfDgD\eta D\bar{\eta} e^{-I_{\mathcal{N}=4 \text{ super-Schw.}}},$$

$$f(\tau + 2\pi) = f(\tau), g(\tau + 2\pi) = e^{i2\pi\alpha\sigma_3} g(\tau), \eta(\tau + 2\pi) = -e^{i2\pi\alpha\sigma_3}\eta(\tau)$$

$$\alpha = \frac{\beta\Omega_E}{4\pi},$$

$$f(\tau) = \tan\left(\frac{\tau}{2}\right), g(\tau) = e^{i\tau[n+\alpha]\sigma_3}, \eta(\tau) = \bar{\eta}(\tau) = 0$$

$$Z_{\text{Disk}}^{\mathcal{N}=4 \text{ Schw.}}(\beta, \alpha) = \sum_{n \in \mathbb{Z}} \frac{E_{\text{gap}}}{T} \frac{2\cot(\pi\alpha)(\alpha + n)}{\pi^3(1 - 4(n + \alpha)^2)^2} e^{2\pi^2 \frac{T}{E_{\text{gap}}}(1 - 4(n+\alpha)^2)}$$

$$Z_{\text{Disk}}^{\mathcal{N}=4 \text{ Schw.}}\left(\beta, \alpha = \frac{1}{2}\right) = 1$$

$$\mathrm{Vol}\left[\frac{\mathrm{Diff}\left(S^{1|4}\right)}{H_{\mathrm{Disk}}}\right]=\lim_{T\to 0}\left[Z_{\mathrm{Disk}}^{\mathcal{N}=4\ \mathrm{Schw.}}\left(\beta,\alpha=\frac{1}{2}\right)\right]=1$$

$$(\tau,\psi)\sim\left(\tau+\frac{2\pi}{c},\psi+\frac{2\pi}{c}\right)\sim(\tau,\psi+2\pi).$$

$$f(\tau+2\pi)=\frac{f(\tau)+\tan(\pi/c)}{1-f(\tau)\tan(\pi/c)},g(\tau+2\pi)=\mathrm{e}^{2\pi i\alpha\sigma_3}g(\tau)\mathrm{e}^{2\pi i\frac{2\pi}{c}\sigma_3},\eta(\tau+2\pi)=-e^{2\pi i\alpha\sigma_3}\eta(\tau)$$

$$f(\tau)=\tan\left(\frac{\tau}{2c}\right),g(\tau)=\mathrm{e}^{i\tau\left[n+\alpha\pm\frac{1}{2}\left(1-\frac{1}{c}\right)\right]\sigma_3},\eta(\tau)=\bar{\eta}(\tau)=0$$

$$Z_{(c)}^{\mathcal{N}=4\ \mathrm{Schw.}}(\beta,\alpha)=\sum_{n\in\mathbb{Z}}\frac{E_{\mathrm{gap}}}{T}\frac{\cot(\pi\alpha)}{4\pi^3}\left(\frac{e^{2\pi^2\frac{T}{E_{\mathrm{gap}}}\left[\frac{1}{c^2}-4\left(n+\frac{1}{2}-\alpha-\frac{1}{2c}\right)^2\right]}}{(1-2\alpha+2n)^2}-\frac{e^{2\pi^2\frac{T}{E_{\mathrm{gap}}}\left[\frac{1}{c^2}-4\left(n+\frac{1}{2}+\alpha-\frac{1}{2c}\right)^2\right]}}{(1+2\alpha+2n)^2}\right)$$

$$Z_{(c)}^{\mathcal{N}=4\ \mathrm{Schw.}}\left(\beta,\alpha=\frac{1}{2}\right)=\frac{1}{c}$$

$$\mathrm{Vol}\left[\frac{\mathrm{Diff}\left(S^{1|4}\right)}{H_{\mathrm{Orb}}}\right]=\lim_{T\to 0}\left[Z_{(c)}^{\mathcal{N}=4\ \mathrm{Schw.}}\left(\beta,\alpha=\frac{1}{2}\right)\right]=\frac{1}{c}$$

$$I[\phi]=\frac{\pi q_\Lambda\phi^\Lambda}{c}-\frac{4\pi}{c}\mathrm{Im}F\left(\frac{\phi^\Lambda+\mathrm{i}p^\Lambda}{2}\right)$$
$$=\frac{\pi C_{IJK}p^Ip^Jp^K}{c\phi^0}-\frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}+\frac{\pi q_0\phi^0}{c}+\frac{\pi q_I\phi^I}{c}$$

$$Z_{\text{non-zero},c}^{Q_{\mathrm{eq}}\mathcal{V}}=\begin{cases}\mathrm{e}^{5\mathcal{K}}\mathrm{e}^{-\frac{17}{2}\mathcal{K}}, & c=1,\\ \mathrm{e}^{5\mathcal{K}}\mathrm{e}^{-\frac{15}{2}\mathcal{K}}, & c>1,\end{cases},Z_{\text{zero-mode},c}^{Q_{\mathrm{eq}}\mathcal{V}}=\begin{cases}\mathrm{e}^{\frac{15}{2}\mathcal{K}}, & c=1,\\ \frac{1}{c}\mathrm{e}^{\frac{13}{2}\mathcal{K}}, & c>1.\end{cases}$$

$$Z_{1-\mathrm{loop},c}^{Q_{\mathrm{eq}}\mathcal{V}}(\phi)=Z_{\text{non-zero},c}^{Q_{\mathrm{eq}}\mathcal{V}}(\phi)Z_{\mathrm{zero-modes},c}^{Q_{\mathrm{eq}}\mathcal{V}}(\phi)=\frac{e^{4\mathcal{K}}}{c}$$
$$=\frac{1}{c}\left(\frac{\phi^0}{-2C_{IJK}p^Ip^Jp^K}\right)^4$$

$$W^{(c)}(q,p)=\sqrt{c}K_c(\Delta)\int\mathrm{d}^{16}\phi\sqrt{\frac{-2C_{IJK}p^Ip^Jp^K\det\left(3C_{IJ}\right)}{c^{16}(\phi^0)^{18}}}\frac{1}{c}\left(\frac{\phi^0}{-2C_{IJK}p^Ip^Jp^K}\right)^4$$
$$\times\exp\left(-\frac{\pi C_{IJK}p^Ip^Jp^K}{c\phi^0}+\frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}-\frac{\pi q_0\phi^0}{c}-\frac{\pi q_I\phi^I}{c}\right)$$

$$\int d^{15}\phi\exp\left(\frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}-\frac{\pi q_I\phi^I}{c}\right)=\sqrt{-\frac{c^{15}(\phi^0)^{15}}{\det\left(3C_{IJ}\right)}}\exp\left(-\frac{\pi}{12c}\phi^0q_IC^{IJ}q_J\right)$$

$$W^{(c)}(q,p)=K_c(\Delta)\int d\phi^0\frac{1}{c}\frac{(\phi^0)^{5/2}}{\mathrm{i}\left(-2C_{IJK}p^Ip^Jp^K\right)^{7/2}}\exp\left(-\frac{\pi C_{IJK}p^Ip^Jp^K}{c\phi^0}-\frac{\pi\phi^0}{c}\left(q^0+\frac{1}{12}q_IC^{IJ}q_J\right)\right)$$

$$\Delta = 4C_{IJK}p^I p^J p^K \left(q^0 + \frac{1}{12} q_I C^{IJ} q_J\right)$$

$$\sigma = \frac{-\pi C_{IJK} p^I p^J p^K}{c\phi^0}$$

$$W^{(c)}(q,p) = K_c(\Delta)\frac{2\pi}{c^{9/2}}\left(\frac{\pi}{2}\right)^{7/2}\frac{1}{2\pi \mathrm{i}}\int \frac{d\sigma}{\sigma^{9/2}} \exp\left(\sigma + \frac{\pi^2\Delta}{4c^2\sigma}\right)$$

$$W(q,p) = \sum_c W^{(c)}(q,p) = \sum_{c=1}^{\infty} \frac{2\pi}{c^{9/2}}\left(\frac{\pi}{2}\right)^{7/2} K_c(\Delta)\tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right) = W_{\text{micro}}\,(\Delta)$$

$$Z^{T=0}_{\frac{1}{8}-\text{BPSBH},\Sigma}\left(\Delta, \Omega_E = \frac{2\pi}{\beta}\right) = Z^{T=0}_{\frac{1}{8}-\text{BPSBH},\Sigma}(\Delta, J=0),$$

$$\lim_{T\to 0} Z^{\mathcal{N}=4\text{ Schw.}}_{\text{Disk}}(\beta,\alpha) = 1, \forall\alpha$$

$$\lim_{T\to 0} Z^{\mathcal{N}=4\text{ Schw.}}_{\text{Orb}}(\beta,\alpha) = \frac{1}{c}, \forall\alpha$$

$$\lim_{T\to 0} Z^{\mathcal{N}=4\text{ Schw.}}_{\text{Disk}}(\beta,J) = \delta_{J,0}, \lim_{T\to 0} Z^{\mathcal{N}=4\text{ Schw.}}_{\text{Orb}}(\beta,J) = \frac{\delta_{J,0}}{c}$$

$$Z^{T=0}_{\frac{1}{8}-\text{BPSBH,disk/orb}}\left(\Delta, \Omega_E = \frac{2\pi}{\beta}\right) = Z^{T=0}_{\frac{1}{8}-\text{BPSBH,disk/orb}}(\Delta, J=0).$$

$$(\tau,\psi) \sim \left(\tau + \frac{2\pi}{c}, \psi - 4\pi\varphi\right) \sim (\tau, \psi + 2\pi)$$

$$g(\tau+\beta) = \mathrm{e}^{i2\pi\alpha\sigma_3} g(\tau) \mathrm{e}^{-i2\pi\varphi\sigma_3}$$

$$Z^{\mathcal{N}=4\text{ Schw.}}_{(c,\varphi)}(\beta,\alpha) = \sum_{n\in\mathbb{Z}} \frac{T}{E_{\text{gap}}} \frac{\cot(\pi\alpha)}{2\pi}\left(\mathrm{e}^{2\pi^2\frac{T}{E_{\text{gap}}}\left(\frac{1}{c^2}-4(n-\alpha+\varphi)^2\right)} - e^{2\pi^2\frac{T}{E_{\text{gap}}}\left(\frac{1}{c^2}-4(n+\alpha+\varphi)^2\right)}\right)$$

$$\lim_{T\to 0} Z^{T=0}_{\frac{1}{8}-\text{BPSBH},(b,\varphi)}(\Delta, J) = 0$$

$$f(\tau+2\pi) = \mathrm{e}^{\frac{b}{2}} f(\tau), g(\tau+2\pi) = \mathrm{e}^{2\pi i\alpha\sigma_3} g(\tau) \mathrm{e}^{2\pi i\varphi\sigma_3}, \eta(\tau+2\pi) = -e^{i2\pi i\alpha\sigma_3}\eta(\tau)$$

$$\lim_{T\to 0} Z^{\mathcal{N}=4\text{ Schw.}}_{\text{Trumpet }(b,\varphi)}(\beta,\alpha) = 0, \forall\alpha$$

$$E_{\frac{1}{8}\text{-BPS BH}} \sim \Delta^{1/4}$$

$$\rho_{\text{trumpet},(b,\varphi)} \sim \Theta\left(E - E_{\frac{1}{8}\text{-BPS BH}} - E_{\text{gap}}\right).$$

$$\lim_{T\to 0} \int d\mu[b] d\mu[\varphi] Z^{\mathcal{N}=4\text{ Schw.}}_{\text{Trumpet }(b,\varphi)}(\beta,J) Z^{\text{matter}}_{(b,\varphi)} \text{Vol}_{g,m} = 0$$

$$W(p,q) = Z^{T=0}_{\frac{1}{8}\text{-BPSBH}}(\Delta, \Omega_E \beta = 2\pi) = Z^{T=0}_{\frac{1}{8}\text{-BPSBH}}(\Delta, \Omega_E = 0) = d_{\frac{1}{8}\text{-BPSBH}}.$$

$$E = E_{\frac{1}{8}\text{-BPS BH}} + E_{\text{gap}}$$

$$\rho^{\text{cont.}}_{\text{Disk},(c,\varphi)} \sim \Theta\left(E - E_{\frac{1}{8}\text{-BPS BH}} - E_{\text{gap}}\right), \rho^{\text{cont.}}_{\text{Orbifold},c} \sim \Theta\left(E - E_{\frac{1}{8}\text{-BPS BH}} - E_{\text{gap}}\right)$$

$$E_{\text{gap}} \sim \frac{1}{\Delta^{3/2}}, \Delta \to \infty$$

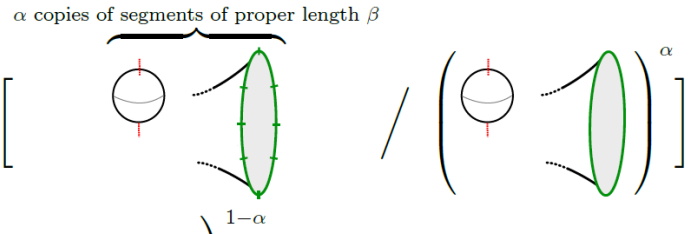
$$\rho_{\text{orbifold, }(c,\varphi)} \sim \Theta\left(E - E_{\frac{1}{8}\text{-BPS BH}} - E_{\text{gap}}\right), \rho_{\text{trumpet, }(b,\varphi)} \sim \Theta\left(E - E_{\frac{1}{8}\text{-BPS BH}} - E_{\text{gap}}\right),$$

$$E_{\frac{1}{8}\text{-BPS BH}} + E_{\text{gap}}\,(1 + O(1/\Delta))$$

$$\int_{E_{\frac{1}{8}\text{-BPSBH}}}^{E_{\frac{1}{8}\text{-BPSBH}} + \#E_{\text{gap}}} dE\rho(E) = d_{\frac{1}{8}\text{-BPSBH}}$$

$$\rho_{\text{TFD}} = |\text{TFD}(T=0)\rangle\langle\text{TFD}(T=0)|, \rho_L = \text{Tr}_R(\rho_{\text{TFD}})$$

$$S_\alpha = \frac{1}{1-\alpha}\frac{\text{Tr}\,(\rho_L)^\alpha}{(\text{Tr}\,\rho_L)^\alpha} = \frac{1}{1-\alpha}$$



$$= \frac{\left(\quad + \quad\right)}{1-\alpha} = \frac{(d_{\frac{1}{8}\text{-BPS BH}})^{1-\alpha}}{1-\alpha}.$$

$$\vartheta_1(\tau, z) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} (-1)^n q^{n^2/2} \zeta^n$$

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \mathrm{e}^{\frac{2\pi i m c z^2}{c\tau + d}} \varphi(\tau, z)\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$$

$$\varphi(\tau, z + \lambda\tau + \mu) = \mathrm{e}^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(\tau, z)\ \forall\, \lambda, \mu \in \mathbb{Z}$$

$$\varphi(\tau, z) = \sum_{n,r \in \mathbb{Z}} c(n, r) q^n \zeta^r$$

$$c(n,r) = C_r(4nm - r^2) \therefore C_r(\Delta) \parallel \ r \bmod 2m.$$

$$\varphi(\tau, z) = \sum_{n,\ell \in \mathbb{Z}} c(n,\ell) q^n \zeta^\ell$$

$$c(n,\ell) = C_\nu(\Delta), \Delta = 4mn - \ell^2, \nu = \ell \bmod 2m$$

$$C_\nu(\Delta) = \sum_{c=1}^{\infty} \left(\frac{c}{2\pi}\right)^{k-\frac{5}{2}} \sum_{\widetilde{\Delta}<0} C_\mu(\widetilde{\Delta}) \mathrm{Kl}(\Delta, \nu, \widetilde{\Delta}, \tilde{\nu}; c) \left|\frac{\widetilde{\Delta}}{4k}\right|^{\frac{3}{2}-k} I_{\frac{3}{2}-k}\left(\frac{\pi}{c}\sqrt{|\widetilde{\Delta}|\Delta}\right)$$

$$\tilde{I}_\rho(z) = \frac{1}{2\pi \mathrm{i}} \int_{\epsilon - \mathrm{i}\infty}^{\epsilon + \mathrm{i}\infty} \frac{d\sigma}{\sigma^{\rho+1}} \exp\left(\sigma + \frac{z^2}{4\sigma}\right)$$

$$\tilde{I}_\rho(z) = \left(\frac{z}{2}\right)^{-\rho} I_\rho(z)$$

$$I_\rho(z) \sim \frac{\mathrm{e}^z}{\sqrt{2\pi z}}\left(1 - \frac{(\mu-1)}{8z} + \frac{(\mu-1)(\mu-3^2)}{2!\,(8z)^3} - \frac{(\mu-1)(\mu-3^2)(\mu-5^2)}{3!\,(8z)^5} + \cdots\right).$$

$$\mathrm{Kl}(\Delta, \nu; \widetilde{\Delta}, \tilde{\nu}; c) := \mathrm{e}^{\frac{\pi \mathrm{i}}{2}\left(\frac{1}{2}-k\right)} \sum_{\substack{-c \le d < 0 \\ (d,c)=1}} \mathrm{e}^{2\pi \mathrm{i} \frac{d}{c}(\Delta/4m)} M\left(\gamma_{c,d}\right)^{-1}_{\nu\tilde{\nu}} \mathrm{e}^{2\pi \mathrm{i} \frac{a}{c}(\widetilde{\Delta}/4m)}$$

$$\vartheta_{m,\mu}(\tau, z) = \sum_{\ell = \mu \bmod 2m} q^{\ell^2/4m} \zeta^\ell$$

$$\vartheta_{m,\mu}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^{\frac{1}{2}} \mathrm{e}^{2\pi i m \frac{cz^2}{c\tau + d}} M(\gamma)^{-1}_{\nu\mu} \vartheta_{\mu,\ell}(\tau, z)$$

$$M^{-1}(\gamma)_{\nu\mu} = \frac{\mathrm{e}^{\frac{\pi \mathrm{i}}{4}} \mathrm{sign}(c)}{\sqrt{2r|c|}} \exp\left(-\frac{\mathrm{i}\pi}{6}\Phi(\gamma)\right) \sum_{\varepsilon = \pm} \sum_{n=0}^{c-1} \varepsilon \exp\left(\frac{\mathrm{i}\pi}{2rc}\left(d(\nu+1)^2 - 2(\nu+1)(2rn + \varepsilon(\mu+1)) + a(2rn + \varepsilon(\mu+1))^2\right)\right)$$

$$\Phi(\gamma) = \frac{a+d}{c} - 12 \mathrm{sign}(c) s(a, |c|)$$

$$s(a,c) = \frac{1}{4c} \sum_{j=1}^{c-1} \cot\left(\frac{\pi j}{c}\right) \cot\left(\frac{\pi j a}{c}\right).$$

$$\delta_\xi \phi^i = R(\phi)^i_\alpha \xi^\alpha,$$

$$\delta_{\xi_1} \delta_{\xi_2} - \delta_{\xi_2} \delta_{\xi_1} = \delta_{\xi_3}$$

$$\xi_3^\alpha = f(\phi)_{\beta\gamma}{}^\alpha \xi_1^\beta \xi_2^\gamma$$

$$\delta_{\text{brst}}\phi^i = R(\phi)^i_\alpha \Lambda c^\alpha, \delta_{\text{brst}} c^\alpha = \frac{1}{2} f(\phi)_{\beta\gamma}{}^\alpha c^\beta \Lambda c^\alpha$$

$$\delta^2_{\text{brst}} = 0$$

$$\delta_{\text{brst}}\,\tilde{\phi}^i = \delta_{\text{brst}}\,\phi^i - \delta_{\text{brst}}\,\overline{\phi^i}, \delta_{\text{brst}}\,\tilde{c}^\alpha = \delta_{\text{brst}}\,c^\alpha - \delta_{\text{brst}}\,\bar{c}^\alpha$$

$$\delta_{\text{brst}}\overline{\phi^i} = \Lambda \bar{c}^\alpha R(\bar{\phi})^i_\alpha = 0$$

$$\delta_{\text{brst}} \Rightarrow \delta_{\text{eq}}$$

$$\delta_{\text{eq}}\bar{\phi}^i = 0, \quad \delta_{\text{eq}}\tilde{\phi}^i = R(\phi)^i_\alpha \Lambda(\bar{c} + \tilde{c})^\alpha$$
$$\delta_{\text{eq}}\bar{c}^\alpha = 0, \quad \delta_{\text{eq}}\tilde{c}^\alpha = \frac{1}{2} f(\phi)^\alpha_{\beta\gamma}(\bar{c} + \tilde{c})^\beta \Lambda(\bar{c} + \tilde{c})^\alpha - \frac{1}{2} f(\bar{\phi})^\alpha_{\beta\gamma} \bar{c}^\beta \Lambda \bar{c}^\alpha$$

$$\delta^2_{\text{eq}} = \delta_{\bar{\xi}} \neq 0, \left[\delta_{\text{eq}}, \delta_{\bar{\xi}}\right] = 0$$

$$\delta_{\bar{\xi}}\bar{\phi} = 0, \delta_{\bar{\xi}}\tilde{\phi}^i = \bar{\xi}^\alpha R\left(\phi^i\right)^i_\alpha, \delta_{\bar{\xi}}\bar{c}^\alpha = 0, \delta_{\bar{\xi}}\tilde{c}^\alpha = f(\phi)^\alpha_{\beta\gamma}(\bar{c} + \tilde{c})^\beta \bar{\xi}^\gamma$$

$$\tilde{\xi} = \Lambda_{[2} f(\bar{\phi})_{\beta\gamma}{}^\alpha \bar{c}^\beta \Lambda_{1]} \bar{c}^\gamma$$

$$\delta_{\text{eq}}\tilde{b}_\alpha = \Lambda B_\alpha, \delta_{\text{eq}} B^\alpha = \frac{1}{2} f(\bar{\phi})_{\delta\epsilon}{}^\beta \bar{c}^\delta \Lambda \bar{c}^\epsilon f(\bar{\phi})^\gamma_{\alpha\beta}$$

$$\delta^2_{\text{eq}}\, b_\alpha = \delta_{\bar{\xi}} b_\alpha = \bar{\xi} f(\phi)_{\beta\alpha}{}^\gamma b_\gamma, \delta^2_{\text{eq}}\, B_\alpha = \delta_{\bar{\xi}} B_\alpha = \bar{\xi} f(\phi)_{\beta\alpha}{}^\gamma B_\gamma$$

$$\Sigma^{45} \to X^1, \Sigma^{67} \to X^2, \Sigma^{68} \to X^3, \Sigma^{69} \to X^4, \Sigma^{78} \to X^5, \Sigma^{79} \to X^6, \Sigma^{89} \to X^7$$
$$\Sigma^{46} \to X^8, \Sigma^{47} \to X^9, \Sigma^{48} \to X^{10}, \Sigma^{49} \to X^{11}, \Sigma^{56} \to X^{12}$$
$$\Sigma^{57} \to X^{13}, \Sigma^{58} \to X^{14}, \Sigma^{59} \to X^{15}$$

$$F_{\text{trunc.}}\,(X) = -\frac{1}{2}\frac{X^1 C_{ab} X^a X^b}{X^0}, a = 2, \ldots, 7$$

$$\int \prod_{I=8}^{15} \mathrm{d}\phi^I \exp\left(\sum_{I,J=8}^{15} \frac{3\pi C_{IJK}\phi^I \phi^J p^K}{c\phi^0}\right)\Bigg|_{p^8=\cdots=p^{15}=0} = \frac{4c^4(\phi^0)^4}{(P^2)^2}, P^2 = p^a C_{ab} p^b$$

$$[\mathrm{d}^8\phi]_c \equiv\ \mathrm{d}^8\phi \sqrt{2\det{}'\left(\frac{1}{2c}\mathrm{Im}F_{\Lambda\Sigma}\right)}$$

$$W^{(c)}(q,p) = \sqrt{c} K_c(\Delta) \int\ [\mathrm{d}^8\phi]_c c^{-1} \mathrm{e}^{4\mathcal{K}(\phi)} e^{\frac{4\pi}{c}\mathrm{Im}F_{\text{trunc}}\left(\frac{\phi^\Lambda + ip^\Lambda}{2}\right) - \frac{\pi}{c} q_\Lambda \phi^\Lambda}$$

$$A^\pm = \left(\omega^a \pm \frac{e^a}{l}\right) T_a$$

$$\omega_a \equiv \frac{1}{2}\epsilon_{abc}\omega^{bc}$$

$$I = I_{CS}(A^+) - I_{CS}(A^-)$$

$$I_{CS}(A^\pm) = \frac{\mathrm{k}}{4\pi}\int \mathrm{Tr}\left[A^\pm \wedge dA^\pm + \frac{2}{3}(A^\pm)^3\right] + \mathcal{B}_\infty(A^\pm)$$

$$\mathrm{k} = \frac{l}{4G}$$

$$\mathrm{Tr}(\mathrm{L}_0\ \mathrm{L}_0) = \frac{1}{2}, \mathrm{Tr}(\ \mathrm{L}_1\ \mathrm{L}_{-1}) = -1$$

$$g_{\mu\nu} = \frac{l^2}{2}\mathrm{Tr}\big((A^+ - A^-)_\mu (A^+ - A^-)_\nu\big)$$

$$A^\pm = b_\pm^{-1}(d + a^\pm)b_\pm$$

$$a^\pm(t,\theta) = (\xi_\pm(t,\theta)dt \pm p_\pm(t,\theta)d\theta)\mathrm{L}_0$$

$$dA^\pm + A^{\pm 2} = 0$$

$$\dot{p}_\pm(t,\theta) = \pm\xi'_\pm(t,\theta)$$

$$\delta\mathcal{B}_\infty^\pm = -\frac{\mathrm{k}}{2\pi}\int dtd\theta\mathrm{Tr}\big(A_t^\pm \delta A_\theta^\pm\big) = \mp\frac{\mathrm{k}}{4\pi}\int dtd\theta\xi_\pm\delta p_\pm$$

$$\xi_\pm = -\frac{4\pi}{\mathrm{k}}\frac{\delta H^\pm}{\delta p_\pm}$$

$$H^\pm = \int d\theta\mathcal{H}^\pm\big(p_\pm, p'_\pm, \cdots\big)$$

$$\mathcal{B}_\infty^\pm = \pm\int dtd\theta\mathcal{H}^\pm = \pm\int dtH^\pm$$

$$\dot{p}_\pm(t,\theta) = \mp\frac{4\pi}{\mathrm{k}}\frac{\partial}{\partial\theta}\left(\frac{\delta H^\pm}{\delta p_\pm}\right)$$

$$\dot{p}_\pm(t,\theta) = \{p_\pm(t,\theta), H^\pm\}_{PB}$$

$$\{p_\pm(t,\theta), p_\pm(t,\theta')\}_{PB} = \mp\frac{4\pi}{\mathrm{k}}\frac{\partial}{\partial\theta}\delta(\theta - \theta')$$

$$H_{CFT} = \int d\theta\sigma(t,\theta)\left[\frac{1}{2}\left(\frac{\partial\Pi(t,\theta)}{\partial\theta}\right)^2 + \frac{\pi^2}{6}\sigma^2(t,\theta) + W(\theta)\right]$$

$$\partial_t\sigma(t,\theta) + \partial_\theta(\sigma(t,\theta)v(t,\theta)) = 0$$

$$\partial_t v(t,\theta) + v(t,\theta)\partial_\theta v(t,\theta) = -\partial_\theta\left(\frac{\pi^2}{2}\sigma^2(t,\theta) + W(\theta)\right)$$

$$v(t,\theta) = \partial_\theta\Pi(t,\theta)$$

$$p_\pm(t,\theta) = v(t,\theta) \pm \pi\sigma(t,\theta)$$

$$\partial_t p_\pm(t,\theta) + p_\pm(t,\theta)\partial_\theta p_\pm(t,\theta) + W'(\theta) = 0$$

$$H_{CFT} = H^+_{CFT} + H^-_{CFT}, \text{ where } H^\pm_{CFT} = \pm\frac{1}{2\pi}\int d\theta\left(\frac{p^3_\pm(t,\theta)}{6} + W(\theta)p_\pm(t,\theta)\right)$$

$$\delta a^\pm = d\lambda^\pm + [a^\pm, \lambda^\pm]$$

$$\delta p_\pm = \eta'_\pm, \text{ and } \delta\xi_\pm = \dot{\eta}_\pm$$

$$Q^\pm_n = \pm\frac{\mathrm{k}}{4\pi}\int d\theta \sum_{k=0}^{n} \frac{{}^nC_k}{2^k(2k+1)} p^{2k+1}_\pm(t,\theta) W^{n-k}(\theta)$$

$$\left\{Q^\pm_m, Q^\pm_n\right\}_{PB} = 0, \forall m,n \geq 1.$$

$$H^\pm_{(1)} = \frac{\mathcal{C}_\pm \mathrm{k}}{2} H^\pm_{CFT} = \pm\frac{\mathcal{C}_\pm \mathrm{k}}{4\pi}\int d\theta\left(\frac{p^3_\pm(t,\theta)}{6} + W(\theta)p_\pm(t,\theta)\right)$$

$$\xi^\pm = \mp\mathcal{C}_\pm\left(\frac{p^2_\pm(t,\theta)}{2} + W(\theta)\right)$$

$$p_\pm(\theta)\partial_\theta p_\pm(\theta) + W'(\theta) = 0$$

$$\frac{\partial}{\partial\theta}(v(\theta)\sigma(\theta)) = 0, \text{ and } \frac{\partial}{\partial\theta}\left(\frac{\sigma^2(\theta) + v^2(\theta)}{2}\right) + W'(\theta) = 0.$$

$$\mathrm{Hol}_\tau(A^\pm) \sim \exp(\beta_\pm\xi^\pm) \sim \mathbb{I} \Rightarrow \beta_\pm\xi^\pm = \mp 2\pi$$

$$\sigma = \frac{(\beta_- + \beta_+)l}{\beta_-\beta_+}, v = \frac{\pi l(\beta_- - \beta_+)}{\beta_-\beta_+}$$

$$b_\pm(r) = \exp\left(\pm\frac{r}{2l}(L_+ - L_-)\right)$$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\phi + \frac{A_\phi}{r^2}dt\right)^2$$

$$f(r) = \frac{r^2}{l^2} - \frac{2\pi^2 l^2(\beta^2_- + \beta^2_+)}{\beta^2_-\beta^2_+} + \frac{A^2_\phi}{r^2}, \text{ and } A_\phi = \pi^2 l^3\left(\frac{1}{\beta^2_+} - \frac{1}{\beta^2_-}\right)$$

$$p_\pm = \pm\frac{2\pi l}{\beta_\pm}$$

$$\mathcal{C}_\pm = \frac{\beta_\pm}{\pi l^2}$$

$$r_\pm = \pi l^2\left(\frac{1}{\beta_+} \pm \frac{1}{\beta_-}\right)$$

$$M = \frac{\pi^2\sigma^2 + v^2}{8G}, \text{ and } J = \frac{\pi l\sigma v}{4G}$$

$$|v| \leq \pi\sigma.$$

$$v\sigma = \frac{4GJ}{\pi l} \text{ and } \sigma^2 = \frac{4G\left(\sqrt{l^2M^2 - J^2} + lM\right)}{\pi^2 l}.$$

$$r_+ = l\pi\sigma \text{ and } r_- = |lv|.$$

$$\beta = \frac{\beta_+ + \beta_-}{2},$$

$$T = \frac{1}{\beta} = \frac{\pi^2\sigma^2 - v^2}{2\pi^2 l\sigma}$$

$$\frac{\beta_- - \beta_+}{2} = \frac{2\pi l v}{\pi^2\sigma^2 - v^2} = \frac{v}{\pi\sigma}\beta$$

$$H^{\pm}_{(2)} = \pm\frac{\mathcal{C}_\pm k}{4\pi}\int d\theta\left(\frac{p^2_\pm(t,\theta)}{2} + W(\theta)p_\pm(t,\theta)\right)$$

$$S = \frac{2\pi r_+}{4G} = \frac{\pi^2 l^2}{2G}\left(\frac{1}{\beta_+} + \frac{1}{\beta_-}\right)$$

$$\mathcal{H} = \mathcal{H}^+ \otimes \mathcal{H}^-.$$

$$p_\pm(t,\theta) = 2\pi\sqrt{\frac{2}{\mathrm{k}}}\tilde{p}_\pm(t,\theta)$$

$$\{\tilde{p}_\pm(t,\theta), \tilde{p}_\pm(t,\theta')\}_{PB} = \mp\frac{1}{2\pi}\frac{\partial}{\partial\theta}\delta(\theta - \theta')$$

$$c = \frac{3l}{2G},$$

$$[\tilde{p}_\pm(t,\theta), \tilde{p}_\pm(t,\theta')] = \mp\frac{i}{2\pi c}\frac{\partial}{\partial\theta}\delta(\theta - \theta')$$

$$\tilde{p}_\pm(t,\theta) = \frac{1}{2\pi\sqrt{c}}\sum_n \alpha^\pm_{\pm n}e^{in\theta}$$

$$\left[\alpha^\pm_m, \alpha^\pm_n\right] = m\delta_{m+n}$$

$$\psi(t,\theta) = \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\psi_{m-1/2}e^{im\theta}, \psi^\dagger(t,\theta) = \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\psi^\dagger_{m-1/2}e^{-im\theta}$$

$$\{\psi_q, \psi_r\} = 0, \{\psi^\dagger_q, \psi^\dagger_r\} = 0, \{\psi_q, \psi^\dagger_r\} = \delta_{q,r}$$

$$\{\psi(t,\theta), \psi(t,\phi)\} = 0, \{\psi^\dagger(t,\theta), \psi^\dagger(t,\phi)\} = 0$$
$$\{\psi(t,\theta), \psi^\dagger(t,\phi)\} = \delta(\theta - \phi)$$

$$:\psi_q^\dagger\psi_r:=\begin{cases}\psi_q^\dagger\psi_r & r>0\\ -\psi_r\psi_q^\dagger & r<0\end{cases}$$

$$:\psi_q^\dagger\psi_r:=\psi_q^\dagger\psi_r-\langle\psi_q^\dagger\psi_r\rangle$$

$$B_n^K=\sum_{m\in\mathbb{Z}} m^K:\psi_{m-n-1/2}^\dagger\psi_{m-1/2}:$$

$$\left[B_m^{K_1},B_n^{K_2}\right]=\sum_{q\in\mathbb{Z}}((q-n)^{K_1}q^{K_2}-(q-m)^{K_2}q^{K_1}):\psi_{q-m-n-1/2}^\dagger\psi_{q-1/2}:$$
$$+\delta_{m+n}\sum_{l=0}^{m-1} l^{K_1}(l-m)^{K_2}$$

$$:\psi^\dagger(t,\theta)\psi(t,\theta):=\frac{1}{2\pi}\sum_{m,n\in\mathbb{Z}}:\psi_{m-n-1/2}^\dagger\psi_{m-1/2}:e^{in\theta}=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}B_n^0e^{in\theta}.$$

$$[B_m^0,B_n^0]=m\delta_{m+n}.$$

$$:\psi^\dagger(t,\theta)\psi(t,\theta):\equiv\sqrt{c}\tilde{p}_\pm(t,\theta).$$

$$\psi(\theta)=\frac{1}{\sqrt{2\pi}}:e^{-2\pi i\varphi(\theta)}$$

$$\tilde{p}(\theta)=\frac{\varphi'(\theta)}{\sqrt{c}}.$$

$$2\pi c:\frac{\tilde{p}^2(\theta)}{2}:=-\frac{i}{2}:\left(\psi^\dagger\partial_\theta\psi-\partial_\theta\psi^\dagger\psi\right):$$
$$=\frac{1}{2\pi}\sum_p\left(B_p^1-\frac{p}{2}B_p^0\right)e^{ip\theta}$$

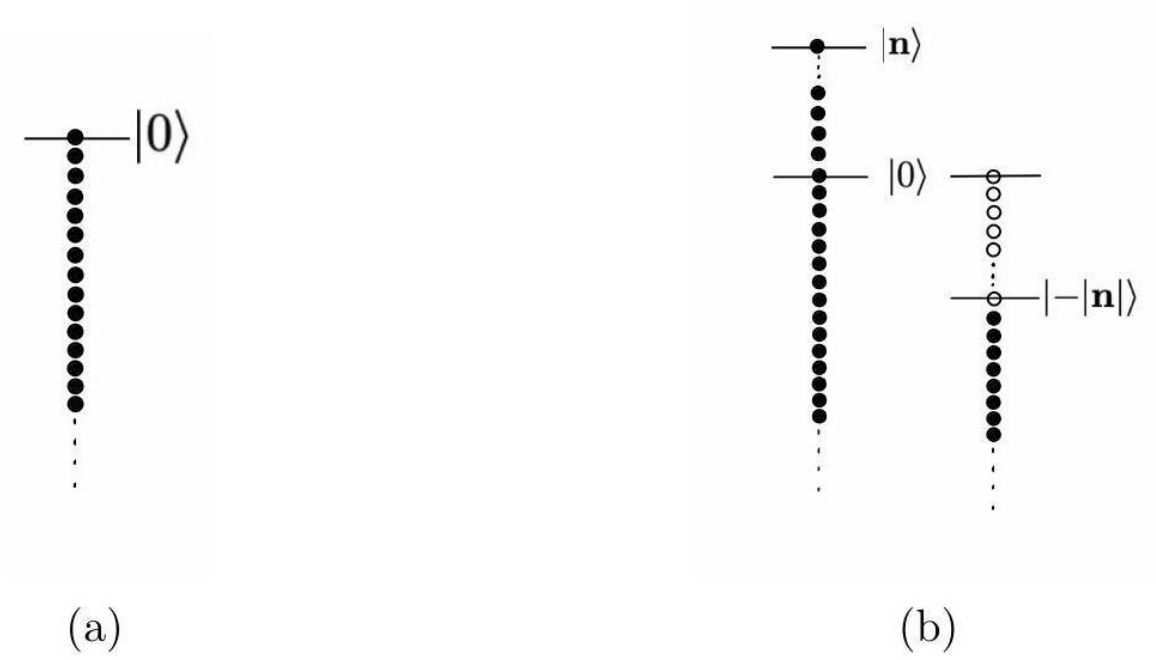
$$(2\pi)^2c^{3/2}:\frac{\tilde{p}^3(\theta)}{3}:=-\frac{1}{6}:\partial_\theta^2\psi^\dagger\psi+\psi^\dagger\partial_\theta^2\psi-4\partial_\theta\psi^\dagger\partial_\theta\psi:$$
$$=\frac{1}{2\pi}\sum_p\left(B_p^2-pB_p^1+\frac{p^2}{6}B_p^0\right)e^{ip\theta}$$

$$\hat{p}_\pm^2(t,\theta):=\frac{24}{c^2}\sum_p\left(B_p^1-\frac{p}{2}B_p^0\right)e^{ip\theta}$$

$$H^\pm=\pm\mathcal{C}_\pm\frac{\sqrt{3}}{c^2}B_0^2$$

$$Q_n^\pm=\pm\frac{1}{2\sqrt{3}}\left(\frac{6}{c^2}\right)^nB_0^{2n}$$

$$\psi_{k+\frac{1}{2}}|0\rangle = 0, \forall k \geq 0$$
$$\psi^{\dagger}_{k-\frac{1}{2}}|0\rangle = 0, \forall k \leq 0$$



(a) (b)

$$\prod_{i=1}^{n_p} \prod_{j=1}^{n_h} \psi^{\dagger}_{n_i+\frac{1}{2}} \psi_{-m_j-\frac{1}{2}}|0\rangle.$$

$$|\mathbf{n}\rangle = \begin{cases} \psi^{\dagger}_{\mathbf{n}-\frac{1}{2}} \cdots \psi^{\dagger}_{\frac{1}{2}}|0\rangle & \mathbf{n} > 0, \\ \psi_{\mathbf{n}+\frac{1}{2}} \cdots \psi_{-\frac{1}{2}}|0\rangle & \mathbf{n} < 0. \end{cases}$$

$$\left[B^K_m, \psi_{k-\frac{1}{2}}\right] = -(k+m)^K \psi_{k+m-\frac{1}{2}}, \left[B^K_m, \psi^{\dagger}_{k-\frac{1}{2}}\right] = (k-m)^K \psi^{\dagger}_{k-m-\frac{1}{2}},$$

$$B^0_0|\mathbf{n}\rangle = \mathbf{n}|\mathbf{n}\rangle, B^0_m|\mathbf{n}\rangle = 0, \forall m > 0.$$

$$|\vec{k}; \mathbf{n}\rangle = \prod_{m>0} (B^0_{-m})^{k_m}|\mathbf{n}\rangle$$

$$\langle \vec{k}; \mathbf{n} \mid \vec{k}'; \mathbf{n}\rangle = z_{\vec{k}} \delta_{\vec{k},\vec{k'}}, z_{\vec{k}} = \prod_m k_m! \, m^{k_m}$$

$$\mathcal{H} = \bigoplus_{\mathbf{n}\in\mathbb{Z}} \mathcal{H}_{\mathbf{n}}.$$

$$|\vec{m}; \vec{n}; \mathbf{n}\rangle = \prod_i \psi^{\dagger}_{\mathbf{n}+n_i+\frac{1}{2}} \psi_{\mathbf{n}-m_i-\frac{1}{2}}|\mathbf{n}\rangle, n_i, m_i \geq 0$$

$$\langle R'; \mathbf{n} \mid R; \mathbf{n}\rangle = \delta_{RR'}$$

$$|\vec{m};\vec{n};\mathbf{n}\rangle = |R;\mathbf{n}\rangle = \prod_i \psi^{\dagger}_{\mathbf{n}+n_i+\frac{1}{2}}\psi_{\mathbf{n}-m_i-\frac{1}{2}}|\mathbf{n}\rangle, n_i, m_i \geq 0$$

$$\mathcal{E}^K = \mathcal{E}^K_R(n_i, m_i) + g_K(\mathbf{n}),$$

$$\mathcal{E}^K_R(n_i, m_i) = \sum_i ((\mathbf{n}+n_i+1)^K - (\mathbf{n}-m_i)^K),$$

$$\mathcal{E}^0_R = 0, g_0(\mathbf{n}) = \mathbf{n}$$

$$\mathcal{E}^1_R = \sum_i (n_i + m_i + 1), g_1(\mathbf{n}) = \frac{\mathbf{n}(\mathbf{n}+1)}{2}$$

$$\mathcal{E}^2_R = \sum_i \left(\left(n_i+\frac{1}{2}\right)^2 - \left(m_i+\frac{1}{2}\right)^2\right) + (2\mathbf{n}+1)\sum_i (n_i + m_i + 1),$$

$$g_2(\mathbf{n}) = \frac{\mathbf{n}(\mathbf{n}+1)(2\mathbf{n}+1)}{3}$$

$$\sum_i (n_i + m_i + 1) = \sum_{i=1}^{v_1} l_i = |R| \equiv \Re$$

$$\mathcal{E}^1_R = |R|.$$

$$\sum_i \left(\left(n_i+\frac{1}{2}\right)^2 - \left(m_i+\frac{1}{2}\right)^2\right) = \sum_{i=1}^{v_1} l_i(l_i - 2i) + |R| \equiv \kappa_R$$

$$\mathcal{E}^2_R = \kappa_R + (2\mathbf{n}+1)|R|.$$

$$\mathcal{E}^2_R = -\kappa_R - (2|\mathbf{n}|-1)|R|, \text{ for } \mathbf{n} < 0.$$

$$\langle\Psi|\colon \hat{v}(t,\theta)\hat{\sigma}(t,\theta)\colon|\Psi\rangle = \frac{4GJ}{\pi l} \text{ and } \langle\Psi|\colon \pi^2\hat{\sigma}^2(t,\theta) + \hat{v}^2(t,\theta)\colon|\Psi\rangle = 8GM$$

$$\langle\Psi^{\pm}|\colon \hat{p}^2_{\pm}(t,\theta)\colon|\Psi^{\pm}\rangle = \frac{12}{c}(Ml \pm J) = \frac{4\pi^2 l^2}{\beta^2_{\pm}}$$

$$\langle\mathbf{n}_{\pm}|\hat{p}_{\pm}|\mathbf{n}_{\pm}\rangle = \frac{2\sqrt{3}}{c}\mathbf{n}_{\pm}, \langle\mathbf{n}_{\pm}|\hat{p}^2_{\pm}|\mathbf{n}_{\pm}\rangle = \frac{12}{c^2}(\mathbf{n}_{\pm})^2$$

$$\langle R^{\pm};\mathbf{n}_{\pm}|B^0_0|R^{\pm};\mathbf{n}_{\pm}\rangle = \mathbf{n}_{\pm}, \langle R^{\pm};\mathbf{n}_{\pm}|B^1_0|R^{\pm};\mathbf{n}_{\pm}\rangle = \frac{\mathbf{n}_{\pm}(\mathbf{n}_{\pm}+1)}{2} + |R^{\pm}|,$$

$$\langle R^{\pm};\mathbf{n}_{\pm}|\hat{p}_{\pm}|R^{\pm};\mathbf{n}_{\pm}\rangle = \frac{2\sqrt{3}}{c}\mathbf{n}_{\pm}, \langle R^{\pm};\mathbf{n}_{\pm}|\hat{p}^2_{\pm}|R^{\pm};\mathbf{n}_{\pm}\rangle = \frac{24}{c^2}\left(\frac{\mathbf{n}_{\pm}(\mathbf{n}_{\pm}+1)}{2} + |R^{\pm}|\right).$$

$$\langle R^{\pm}|\colon \hat{p}^2_{\pm}(t,\theta)\colon|R^{\pm}\rangle = \frac{24}{c^2}|R^{\pm}|$$

$$|R^{\pm}| = \frac{c}{2}(Ml \pm J) = \frac{c^2\pi^2 l^2}{6\beta^2_{\pm}}$$

$$d(p_+, p_-) = e^{2\pi\sqrt{\frac{|R^+|}{6}}+2\pi\sqrt{\frac{|R^-|}{6}}}$$

$$S_{\text{stat}} = \ln\, d(p_+, p_-) = 2\pi\sqrt{\frac{|R^+|}{6}} + 2\pi\sqrt{\frac{|R^-|}{6}} = \frac{\pi^2 l^2}{2G}\left(\frac{1}{\beta_+} + \frac{1}{\beta_-}\right)$$

$$S = \log\, \mathcal{Z} + \beta\bar{E}$$

$$\mathcal{Z} = \text{Trexp}\left(-\beta_+ H^+_{(1)}\right)\text{Trexp}\left(-\beta_- H^-_{(1)}\right)$$

$$\mathbf{n}_\pm = \pm\frac{\pi c l}{\sqrt{3}\beta_\pm}$$

$$\mathcal{Z} = \sum_{R^+} \exp\left[-\frac{\sqrt{3}}{\pi c^2}\left(\frac{\beta_+}{l}\right)^2 \langle R^+; \mathbf{n}_+|B_0^2|R^+; \mathbf{n}_+\rangle\right] \cdot \sum_{R^-} \exp\left[\frac{\sqrt{3}}{\pi c^2}\left(\frac{\beta_-}{l}\right)^2 \langle R^-; \mathbf{n}_-|B_0^2|R^-; \mathbf{n}_-\rangle\right].$$

$$\begin{aligned}\left\langle R^\pm; \mathbf{n}_\pm\middle|B_0^2\middle|R^\pm; \mathbf{n}_\pm\right\rangle &= \pm\left(\kappa_{R^\pm} + 2\left|\mathbf{n}_\pm\right|\left|R^\pm\right|\right) + g_2(\mathbf{n}_\pm) \\ &= \pm\left(\sum_i l_i^\pm\left(l_i^\pm - 2i\right) + \left|R^\pm\right| + \left(2\left|\mathbf{n}_\pm\right| \pm 1\right)\left|R^\pm\right|\right) + g_2(\mathbf{n}_\pm)\end{aligned}$$

$$\mathcal{Z}_\pm = \sum_{R^\pm} \exp\left[-\frac{\sqrt{3}}{\pi c^2}\left(\frac{\beta_\pm}{l}\right)^2\left(\sum_i l_i^\pm\left(l_i^\pm + 1 - 2i\right) + \left(2\left|\mathbf{n}_\pm\right| \pm 1\right)\left|R^\pm\right|\right)\right].$$

$$\mathcal{Z}_\pm = \sum_{R^\pm} \exp\left[-\frac{\tilde{A}_\pm}{2\left|\mathbf{n}_\pm\right|}\sum_i l_i^\pm\left(l_i^\pm + 1 - 2i\right) - \tilde{A}_\pm\sum_i l_i^\pm\right]$$

$$\tilde{A}_\pm = 2\left(\frac{\beta_\pm}{cl}\right)$$

$$\mathcal{F}^\pm = \ln\, \mathcal{Z}_\pm$$

$$\mathcal{F}^\pm = \sum_{g=0}^{\infty} N_\pm^{2-2g} F_g^\pm\left(\tilde{A}_\pm\right)$$

$$F_0^\pm\left(\tilde{A}_\pm\right) = -\frac{\tilde{A}_\pm}{48}, F_1^\pm\left(\tilde{A}_\pm\right) = -\log\, \eta(Q_\pm)$$

$$\eta(Q_\pm) := Q_\pm^{1/24}\prod_{n=1}^{\infty}\left(1 - Q_\pm^n\right)$$

$$F_g\left(\tilde{A}_\pm\right) \sim \left(\tilde{A}_\pm\right)^{-2g+1}\pi^{2g}\mathbf{c}_g, g \geq 2$$

$$F_1(\tilde{A}_\pm) = \frac{\pi^2}{3\tilde{A}_\pm} + \frac{1}{2}\ln\ \tilde{A}_\pm$$

$$\mathcal{F}^\pm = \mathcal{F}_0^\pm + \frac{1}{2}\ln\ \tilde{A}_\pm$$

$$\mathcal{F}_0^\pm = \frac{\pi^2}{3\tilde{A}_\pm}\sum_{g\geq 0}\frac{3^g \mathbf{c}_g}{16^{g-1}},\ \text{with}\ \mathbf{c}_0 = -\frac{1}{192},\ \text{and}\ \mathbf{c}_1 = \frac{1}{3}$$

$$\mathcal{F}_0^\pm = \frac{\pi^2}{3\tilde{A}_\pm} - \frac{\pi^2}{3\tilde{A}_\pm}\left(\frac{1}{12} - 16\sum_{g\geq 2}\left(\frac{3}{16}\right)^g \mathbf{c}_g\right)$$

$$\Delta S = \frac{1}{2}\ln\left(\frac{\beta_+\beta_-}{c^2 l^2}\right) = -\frac{1}{2}\ln\left(\frac{9l^4}{4G^2\beta^2(1-l^2\Omega^2)}\right)$$

$$\mathcal{Z}_\pm = \mathrm{Tr}\left[\exp\left(-\frac{k}{4\pi}\frac{\beta_\pm}{l}\int d\theta\,\frac{p_\pm^2(t,\theta)}{2}\right)\right] = \mathrm{Trexp}\left(-\frac{\beta_\pm}{cl}B_0^1(p_\pm)\right)$$

$$B_0^1\left|R^\pm,\mathbf{n}\right\rangle = \left(\left|R^\pm\right| + \frac{1}{2}\mathbf{n}(\mathbf{n}+1)\right)\left|R^\pm,\mathbf{n}\right\rangle.$$

$$\mathcal{Z}_\pm = \sum_{R^\pm}\exp\left(-\frac{\beta_\pm}{cl}\left|R^\pm\right|\right)$$

$$\mathcal{Z}_\pm = \sum_{n=0}^{\infty} p(n)e^{-\mu_\pm n}$$

$$\mathcal{Z}_\pm = \prod_{k=1}^{\infty}\frac{1}{1-e^{-\mu_\pm k}}, \mu_\pm > 0$$

$$\log\ \mathcal{Z}_\pm = \frac{\pi^2}{6\mu_\pm} + \frac{1}{2}\log\ \mu_\pm + \cdots$$

$$\mathcal{F}_0 = \frac{l^2\pi^2}{2\beta G(1-l^2\Omega^2)}$$

$$\Delta S = -\frac{1}{2}\log\left(\frac{9l^4}{4G^2\beta^2(1-l^2\Omega^2)}\right)$$

$$\xi^\pm = \mp\mathcal{C}_\pm\frac{p_\pm^2}{2}$$

$$ds^2=\frac{\pi^4 l^6}{\beta_-^4\beta_+^4}\left(\beta_-^4\mathcal{C}_+^2+2\beta_-^2\beta_+^2\mathcal{C}_+\mathcal{C}_-\cosh\left(\frac{2\rho}{l}\right)+\beta_+^4\mathcal{C}_-^2\right)dt^2+d\rho^2$$
$$-\frac{2\pi^3 l^5}{\beta_-^3\beta_+^3}\left(\beta_-^3\mathcal{C}_+-\beta_-\beta_+\cosh\left(\frac{2\rho}{l}\right)(\beta_-\mathcal{C}_+-\beta_+\mathcal{C}_-)-\beta_+^3\mathcal{C}_-\right)dtd\theta$$
$$+\frac{\pi^2 l^4}{\beta_-^2\beta_+^2}\left(\beta_-^2+\beta_+^2-2\beta_-\beta_+\cosh\left(\frac{2\rho}{l}\right)\right)d\theta^2$$

$$p_\pm(t,\theta)=v\pm\pi\sigma=\pm\frac{2\pi l}{\beta_\pm}$$

$$\rho=\frac{l}{2}\cosh^{-1}\left(\frac{\pi^2 l^4(\beta_-^2+\beta_+^2)-\beta_-^2\beta_+^2 r^2}{2\pi^2\beta_-\beta_+ l^4}\right).$$

$$ds^2=\frac{\pi^2 l^2}{\beta_-^4\beta_+^4}(\pi^2\beta_-^4\mathcal{C}_+^2 l^4+\beta_-\beta_+\mathcal{C}_+\mathcal{C}_-(\pi^2 l^4(\beta_-^2+\beta_+^2)-\beta_-^2\beta_+^2 r^2)+\pi^2\beta_+^4\mathcal{C}_-^2 l^4)dt^2$$
$$+r^2 d\theta^2+\left(\frac{l^2 r^2\beta_-^4\beta_+^4}{\pi^4 l^8(\beta_-^2-\beta_+^2)^2-2\pi^2\beta_-^2\beta_+^2 l^4 r^2(\beta_-^2+\beta_+^2)+\beta_-^4\beta_+^4 r^4}\right)dr^2$$
$$-\frac{\pi l}{\beta_-^3\beta_+^3}\left(\mathcal{C}_+(\pi^2\beta_- l^4(\beta_-^2-\beta_+^2)+\beta_-^3\beta_+^2 r^2)-\beta_+\mathcal{C}_-(\pi^2 l^4(\beta_+^2-\beta_-^2)+\beta_-^2\beta_+^2 r^2)\right)dtd\theta$$

$$t\to t,\theta\to\theta+\left(\frac{2A_\phi}{r^2}\right)t,\mathcal{C}_\pm=\pm\frac{2}{l(v\pm\pi\sigma)}=\pm\frac{2}{lp_\pm}=\frac{\beta_\pm}{\pi l^2}$$

$$\xi_\pm=\mp\mathcal{C}_\pm p_\pm$$

$$ds^2=\frac{\pi^2 l^4}{\beta_-^2\beta_+^2}\left(\beta_-^2\mathcal{C}_+^2-2\beta_-\beta_+\mathcal{C}_+\mathcal{C}_-\cosh\left(\frac{2\rho}{l}\right)+\beta_+^2\mathcal{C}_-^2\right)dt^2+d\rho^2$$
$$-\frac{2\pi^2 l^4}{\beta_-^2\beta_+^2}\left(\beta_-^2\mathcal{C}_+-\beta_-\beta_+(\mathcal{C}_++\mathcal{C}_-)\cosh\left(\frac{2\rho}{l}\right)+\beta_+^2\mathcal{C}_-\right)dtd\theta$$
$$+\frac{\pi^2 l^4}{\beta_-^2\beta_+^2}\left(\beta_-^2+\beta_+^2-2\beta_-\beta_+\cosh\left(\frac{2\rho}{l}\right)\right)d\theta^2$$

$$\rho=\rho=\frac{l}{2}\cosh^{-1}\left(\frac{\pi^2 l^4(\beta_-^2+\beta_+^2)-\beta_-^2\beta_+^2 r^2}{2\pi^2\beta_-\beta_+ l^4}\right).$$

$$ds^2=\frac{1}{\beta_-^2\beta_+^2}\left(\pi^2\beta_-^2\mathcal{C}_+^2 l^4+\mathcal{C}_+\mathcal{C}_-\left(\beta_-^2\beta_+^2 r^2-\pi^2 l^4(\beta_-^2+\beta_+^2)\right)+\pi^2\beta_+^2\mathcal{C}_-^2 l^4\right)dt^2$$
$$+r^2 d\theta^2+\frac{l^2 r^2\beta_-^4\beta_+^4}{\pi^4 l^8(\beta_-^2-\beta_+^2)^2-2\pi^2\beta_-^2\beta_+^2 l^4 r^2(\beta_-^2+\beta_+^2)+\beta_-^4\beta_+^4 r^4}dr^2$$
$$-\frac{1}{\beta_-^2\beta_+^2}(\pi^2\mathcal{C}_+ l^4(\beta_-^2-\beta_+^2)+\beta_-^2\beta_+^2\mathcal{C}_+ r^2+\pi^2\mathcal{C}_- l^4(\beta_+^2-\beta_-^2)+\beta_-^2\beta_+^2\mathcal{C}_- r^2)dtd\theta$$

$$\mathcal{C}_\pm=\pm\frac{1}{l}.$$

$$\prod_{k=1}^{\infty}\frac{1}{1-e^{-\mu k}}$$

$$q = e^{-\mu}, \text{ and } \tau = \frac{i\mu}{2\pi}$$

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} \left(1 - q^k\right)$$

$$\prod_{k=1}^{\infty} \frac{1}{1 - q^k} = e^{-\mu/24} \eta\left(\frac{i\mu}{2\pi}\right)^{-1}$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)$$

$$\eta\left(\frac{i\mu}{2\pi}\right) = \left(\frac{\mu}{2\pi}\right)^{-1/2} \eta\left(\frac{2\pi i}{\mu}\right).$$

$$\eta\left(\frac{2\pi i}{\mu}\right)$$

$$\eta\left(\frac{2\pi i}{\mu}\right) = q'^{1/24} \prod_{k=1}^{\infty} \left(1 - q'^k\right)$$
$$= e^{\frac{2\pi i(2\pi i/\mu)}{24}} = e^{\pi^2/6\mu}$$

$$\prod_{k=1}^{\infty} \frac{1}{(1 - e^{-\mu k})} = \sqrt{\frac{\mu}{2\pi}} e^{-\frac{t}{24}} e^{\frac{\pi^2}{6\mu}}$$

$$\mathcal{S} = \frac{1}{2} \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{-g} \left[\mathcal{R} - \frac{(n-2)(n-1)}{3}\Lambda + \alpha_2 \mathcal{L}_{\mathrm{GB}} - \mathcal{F}^q\right],$$

$$\mathcal{L}_{\mathrm{GB}} = \mathcal{R}_{\gamma\zeta\varpi\delta}\mathcal{R}^{\gamma\zeta\varpi\delta} - 4\mathcal{R}_{\gamma\zeta}\mathcal{R}^{\gamma\zeta} + \mathcal{R}^2$$

$$\mathcal{F} = \sum_{a=1}^{\frac{(n-1)(n-2)}{2}} \mathrm{Tr}\left(F_{\gamma\zeta}^{(a)} F^{\gamma\zeta(a)}\right)$$

$$\mathbf{F}^{(a)} = \mathbf{d}\mathbf{A}^{(a)} + \frac{1}{2\sigma} C^{(a)}_{(b)(c)} \mathbf{A}^{(b)} \wedge \mathbf{A}^{(c)},$$

$$F^{(a)}_{\gamma\zeta} = \partial_\gamma A^{(a)}_\zeta - \partial_\zeta A^{(a)}_\gamma + \frac{1}{2\sigma} C^{(a)}_{(b)(c)} A^{(b)}_\gamma A^{(c)}_\zeta$$

$$\mathbf{d}\left({}^\star\mathbf{F}^{(a)} \mathcal{F}^{q-1}\right) + \frac{1}{\sigma} C^{(a)}_{(b)(c)} \mathcal{F}^{q-1} \mathbf{A}^{(b)} \wedge {}^\star\mathbf{F}^{(c)} = 0$$

$$\mathcal{G}_{\mu\nu} + \alpha_2 \mathcal{H}_{\mu\nu} + \frac{(n-2)(n-1)}{6}\Lambda g_{\mu\nu} = \mathcal{T}_{\mu\nu},$$

$$\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}$$

$$\mathcal{H}_{\mu\nu} = 2\left(-\mathcal{R}_{\mu\sigma\kappa\tau}\mathcal{R}^{\kappa\tau\sigma}{}_\nu - 2\mathcal{R}_{\mu\rho\nu\sigma}\mathcal{R}^{\rho\sigma} - 2\mathcal{R}_{\mu\sigma}\mathcal{R}^{\sigma}{}_\nu + \mathcal{R}\mathcal{R}_{\mu\nu}\right) - \frac{1}{2} g_{\mu\nu}\mathcal{L}_{\mathrm{GB}}$$

$$\mathcal{T}^{\mu}{}_{\nu} = -\frac{1}{2}\delta^{\mu}{}_{\nu}\mathcal{F}^{q} + 2q\mathrm{Tr}\left(F_{\nu\lambda}{}^{(a)}F^{\mu\lambda(a)}\right)\mathcal{F}^{q-1}$$

$$\mathbf{A}^{(a)} = \frac{\mathcal{Q}}{r^2}C^{(a)}_{(i)(j)}\chi^{i}\,\mathrm{d}\chi^{j}$$

$$\begin{aligned}
\chi_1 &= r\cos\,\theta_{n-3}\sin\,\theta_{n-4}\cdots\sin\,\theta_1\\
\chi_2 &= r\sin\,\theta_{n-3}\sin\,\theta_{n-4}\cdots\sin\,\theta_1\\
\chi_3 &= r\cos\,\theta_{n-4}\sin\,\theta_{n-5}\cdots\sin\,\theta_1\\
\chi_4 &= r\sin\,\theta_{n-4}\sin\,\theta_{n-5}\cdots\sin\,\theta_1\\
&\vdots\\
\chi_{n-2} &= r\cos\,\theta_1
\end{aligned}$$

$$\mathcal{F} = \frac{(n-2)(n-3)\mathcal{Q}^2}{r^4}$$

$$\mathrm{Tr}\left(F^{(a)}_{\theta_i\lambda}F^{\theta_i\lambda(a)}\right) = \frac{1}{(n-2)}\mathcal{F} = \frac{(n-3)\mathcal{Q}^2}{r^4}$$

$$\mathcal{T}^{a}{}_{b} = -\frac{1}{2}\mathcal{F}^{q}\mathrm{diag}\left[1,1,\left(1-\frac{4q}{n-2}\right),\left(1-\frac{4q}{n-2}\right),\cdots,\left(1-\frac{4q}{n-2}\right)\right]$$

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\sum_{i=2}^{n-2}\mathfrak{h}_{ij}\,\mathrm{d}x^{i}\,\mathrm{d}x^{j},$$

$$f_{\pm}(r) = \begin{cases} 1+\dfrac{r^2}{2\tilde{\alpha}_2}\left(1\pm\sqrt{1+\dfrac{16\mathfrak{M}\tilde{\alpha}_2}{(n-2)r^{n-1}}+\dfrac{4\tilde{\alpha}_2 Q_1}{r^{4q}}}\right), & q\neq\dfrac{n-1}{4};\\ 1+\dfrac{r^2}{2\tilde{\alpha}_2}\left(1\pm\sqrt{1+\dfrac{16\mathfrak{M}\tilde{\alpha}_2}{(n-2)r^{n-1}}+\dfrac{4\tilde{\alpha}_2 Q_2\ln(r)}{r^{n-1}}}\right), & q=\dfrac{n-1}{4}, \end{cases}$$

$$Q_1 = \frac{[(n-3)(n-2)\mathcal{Q}^2]^q}{(n-2)(n-1-4q)}$$

$$Q_2 = \frac{[(n-3)(n-2)\mathcal{Q}^2]^{\frac{(n-1)}{4}}}{(n-2)}$$

$$\mathfrak{M} = \frac{M}{2\Sigma_{(n-2)}}$$

$$\Sigma_{(n-2)} = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\dfrac{n-1}{2}\right)}$$

$$f_{-}(r) = \begin{cases} 1+\dfrac{r^2}{2\tilde{\alpha}_2}\left(1-\sqrt{1+\dfrac{64\pi\mathcal{G}_M\tilde{\alpha}_2}{(n-2)\Sigma_{(n-2)}r^{n-1}}+\dfrac{4\tilde{\alpha}_2 Q_1}{r^{4q}}}\right), & \dfrac{n-1}{4}<q<\dfrac{n-1}{2};\\ 1+\dfrac{r^2}{2\tilde{\alpha}_2}\left(1-\sqrt{1+\dfrac{64\pi\mathcal{G}_M\tilde{\alpha}_2}{(n-2)\Sigma_{(n-2)}r^{n-1}}+\dfrac{4\tilde{\alpha}_2 Q_2\ln(r)}{r^{n-1}}}\right), & q=\dfrac{n-1}{4}. \end{cases}$$

$$f(r) = 1 - \frac{16\pi \mathcal{G} M}{(n-2)\Sigma_{(n-2)} r^{n-3}}.$$

$$f(r) = 1 + \frac{r^2}{2\tilde{\alpha}_2}\left(1 - \sqrt{1 + \frac{64\pi \mathcal{G} M \tilde{\alpha}_2}{(n-2)\Sigma_{(n-2)} r^{n-1}}}\right).$$

$$f(r) = \begin{cases} 1 - \dfrac{8\mathcal{G}M}{3\pi r^2} - \dfrac{2Q^2 \ln(r)}{r^2}, & (n = 5); \\ 1 - \dfrac{16\pi \mathcal{G} M}{(n-2)\Sigma_{(n-2)} r^{n-3}} - \dfrac{(n-3)Q^2}{(n-5)r^2}, & (n > 5). \end{cases}$$

$$1 + \frac{16\mathfrak{M}\tilde{\alpha}_2}{(n-2)r^{n-1}} + \frac{4\tilde{\alpha}_2 Q_1}{r^{4q}} > 0,$$
$$1 + \frac{16\mathfrak{M}\tilde{\alpha}_2}{(n-2)r^{n-1}} + \frac{4\tilde{\alpha}_2 Q_2 \ln(r)}{r^{n-1}} > 0.$$

$$\mathrm{d}s^2 = -h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + k(r)\mathrm{d}\Omega_{n-2}^2$$

$$\begin{aligned} \mathrm{d}\Omega_{n-2}^2 &= \mathrm{d}\theta_1^2 + \sin^2\theta_1\, \mathrm{d}\theta_2^2 + \cdots + (\sin^2\theta_1 \cdots \sin^2\theta_{n-3})\mathrm{d}\theta_{n-2}^2 \\ &= \mathrm{d}\theta_1^2 + \sin^2\theta_1\, \mathrm{d}\theta_2^2 + \cdots + \left(\prod_{i=1}^{n-3} \sin^2\theta_i\right)\mathrm{d}\theta_{n-2}^2, \end{aligned}$$

$$g_{a\,b} = \begin{pmatrix} g_{0\,0} & & & & & & \\ & g_{1\,1} & & & & & \\ & & g_{2\,2} & & & & \\ & & & g_{3\,3} & & & \\ & & & & \ddots & & \\ & & & & & g_{\chi+1\,\chi+1} & \\ & & & & & & \ddots \\ & & & & & & g_{n-1\,n-1} \end{pmatrix}_{n\times n}$$

$$= \begin{pmatrix} -h(r) & & & & & & \\ & \frac{1}{f(r)} & & & & & \\ & & k(r) & & & & \\ & & & k(r)\sin^2\theta_1 & & & \\ & & & & \ddots & & \\ & & & & & k(r)\prod_{i=1}^{\chi-1}\sin^2\theta_i & \\ & & & & & & \ddots \\ & & & & & & k(r)\prod_{i=1}^{n-3}\sin^2\theta_i \end{pmatrix}_{n\times n}$$

$$\tilde{\mathcal{L}} = \frac{1}{2}(g_{ab})\dot{x}^a\dot{x}^b = \frac{1}{2}(g_{ab})\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}$$

$$\begin{aligned}\tilde{\mathcal{L}} =& \frac{1}{2}\left[(g_{00})\dot{t}^2 + (g_{11})\dot{r}^2 + (g_{22})\dot{\theta}_1^2 + (g_{33})\dot{\theta}_2^2\right.\\ &\left. + \cdots + \left(g_{\chi+1\chi+1}\right)\dot{\theta}_\chi^2 + \cdots + (g_{n-1n-1})\dot{\theta}_{n-2}^2\right]\\ =& \frac{1}{2}\left[(-h(r))\dot{t}^2 + \left(\frac{1}{f(r)}\right)\dot{r}^2 + (k(r))\dot{\theta}_1^2 + (k(r)\sin^2\,\theta_1)\dot{\theta}_2^2\right.\\ &\left. + \cdots + \left(k(r)\prod_{i=1}^{\chi-1}\sin^2\,\theta_i\right)\dot{\theta}_\chi^2 + \cdots + \left(k(r)\prod_{i=1}^{n-3}\sin^2\,\theta_i\right)\dot{\theta}_{n-2}^2\right].\end{aligned}$$

$$P_a = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}^a} = (g_{ab})\dot{x}^b = \dot{x}_a$$

$$\begin{aligned}P_0 = P_t &= \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}^0} = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{t}} = (-h(r))\dot{t} \equiv -\mathrm{E},\\ P_1 = P_r &= \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}^1} = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{r}} = \left(\frac{1}{f(r)}\right)\dot{r},\\ P_2 = P_{\theta_1} &= \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}^2} = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\theta}_1} = (k(r))\dot{\theta}_1,\\ P_3 = P_{\theta_2} &= \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}^3} = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\theta}_2} = (k(r)\sin^2\,\theta_1)\dot{\theta}_2,\\ &\vdots\\ P_{\chi+1} = P_{\theta_\chi} &= \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}^{\chi+1}} = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\theta}_\chi} = \left(k(r)\prod_{i=1}^{\chi-1}\sin^2\,\theta_i\right)\dot{\theta}_\chi,\\ &\vdots\\ P_{n-1} = P_{\theta_{n-2}} &= \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}^{n-1}} = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\theta}_{n-2}} = \left(k(r)\prod_{i=1}^{n-3}\sin^2\,\theta_i\right)\dot{\theta}_{n-2} \equiv \text{Ł},\end{aligned}$$

$$\frac{\partial S}{\partial\tau} = -\mathcal{H} = -\frac{1}{2}\left(g^{ab}\right)\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}$$

$$\mathcal{H} = P_a\dot{x}^a - \tilde{\mathcal{L}} = \frac{1}{2}(g_{ab})\dot{x}^a\dot{x}^b = -\frac{1}{2}m^2$$

$$\begin{aligned}-2\frac{\partial S}{\partial\tau} =& -\frac{1}{h(r)}\left(\frac{\partial S_t}{\partial t}\right)^2 + f(r)\left(\frac{\partial S_r}{\partial r}\right)^2\\ &+ \sum_{\chi=1}^{n-3}\frac{1}{\left(k(r)\prod_{i=1}^{\chi-1}\sin^2\,\theta_i\right)}\left(\frac{\partial S_{\theta_\chi}}{\partial\theta_\chi}\right)^2\\ &+ \frac{1}{\left(k(r)\prod_{i=1}^{n-3}\sin^2\,\theta_i\right)}\left(\frac{\partial S_{\theta_{n-2}}}{\partial\theta_{n-2}}\right)^2\end{aligned}$$

$$S = \frac{1}{2}m^2\tau - \mathrm{E}t + S_r(r) + \sum_{\chi=1}^{n-3}S_{\theta_\chi}(\theta_\chi) + \text{Ł}\theta_{n-2}$$

$$\frac{\partial S_\tau}{\partial \tau} = -\frac{1}{2}\left(g^{ab}\right)P_a P_b = \frac{1}{2}m^2,$$
$$\frac{\partial S_t}{\partial t} = P_t = -h(r)\dot{t} = -\mathrm{E},$$
$$\frac{\partial S_r}{\partial r} = P_r = \frac{1}{f(r)}\dot{r},$$
$$\frac{\partial S_{\theta_\chi}}{\partial \theta_\chi} = P_{\theta_\chi} = k(r)\left(\prod_{i=1}^{\chi-1}\sin^2\,\theta_i\right)\dot{\theta}_\chi,$$
$$\frac{\partial S_{\theta_{n-2}}}{\partial \theta_{n-2}} = P_{\theta_{n-2}} = k(r)\left(\prod_{i=1}^{n-3}\sin^2\,\theta_i\right)\dot{\theta}_{n-2} = Ł.$$

$$k(r)m^2 = \frac{k(r)}{h(r)}(-\mathrm{E})^2 - \frac{k(r)}{f(r)}(\dot{r})^2 - k^2(r)\sum_{\chi=1}^{n-3}\left[\left(\prod_{i=1}^{\chi-1}\sin^2\,\theta_i\right)\left(\dot{\theta}_\chi\right)^2\right] - \left[\prod_{i=1}^{n-3}(1+\cot^2\,\theta_i)\right](Ł)^2$$

$$\prod_{i=1}^{n-3}(1+\cot^2\,\theta_i) = 1 + \prod_{i=1}^{n-3}\cot^2\,\theta_i + \mathfrak{O}(\theta_i)$$

$$0 = \left\{\frac{k(r)}{h(r)}\mathrm{E}^2 - \frac{k(r)}{f(r)}(\dot{r})^2 - \mathrm{L}^2 - k(r)m^2\right\} - \left\{k^2(r)\sum_{\chi=1}^{n-3}\left[\left(\prod_{i=1}^{\chi-1}\sin^2\,\theta_i\right)\left(\dot{\theta}_\chi\right)^2\right] + \left[\prod_{i=1}^{n-3}\cot^2\,\theta_i + \mathfrak{O}(\theta_i)\right]Ł^2\right\}$$

$$\mathcal{K} = \left\{\frac{k(r)}{h(r)}\mathrm{E}^2 - \frac{k(r)}{f(r)}(\dot{r})^2 - Ł^2 - k(r)m^2\right\}$$
$$\mathcal{K} = \left\{k^2(r)\sum_{\chi=1}^{n-3}\left[\left(\prod_{i=1}^{\chi-1}\sin^2\,\theta_i\right)\left(\dot{\theta}_\chi\right)^2\right] + \left[\prod_{i=1}^{n-3}\cot^2\,\theta_i + \mathfrak{O}(\theta_i)\right]Ł^2\right\}$$

$$\dot{t} = \frac{1}{h(r)}\mathrm{E}$$
$$\dot{r} = \pm\sqrt{\frac{f(r)}{h(r)}\mathrm{E}^2 - \frac{f(r)}{k(r)}(\mathcal{K}+\mathrm{L}^2) - f(r)m^2}$$

$$\sqrt{\sum_{\chi=1}^{n-3}\prod_{i=1}^{\chi-1}\sin^2\,\theta_i\left(\dot{\theta}_\chi\right)^2} = \pm\sqrt{\frac{\mathcal{K} - \left[\prod_{i=1}^{n-3}\cot^2\,\theta_i + \mathfrak{O}(\theta_i)\right]Ł^2}{k^2(r)}}$$

$$\dot{\theta}_{n-2} = \frac{1}{k(r)\left(\prod_{i=1}^{n-3}\sin^2\,\theta_i\right)}Ł$$

$$\dot{t}=\frac{1}{h(r)}\mathrm{E}$$

$$\dot{r}=\pm\frac{1}{k(r)}\sqrt{\Re}$$

$$\sqrt{\sum_{\chi=1}^{n-3}\prod_{i=1}^{\chi-1}\sin^2\,\theta_i\left(\dot{\theta}_\chi\right)^2}=\pm\frac{1}{k(r)}\sqrt{\Theta_\chi}$$

$$\dot{\theta}_{n-2}=\frac{1}{k(r)\left(\prod_{i=1}^{n-3}\sin^2\,\theta_i\right)}Ł$$

$$\Re=k^2(r)\frac{f(r)}{h(r)}\mathrm{E}^2-k(r)f(r)(\mathcal{K}+Ł^2)-k^2(r)f(r)m^2$$

$$\Theta_\chi=\mathcal{K}-\left[\prod_{i=1}^{n-3}\cot^2\,\theta_i+\mathfrak{O}(\theta_i)\right]Ł^2$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2+V_{\mathrm{eff}}=0$$

$$V_{\mathrm{eff}}=\frac{f(r)}{k(r)}(\mathcal{K}+Ł^2)-\frac{f(r)}{h(r)}\mathrm{E}^2$$

$$\Re=k^2(r)\frac{f(r)}{h(r)}\mathrm{E}^2-k(r)f(r)(\mathcal{K}+Ł^2).$$

$$\xi=\frac{Ł}{\mathrm{E}}$$

$$\eta=\frac{\mathcal{K}}{\mathrm{E}^2}$$

$$V_{\mathrm{eff}}=\mathrm{E}^2\left\{\frac{f(r)}{k(r)}(\eta+\xi^2)-\frac{f(r)}{h(r)}\right\}$$

$$\Re=\mathrm{E}^2\left\{k^2(r)\frac{f(r)}{h(r)}\mathrm{E}^2-k(r)f(r)(\eta+\xi^2)\right\}$$

$$\begin{array}{lll} V_{\mathrm{eff}}=0, & & \Re=0 \\ \dfrac{\partial V_{\mathrm{eff}}}{\partial r}=0, & \text{or} & \dfrac{\partial\Re}{\partial r}=0 \\ \dfrac{\partial^2 V_{\mathrm{eff}}}{\partial r^2}<0, & & \dfrac{\partial^2\Re}{\partial r^2}>0 \end{array}$$

$$k(r_p)h'(r_p)-h(r_p)k'(r_p)=0$$

$$(\eta+\xi^2)=\frac{h(r)k^2(r)f'(r)-f(r)k^2(r)h'(r)+2f(r)h(r)k(r)k'(r)}{h^2(r)[k(r)f'(r)+f(r)k'(r)]}$$

$$(\eta+\xi^2)=\frac{k^2(r_p)}{h(r_p)[k(r_p)f'(r_p)+f(r_p)k'(r_p)]}\left\{f'(r_p)-f(r_p)\left[\frac{h'(r_p)}{h(r_p)}-\frac{2k'(r_p)}{k(r_p)}\right]\right\}$$

$$e_{\hat{\alpha}}=\left\{e_{\hat{t}},e_{\hat{r}},\cdots,e_{\hat{\theta}_\chi},\cdots,e_{\hat{\theta}_{n-2}}\right\}$$

$$e_{\mu}=\left\{e_{t},e_{r},\cdots,e_{\theta_\chi},\cdots,e_{\theta_{n-2}}\right\}$$

$$e_{\hat{\alpha}}=e^{\mu}_{\hat{\alpha}}e_{\mu}$$

$$e^{\mu}_{\hat{\alpha}}e^{\nu}_{\hat{\beta}}g_{\mu\nu}=\eta_{\hat{\alpha}\hat{\beta}}$$

$$\begin{aligned}-1&=e_{\hat{t}}\cdot e_{\hat{t}}\\1&=e_{\hat{r}}\cdot e_{\hat{r}}\\1&=e_{\hat{r}}\cdot e_{\hat{r}}\\1&=e_{\hat{\theta}_{n-2}}\cdot e_{\hat{\theta}_{n-2}},\end{aligned}$$

$$\begin{aligned}e_{\hat{t}}&=\zeta e_t+\gamma e_{\theta_{n-2}},\\e_{\hat{r}}&=\iota e_r,\\&\vdots\\e_{\hat{\theta}_\chi}&=\varsigma e_{\theta_\chi},\\&\vdots\\e_{\hat{\theta}_{n-2}}&=\varrho e_{\theta_{n-2}},\end{aligned}$$

$$\zeta=e^{t}_{\hat{t}}=\sqrt{\frac{g_{n-1n-1}}{(g_{0n-1})^2-g_{00}g_{n-1n-1}}},$$

$$\gamma=e^{\theta_{n-2}}_{\hat{t}}=-\frac{g_{0n-1}}{g_{n-1n-1}}\sqrt{\frac{g_{n-1n-1}}{(g_{0n-1})^2-g_{00}g_{n-1n-1}}},$$

$$\iota=e^{r}_{\hat{r}}=\frac{1}{\sqrt{g_{11}}},$$

$$\varsigma=e^{\theta_\chi}_{\hat{\theta}_\chi}=\frac{1}{\sqrt{g_{\chi+1\chi+1}}},$$

$$\varrho=e^{\theta_{n-2}}_{\hat{\theta}_{n-2}}=\frac{1}{\sqrt{g_{n-1n-1}}}.$$

$$\begin{aligned}P^{\hat{t}}&=-\left(\zeta P_t+\gamma P_{\theta_{n-2}}\right)\\P^{\hat{r}}&=\iota P_r\\P^{\hat{\theta}\chi}&=\varsigma P_{\theta_\chi}\\P^{\hat{\theta}_{n-2}}&=\varrho P_{\theta_{n-2}}\end{aligned}$$

$$P^{\hat{t}} = \zeta \mathrm{E} - \gamma \mathrm{E} = \left(\frac{1}{h(r)}\right)^{\frac{1}{2}} \mathrm{E},$$

$$P^{\hat{r}} = \frac{1}{\sqrt{g_{11}}}\left(\frac{1}{f(r)}\right)\dot{r} = \left(\frac{1}{f(r)}\right)^{\frac{1}{2}} |\dot{r}|,$$

$$P^{\hat{\theta}_\chi} = \frac{1}{\sqrt{g_{\chi+1\chi+1}}}\left(k(r)\prod_{i=1}^{\chi-1} \sin^2\,\theta_i\right)\dot{\theta}_\chi = \left(k(r)\prod_{i=1}^{\chi-1} \sin^2\,\theta_i\right)^{\frac{1}{2}}\dot{\theta}_\chi,$$

$$P^{\hat{\theta}_{n-2}} = \frac{1}{\sqrt{g_{n-1n-1}}}\,\mathrm{L} = \left(\frac{1}{k(r)\prod_{i=1}^{n-3} \sin^2\,\theta_i}\right)^{\frac{1}{2}} \text{Ł}.$$

$$\boldsymbol{P} = P^{\hat{r}} e_{\hat{r}} + \sum_{\chi=1}^{n-3} P^{\hat{\theta}_\chi} e_{\hat{\theta}_\chi} + P^{\hat{\theta}_{n-2}} e_{\hat{\theta}_{n-2}}$$

$$\left(P^{\hat{t}}\right)^2 = \boldsymbol{P}^2 = \left(P^{\hat{r}}\right)^2 + \sum_{\chi=1}^{n-3}\left(P^{\hat{\theta}_\chi}\right)^2 + \left(P^{\hat{\theta}_{n-2}}\right)^2$$

$$P^{\hat{r}} = |\boldsymbol{P}|\cos\,\alpha\cos\,\beta,$$
$$P^{\hat{\theta}_\chi} = |\boldsymbol{P}|\sin\,\alpha,$$
$$P^{\hat{\theta}_{n-2}} = |\boldsymbol{P}|\cos\,\alpha\sin\,\beta,$$

$$\tan\,\beta = \frac{P^{\hat{\theta}_{n-2}}}{P^{\hat{r}}},$$
$$\sin\,\alpha = \frac{P^{\hat{\theta}_\chi}}{P^{\hat{t}}}.$$

$$\mathrm{X} \equiv -r_o\beta$$
$$\mathrm{Y} \equiv r_o\alpha$$

$$\mathrm{X} \equiv \lim_{r_o\to\infty}\left[-r_o\left(\frac{P^{\hat{\theta}_{n-2}}}{P^{\hat{r}}}\right)\Bigg|_{(r=r_o)}\right],$$
$$\mathrm{Y} \equiv \lim_{r_o\to\infty}\left[r_o\left(\frac{\left\|\boldsymbol{P}^{\hat{\theta}_\chi}\right\|}{P^{\hat{t}}}\right)\Bigg|_{(r=r_o)}\right],$$

$$\left\|\boldsymbol{P}^{\hat{\theta}_\chi}\right\| \equiv \sqrt{\sum_{\chi=1}^{n-3}\left|P^{\hat{\theta}_\chi}\right|^2} = \sqrt{\sum_{\chi=1}^{n-3}\left(k(r)\prod_{i=1}^{\chi-1} \sin^2\,\theta_i\right)\left(\dot{\theta}_\chi\right)^2}.$$

$$\mathrm{X}=\lim_{r_o\to\infty}\left[-r_o\left(\frac{\left(\frac{1}{k(r)\prod_{i=1}^{n-3}\sin^2\theta_i}\right)^{\frac{1}{2}}Ł}{\left(\frac{1}{f(r)}\right)^{\frac{1}{2}}|\dot{r}|}\right)\Bigg|_{(r=r_o)}\right]$$

$$=\lim_{r_o\to\infty}\left[-r_o\left(\frac{\left(\frac{1}{\sqrt{\prod_{i=1}^{n-3}\sin^2\theta_i}}\right)Ł}{\sqrt{\frac{k(r)}{h(r)}\mathrm{E}^2-(\mathcal{K}+Ł^2)}}\right)\Bigg|_{(r=r_o)}\right]$$

$$=\lim_{r_o\to\infty}\left[-r_o\left(\frac{1}{\sqrt{\frac{k(r)}{h(r)}-(\eta+\xi^2)}}\cdot\frac{\xi}{\prod_{i=1}^{n-3}\sin\theta_i}\right)\Bigg|_{(r=r_o)}\right]$$

$$=-\frac{1}{\sqrt{£}}\left(\frac{\xi}{\prod_{i=1}^{n-3}\sin\theta_i}\right),$$

$$\mathrm{Y}=\lim_{r_o\to\infty}\left[r_o\left(\frac{\sqrt{k(r)\sum_{\chi=1}^{n-3}\prod_{i=1}^{\chi-1}\sin^2\theta_i\left(\dot{\theta}_\chi\right)^2}}{\mathrm{E}\sqrt{\frac{1}{h(r)}}}\right)\Bigg|_{(r=r_o)}\right]$$

$$=\lim_{r_o\to\infty}\left[r_o\left(\pm\sqrt{\frac{\mathcal{K}-\left[\prod_{i=1}^{n-3}\cot^2\theta_i+\mathfrak{O}(\theta_i)\right]\mathrm{E}^2}{\mathrm{E}^2}}\cdot\sqrt{\frac{h(r)}{k(r)}}\right)\Bigg|_{(r=r_o)}\right]$$

$$=\lim_{r_o\to\infty}\left[r_o\left(\pm\sqrt{\eta-\xi^2\left[\prod_{i=1}^{n-3}\cot^2\theta_i+\mathfrak{O}(\theta_i)\right]}\cdot\sqrt{\frac{h(r)}{k(r)}}\right)\Bigg|_{(r=r_o)}\right]$$

$$=\pm\frac{1}{\sqrt{£}}\sqrt{\eta-\xi^2\left[\prod_{i=1}^{n-3}\cot^2\theta_i+\mathfrak{O}(\theta_i)\right]}$$

$$\mathrm{X}=-\frac{\xi}{\sqrt{£}}$$

$$\mathrm{Y}=\pm\sqrt{\frac{\eta}{£}}$$

$$(R_s)^2\equiv\mathrm{X}^2+\mathrm{Y}^2=\frac{(\xi^2+\eta)}{£}$$

$$R_s = \sqrt{\frac{(r_p)^4 £^2}{£h(r_p)\left[(r_p)^2 £ f'(r_p) + 2r_p £ f(r_p)\right]}\left\{f'(r_p) - f(r_p)\left[\frac{h'(r_p)}{h(r_p)} - \frac{4r_p £}{(r_p)^2 £}\right]\right\}}$$

$$= \sqrt{\frac{(r_p)^3}{h(r_p)[r_p f'(r_p) + 2f(r_p)]}\left\{f'(r_p) - f(r_p)\left[\frac{h'(r_p)}{h(r_p)} - \frac{4}{r_p}\right]\right\}}$$

$$\mathrm{X} = \lim_{r_o \to \infty}\left[-r_o\left(\frac{P^{\hat{\theta}_{n-2}}}{P^{\hat{t}}}\right)\Big|_{(r=r_o)}\right]$$

$$= \lim_{r_o \to \infty}\left[-r_o\left(\frac{\xi}{\prod_{i=1}^{n-3}\sin\theta_i}\cdot\sqrt{\frac{h(r)}{k(r)}}\right)\Big|_{(r=r_o)}\right]$$

$$= -\frac{1}{\sqrt{£}}\left(\frac{\xi}{\prod_{i=1}^{n-3}\sin\theta_i}\right).$$

$$\theta_{\mathrm{g}} = \frac{\mathrm{G}M}{c^2 D}$$

$$\delta \equiv \frac{\theta_{\mathrm{dyn}}}{\theta_{\mathrm{g}}} - 1$$

$$\theta_{\mathrm{dyn}} = \frac{\mathrm{G}M_{\mathrm{dyn}}}{c^2 D_{\mathrm{dyn}}}$$

$$\theta_{\mathrm{sh}} = \frac{r_{\mathrm{sh}}}{D},$$

$$r_{\mathrm{sh}} = \frac{3\sqrt{3}GM_{\mathrm{dyn}}}{c^2},$$

$$\theta_{\mathrm{sh}} = \frac{r_{\mathrm{sh}}}{D_{\mathrm{dyn}}} = \frac{\left(\frac{3\sqrt{3}GM_{\mathrm{dyn}}}{c^2}\right)}{D_{\mathrm{dyn}}} = 3\sqrt{3}\left(\frac{GM_{\mathrm{dyn}}}{c^2 D_{\mathrm{dyn}}}\right)$$

$$\theta_{\mathrm{sh}} = 3\sqrt{3}\theta_{\mathrm{dyn}}$$

$$\delta = \frac{\left(\frac{\theta_{\mathrm{sh}}}{3\sqrt{3}}\right)}{\theta_{\mathrm{g}}} - 1$$

$$\theta_{\mathrm{sh}} = 3\sqrt{3}(\delta + 1)\theta_{\mathrm{g}}$$

$$\theta_{\mathrm{sh}} = \frac{r_{\mathrm{sh}}}{D} = \frac{r_{\mathrm{sh}}}{\left(\frac{GM}{c^2\theta_{\mathrm{g}}}\right)} = 3\sqrt{3}(\delta + 1)\theta_{\mathrm{g}}$$

$$r_{\mathrm{sh}} = 3\sqrt{3}(\delta+1)GM$$

$$r_{\mathrm{sh}} = \sqrt{\frac{n-1}{n-3}\left[\frac{16\pi \mathcal{G} M_{\mathrm{dyn}}(n-1)}{2(n-2)\Sigma_{(n-2)}}\right]^{\frac{2}{n-3}}}$$

$$\theta_{\mathrm{sh}} = \frac{r_{\mathrm{sh}}}{D_{\mathrm{dyn}}}$$

$$= \left\{\left(\frac{n-1}{n-3}\right)^{\frac{1}{2}}\left[\frac{16\pi(n-1)}{2(n-2)\Sigma_{(n-2)}}\right]^{\frac{1}{n-3}} \cdot \left(\mathcal{G}M_{\mathrm{dyn}}\right)^{\frac{1}{n-3}}\right\} \cdot \frac{1}{D_{\mathrm{dyn}}}$$

$$\mathfrak{C}(n) \equiv \left(\frac{n-1}{n-3}\right)^{\frac{1}{2}}\left[\frac{16\pi(n-1)}{2(n-2)\Sigma_{(n-2)}}\right]^{\frac{1}{n-3}}$$

$$\theta_{\mathrm{sh}} = \mathfrak{C}(n) \cdot \left(\mathcal{G}M_{\mathrm{dyn}}\right)^{\frac{1}{n-3}} \cdot \frac{1}{D_{\mathrm{dyn}}}$$

$$= \mathfrak{C}(n) \cdot \left(\mathcal{G}M_{\mathrm{dyn}}\right)^{\frac{1}{n-3}-1} \cdot \left(\frac{\mathcal{G}M_{\mathrm{dyn}}}{D_{\mathrm{dyn}}}\right)$$

$$= \mathfrak{C}(n) \cdot \left(\mathcal{G}M_{\mathrm{dyn}}\right)^{\frac{1}{n-3}-1} \cdot \theta_{\mathrm{dyn}}$$

$$\delta = \frac{\left[\dfrac{\theta_{\mathrm{sh}}}{\mathfrak{C}(n) \cdot \left(\mathcal{G}M_{\mathrm{dyn}}\right)^{\frac{1}{n-3}-1}}\right]}{\theta_{\mathrm{g}}} - 1$$

$$\theta_{\mathrm{sh}} = \left[\mathfrak{C}(n) \cdot \left(\mathcal{G}M_{\mathrm{dyn}}\right)^{\frac{1}{n-3}-1}\right](\delta+1)\theta_{\mathrm{g}}$$

$$\theta_{\mathrm{sh}} = \frac{r_{\mathrm{sh}}}{D} = \frac{r_{\mathrm{sh}}}{\left(\frac{\mathcal{G}M}{\theta_{\mathrm{g}}}\right)} = \left[\mathfrak{C}(n) \cdot \left(\mathcal{G}M_{\mathrm{dyn}}\right)^{\frac{1}{n-3}-1}\right](\delta+1)\theta_{\mathrm{g}}$$

$$r_{\mathrm{sh}} = \left[\mathfrak{C}(n) \cdot (\mathcal{G})^{\frac{1}{n-3}} \cdot \left(M_{\mathrm{dyn}}\right)^{\frac{1}{n-3}-1}\right](\delta+1)M$$

$$G = \frac{\mathcal{G}}{\mathcal{V}_{(n-4)}}$$

$$\mathcal{G} = G(2\pi\mathcal{R})^{n-4},$$

$$r_{\mathrm{sh}} = \left[\mathfrak{C}(n) \cdot (G)^{\frac{1}{n-3}} \cdot \left(\frac{M_{\mathrm{dyn}}}{2\pi\mathcal{R}}\right)^{\frac{1}{n-3}-1}\right](\delta+1)M$$

$$r_{\mathrm{sh}} = \left[\mathfrak{C}(n) \cdot (G)^{\frac{1}{n-3}}\right](\delta+1)M$$

$$\mathcal{S}^{(0)} = \int \mathrm{d}^n x\sqrt{-g}\left[-\frac{1}{2}(\nabla\Phi)^2 - \frac{1}{2}m^2\Phi^2\right]$$

$$\Phi(x) = \sum_{\ell,\mathbf{m}} \frac{\Psi^{(0)}_{\ell,\mathbf{m}}(t,r)}{r^{(n-2)/2}} Y_{\ell,\mathbf{m}}(\theta)$$

$$\mathcal{S}^{(0)} = \sum_{\ell,\mathbf{m}} \mathcal{S}^{(0)}_{\ell,\mathbf{m}}$$

$$\mathcal{S}^{(0)}_{\ell,\mathbf{m}} = \int \mathrm{d}^2 x\sqrt{-g^{(2)}}\left[-\frac{1}{2}D_a\bar{\Psi}^{(0)}_{\ell,\mathbf{m}}D^a\Psi^{(0)}_{\ell,\mathbf{m}} - \frac{1}{2f_t}V^{(0)}_\ell(r)\left|\Psi^{(0)}_{\ell,\mathbf{m}}\right|^2\right]$$

$$V^{(0)}_\ell(r) \equiv f_t\left\{\frac{\ell(\ell+n-3)}{r^2} + \frac{(n-2)(n-4)}{4r^2}f_r + \frac{n-2}{2r}\left[\frac{(f_t f_r)'}{2f_t}\right] + m^2\right\}.$$

$$\mathrm{d}r_* = \mathrm{d}r/\sqrt{f_t f_r}$$

$$\mathcal{S}^{(0)}_{\ell,\mathbf{m}} = \int \mathrm{d}t\,\mathrm{d}r_*\left[\frac{1}{2}\left|\partial_t\Psi^{(0)}_{\ell,\mathbf{m}}\right|^2 - \frac{1}{2}\left|\partial_{r_*}\Psi^{(0)}_{\ell,\mathbf{m}}\right|^2 - \frac{1}{2}V^{(0)}_\ell(r)\left|\Psi^{(0)}_{\ell,\mathbf{m}}\right|^2\right],$$

$$\left[\partial^2_{r_*} - \partial^2_t - V^{(0)}_\ell(r)\right]\Psi^{(0)}_{\ell,\mathbf{m}}(r_*) = 0$$

$$\mathcal{S}^{(1)} = \int \mathrm{d}^n x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu(x) = \begin{pmatrix} A_a(x) \\ D_A A^{(\mathrm{L})}(x) + A^{(\mathrm{T})}_A(x)\end{pmatrix}.$$

$$\begin{aligned}\delta_\Lambda A_a &= \partial_a\Lambda,\\ \delta_\Lambda A^{(\mathrm{L})} &= \Lambda,\\ \delta_\Lambda A^{(\mathrm{T})}_A &= 0.\end{aligned}$$

$$\mathcal{A}_a = A_a - D_a A^{(\mathrm{L})}$$

$$\begin{aligned}F_{ab} &= D_a\mathcal{A}_b - D_b\mathcal{A}_a\\ F_{aA} &= D_a A^{(\mathrm{T})}_A - D_A\mathcal{A}_a\\ F_{AB} &= D_A A^{(\mathrm{T})}_B - D_B A^{(\mathrm{T})}_A\end{aligned}$$

$$\begin{aligned}\mathcal{A}^a(x) &= \sum_{\ell,\mathbf{m}} \mathcal{A}^a_{\ell,\mathbf{m}}(t,r)\mathcal{Y}_{\ell,\mathbf{m}}(\theta)\\ A^{(\mathrm{T})}_A(x) &= \sum_{\ell,\mathbf{m}} A^{(\mathrm{V})}_{\ell,\mathbf{m}}(t,r)\mathcal{Y}^{(\mathrm{T})}_{A;\ell,\mathbf{m}}(\theta)\end{aligned}$$

$$\mathcal{S}^{(1)}=\sum_{\ell,\mathbf{m}}\left(\mathcal{S}^{(\mathrm{V})}_{\ell,\mathbf{m}}+\mathcal{S}^{(\mathrm{S})}_{\ell,\mathbf{m}}\right)$$

$$\mathcal{S}^{(\mathrm{V})}_{\ell,\mathbf{m}}=\int \mathrm{d}^2x\sqrt{-g^{(2)}}r^{n-4}\left[-\frac{1}{2}D_a\bar{A}^{(\mathrm{V})}_{\ell,\mathbf{m}}D^aA^{(\mathrm{V})}_{\ell,\mathbf{m}}-\frac{1}{2}\frac{(\ell+1)(\ell+n-4)}{r^2}\left|A^{(\mathrm{V})}_{\ell,\mathbf{m}}\right|^2\right]$$

$$\mathcal{S}^{(\mathrm{S})}_{\ell,\mathbf{m}}=\int \mathrm{d}^2x\sqrt{-g^{(2)}}r^{n-2}\left[-\frac{1}{4}\bar{\mathfrak{f}}_{ab;\ell,\mathbf{m}}\mathfrak{f}^{ab}_{\ell,\mathbf{m}}-\frac{1}{2}\frac{\ell(\ell+n-3)}{r^2}\overline{\mathcal{A}}_{a;\ell,\mathbf{m}}\mathcal{A}^a_{\ell,\mathbf{m}}\right]$$

$$\mathfrak{f}^{ab}_{\ell,\mathbf{m}}\equiv D^a\mathcal{A}^b_{\ell,\mathbf{m}}-D^b\mathcal{A}^a_{\ell,\mathbf{m}}$$

$$A^{(\mathrm{V})}_{\ell,\mathbf{m}}(t,r)=\frac{\Psi^{(\mathrm{V})}_{\ell,\mathbf{m}}(t,r)}{r^{(n-4)/2}}$$

$$\mathcal{S}^{(\mathrm{V})}_{\ell,\mathbf{m}}=\int \mathrm{d}^2x\sqrt{-g^{(2)}}\left[-\frac{1}{2}D_a\bar{\Psi}^{(\mathrm{V})}_{\ell,\mathbf{m}}D^a\Psi^{(\mathrm{V})}_{\ell,\mathbf{m}}-\frac{1}{2f_t}V^{(\mathrm{V})}_\ell(r)\left|\Psi^{(\mathrm{V})}_{\ell,\mathbf{m}}\right|^2\right]$$

$$V^{(\mathrm{V})}_\ell(r)\equiv f_t\left\{\frac{(\ell+1)(\ell+n-4)}{r^2}+\frac{(n-4)(n-6)}{4r^2}f_r+\frac{n-4}{2r}\left[\frac{(f_tf_r)'}{2f_t}\right]\right\}$$

$$\mathcal{S}^{(\mathrm{V})}_{\ell,\mathbf{m}}=\int \mathrm{d}t\,\mathrm{d}r_*\left[\frac{1}{2}\left|\partial_t\Psi^{(\mathrm{V})}_{\ell,\mathbf{m}}\right|^2-\frac{1}{2}\left|\partial_{r_*}\Psi^{(\mathrm{V})}_{\ell,\mathbf{m}}\right|^2-\frac{1}{2}V^{(\mathrm{V})}_\ell(r)\left|\Psi^{(\mathrm{V})}_{\ell,\mathbf{m}}\right|^2\right]$$

$$\left[\partial^2_{r_*}-\partial^2_t-V^{(\mathrm{V})}_\ell(r)\right]\Psi^{(\mathrm{V})}_{\ell,\mathbf{m}}(r_*)=0$$

$$\tilde{\mathcal{S}}^{(\mathrm{S})}_{\ell,\mathbf{m}}=\int \mathrm{d}^2x\sqrt{-g^{(2)}}\left[\frac{1}{2}\sqrt{\ell(\ell+n-3)}r^{\frac{(n-4)}{2}}\mathrm{Re}\left\{\bar{\Psi}^{(\mathrm{S})}_{\ell,\mathbf{m}}\varepsilon_{ab}F^{ab}_{\ell,\mathbf{m}}\right\}-\frac{1}{2}\frac{\ell(\ell+n-3)}{r^2}\left(\left|\Psi^{(\mathrm{S})}_{\ell,\mathbf{m}}\right|^2+r^{n-2}\overline{\mathcal{A}}_{a;\ell,\mathbf{m}}\mathcal{A}^a_{\ell,\mathbf{m}}\right)\right].$$

$$\Psi^{(\mathrm{S})}_{\ell,\mathbf{m}}=\frac{1}{2}\frac{r^{\frac{n}{2}}}{\sqrt{\ell(\ell+n-3)}}\varepsilon_{ab}\mathfrak{f}^{ab}_{\ell,\mathbf{m}}$$

$$\mathcal{A}^a_{\ell,\mathbf{m}}=\frac{r^{-\frac{(n-4)}{2}}}{\sqrt{\ell(\ell+n-3)}}\left\{\varepsilon^{ab}D_b-t^a\frac{n-4}{2r}\sqrt{\frac{f_r}{f_t}}\right\}\Psi^{(\mathrm{S})}_{\ell,\mathbf{m}}$$

$$\tilde{\mathcal{S}}^{(\mathrm{S})}_{\ell,\mathbf{m}}=\int \mathrm{d}^2x\sqrt{-g^{(2)}}\left[-\frac{1}{2}D_a\bar{\Psi}^{(\mathrm{S})}_{\ell,\mathbf{m}}D^a\Psi^{(\mathrm{S})}_{\ell,\mathbf{m}}-\frac{1}{2f_t}V^{(\mathrm{S})}_\ell(r)\left|\Psi^{(\mathrm{S})}_{\ell,\mathbf{m}}\right|^2\right]$$
$$=\int \mathrm{d}t\,\mathrm{d}r_*\left[\frac{1}{2}\left|\partial_t\Psi^{(\mathrm{S})}_{\ell,\mathbf{m}}\right|^2-\frac{1}{2}\left|\partial_{r_*}\Psi^{(\mathrm{S})}_{\ell,\mathbf{m}}\right|^2-\frac{1}{2}V^{(\mathrm{S})}_\ell(r)\left|\Psi^{(\mathrm{S})}_{\ell,\mathbf{m}}\right|^2\right]$$

$$\left[\partial^2_{r_*}-\partial^2_t-V^{(\mathrm{S})}_\ell(r)\right]\Psi^{(\mathrm{S})}_{\ell,\mathbf{m}}(r_*)=0,$$

$$V^{(\mathrm{S})}_\ell(r)\equiv f_t\left\{\frac{\ell(\ell+n-3)}{r^2}+\frac{(n-2)(n-4)}{4r^2}f_r-\frac{n-4}{2r}\left[\frac{(f_tf_r)'}{2f_t}\right]\right\}$$

$$\mathbb{A}^{(p)}=\frac{1}{p!}A_{\mu_1\mu_2\cdots\mu_p}\,\mathrm{d}x^{\mu_1}\wedge\,\mathrm{d}x^{\mu_2}\wedge\cdots\wedge\,\mathrm{d}x^{\mu_p}$$

$$\mathcal{S}^{(p)} = -\frac{1}{2}\int \mathbb{F}^{(p+1)} \wedge^{\star} \mathbb{F}^{(p+1)}$$

$$F_{\mu_1\mu_2\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]}$$

$$\mathcal{S}^{(p)} = -\frac{1}{2(p+1)!}\int \mathrm{d}^n x\sqrt{-g} F_{\mu_1\mu_2\cdots\mu_{p+1}} F^{\mu_1\mu_2\cdots\mu_{p+1}}$$

$$\hat{\mathbb{A}}^{(p)} = \frac{1}{p!} A_{A_1A_2\cdots A_p} \mathcal{D}^{A_1A_2\cdots A_p},$$

$$\mathcal{D}^{A_1A_2\cdots A_p} \equiv \mathrm{d}x^{A_1} \wedge \mathrm{d}x^{A_2} \wedge \cdots \wedge \mathrm{d}x^{A_p}$$

$$\begin{aligned}\mathbb{A}^{(p)} =& \frac{1}{2}\frac{1}{(p-2)!} A_{abA_1\cdots A_{p-2}}\, \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathcal{D}^{A_1\cdots A_{p-2}} \\ &+ \frac{1}{(p-1)!} A_{aA_1\cdots A_{p-1}}\, \mathrm{d}x^a \wedge \mathcal{D}^{A_1\cdots A_{p-1}} \\ &+ \frac{1}{p!} A_{A_1\cdots A_p} \mathcal{D}^{A_1\cdots A_p}\end{aligned}$$

$$\begin{aligned}(T_{ab})_{A_1\cdots A_{p-2}} &\equiv A_{abA_1\cdots A_{p-2}} \\ (V_a)_{A_1\cdots A_{p-1}} &\equiv A_{aA_1\cdots A_{p-1}} \\ X_{A_1\cdots A_p} &\equiv A_{A_1\cdots A_p}\end{aligned}$$

$$\mathbb{A}^{(p)} = \frac{1}{2}\,\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \hat{\mathbb{T}}^{(p-2)}_{ab} + \mathrm{d}x^a \wedge \hat{\mathbb{V}}^{(p-1)}_a + \hat{\mathbb{X}}^{(p)}$$

$$\begin{aligned}\hat{\mathbb{A}}^{(p)} &= \hat{\mathrm{d}}\hat{\mathbb{A}}^{(p-1)} + \hat{\mathcal{A}}^{(p)} \\ &= \hat{\mathrm{d}}\left(\hat{\mathrm{d}}\hat{\mathbb{A}}^{(p-2)} + \hat{\mathcal{A}}^{(p-1)}\right) + \hat{\mathcal{A}}^{(p)} \\ &= \hat{\mathrm{d}}\hat{\mathcal{A}}^{(p-1)} + \hat{\mathcal{A}}^{(p)},\end{aligned}$$

$$D^{A_1}\mathcal{A}_{A_1A_2\cdots A_{p+1}} = 0$$

$$\begin{aligned}\mathbb{A}^{(p)} =& \frac{1}{2}\,\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \left(\hat{\mathrm{d}}\hat{\mathcal{T}}^{(p-3)}_{ab} + \hat{\mathcal{T}}^{(p-2)}_{ab}\right) \\ &+ \mathrm{d}x^a \wedge \left(\hat{\mathrm{d}}\hat{\mathcal{V}}^{(p-2)}_a + \hat{\mathcal{V}}^{(p-1)}_a\right) \\ &+ \left(\hat{\mathrm{d}}\hat{\mathcal{X}}^{(p-1)} + \hat{\mathcal{X}}^{(p)}\right),\end{aligned}$$

$$\begin{aligned}\mathrm{d}\Lambda^{(p-1)} =& \delta_\Lambda \mathbb{A}^{(p)} \\ =& \frac{1}{2}\,\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \left(\hat{\mathrm{d}}\hat{\Lambda}^{(p-3)}_{ab} + 2D_{[a}\hat{\Lambda}^{(p-2)}_{b]}\right) \\ &+ \mathrm{d}x^a \wedge \left(-\hat{\mathrm{d}}\hat{\Lambda}^{(p-2)}_a + D_a\hat{\Lambda}^{(p-1)}\right) \\ &+ \hat{\mathrm{d}}\hat{\Lambda}^{(p-1)}\end{aligned}$$

$$\begin{aligned}&\delta_\Lambda\hat{\mathcal{T}}^{(p-3)}_{ab} = \hat{\Lambda}^{(p-3)}_{ab}, \quad && \delta_\Lambda\hat{\mathcal{T}}^{(p-2)}_{ab} = 2D_{[a}\hat{\Lambda}^{(p-2)}_{b]} \\ &\delta_\Lambda\hat{\mathcal{V}}^{(p-2)}_a = -\hat{\Lambda}^{(p-2)}_a, && \delta_\Lambda\hat{\mathcal{V}}^{(p-1)}_a = D_a\hat{\Lambda}^{(p-1)} \\ &\delta_\Lambda\hat{\mathcal{X}}^{(p-1)} = \hat{\Lambda}^{(p-1)}, && \delta_\Lambda\hat{\mathcal{X}}^{(p)} = 0\end{aligned}$$

$$\hat{\mathcal{H}}_{ab}^{(p-2)} = \hat{\mathcal{F}}_{ab}^{(p-2)} + 2D_{[a}\hat{\mathcal{V}}_{b]}^{(p-2)}$$
$$\hat{\mathcal{A}}_{a}^{(p-1)} = \hat{\mathcal{V}}_{a}^{(p-1)} - D_a\hat{\mathcal{X}}^{(p-1)}$$

$$\begin{aligned}\mathbb{F}^{(p+1)} =& \frac{1}{2}\,\mathrm{d}x^a \wedge\, \mathrm{d}x^b \wedge \left(2D_{[a}\hat{\mathcal{A}}_{b]}^{(p-1)} + (p-2)!\,\hat{\mathrm{d}}\hat{\mathcal{H}}_{ab}^{(p-2)}\right)\\ &+\mathrm{d}x^a \wedge \left(D_a\hat{\mathcal{X}}^{(p)} - \hat{\mathrm{d}}\hat{\mathcal{A}}_{a}^{(p-1)}\right)\\ &+\hat{\mathrm{d}}\hat{\mathcal{X}}^{(p)}\end{aligned}$$

$$\left(\mathcal{H}^{ab}\right)^{A_1\cdots A_{p-2}}(x) = \sum_{\ell,\mathbf{m}} \mathcal{H}_{\ell,\mathbf{m}}^{ab}(t,r)\mathcal{Y}_{\ell,\mathbf{m}}^{(\mathrm{T}))A_1\cdots A_{p-2}}(\theta)$$
$$\left(\mathcal{A}^{a}\right)^{A_1\cdots A_{p-1}}(x) = \sum_{\ell,\mathbf{m}} \mathcal{A}_{\ell,\mathbf{m}}^{a}(t,r)\mathcal{Y}_{\ell,\mathbf{m}}^{(\mathrm{T}))A_1\cdots A_{p-1}}(\theta)$$
$$\mathcal{X}^{A_1\cdots A_{p}}(x) = \sum_{\ell,\mathbf{m}} \mathcal{X}_{\ell,\mathbf{m}}(t,r)\mathcal{Y}_{\ell,\mathbf{m}}^{(\mathrm{T}))A_1\cdots A_{p}}(\theta)$$

$$\hat{\mathcal{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)} = \frac{1}{p!}\mathcal{Y}_{A_1\cdots A_p;\ell,\mathbf{m}}^{(\mathrm{T})}\mathcal{D}^{A_1\cdots A_p},$$

$$\hat{\Delta}\hat{\mathcal{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)} = \lambda_p\hat{\mathcal{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)}$$
$$\hat{\delta}_{\Lambda}\hat{\mathcal{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)} = 0$$

$$\hat{\Delta} \equiv \hat{\delta}\hat{\mathrm{d}} + \hat{\mathrm{d}}\hat{\delta}$$

$$\lambda_p \equiv (\ell + p)(\ell + n - p - 3)$$

$$\hat{\delta}\hat{\mathrm{d}}\hat{\mathcal{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)} = \lambda_p\hat{\mathcal{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)}$$

$$\tilde{\Delta}\mathcal{Y}_{A_1\cdots A_p;\ell,\mathbf{m}}^{(\mathrm{T})} = -\gamma_p\mathcal{Y}_{A_1\cdots A_p;\ell,\mathbf{m}}^{(\mathrm{T})}$$

$$\tilde{\Delta}\mathcal{Y}_{A_1\cdots A_p;\ell,\mathbf{m}}^{(\mathrm{T})} = \mathcal{Y}^{\star}{}_{A_1\cdots A_p;\ell,\mathbf{m}} :: \blacksquare^{\eta}$$

$$\gamma_p = \ell(\ell + n - 3) - p$$

$$\mathcal{S}^{(p)} = \sum_{\ell,\mathbf{m}} \left(\mathcal{S}_{\ell,\mathbf{m}}^{(p)} + \mathcal{S}_{\ell,\mathbf{m}}^{(p-1)} + \mathcal{S}_{\ell,\mathbf{m}}^{(p-2)}\right),$$
$$\mathcal{S}_{\ell,\mathbf{m}}^{(p)} = \int \mathrm{d}^2x\sqrt{-g^{(2)}}r^{n-2p-2}\left[-\frac{1}{2p!}D_a\overline{\mathcal{X}}_{\ell,\mathbf{m}}D^a\mathcal{X}_{\ell,\mathbf{m}} - \frac{1}{2p!}\frac{(\ell+p)(\ell+n-p-3)}{r^2}\left|\mathcal{X}_{\ell,\mathbf{m}}\right|^2\right],$$
$$\begin{aligned}\mathcal{S}_{\ell,\mathbf{m}}^{(p-1)} =& \int \mathrm{d}^2x\sqrt{-g^{(2)}}r^{n-2p}\left[-\frac{1}{4(p-1)!}\overline{\mathfrak{F}}_{ab;\ell,\mathbf{m}}\mathfrak{F}_{\ell,\mathbf{m}}^{ab}\right.\\ &\left.-\frac{1}{2(p-1)!}\frac{(\ell+p-1)(\ell+n-p-2)}{r^2}\overline{\mathcal{A}}_{a;\ell,\mathbf{m}}\mathcal{A}_{\ell,\mathbf{m}}^{a}\right],\end{aligned}$$
$$\mathcal{S}_{\ell,\mathbf{m}}^{(p-2)} = \int \mathrm{d}^2x\sqrt{-g^{(2)}}r^{n-2p+2}\left[-\frac{(p-2)!}{4}\frac{(\ell+p-2)(\ell+n-p-1)}{r^2}\overline{\mathcal{H}}_{ab;\ell,\mathbf{m}}\mathcal{H}_{\ell,\mathbf{m}}^{ab}\right],$$

$$\mathfrak{F}_{\ell,\mathbf{m}}^{ab} \equiv D^a\mathcal{A}_{\ell,\mathbf{m}}^{b} - D^b\mathcal{A}_{\ell,\mathbf{m}}^{a}.$$

$$\mathcal{X}_{\ell,\mathbf{m}}(t,r)=\sqrt{p!}\,\frac{\Psi^{(p)}_{\ell,\mathbf{m}}(t,r)}{r^{(n-2p-2)/2}},$$

$$\mathcal{S}^{(p)}_{\ell,\mathbf{m}}=\int \mathrm{d}^2x\sqrt{-g^{(2)}}\left[-\frac{1}{2}D_a\bar{\Psi}^{(p)}_{\ell,\mathbf{m}}D^a\Psi^{(p)}_{\ell,\mathbf{m}}-\frac{1}{2f_t}V^{(p)}_{\ell}(r)\left|\Psi^{(p)}_{\ell,\mathbf{m}}\right|^2\right]$$

$$V^{(p)}_{\ell}(r)\equiv f_t\left\{\frac{(\ell+p)(\ell+n-p-3)}{r^2}+\frac{(n-2p-2)(n-2p-4)}{4r^2}f_r+\frac{n-2p-2}{2r}\left[\frac{(f_tf_r)'}{2f_t}\right]\right\}.$$

$$\tilde{\mathcal{S}}^{(p-1)}_{\ell,\mathbf{m}}=\int \mathrm{d}^2x\sqrt{-g^{(2)}}\left[\frac{1}{2}\sqrt{\frac{(\ell+p-1)(\ell+n-p-2)}{(p-1)!}}r^{\frac{(n-2p-2)}{2}}\mathrm{Re}\left\{\bar{\Psi}^{(\tilde{p})}_{\ell,\mathbf{m}}\varepsilon_{ab}\mathfrak{V}^{ab}_{\ell,\mathbf{m}}\right\}-\frac{1}{2}\frac{(\ell+p-1)(\ell+n-p-2)}{r^2}\left(\left|\Psi^{(\tilde{p})}_{\ell,\mathbf{m}}\right|^2+\frac{r^{n-2p}}{(p-1)!}\overline{\mathcal{A}}_{a;\ell,\mathbf{m}}\mathcal{A}^a_{\ell,\mathbf{m}}\right)\right],$$

$$\Psi^{(\tilde{p})}_{\ell,\mathbf{m}}=\frac{1}{2}\frac{r^{\frac{(n-2p+2)}{2}}}{\sqrt{(p-1)!\,(\ell+p-1)(\ell+n-p-2)}}\varepsilon_{ab}\mathfrak{V}^{ab}_{\ell,\mathbf{m}}$$

$$\mathcal{A}^a_{\ell,\mathbf{m}}=\sqrt{\frac{(p-1)!}{(\ell+p-1)(\ell+n-p-2)}}r^{-\frac{(n-2p-2)}{2}}\left\{\varepsilon^{ab}D_b-t^a\frac{n-2p-2}{2r}\sqrt{\frac{f_r}{f_t}}\right\}\Psi^{(\tilde{p})}_{\ell,\mathbf{m}}$$

$$\tilde{\mathcal{S}}^{(p-1)}_{\ell,\mathbf{m}}=\int \mathrm{d}^2x\sqrt{-g^{(2)}}\left[-\frac{1}{2}D_a\bar{\Psi}^{(\tilde{p})}_{\ell,\mathbf{m}}D^a\Psi^{(\tilde{p})}_{\ell,\mathbf{m}}-\frac{1}{2f_t}V^{(\tilde{p})}_{\ell}(r)\left|\Psi^{(\tilde{p})}_{\ell,\mathbf{m}}\right|^2\right]$$

$$V^{(\tilde{p})}_{\ell}(r)\equiv f_t\left\{\frac{(\ell+p-1)(\ell+n-p-2)}{r^2}+\frac{(n-2p)(n-2p-2)}{4r^2}f_r-\frac{(n-2p-2)}{2r}\left[\frac{(f_tf_r)'}{2f_t}\right]\right\}.$$

$$V^{(\varkappa)}_{\ell}(r)\equiv f_t\left\{\frac{(\ell+\varkappa)(\ell+n-\varkappa-3)}{r^2}+\frac{(n-2\varkappa-2)(n-2\varkappa-4)}{4r^2}f_r+\frac{n-2\varkappa-2}{2r}\left[\frac{(f_tf_r)'}{2f_t}\right]\right\}.$$

$$V^{(\varkappa)}_{\ell}(r)\Rightarrow\begin{cases}V^{(0)}_{\ell}(r), & \varkappa=0;\\ V^{(\mathrm{V})}_{\ell}(r), & \varkappa=1;\\ V^{(\mathrm{S})}_{\ell}(r), & \varkappa=n-3;\\ V^{(p)}_{\ell}(r), & \varkappa=p;\\ V^{(\tilde{p})}_{\ell}(r), & \varkappa=\tilde{p}=n-p-2.\end{cases}$$

$$h_{ab}(x)=H_{ab}(x)$$
$$h_{aA}(x)=D_AH^{(\mathrm{S})}_a(x)+h^{(\mathrm{V})}_{aA}(x)$$
$$h_{AB}(x)=r^2\left[K(x)\Omega_{AB}+D_{\langle A}D_{B\rangle}G(x)+D_{(A}h^{(\mathrm{V})}_{B)}(x)+h^{(\mathrm{TT})}_{AB}(x)\right]$$

$$H_{ab}(x), H_a^{(\mathrm{L})}(x), K(x), G(x).$$

$$h_{aA}^{(\mathrm{V})}(x), h_A^{(\mathrm{V})}(x);$$

$$D^A h_{aA}^{(\mathrm{V})}(x) = 0$$
$$D^A h_A^{(\mathrm{V})}(x) = 0$$

$$h_{AB}^{(\mathrm{TT})}(x);$$

$$D^A h_{AB}^{(\mathrm{TT})}(x) = 0,$$
$$\Omega^{AB} h_{AB}^{(\mathrm{TT})}(x) = 0.$$

$$\delta_\xi h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$$\xi_a(x) = D_\alpha \xi^{(\mathrm{S})}(x) + \xi_\alpha^{(\mathrm{V})}(x)$$
$$\xi_A(x) = D_A \xi^{(\mathrm{S})}(x) + \xi_A^{(\mathrm{V})}(x)$$

$$\delta_\xi H_{ab} = D_a \xi_b + D_b \xi_a$$
$$\delta_\xi H_a^{(\mathrm{S})} = \xi_a + D_a \xi^{(\mathrm{S})} - \frac{2}{r} r_a \xi^{(\mathrm{S})}$$
$$\delta_\xi K = \frac{2}{r} r^a \xi_a + \frac{2}{n-2} \frac{1}{r^2} D_A D^A \xi^{(\mathrm{S})}$$
$$\delta_\xi G = \frac{2}{r^2} \xi^{(\mathrm{S})}$$
$$\delta_\xi h_{aA}^{(\mathrm{V})} = D_a \xi_A^{(\mathrm{V})} - \frac{2}{r} r_a \xi_A^{(\mathrm{V})}$$
$$\delta_\xi h_A^{(\mathrm{V})} = \frac{2}{r^2} \xi_A^{(\mathrm{V})}$$
$$\delta_\xi h_{AB}^{(\mathrm{TT})} = 0$$

$$h_{AB}^{(\mathrm{TT})} \otimes \oint \mathfrak{d}^4 \mathfrak{S}_{gravity}$$

$$\mathcal{H}_{aA}^{(\mathrm{V})} = h_{aA}^{(\mathrm{V})} - \frac{1}{2} r^2 D_a h_A^{(\mathrm{V})}$$

$$\mathcal{H}_{ab} = H_{ab} - 2D_{(a} H_{b)}^{(\mathrm{S})} + D_{(a}\left(r^2 D_{b)} G\right)$$
$$\mathcal{K} = K - \frac{1}{n-2} D_A D^A G + r r^a D_a G - \frac{2}{r} r^a H_a^{(\mathrm{S})}$$

$$\mathcal{H}_{ab}(x) = \sum_{\ell,\mathbf{m}} \mathcal{H}_{ab;\ell,\mathbf{m}}(t,r) \mathcal{Y}_{\ell,\mathbf{m}}(\theta)$$
$$\mathcal{K}(x) = \sum_{\ell,\mathbf{m}} \mathcal{K}_{\ell,\mathbf{m}}(t,r) \mathcal{Y}_{\ell,\mathbf{m}}(\theta)$$
$$\mathcal{H}_{aA}^{(\mathrm{V})}(x) = \sum_{\ell,\mathbf{m}} \mathcal{H}_{a;\ell,\mathbf{m}}(t,r) \mathcal{Y}_{A;\ell,\mathbf{m}}^{(\mathrm{T})}(\theta)$$
$$h_{AB}^{(\mathrm{TT})}(x) = \sum_{\ell,\mathbf{m}} h_{\ell,\mathbf{m}}^{(\mathrm{T})}(t,r) \mathcal{Y}_{AB;\ell,\mathbf{m}}^{(\mathrm{TT})}(\theta)$$

$$\mathcal{S}^{(\mathrm{gr})} = \int \mathrm{d}^n x\sqrt{-g}\left[-\frac{1}{2}\nabla_\rho h_{\mu\nu}\nabla^\rho h^{\mu\nu} + \nabla_\rho h_{\mu\nu}\nabla^\nu h^{\mu\rho} - \nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{2}\nabla_\mu h\nabla^\mu h\right]$$

$$g_{\mu\nu}^{\mathrm{full}} = g_{\mu\nu} + \sqrt{32\pi\mathcal{G}}h_{\mu\nu}$$

$$\begin{aligned}
\mathcal{S}^{(\mathrm{gr})} =& \sum_{\ell,\mathbf{m}}\left(\mathcal{S}^{(\mathrm{T})}_{\ell,\mathbf{m}} + \mathcal{S}^{(\mathrm{RW})}_{\ell,\mathbf{m}} + \mathcal{S}^{(\mathrm{Z})}_{\ell,\mathbf{m}}\right)\\
\mathcal{S}^{(\mathrm{T})}_{\ell,\mathbf{m}} =& \int \mathrm{d}^2x\sqrt{-g^{(2)}}r^{n-2}\left[-\frac{1}{2}D_a\bar{h}^{(\mathrm{T})}_{\ell,\mathbf{m}}D^a h^{(\mathrm{T})}_{\ell,\mathbf{m}} - \frac{1}{r}r^a D_a\left|h^{(\mathrm{T})}_{\ell,\mathbf{m}}\right|^2\right.\\
&\left.-\frac{1}{2}\frac{\ell(\ell+n-3)+2(n-3)}{r^2}\left|h^{(\mathrm{T})}_{\ell,\mathbf{m}}\right|^2\right]\\
\mathcal{S}^{(\mathrm{RW})}_{\ell,\mathbf{m}} =& \int \mathrm{d}^2x\sqrt{-g^{(2)}}2r^{n-4}\left[-\frac{1}{4}\overline{\mathcal{F}}_{ab;\ell,\mathbf{m}}\mathcal{F}^{ab}_{\ell,\mathbf{m}} - \frac{2}{r}r_a\mathrm{Re}\left\{\overline{\mathcal{H}}^b_{\ell,\mathbf{m}}D_b\mathcal{H}^a_{\ell,\mathbf{m}}\right\}\right.\\
&\left.-\frac{1}{2}\left(\frac{(\ell+1)(\ell+n-4)}{r^2}\overline{\mathcal{H}}_{a;\ell,\mathbf{m}}\mathcal{H}^a_{\ell,\mathbf{m}} - 4\left|r_a\mathcal{H}^a_{\ell,\mathbf{m}}\right|^2\right)\right]\\
\mathcal{S}^{(\mathrm{Z})}_{\ell,\mathbf{m}} =& \int \mathrm{d}^2x\sqrt{-g^{(2)}}r^{n-2}\left[-\frac{1}{2}D_c\overline{\mathcal{H}}_{ab;\ell,\mathbf{m}}D^c\mathcal{H}^{ab}_{\ell,\mathbf{m}} + D_c\overline{\mathcal{H}}_{ab;\ell,\mathbf{m}}D^b\mathcal{H}^{ac}_{\ell,\mathbf{m}}\right.\\
&-\mathrm{Re}\left\{D_a\overline{\mathcal{H}}_{\ell,\mathbf{m}}D_b\mathcal{H}^{ab}_{\ell,\mathbf{m}}\right\}\\
&+\frac{1}{2}D_a\overline{\mathcal{H}}_{\ell,\mathbf{m}}D^a\mathcal{H}_{\ell,\mathbf{m}} + \frac{(n-2)(n-3)}{2}D_a\overline{\mathcal{K}}_{\ell,\mathbf{m}}D^a\mathcal{K}_{\ell,\mathbf{m}}\\
&-(n-2)\mathrm{Re}\left\{D_a\overline{\mathcal{K}}_{\ell,\mathbf{m}}\left(D_b\mathcal{H}^{ab}_{\ell,\mathbf{m}} - D^a\mathcal{H}_{\ell,\mathbf{m}}\right)\right\}\\
&-\frac{n-2}{r}\mathrm{Re}\left\{\left(D_a\overline{\mathcal{H}}_{\ell,\mathbf{m}} + (n-4)D_a\overline{\mathcal{K}}_{\ell,\mathbf{m}}\right)\left(r_b\mathcal{H}^{ab}_{\ell,\mathbf{m}} - r^a\mathcal{K}_{\ell,\mathbf{m}}\right)\right\}\\
&-\frac{1}{2}\frac{\ell(\ell+n-3)}{r^2}\left(\overline{\mathcal{H}}_{ab;\ell,\mathbf{m}}\mathcal{H}^{ab}_{\ell,\mathbf{m}} - \left|\mathcal{H}_{\ell,\mathbf{m}}\right|^2 - (n-3)(n-4)\left|\mathcal{K}_{\ell,\mathbf{m}}\right|^2\right.\\
&\left.\left.-2(n-3)\mathrm{Re}\left\{\overline{\mathcal{H}}_{\ell,\mathbf{m}}\mathcal{K}_{\ell,\mathbf{m}}\right\}\right)\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}^{ab}_{\ell,\mathbf{m}} &\equiv D^a\mathcal{H}^b_{\ell,\mathbf{m}} - D^b\mathcal{H}^a_{\ell,\mathbf{m}}\\
\mathcal{H}_{\ell,\mathbf{m}} &\equiv g_{ab}\mathcal{H}^{ab}_{\ell,\mathbf{m}}
\end{aligned}$$

$$h^{(\mathrm{T})}_{\ell,\mathbf{m}} = \frac{\Psi^{(\mathrm{T})}_{\ell,\mathbf{m}}}{r^{(n-2)/2}}$$

$$\begin{aligned}
\mathcal{S}^{(\mathrm{T})}_{\ell,\mathbf{m}} &= \int \mathrm{d}^2x\sqrt{-g^{(2)}}\left[-\frac{1}{2}D_a\bar{\Psi}^{(\mathrm{T})}_{\ell,\mathbf{m}}D^a\Psi^{(\mathrm{T})}_{\ell,\mathbf{m}} - \frac{1}{2f_t}V^{(\mathrm{T})}_\ell(r)\left|\Psi^{(\mathrm{T})}_{\ell,\mathbf{m}}\right|^2\right]\\
&= \int \mathrm{d}t\,\mathrm{d}r_*\left[\frac{1}{2}\left|\partial_t\Psi^{(\mathrm{T})}_{\ell,\mathbf{m}}\right|^2 - \frac{1}{2}\left|\partial_{r_*}\Psi^{(\mathrm{T})}_{\ell,\mathbf{m}}\right|^2 - \frac{1}{2}V^{(\mathrm{T})}_\ell(r)\left|\Psi^{(\mathrm{T})}_{\ell,\mathbf{m}}\right|^2\right]
\end{aligned}$$

$$V^{(\mathrm{T})}_\ell(r) \equiv f_t\left\{\frac{\ell(\ell+n-3)+2(n-3)}{r^2} + \frac{n^2-14n+32}{4r^2}f_r + \frac{n-6}{2r}\left[\frac{(f_tf_r)'}{2f_t}\right]\right\}$$

$$\begin{aligned}
\tilde{\mathcal{S}}^{(\mathrm{RW})}_{\ell,\mathbf{m}} =& \int \mathrm{d}^2x\sqrt{-g^{(2)}}\left[\sqrt{\frac{\mathbb{F}_\ell(r)}{2}}r^{\frac{(n-6)}{2}}\mathrm{Re}\left\{\bar{\Psi}^{(\mathrm{RW})}_{\ell,\mathbf{m}}\left(\varepsilon_{ab}\mathcal{F}^{ab}_{\ell,\mathbf{m}} - \frac{4}{r}\sqrt{\frac{f_r}{f_t}}t_a\mathcal{H}^a_{\ell,\mathbf{m}}\right)\right\}\right.\\
&\left.-\frac{1}{2}\frac{\mathbb{F}_\ell(r)}{r^2}\left(\left|\Psi^{(\mathrm{RW})}_{\ell,\mathbf{m}}\right|^2 + 2r^{n-4}\overline{\mathcal{H}}_{a;\ell,\mathbf{m}}\mathcal{H}^a_{\ell,\mathbf{m}}\right)\right]
\end{aligned}$$

$$\mathbb{F}_\ell(r) \equiv (\ell+1)(\ell+n-4) - 2(n-3)r_ar^a - 2rD_ar^a$$

$$\Psi^{(\mathrm{RW})}_{\ell,\mathbf{m}} = \frac{r^{\frac{(n-2)}{2}}}{\sqrt{2\mathbb{F}_\ell(r)}}\left(\varepsilon_{ab}\mathcal{F}^{ab}_{\ell,\mathbf{m}} - \frac{4}{r}\sqrt{\frac{f_r}{f_t}}t_a\mathcal{H}^a_{\ell,\mathbf{m}}\right)$$

$$\mathcal{H}^a_{\ell,\mathbf{m}} = \frac{r^{-\frac{(n-6)}{2}}}{\sqrt{2\mathbb{F}_\ell(r)}}\left\{\varepsilon^{ab}D_b - t^a\left[\frac{n-2}{2r} + \frac{\mathbb{F}'_\ell(r)}{2\mathbb{F}_\ell(r)}\right]\sqrt{\frac{f_r}{f_t}}\right\}\Psi^{(\mathrm{RW})}_{\ell,\mathbf{m}}$$

$$\begin{aligned}\tilde{\mathcal{S}}^{(\mathrm{RW})}_{\ell,\mathbf{m}} &= \int \mathrm{d}^2x\sqrt{-g^{(2)}}\left[-\frac{1}{2}D_a\bar{\Psi}^{(\mathrm{RW})}_{\ell,\mathbf{m}}D^a\Psi^{(\mathrm{RW})}_{\ell,\mathbf{m}} - \frac{1}{2f_t}V^{(\mathrm{RW})}_\ell(r)\left|\Psi^{(\mathrm{RW})}_{\ell,\mathbf{m}}\right|^2\right]\\ &= \int \mathrm{d}t\,\mathrm{d}r_*\left[\frac{1}{2}\left|\partial_t\Psi^{(\mathrm{RW})}_{\ell,\mathbf{m}}\right|^2 - \frac{1}{2}\left|\partial_{r_*}\Psi^{(\mathrm{RW})}_{\ell,\mathbf{m}}\right|^2 - \frac{1}{2}V^{(\mathrm{RW})}_\ell(r)\left|\Psi^{(\mathrm{RW})}_{\ell,\mathbf{m}}\right|^2\right]\end{aligned}$$

$$\begin{aligned}V^{(\mathrm{RW})}_\ell(r) \equiv & f_t\left\{\frac{(\ell+1)(\ell+n-4)}{r^2} + \frac{(n-4)(n-6)}{4r^2}f_r - \frac{n+2}{2r}\left[\frac{(f_tf_r)'}{2f_t}\right]\right.\\ &\left. + \left[\frac{\mathbb{F}'_\ell}{2\mathbb{F}_\ell}\left(\frac{n-4}{r} + \frac{\mathbb{F}'_\ell}{2\mathbb{F}_\ell}\right) - \left(\frac{\mathbb{F}'_\ell}{2\mathbb{F}_\ell}\right)'\right]f_r - \frac{\mathbb{F}'_\ell}{2\mathbb{F}_\ell}\left[\frac{(f_tf_r)'}{2f_t}\right]\right\}\end{aligned}$$

$$\mathbb{F}_\ell(r) \equiv (\ell+1)(\ell+n-4) - 2(n-3)f_r - 2r\left[\frac{(f_tf_r)'}{2f_t}\right]$$

$$\begin{aligned}V^{(\mathrm{Z})}_\ell(r) \equiv & \left\{\frac{\ell(\ell+n-3)}{r^2} + \frac{(n-2)(n-4)}{4r^2}\mathrm{f}(r) + \frac{n-2}{2r}\mathrm{f}'(r)\right\}\mathrm{f}(r)\\ & - \left\{\frac{2\mathrm{f}'(r)}{r}\frac{[2\lambda_\ell + (n-1)(n-2)][\mathrm{H}_\ell(r) + 2(n-3)\lambda_\ell]}{\mathrm{H}^2_\ell(r)}\right\}\mathrm{f}(r)\end{aligned}$$

$$\mathrm{H}_\ell(r) = 2\ell(\ell+n-3) - 2(n-2)\mathrm{f}(r) + (n-2)r\mathrm{f}'(r)$$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + [\omega^2 - V(x)]\Psi(x) = 0$$

$$\Psi(x \to \pm\infty) \propto e^{\pm i\omega x}$$

$$\begin{aligned}\omega^2 = & V_0 + A_2(\mathcal{K}^2) + A_4(\mathcal{K}^2) + A_6(\mathcal{K}^2) + \cdots\\ & -i\mathcal{K}\sqrt{-2V_2}[1 + A_3(\mathcal{K}^2) + A_5(\mathcal{K}^2) + A_7(\mathcal{K}^2) + \cdots],\end{aligned}$$

$$\mathcal{K} = \begin{cases}+\mathrm{n} + \frac{1}{2}, \mathrm{Re}(\omega) > 0\\ -\mathrm{n} - \frac{1}{2}, \mathrm{Re}(\omega) < 0\end{cases}$$

$$\begin{aligned}P_k(\epsilon) = & V_0 + A_2(\mathcal{K}^2)\epsilon^2 + A_4(\mathcal{K}^2)\epsilon^4 + A_6(\mathcal{K}^2)\epsilon^6 + \cdots\\ & -i\mathcal{K}\sqrt{-2V_2}[\epsilon + A_3(\mathcal{K}^2)\epsilon^3 + A_5(\mathcal{K}^2)\epsilon^5 + A_7(\mathcal{K}^2)\epsilon^7 + \cdots],\end{aligned}$$

$$\omega^2 = P_k(1)$$

$$P_{\tilde{n}/\tilde{m}}(\epsilon) = \frac{Q_0 + Q_1\epsilon + \cdots + Q_{\tilde{n}}\epsilon^{\tilde{n}}}{R_0 + R_1\epsilon + \cdots + R_{\tilde{m}}\epsilon^{\tilde{m}}}$$

$$P_{\tilde{n}/\tilde{m}}(\epsilon) - P_k(\epsilon) = \mathcal{O}\left(\epsilon^{k+1}\right)$$

$$\chi'' = \lambda_0(x)\chi' + s_0(x)\chi$$

$$\chi''' = \lambda_1(x)\chi' + s_1(x)\chi,$$

$$\chi^{(n+2)} = \lambda_n(x)\chi' + s_n(x)\chi$$

$$\begin{aligned}\lambda_n(x) &= \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x)\\ s_n(x) &= s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x)\end{aligned}$$

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = \mathfrak{Bh}$$

$$\lambda_n(x') = \sum_{i=0}^{\infty} c_n^i (x - x')^i$$

$$s_n(x') = \sum_{i=0}^{\infty} d_n^i (x - x')^i$$

$$c_n^i = \sum_{k=0}^{i} c_0^k c_{n-1}^{i-k} + (i+1)c_{n-1}^{i+1} + d_{n-1}^i$$

$$d_n^i = \sum_{k=0}^{i} d_0^k c_{n-1}^{i-k} + (i+1)d_{n-1}^{i+1}$$

$$d_n^0 c_{n-1}^0 - c_n^0 d_{n-1}^0 = 0$$

$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} + V(t,x)\Psi = 0$$

$$\left[4\frac{\partial^2}{\partial u \partial v} + V(u,v)\right]\Psi(u,v) = 0$$

$$\Psi(u = u_0, v) = \exp\left[-\frac{(v - v_c)^2}{2\sigma^2}\right], \Psi(u, v = v_0) = \Psi(u = u_0, 0)$$

$$\Psi(N) = \Psi(W) + \Psi(E) - \Psi(S) - \frac{\Delta^2}{8}[\Psi(W) + \Psi(E)]V(S) + \mathcal{O}(\Delta^4)$$

$$N = (u + \Delta, v + \Delta), W = (u + \Delta, v), E = (u, v + \Delta) \text{ and } S = (u, v)$$

$$\Psi\left(t = \frac{v+u}{2}, x = \frac{v-u}{2}\right)$$

$$\{t_0, t_0 + \Delta, t_0 + 2\Delta, \cdots, t_0 + \infty\Delta\}$$

$$\hat{g} := -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\,\theta d\phi^2)$$

$$\tau = \frac{t}{4M} \in \mathbb{R}, x = \log\left(\frac{r}{M} - 1\right) \in \mathbb{R}, \hat{g} = 16M^2(1 + e^{-x})^{-2} g$$

$$g = -dt^2 + \cosh^4\left(\frac{x}{2}\right)(dx^2 + d\theta^2 + \sin^2\,\theta d\varphi^2)$$

$$A(t,x) = \varphi(t,x)\omega(\tau_1,\tau_2) + \tau_3\cos\,\theta d\varphi$$

$$\omega(\tau_1,\tau_2) = \tau_1 d\theta + \tau_2\sin\,\theta d\varphi, \varphi = \varphi(t,x)$$

$$\mathcal{L}[x,\varphi,\partial_x\varphi,\partial_t\varphi] = -\frac{1}{2}\cosh^2\left(\frac{x}{2}\right)(\partial_t\varphi)^2 + \frac{1}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)\left((\partial_x\varphi)^2 + \frac{1}{2}(1-\varphi^2)^2\right)$$

$$\partial_t^2\varphi - Q\partial_x(Q\partial_x\varphi) + Q^2(\varphi^2-1)\varphi = 0$$

$$\varphi(t,x) \mapsto \varphi\left(\frac{3}{2}t, x\right)$$

$$Q(x) := \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)$$

$$\mathcal{E}[\varphi,\varphi_t] = \int\left(\frac{1}{2}Q^{-1}(\partial_t\varphi)^2 + \frac{1}{2}Q\left((\partial_x\varphi)^2 + \frac{1}{2}(1-\varphi^2)^2\right)\right)dx$$

$$Q^{-1}\partial_t(\partial_t\varphi)^2 + Q\partial_t\left((\partial_x\varphi)^2 + \frac{1}{2}(1-\varphi^2)^2\right) + \partial_x(Q\partial_t\varphi\partial_x\varphi) = 0$$

$$\mathcal{P}[\varphi,\partial_t\varphi] = \int Q^{-1}\partial_t\varphi\partial_x\varphi dx$$

$$H'' - \tanh\left(\frac{x}{2}\right)H' + H(1-H^2) = 0, x \in \mathbb{R}$$

$$H(x) = \tanh\left(\frac{x}{2}\right)$$

$$\alpha(x) = \frac{1}{3}(\sinh\,x + x),$$

$$\tilde{Q}(x) = Q(\alpha^{-1}(x)), \tilde{H}(x) = H(\alpha^{-1}(x)),$$

$$0 \le \tilde{Q}(x) \lesssim \frac{1}{|x|}, |\tilde{H}(x) \mp 1| \lesssim \frac{1}{|x|}, \text{ as } x \to \pm\infty.$$

$$\partial_t^2\phi - \partial_x^2\phi - \tilde{Q}^2(\phi - \phi^3) = 0$$

$$\begin{cases}\partial_t\phi_1 = \phi_2 \\ \partial_t\phi_2 = \partial_x^2\phi_1 + \tilde{Q}^2(x)(1-\phi_1^2)\phi_1\end{cases}$$

$$E[\phi_1,\phi_2] = \frac{1}{2}\int\left(\phi_2^2 + (\partial_x\phi_1)^2 + \frac{1}{2}\tilde{Q}^2(1-\phi_1^2)^2\right)dx$$

$$E[\phi_1,\phi_2]: 0 = E[\pm 1,0] < E[\tilde{H},0] = \frac{1}{2}\int QH'^2 + \frac{1}{4}\int Q(1-H^2)^2 = \frac{6}{5}$$

$$\mathbf{E}:=\left\{\phi=(\phi_1,\phi_2)\in\left(L^1_{\mathrm{loc}}(\mathbb{R})\right)^2:\partial_x\phi_1\in L^2(\mathbb{R}),\tilde{Q}(1-\phi_1^2)\in L^2(\mathbb{R}),\phi_2\in L^2(\mathbb{R})\right\}$$

$$H_0(\mathbb{R}):=\left\{\phi\in L^1_{\mathrm{loc}}(\mathbb{R}):\partial_x\phi\in L^2(\mathbb{R}),\tilde{Q}\phi\in L^2(\mathbb{R})\right\}$$

$$\|\phi\|^2_{H_0(\mathbb{R})}:=\|\partial_x\phi\|^2_{L^2(\mathbb{R})}+\|\tilde{Q}\phi\|^2_{L^2(\mathbb{R})}$$

$$|\phi(x)|\leq|\phi(0)|+\|\partial_x\phi\|_{L^2}|x|^{\frac{1}{2}}$$

$$H_0(\mathbb{R})\times L^2(\mathbb{R})$$

$$\mathbf{E}=\left\{\phi=(\phi_1,\phi_2)\in H_0(\mathbb{R})\times L^2(\mathbb{R}):\tilde{Q}(1-\phi_1^2)\in L^2(\mathbb{R})\right\}$$

$$\|\phi\|^2_{(H_0\times L^2)(\mathbb{R})}:=\|\phi_1\|^2_{H_0(\mathbb{R})}+\|\phi_2\|^2_{L^2(\mathbb{R})}$$

$$\sup_{t\geq0}\|\phi(t)-\widetilde{\mathbf{H}}\|_{(H_0\times L^2)(\mathbb{R})}<\delta$$

$$\lim_{t\to\infty}\|\phi(t)-\widetilde{\mathbf{H}}\|_{(H^1\times L^2)(I)}=0$$

$$\partial_t^2\phi-\partial_r^2\phi-\frac{1}{r}\partial_r\phi-\frac{2}{r^2}(\phi-\phi^3)=0,t\in\mathbb{R},r>0$$

$$H(r)=\frac{1-r^2}{1+r^2}$$

$$\phi_1(t,x)=\widetilde{H}(x)+w_1(t,x),\phi_2(t,x)=w_2(t,x)$$

$$\begin{cases}\partial_t w_1=w_2\\ \partial_t w_2=-Lw_1-\tilde{Q}^2\left(3\widetilde{H}w_1^2+w_1^3\right)\end{cases}$$

$$Lw=-\partial_x^2w+V(x)w,\ \text{with}\ V(x)=2\tilde{Q}^2(1-\tilde{Q})$$

$$L\phi_0=-\mu_0^2\phi_0,\left|\partial_x^k\phi_0(x)\right|\lesssim e^{-\frac{\sqrt{2}}{2}\mu_0x},k=0,1,2.$$

$$\mathbf{Y}_\pm=\begin{pmatrix}\phi_0\\ \pm\mu_0\phi_0\end{pmatrix},\mathbf{Z}_\pm=\begin{pmatrix}\phi_0\\ \pm\mu_0^{-1}\phi_0\end{pmatrix}$$

$$\mathcal{A}_0=\left\{\boldsymbol{\varepsilon}\in H_0(\mathbb{R})\times L^2(\mathbb{R})\ \text{such that}\ \|\boldsymbol{\varepsilon}\|_{(H_0\times L^2)(\mathbb{R})}<\delta_0\ \text{and}\ \langle\boldsymbol{\varepsilon},\mathbf{Z}_+\rangle=0\right\}$$

$$|h(\varepsilon)|\leq C\|\varepsilon\|^{3/2}_{H_0\times L^2}$$

$$\mathcal{M}=\left\{\tilde{\mathbf{H}}+\varepsilon+h(\varepsilon)\mathbf{Y}_+\text{with}\ \varepsilon\in\mathcal{A}_0\right\}$$

$$\|\boldsymbol{\phi}(t)-\widetilde{\mathbf{H}}\|_{(H_0\times L^2)(\mathbb{R})}\leq C\|\boldsymbol{\phi}(0)-\widetilde{\mathbf{H}}\|_{(H_0\times L^2)(\mathbb{R})}$$

$$\|\phi(t)-\widetilde{\mathbf{H}}\|_{(H_0\times L^2)(\mathbb{R})}\leq\frac{\delta_0}{2},$$

$$(\phi,\partial_t\phi)(0)=(1+a(0))\widetilde{\mathbf{H}}+(\tilde{u}_1,u_2)(0)+b_-(0)\mathbf{Y}_-+h(\boldsymbol{\varepsilon})\mathbf{Y}_+,$$

$$\|u_2\|_2^2 + \|\tilde{u}_1\|_{H_0}^2 + a^2 \lesssim \|u_2\|_2^2 + \langle \tilde{L}\tilde{u}_1, \tilde{u}_1 \rangle + \int \tilde{Q}^2 \left(u_1^2 + 2\tilde{H}u_1\right)^2 \lesssim \delta_0^2$$

$$\partial_x \alpha(x) = Q^{-1}(x), \partial_x \alpha^{-1}(x) = \tilde{Q}(x)$$

$$\partial_x \tilde{Q}(x) = -\tilde{Q}^2(x)\tilde{H}(x), \partial_x^2 \tilde{Q}(x) = 2\tilde{Q}^3(x) - \frac{5}{3}\tilde{Q}^4(x)$$

$$\alpha'(x) = \frac{1}{3}(\cosh\, x + 1) = \frac{2}{3}\cosh^2\left(\frac{x}{2}\right) = \frac{1}{Q(x)}$$

$$(\alpha^{-1})'(x) = \frac{1}{\alpha'(\alpha^{-1}(x))} = Q(\alpha^{-1}(x)) = \tilde{Q}(x)$$

$$Q'(x) = -\frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right) = -Q(x)H(x)$$

$$\partial_x \tilde{Q}(x) = Q'(\alpha^{-1}(x))(\alpha^{-1})'(x) = -\tilde{Q}^2(x)\tilde{H}(x).$$

$$\tilde{H}'(x) = \tfrac{1}{3}\tilde{Q}^2(x) \text{ and } \tilde{H}^2 = 1 - \tfrac{2}{3}\tilde{Q}$$

$$\begin{aligned}\partial_x^2 \tilde{Q}(x) &= -2\tilde{Q}(x)\tilde{Q}'(x)\tilde{H}(x) - \tilde{Q}^2(x)\tilde{H}'(x) \\ &= 2\tilde{Q}^3(x)\tilde{H}^2(x) - \frac{1}{3}\tilde{Q}^4(x) = 2\tilde{Q}^3(x) - \frac{5}{3}\tilde{Q}^4(x)\end{aligned}$$

$$\alpha^{-1}(x) = \frac{3}{2}x + \mathcal{O}(x^2), \tilde{Q}(x) = \frac{3}{2} - \frac{27}{32}x^2 + \mathcal{O}(x^4), \tilde{H}(x) = \frac{3}{4}x - \frac{27}{16}x^3 + \mathcal{O}(x^5).$$

$$\lim_{x\to\pm\infty} \frac{|\alpha^{-1}(x)|}{\ln(|x|)} = 1, \lim_{x\to\pm\infty} (1+|x|)\tilde{Q}(x) = 1, \lim_{x\to\pm\infty} (1+|x|)|\tilde{H}(x) \mp 1| = -\frac{1}{3}$$

$$\int \tilde{Q}^{1+\varepsilon} dx$$

$$\begin{aligned}\lim_{x\to\pm\infty} x\tilde{Q}(x) &= \lim_{y\to\pm\infty} \alpha(y)Q(y) = \lim_{y\to\pm\infty} \frac{1}{2}\frac{\sinh\, y + y}{\cosh^2\left(\frac{y}{2}\right)} \\ &= \lim_{y\to\pm\infty} \frac{\cosh\, y + 1}{\sinh\, y} = \pm 1\end{aligned}$$

$$\lim_{x\to\pm\infty} \frac{|\alpha^{-1}(x)|}{\ln(|x|)} = \lim_{x\to\pm\infty} |x|\tilde{Q}(x) = 1$$

$$\operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{\cosh^2\left(\frac{x}{2}\right)} = \frac{4}{e^x + 2 + e^{-x}} = \frac{4}{3e^{-x} + 2 - 2x + 6\alpha(x)}$$

$$\tilde{Q}(x) = \frac{6}{3e^{-\alpha^{-1}(x)} + 2 - 2\alpha^{-1}(x) + 6x} \leq \frac{3}{1 - \alpha^{-1}(x) + 3x}$$

$$\tilde{Q}(x) \geq \frac{3}{1 - \alpha^{-1}(x) + 3x} \geq \frac{3}{1 + 3x}$$

$$\lim_{x\to+\infty}(1+x)\tilde{Q}(x)=1$$

$$\lim_{x\to\pm\infty}x(\tilde{H}(x)-1)=\lim_{y\to\pm\infty}\alpha(y)\left(\tanh\left(\frac{y}{2}\right)\mp 1\right)=-\frac{1}{3}$$

$$\alpha^{-1}(x)=\alpha^{-1}(0)+\partial_x\alpha^{-1}(0)x+\mathcal{O}(x^2)=\frac{3}{2}x+\mathcal{O}(x^2)$$

$$\begin{aligned}\tilde{Q}(x)&=\tilde{Q}(0)+\tilde{Q}'(0)x+\frac{1}{2}\tilde{Q}''(0)x^2+\frac{1}{6}\tilde{Q}'''(0)x^3+\mathcal{O}(x^4)\\&=\frac{3}{2}-\frac{27}{32}x^2+\mathcal{O}(x^4)\end{aligned}$$

$$\begin{aligned}\tilde{H}(x)&=\tilde{H}(0)+\tilde{H}'(0)x+\frac{1}{2}\tilde{H}''(0)x^2+\frac{1}{6}\tilde{H}'''(0)x^3+\frac{1}{24}\tilde{H}''''(0)x^4+\mathcal{O}(x^5)\\&=\frac{3}{4}x-\frac{27}{16}x^3+\mathcal{O}(x^5)\end{aligned}$$

$$\tilde{Q}'(x)=-\tilde{Q}^2(x)\tilde{H}(x),\tilde{Q}''(x)=\tilde{Q}^2(x)(1-\tilde{Q}(x)),\tilde{H}'(x)=\tfrac{1}{3}\tilde{Q}^2(x),\tilde{H}''(x)=-\tfrac{2}{3}\tilde{Q}^3(x)\tilde{H}(x)$$

$$\tilde{H}'''(x)=2\tilde{Q}^4\tilde{H}^2(x)-\frac{2}{9}\tilde{Q}^5$$

$$\int\tilde{Q}^{1+\varepsilon}(x)dx=\int Q^{\varepsilon}(s)ds<+\infty$$

$$\phi_1(t,x)=\tilde{H}+\bar{w}_1(t,x),\phi_2(t,x)=\bar{w}_2(t,x)$$

$$(1-\phi_1^2)^2=\left(1-\tilde{H}^2\right)^2-2\left(1-\tilde{H}^2\right)\left(2\tilde{H}\bar{w}_1+\bar{w}_1^2\right)+\left(2\tilde{H}\bar{w}_1+\bar{w}_1^2\right)^2.$$

$$\begin{aligned}E[\phi_1,\phi_2]&=\int\left(\frac{1}{2}\phi_2^2+\frac{1}{2}(\partial_x\phi_1)^2+\frac{1}{4}\tilde{Q}^2(1-\phi_1^2)^2\right)dx\\&=E[\tilde{H},0]+\frac{1}{2}\int\bar{w}_2^2+\frac{1}{2}\int(\partial_x\bar{w}_1)^2-\int\tilde{H}''\bar{w}_1\\&\quad-\frac{1}{2}\int\tilde{Q}^2\left(1-\tilde{H}^2\right)\left(2\tilde{H}\bar{w}_1+\bar{w}_1^2\right)+\frac{1}{4}\int\tilde{Q}^2\left(2\tilde{H}\bar{w}_1+\bar{w}_1^2\right)^2\\&=E[\tilde{H},0]+\frac{1}{2}\int\bar{w}_2^2+\frac{1}{2}\int\bar{w}_1\left(-\partial_x^2\bar{w}_1+2\tilde{Q}^2(1-\tilde{Q})\bar{w}_1\right)\\&\quad+\frac{1}{4}\int\tilde{Q}^2\left(4\tilde{H}\bar{w}_1^3+\bar{w}_1^4\right)\end{aligned}$$

$$2\left\{E(\phi_1,\phi_2)-E(\tilde{H},0)\right\}=\int\bar{w}_2^2+\langle L\bar{w}_1,\bar{w}_1\rangle+\frac{1}{2}\int\tilde{Q}^2\left(4\tilde{H}\bar{w}_1^3+\bar{w}_1^4\right)$$

$$\left\langle\begin{array}{l}\phi(t,x)-\tilde{H}=a_1(t)\phi_0(x)+u_1(t,x)\\\partial_t\phi(t,x)=\mu_0a_2(t)\phi_0(x)+u_2(t,x)\end{array}\right.$$

$$a_1(t) = \langle \phi(t) - \widetilde{H}, \phi_0 \rangle = -\frac{1}{\mu_0^2} \langle \phi(t) - \widetilde{H}, L[\phi_0] \rangle,$$
$$a_2(t) = \frac{1}{\mu_0} \langle \partial_t \phi(t), \phi_0 \rangle = -\frac{1}{\mu_0^3} \langle \partial_t \phi(t), L[\phi_0] \rangle,$$

$$\langle u_1(t), \phi_0 \rangle = 0 = \langle u_2(t), \phi_0 \rangle.$$

$$b_+ = \frac{1}{2}(a_1 + a_2), b_- = \frac{1}{2}(a_1 - a_2).$$

$$\|u_1(t)\|_{H_0} + \|u_2(t)\|_{L^2} + |a_1(t)| + |a_2(t)| + |b_+(t)| + |b_-(t)| \leq C\delta.$$

$$\begin{aligned} \|\phi_2\|_{L^2}^2 &= \mu_0^2 |a_2|^2 \|\phi_0\|_{L^2}^2 + \mu_0 a_2 \langle u_2(t), \phi_0 \rangle + \|u_2(t)\|_{L^2}^2 \\ &= \mu_0^2 |a_2|^2 \|\phi_0\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2 \leq \delta^2. \end{aligned}$$

$$|a_2|, \|u_2(t)\|_{L^2} \lesssim \delta$$

$$\left\|\partial_x(\phi_1 - \widetilde{H})\right\|_{L^2(\mathbb{R})}^2 + \left\|\tilde{Q}(\phi_1 - \widetilde{H})\right\|_{L^2(\mathbb{R})}^2 \leq \delta^2$$

$$\int_{-R}^{R} (a_1\phi_0 + u_1)^2 \lesssim R^2\delta^2$$

$$a_1^2 + \int_{-R}^{R} u_1^2 \leq CR^2\delta^2 + C|a_1| \left| \int_{-R}^{R} \phi_0 u_1 \right|$$

$$\begin{aligned} &C|a_1| \left| \int_{-R}^{R} \phi_0 u_1 \right| \\ &\leq C|a_1| \left\|\tilde{Q}^{-1}\phi_0\right\|_{L^2(|x|\geq R)} \left\|\tilde{Q}(a_1\phi_0 + u_1)\right\|_{L^2(|x|\geq R)} + C|a_1|^2 \|\phi_0\|_{L^2(|x|\geq R)}^2 \\ &\leq \delta^2 + Ce^{-2c_0R}a_1^2 \end{aligned}$$

$$|a_1(t)| \leq C\delta, |b_+(t)| + |b_-(t)| \leq C\delta$$

$$\|u_1(t)\|_{L^2(-R,R)} \leq CR^2\delta$$

$$\left\|\tilde{Q}(a_1\phi_0 + u_1)\right\|_{L^2(\mathbb{R})} \leq \delta$$

$$\left\|\tilde{Q}u_1\right\|_{L^2(\mathbb{R})} \leq C\delta$$

$$\|a_1\phi_0' + \partial_x u_1\|_{L^2(\mathbb{R})} \leq \delta$$

$$\|\partial_x u_1\|_{L^2(\mathbb{R})} \leq C\delta$$

$$\left\|\tilde{Q}^{\frac{p+2}{2p}}u\right\|_{L^p} \leq \sqrt{2}\|u\|_{H_0}$$

$$\|u\|^2 := \|\partial_x u\|_{L^2(\mathbb{R})}^2 + \left\|\tilde{Q}^{\frac{3}{2}}u\right\|_{L^2(\mathbb{R})}^2$$

$$u(x) = v(\alpha^{-1}(x))$$

$$\partial_x u(x) = \partial_y v(\alpha^{-1}(x)) Q(\alpha^{-1}(x))$$

$$\int (\partial_x u)^2 = \int Q(\partial_y v)^2, \int \tilde{Q}^k u^2 = \int Q^{k-1} v^2, \text{ for } k = 2,3$$

$$\|u\|_{\dot{H}_0}^2 = \int \left((\partial_y g)^2 + Hg\partial_y g + \frac{1}{4} H^2 g^2 + g^2 \right) = \int (\partial_y g)^2 + \left(\frac{5}{4} - \frac{1}{3} Q \right) g^2 \geq \frac{1}{2} \|g\|_{H^1}^2$$

$$\left\| \tilde{Q}^{\frac{p+2}{2p}} u \right\|_{L^p} = \|g\|_{L^p} \leq \|g\|_{H^1}$$

$$\frac{1}{2} \int (\partial_x u)^2 + \int \tilde{Q}^3 u^2 = \frac{1}{2} \int (\partial_y g)^2 + \left(\frac{1}{8} + \frac{5}{6} Q \right) g^2 \geq \frac{1}{8} \|g\|_{H^1}^2$$

$$\|\tilde{Q} u\|_{L^2(\mathbb{R})} \leq \|g\|_{H^1}$$

$$\|u\|_{\dot{H}_0} \lesssim \|u\|$$

$$\begin{cases} \dot{a}_1(t) = \mu_0 a_2(t) \\ \dot{a}_2(t) = \mu_0 a_1(t) - \dfrac{N_0}{\mu_0}, \end{cases} \text{ or equivalently } \begin{cases} \dot{b}_+(t) = \mu_0 b_+(t) - \dfrac{N_0}{2\mu_0} \\ \dot{b}_-(t) = -\mu_0 b_-(t) + \dfrac{N_0}{2\mu_0} \end{cases}$$

$$N = \tilde{Q}^2 \left(3\tilde{H}(a_1 \phi_0 + u_1)^2 + (a_1 \phi_0 + u_1)^3 \right)$$

$$N^{\perp} = N - N_0 \phi_0, \text{ and } N_0 = \langle N, \phi_0 \rangle.$$

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = -L u_1 - N^{\perp} \end{cases}$$

$$\|\phi(0) - \tilde{\mathbf{H}}\|_{(H_0 \times L^2)(\mathbb{R})} \leq \delta$$

$$\phi_1(t,x) = \tilde{H}(x) + w_1(t,x), \phi_2(t,x) = w_2(t,x)$$

$$\begin{cases} \partial_t w_1 = w_2 \\ \partial_t w_2 = \partial_x^2 w_1 - F(t,x,w_1) \end{cases}$$

$$F(t,x,w_1) = 2\tilde{Q}^2(1 - \tilde{Q}) w_1 + \tilde{Q}^2 \left(3\tilde{H} w_1^2 + w_1^3 \right)$$

$$w_1, \tilde{w}_1, \text{ if } \left\| \tilde{Q}^{1/2} w_1 \right\|_{L^\infty} \leq 1 \text{ and } \left\| \tilde{Q}^{1/2} \tilde{w}_1 \right\|_{L^\infty} \leq 1$$

$$|F(t,x,w_1) - F(t,x,\tilde{w}_1)| \leq C \tilde{Q} |w_1 - \tilde{w}_1|$$

$$\left| f^{(k)}(x) \right| \leq C_k \tilde{Q}(x)^3 \text{ for all } x \in \mathbb{R}$$

$$\chi(x) = 1 \text{ for } |x| \leq 1, \chi(x) = 0 \text{ for } |x| \geq 2, \chi'(x) \leq 0 \text{ for } x \geq 0.$$

$$\zeta_A^2(x) = \exp\left(-\frac{1}{A} |\alpha^{-1}(x)| (1 - \chi(x)) \right), \varphi_A(x) = \int_0^x \tilde{Q} \zeta_A^2(y) dy, x \in \mathbb{R}$$

$$\sigma_A(x) = \operatorname{sech}\left(\frac{1}{A}\alpha^{-1}(x)\right)$$

$$\zeta_B^2(x) = \exp\left(-\frac{1}{B}|\alpha^{-1}(x)|(1-\chi(x))\right), \varphi_B(x) = \int_0^x \tilde{Q}\zeta_B^2(y)dy, x \in \mathbb{R}$$

$$\psi_{A,B}(x) = \tilde{\chi}_A^2(x)\varphi_B(x), \tilde{\chi}_A(x) = \chi\left(\frac{\alpha^{-1}(x)}{A}\right), \tilde{\chi}_B(x) = \chi\left(\frac{\alpha^{-1}(x)}{B^2}\right)$$

$$1 \ll B \ll B^2 \ll A.$$

$$\mathcal{I} = \int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) u_2$$

$$w_i = \zeta_A u_i, i = 1,2$$

$$A = \delta^{-1/4}$$

$$\frac{d}{dt}\mathcal{I} \leq -\frac{1}{2}C_0 \int \tilde{Q}\left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2\right] + C\int \tilde{Q}^7 u_1^2 + C|a_1|^4$$

$$\frac{d}{dt}\mathcal{I} = -\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 - \int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) N^{\perp}$$

$$\mathcal{I}_1 = \int \varphi_A u_2 \partial_x u_1, \mathcal{I}_2(t) = \frac{1}{2}\int \varphi_A' u_1 u_2$$

$$\frac{d}{dt}\mathcal{I}_1(t) = \int \varphi_A(\dot{u}_2 \partial_x u_1 + u_2 \partial_x \dot{u}_1) = -\int \varphi_A(L[u_1] + N^{\perp})\partial_x u_1 + \int \varphi_A u_2 \partial_x u_2$$
$$= -\int \varphi_A L[u_1]\partial_x u_1 - \int \varphi_A \partial_x u_1 N^{\perp} - \frac{1}{2}\int \varphi_A' u_2^2 = -\int \varphi_A L[u_1]\partial_x u_1 - \int \varphi_A \partial_x u_1 N^{\perp} - \frac{1}{2}\int \varphi_A' u_2^2$$

$$\int \varphi_A L[u_1]\partial_x u_1 = \int \varphi_A(-\partial_x^2 u_1 + V u_1)\partial_x u_1 = -\frac{1}{2}\int \varphi_A \partial_x(\partial_x u_1)^2 + \frac{1}{2}\int \varphi_A V \partial_x u_1^2$$
$$= \frac{1}{2}\int \varphi_A'(\partial_x u_1)^2 - \frac{1}{2}\int \varphi_A' V u_1^2 - \frac{1}{2}\int \varphi_A V' u_1^2$$

$$\frac{d}{dt}\mathcal{I}_1 = -\frac{1}{2}\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{2}\int \varphi_A' V u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 - \frac{1}{2}\int \varphi_A' u_2^2 - \int \varphi_A \partial_x u_1 N^{\perp}$$

$$\frac{d}{dt}\mathcal{I}_2 = \frac{1}{2}\int \varphi_A'(\dot{u}_1 u_2 + u_1 \dot{u}_2) = \frac{1}{2}\int \varphi_A' u_2^2 - \frac{1}{2}\int \varphi_A' u_1(L[u_1] + N^{\perp})$$
$$= \frac{1}{2}\int \varphi_A' u_2^2 - \frac{1}{2}\int \varphi_A' u_1 L[u_1] - \frac{1}{2}\int \varphi_A' u_1 N^{\perp}$$

$$\int \varphi_A' u_1 L[u_1] = \int \varphi_A' u_1(-\partial_x^2 u_1 + V)u_1\Big) = \int (\varphi_A' u_1)_x \partial_x u_1 + \int \varphi_A' V u_1^2$$
$$= \int \varphi_A'' u_1 \partial_x u_1 + \int \varphi_A'(\partial_x u_1)^2 + \int \varphi_A' V u_1^2 = -\frac{1}{2}\int \varphi_A''' u_1^2 + \int \varphi_A'(\partial_x u_1)^2 + \int \varphi_A' V u_1^2$$

$$\frac{d}{dt}\mathcal{I}_2(t) = \frac{1}{2}\int \varphi_A' u_2^2 + \frac{1}{4}\int \varphi_A''' u_1^2 - \frac{1}{2}\int \varphi_A'(\partial_x u_1)^2 - \frac{1}{2}\int \varphi_A' V u_1^2 - \frac{1}{2}\int \varphi_A' u_1 N^{\perp}$$

$$\frac{d}{dt}\mathcal{I}(t)=-\int\ \varphi_A'(\partial_x u_1)^2+\frac{1}{4}\int\ \varphi_A'''u_1^2+\frac{1}{2}\int\ \varphi_A V'u_1^2-\int\ \left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)N^{\perp}$$

$$-\int\ \left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)N^{\perp}$$

$$-\int\ \left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)\left(\tilde{Q}^2\left(3\tilde{H}(a_1\phi_0+u_1)^2+(a_1\phi_0+u_1)^3\right)-N_0\phi_0\right)$$

$$\leq Ca_1^4+C\int\ \tilde{Q}^7u_1^2+CA\left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\int\ \tilde{Q}^3w_1^2+\frac{1}{72}\int\ \tilde{Q}^3\tilde{H}^2w_1^2+\frac{4}{9}\int\ \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2$$

$$\int\ \left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)\tilde{Q}^2\left(3\tilde{H}(a_1\phi_0+u_1)^2+(a_1\phi_0+u_1)^3\right)$$

$$=a_1^2\int\ \tilde{Q}^2(3\tilde{H}+a_1\phi_0)\phi_0^2\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)+3a_1\int\ \tilde{Q}^2(2\tilde{H}+a_1\phi_0)\phi_0u_1\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)$$

$$+3\int\ \tilde{Q}^2(\tilde{H}+a_1\phi_0)u_1^2\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)+\int\ \tilde{Q}^2u_1^3\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)$$

$$=I_1+I_2+I_3+I_4$$

$$|I_1|\leq a_1^2\int\ \left|\partial_x\left(\tilde{Q}^2(3\tilde{H}+a_1\phi_0)\phi_0^2\right)\varphi_Au_1\right|+\frac{1}{2}a_1^2\int\ \left|\tilde{Q}^2(3\tilde{H}+a_1\phi_0)\phi_0^2\varphi_A'u_1\right|$$

$$\lesssim a_1^2\left[\left(\int\ \tilde{Q}\phi_0^4|\alpha^{-1}|^2\right)^{\frac{1}{2}}+\left(\int\ \tilde{Q}^3|\phi_0\phi_0'\alpha^{-1}|^2\right)^{\frac{1}{2}}\right]\left(\int\ \tilde{Q}^7u_1^2\right)^{\frac{1}{2}}\lesssim a_1^4+\int\ \tilde{Q}^7u_1^2$$

$$|I_2|=\frac{3}{2}|a_1|\left|\int\ \partial_x\left(\tilde{Q}^2(2\tilde{H}+a_1\phi_0)\phi_0\right)\varphi_Au_1^2\right|\lesssim|a_1|\int\ |\alpha^{-1}(x)|(\tilde{Q}\phi_0+\phi_0')\tilde{Q}^2u_1^2\lesssim\int\ \tilde{Q}^7u_1^2$$

$$I_4=\int\ \tilde{Q}^2u_1^3\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)=\frac{1}{4}\int\ \tilde{Q}^2\varphi_A'u_1^4+\frac{1}{2}\int\ \tilde{Q}^3\tilde{H}\varphi_Au_1^4=\frac{1}{4}\int\ \tilde{Q}^2(2\varphi_A'+\tilde{Q}\tilde{H}\varphi_A)u_1^4$$

$$I_{3,1}:=\int\ \tilde{Q}^2\tilde{H}u_1^2\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right),I_{3,2}:=a_1\int\ \tilde{Q}^2\phi_0u_1^2\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)$$

$$|I_{3,2}|\lesssim A|a_1|\int\ \tilde{Q}^2(|\phi_0|+|\phi_0'|)|u_1|^3\lesssim A\left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\int\ \tilde{Q}^3w_1^2$$

$$I_{3,1}=\int\ \left(\frac{1}{6}\varphi_A'\tilde{Q}^2\tilde{H}-\frac{1}{3}\varphi_A\left(\tilde{Q}^2\tilde{H}\right)'\right)u_1^3=\int\ \left(\frac{1}{6}\varphi_A'\tilde{Q}^2\tilde{H}-\frac{1}{3}\varphi_A\tilde{Q}^3\left(\frac{1}{3}\tilde{Q}-2\tilde{H}^2\right)\right)u_1^3$$

$$3I_{3,1}+I_4=\frac{1}{6}\int\ \varphi_A'\tilde{Q}^2\tilde{H}u_1^3+\frac{1}{2}\int\ \tilde{Q}^2\varphi_A'u_1^4-\frac{1}{9}\int\ \varphi_A\tilde{Q}^4u_1^3+\frac{2}{3}\int\ \tilde{Q}^3\tilde{H}|\varphi_A\tilde{H}|u_1^3+\frac{1}{4}\int\ \tilde{Q}^3|\varphi_A\tilde{H}|u_1^4$$

$$=\frac{1}{2}\int\ \varphi_A'\tilde{Q}^2u_1^2\left(u_1+\frac{1}{6}\tilde{H}\right)^2-\frac{1}{72}\int\ \varphi_A'\tilde{Q}^2\tilde{H}^2u_1^2$$

$$-\frac{1}{9}\int\ \varphi_A\tilde{Q}^4u_1^3+\frac{1}{4}\int\ \tilde{Q}^3|\varphi_A\tilde{H}|u_1^2\left(u_1+\frac{4}{3}\tilde{H}\right)^2-\frac{4}{9}\int\ \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2$$

$$\int\ \varphi_A\tilde{Q}^4u_1^3\lesssim A\left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\int\ \tilde{Q}^{7/2}u_1^2$$

$$3I_{3,1}+I_4\geq -\frac{1}{72}\int \tilde{Q}^3\tilde{H}^2w_1^2-\frac{1}{9}\int \varphi_A\tilde{Q}^4u_1^3-\frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2$$
$$\geq -\frac{1}{72}\int \tilde{Q}^3\tilde{H}^2w_1^2-\frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2-CA\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\int \tilde{Q}^{7/2}\zeta_A^{-2}w_1^2$$

$$N_0\int \phi_0\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)$$

$$N=\tilde{Q}^2\left(3\tilde{H}(a_1^2\phi_0^2+2a_1\phi_0u_1+u_1^2)+a_1^3\phi_0^3+3a_1^2\phi_0^2u_1+3a_1\phi_0u_1^2+u_1^3\right)$$

$$|N|\lesssim \tilde{Q}^2(a_1^2\phi_0^2+u_1^2+|u_1|^3+u_1^4)$$

$$\phi_0,\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\lesssim\|\tilde{Q}^{1/2}u_1\|_{H^1}\lesssim 1,A\geq 2$$

$$|N_0|=|\langle\phi_0,N\rangle|\lesssim a_1^2+\int \tilde{Q}^2\phi_0u_1^2\lesssim a_1^2+\int \tilde{Q}^7u_1^2$$

$$-\int \phi_0\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)=\int u_1\left(\varphi_A\phi_0'+\frac{1}{2}\varphi_A'\phi_0\right)$$

$$|\varphi_A\phi_0'+\varphi_A'\phi_0|\lesssim\alpha^{-1}(x)\phi_0'+\tilde{Q}\zeta_A^2\phi_0\lesssim\tilde{Q}^7$$

$$\left|N_0\int \phi_0\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)\right|\lesssim\left(a_1^2+\int \tilde{Q}^7u_1^2\right)\int \tilde{Q}^7|u_1|$$
$$\lesssim\left(a_1^2+\int \tilde{Q}^7u_1^2\right)\left(\int \tilde{Q}^7u_1^2\right)^{\frac{1}{2}}\left(\int \tilde{Q}^7\right)^{\frac{1}{2}}\lesssim a_1^4+\int \tilde{Q}^7u_1^2$$

$$-\int \left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)\left(\tilde{Q}^2\left(3\tilde{H}(a_1\phi_0+u_1)^2+(a_1\phi_0+u_1)^3\right)-N_0\phi_0\right)$$
$$\leq Ca_1^4+C\int \tilde{Q}^7u_1^2+\frac{1}{72}\int \tilde{Q}^3\tilde{H}^2w_1^2+\frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2$$
$$+CA\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\int \tilde{Q}^3w_1^2+CA\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\int \tilde{Q}^{7/2}\zeta_A^{-2}w_1^2$$
$$\leq Ca_1^4+C\int \tilde{Q}^7u_1^2+\frac{1}{72}\int \tilde{Q}^3\tilde{H}^2w_1^2+\frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2+CA\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\int \tilde{Q}^3w_1^2$$

$$-\int \varphi_A'(\partial_x u_1)^2+\frac{1}{4}\int \varphi_A'''u_1^2+\frac{1}{2}\int \varphi_AV'u_1^2+\frac{1}{72}\int \tilde{Q}^3\tilde{H}^2w_1^2+\frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2$$
$$=-\int \tilde{Q}(\partial_x w_1)^2+\frac{1}{2}\int \left[\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]\tilde{Q}w_1^2+\frac{1}{72}\int \tilde{Q}^3\tilde{H}^2w_1^2$$
$$+\frac{1}{4}\int \tilde{Q}''w_1^2+\frac{1}{2}\int \varphi_AV'u_1^2+\frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2$$

$$\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2=\frac{1}{A}\left[\chi''|\alpha^{-1}|+2\chi'\tilde{Q}\mathrm{sgn}(\alpha^{-1})+(1-\chi)\tilde{Q}^2\tilde{H}\mathrm{sgn}(\alpha^{-1})\right]$$

$$\left|\frac{\zeta_A'}{\zeta_A}\right|\lesssim\frac{1}{A}\tilde{Q}\mathbf{1}_{\{|x|\geq 1\}},\left|\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right|\lesssim\frac{1}{A}\tilde{Q}^2\mathbf{1}_{\{|x|\geq 1\}}$$

$$\frac{1}{4}\left(\tilde{Q}''(x)+\frac{1}{18}\tilde{Q}^3(x)\tilde{H}^2(x)\right)\tilde{Q}^{-1}(x)\varphi_A'(x)+\frac{1}{2}\varphi_A(x)V'(x)+\frac{4}{9}\left|\varphi_A(x)\tilde{H}(x)\right|\tilde{H}^2\tilde{Q}^3(x)\leq -C\tilde{Q}^3(x)$$

$$\frac{1}{4}\int \tilde{Q}''w_1^2$$

$$\begin{aligned}\int \varphi_A'(\partial_x u_1)^2 &= \int \tilde{Q}\left(\partial_x w_1-\frac{\zeta_A'}{\zeta_A}w_1\right)^2=\int \tilde{Q}(\partial_x w_1)^2-2\int \tilde{Q}\frac{\zeta_A'}{\zeta_A}w_1\partial_x w_1+\int \left(\frac{\zeta_A'}{\zeta_A}\right)^2\tilde{Q}w_1^2\\ &=\int \tilde{Q}(\partial_x w_1)^2+\int \left[\left(\frac{\tilde{Q}\zeta_A'}{\zeta_A}\right)'+\tilde{Q}\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]w_1^2=\int \tilde{Q}(\partial_x w_1)^2+\int \frac{(\tilde{Q}\zeta_A')'}{\zeta_A}w_1^2\end{aligned}$$

$$\int \varphi_A'''u_1^2=\int \left[\tilde{Q}''+2\tilde{Q}'\left(\frac{\zeta_A'}{\zeta_A}\right)+2\frac{(\tilde{Q}\zeta_A')'}{\zeta_A}+2\tilde{Q}\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]w_1^2$$

$$\begin{aligned}-\int \varphi_A'(\partial_x u_1)^2+\frac{1}{4}\int \varphi_A'''u_1^2 &=-\int \tilde{Q}(\partial_x w_1)^2+\frac{1}{4}\int \tilde{Q}''w_1^2+\frac{1}{2}\int \left[\tilde{Q}'\left(\frac{\zeta_A'}{\zeta_A}\right)-\frac{(\tilde{Q}\zeta_A')'}{\zeta_A}+\tilde{Q}\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]w_1^2\\ &=-\int \tilde{Q}(\partial_x w_1)^2+\frac{1}{4}\int \tilde{Q}''w_1^2+\frac{1}{2}\int \tilde{Q}\left[\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]w_1^2\end{aligned}$$

$$\frac{\zeta_A'}{\zeta_A}=\frac{1}{A}[\chi'|\alpha^{-1}|-\operatorname{sgn}(\alpha^{-1})(\alpha^{-1})'(1-\chi)]$$

$$\frac{\zeta_A''}{\zeta_A}=\left(\frac{\zeta_A'}{\zeta_A}\right)^2+\frac{1}{A}[\chi''|\alpha^{-1}|+2\chi'(\alpha^{-1})'\operatorname{sgn}(\alpha^{-1})-(1-\chi)(\alpha^{-1})''\operatorname{sgn}(\alpha^{-1})]$$

$$\left|\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right|\lesssim\frac{1}{A}$$

$$\left|\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right|=\frac{1}{A}\tilde{Q}^2|\tilde{H}|\leq\frac{1}{A}\tilde{Q}^2(x)$$

$$\left|\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right|\lesssim\frac{\tilde{Q}^2\mathbf{1}_{\{|x|\geq 1\}}}{A}$$

$$\begin{aligned}&\frac{1}{4}\left(\tilde{Q}''+\frac{1}{18}\tilde{H}^2\tilde{Q}^3\right)\varphi_A'+\frac{1}{2}\varphi_A V'+\frac{4}{9}(\varphi_A\tilde{H})\tilde{H}^2\tilde{Q}^3\\ &\quad=\left[\left(\frac{1}{2}-\frac{5}{12}\tilde{Q}+\frac{1}{72}\tilde{H}^2\right)\varphi_A'-\left(2-3\tilde{Q}-\frac{4}{9}\tilde{H}^2\right)\varphi_A\tilde{H}\right]\tilde{Q}^3\\ &\quad=\left[\frac{1}{18}\left(\frac{37}{4}-\frac{23}{3}\tilde{Q}\right)\zeta_A^2-\frac{1}{9}\left(14-\frac{73}{3}\tilde{Q}\right)\varphi_A\tilde{H}\right]\tilde{Q}^3\end{aligned}$$

$$14-\frac{73}{3}\tilde{Q}\geq\frac{53}{9}>\frac{47}{8},\left(\frac{37}{8}-\frac{23}{6}\tilde{Q}\right)\zeta_A^2\leq\frac{37}{8}$$

$$\varphi_A=\int_0^{\alpha^{-1}(x)}e^{-\frac{2}{A}s(1-\chi(\alpha(s)))}ds\geq\int_0^1 ds+\int_1^{\alpha^{-1}(x)}e^{-\frac{2}{A}s}ds\geq 1$$

$$\frac{1}{4}\left(\tilde{Q}'' + \frac{1}{18}\tilde{H}^2\tilde{Q}^3\right)\varphi_A' + \frac{1}{2}\varphi_A V' + \frac{4}{9}(\varphi_A\tilde{H})\tilde{H}^2\tilde{Q}^3 \le \frac{37}{72}\left(1 - \frac{47}{37}\varphi_A\tilde{H}\right)\tilde{Q}^3 \le \frac{37}{72}R(\alpha^{-1}(x))\tilde{Q}^3$$

$$R(s) := 1 - \frac{6}{5}H(s)$$

$$-\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 + \frac{1}{72}\int \tilde{Q}^3\tilde{H}^2 w_1^2 + \frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2 u_1^2$$
$$\le -C_0\int \tilde{Q}\left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2\right] + C'\int \tilde{Q}^7 u_1^2$$

$$\int \tilde{Q}^7 u_1^2$$

$$\frac{1}{4}\int \tilde{Q}'' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2$$

$$-\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 + \frac{1}{72}\int \tilde{Q}^3\tilde{H}^2 w_1^2 + \frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2 u_1^2$$
$$\le -\int \tilde{Q}(\partial_x w_1)^2 + \frac{C'}{A}\int_{|x|\ge 1}\tilde{Q}^3 w_1^2 - C\int_{|x|\ge\tilde{x}}\tilde{Q}^3 u_1^2 + C\int_{|x|\le\tilde{x}}\tilde{Q}^2 u_1^2$$

$$\left|\frac{1}{4}\left(\tilde{Q}'' + \frac{1}{18}\tilde{H}^2\tilde{Q}^3\right)\tilde{Q}^{-1}\varphi_A' + \frac{1}{2}\varphi_A V' + \frac{4}{9}(\varphi_A\tilde{H})\tilde{H}^2\tilde{Q}^3\right| \lesssim \tilde{Q}^2$$

$$-\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 + \frac{1}{72}\int \tilde{Q}^3\tilde{H}^2 w_1^2 + \frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2 u_1^2$$
$$\le -C_0\int \tilde{Q}\left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2\right] + C\int \tilde{Q}^7 u_1^2$$

$$\frac{d}{dt}\mathcal{I} = -\int \tilde{Q}(\partial_x w_1)^2 - \frac{1}{2}\int \left[\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]\tilde{Q} w_1^2 + \frac{1}{4}\int \tilde{Q}'' u_1^2 + \int \varphi_A V' u_1^2$$
$$-\int \left(\varphi_A\partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right)N^{\perp}$$
$$\le -C_0\int \tilde{Q}\left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2\right] + C\int \tilde{Q}^7 u_1^2 + C|a_1|^4 + CA\left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\int \tilde{Q}^3 w_1^2$$

$$L_0 = -\partial_x^2 + V_0, \text{ with } V_0 := 2\left(\frac{\partial_x\phi_0}{\phi_0}\right)^2 - 2\mu_0^2 - V,$$
$$U = \phi_0\cdot\partial_x\cdot\phi_0^{-1}, U^* = -\phi_0^{-1}\cdot\partial_x\cdot\phi_0$$

$$L = U^*U - \mu_0^2, L_0 = UU^* - \mu_0^2$$

$$UL = L_0U$$

$$\begin{cases}\dot{v}_1 = v_2\\ \dot{v}_2 = -L_0[v_1]\end{cases}$$

$$X_\gamma : L^2(\mathbb{R}) \to H^2(\mathbb{R}), X_\gamma = (1 - \gamma\partial_x^2)^{-1}$$

$$\widehat{X_\gamma h}(\xi) = \frac{\hat{h}(\xi)}{1+\gamma\xi^2}.$$

$$X_\gamma^{-1}\left[fX_\gamma g\right] = fg + \gamma X_\gamma^{-1}\left[\partial_x\left((\partial_x f)g\right)\right] + \gamma X_\gamma^{-1}[(\partial_x f)(\partial_x g)]$$

$$X_\gamma^{-1}\left[fX_\gamma g\right] = fg + h$$

$$X_\gamma h = fX_\gamma g - X_\gamma(fg) = \gamma(\partial_x^2 fg + 2\partial_x f\partial_x g + f\partial_x^2 g - f\partial_x^2 g) = \gamma\left(\partial_x\left((\partial_x f)g\right) + \partial_x f\partial_x g\right)$$

$$\begin{cases} v_1 = (1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}_B u_1) \\ v_2 = (1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}_B u_2) \end{cases}$$

$$\begin{cases} \dot{v}_1 = v_2 \\ \dot{v}_2 = -(1-\gamma\partial_x^2)^{-1}ULu_1 - (1-\gamma\partial_x^2)^{-1}U(N^\perp) \end{cases}$$

$$\tilde{\chi}_B Lu_1 = L(\tilde{\chi}_B u_1) + 2\tilde{\chi}_B'\partial_x u_1 + \tilde{\chi}_B'' u_1.$$

$$\begin{aligned} -(1-\gamma\partial_x^2)^{-1}UL(\tilde{\chi}_B u_1) &= -(1-\gamma\partial_x^2)^{-1}L_0 U(\tilde{\chi}_B u_1) = -(1-\gamma\partial_x^2)^{-1}L_0[(1-\gamma\partial_x^2)v_1] \\ &= -(1-\gamma\partial_x^2)^{-1}(-\partial_x^2 + V_0)(1-\gamma\partial_x^2)v_1 = \partial_x^2 v_1 - (1-\gamma\partial_x^2)^{-1}[V_0(1-\gamma\partial_x^2)v_1] \end{aligned}$$

$$\begin{aligned} (1-\gamma\partial_x^2)[V_0 v_1] &= V_0 v_1 - \gamma(V_0'' v_1 + 2V_0'\partial_x v_1 + V_0\partial_x^2 v_1) \\ &= V_0(1-\gamma\partial_x^2)v_1 - \gamma(V_0'' v_1 + 2V_0'\partial_x v_1) \end{aligned}$$

$$-(1-\gamma\partial_x^2)^{-1}UL(\tilde{\chi}_B u_1) = -L_0 v_1 - \gamma(1-\gamma\partial_x^2)^{-1}(V_0'' v_1 + 2V_0'\partial_x v_1)$$

$$\left\| \begin{array}{c} \dot{v}_1 = v_2 \\ \dot{v}_2 = -L_0 v_1 - \gamma(1-\gamma\partial_x^2)^{-1}(V_0'' v_1 + 2V_0'\partial_x v_1) \\ -(1-\gamma\partial_x^2)^{-1}U[2\tilde{\chi}_B'\partial_x u_1 + \tilde{\chi}_B'' u_1] - (1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}_B N^\perp) \end{array} \right\|$$

$$\mathcal{J}(t) = \int \left(\psi_{A,B}(x)\partial_x v_1(t,x) + \frac{1}{2}\psi_{A,B}'(x)v_1(t,x)\right)v_2(t,x)dx$$

$$z := \tilde{\chi}_A\zeta_B v_1$$

$$B = \alpha^{-1}\left(\delta^{-1/8}\right)$$

$$\frac{d}{dt}\mathcal{J} \leq -C_2\int \tilde{Q}\left[(\partial_x z)^2 + \tilde{Q}^2 z^2\right] + C\ln\left(\delta^{-\frac{1}{8}}\right)^{-1}\int \tilde{Q}\left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2\right] + C\delta^{\frac{1}{2}}|a_1|^3$$

$$\frac{d}{dt}\mathcal{J}=\int\left(\psi_{A,B}\partial_x v_1+\frac{1}{2}\psi'_{A,B}v_1\right)\dot{v}_2$$
$$=-\int\left(\psi_{A,B}\partial_x v_1+\frac{1}{2}\psi'_{A,B}v_1\right)L_0 v_1$$
$$-\gamma\int\left(\psi_{A,B}\partial_x v_1+\frac{1}{2}\psi'_{A,B}v_1\right)(1-\gamma\partial_x^2)^{-1}(V_0''v_1+2V_0'\partial_x v_1)$$
$$-\int\left(\psi_{A,B}\partial_x v_1+\frac{1}{2}\psi'_{A,B}v_1\right)(1-\gamma\partial_x^2)^{-1}U[2\tilde{\chi}'_B\partial_x u_1+\tilde{\chi}''_B u_1]$$
$$-\int\left(\psi_{A,B}\partial_x v_1+\frac{1}{2}\psi'_{A,B}v_1\right)(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}_B N^{\perp})$$
$$=:J_1+J_2+J_3+J_4$$

$$J_1=-\int\psi'_{A,B}(\partial_x v_1)^2+\frac{1}{4}\int\psi'''_{A,B}v_1^2-\int\left(\psi_{A,B}\partial_x v_1+\frac{1}{2}\psi'_{A,B}v_1\right)V_0 v_1$$

$$\psi'_{A,B}=\tilde{Q}\tilde{\chi}_A^2\zeta_B^2+(\tilde{\chi}_A^2)'\varphi_B$$
$$\psi''_{A,B}=\tilde{Q}'\tilde{\chi}_A^2(\zeta_B^2)+\tilde{Q}\tilde{\chi}_A^2(\zeta_B^2)'+2\tilde{Q}(\tilde{\chi}_A^2)'\zeta_B^2+(\tilde{\chi}_A^2)''\varphi_B$$
$$\psi'''_{A,B}=\tilde{Q}''\tilde{\chi}_A^2\zeta_B^2+3\tilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2+2\tilde{Q}'\tilde{\chi}_A^2(\zeta_B^2)'+3\tilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)'$$
$$+3\tilde{Q}(\tilde{\chi}_A^2)''\zeta_B^2+\tilde{Q}\tilde{\chi}_A^2(\zeta_B^2)''+(\tilde{\chi}_A^2)'''\varphi_B$$

$$-\int\psi'_{A,B}(\partial_x v_1)^2+\frac{1}{4}\int\psi'''_{A,B}v_1^2$$
$$=-\int\tilde{Q}^2\zeta_B^2(\partial_x v_1)^2+\frac{1}{4}\int\tilde{Q}''\tilde{\chi}_A^2\zeta_B^2v_1^2+\frac{1}{4}\int\tilde{Q}\tilde{\chi}_A^2(\zeta_B^2)''v_1^2$$
$$+\frac{3}{4}\int\tilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2v_1^2+\frac{3}{4}\int\tilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)'v_1^2+\frac{3}{4}\int\tilde{Q}(\tilde{\chi}_A^2)''\zeta_B^2v_1^2$$
$$+\frac{1}{2}\int\tilde{Q}'\tilde{\chi}_A^2(\zeta_B^2)'v_1^2-\int(\tilde{\chi}_A^2)'\varphi_B(\partial_x v_1)^2+\frac{1}{4}\int(\tilde{\chi}_A^2)'''\varphi_B v_1^2$$

$$\int\tilde{Q}\tilde{\chi}_A^2\zeta_B^2(\partial_x v_1)^2=\int\tilde{Q}(\partial_x z)^2+\int\left(\tilde{Q}(\tilde{\chi}_A\zeta_B)'\right)'\tilde{\chi}_A\zeta_B v_1^2$$
$$=\int\tilde{Q}(\partial_x z)^2+\int\tilde{Q}\frac{\zeta_B''}{\zeta_B}z^2+\int\tilde{Q}\tilde{\chi}_A''\tilde{\chi}_A\zeta_B^2v_1^2+\frac{1}{2}\int\tilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)'v_1^2$$
$$+\frac{1}{2}\int\tilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2v_1^2+\frac{1}{2}\int\tilde{Q}'\tilde{\chi}_A^2(\zeta_B^2)'v_1^2$$

$$\frac{1}{4}\int\tilde{Q}\tilde{\chi}_A^2(\zeta_B^2)''v_1^2=\frac{1}{2}\int\tilde{Q}\left(\frac{\zeta_B''}{\zeta_B}+\frac{(\zeta_B')^2}{\zeta_B^2}\right)z^2$$

$$-\int\psi'_{A,B}(\partial_x v_1)^2+\frac{1}{4}\int\psi'''_{A,B}v_1^2=-\left\{\int\tilde{Q}(\partial_x z)^2-\frac{1}{4}\int\tilde{Q}''z^2+\frac{1}{2}\int\tilde{Q}\left(\frac{\zeta_B''}{\zeta_B}-\frac{(\zeta_B')^2}{\zeta_B^2}\right)z^2\right\}+\tilde{J}_1$$

$$\tilde{J}_1=\frac{1}{4}\int\tilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)'v_1^2+\frac{1}{4}\int\tilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2v_1^2+\frac{1}{2}\int\tilde{Q}[3(\tilde{\chi}'_A)^2+\tilde{\chi}''_A\tilde{\chi}_A]\zeta_B^2v_1^2$$
$$-\int(\tilde{\chi}_A^2)'\varphi_B(\partial_x v_1)^2+\frac{1}{4}\int(\tilde{\chi}_A^2)'''\varphi_B v_1^2$$

$$\int\left(\psi_{A,B}\partial_x v_1+\frac{1}{2}\psi'_{A,B}v_1\right)V_0 v_1=\frac{1}{2}\int V_0\partial_x\left(\psi_{A,B}v_1^2\right)=-\frac{1}{2}\int\frac{\varphi_B}{\zeta_B^2}V_0'z^2$$

$$V_B = -\frac{1}{4}\tilde{Q}'' + \frac{1}{2}\tilde{Q}\left(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\right) - \frac{1}{2}\frac{\varphi_B}{\zeta_B^2}V_0'$$

$$V_B = \left[\frac{1}{2}\tilde{Q}\left(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\right) - \frac{1}{10}\frac{\varphi_B}{\zeta_B^2}V_0'\right] + \left[-\frac{1}{4}\tilde{Q}'' - \frac{2}{5}\frac{\varphi_B}{\zeta_B^2}V_0'\right] =: V_{\otimes}^{\dagger} + V_{\boxtimes}^{\triangle}$$

$$J_1 = -\int \left[\tilde{Q}(\partial_x z)^2 + V_{\boxplus}^{\star} z^2 + V_{\sqcup}^{\odot} z^2\right] + \tilde{J}_1$$

$$V_B^I \geq V_1 \text{ where } V_1 = C_1'\tilde{Q}^3(x)\mathbf{1}_{\{|x|\geq 1\}}(x),$$

$$\left|\frac{\zeta_B''}{\zeta_B} - \left(\frac{\zeta_B'}{\zeta_B}\right)^2\right| \leq \frac{C}{B}\tilde{Q}^2(x)\mathbf{1}_{\{|x|\geq 1\}}(x)$$

$$\frac{\varphi_B}{\zeta_B^2} = \frac{1}{\zeta_B^2}\int_0^{\alpha^{-1}(x)} \zeta_B^2(\alpha(s))ds \geq \alpha^{-1}(x)$$

$$\begin{aligned} V_B^{\mathrm{I}}(x) \geq & \frac{1}{10}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\tilde{Q}^3(x)\mathbf{1}_{\{|x|\geq 1\}}(x) \\ \geq & \left(\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\tilde{Q}^3(x)\right)\mathbf{1}_{\{1\leq x\leq x_{2,2}\}}(x) + \frac{1}{20}\alpha^{-1}(x)|V_0'(x)| \\ & + \left(\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\tilde{Q}^3(x)\right)\mathbf{1}_{\{x\geq x_{2,2}\}}(x) \end{aligned}$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| \geq \tilde{C}$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\tilde{Q}^3 \geq \tilde{C} - \frac{27}{8}\frac{C}{B} \geq \frac{1}{2}\tilde{C} > 0$$

$$\tilde{Q}^3 \lesssim |V_0'| \lesssim \tilde{Q}^3$$

$$C'\tilde{Q}^3 \leq |V_0'(x)|$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\tilde{Q}^3 \geq \left(\frac{C'}{20}\alpha^{-1}(x) - \frac{C}{B}\right)\tilde{Q}^3$$

$$x \in \left[x_{2,2}, +\infty\right) \mapsto \alpha^{-1}(x)$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\tilde{Q}^3 \geq \left(\frac{C'}{20}\alpha^{-1}\left(x_{2,2}\right) - \frac{C}{B}\right)\tilde{Q}^3$$

$$B_2 = \frac{10}{\alpha^{-1}\left(x_{2,2}\right)}\frac{C}{C'}$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\tilde{Q}^3 \geq \frac{1}{2}C'\tilde{Q}^3$$

$$\alpha^{-1}\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$$

$$V_{\odot}^{\dagger}(x) \geq \frac{1}{2}\tilde{C}\mathbf{1}_{\{1\leq x\leq \tilde{x}\}}(x) + \frac{1}{2}C'\tilde{Q}^3\mathbf{1}_{\{x\geq \tilde{x}\}}(x) + \frac{1}{20}\alpha^{-1}(x)|V_0'(x)|$$
$$\geq \frac{1}{2}\tilde{C}\mathbf{1}_{\{1\leq x\leq \tilde{x}\}}(x) + \frac{1}{2}C'\tilde{Q}^3\mathbf{1}_{\{x\geq \tilde{x}\}}(x)$$

$$V_B^{\mathrm{I}}(x) \geq C_1'\tilde{Q}^3\mathbf{1}_{\{x\geq 1\}}(x),$$

$$V_{\triangleq}^{\star} \geq V_2 \text{ where } V_2 = C_1''\tilde{Q}^3(x),$$

$$-\frac{1}{4}\tilde{Q}'' = \frac{1}{2}\tilde{Q}^3\left(\frac{5}{6}\tilde{Q} - 1\right)$$

$$\bar{x} = \alpha\left(2\operatorname{arccosh}\left(\sqrt{\frac{4}{5}}\right)\right) \sim 0.576$$

$$V_{\wedge}^{\boxdot}(x) > 0$$

$$V_{\bowtie}^{\vee}(x) \geq -\frac{1}{4}\tilde{Q}'' - \frac{1}{5}\alpha^{-1}(x)V'(x) = \frac{1}{2}\tilde{Q}^3\left(\frac{5}{6}\tilde{Q} - 1\right) + \frac{2}{5}\alpha^{-1}(x)(2 - 3\tilde{Q})\tilde{Q}^3\tilde{H}$$
$$= \left(\frac{4}{5}\alpha^{-1}(x)\tilde{H} - \frac{1}{2}\right)\tilde{Q}^3 + \left(\frac{5}{12} - \frac{6}{5}\alpha^{-1}(x)\tilde{H}\right)\tilde{Q}^4 = k(\alpha^{-1}(x))\tilde{Q}^3$$

$$k\colon \mathbb{R}_+ \mapsto \mathbb{R}$$

$$k(s) := \frac{4}{5}sH(s) - \frac{1}{2} + \left(\frac{5}{12} - \frac{6}{5}sH(s)\right)Q(s).$$

$$V_{\sqcap}^{\triangleq}(x) \geq C\mathbf{1}_{|x|\leq x_{2,2}}(x) + \tilde{C}\tilde{Q}^3\mathbf{1}_{|x|\geq x_{2,2}}(x)$$

$$V_B^{\sqcup}(x) \geq C_1''\tilde{Q}^3(x)$$

$$\frac{d}{dt}\mathcal{J} \leq -\int \tilde{Q}\left[(\partial_x z)^2 + C_1\tilde{Q}^2 z^2\right] + \tilde{J}_1 + J_2 + J_3 + J_4$$

$$\|(1-\gamma\partial_x^2)^{-1}f\|_{L^2} \leq \|f\|_{L^2}, \|(1-\gamma\partial_x^2)^{-1}\partial_x f\|_{L^2} \leq \gamma^{-\frac{1}{2}}\|f\|_{L^2}$$
$$\|(1-\gamma\partial_x^2)^{-1}\partial_x^2 f\|_{L^2} \leq \gamma^{-1}\|f\|_{L^2}$$

$$\|\operatorname{sech}(K\alpha^{-1}(x))(1-\gamma\partial_x^2)^{-1}f\|_{L^2} \leq (1+m_0)\|(1-\gamma\partial_x^2)^{-1}[\operatorname{sech}(K\alpha^{-1}(x))f]\|_{L^2},$$

$$\|\cosh(K\alpha^{-1}(x))(1-\gamma\partial_x^2)^{-1}f\|_{L^2} \lesssim \|(1-\gamma\partial_x^2)^{-1}[\cosh(K\alpha^{-1}(x))f]\|_{L^2}$$

$$g = \operatorname{sech}(K\alpha^{-1})(1-\gamma\partial_x^2)^{-1}f \text{ and } k = (1-\gamma\partial_x^2)^{-1}[\operatorname{sech}(K\alpha^{-1})f]$$

$$\begin{aligned} f =& \cosh(K\alpha^{-1})(1-\gamma\partial_x^2)k = (1-\gamma\partial_x^2)[\cosh(K\alpha^{-1})g] \\ =& \cosh(K\alpha^{-1})g - \gamma[\cosh(K\alpha^{-1})''g + 2\cosh(K\alpha^{-1})'\partial_x g + \cosh(K\alpha^{-1})\partial_x^2 g] \\ =& \cosh(K\alpha^{-1})(1-\gamma\partial_x^2)g - \gamma K\cosh(K\alpha^{-1})\tilde{Q}^2\left[K - \tilde{H}\tanh(K\alpha^{-1})\right]g \\ & -2\gamma K\cosh(K\alpha^{-1})\tilde{Q}\tanh(K\alpha^{-1})\partial_x g \end{aligned}$$

$$(1-\gamma\partial_x^2)k = (1-\gamma\partial_x^2)g - \gamma K\tilde{Q}^2\left[K - \tilde{H}\tanh(K\alpha^{-1})\right]g - 2\gamma K\tilde{Q}\tanh(K\alpha^{-1})\partial_x g$$

$$\begin{aligned} g =& k + \gamma K(1-\gamma\partial_x^2)^{-1}\{\tilde{Q}^2[K-\tilde{H}\tanh(K\alpha^{-1})]g\} \\ &+2\gamma K(1-\gamma\partial_x^2)^{-1}[\tilde{Q}\tanh(K\alpha^{-1})\partial_x g] \end{aligned}$$

$$\|(1-\gamma\partial_x^2)^{-1}\|_{\mathcal{L}(L^2,L^2)} \le 1, \|(1-\gamma\partial_x^2)^{-1}\partial_x\|_{\mathcal{L}(L^2,L^2)} \le \gamma^{-\frac{1}{2}}$$

$$\begin{aligned} &\gamma K\|(1-\gamma\partial_x^2)^{-1}\{\tilde{Q}^2[K-\tilde{H}\tanh(K\alpha^{-1})]g\}\|_{L^2} \\ &\quad \le \gamma K\left\|\tilde{Q}^2[K-\tilde{H}\tanh(K\alpha^{-1})]g\right\|_{L^2} \le (1+K)\gamma K\left\|\tilde{Q}^2 g\right\|_{L^2} \end{aligned}$$

$$\begin{aligned} &\|(1-\gamma\partial_x^2)^{-1}[\tilde{Q}\tanh(K\alpha^{-1})\partial_x g]\|_{L^2} \\ &\quad \le \left\|(1-\gamma\partial_x^2)^{-1}\partial_x[\tilde{Q}\tanh(K\alpha^{-1})g]\right\|_{L^2} + \left\|(1-\gamma\partial_x^2)^{-1}\left[\partial_x\left(\tilde{Q}\tanh(K\alpha^{-1})\right)g\right]\right\|_{L^2} \\ &\quad \le \gamma^{-\frac{1}{2}}\left\|\tilde{Q}\tanh(K\alpha^{-1})g\right\|_{L^2} + \left\|\tilde{Q}^2\left(K\operatorname{sech}^2(K\alpha^{-1}) - \tilde{H}\tanh(K\alpha^{-1})\right)g\right\|_{L^2} \\ &\quad \le \gamma^{-\frac{1}{2}}K\left\|\tilde{Q}^2 g\right\|_{L^2} + \left\|\tilde{Q}^2 g\right\|_{L^2} \le 3\gamma^{-\frac{1}{2}}\left\|\tilde{Q}^2 g\right\|_{L^2} \end{aligned}$$

$$\|g\|_{L^2} \le \|k\|_{L^2} + (1+K)\gamma K\left\|\tilde{Q}^2 g\right\|_{L^2} + 6K\gamma^{\frac{1}{2}}\left\|\tilde{Q}^2 g\right\|_{L^2}.$$

$$\|g\|_{L^2} \le (1+m_0)\|k\|_{L^2},$$

$$g = \cosh(K\alpha^{-1})(1-\gamma\partial_x^2)^{-1}f \text{ and } K = (1-\gamma\partial_x^2)^{-1}[\cosh(K\alpha^{-1})f]$$

$$\begin{aligned} f &= \operatorname{sech}(K\alpha^{-1})(1-\gamma\partial_x^2)k = (1-\gamma\partial_x^2)[\operatorname{sech}(K\alpha^{-1})g] \\ &= \operatorname{sech}(K\alpha^{-1})g - \gamma[\operatorname{sech}(K\alpha^{-1})''g + 2\operatorname{sech}(K\alpha^{-1})'\partial_x g + \operatorname{sech}(K\alpha^{-1})\partial_x^2 g] \\ &= \operatorname{sech}(K\alpha^{-1})(1-\gamma\partial_x^2)g - \gamma K\tilde{Q}\operatorname{sech}(K\alpha^{-1})\left[K\tilde{Q}\left(1-2\operatorname{sech}^2(K\alpha^{-1})\right)g - 2\tanh(K\alpha^{-1})\partial_x g\right] \end{aligned}$$

$$\begin{aligned} g =& k + \gamma K^2(1-\gamma\partial_x^2)^{-1}\left[\tilde{Q}^2\left(1-2\operatorname{sech}^2(K\alpha^{-1})\right)g\right] \\ &-2\gamma K(1-\gamma\partial_x^2)^{-1}[\tilde{Q}\tanh(K\alpha^{-1})\partial_x g] \end{aligned}$$

$$\left\|(1-\gamma\partial_x^2)^{-1}\left[\tilde{Q}^2\left(1-2\operatorname{sech}^2(K\alpha^{-1})\right)g\right]\right\|_{L^2} \lesssim \left\|\tilde{Q}^2\left(1-2\operatorname{sech}^2(K\alpha^{-1})\right)g\right\|_{L^2} \lesssim \|g\|_{L^2}$$

$$\begin{aligned} &\left\|(1-\gamma\partial_x^2)^{-1}[\tilde{Q}\tanh(K\alpha^{-1})\partial_x g]\right\|_{L^2} \\ &\lesssim \left\|(1-\gamma\partial_x^2)^{-1}\partial_x[\tilde{Q}\tanh(K\alpha^{-1})g]\right\|_{L^2} + \left\|(1-\gamma\partial_x^2)^{-1}\left[\partial_x\left(\tilde{Q}\tanh(K\alpha^{-1})\right)g\right]\right\|_{L^2} \\ &\lesssim \gamma^{-\frac{1}{2}}\left\|\tilde{Q}\tanh(K\alpha^{-1})g\right\|_{L^2} + \left\|\tilde{Q}^2[K\operatorname{sech}^2(K\alpha^{-1}) - \tilde{H}\tanh(K\alpha^{-1})]g\right\|_{L^2} \\ &\lesssim \gamma^{-\frac{1}{2}}\|g\|_{L^2}. \end{aligned}$$

$$\|g\|_{L^2} \le \|k\|_{L^2} + \tilde{C}\gamma^{\frac{1}{2}}\|g\|_{L^2}$$

$$\left\|\operatorname{sech}\left(\left(\frac{3}{2}+\frac{1}{A}\right)\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}f\right\| \lesssim \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\left(\frac{3}{2}+\frac{1}{A}\right)\alpha^{-1}\right)f\right]\right\|,$$

$$\left\|\operatorname{sech}\left(\left(\frac{1}{2}+\frac{1}{A}\right)\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}f\right\| \lesssim \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\left(\frac{1}{2}+\frac{1}{A}\right)\alpha^{-1}\right)f\right]\right\|$$

$$\sigma_A\tilde{Q}^{-\frac{1}{2}} \lesssim \cosh\left(\frac{2-A}{2A}\alpha^{-1}\right) \lesssim \sigma_A\tilde{Q}^{-\frac{n}{2}}$$

$$\left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}(1-\gamma\partial_x^2)^{-1}f\right\| \lesssim \left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}f\right\|.$$
$$\left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}(1-\gamma\partial_x^2)^{-1}\partial_x f\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}f\right\|.$$

$$v=(1-\gamma\partial_x^2)^{-1}Uu$$

$$\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}v\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}u\right\|$$

$$\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x v\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x u\right\| + \left\|\sigma_A\tilde{Q}^{\frac{5}{2}}u\right\|$$

$$\sigma_A\tilde{Q}^{\frac{n}{2}} \lesssim \operatorname{sech}\left(\left(\frac{n}{2}+\frac{1}{A}\right)\alpha^{-1}\right) \lesssim \sigma_A\tilde{Q}^{\frac{n}{2}}$$

$$\begin{aligned}
\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}v\right\| \lesssim& \left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)v\right\| \\
\lesssim& \left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}\partial_x u\right\| + \left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}[h_0 u]\right\| \\
\lesssim& \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)\partial_x u\right]\right\| + \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)h_0 u\right]\right\| \\
\lesssim& \left\|(1-\gamma\partial_x^2)^{-1}\partial_x\left[\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)u\right]\right\| \\
&+\frac{3A+2}{2A}\left\|(1-\gamma\partial_x^2)^{-1}\left[\tilde{Q}\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)u\right]\right\| + \left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)h_0 u\right\| \\
\lesssim& \gamma^{-\frac{1}{2}}\left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)u\right\| + \left\|\tilde{Q}\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)u\right\| + \left\|\sigma_A\tilde{Q}^{\frac{3}{2}}h_0 u\right\| \\
\lesssim& \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}u\right\| + \left\|\sigma_A\tilde{Q}^{\frac{3}{2}}u\right\| + \left\|\sigma_A\tilde{Q}^{\frac{3}{2}}h_0 u\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}u\right\|.
\end{aligned}$$

$$\partial_x U = \partial_x^2 - h_0\partial_x - h_0'$$

$$|h_0'| \lesssim |V| \lesssim \tilde{Q}^2$$

$$\begin{aligned}
\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x v\right\| \lesssim& \left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)\partial_x v\right\| \\
\lesssim& \left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}\partial_x^2 u\right\| + \left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}[h_0\partial_x u]\right\| \\
&+\left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}[h_0' u]\right\| \\
\lesssim& \left\|(1-\gamma\partial_x^2)^{-1}\partial_x\left[\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)\partial_x u\right]\right\| + \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)'\partial_x u\right]\right\| \\
&+\left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)h_0\partial_x u\right]\right\| + \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)h_0' u\right]\right\| \\
\lesssim& \gamma^{-\frac{1}{2}}\left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)\partial_x u\right\| + \left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)\tilde{Q}^2 u\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x u\right\| + \left\|\sigma_A\tilde{Q}^{\frac{5}{2}}u\right\|
\end{aligned}$$

$$\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x(\tilde{\chi}_B u_1)\right\| \lesssim \left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|.$$

$$\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}v_1\right\|_{L^2} \lesssim \gamma^{-\frac{1}{2}}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|$$
$$\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x v_1\right\| \lesssim \gamma^{-\frac{1}{2}}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)$$

$$\begin{aligned}\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x(\tilde{\chi}_B u_1)\right\| &\lesssim \left\|\zeta_A\tilde{Q}^{\frac{1}{2}}\partial_x u_1\right\| + \left\|\zeta_A\tilde{Q}^{\frac{3}{2}}u_1\right\| \\ &\lesssim \left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\| + \left\|\tilde{Q}^{\frac{1}{2}}\zeta_A' u_1\right\| \lesssim \left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|.\end{aligned}$$

$$\begin{aligned}\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x v_1\right\| &\lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x(\tilde{\chi}_B u_1)\right\| + \left\|\sigma_A\tilde{Q}^{\frac{5}{2}}\tilde{\chi}_B u_1\right\| \\ &\lesssim \gamma^{-\frac{1}{2}}\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \gamma^{-\frac{1}{2}}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\| + \left\|\sigma_A\tilde{Q}^{\frac{5}{2}}u_1\right\| \lesssim \gamma^{-\frac{1}{2}}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right),\end{aligned}$$

$$\zeta_B(x) \lesssim e^{-\frac{1}{B}|\alpha^{-1}(x)|}, |\zeta_B'(x)| \lesssim \frac{1}{B}\tilde{Q}e^{-\frac{1}{B}|\alpha^{-1}(x)|}, |\varphi_B| \lesssim B$$
$$|\tilde{\chi}_A'| \lesssim \frac{1}{A}\tilde{Q}, |(\tilde{\chi}_A^2)'| \lesssim \frac{1}{A}\tilde{Q}, |\tilde{\chi}_A''| \lesssim \frac{1}{A}\tilde{Q}^2, |(\tilde{\chi}_A^2)'''| \lesssim \frac{1}{A}\tilde{Q}^3$$

$$\tilde{\chi}_A'(x) = \tilde{\chi}_A''(x) = \tilde{\chi}_A'''(x) = 0$$

$$\begin{aligned}\tilde{J}_1 =& \frac{1}{4}\int \tilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)' v_1^2 + \frac{1}{4}\int \tilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2 v_1^2 + \frac{1}{2}\int \tilde{Q}[3(\tilde{\chi}_A')^2 + \tilde{\chi}_A''\tilde{\chi}_A]\zeta_B^2 v_1^2 \\ &+ \frac{1}{4}\int (\tilde{\chi}_A^2)'''\varphi_B v_1^2 - \int (\tilde{\chi}_A^2)'\varphi_B(\partial_x v_1)^2 \\ =& J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5}\end{aligned}$$

$$|(\tilde{\chi}_A^2)'(\zeta_B^2)'| \lesssim \frac{1}{AB}e^{-2\frac{A}{B}}\sigma_A^2\tilde{Q}^2, |(\tilde{\chi}_A^2)'\zeta_B^2| \lesssim \frac{1}{A}e^{-2\frac{A}{B}}\sigma_A^2\tilde{Q}$$
$$(\tilde{\chi}_A')^2\zeta_B^2 + |\tilde{\chi}_A''\tilde{\chi}_A|\zeta_B^2 \lesssim \frac{1}{A}e^{-2\frac{A}{B}}\sigma_A^2\tilde{Q}^2, |(\tilde{\chi}_A^2)'''\varphi_B| \lesssim \frac{B}{A}\sigma_A^2\tilde{Q}^3, |(\tilde{\chi}_A^2)'\varphi_B| \lesssim \frac{B}{A}\sigma_A^2\tilde{Q}$$

$$|J_{1,1}| + |J_{1,2}| + |J_{1,3}| + |J_{1,4}| \lesssim \frac{B}{A}\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}v_1\right\|^2 \lesssim \gamma^{-1}\frac{B}{A}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|^2$$

$$|J_{1,5}| \lesssim \frac{B}{A}\left\|\sigma_A\tilde{Q}^{\frac{1}{2}}\partial_x v_1\right\|^2 \lesssim \gamma^{-1}\frac{B}{A}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\|^2 + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|^2\right)$$

$$|\tilde{J}_1| \lesssim \gamma^{-1}\frac{B}{A}\left(\int \tilde{Q}(\partial_x w_1)^2 + \int \tilde{Q}^3 w_1^2\right)$$

$$|J_2| \lesssim \gamma\left\|\tilde{Q}(1-\gamma\partial_x^2)^{-1}\left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi_{A,B}' v_1\right)\right\| \left\|\tilde{Q}^{-1}(V_0'' v_1 + V_0'\partial_x v_1)\right\|$$

$$\begin{aligned}\left\|\tilde{Q}(1-\gamma\partial_x^2)^{-1}(\psi_{A,B}\partial_x v_1)\right\| &\lesssim \left\|\operatorname{sech}(\alpha^{-1})(1-\gamma\partial_x^2)^{-1}(\psi_{A,B}\partial_x v_1)\right\| \lesssim \left\|(1-\gamma\partial_x^2)^{-1}(\operatorname{sech}(\alpha^{-1})\psi_{A,B}\partial_x v_1)\right\| \\ &\lesssim \left\|\operatorname{sech}(\alpha^{-1})\psi_{A,B}\partial_x v_1\right\| \lesssim \left\|\tilde{Q}\psi_{A,B}\partial_x v_1\right\|.\end{aligned}$$

$$\partial_x z = \tilde{\chi}_A\zeta_B\partial_x v_1 + (\tilde{\chi}_A\zeta_B)' v_1 \implies \tilde{\chi}_A^2\zeta_B^2(\partial_x v_1)^2 \lesssim (\partial_x z)^2 + |(\tilde{\chi}_A\zeta_B)' v_1|^2$$

$$|(\tilde{\chi}_A\zeta_B)' v_1|^2\tilde{\chi}_A^2 \lesssim \left(\frac{1}{A}+\frac{1}{B}\right)^2\tilde{Q}^2\tilde{\chi}_A^2\zeta_B^2 v_1^2 \lesssim \frac{1}{B^2}\tilde{Q}^2 z^2$$

$$\tilde{\chi}_A^4\zeta_B^2(\partial_x v_1)^2 \lesssim \tilde{\chi}_A^2(\partial_x z)^2 + \frac{1}{B^2}\tilde{Q}^2 z^2$$

$$\left|\psi_{A,B}\right| \leq |\alpha^{-1}(x)|\tilde{\chi}_A^2$$

$$\tilde{Q}^2\left|\psi_{A,B}\partial_x v_1\right|^2 \lesssim |\alpha^{-1}(x)|^2\tilde{Q}^2\tilde{\chi}_A^4(\partial_x v_1)^2 \lesssim \tilde{Q}\tilde{\chi}_A^4\zeta_B^2(\partial_x v_1)^2 \lesssim \tilde{Q}(\partial_x z)^2 + \frac{1}{B^2}\tilde{Q}^3 z^2$$

$$\left\|\tilde{Q}\psi_{A,B}\partial_x v_1\right\| \lesssim \left(\tilde{Q}(\partial_x z)^2 + \frac{1}{B^2}\tilde{Q}^3 z^2\right)^{\frac{1}{2}}$$

$$\left\|\tilde{Q}(1-\gamma\partial_x^2)^{-1}\left(\psi'_{A,B}v_1\right)\right\| \lesssim \left\|\tilde{Q}\psi'_{A,B}v_1\right\|$$

$$\left(\psi'_{A,B}\right)^2 \leq \frac{21}{10}\tilde{Q}^2\tilde{\chi}_A^2$$

$$|\alpha^{-1}(x)| \geq 2A$$

$$\left(\psi'_{A,B}\right)^2 = \left[(\tilde{\chi}_A^2)'\varphi_B + \tilde{\chi}_A^2\tilde{Q}\zeta_B^2\right]^2 \leq 8(\tilde{\chi}_A\tilde{\chi}'_A\varphi_B)^2 + 2\tilde{Q}^2\tilde{\chi}_A^4\zeta_B^4 \leq \tilde{Q}^2\tilde{\chi}_A^2\left(\frac{C}{A^2}B^2 + 2\right) \leq \frac{21}{10}\tilde{Q}^2\tilde{\chi}_A^2$$

$$\left|\tilde{Q}^2\left(\psi'_{A,B}\right)^2 v_1^2\right| \leq \frac{21}{10}\tilde{Q}^4\tilde{\chi}_A^2 v_1^2 \leq \frac{63}{20}\tilde{Q}^3 z^2$$

$$\left\|\tilde{Q}\psi_{A,B}v_1\right\|^2 \leq 4\int \tilde{Q}^3 z^2$$

$$\left\|\tilde{Q}(1-\gamma\partial_x^2)^{-1}\left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B}v_1\right)\right\| \lesssim \left(\int \tilde{Q}(\partial_x z)^2 + \tilde{Q}^3 z^2\right)^{\frac{1}{2}}$$

$$|h_0| \lesssim 1, h'_0 = \frac{1}{4h_0}(V'_0 + V'), h''_0 = V' - 2h_0 h'_0$$

$$V''_0 = 4(h'_0)^2 + 4h_0 h''_0 - V''$$

$$|V''_0 v_1| + |V'_0\partial_x v_1| \lesssim \tilde{Q}^3|v_1| + \tilde{Q}^3|\partial_x v_1|$$

$$\left\|\tilde{Q}^{-1}(V''_0 v_1 + V'_0\partial_x v_1)\right\| \lesssim \left\|\tilde{Q}^2 v_1\right\| + \left\|\tilde{Q}^2\partial_x v_1\right\|$$

$$\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^2 v_1^2 \lesssim \tilde{\chi}_A^2\zeta_B^2 v_1^2 = z^2$$

$$\tilde{Q}^4 v_1^2 = \tilde{Q}^4 v_1^2\tilde{\chi}_A^2 + \tilde{Q}^4 v_1^2(1-\tilde{\chi}_A^2) \lesssim \tilde{Q}^{\frac{7}{2}}z^2 + e^{-\frac{A}{2}}\tilde{Q}^{\frac{7}{2}}v_1^2.$$

$$\left\|\tilde{Q}^2 v_1\right\| \lesssim \left\|\tilde{Q}^{\frac{7}{4}}z\right\| + e^{-\frac{A}{2}}\left\|\tilde{Q}^{\frac{7}{4}}v_1\right\| \lesssim \left\|\tilde{Q}^{\frac{3}{2}}z\right\| + e^{-\frac{A}{4}}\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}v_1\right\|.$$

$$\left\|\tilde{Q}^2 v_1\right\| \lesssim \left\|\tilde{Q}^{\frac{3}{2}}z\right\| + \gamma^{-\frac{1}{2}}e^{-\frac{A}{4}}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|.$$

$$\tilde{\chi}_A\zeta_B\partial_x v_1 = \partial_x z - \frac{\zeta'_B}{\zeta_B}z - \tilde{\chi}'_A\zeta_B v_1$$

$$|\tilde{\chi}_A\zeta_B\partial_x v_1| \lesssim \partial_x z + \frac{1}{B}\tilde{Q}z$$

$$\tilde{Q}^4(\partial_x v_1)^2 = \tilde{Q}^4(\partial_x v_1)^2\tilde{\chi}_A^2 + \tilde{Q}^4(\partial_x v_1)^2(1-\tilde{\chi}_A^2) \lesssim \tilde{Q}^{\frac{7}{2}}(\partial_x z)^2 + \frac{1}{B}\tilde{Q}^{\frac{11}{2}}z^2 + e^{-A}\tilde{Q}^3(\partial_x v_1)^2$$

$$\begin{aligned}\|\tilde{Q}^2\partial_x v_1\| &\lesssim \left\|\tilde{Q}^{\frac{7}{4}}\partial_x z\right\| + \frac{1}{\sqrt{B}}\left\|\tilde{Q}^{\frac{11}{4}}z\right\| + e^{-\frac{A}{2}}\left\|\tilde{Q}^{\frac{7}{4}}\partial_x v_1\right\| \\ &\lesssim \left\|\tilde{Q}^{\frac{1}{2}}\partial_x z\right\| + \frac{1}{\sqrt{B}}\left\|\tilde{Q}^{\frac{3}{2}}z\right\| + e^{-\frac{A}{2}}\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}\partial_x v_1\right\| \\ &\lesssim \left\|\tilde{Q}^{\frac{1}{2}}\partial_x z\right\| + \frac{1}{\sqrt{B}}\left\|\tilde{Q}^{\frac{3}{2}}z\right\| + \gamma^{-\frac{1}{2}}e^{-\frac{A}{2}}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)\end{aligned}$$

$$\|\tilde{Q}^2 v_1\| + \|\tilde{Q}^2\partial_x v_1\| \lesssim \left\|\tilde{Q}^{\frac{1}{2}}\partial_x z\right\| + \left\|\tilde{Q}^{\frac{3}{2}}z\right\| + \gamma^{-\frac{1}{2}}e^{-\frac{A}{4}}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)$$

$$|J_2| \lesssim \gamma\left(\int \tilde{Q}(\partial_x z)^2 + \tilde{Q}^3 z^2\right) + e^{-\frac{A}{4}}\left(\int \tilde{Q}(\partial_x w_1)^2 + \tilde{Q}^3 w_1^2\right)$$

$$|J_3| \lesssim \left(\left\|\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^{-1}\psi_{A,B}\partial_x v_1\right\| + \left\|\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^{-1}\psi'_{A,B}v_1\right\|\right)\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}'_B\partial_x u_1 + \tilde{\chi}''_B u_1)\right\|$$

$$\left\|\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^{-1}\psi_{A,B}\partial_x v_1\right\| \lesssim B\left\|\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A\partial_x v_1\right\| \lesssim \gamma^{-\frac{1}{2}}B\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)$$

$$|\psi'_{A,B}| \leq |(\tilde{\chi}_A^2)'\varphi_B| + \tilde{Q}\tilde{\chi}_A^2\zeta_B^2 \lesssim \frac{B}{A}\tilde{Q}\tilde{\chi}_A 1_{\{A\leq|\alpha^{-1}(x)|\leq 2A\}} + \tilde{Q}\tilde{\chi}_A^2\zeta_B^2$$

$$\left\|\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^{-1}\psi'_{A,B}v_1\right\| \lesssim \frac{B}{A}\left\|\tilde{Q}^{\frac{3}{2}}1_{\{A\leq|x|\leq 2A\}}v_1\right\| + \|\tilde{Q}^{3/2}z\| \lesssim \gamma^{-\frac{1}{2}}\frac{B}{A}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\| + \|\tilde{Q}^{3/2}z\|$$

$$\left\|\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^{-1}\psi_{A,B}\partial_x v_1\right\| + \left\|\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^{-1}\psi'_{A,B}v_1\right\| \lesssim \gamma^{-\frac{1}{2}}B\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right) + \|\tilde{Q}^{3/2}z\|$$

$$1 \lesssim \sigma_A \lesssim \zeta_A[-2\alpha(A), 2\alpha(A)], U = \partial_x - h_0$$

$$\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}''_B u_1)\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}''_B u_1\right\|$$

$$\tilde{\chi}''_B \lesssim B^{-4}\tilde{Q}^2[-2\alpha(B^2), 2\alpha(B^2)]$$

$$\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}''_B u_1)\right\| \lesssim \gamma^{-\frac{1}{2}}B^{-4}\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}u_1\right\| \lesssim \gamma^{-\frac{1}{2}}B^{-4}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|$$

$$\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}'_B\partial_x u_1)\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}'_B\partial_x u_1\right\| \lesssim \gamma^{-\frac{1}{2}}B^{-2}\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\|$$

$$\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}'_B\partial_x u_1 + \tilde{\chi}''_B u_1)\right\| \lesssim \gamma^{-\frac{1}{2}}B^{-2}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)$$

$$|J_3| \lesssim \gamma^{-1}B^{-1}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)^2 + \gamma^{-\frac{1}{2}}B^{-2}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)\|\tilde{Q}^{3/2}z\|$$

$$N = N_g + N_b$$

$$N_g := \tilde{Q}^2\big(3\tilde{H}(a_1^2\phi_0^2 + 2a_1\phi_0 u_1) + a_1^3\phi_0^3 + 3a_1^2\phi_0^2 u_1 + 3a_1\phi_0 u_1^2\big)$$
$$N_b := \tilde{Q}^2 u_1^2\big(3\tilde{H} + u_1\big)$$

$$N^{\perp} = N - N_0\phi_0 = \big(N_g - N_0\phi_0\big) + N_b := N_g^{\perp} + N_b$$

$$J_{4,g} = -\int \left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B}v_1\right)(1-\gamma\partial_x^2)^{-1}U\big(\tilde{\chi}_B N_g^{\perp}\big)$$

$$|J_{4,g}| \lesssim \left(\left\|\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^{-1}\psi_{A,B}\partial_x v_1\right\| + \left\|\tilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^{-1}\psi'_{A,B}v_1\right\|\right)\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U\big(\tilde{\chi}_B N_g^{\perp}\big)\right\|$$

$$\begin{aligned}&\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U\big(\tilde{\chi}_B N_g^{\perp}\big)\right\| \\ &\le \left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U\big(\tilde{\chi}_B N_g\big)\right\| + |N_0|\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}_B\phi_0)\right\|.\end{aligned}$$

$$|N_g| \lesssim \tilde{Q}^2\phi_0(a_1^2 + u_1^2)$$
$$|N_0| \lesssim a_1^2 + \int \tilde{Q}^2\phi_0 u_1^2 \lesssim a_1^2 + \left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|.$$

$$\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U\big(\tilde{\chi}_B N_g\big)\right\| \lesssim \left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}(1-\gamma\partial_x^2)^{-1}U\big(\tilde{\chi}_B N_g\big)\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}N_g\right\|$$

$$\begin{aligned}\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U\big(\tilde{\chi}_B N_g\big)\right\| &\lesssim \gamma^{-\frac{1}{2}}\left(a_1^2\left\|\sigma_A\tilde{Q}^{\frac{3}{2}}\phi_0\right\| + \left\|\sigma_A\tilde{Q}^{\frac{3}{2}}\phi_0 u_1^2\right\|\right) \\ &\lesssim \gamma^{-\frac{1}{2}}\left(a_1^2 + \left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right).\end{aligned}$$

$$\left\|\tilde{Q}^{-\frac{1}{2}}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U\phi_0\right\| \lesssim \left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}(1-\gamma\partial_x^2)^{-1}U\phi_0\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\tilde{Q}^{-\frac{1}{2}}\phi_0\right\| \lesssim \gamma^{-\frac{1}{2}}$$

$$\begin{aligned}|J_{4,g}| \lesssim& \gamma^{-1}B\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)\left(a_1^2 + \left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right) \\ &+\gamma^{-\frac{1}{2}}\left\|\tilde{Q}^{\frac{3}{2}}z\right\|\left(a_1^2 + \left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)\end{aligned}$$

$$J_{4,b} = \int \left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B}v_1\right)(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}_B N_b)$$

$$\left\|\tilde{Q}^{1/2}\tilde{\chi}_A^{-1}\left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B}v_1\right)\right\| \lesssim \gamma^{-\frac{1}{2}}B\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right) + \left\|\tilde{Q}^{3/2}z\right\|$$

$$\begin{aligned}\left\|\tilde{Q}^{-1/2}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}_B N_b)\right\| &\lesssim \left\|\sigma_A\tilde{Q}^{-1/2}(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}_B N_b)\right\| \\ &\lesssim \gamma^{-\frac{1}{2}}\left(\left\|\sigma_A\tilde{\chi}_B\tilde{Q}^{3/2}u_1^2\right\| + \left\|\sigma_A\tilde{\chi}_B\tilde{Q}^{3/2}u_1^3\right\|\right).\end{aligned}$$

$$\begin{aligned}\left\|\tilde{Q}^{-1/2}\tilde{\chi}_A(1-\gamma\partial_x^2)^{-1}U(\tilde{\chi}_B N_b)\right\| &\lesssim \gamma^{-\frac{1}{2}}\alpha(B)\left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\left\|\sigma_A\tilde{Q}^{3/2}u_1\right\| + \gamma^{-\frac{1}{2}}\alpha(B)^2\left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}^2\left\|\sigma_A\tilde{Q}^{3/2}u_1\right\| \\ &\lesssim \gamma^{-\frac{1}{2}}\alpha(B)\left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\left(1+\alpha(B)\left\|\tilde{Q}^{1/2}u_1\right\|_{L^\infty}\right)\left\|\tilde{Q}^{3/2}w_1\right\|.\end{aligned}$$

$$|J_{4,b}| \lesssim \gamma^{-1}B\alpha(B)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\left(1+\alpha(B)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\right)\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\|+\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)\|\tilde{Q}^{3/2}w_1\|$$
$$+\gamma^{-\frac{1}{2}}B\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\left(1+\alpha(B)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\right)\|\tilde{Q}^{3/2}z\|\|\tilde{Q}^{3/2}w_1\|$$

$$|J_4| \lesssim \gamma^{-1}B\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\|+\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)\left(a_1^2+\alpha(B)\left(1+\alpha(B)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\right)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)$$
$$+\gamma^{-\frac{1}{2}}\left\|\tilde{Q}^{\frac{3}{2}}z\right\|\left(a_1^2+B\left(1+\alpha(B)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\right)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)$$

$$\frac{d}{dt}\mathcal{J} \le -4C_2\int \tilde{Q}\left[(\partial_x z)^2+\tilde{Q}^2z^2\right]+\gamma^{-1}\frac{CB}{A}\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]$$
$$+\gamma C\int \tilde{Q}\left[(\partial_x z)^2+\tilde{Q}^2z^2\right]+Ce^{-\frac{A}{4}}\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]$$
$$+\gamma^{-1}B^{-1}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\|+\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)^2+\gamma^{-\frac{1}{2}}B^{-2}\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\|+\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)\|\tilde{Q}^{3/2}z\|$$
$$+\gamma^{-1}CB\left(\left\|\tilde{Q}^{\frac{1}{2}}\partial_x w_1\right\|+\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)\left(a_1^2+\alpha(B)\left(1+\alpha(B)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\right)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)$$
$$+\gamma^{-\frac{1}{2}}C\left\|\tilde{Q}^{\frac{3}{2}}z\right\|\left(a_1^2+B\left(1+\alpha(B)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\right)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)$$

$$\frac{d}{dt}\mathcal{J} \le -C_2\int \tilde{Q}\left[(\partial_x z)^2+\tilde{Q}^2z^2\right]+C\left(\frac{1}{B}+\frac{B}{A}+e^{-\frac{A}{4}}\right)\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]$$
$$+CB\alpha(B)\left(a_1^2+B\left(1+\alpha(B)\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\right)\|u_1\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)^2$$

$$\alpha(B)\|\tilde{Q}^{1/2}u_1\|_{L^\infty} \lesssim \delta^{\frac{7}{8}} \lesssim 1, B^{-1}+A^{-1}B+e^{-\frac{A}{4}} \lesssim \ln\left(\delta^{-\frac{1}{8}}\right)^{-1}$$
$$B\alpha(B)\left(\alpha(B)\|u_1\|_{L^\infty}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|\right)^2 \lesssim \delta^{-\frac{1}{2}}\|u_1\|_{L^\infty}^2\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|^2 \lesssim \delta^{\frac{3}{2}}\left\|\tilde{Q}^{\frac{3}{2}}w_1\right\|^2.$$

$$\frac{d}{dt}\mathcal{J} \le -C_2\int \tilde{Q}\left[(\partial_x z)^2+\tilde{Q}^2z^2\right]+C\ln\left(\delta^{-\frac{1}{8}}\right)^{-1}\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]+C\delta^{\frac{1}{2}}|a_1|^3$$

$$\int \tilde{Q}^7u_1^2$$

$$v=(1-\gamma\partial_x^2)^{-1}Uu$$

$$\int \tilde{Q}^7u^2 \lesssim \int \tilde{Q}^{\frac{9}{2}}[(\partial_x v)^2+v^2]$$

$$v-\gamma\partial_x^2 v=\phi_0\partial_x\left(\frac{u}{\phi_0}\right)$$

$$\partial_x\left(\frac{u}{\phi_0}+\gamma\frac{\partial_x v}{\phi_0}\right)=\frac{1}{\phi_0}(v-\gamma h_0\partial_x v)$$

$$\frac{u}{\phi_0}+\gamma\frac{\partial_x v}{\phi_0}=a+\int_0^x\frac{1}{\phi_0}(v-\gamma h_0\partial_x v)$$

$$u=a\phi_0-\gamma\partial_x v+\tilde{u}$$

$$\tilde{u} = \phi_0 \int_0^x \frac{1}{\phi_0}(v - \gamma h_0 \partial_x v)$$

$$\phi_0 \int_0^x \frac{|v|}{\phi_0} \lesssim \phi_0 \left(\int \tilde{Q}^{\frac{9}{2}} v^2 \right)^{\frac{1}{2}} \left(\int_0^x \frac{1}{\tilde{Q}^{\frac{9}{2}} \phi_0^2} \right)^{\frac{1}{2}} \lesssim \left\| \tilde{Q}^{\frac{9}{4}} v \right\| \left(\int_0^{\alpha^{-1}(x)} \frac{1}{Q^{\frac{11}{2}}} \right)^{\frac{1}{2}} \lesssim \tilde{Q}^{-\frac{11}{4}} \left\| \tilde{Q}^{\frac{9}{4}} v \right\|$$

$$\phi_0 \int_0^x \frac{|h_0 \partial_x v|}{\phi_0} \lesssim \phi_0 \left(\int \tilde{Q}^{\frac{9}{2}} (h_0 \partial_x v)^2 \right)^{\frac{1}{2}} \left(\int_0^x \frac{1}{\tilde{Q}^{\frac{9}{2}} \phi_0^2} \right)^{\frac{1}{2}} \lesssim \tilde{Q}^{-\frac{11}{4}} \left\| \tilde{Q}^{\frac{9}{4}} \partial_x v \right\|$$

$$\tilde{Q}^{\frac{11}{2}} \tilde{u}^2 \lesssim \int \tilde{Q}^{\frac{9}{2}} [(\partial_x v)^2 + v^2]$$

$$\int \tilde{Q}^7 \tilde{u}^2 \lesssim \left(\int \tilde{Q}^{\frac{3}{2}} \right) \left(\int \tilde{Q}^{\frac{9}{2}} [(\partial_x v)^2 + v^2] \right) \lesssim \int \tilde{Q}^{\frac{9}{2}} [(\partial_x v)^2 + v^2]$$

$$a = \gamma \langle \partial_x v, \phi_0 \rangle - \langle \tilde{u}, \phi_0 \rangle.$$

$$a^2 \lesssim \left(\int \phi_0 \partial_x v \right)^2 + \left(\int \phi_0 \tilde{u} \right)^2 \lesssim \int \tilde{Q}^{\frac{9}{2}} (\partial_x v)^2 + \int \tilde{Q}^7 \tilde{u}^2 \lesssim \int \tilde{Q}^{\frac{7}{2}} [(\partial_x v)^2 + v^2]$$

$$\int \tilde{Q}^7 u_1^2 \lesssim \int \tilde{Q}^2 \left[(\partial_x z)^2 + \tilde{Q}^2 z^2 \right] + e^{-\frac{A}{4}} \int \tilde{Q} \left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2 \right]$$

$$\int \tilde{Q}^7 u_1^2 \lesssim \int \tilde{Q}^{\frac{9}{2}} [(\partial_x v_1)^2 + v_1^2]$$

$$\tilde{Q} \lesssim e^{-|\alpha^{-1}(x)|}, \tilde{Q}^{\frac{1}{4}} \lesssim \zeta_B^2$$

$$\begin{aligned} \tilde{Q}^{\frac{9}{2}} [(\partial_x v_1)^2 + v_1^2] &\lesssim \int \tilde{Q}^4 \zeta_B^2 (\partial_x v_1)^2 + \int \tilde{Q}^4 \zeta_B^2 v_1^2 \\ &\lesssim \int \tilde{Q}^4 \left[(\partial_x z)^2 + \tilde{Q}^2 z^2 \right] + \int \tilde{Q}^4 z^2 + e^{-3A} \int \tilde{Q} \zeta_B^2 (1 - \tilde{\chi}_A^2) (\partial_x v_1)^2 + e^{-A} \int \tilde{Q}^3 \zeta_B^2 (1 - \tilde{\chi}_A^2) v_1^2 \end{aligned}$$

$$\begin{aligned} \tilde{Q}^{\frac{9}{2}} [(\partial_x v_1)^2 + v_1^2] &\lesssim \int \tilde{Q}^2 \left[(\partial_x z)^2 + \tilde{Q}^2 z^2 \right] + e^{-3A} \int \sigma_A^2 \tilde{Q} (\partial_x v_1)^2 + e^{-A} \int \sigma_A^2 \tilde{Q}^3 v_1^2 \\ &\lesssim \int \tilde{Q}^2 \left[(\partial_x z)^2 + \tilde{Q}^2 z^2 \right] + e^{-A} \int \tilde{Q} \left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2 \right] \end{aligned}$$

$$\mathcal{H} = \mathcal{J} + 8 C C_0^{-1} \ln \left(\delta_3^{-\frac{1}{8}} \right)^{-1} \mathcal{J}$$

$$\frac{d}{dt} \mathcal{H} \leq -C_3 \int \tilde{Q} \left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2 \right] + |a_1|^3$$

$$\begin{aligned} \frac{d}{dt} \mathcal{J} &\leq -\frac{1}{2} C_0 \int \tilde{Q} \left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2 \right] + C \int \tilde{Q} \left[(\partial_x z)^2 + \tilde{Q}^2 z^2 \right] + C e^{-\frac{A}{4}} \int \tilde{Q} \left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2 \right] + C \delta_3 |a_1|^3 \\ &\leq -\frac{1}{4} C_0 \int \tilde{Q} \left[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2 \right] + C \int \tilde{Q} \left[(\partial_x z)^2 + \tilde{Q}^2 z^2 \right] + |a_1|^3 \end{aligned}$$

$$\frac{d}{dt}\mathcal{J} \le -C_2\int \tilde{Q}\left[(\partial_x z)^2+\tilde{Q}^2z^2\right]+C\ln\left(\delta_3^{-\frac{1}{8}}\right)^{-1}\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]+C\delta^{\frac{1}{2}}|a_1|^3$$

$$\frac{d}{dt}\mathcal{H} \le \left(-C_2+8C^2C_0^{-1}\ln\left(\delta_3^{-\frac{1}{8}}\right)^{-1}\right)\int \tilde{Q}\left[(\partial_x z)^2+\tilde{Q}^2z^2\right]$$
$$-C\ln\left(\delta_3^{-\frac{1}{8}}\right)^{-1}\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]+C\left(\delta^{\frac{1}{2}}+8C_0^{-1}\ln\left(\delta_3^{-\frac{1}{8}}\right)^{-1}\right)|a_1|^3$$

$$\frac{d}{dt}\mathcal{H} \le -\frac{C_2}{2}\int \tilde{Q}\left[(\partial_x z)^2+\tilde{Q}^2z^2\right]-C\ln\left(\delta_3^{-\frac{1}{8}}\right)^{-1}\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]+|a_1|^3$$

$$C_3 = C\ln\left(\delta_3^{-\frac{1}{8}}\right)^{-1} > 0$$

$$\mathcal{B} = b_+^2 - b_-^2$$

$$\left|\dot{b}_+ - \mu_0 b_+\right| + \left|\dot{b}_- + \mu_0 b_-\right| \le C_4\left(b_+^2+b_-^2+\int \tilde{Q}^3w_1^2\right)$$

$$\left|\frac{d}{dt}(b_+^2)-2\mu_0 b_+^2\right| + \left|\frac{d}{dt}(b_-^2)+2\mu_0 b_-^2\right| \le C_4\left(b_+^2+b_-^2+\int \tilde{Q}^3w_1^2\right)^{\frac{3}{2}}$$

$$\frac{d}{dt}\mathcal{B} \ge \mu_0(b_+^2+b_-^2)-C_4\int \tilde{Q}^3w_1^2 = \frac{\mu_0}{2}(a_1^2+a_2^2)-C_4\int \tilde{Q}^3w_1^2$$

$$|N_0| \lesssim a_1^2+\int \tilde{Q}^3w_1^2 \lesssim b_+^2+b_-^2+\int \tilde{Q}^3w_1^2$$

$$\frac{d}{dt}(\mathcal{B}-2C_4C_3^{-1}\mathcal{H}) \ge \frac{\mu_0}{2}(a_1^2+a_2^2)+C_4\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]-2C_4|a_1|^3$$

$$\frac{d}{dt}(\mathcal{B}-2C_4C_3^{-1}\mathcal{H}) \ge \frac{\mu_0}{4}(a_1^2+a_2^2)+C_4\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]$$

$$|\mathcal{I}| \le \left|\int \left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A' u_1\right)u_2\right| \lesssim A\left(\|\partial_x u_1\|_{L^2}+\left\|\tilde{Q}u_1\right\|_{L^2}\right)\|u_2\|_{L^2} \lesssim \delta$$

$$U = \partial_x - h_0, \psi_{A,B}' = \tilde{Q}\tilde{\chi}_A^2\zeta_B^2+(\tilde{\chi}_A^2)'\varphi_B$$

$$|\mathcal{J}| = \left|\int \left(\psi_{A,B}(x)\partial_x v_1(t,x)+\frac{1}{2}\psi_{A,B}'(x)v_1(t,x)\right)v_2(t,x)dx\right| \lesssim B\left(\|\partial_x v_1\|_{L^2}+\left\|\tilde{Q}v_1\right\|_{L^2}\right)\|v_2\|_{L^2}$$
$$\lesssim \delta$$

$$|\mathcal{H}| \lesssim \delta$$

$$\int_0^\infty \left\{a_1^2+a_2^2+\int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2w_1^2\right]\right\}dt \lesssim \delta$$

$$\int \tilde{Q}^2\left[(\partial_x u_1)^2+\tilde{Q}^{\frac{3}{2}}u_1^2\right]\lesssim \int \tilde{Q}\left[(\partial_x w_1)^2+\tilde{Q}^2 w_1^2\right]$$

$$\int_0^\infty\left\{a_1^2+a_2^2+\int \tilde{Q}^2\left[(\partial_x u_1)^2+\tilde{Q}^{\frac{3}{2}}u_1^2\right]\right\}dt\lesssim \delta$$

$$\mathcal{K}=\int u_1u_2\tilde{Q}^2 \text{ and } \mathcal{G}=\frac{1}{2}\int\left[(\partial_x u_1)^2+\tilde{Q}^{\frac{3}{2}}u_1^2+u_2^2\right]\tilde{Q}^2$$

$$\frac{d}{dt}\mathcal{K}=\int [\dot{u}_1u_2+u_1\dot{u}_2]\tilde{Q}^2=\int [u_2^2-u_1(Lu_1+N^\perp)]\tilde{Q}^2$$
$$=\int [u_2^2-(\partial_x u_1)^2-2\tilde{Q}^2(1-\tilde{Q})u_1^2]\tilde{Q}^2+\frac{1}{2}\int (\tilde{Q}^2)''u_1^2-\int N^\perp\tilde{Q}^2u_1$$

$$\int N^\perp\tilde{Q}^2u_1\lesssim a_1^2+\int \tilde{Q}^{\frac{7}{2}}u_1^2$$

$$\int \tilde{Q}^2u_2^2\leq\frac{d}{dt}\mathcal{K}+Ca_1^2+C\int \tilde{Q}^2\left[(\partial_x u_1)^2+\tilde{Q}^{\frac{3}{2}}u_1^2\right]$$

$$\int_0^\infty [a_1^2+a_2^2+\mathcal{G}]dt\lesssim\delta$$

$$\frac{d}{dt}\mathcal{G}=\int\left[(\partial_x\dot{u}_1)(\partial_x u_1)+\tilde{Q}^{\frac{3}{2}}\dot{u}_1u_1+\dot{u}_2u_2\right]\tilde{Q}^2=\int\left[(\partial_x u_2)(\partial_x u_1)+\tilde{Q}^{\frac{3}{2}}u_2u_1-(Lu_1+N^\perp)u_2\right]\tilde{Q}^2$$
$$=-2\int \tilde{Q}\tilde{Q}'u_2\partial_x u_1+\int\left(2\tilde{Q}^{\frac{3}{2}}-2\tilde{Q}^{\frac{1}{2}}+1\right)\tilde{Q}^{\frac{7}{2}}u_1u_2-\int \tilde{Q}^2N^\perp u_2$$

$$\left|\frac{d}{dt}\mathcal{G}\right|\lesssim a_1^2+\mathcal{G}$$

$$\lim_{n\to\infty}[a_1^2(t_n)+a_2^2(t_n)+\mathcal{G}(t_n)]=0$$

$$\mathcal{G}(t)\lesssim\int_t^\infty [a_1^2+\mathcal{G}]dt$$

$$\left|\frac{d}{dt}(a_1^2)\right|+\left|\frac{d}{dt}(a_2^2)\right|\lesssim a_1^2+a_2^2+\int \tilde{Q}^{\frac{7}{2}}u_1^2$$

$$a_1^2(t)+a_2^2(t)\lesssim\int_t^\infty [a_1^2+a_2^2+\mathcal{G}]dt$$

$$\lim_{t\to\infty}(|a_1(t)|+|a_2(t)|)=0$$

$$\widehat{H}(x):=(\sinh(\alpha^{-1}(x))+3\alpha^{-1}(x))\widetilde{H}(x)-4=:\beta(x)\widetilde{H}(x)-4$$
$$\beta(x)=3x+2\alpha^{-1}(x)$$

$$\langle\tilde{L}u_1,u_1\rangle\geq 0$$

$$\phi_1=\alpha_{00}\widetilde{H}+\beta_{00}\widehat{H}+\frac{1}{3}\left(\widehat{H}\int_x^\infty\phi_0\widetilde{H}+\widetilde{H}\int_0^x\phi_0\widehat{H}\right)$$

$$\phi_1 = \frac{1}{3}\left(\widehat{H}\int_x^\infty \phi_0\widetilde{H} + \widetilde{H}\int_0^x \phi_0\widehat{H}\right)$$

$$\phi_{1,x} = \frac{1}{3}\left(\widehat{H}'\int_x^\infty \phi_0\widetilde{H} + \frac{1}{3}\tilde{Q}^2\int_0^x \phi_0\widehat{H}\right) \in L^2(\mathbb{R})$$

$$\lim_\infty \phi_{1,x}\widehat{H} = \lim_\infty \phi_1\widehat{H}_x = 0$$

$$\langle\phi_0, \widehat{H}\rangle = \langle\tilde{L}\phi_1, \widehat{H}\rangle = \langle\phi_1, \tilde{L}\widehat{H}\rangle = 0$$

$$\int_0^\infty \phi_0\widehat{H} = 0$$

$$[0,\infty) \ni x \longmapsto g(x) := \int_0^x \phi_0\widehat{H}$$

$$\langle\phi_0, \phi_1\rangle = \frac{2}{3}\int_0^\infty \phi_0\left(\widehat{H}\int_x^\infty \phi_0\widetilde{H} + \widetilde{H}\int_0^x \phi_0\widehat{H}\right) = \frac{4}{3}\int_0^\infty \phi_0\widetilde{H}\int_0^x \phi_0\widehat{H} = \frac{4}{3}\int_0^\infty \phi_0\widetilde{H}g < 0$$

$$\langle\phi_0, u\rangle = \langle\tilde{Q}^2\widetilde{H}^3, u\rangle = 0$$

$$\langle\tilde{L}u, u\rangle \geq c_0\|u\|_{H_0}^2$$

$$\tau = \inf\{\langle\tilde{L}u, u\rangle : \|u\|_{H_0} = 1, \langle\phi_0, u\rangle = \langle\tilde{Q}^2\widetilde{H}^3, u\rangle = 0\}.$$

$$0 < \int (\partial_x u_n)^2 + \tilde{Q}^3 u_n^2 \leq \frac{5}{3}\tilde{Q}^3 u_n^2 + \eta$$

$$\int \tilde{Q}^3(u_n - u_\infty)^2 \to 0, \text{ as } n \to +\infty$$

$$v_\infty = u_\infty / \|u_\infty\|_{H_0}$$

$$\langle\tilde{L}u_\infty, u_\infty\rangle \leq \liminf_{n\to\infty}\langle\tilde{L}u_n, u_n\rangle = 0$$

$$\tilde{L}u = \alpha\left(-\partial_x^2 u + \tilde{Q}^2 u\right) + \beta\phi_0 + \gamma\tilde{Q}^2\widetilde{H}^3$$

$$\|u\|_{H_0} = 1, \langle\phi_0, u\rangle = \langle\tilde{Q}^2\widetilde{H}^3, u\rangle = 0$$

$$\alpha = \langle\tilde{L}u, u\rangle, \text{ so } \alpha = \tau = 0$$

$$\tilde{L}u_\infty = \beta\phi_0 + \gamma\tilde{Q}^2\widetilde{H}^3$$

$$u_\infty = \frac{\beta}{3}\left(\widehat{H}\int_x^\infty \phi_0\widetilde{H} + \widetilde{H}\int_0^x \phi_0\widehat{H}\right)$$

$$\langle Lu, u\rangle \geq C_5\left(\int (\partial_x u)^2 + \int \tilde{Q}^2 u^2\right)$$

$$L = \varepsilon_0\left(-\partial_x^2 + \tilde{Q}^2\right) + L_{\varepsilon_0}$$

$$L_{\varepsilon_0} := -(1-\varepsilon_0)\partial_x^2 + 2\tilde{Q}^2\left(1-\frac{1}{2}\varepsilon_0 - \tilde{Q}\right).$$

$$\begin{aligned}\left\langle L_{\varepsilon_0} f, f\right\rangle &= (1-\varepsilon_0)\int f'^2 + 2\int f^2\tilde{Q}^2\left(1-\frac{1}{2}\varepsilon_0-\tilde{Q}\right)\\ &= (1-\varepsilon_0)\int f'^2(x)dx + 2\int f^2(\alpha(y))Q(1-Q)(y)dy - \varepsilon_0\int f^2(\alpha(y))Q(y)dy\\ &\sim -0.6564\end{aligned}$$

$$L \geq L_{\varepsilon_0} \geq (1-\varepsilon_0)\left(-\partial_x^2 - 0.845Q_p^{7/2}\right)$$

$$0.808 \leq \mu_{\varepsilon_0} \leq 0.882$$

$$L_{\varepsilon_0}\phi_0 = -(1-\varepsilon_0)\mu_0^2\phi_0 - \varepsilon_0\tilde{Q}^3\phi_0$$

$$L_{\varepsilon_0}^{-1}\phi_0 = -\frac{\phi_0 + \varepsilon_0 L_{\varepsilon_0}^{-1}(\tilde{Q}^3\phi_0)}{\mu_0^2(1-\varepsilon_0)}$$

$$\left\langle L_{\varepsilon_0}^{-1}\phi_0, \phi_0\right\rangle = -\frac{1+\varepsilon_0\left\langle L_{\varepsilon_0}^{-1}(\tilde{Q}^3\phi_0), \phi_0\right\rangle}{\mu_0^2(1-\varepsilon_0)}$$

$$u_\varepsilon = L_\varepsilon^{-1}(\tilde{Q}^3\phi_0) \in L^\infty(\mathbb{R})$$

$$-(1-\varepsilon)\partial_x^2 u + 2\tilde{Q}^2\left(1-\frac{\varepsilon}{2}-\tilde{Q}\right)u = \tilde{Q}^3\phi_0$$

$$-(1-\varepsilon)\langle\partial_x^2 u_\varepsilon, \partial_x^{-2}\phi_0\rangle + 2\left\langle\tilde{Q}^2\left(1-\frac{\varepsilon}{2}-\tilde{Q}\right)u_\varepsilon, \partial_x^{-2}\phi_0\right\rangle = \left\langle\tilde{Q}^3\phi_0, \partial_x^{-2}\phi_0\right\rangle$$

$$\left\langle L_{\varepsilon_0}^{-1}\phi_0, \phi_0\right\rangle \leq -\frac{1-\varepsilon_0 C_{\varepsilon_1}}{\mu_0^2(1-\varepsilon_0)} < 0$$

$$\begin{aligned}&2\{E(\phi_1,\phi_2) - E(\tilde{H},0)\}\\ &=\mu_0^2(a_2^2-a_1^2) + \|u_2\|_{L^2}^2 + \langle Lu_1, u_1\rangle + 2\int \tilde{Q}^2\tilde{H}(a_1\phi_0+u_1)^3 + \frac{1}{2}\int \tilde{Q}^2(a_1\phi_0+u_1)^4\\ &=\mu_0^2(a_2^2-a_1^2) + \|u_2\|_{L^2}^2 + \langle Lu_1, u_1\rangle + \frac{1}{2}\int \tilde{Q}^2\left(u_1^2+4\tilde{H}u_1\right)u_1^2\\ &\quad +2a_1\int \tilde{Q}^2\phi_0 u_1^3 + 3a_1\int \tilde{Q}^2\left(2\tilde{H}+a_1\phi_0\right)\phi_0 u_1^2 + 2a_1^2\int \tilde{Q}^2\left(3\tilde{H}+a_1\phi_0\right)\phi_0^2 u_1\\ &\quad +\frac{1}{2}a_1^3\int \tilde{Q}^2\left(4\tilde{H}+a_1\phi_0\right)\phi_0^3\\ &\quad \doteq I_1 + I_2 + I_3\end{aligned}$$

$$\frac{1}{2}\int \tilde{Q}^2\left(u_1^2+4\tilde{H}u_1+(2\tilde{H})^2-(2\tilde{H})^2\right)u_1^2 = \frac{1}{2}\int \tilde{Q}^2\left(u_1+2\tilde{H}\right)^2 u_1^2 - 2\int \tilde{Q}^2\tilde{H}^2u_1^2$$

$$V - 2\tilde{Q}^2\tilde{H}^2 = 2\tilde{Q}^2\left(1-\tilde{Q}-\tilde{H}^2\right) = 2\tilde{Q}^2\left(\frac{2}{3}\tilde{Q}-\tilde{Q}\right) = -\frac{2}{3}\tilde{Q}^3,$$

$$I_1 = \mu_0^2(a_2^2 - a_1^2) + \|u_2\|_{L^2}^2 + \int \underbrace{\left(-\partial_x^2 u_1 - \frac{2}{3}\tilde{Q}^3 u_1\right)}_{\tilde{L}u_1} u_1 + \frac{1}{2}\int \tilde{Q}^2\left(u_1 + 2\tilde{H}\right)^2 u_1^2$$

$$\left|2a_1 \int \tilde{Q}^2 \phi_0 u_1^3\right| \lesssim |a_1|\left\|\tilde{Q}u_1\right\|_{L^2}^3,$$

$$\left|3a_1 \int \tilde{Q}^2\left(2\tilde{H} + a_1\phi_0\right)\phi_0 u_1^2\right| \lesssim |a_1|\left\|\tilde{Q}u_1\right\|_{L^2}^2,$$

$$\left|2a_1^2 \int \tilde{Q}^2\left(3\tilde{H} + a_1\phi_0\right)\phi_0^2 u_1\right| \lesssim C_\epsilon a_1^4 + \epsilon\left\|\tilde{Q}u_1\right\|_{L^2}^2$$

$$|I_2| \le C|a_1|^3 + \epsilon\left\|\tilde{Q}u_1\right\|_{L^2}^2$$

$$|I_3| = \left|\frac{1}{2}a_1^3 \int \tilde{Q}^2\left(4\tilde{H} + a_1\phi_0\right)\phi_0^3\right| \lesssim |a_1|^3.$$

$$-4\mu_0^2 b_+ b_- + \|u_2\|_{L^2}^2 + \langle \tilde{L}u_1, u_1\rangle + \int \tilde{Q}^2\left(u_1 + 2\tilde{H}\right)^2 u_1^2 \le C\{E(\phi_1, \phi_2) - E(\tilde{H}, 0)\} + C|a_1|^3 + \epsilon\|u_1\|_{H_0}^2$$

$$u_1 = a(t)\tilde{H} + \tilde{u}_1, \langle \phi_0, \tilde{u}_1\rangle = \langle \tilde{Q}^2\tilde{H}^3, \tilde{u}_1\rangle = 0$$

$$a(t) = \frac{\langle u_1, \tilde{Q}^2\tilde{H}^3\rangle}{\langle \tilde{H}, \tilde{Q}^2\tilde{H}^3\rangle}$$

$$\langle \tilde{L}u_1, u_1\rangle = \langle \tilde{L}\tilde{u}_1, \tilde{u}_1\rangle$$

$$a^2\|\tilde{H}\|_{H_0}^2 + \|\tilde{u}_1\|_{H_0}^2 = \|u_1\|_{H_0}^2$$

$$\delta_0^2 = b_+^2(0) + b_-^2(0) + \|u_2(0)\|_{L^2}^2 + \|u_1(0)\|_{H_0}^2 + \left\|\tilde{Q}\left(u_1(0) + 2\tilde{H}\right)u_1(0)\right\|_{L^2}^2$$

$$\left|E(\phi_1, \phi_2) - E(\tilde{H}, 0)\right| \lesssim \delta_0^2$$

$$\|u_2\|_2^2 + \langle \tilde{L}\tilde{u}_1, \tilde{u}_1\rangle + \int \tilde{Q}^2\left(u_1 + 2\tilde{H}\right)^2 u_1^2 \lesssim \delta_0^2 + b_+^2 + b_-^2 + |a_1|^3 + \epsilon\|u_1\|_{H_0}^2$$

$$\begin{aligned}\int \tilde{Q}^2\left(u_1 + 2\tilde{H}\right)^2 u_1^2 &= \int \tilde{Q}^2\left(a(2+a)\tilde{H}^2 + 2(1+a)\tilde{H}\tilde{u}_1 + \tilde{u}_1^2\right)^2 \\ &\ge a^2(2+a)^2 \int \tilde{Q}^2\tilde{H}^4 + 4a(1+a)(2+a)\int \tilde{Q}^2\tilde{H}^3\tilde{u}_1 + 2a(2+a)\int \tilde{Q}^2\tilde{H}^2\tilde{u}_1^2 \\ &\ge a^2(2+a)^2 \int \tilde{Q}^2\tilde{H}^4 - C|a|\|u_1\|_{H_0}^2\end{aligned}$$

$$\|u_2\|_2^2 + \langle \tilde{L}\tilde{u}_1, \tilde{u}_1\rangle + a^2(2+a)^2 \lesssim \delta_0^2 + b_+^2 + b_-^2 + |a_1|^3 + \epsilon a^2 + \epsilon\|\tilde{u}_1\|_{H_0}^2 + |\,a\|\tilde{u}_1\|_{H_0}^2.$$

$$a^2 + \|u_2\|_{L^2}^2 + \|\tilde{u}_1\|_{H_0}^2 \lesssim |b_+|^2 + |b_-|^2 + \delta_0^2 + O\left(|b_+|^3, |b_-|^3, |a|^3, \|\tilde{u}_1\|_{H_0}^3\right)$$

$$\langle \varepsilon_1, \phi_0\rangle + \langle \varepsilon_2, \mu_0^{-1}\phi_0\rangle = 0$$

$$b_-(0) = \langle \varepsilon_1, \phi_0\rangle = -\langle \varepsilon_2, \mu_0^{-1}\phi_0\rangle$$

$$\varepsilon_1 = b_-(0)\phi_0 + a(0)\tilde{H} + \tilde{u}_1(0), \varepsilon_2 = -b_-(0)\mu_0\phi_0 + u_2(0).$$

$$\langle \tilde{u}_1(0), \phi_0 \rangle = \langle \tilde{u}_1(0), \tilde{Q}^3\tilde{H} \rangle = \langle u_2(0), \phi_0 \rangle = 0$$

$$(\phi, \partial_t\phi)(0) = (1 + a(0))\tilde{\mathbf{H}} + (\tilde{u}_1, u_2)(0) + b_-(0)\mathbf{Y}_- + h(\boldsymbol{\varepsilon})\mathbf{Y}_+.$$

$$h(\varepsilon) := b_+(0)$$

$$\|u_1\|_{H_0}^2 = \|\partial_x u_1\|_{L^2}^2 + \left\|\tilde{Q}u_1\right\|_{L^2}^2.$$

$$|a| \le K^2\delta_0, \|\tilde{u}_1\|_{H_0} \le K^2\delta_0 \text{ and } \|u_2\|_{L^2} \le K^2\delta_0$$
$$|b_-| \le K\delta_0$$
$$|b_+| \le K^5\delta_0^2$$

$$(\tilde{u}_1(0), u_2(0)), b_+(0), b_-(0)$$

$$|a(0)| \le \delta_0, \|\tilde{u}_1(0)\|_{H_0} \le \delta_0, \left\|\tilde{Q}\left(u_1(0) + 2\tilde{H}\right)u_1(0)\right\|_{L^2} \le \delta_0, \|u_2(0)\|_{L^2} \le \delta_0, |b_-(0)| \le \delta_0,$$

$$|b_+(0)| \le K^5\delta_0$$

$$b_+(0) \in [-K^5\delta_0^2, K^5\delta_0^2]$$

$$b_+(0) \in [-K^5\delta_0^2, K^5\delta_0^2]$$

$$a^2 + \|\tilde{u}_1\|_{H_0}^2 + \|u_2\|_{L^2}^2 \le 3K^4\delta_0^2$$

$$a^2 + \|\tilde{u}_1\|_{H_0}^2 + \|u_2\|_{L^2}^2 \le C_6(K^2\delta_0^2 + K^{10}\delta_0^4 + \delta_0^2)$$

$$C_6 \le \frac{1}{4}K^2, K^4\delta_0 \le 1$$

$$a^2 + \|\tilde{u}_1\|_{H_0}^2 + \|u_2\|_{L^2}^2 \le \frac{3}{4}K^4\delta_0^2$$

$$\left|\frac{d}{dt}(e^{2\mu_0 t}b_-^2)\right| \le C_7(K^{15}\delta_0^6 + K^6\delta_0^3)e^{2\mu_0 t}$$

$$b_-^2 \le \frac{C_7}{2\mu_0}(K^{15}\delta_0^6 + K^6\delta_0^3) + \delta_0^2$$

$$\frac{C_7}{2\mu_0}K^{15}\delta_0^4 \le \frac{1}{4}K^2, \frac{C_7}{2\mu_0}K^6\delta_0 \le \frac{1}{4}K^2, 1 \le \frac{1}{4}K^4$$

$$\frac{d}{dt}(b_+^2) \ge 2\mu_0 b_+^2 - 2C_4|b_+|\left(b_+^2 + b_-^2 + \int \tilde{Q}^3 w_1^2\right) \ge 2\mu_0 b_+^2 - 2C_4|b_+|(b_+^2 + K^2\delta_0^2 + K^4\delta_0^2)$$
$$\ge 2\mu_0 K^{10}\delta_0^4 - C_8K^5\delta_0(K^{10}\delta_0^4 + K^4\delta_0^2)$$

$$C_8K^{15}\delta_0^2 \le \frac{1}{2}\mu_0 K^{10}, C_8K^9 \le \frac{1}{2}\mu_0 K^{10}$$

$$\frac{d}{dt}(b_+^2) \ge \mu_0 K^{10}\delta_0^4 > 0$$

$$b_+(0) \in [-K^5\delta_0^2, K^5\delta_0^2] \longmapsto b_+(T) \in \{-K^5\delta_0^2, K^5\delta_0^2\}$$

$$\left\|(\phi, \partial_t\phi)(t) - (\widetilde{H}, 0)\right\|_{H_0\times L^2} < \delta, \left\|\left(\tilde{\phi}, \partial_t\tilde{\phi}\right)(t) - (\widetilde{H}, 0)\right\|_{H_0\times L^2} < \delta$$

$$(\phi, \partial_t\phi) = (\widetilde{H}, 0) + \varepsilon + b_+(0)\mathbf{Y}_+, \left(\tilde{\phi}, \partial_t\tilde{\phi}\right) = (\tilde{H}, 0) + \tilde{\varepsilon} + \tilde{b}_+(0)\mathbf{Y}_+$$

$$\left|b_+(0) - \tilde{b}_+(0)\right| \le C\delta^{\frac{1}{2}}\|\varepsilon - \tilde{\varepsilon}\|_{H_0\times L^2}$$

$$\begin{aligned}&\|\partial_x u_1(t)\|_{L^2} + \|\partial_x\tilde{u}_1(t)\|_{L^2} + \left\|\tilde{Q}u_1\right\|_{L^2} + \left\|\tilde{Q}\tilde{u}_1\right\|_{L^2}\\ &+\|u_2(t)\|_{L^2} + \|\tilde{u}_2(t)\|_{L^2} + \left|b_\pm(t)\right| + \left|\tilde{b}_\pm(t)\right| \le C_0\delta.\end{aligned}$$

$$\begin{aligned}&\check{a}_1 = a_1 - \tilde{a}_1, \check{a}_2 = a_2 - \tilde{a}_2, \check{b}_+ = b_+ - \tilde{b}_+, \check{b}_- = b_- - \tilde{b}_-,\\ &\check{u}_1 = u_1 - \tilde{u}_1, \check{u}_2 = u_2 - \tilde{u}_2, \check{N} = N - \tilde{N}, \check{N}^\perp = N^\perp - \tilde{N}^\perp, \check{N}_0^\perp = N_0 - \tilde{N}_0.\end{aligned}$$

$$\begin{cases}\dot{\check{b}}_+(t) = \mu_0\check{b}_+(t) - \dfrac{\check{N}_0}{2\mu_0}\\ \dot{\check{b}}_-(t) = -\mu_0\check{b}_-(t) + \dfrac{\check{N}_0}{2\mu_0},\end{cases} \text{ and } \begin{cases}\dot{\check{u}}_1 = \check{u}_2\\ \dot{\check{u}}_2 = -L\check{u}_1 - \check{N}^\perp\end{cases}$$

$$\left|\check{N}_0\right| + \left\|\check{N}^\perp\right\|_{L^2} \le C\delta\left(\left|\check{b}_+\right| + \left|\check{b}_-\right| + \left\|\tilde{Q}u_1\right\|_{L^2}\right).$$

$$|\check{N}| \lesssim \tilde{Q}^2(|\check{a}_1|\phi_0 + |\check{u}_1|)(|a_1|\phi_0 + |\tilde{a}_1|\phi_0 + |u_1| + |\tilde{u}_1|).$$

$$\|\check{N}\|_{L^2} \le \delta\left(|\check{a}_1| + \left\|\tilde{Q}u_1\right\|_{L^2}\right)$$

$$\beta_+ = \check{b}_+^2, \beta_- = \check{b}_-^2, \beta_c = \langle L\check{u}_1, \check{u}_1\rangle + \langle\check{u}_2, \check{u}_2\rangle.$$

$$\dot{\beta}_c = -2\langle\check{N}^\perp, \check{u}_2\rangle, \dot{\beta}_+ - 2\mu_0\beta_+ = -\frac{1}{\mu_0}\check{b}_+\check{N}_0, \dot{\beta}_- + 2\mu_0\beta_- = \frac{1}{\mu_0}\check{b}_-\check{N}_0$$

$$\left|\dot{\beta}_c\right| + \left|\dot{\beta}_+ - 2\mu_0\beta_+\right| + \left|\dot{\beta}_- + 2\mu_0\beta_-\right| \le K\delta(\beta_c + \beta_+ + \beta_-),$$

$$0 \le K\delta(\beta_c(0) + \beta_+(0) + \beta_-(0)) < \frac{\mu_0}{10}\beta_+(0).$$

$$K\delta(\beta_c + \beta_+ + \beta_-) \le \mu_0\beta_+.$$

$$\mu_0\beta_+ \le 2\mu_0\beta_+ - K\delta(\beta_c + \beta_+ + \beta_-) \le \dot{\beta}_+$$

$$\dot{\beta}_c \le \mu_0\beta_+ \le \dot{\beta}_+,$$

$$\beta_c(t) \le \beta_c(0) + \beta_+(t) - \beta_+(0) \le \beta_c(0) + \beta_+(t)$$

$$K\delta\beta_c(t) \le K\delta(\beta_c(0) + \beta_+(t)) \le \frac{\mu_0}{10}\beta_+(0) + K\delta\beta_+(t) \le \frac{\mu_0}{5}\beta_+(t)$$

$$\dot{\beta}_- \le -2\mu_0\beta_- + \mu_0\beta_+,$$

$$\beta_-(t) \le e^{-2\mu_0 t}\beta_-(0) + \mu_0\beta_+(t)e^{-2\mu_0 t}\int_0^t e^{2\mu_0 s}ds \le \beta_-(0) + \frac{1}{2}\beta_+(t)$$

$$K\delta\beta_-(t) \le K\delta(\beta_-(0) + \beta_+(t)) \le \frac{\mu_0}{10}\beta_+(0) + K\delta\beta_+(t) \le \frac{\mu_0}{5}\beta_+(t)$$

$$K\delta(\beta_c(t) + \beta_+(t) + \beta_-(t)) \le \frac{3}{5}\mu_0\beta_+(t).$$

$$\varepsilon = \mathbf{u}(0) + b_-(0)\mathbf{Y}_-, \tilde{\varepsilon} = \tilde{\mathbf{u}}(0) + \tilde{b}_-(0)\mathbf{Y}_-,$$

$$V(x) = 2\tilde{Q}^2(x)(1 - \tilde{Q}(x))$$

$$\tilde{Q}(x) = Q(\alpha^{-1}(x)) \text{ with } \alpha(x) = \frac{1}{3}(\sinh x + x).$$

$$L\phi := -\partial_x^2\phi + V(x)\phi, \text{ with } V(x) = 2\tilde{Q}^2(x)(1 - \tilde{Q}(x)),$$

$$L\phi_0 = \lambda_0\phi_0, \phi_0 \in H^2(\mathbb{R}).$$

$$L\tilde{Q} = -\partial_x^2\tilde{Q} + 2\tilde{Q}^3(1 - \tilde{Q}) = \partial_x\left(\tilde{Q}^2\tilde{H}\right) + 2\tilde{Q}^3(1 - \tilde{Q}) = -2\tilde{Q}^3\tilde{H}^2 + \frac{1}{3}\tilde{Q}^4 + 2\tilde{Q}^3(1 - \tilde{Q}) = -\frac{5}{3}\tilde{Q}^4,$$

$$\langle L\tilde{Q}, \tilde{Q}\rangle = -\frac{5}{3}\int \tilde{Q}^5(x)dx = -\frac{5}{3}\int Q^4(y)dy < 0$$

$$\begin{cases} -L_c v(y) = f(y), y \in \mathbb{R} \\ v \in H^2(\mathbb{R}), \end{cases}$$

$$L_c^{-1}: \mathcal{C}^1(\mathbb{R}) \to \mathcal{C}^1(\mathbb{R})$$

$$-L\phi_0(x) = \lambda_0\phi_0(x), x \in \mathbb{R}$$

$$\int_0^\infty |V(x)|dx = 2\int_0^\infty \tilde{Q}^2(x)|1 - \tilde{Q}(x)|dx = 2\int_0^\infty Q(s)|1 - Q(s)|ds \le \int_0^\infty Q(s)ds < +\infty$$

$$0.808 \le \mu_0 \le 0.883$$

$$\lambda_0 = \inf_{\|f\|_{L^2}=1}(Lf, f)$$

$$f(x) := c_0 e^{-\frac{1}{2}x^2}(a_4x^4 + a_2x^2 + a_0),$$

$$a_4 := -0.0574167, a_2 := 0.115416, a_0 := -0.761391.$$

$$1 = \int f^2 = c_0^2\int e^{-x^2}(a_4^2x^8 + 2a_4a_2x^6 + (a_2^2 + 2a_4a_0)x^4 + 2a_2a_0x^2 + a_0^2)$$

$$\int e^{-x^2} = \sqrt{\pi}, \int x^2e^{-x^2} = \frac{\sqrt{\pi}}{2}, \int x^4e^{-x^2} = \frac{3\sqrt{\pi}}{4}$$

$$\int x^6e^{-x^2} = \frac{15\sqrt{\pi}}{8}, \int x^8e^{-x^2} = \frac{105\sqrt{\pi}}{16}$$

$$(Lf,f)=\int f'^2+2\int f^2\tilde{Q}^2(1-\tilde{Q})=\int f'^2(x)dx+2\int f^2(\alpha(y))Q(1-Q)(y)dy\sim -0.652$$

$$Q_p=\left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{1/(p-1)},p=9/2$$

$$2\tilde{Q}^2(x)(1-\tilde{Q}(x))\geq -0.845Q_p^{7/2}(x)=-\frac{2.32375}{\cosh^2\left(\frac{7}{4}x\right)}$$

$$|\phi_0(x)|,|\partial_x\phi_0(x)|,|\partial_x^2\phi_0(x)|\lesssim e^{-\frac{\sqrt{2}}{2}\mu_0 x}$$

$$\partial_x^2\phi_0=q(x)\phi_0$$

$$q(x)\geq\mu_0^2,$$

$$\frac{1}{2}\partial_x^2v(x)=(\partial_xu)^2(x)+q(x)u^2(x)\geq\mu_0^2v^2(x),$$

$$z=e^{-\sqrt{2}\mu_0x}\left(\partial_xv+\sqrt{2}\mu_0v\right)$$

$$\partial_xz=e^{-\sqrt{2}\mu_0x}(\partial_x^2v-2\mu_0^2v)\geq 0$$

$$z(x)\geq z(x_0)>0,$$

$$\partial_xv+\sqrt{2}\mu_0v\geq z(x_0)e^{\sqrt{2}\mu_0},$$

$$\partial_x\left(e^{\sqrt{2}\mu_0x}v\right)=e^{2\sqrt{2}\mu_0x}z\leq 0\ \text{ for } x\geq x_r,$$

$$|\phi_0(x)|\lesssim e^{-\frac{\sqrt{2}}{2}\mu_0x}.$$

$$\mu_0^2\leq q(x)\leq\mu_0^2+1,$$

$$\mu_0^2\int_{x_1}^{x_2}\phi_0\leq\partial_x\phi_0(x_2)-\partial_x\phi_0(x_1)\leq(\mu_0^2+1)\int_{x_1}^{x_2}\phi_0,$$

$$|\partial_x\phi_0(x)|\leq(\mu_0^2+1)\int_x^\infty|\phi_0|\lesssim e^{-\frac{\sqrt{2}}{2}\mu_0x}$$

$$\phi'(x)=\int_0^x(\mu_0^2+V(y))\phi_0(y)dy<0.$$

$$\phi_0'(x)=-\int_x^\infty(\mu_0^2+V(y))\phi_0(y)dy<0$$

$$h_0(x)=\frac{\phi_0'(x)}{\phi_0(x)}.$$

$$\phi_0(x) = \phi_0(0)\exp\left(\int_0^x h_0(y)dy\right)$$

$$\begin{cases} h_0'(x) + h_0^2(x) = \mu_0^2 + V(x), \text{ for } x \geq 0 \\ h_0(0) = 0 \end{cases}$$

$$h_0'(x) = -\frac{1}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy$$

$$\phi_0''(x) = (\mu_0^2 + V(x))\phi_0$$

$$\phi_0(x) = \phi_0(0)\exp\left(\int_0^x h_0(y)dy\right)$$

$$\begin{aligned}(\phi_0'(x))^2 &= -\int_x^\infty (\mu_0^2 + V(y))(\phi_0^2)'(y)dy \\ &= (\mu_0^2 + V(x))\phi_0^2(x) + \int_x^\infty V'(y)\phi_0^2(y)dy\end{aligned}$$

$$h_0^2(x) = \mu_0^2 + V(x) + \frac{1}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy$$

$$\begin{aligned} L_0 &= -\partial_x^2 + 2(h_0^2 - \mu_0^2) - V, \\ U &= \partial_x - h_0, U^* = -\partial_x - h_0. \end{aligned}$$

$$\begin{cases} V(x) \leq 0 & \text{for } 0 \leq x \leq x_0, \\ V(x) \geq 0 & \text{for } x \geq x_0 \end{cases} \quad \begin{cases} V'(x) \geq 0 & \text{for } 0 \leq x \leq x_1 \\ V'(x) \leq 0 & \text{for } x \geq x_1 \end{cases}$$

$$\begin{cases} V''(x) \geq 0 & \text{for } 0 \leq x \leq x_{2,1} \\ V''(x) \leq 0 & \text{for } x_{2,1} \leq x \leq x_{2,2} \\ V''(x) \geq 0 & \text{for } x \geq x_{2,2} \end{cases}$$

$$V(x) = 2\tilde{Q}^2(x)(1 - \tilde{Q}(x))$$

$$\begin{aligned} V'(x) &= 4\tilde{Q}(x)\tilde{Q}'(x) - 6\tilde{Q}^2(x)\tilde{Q}'(x) \\ &= 2\tilde{Q}^2(x)Q'(\alpha^{-1}(x))(2 - 3\tilde{Q}(x)) \end{aligned}$$

$$V(x_0) = 2\tilde{Q}^2(x_0)\left(1 - \tilde{Q}(x_0)\right) = 0$$

$$V'(x_0) = 2\tilde{Q}^2(x_0)Q'\left(\alpha^{-1}(x_0)\right)\left(2 - 3\tilde{Q}(x_0)\right) = -2\tilde{Q}^3(x_0)Q'\left(\alpha^{-1}(x_0)\right) > 0$$

$$\begin{aligned} V''(x) &= 8\tilde{Q}^2(x)Q'^2(\alpha^{-1}(x)) + 4\tilde{Q}^3(x)Q''(\alpha^{-1}(x)) - 18\tilde{Q}^3(x)Q'^2(\alpha^{-1}(x)) \\ &\quad - 6\tilde{Q}^4(x)Q''(\alpha^{-1}(x)) \end{aligned}$$

$$V''(x) = 2\tilde{Q}^4(x)\left(6 - \frac{50}{3}\tilde{Q}(x) + 9\tilde{Q}^2(x)\right)$$

$$V''(0) = 2\left(\frac{3}{2}\right)^4\left(6 - \frac{50}{3}\left(\frac{3}{2}\right) + 9\left(\frac{3}{2}\right)^2\right) \sim 12.65, V''(x_0) = -\frac{5}{3} < 0$$

$$V''(x_1) = 2\left(\frac{2}{3}\right)^4\left(6 - \frac{50}{3}\left(\frac{2}{3}\right) + 9\left(\frac{2}{3}\right)^2\right) \sim -0.44$$

$$\tilde{\mu}_0 := \sqrt{\mu_0^2 + \max_{y>0} V(y)},\ \max_{y>0} V(y) = \frac{8}{27},$$

$$-\tilde{\mu}_0 \le h_0(x)$$

$$h_0(x) \le -\mu_0$$

$$\lim_{x\to+\infty} h_0(x) = -\mu_0$$

$$h_0'(0) = \mu_0^2 + V(0) = \mu_0^2 - \frac{9}{4} \sim -1.59.$$

$$h_0^2(x) \le h_0^2(\bar{x}) = \mu_0^2 + V(\bar{x}) \le \mu_0^2 + \max_{y>0} V(y) = \tilde{\mu}_0^2$$

$$\mu_0^2 - h_0^2(x) = -\frac{1}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy - V(x)$$
$$\le -\int_x^\infty V'(y)dy - V(x) = 0$$

$$\mu_0^2 - h_0^2(x) = -\frac{1}{\phi_0^2(x)}\int_x^{x_1} V'(y)\phi_0^2(y)dy + \frac{1}{\phi_0^2(x)}\int_{x_1}^\infty |V'(y)|\phi_0^2(y)dy - V(x)$$
$$\le -\frac{\phi_0^2(x_1)}{\phi_0^2(x)}\int_x^{x_1} V'(y)dy - \frac{\phi_0^2(x_1)}{\phi_0^2(x)}\int_{x_1}^\infty V'(y)dy - V(x)$$
$$= -\left(1 - \frac{\phi_0^2(x_1)}{\phi_0^2(x)}\right)V(x) \le 0$$

$$\left|h_0^2(x) - \mu_0^2\right| \le \frac{1}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy + |V(x)| \le 2|V(x)|$$

$$(\mu_0^2 - \tilde{\mu}_0^2)x - 2\tilde{Q}(x)\tilde{H}(x) \le h_0(x) \le \mu_0^2 x - 2\tilde{Q}(x)\tilde{H}(x),$$

$$\mu_0^2 x - R(x) \le h_0 \ \text{ for all } x > 0$$

$$R(x) := 2\ln\left(\frac{3}{2}\right) - 2\ln(\tilde{Q}) + 2\tilde{Q}\tilde{H} + \frac{\mu_0^2 - \tilde{\mu}_0^2}{2}x^2$$

$$\begin{cases} h_1' = \mu_0^2 + V \\ h_1(0) = 0 \end{cases}$$

$$\int_0^x V(y)dy = 2\int_0^x \tilde{Q}^2(y)(1 - \tilde{Q}(y))dy = 2\int_0^{\alpha^{-1}(x)} Q(s)(1 - Q(s))ds$$
$$= 2\int_0^{\alpha^{-1}(x)} Q''(s)ds = 2Q'(\alpha^{-1}(x)) = -2\tilde{Q}(x)\tilde{H}(x)$$

$$h_1(x) = \int_0^x (\mu_0^2 + V(y))dy = \mu_0^2 x + \int_0^x V(y)dy = \mu_0^2 x - 2\tilde{Q}(x)\tilde{H}(x)$$

$$h_0(x) \leq \mu_0^2 x - 2\tilde{Q}(x)\tilde{H}(x)$$

$$\begin{cases} h_2' = \mu_0^2 - \tilde{\mu}_0^2 + V \\ h_2(0) = 0 \end{cases}$$

$$h_2(x) = \int_0^x (\mu_0^2 - \tilde{\mu}_0^2 + V(y))dy = (\mu_0^2 - \tilde{\mu}_0^2)x + \int_0^x V(y)dy$$
$$= (\mu_0^2 - \tilde{\mu}_0^2)x - 2\tilde{Q}(x)\tilde{H}(x)$$

$$h_2'(x) \leq \mu_0^2 + V(x) - h_0^2(x) = h_0'(x)$$

$$(\mu_0^2 - \tilde{\mu}_0^2)x - 2\tilde{Q}(x)\tilde{H}(x) \leq h_0(x)$$

$$\begin{cases} h_3' = \mu_0^2 - h_2^2 + V \\ h_3(0) = 0 \end{cases}$$

$$h_3(x) = \mu_0^2 x - 2\ln\left(\frac{3}{2}\right) + 2\ln(\tilde{Q}) - 2\tilde{Q}\tilde{H} - \frac{\mu_0^2 - \tilde{\mu}_0^2}{2}x^2$$

$$\frac{4}{3}\left(\mu_0^2 - \frac{9}{4}\right)\tilde{H}(x) \leq h_0(x)$$

$$(\mu_0^2 - \tilde{\mu}_0^2)(x - x_0) - \tilde{\mu}_0 \leq h_0(x) \leq -\frac{\mu_0}{x_0}x.$$

$$g(x) = h_0(x) - \frac{4}{3}\left(\mu_0^2 - \frac{9}{4}\right)\tilde{H}(x).$$

$$g(x) = g'(\xi)x$$

$$h_0^2 \leq \mu_0^2 + V - \frac{4}{9}\left(\mu_0^2 - \frac{9}{4}\right)\tilde{Q}^2$$

$$h_0^2 \leq \mu_0^4 x^2 - 2\mu_0^2 xR(x) + R^2(x)$$

$$\mu_0^4 x^2 - 2\mu_0^2 xR(x) + R^2(x) \leq \mu_0^2 + V - \frac{4}{9}\left(\mu_0^2 - \frac{9}{4}\right)\tilde{Q}^2$$

$$\mu_0^4 x^2 + \left(\frac{4}{9}\tilde{Q}^2 - 2xR(x) - 1\right)\mu_0^2 \leq V + \tilde{Q}^2 - R^2$$

$$\mu_0^4 x^2 + \left(\frac{4}{9}\tilde{Q}^2 - 2xR(x) - 1\right)\mu_0^2 \leq G(\alpha^{-1}(x))$$

$$G(s) := (0.883)^4\alpha(s)^2 + \left(\frac{4}{9}\tilde{Q}^2 - 2\alpha(s)R(\alpha(s)) - 1\right)(0.808)^2$$

$$G(s) \leq V(\alpha(s)) + Q^2(s) - R^2(\alpha(s))$$

$$h_0'(x_0) = \mu_0^2 - h_0^2(x_0) \leq \mu_0^2 - \mu_0^2 = 0.$$

$$\mu_0^2 - \tilde{\mu}_0^2 + V(x) \leq \mu_0^2 - h_0^2 + V = h_0'(x) \leq 0.$$

$$h_0'' = V' - 2h_0h_0' \geq V' - 2(\mu_0^2 x - R(x))h_0'$$
$$\geq V' - 2(\mu_0^2 x - R(x))(\mu_0^2 - \tilde{\mu}_0^2 + V) \geq V' - 2(0.808^2 x - R(x))\left(-\frac{8}{27} + V\right) =: j_1(\alpha^{-1}(x))$$

$$h_0''(x) \geq j_1(\alpha^{-1}(x)) > 0$$

$$h_0(x) \geq h_0'(x_0)(x - x_1) + h_0(x_0)$$
$$= \left(\mu_0^2 - h_0^2(x_0)\right)(x - x_0) + h_0(x_0) \geq (\mu_0^2 - \tilde{\mu}_0^2)(x - x_0) - \tilde{\mu}_0$$

$$h_0'' = V' - 2h_0h_0'$$
$$\geq V' - 2\left(\frac{4\mu_0^2 - 9}{3}\right)\left(\mu_0^2 + V - \left(\frac{4\mu_0^2 - 9}{3}\right)^2 \widetilde{H}^2\right)\widetilde{H}$$
$$\geq V' - 2\left(\frac{4(0.808)^2 - 9}{3}\right)\left((0.808)^2 + V - \left(\frac{4(0.808)^2 - 9}{3}\right)^2 \widetilde{H}^2\right)\widetilde{H} =: j_2(\alpha^{-1}(x))$$

$$j_2(s) = 2Q^3H(3Q - 2)$$
$$-2\left(\frac{4(0.808)^2 - 9}{3}\right)\left((0.808)^2 - \left(\frac{4(0.808)^2 - 9}{3}\right)^2 + \frac{2}{3}\left(\frac{4(0.808)^2 - 9}{3}\right)^2 Q + 2Q^2(1 - Q)\right)$$
$$= -2\left(\frac{4(0.808)^2 - 9}{3}\right)\left(\mu_0^2 - \left(\frac{4(0.808)^2 - 9}{3}\right)^2 - \frac{4}{3}\left(\frac{4(0.808)^2 - 9}{3}\right)\right)^3 Q$$
$$-4\left(\frac{4(0.808)^2 - 9}{3}\right)Q^2 + 4\left(\frac{4(0.808)^2 - 9}{3} - H\right)Q^3 + 6HQ^4.$$

$$h_0(x) \leq \frac{h_0(x_0)}{x_0}x \leq -\frac{\mu_0}{x_0}x.$$

$$V_0(x) = V(x) + \frac{2}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy$$

$$V_0(x) = \frac{1}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy - \frac{1}{\phi_0^2(x)}\int_x^\infty V(y)(\phi_0^2(y))'dy$$

$$= \frac{1}{\phi_0^2(x)}\int_x^\infty [V'(y) - 2h_0(y)V(y)]\phi_0^2(y)dy$$

$$V_0(x) = \frac{1}{\phi_0^2(x)}\int_x^\infty K(y)\phi_0^2(y)dy$$

$$K(y) := V'(y) - 2h_0(y)V(y)$$

$$K(y) = V' - 2h_0V \geq V' - 2(\mu_0^2 x - R)V \geq V' - 2(0.808^2 x - R)V$$
$$= 2\tilde{Q}^2\left[2(0.808^2 x - R) - 2\left((0.808^2 x - R) + \widetilde{H}\right)\tilde{Q} + 3\widetilde{H}\tilde{Q}^2\right] =: 2\tilde{Q}^2 k_1(\alpha^{-1}(y)).$$

$$K(y) \geq V' + \frac{8}{3}\left(\mu_0^2 - \frac{9}{4}\right)\tilde{H}V$$
$$= 2\tilde{Q}^2\tilde{H}\left[\frac{2}{3}(4\mu_0^2 - 9) + 2\left(1 - \frac{4}{3}\mu_0^2\right)\tilde{Q} + 3\tilde{Q}^2\right]$$
$$\geq 4\tilde{Q}^2\tilde{H}\left[\frac{1}{3}(4(0.808)^2 - 9) + \left(1 - \frac{4}{3}(0.883)^2\right)\tilde{Q} + \frac{3}{2}\tilde{Q}^2\right] =: 4\tilde{Q}^2\tilde{H}k_2(\alpha^{-1}(y)),$$

$$L = (-\partial_x - h_0)(\partial_x + h_0) - \mu_0^2 = U^*U - \mu_0^2$$

$$L_0 = (\partial_x + h_0)(-\partial_x - h_0) - \mu_0^2 = UU^* - \mu_0^2$$

$$UL = L_0U$$

$$\sigma_p(L_0) = \sigma_p(L) \setminus \{-\mu_0^2\}$$

$$V_0'(x) = 4h_0(x)h_0'(x) - V'(x)$$
$$= -\frac{2h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy - \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \frac{V'(y)}{h_0(y)}(\phi_0^2(y))'dy - V'(x)$$
$$= -\frac{2h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy + \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{V'(y)}{h_0(y)}\right)'\phi_0^2(y)dy$$
$$-\frac{h_0(x)V'(y)\phi_0^2(y)}{\phi_0^2(x)h_0(y)}\Bigg|_x^\infty - V'(x)$$
$$= \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{V'(y)}{h_0(y)} - 2V(y)\right)'\phi_0^2(y)dy = \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{V''(y)}{h_0(y)} - \frac{V'(y)h_0'(y)}{h_0^2(y)} - 2V'(y)\right)\phi_0^2(y)dy$$
$$= \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty (V''(y)h_0(y) - V'(y)h_0'(y) - 2V'(y)h_0^2(y))\left(\frac{\phi_0}{h_0}\right)^2(y)dy$$

$$V_0'(x) = \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty I(y)\left(\frac{\phi_0}{h_0}\right)^2(y)dy$$

$$I(y) = V''(y)h_0(y) - V'(y)(h_0^2(y) + \mu_0^2 + V(y)).$$

$$I(y) = -V''(y)|h_0(y)| + |V'(y)|(h_0^2(y) + \mu_0^2 + V(y))$$
$$\geq -\tilde{\mu}_0V''(y) + |V'(y)|(2\mu_0^2 + V(y)) \geq -1.038V''(y) + (2 \cdot 0.808^2 + V(y))|V'(y)|$$

$$I(\alpha(s)) \geq -2.075Q^4\left(6 - \frac{50}{3}Q + 9Q^2\right) + 4(2 - 3Q)(0.652 + Q^2 - Q^3)Q^3H$$
$$= 2Q^3\left[2.611 - 6(1.038 + 0.652H)Q + \left(\frac{50}{3}1.038 + 4H\right)Q^2 - (9.342 + 20H)Q^3 + 6Q^4H\right] =: 2Q^3i_1(s)$$

$$I(y) = |V''(y)h_0| + |V'(y)|(h_0^2(y) + \mu_0^2 + V(y)) \geq \mu_0|V''(y)| + |V'(y)|(2\mu_0^2 + V(y))$$
$$\geq 0.808|V''(y)| + |V'(y)|(2 \cdot 0.808^2 + V(y))$$

$$I(\alpha(s)) = -2\mu_0Q^4\left(6 - \frac{50}{3}Q + 9Q^2\right) + 4(2 - 3Q)(0.808^2 + Q^2 - Q^3)Q^3H$$
$$= 2Q^3H\left[4 \cdot 0.808^2H - 6 \cdot 0.808(1 + 0.808H)Q + \left(\frac{50}{3} \cdot 0.808 + 4H\right)Q^2\right.$$
$$\left.-(10H + 9 \cdot 0.808)Q^3 + 6HQ^4\right] =: 2Q^3Hi_2(s)$$

$$I(y) = |V''(y)h_0(y)| - V'(y)(h_0^2(y) + \mu_0^2 + V(y))$$
$$\geq \mu_0|V''(y)| - V'(y)(\tilde{\mu}_0^2 + \mu_0^2 + V(y)) \geq 0.808|V''(y)| - V'(y)(1.959 + V(y))$$

$$I(\alpha(s)) \geq -2 \cdot 0.808Q^4\left(6 - \frac{50}{3}Q + 9Q^2\right) + 2(2-3Q)(1.959 + 2Q^2 - 2Q^3)Q^3H$$
$$= 2Q^3\left[3.842H - 3(1.959 + 1.616H)Q + \left(\frac{50}{3} \cdot 0.808 + 4H\right)Q^2 - (7.272 + 10H)Q^3 + 6HQ^4\right] =: 2Q^3 i_3(s$$

$$\left(\alpha^{-1}(x_0), \alpha^{-1}(x_1)\right), \left(\alpha^{-1}(x_1), \alpha^{-1}\left(x_{2,2}\right)\right)$$

$$\left(\alpha^{-1}\left(x_{2,2}\right), \infty\right)$$

$$\left(0, \alpha^{-1}\left(x_{2,1}\right)\right) \text{ and } \left(\alpha^{-1}\left(x_{2,1}\right), \alpha^{-1}(x_0)\right)$$

$$I(y) = |V''(y)h_0(y)| + \left|V'(y)(h_0^2(y) + \mu_0^2 + V(y))\right| \geq 0.$$

$$h_0^2(y) + \mu_0^2 + V(y) \geq 0$$

$$h_0^2(y) + \mu_0^2 + V(y) \geq \left(\frac{8}{27}(x - x_0) + \tilde{\mu}_0\right)^2 + \mu_0^2 + V(y)$$

$$\geq \left(\frac{8}{27}(x - x_0) + 0.974\right)^2 + 0.652 + V(y).$$

$$I(y) \geq -\frac{\mu_0}{x_0}yV''(y) - V'(y)\left(\left(\frac{8}{27}(y - x_0) + 0.974\right)^2 + 0.652 + V(y)\right)$$

$$I(\alpha(s)) \geq -2\frac{0.808}{x_0}\alpha(s)Q^4\left(6 - \frac{50}{3}Q + 9Q^2\right)$$
$$+2(2-3Q)Q^3H\left(\left(\frac{8}{27}(\alpha(s) - x_0) + 0.974\right)^2 + 0.652 + 2Q^2(1-Q)\right) =: m(s),$$

$$s \in \left(\alpha^{-1}\left(x_{2,1}\right), \alpha^{-1}(x_0)\right)$$

$$h_0^2(y) + \mu_0^2 + V(y) \leq \left(\mu_0^2 - \frac{9}{4}\right)^2 y^2 + \mu_0^2 + V(y) \leq \left(0.652 - \frac{9}{4}\right)^2 y^2 + 0.78 + V(y) \leq 0,$$

$$I(y) = V''(y)|h_0(y)| + V'(y)\left|h_0^2(y) + \mu_0^2 + V(y)\right|.$$

$$I(y) \geq \left(0.652 - \frac{9}{4}\right)yV''(y) - V'(y)\left(\left(0.652 - \frac{9}{4}\right)^2 y^2 + 0.652 + V(y)\right)$$

$$I(\alpha(s)) \geq 2\left(0.652 - \frac{9}{4}\right)\alpha(s)Q^4\left(6 - \frac{50}{3}Q + 9Q^2\right)$$
$$+2Q^3H(2-3Q)\left(\left(0.652 - \frac{9}{4}\right)^2 \alpha(s)^2 + 0.652 + 2Q^2(1-Q)\right) =: \hat{m}(s)$$

$$V_0'(x) = \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty (V''(y)h_0(y) - V'(y)h_0'(y) - 2V'(y)h_0^2(y))\left(\frac{\phi_0}{h_0}\right)^2(y)dy \leq 0$$

$$3V'(x) \le V_0'(x) \le \frac{1}{2}V'(x),$$

$$|V_0'(x)| \le \left|\frac{4h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy\right| + |V'(x)|$$
$$= \left|\frac{4h_0(x)}{\phi_0^2(x)}\int_x^\infty \frac{V'(y)}{2h_0(y)}(\phi_0^2(y))'dy\right| + |V'(x)| \le \left|\frac{2\mu_0^{-1}h_0(x)V'(x)}{\phi_0^2(x)}\int_x^\infty (\phi_0^2(y))'dy\right| + |V'(x)| \le 3|V'(x)|$$

$$\begin{aligned}
V_0'(x) =& 4h_0(x)h_0'(x) - \frac{3}{2}V'(x) + \frac{1}{2}V'(x)\\
=& h_0(x)h_0'(x) + 3h_0(x)h_0'(x) - \frac{3}{2}V'(x) + \frac{1}{2}V'(x)\\
=& -\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy - \frac{3}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \frac{V'(y)}{h_0(y)}(\phi_0^2(y))'dy - \frac{3}{2}V'(x) + \frac{1}{2}V'(x)\\
=& -\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy + \frac{3}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{V'(y)}{h_0(y)}\right)'\phi_0^2(y)dy\\
& -\frac{3}{2}\frac{h_0(x)V'(y)\phi_0^2(y)}{\phi_0^2(x)h(y)}\Bigg|_x^\infty - \frac{3}{2}V'(x) + \frac{1}{2}V'(x)\\
=& \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{3}{2}\frac{V'(y)}{h_0(y)} - V(y)\right)'\phi_0^2(y)dy + \frac{1}{2}V'(x)\\
=& \frac{1}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(3\frac{V''(y)}{h_0(y)} - 3\frac{V'(y)h_0'(y)}{h_0^2(y)} - 2V'(y)\right)\phi_0^2(y)dy + \frac{1}{2}V'(x)\\
=& \frac{1}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty (3V''(y)h_0(y) - 3V'(y)h_0'(y) - 2V'(y)h_0^2(y))\left(\frac{\phi_0}{h_0}\right)^2(y)dy + \frac{1}{2}V'(x)
\end{aligned}$$

$$V_0'(x) = \frac{1}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty J(y)\left(\frac{\phi_0}{h_0}\right)^2(y)dy + \frac{1}{2}V'(x)$$

$$J(y) = 3V''(y)h_0(y) - V'(y)(3\mu_0^2 - h_0^2(y) + 3V(y)).$$

$$3\mu_0^2 - h_0^2(y) + 3V(y) \ge 3\mu_0^2 - \tilde{\mu}_0^2 + 3V(y) = 2\mu_0^2 - \frac{8}{27} > 0.$$

$$J(y) \ge -3V''(y) - (3\mu_0^2 - \tilde{\mu}_0^2 + 3V(y))V'(y) \ge -3V''(y) - (1.3 + 3V(y))V'(y) \ge 0$$

$$V_0'(x) = \frac{1}{2}\underbrace{\frac{h_0(x)}{\phi_0^2(x)}}_{\le 0}\underbrace{\int_x^\infty J(y)\left(\frac{\phi_0}{h_0}\right)^2(y)dy}_{\ge 0} + \frac{1}{2}V'(x) \le \frac{1}{2}V'(x) \le 0$$

$$E \ge R_1 + R_2 + R_3 + J_1 + J_2,$$

$$j \equiv 2(R_1 + R_2 + R_3) + 3(J_1 + J_2).$$

$$Q\phi^m = 0, Q\psi_m = -\frac{i}{2}\epsilon_{mnp}[\phi^n, \phi^p], Qf = -i[\phi^m, \psi_m].$$

$$\mathcal{I} = \mathcal{I}_{\text{s.c.d.}} \cdot \mathcal{I}_{w_2} \cdot \sum_i (-1)^{F_i}\chi(\mathbf{R}_i)t^{j_i}.$$

$$j \equiv 2(R_1 + R_2 + R_3) + 3(J_1 + J_2) = 24 + 2n$$

$$\mathcal{O}_{24}^2 = \epsilon^{pqr} v^m{}_q v^n{}_r \mathrm{tr}\left(\psi_{(m}\psi_n\psi_{p)}\right)$$

$$v^m{}_n = \mathrm{tr}(\phi^m\psi_n) - \frac{1}{3}\delta_n^m \mathrm{tr}\left(\phi^p\psi_p\right)$$

$$u_2{}^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right), \quad u_3{}^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^j\phi^{k)}\right)$$

$$v_2{}^i{}_j \equiv \mathrm{tr}\left(\phi^i\psi_j\right) - \frac{1}{3}\delta_j^i \mathrm{tr}(\phi^a\psi_a), \quad v_3{}^{ij}{}_k \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_k\right) - \frac{1}{4}\delta_k^i \mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_a\right) - \frac{1}{4}\delta_k^j \mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_a\right)$$

$$w_2{}^i \equiv \mathrm{tr}\left(f\phi^i + \frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right), \quad w_3{}^{ij} \equiv \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right)$$

$$\sum_a f_a(g_I) R_a(g_I) = 0$$

$$\sum_a f_a(g_I) r_a$$

$$\phi^m = \phi^m{}_1 T^1 + \phi^m{}_2 T^2 + \phi^m{}_3 T^3 + \sqrt{3}\hat{\phi}^m T^8$$
$$\psi_m = \psi_{m_1} T^1 + \psi_{m_2} T^2 + \psi_{m_3} T^3$$
$$f = f_1 T^1 + f_2 T^2 + f_3 T^3$$

$$\phi^m = \begin{pmatrix} \phi^m|_{SU(2)} & 0 \\ 0 & 0 \end{pmatrix} + \hat{\phi}^m \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \psi_m = \begin{pmatrix} \psi_m|_{SU(2)} & 0 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} f|_{SU(2)} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{tr}\left[\phi^{(i}\phi^j\phi^{k)}\right] \overset{SU(2)\times U(1)}{\rightarrow} \mathrm{tr}\left[\phi^{(i}\phi^j\phi^{k)}\right]_{SU(2)} + 3\hat{\phi}^{(i}\mathrm{tr}\left[\phi^j\phi^{k)}\right]_{SU(2)} - 6\hat{\phi}^{(i}\hat{\phi}^j\hat{\phi}^{k)}$$

$$\mathcal{O}^3[R] \overset{SU(2)\times U(1)}{\rightarrow} \sum_{R_1 \otimes R_2 \supset R} \hat{\mathcal{O}}[R_1]\mathcal{O}^2[R_2]$$

$$Q\Psi = \begin{pmatrix} Q\Psi|_{SU(2)} & 0 \\ 0 & 0 \end{pmatrix}$$

$$Q\Psi = \begin{pmatrix} Q\Psi|_{SU(2)} & 0 \\ 0 & Q\Psi|_{U(1)} = 0 \end{pmatrix}$$

$$Q\mathcal{O}^3[R] \overset{SU(2)\times U(1)}{\rightarrow} \sum_{R_1 \otimes R_2 \supset R} \hat{\mathcal{O}}[R_1]\mathcal{O}^2[R_2]$$

$$\Rightarrow \mathcal{O}^3[R] \overset{SU(2)\times U(1)}{\rightarrow} \sum_{R_1 \otimes R_2 \supset R} \hat{\mathcal{O}}[R_1]\mathcal{O}^2[R_2], \text{ where } \mathcal{O}^2[R_2] = Q\mathcal{O}^2[R_2].$$

$$([1,1] \otimes [1,1])_A = [3,0] + [0,3] + [1,1].$$

$$[2,0] \otimes ([1,1] \otimes [1,1])_A \otimes [1,3] = \cdots + (7 \times [3,0]) + \cdots$$

$$([2,1] \otimes [2,1])_A \otimes [1,3] = \cdots + (4 \times [3,0]) + \cdots$$

$$0 = 24s_1 - \frac{16}{15}s_3 + \frac{4}{15}s_4 - \frac{32}{15}s_5 + 16s_9 + 85s_{10} + 5s_{11} - 40s_{12} - \frac{5}{3}s_{13} + \frac{5}{3}s_{14} - 40s_{15}$$
$$+ \frac{125}{7}s_{17} + 31s_{18} + \frac{24}{7}s_{19} + \frac{543}{7}s_{20} + \frac{8}{5}s_{22} + \frac{16}{15}s_{23} + \frac{251}{7}s_{30} - s_{31} - 9s_{32} + \frac{174}{7}s_{33} - \frac{17}{2}s_{34}$$

$$\mathcal{O}^{ijk} \overset{SU(2)\times U(1)}{\rightarrow} \hat{\phi}^a\hat{\phi}^b\hat{\phi}^c\tilde{\mathcal{T}}^{(ijk)}_{(abc)}$$

$$\tilde{\mathcal{T}}^{(ijk)}_{(abc)} = \mathcal{T}^{(ijk)}_{(abc)} + \delta^i_a\mathcal{T}^{(jk)}_{(bc)} + \delta^{(ij)}_{(ab)}\mathcal{T}^k_c + \delta^{(ijk)}_{(abc)}\mathcal{T}$$

$$[3,0] \otimes [0,3] = [3,3] \oplus [2,2] \oplus [1,1] \oplus [0,0]$$

$$\mathcal{O}^{ijk} \overset{SU(2)\times U(1)}{\rightarrow} \hat{\phi}^i\hat{\phi}^j\hat{\phi}^k\mathcal{T} + \hat{\phi}^a\hat{\phi}^{(i}\hat{\phi}^j\mathcal{T}^{k)}_a + \hat{\phi}^a\hat{\phi}^b\hat{\phi}^{(i}\mathcal{T}^{jk)}_{ab} + \hat{\phi}^a\hat{\phi}^b\hat{\phi}^c\mathcal{T}^{ijk}_{abc}$$

$$\begin{aligned}
(\mathcal{O}^3_{30})^{ijk} =& 24\epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^c_d v^{|k)}{}_{a_1}v^d_{b_1}\left(r^{(0,2)}_{12}\right)_{a_2b_2c} - \frac{16}{15}\epsilon^{a_1a_2(i}v^{j|}{}_b v^b_c v^{c}_{a_1}\left(r^{(0,2)}_{12}\right)^{|k)}_{a_2} \\
&+ \frac{4}{15}\epsilon^{a_1a_2a_3}v^b{}_{a_1}v^{(i}{}_{a_2}v^j{}_{a_3}\left(r^{(0,2)}_{12}\right)^{k)}_b - \frac{32}{15}\epsilon^{a_1a_2(i}v^j{}_b v^{k)}{}_{a_1}v^b{}_c\left(r^{(0,2)}_{12}\right)^c_{a_2} + 16\epsilon^{a_1a_2(i}v^{j|}{}_d v^b{}_{a_1}v^d{}_c\left(r^{(0,2)}_{12}\right)^{|k)c}_{a_2b} \\
&+ 85\epsilon^{a_1a_2(i|}u^{bc}v_{a_1}v^{|j}{}_b\left(r^{(0,3)}_{14}\right)^{k)}_{a_2cd} + 5\epsilon^{a_1a_2(i|}u^{bc}v^d{}_b v^{|j}{}_{a_1}\left(r^{(0,3)}_{14}\right)^{k)}_{a_2cd} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c_{a_1}v^d{}_b\left(r^{(0,3)}_{14}\right)^{|k)}_{a_2cd} \\
&- \frac{5}{3}\epsilon^{a_1a_2(i|}u^{cd}v^b_{a_1}v^{|j}{}_b\left(r^{(0,3)}_{14}\right)^{k)}_{a_2cd} + \frac{5}{3}\epsilon^{a_1a_2(i}u^{j|c}v^b_{a_1}v^d{}_b\left(r^{(0,3)}_{14}\right)^{|k)}_{a_2cd} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c_{a_1}v^{|k)}{}_d\left(r^{(0,3)}_{14}\right)^d_{a_2bc} \\
&+ \frac{125}{7}\epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{cd}{}_b\left(r^{(0,3)}_{14}\right)^{|k)}_{a_2cd} + 31\epsilon^{a_1a_2(i}v^{bc}{}_{a_1}v^{d|j}{}_b\left(r^{(0,3)}_{14}\right)^{k)}_{a_2cd} + \frac{24}{7}\epsilon^{a_1a_2(i}v^{jk)}{}_b v^{cd}{}_{a_1}\left(r^{(0,3)}_{14}\right)^b_{a_2cd} \\
&+ \frac{543}{7}\epsilon^{a_1a_2a_3}v^{b(}{}_{a_1}v^{j|c}{}_{a_2}\left(r^{(0,3)}_{14}\right)^{|k)}_{a_3bc} + \frac{8}{5}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{|jk)}{}_b\left(r^{(0,2)}_{16}\right)_{a_2} + \frac{16}{15}\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)b}{}_{a_2}\left(r^{(0,3)}_{16}\right)_b \\
&+ \frac{251}{7}\epsilon^{a_1a_2a_3}v^{b_1}{}_{a_1}v^{b_2(i}{}_{a_2}\left(r^{(0,3)}_{16}\right)^{jk)}_{a_3b_1b_2} - \epsilon^{a_1a_2a_3}v^{(i|}{}_{a_1}v^{b_1b_2}{}_{a_2}\left(r^{(0,3)}_{16}\right)^{|jk)}_{a_3b_1b_2} - 9\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{bc}{}_d\left(r^{(0,3)}_{16}\right)^{d|k)}_{a_2bc} \\
&+ \frac{174}{7}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{c|j}{}_d\left(r^{(0,3)}_{16}\right)^{k)d}_{a_2bc} - \frac{17}{2}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{cd}{}_{a_2}\left(r^{(0,3)}_{16}\right)^{|jk)}_{bcd}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{O}^3_{30})^{ijk} =\ & 24\epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^c{}_d v^{|k)}{}_{a_1}v^d{}_{b_1}(r^{(0,2)}_{12})_{a_2b_2c} - \frac{16}{15}\epsilon^{a_1a_2(i}v^{j|}{}_b v^b{}_c v^c{}_{a_1}(r^{(0,2)}_{12})^{|k)}_{a_2} \\
&+ \frac{4}{15}\epsilon^{a_1a_2a_3}v^b{}_{a_1}v^{(i}{}_{a_2}v^j{}_{a_3}(r^{(0,2)}_{12})^{k)}_b - \frac{32}{15}\epsilon^{a_1a_2(i}v^j{}_b v^{k)}{}_{a_1}v^b{}_c(r^{(0,2)}_{12})^c_{a_2} + 16\epsilon^{a_1a_2(i}v^{j|}{}_d v^b{}_{a_1}v^d{}_c(r^{(0,2)}_{12})^{|k)c}_{a_2b} \\
&+ 85\epsilon^{a_1a_2(i|}u^{bc}v^d{}_{a_1}v^{|j}{}_b(r^{(0,3)}_{14})^{k)}_{a_2cd} + 5\epsilon^{a_1a_2(i|}u^{bc}v^d{}_b v^{|j}{}_{a_1}(r^{(0,3)}_{14})^{k)}_{a_2cd} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c{}_{a_1}v^d{}_b(r^{(0,3)}_{14})^{|k)}_{a_2cd} \\
&- \frac{5}{3}\epsilon^{a_1a_2(i|}u^{cd}v^b{}_{a_1}v^{|j}{}_b(r^{(0,3)}_{14})^{k)}_{a_2cd} + \frac{5}{3}\epsilon^{a_1a_2(i}u^{j|c}v^b{}_{a_1}v^d{}_b(r^{(0,3)}_{14})^{|k)}_{a_2cd} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c{}_{a_1}v^{|k)}{}_d(r^{(0,3)}_{14})^d_{a_2bc} \\
&+ \frac{125}{7}\epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{cd}{}_b(r^{(0,3)}_{14})^{|k)}_{a_2cd} + 31\epsilon^{a_1a_2(i}v^{bc}{}_{a_1}v^{d|j}{}_b(r^{(0,3)}_{14})^{k)}_{a_2cd} + \frac{24}{7}\epsilon^{a_1a_2(i}v^{jk)}{}_b v^{cd}{}_{a_1}(r^{(0,3)}_{14})^b_{a_2cd} \\
&+ \frac{543}{7}\epsilon^{a_1a_2a_3}v^{b(i}{}_{a_1}v^{j|c}{}_{a_2}(r^{(0,3)}_{14})^{|k)}_{a_3bc} + \frac{8}{5}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{|jk)}{}_b(r^{(0,2)}_{16})_{a_2} + \frac{16}{15}\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)b}{}_{a_2}(r^{(0,3)}_{16})_b \\
&+ \frac{251}{7}\epsilon^{a_1a_2a_3}v^{b_1}{}_{a_1}v^{b_2(i}{}_{a_2}(r^{(0,3)}_{16})^{jk)}_{a_3b_1b_2} - \epsilon^{a_1a_2a_3}v^{(i|}{}_{a_1}v^{b_1b_2}{}_{a_2}(r^{(0,3)}_{16})^{|jk)}_{a_3b_1b_2} - 9\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{bc}{}_d(r^{(0,3)}_{16})^{d|k)}_{a_2bc}
\end{aligned}$$

$$+ \frac{174}{7}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{c|j}{}_d(r^{(0,3)}_{16})^{k)d}_{a_2bc} - \frac{17}{2}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{cd}{}_{a_2}(r^{(0,3)}_{16})^{|jk)}_{bcd}$$

$$\mathcal{T} = -\frac{648}{5}\mathcal{O}^2_{24}$$

$$
\begin{aligned}
(\mathcal{O}_{32}^{3})^{ijkl} = {} & 162576\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k}{}_{a_1}v^{l)}{}_{a_5}v^{a_5a_6}{}_{a_3}\left(r_{12}^{(0,1)}\right)_{a_2a_4a_6} - 3240\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_2}v^{|l)a_6}{}_{a_3}\left(r_{12}^{(0,\\
& +19080\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_3}v^{|l)a_6}{}_{a_2}\left(r_{12}^{(0,1)}\right)_{a_4a_5a_6} - 183096\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_6}v^{|l)a_6}{}_{a_3}\left(r_{1}\right.\\
& -283500\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{a_5}{}_{a_1}v^{a_6}{}_{a_2}v^{|kl)}{}_{a_3}\left(r_{12}^{(0,1)}\right)_{a_4a_5a_6} + 12600\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}_{a_2}v^{l)a_3}{}_{a_4}\left(r_{12}^{(0,1)}\right)_{a_3}^{a_4}\\
& -16800\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_3}v^{l)a_3}{}_{a_4}\left(r_{12}^{(0,1)}\right)_{a_2}^{a_4} + 75600\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_4}v^{l)a_3}{}_{a_2}\left(r_{12}^{(0,1)}\right)_{a_3}^{a_4}
\end{aligned}
$$

$$
\begin{aligned}
& -46200\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_2}\left(r_{12}^{(0,1)}\right)_{a_3}^{a_4} - 21000\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_3}\left(r_{12}^{(0,1)}\right)_{a_2}^{a_4}\\
& +75600\epsilon^{a_1a_2(i}v^{j|}{}_{a_4}v^{a_3}{}_{a_1}v^{|kl)}{}_{a_2}\left(r_{12}^{(0,1)}\right)_{a_3}^{a_4} + 75600\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|k|a_4}{}_{a_2}\left(r_{12}^{(0,1)}\right)_{a_3}^{|l)}\\
& -37800\epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{|jk}{}_{a_3}\left(r_{12}^{(0,1)}\right)_{a_4}^{l)} - 14000\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k|}{}_{a_2}v^{a_3a_4}{}_{a_5}\left(r_{12}^{(0,1)}\right)_{a_3a_4}^{|l)a_5}\\
& +14000\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_5}\left(r_{12}^{(0,1)}\right)_{a_2a_3}^{a_4a_5} - 28000\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_5}v^{a_4|k}{}_{a_2}\left(r_{12}^{(0,1)}\right)_{a_3a_4}^{l)a_5}\\
& +14000\epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}v^{|jk}{}_{a_2}\left(r_{12}^{(0,1)}\right)_{a_3a_4}^{l)a_5} + 31500\epsilon^{a_1a_2a_3}v^{(i}{}_{a_1}v^{j}{}_{a_2}v^{k}{}_{a_3}\left(r_{14}^{(0,1)}\right)^{l)}\\
& -31500\epsilon^{a_1a_2(i|}\epsilon^{a_3a_4|j}v^{k}{}_{a_1}v^{l)}{}_{a_2}v^{a_5}{}_{a_3}\left(r_{14}^{(0,1)}\right)_{a_4a_5} + 315000\epsilon^{a_1a_2(i|}u^{a_3a_4a_5}v^{|j}{}_{a_1}v^{k}{}_{a_2}\left(r_{14}^{(0,2)}\right)^{l)}_{a_3a_4a_5}\\
& -1055700\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_2}\left(r_{14}^{(0,2)}\right)_{a_3a_4a_5}^{|l|} + 363660\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_3}\left(r_{14}^{(0,2)}\right)_{a_2a_4a_5}^{|l|}\\
& -596400\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_4}v^{a_4a_5}{}_{a_1}\left(r_{14}^{(0,2)}\right)_{a_2a_3a_5}^{|l)} + 1905300\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_2}\left(r_{14}^{(0,2)}\right)_{a_3a_4a_5}^{l)}\\
& +890400\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_3}\left(r_{14}^{(0,2)}\right)_{a_2a_4a_5}^{l)} + 82500\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_5}v^{a_5|k}{}_{a_1}\left(r_{14}^{(0,2)}\right)_{a_2a_3a_4}^{l)}\\
& +2288100\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_2}\left(r_{14}^{(0,2)}\right)_{a_3a_4a_5}^{|l|} - 247560\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_3}\left(r_{14}^{(0,2)}\right)_{a_2a_4a_5}^{|l)}\\
& +1506900\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_5}v^{k|a_5}{}_{a_1}\left(r_{14}^{(0,2)}\right)_{a_2a_3a_4}^{|l|} - 3441300\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_2}\left(r_{14}^{(0,2)}\right)_{a_3a_4a_5}^{l)}\\
& -4275300\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_3}\left(r_{14}^{(0,2)}\right)_{a_2a_4a_5}^{l)} + 30360\epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_1}v^{a_3a_4}{}_{a_5}\left(r_{14}^{(0,2)}\right)_{a_2a_3a_4}^{a_5}\\
& -134400\epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{|l)a_4}{}_{a_5}\left(r_{14}^{(0,2)}\right)_{a_2a_3a_4}^{a_5} + 340680\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k}{}_{a_1}v^{l)a_4}{}_{a_5}\left(r_{14}^{(0,2)}\right)_{a_2a_3a_4}^{a_5}\\
& +1010400\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{|kl)}{}_{a_5}\left(r_{14}^{(0,2)}\right)_{a_2a_3a_4}^{a_5} - 414240\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{kl)}{}_{a_5}\left(r_{14}^{(0,2)}\right)_{a_2a_3a_4}^{a_5}\\
& +48000\epsilon^{a_1a_2(i}v^{jk|}{}_{a_1}v^{a_3a_4}{}_{a_5}\left(r_{16}^{(0,2)}\right)_{a_2a_3a_4}^{|l)a_5} - 58800\epsilon^{a_1a_2(i}v^{jk|}{}_{a_5}v^{a_3a_4}{}_{a_1}\left(r_{16}^{(0,2)}\right)_{a_2a_3a_4}^{|l)a_5}\\
& -369600\epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4|k}{}_{a_5}\left(r_{16}^{(0,2)}\right)_{a_2a_3a_4}^{l)a_5} + 33600\epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4a_5}{}_{a_2}\left(r_{16}^{(0,2)}\right)_{a_3a_4a_5}^{|kl}\\
& +12250\epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}\left(r_{16}^{(0,2)}\right)_{a_2a_3a_4}^{|l)a_5} - 615850\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4}{}_{a_5}\left(r_{16}^{(0,2)}\right)_{a_2a_3a_4}^{|l)a_5}\\
& +369250\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_5}v^{a_4}{}_{a_1}\left(r_{16}^{(0,2)}\right)_{a_2a_3a_4}^{|l)a_5} + 184625\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}\left(r_{16}^{(0,2)}\right)_{a_3a_4a_5}^{|kl|}\\
& -504800\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k}{}_{a_5}\left(r_{16}^{(0,2)}\right)_{a_2a_3a_4}^{l)a_5} - 1085950\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_2}\left(r_{16}^{(0,2)}\right)_{a_3a_4a_5}^{|kl)}\\
& +701100\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_3}\left(r_{16}^{(0,2)}\right)_{a_2a_4a_5}^{|kl)}.
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{O}^3_{32})^{ijkl} &= 162576\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k}{}_{a_1}v^{l)}{}_{a_5}v^{a_5a_6}{}_{a_3}(r^{(0,1)}_{12})_{a_2a_4a_6} - 3240\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_2}v^{|l)a_6}{}_{a_3}(r^{(0,1)}_{12})_{a_4a_5a_6} \\
&+ 19080\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_3}v^{|l)a_6}{}_{a_2}(r^{(0,1)}_{12})_{a_4a_5a_6} - 183096\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_6}v^{|l)a_6}{}_{a_3}(r^{(0,1)}_{12})_{a_2a_4a_5} \\
&- 283500\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{a_5}{}_{a_1}v^{a_6}{}_{a_2}v^{|kl)}{}_{a_3}(r^{(0,1)}_{12})_{a_4a_5a_6} + 126000\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_2}v^{l)a_3}{}_{a_4}(r^{(0,1)}_{12})^{a_4}_{a_3} \\
&- 16800\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_3}v^{l)a_3}{}_{a_4}(r^{(0,1)}_{12})^{a_4}_{a_2} + 75600\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_4}v^{l)a_3}{}_{a_2}(r^{(0,1)}_{12})^{a_4}_{a_3}
\end{aligned}
$$

$$
\begin{aligned}
&- 46200\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_2}(r^{(0,1)}_{12})^{a_4}_{a_3} - 21000\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_3}(r^{(0,1)}_{12})^{a_4}_{a_2} \\
&+ 75600\epsilon^{a_1a_2(i}v^{j|}{}_{a_4}v^{a_3}{}_{a_1}v^{|kl)}{}_{a_2}(r^{(0,1)}_{12})^{a_4}_{a_3} + 75600\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|k|a_4}{}_{a_2}(r^{(0,1)}_{12})^{|l)}_{a_3} \\
&- 37800\epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{|jk}{}_{a_3}(r^{(0,1)}_{12})^{l)}_{a_4} - 14000\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k|}{}_{a_2}v^{a_3a_4}{}_{a_5}(r^{(0,1)}_{12})^{|l)a_5}_{a_3a_4} \\
&+ 14000\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_5}(r^{(0,1)}_{12})^{a_4a_5}_{a_2a_3} - 28000\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_5}v^{a_4|k}{}_{a_2}(r^{(0,1)}_{12})^{l)a_5}_{a_3a_4} \\
&+ 14000\epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}v^{|jk}{}_{a_2}(r^{(0,1)}_{12})^{l)a_5}_{a_3a_4} + 31500\epsilon^{a_1a_2a_3}v^{(i}{}_{a_1}v^{j}{}_{a_2}v^{k}{}_{a_3}(r^{(0,1)}_{14})^{l)} \\
&- 31500\epsilon^{a_1a_2(i|}\epsilon^{a_3a_4|j}v^{k}{}_{a_1}v^{l)}{}_{a_2}v^{a_5}{}_{a_3}(r^{(0,1)}_{14})_{a_4a_5} + 315000\epsilon^{a_1a_2(i|}u^{a_3a_4a_5}v^{|j}{}_{a_1}v^{k}{}_{a_2}(r^{(0,2)}_{14})^{l)}_{a_3a_4a_5} \\
&- 1055700\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_2}(r^{(0,2)}_{14})^{|l)}_{a_3a_4a_5} + 363660\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_3}(r^{(0,2)}_{14})^{|l)}_{a_2a_4a_5} \\
&- 596400\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_4}v^{a_4a_5}{}_{a_1}(r^{(0,2)}_{14})^{|l)}_{a_2a_3a_5} + 1905300\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_2}(r^{(0,2)}_{14})^{l)}_{a_3a_4a_5} \\
&+ 890400\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_3}(r^{(0,2)}_{14})^{l)}_{a_2a_4a_5} + 82500\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_5}v^{a_5|k}{}_{a_1}(r^{(0,2)}_{14})^{l)}_{a_2a_3a_4} \\
&+ 2288100\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_2}(r^{(0,2)}_{14})^{|l)}_{a_3a_4a_5} - 247560\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_3}(r^{(0,2)}_{14})^{|l)}_{a_2a_4a_5} \\
&+ 1506900\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_5}v^{k|a_5}{}_{a_1}(r^{(0,2)}_{14})^{|l)}_{a_2a_3a_4} - 3441300\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_2}(r^{(0,2)}_{14})^{l)}_{a_3a_4a_5} \\
&- 4275300\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_3}(r^{(0,2)}_{14})^{l)}_{a_2a_4a_5} + 30360\epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_1}v^{a_3a_4}{}_{a_5}(r^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} \\
&- 134400\epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{|l)a_4}{}_{a_5}(r^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} + 340680\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k}{}_{a_1}v^{l)a_4}{}_{a_5}(r^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} \\
&+ 1010400\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{|kl)}{}_{a_5}(r^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} - 414240\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{kl)}{}_{a_5}(r^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} \\
&+ 48000\epsilon^{a_1a_2(i}v^{jk|}{}_{a_1}v^{a_3a_4}{}_{a_5}(r^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} - 58800\epsilon^{a_1a_2(i}v^{jk|}{}_{a_5}v^{a_3a_4}{}_{a_1}(r^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} \\
&- 369600\epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4|k}{}_{a_5}(r^{(0,2)}_{16})^{l)a_5}_{a_2a_3a_4} + 33600\epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4a_5}{}_{a_2}(r^{(0,2)}_{16})^{|kl)}_{a_3a_4a_5} \\
&+ 12250\epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}(r^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} - 615850\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4}{}_{a_5}(r^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} \\
&+ 369250\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_5}v^{a_4}{}_{a_1}(r^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} + 184625\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}(r^{(0,2)}_{16})^{|kl)}_{a_3a_4a_5} \\
&- 504800\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k}{}_{a_5}(r^{(0,2)}_{16})^{l)a_5}_{a_2a_3a_4} - 1085950\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_2}(r^{(0,2)}_{16})^{|kl)}_{a_3a_4a_5}
\end{aligned}
$$

$$
+ 701100\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_3}(r^{(0,2)}_{16})^{|kl)}_{a_2a_4a_5}
$$

$$
\mathcal{O}^{ijkl} \overset{SU(2)\times U(1)}{\to} \hat{\phi}^i\hat{\phi}^j\hat{\phi}^k\hat{\phi}^l\mathcal{T} + \hat{\phi}^a\hat{\phi}^{(i}\hat{\phi}^j\hat{\phi}^k\mathcal{T}^{l)}_a + \hat{\phi}^a\hat{\phi}^b\hat{\phi}^{(i}\hat{\phi}^j\mathcal{T}^{kl)}_{ab} + \hat{\phi}^a\hat{\phi}^b\hat{\phi}^c\hat{\phi}^{(i}\mathcal{T}^{jkl)}_{abc} + \hat{\phi}^a\hat{\phi}^b\hat{\phi}^c\hat{\phi}^d\mathcal{T}^{ijkl}_{abcd}.
$$

$$
\mathcal{T} = -1020600\mathcal{O}^2_{24}.
$$

$$
u_2^{ij} \equiv \mathrm{tr}\big(\phi^{(i}\phi^{j)}\big), u_3^{ijk} \equiv \mathrm{tr}\big(\phi^{(i}\phi^{j}\phi^{k)}\big)
$$

$$
\begin{aligned}
(u_2\mathcal{O}^3_{30})^{ijklm} &= u_2^{(ij}(\mathcal{O}^3_{30})^{klm)} \\
(u_2\mathcal{O}^3_{30})^{ijk}_{l} &= \epsilon_{lab}u_2{}^{a(i}(\mathcal{O}^3_{30})^{jk)b} \\
(u_2\mathcal{O}^3_{30})^{i}_{jk} &= \epsilon_{jab}\epsilon_{kcd}u_2{}^{ac}(\mathcal{O}^3_{30})^{ibd}
\end{aligned}
$$

$$
\mathrm{tr}\big[\phi^{(i}\phi^{j)}\big] \overset{SU(2)\times U(1)}{\to} \mathrm{tr}\big[\phi^{(i}\phi^{j)}\big]_{SU(2)} + 6\hat{\phi}^i\hat{\phi}^j.
$$

$$(u_2\mathcal{O}_{30}^3)^{ijklm} \overset{SU(2)\times U(1)}{\to} \hat{\phi}^i\hat{\phi}^j\hat{\phi}^k\hat{\phi}^l\hat{\phi}^m \cdot 6\mathcal{T} + (\mathfrak{Q})$$

$$(u_2\mathcal{O}_{30}^3)_l^{ijk} \overset{SU(2)\times U(1)}{\to} \epsilon_{lab}\hat{\phi}^a\hat{\phi}^i\hat{\phi}^j\hat{\phi}^k\hat{\phi}^b \cdot 6\mathcal{T} + (\mathfrak{O}) = 0 + (\mathfrak{O})$$

$$(u_2\mathcal{O}_{30}^3)_{jk}^{i} \overset{SU(2)\times U(1)}{\to} \epsilon_{jab}\epsilon_{kcd}\hat{\phi}^a\hat{\phi}^c\hat{\phi}^i\hat{\phi}^b\hat{\phi}^d \cdot 6\mathcal{T} + (\mathfrak{O}) = 0 + (\mathfrak{O})$$

$$\left(u\cdots u\mathcal{O}_{24+2k}^3\right)^{i_1\cdots i_n} = u^{(i_1\cdots}\cdots u^{\cdots i_{n-k}}\left(\mathcal{O}_{24+2k}^3\right)^{i_{n-k+1}\cdots i_n)}$$

$$u^{(i_1\cdots}\cdots u^{\cdots i_{n-k})} \overset{SU(2)\times U(1)}{\to} \hat{\phi}^{i_1}\cdots\hat{\phi}^{i_{n-k}}C + \cdots$$

$$\left(u\cdots u\mathcal{O}_{24+2k}^3\right)^{i_1\cdots i_n} \overset{SU(2)\times U(1)}{\to} \hat{\phi}^{i_1}\cdots\hat{\phi}^{i_n}C\mathcal{T} + (\mathfrak{Q}),$$

$$\left(s_{26,1}\right)^i = \epsilon^{a_1a_2a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}\left(R_{14}^{(0,2)}\right)^i_{a_3a_4a_5}$$

$$\left(s_{26,2}\right)^i = v^i{}_a\left(R_{20}^{(0,3)}\right)^a$$

$$\left(s_{26,3}\right)^i = \left(R_{26}^{(0,4)}\right)^i$$

$$\left(s_{28,1}\right)^{ij} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_5}v^{a_4a_5}{}_{a_1}\left(R_{14}^{(0,2)}\right)^{|j)}_{a_2a_3a_4},$$

$$\left(s_{28,2}\right)^{ij} = \epsilon^{a_1a_2a_3}v^{a_4}{}_{a_1}v^{a_5(i}{}_{a_2}\left(R_{14}^{(0,2)}\right)^{j)}_{a_3a_4a_5},$$

$$\left(s_{28,3}\right)^{ij} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_5}v^{|j)a_4}{}_{a_1}\left(R_{14}^{(0,2)}\right)^{a_5}_{a_2a_3a_4},$$

$$\left(s_{28,4}\right)^{ij} = \epsilon^{a_1a_2(i}v^{j)}{}_{a_1}v^{a_3a_4}{}_{a_5}\left(R_{14}^{(0,2)}\right)^{a_5}_{a_2a_3a_4},$$

$$\left(s_{28,5}\right)^{ij} = \epsilon^{a_1a_2(i}v^{j)}{}_{a_5}v^{a_3a_4}{}_{a_1}\left(R_{14}^{(0,2)}\right)^{a_5}_{a_2a_3a_4},$$

$$\left(s_{28,6}\right)^{ij} = \epsilon^{a_1a_2(i}v^{j)}{}_{a_3}v^{a_3}{}_{a_1}\left(R_{16}^{(0,2)}\right)_{a_2},$$

$$\left(s_{28,7}\right)^{ij} = \epsilon^{a_1a_2(i}v^{j)}{}_{a_1}v^{a_3}{}_{a_2}\left(R_{16}^{(0,2)}\right)_{a_3},$$

$$\left(s_{28,8}\right)^{ij} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_4}v^{a_4}{}_{a_1}\left(R_{16}^{(0,2)}\right)^{|j)}_{a_2a_3},$$

$$\left(s_{28,9}\right)^{ij} = \epsilon^{a_1a_2a_3}v^{a_4}{}_{a_1}v^{(i}{}_{a_2}\left(R_{16}^{(0,2)}\right)^{j)}_{a_3a_4},$$

$$\left(s_{28,10}\right)^{ij} = \epsilon^{a_1a_2(i}v^{j)}{}_{a_1}v^{a_3}{}_{a_4}\left(R_{16}^{(0,2)}\right)^{a_4}_{a_2a_3},$$

$$\left(s_{28,11}\right)^{ij} = \epsilon^{a_1a_2a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}\left(R_{16}^{(0,2)}\right)^{ij}_{a_3a_4a_5},$$

$$\left(s_{28,12}\right)^{ij} = v^{ij}{}_a\left(R_{20}^{(0,3)}\right)^a,$$

$$\left(s_{28,13}\right)^{ij} = v^{(i}{}_a\left(R_{22}^{(0,3)}\right)^{j)a}.$$

$$-30s_{28,1} + 30s_{28,3} + \frac{15}{4}s_{28,4} - 15s_{28,5} - \frac{7}{20}s_{28,6} + \frac{21}{80}s_{28,7} + \frac{49}{36}s_{28,8} - \frac{7}{12}s_{28,9} + \frac{7}{24}s_{28,10} + s_{28,11} = 0$$

$$\mathcal{O}^{ij} \overset{SU(2)\times U(1)}{\to} \hat{\phi}^i\hat{\phi}^j\mathcal{T} + \hat{\phi}^a\hat{\phi}^{(i}\mathcal{T}_a^{j)} + \hat{\phi}^a\hat{\phi}^b\mathcal{T}_{ab}^{ij}$$

$$(\mathcal{O}_{24+2n}^3)^{i_1\cdots i_n} \overset{SU(2)\times U(1)}{\to} \hat{\phi}^{i_1}\cdots\hat{\phi}^{i_n}C_n\mathcal{O}_{24}^2 + (\mathfrak{Q}).\ (n \geq 3, C_n \neq 0)$$

$$w_2{}^i \equiv \mathrm{tr}\left(f\phi^i + \frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right)$$

$$[1,0] \otimes [n,0] = [n+1,0] \oplus [n-1,1]$$

$$(w_2\mathcal{O}_{24+2n}^3)^{i_1\cdots i_{n+1}} = w^{(i_1}(\mathcal{O}_{24+2n}^3)^{i_2\cdots i_{n+1})}$$
$$(w_2\mathcal{O}_{24+2n}^3)_j^{i_1\cdots i_{n-1}} = \epsilon_{jab}w^a(\mathcal{O}_{24+2n}^3)^{bi_1\cdots i_{n-1}}$$

$$w_2{}^i \equiv \mathrm{tr}\left(f\phi^i + \frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right)$$

$$(w_2\mathcal{O}_{24+2n}^3)^{i_1\cdots i_{n+1}} \overset{SU(2)\times U(1)}{\to} \hat{\phi}^{(i_2}\cdots\hat{\phi}^{i_{n+1}}C_nw^{i_1)}\mathcal{O}_{24}^2 + (\mathfrak{Q}),$$
$$(w_2\mathcal{O}_{24+2n}^3)_j^{i_1\cdots i_{n-1}} \overset{SU(2)\times U(1)}{\to} \epsilon_{jab}\hat{\phi}^b\hat{\phi}^{i_1}\cdots\hat{\phi}^{i_{n-1}}C_nw^a\mathcal{O}_{24}^2 + (\mathfrak{O}).$$

$$[1,0]_S^{\otimes k} \otimes [n,0] = [k,0] \otimes [n,0].\ [1,0]_S^{\otimes k}$$

$$(w_2)^k\mathcal{O}_{24+2n}^3$$

$$(w_2)^k\mathcal{O}_{24+2n}^3 \overset{SU(2)\times U(1)}{\to} \hat{\phi}^nC_n(w_2)^k\mathcal{O}_{24}^2 + (\mathfrak{Q})$$

$$[2,0] \otimes [n,0] = [n+2,0] \oplus [n,1] \oplus [n-2,2].$$

$$(w_3\mathcal{O}_{24+2n}^3)^{i_1\cdots i_{n+2}} = w^{(i_1i_2}(\mathcal{O}_{24+2n}^3)^{i_3\cdots i_{n+2})}$$
$$(w_3\mathcal{O}_{24+2n}^3)_j^{i_1\cdots i_n} = \epsilon_{ja_1a_2}w^{a_1(i_1}(\mathcal{O}_{24+2n}^3)^{i_2\cdots i_n)a_2}$$
$$(w_3\mathcal{O}_{24+2n}^3)_{jk}^{i_1\cdots i_{n-2}} = \epsilon_{a_1a_2(j}\epsilon_{k)b_1b_2}w^{a_1b_1}(\mathcal{O}_{24+2n}^3)^{a_2b_2i_1\cdots i_{n-2}}$$

$$w_3 = \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right)$$

$$w_3^{ij} \to 2\hat{\phi}^{(i}w_2^{j)}.$$

$$w^{(i_1i_2}(\mathcal{O}_{24+2n}^3)^{i_3\cdots i_{n+2})} \overset{SU(2)\times U(1)}{\to} 2C_n\hat{\phi}^{(i_1\cdots i_{n+1}}w^{i_{n+2})}\mathcal{O}_{24}^2 + (\mathfrak{Q}),$$
$$\epsilon_{ja_1a_2}w^{a_1(i_1}(\mathcal{O}_{24+2n}^3)^{i_2\cdots i_n)a_2} \overset{SU(2)\times U(1)}{\to} -C_n\epsilon_{ja_1a_2}\hat{\phi}^{(i_1\cdots i_n)a_1}w^{a_2}\mathcal{O}_{24}^2 + (\mathfrak{O}),$$
$$\epsilon_{a_1a_2(j}\epsilon_{k)b_1b_2}w^{a_1b_1}(\mathcal{O}_{24+2n}^3)^{a_2b_2i_1\cdots i_{n-2}} \overset{SU(2)\times U(1)}{\to} (\mathfrak{O}).$$

$$(w_3)^k\mathcal{O}_{24+2n}^3$$

$$(w_3)^k\mathcal{O}_{24+2n}^3 \overset{SU(2)\times U(1)}{\to} \hat{\phi}^{n+k}C_n'(w_2)^k\mathcal{O}_{24}^2 + (\mathfrak{Q}),$$

$$[2,0]_S^{\otimes k} \otimes [n,0] = \left([1,0]_{\hat{\phi}} \otimes_S [1,0]_{w_2}\right)_S^{\otimes k} \otimes [n,0]$$

$$(w_2)^k \mathcal{O}^3_{24+2(n+k)}[1,0]^{\otimes k}_S \otimes [n+k,0] = [k,0] \otimes [n+k,0]$$

$$[2,0] \otimes [n,0] = [n+2,0] \oplus [n,1] \oplus [n-2,2]$$

$$[1,0] \otimes [n+1,0] = [n+2,0] \oplus [n,1]$$

$$[2,0] \otimes [n+2,0] = [n+4,0] \oplus [n+2,1] \oplus [n,2]$$

$$(w_2)^2 \mathcal{O}^3_{24+2(n+2)}$$

$$\mathcal{O} \cdot \partial_1^{j_1} \partial_2^{j_2} \mathrm{tr}(X^2)$$

$$\mathcal{O} \cdot \mathrm{tr}(X(z)X(z))$$

$$X(z) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 D_1 + z_2 D_2)^n X(0) = \sum_{n=0}^{\infty} \frac{1}{n!} (z_\alpha D_\alpha)^n X(0)$$

$$Q\phi^m(z) = [\lambda(z), \phi^m(z)], \qquad Q\psi_m(z) = \epsilon_{mnp}[\phi^n(z), \phi^p(z)] + \{\lambda(z), \psi_m(z)\}$$
$$Qf(z) = [\phi^m(z), \psi_m(z)] + [\lambda(z), f(z)], \qquad Q\lambda(z) = \{\lambda(z), \lambda(z)\}$$

$$\phi^m(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (z_\alpha D_\alpha)^n \phi^m(0), \qquad \psi_m(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (z_\alpha D_\alpha)^n \psi_m(0)$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (z_\alpha D_\alpha)^n f(0), \qquad \lambda(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (z_\alpha D_\alpha)^n \big(z_\beta \lambda_\beta(0)\big).$$

$$\mathcal{O} \cdot \left[\mathrm{tr}(X(z)^2) + \sum_{i=1}^{n} a_i \mathrm{tr}(X(z_i)^2) \right]$$

$$R = \frac{N^2}{2} \left(\frac{q}{\ell^2} + \frac{q^2}{2\ell^4} \right), J = \frac{N^2}{2} \left(\frac{3q^2}{2\ell^4} + \frac{q^3}{\ell^6} \right)$$

$$J(R) = -\frac{N^2}{2} - 3R + \frac{(N^2 + 4R)^{\frac{3}{2}}}{2N}$$

$$j \equiv j_1 + j_2 > 2 \left(\frac{2J}{N^2} \right)^{\frac{1}{3}}$$

$$J(R) \sim \frac{4R^{\frac{3}{2}}}{N}$$

$$J_1 \approx J_2 = J \sim \frac{4R^{\frac{3}{2}}}{N}$$

$$\frac{J}{R} \sim \frac{J^{\frac{1}{3}}}{N^{\frac{2}{3}}}$$

$$E - E_{\mathrm{BPS}} = cj^{-\tau} + \cdots,$$

$$u_2^{ij} \equiv \mathrm{tr}\big(\phi^{(i}\phi^{j)}\big), \qquad u_3^{ijk} \equiv \mathrm{tr}\big(\phi^{(i}\phi^{j}\phi^{k)}\big)$$

$$v_2{}^i{}_j \equiv \mathrm{tr}\big(\phi^i\psi_j\big) - \frac{1}{3}\delta^i_j\mathrm{tr}(\phi^a\psi_a), \quad v_3{}^{ij}{}_k \equiv \mathrm{tr}\big(\phi^{(i}\phi^{j)}\psi_k\big) - \frac{1}{4}\delta^i_k\mathrm{tr}\big(\phi^{(j}\phi^{a)}\psi_a\big) - \frac{1}{4}\delta^j_k\mathrm{tr}\big(\phi^{(i}\phi^{a)}\psi_a\big)$$

$$\begin{aligned}
t^{10}[1,2](u_2u_3) &: (R^{(0,0)}_{10})^i_{jk} &&= \epsilon_{a_1a_2(j}\epsilon_{k)b_1b_2}u^{a_1b_1}u^{ia_2b_2}\\
t^{12}[0,0](u_2u_2u_2) &: (R^{(0,0)}_{12}) &&= \epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}u^{a_3b_3}\\
t^{12}[2,2](u_2u_2u_2,u_3u_3) &: (R^{(0,0)}_{12})^{ij}_{kl} &&= \epsilon_{a_1a_2(k}\epsilon_{l)b_1b_2}\left(u^{a_1b_1}u^{a_2b_2}u^{ij} + 6u^{a_1b_1(i}u^{j)a_2b_2}\right)\\
t^{12}[0,3](u_2v_3) &: (R^{(0,1)}_{12})_{ijk} &&= \epsilon_{(i|a_1a_2}\epsilon_{|j|b_1b_2}u^{a_1b_1}v^{a_2b_2}{}_{|k)}\\
t^{12}[1,1](u_2v_3,u_3v_2) &: (R^{(0,1)}_{12})^i_j &&= \epsilon_{ja_1a_2}\left(4u^{a_1b}v^{ia_2}{}_b + 3u^{ia_1b}v^{a_2}{}_b\right)\\
t^{12}[2,2](u_2v_3,u_3v_2) &: (R^{(0,1)}_{12})^{ij}_{kl} &&= \epsilon_{a_1a_2(k}\left(u^{a_1(i}v^{j)a_2}{}_{l)} + u^{ija_1}v^{a_2}{}_{l)}\right)\\
t^{14}[1,0](u_2u_2v_2) &: (R^{(0,1)}_{14})^i &&= \epsilon_{a_1a_2a_3}u^{ia_1}u^{ba_2}v^{a_3}{}_b\\
t^{14}[0,2](u_2u_2v_2,u_3v_3) &: (R^{(0,1)}_{14})_{ij} &&= \epsilon_{a_1a_2(i|}\left(\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}v^{b_3}{}_{|j)} - 2\epsilon_{|j)b_1b_2}u^{a_1b_1c}v^{a_2b_2}{}_c\right)\\
t^{14}[2,1](u_2u_2v_2,u_3v_3) &: (R^{(0,1)}_{14})^{ij}_k &&= \epsilon_{ka_1a_2}\left(3u^{(a_1b}u^{ij)}v^{a_2}{}_b + 4u^{a_1b}u^{a_2(i}v^{j)}{}_b + 24u^{a_1b(i}v^{j)a_2}{}_b\right)\\
t^{14}[1,3](u_2u_2v_2,u_3v_3) &: (R^{(0,1)}_{14})^i_{jkl} &&= \epsilon_{(j|a_1a_2}\epsilon_{|k|b_1b_2}\left(u^{a_1b_1}u^{a_2b_2}v^i{}_{|l)} + 6u^{ia_1b_1}v^{a_2b_2}{}_{|l)}\right)\\
t^{14}[3,2](u_2u_2v_2,u_3v_3) &: (R^{(0,1)}_{14})^{ijk}_{lm} &&= \epsilon_{a_1a_2(l}\left(u^{(a_1i}u^{jk)}v^{a_2}{}_{m)} + 6u^{a_1(ij}v^{k)a_2}{}_{m)}\right)\\
t^{14}[1,3](v_2v_3) &: (R^{(0,2)}_{14})^i_{jkl} &&= \epsilon_{a_1a_2(j}v^{a_1}{}_kv^{ia_2}{}_{l)}\\
t^{16}[0,1](u_2v_2v_2,v_3v_3) &: (R^{(0,2)}_{16})_i &&= \epsilon_{ia_1a_2}\left(12u^{bc}v^{a_1}{}_bv^{a_2}{}_c + 13u^{a_1b}v^{a_2}{}_cv^c{}_b + 12v^{a_1b}{}_cv^{a_2c}{}_b\right)\\
t^{16}[1,2](u_2v_2v_2,v_3v_3) &: (R^{(0,2)}_{16})^i_{jk} &&= \epsilon_{a_1a_2(j}\left(3u^{ib}v^{a_1}{}_{k)}v^{a_2}{}_b - 7u^{ia_1}v^b{}_{k)}v^{a_2}{}_b + 6u^{a_1b}v^i{}_{k)}v^{a_2}{}_b + 24v^{a_1b}{}_{k)}v^{ia_2}{}_b\right)\\
t^{16}[2,3](u_2v_2v_2,v_3v_3) &: (R^{(0,2)}_{16})^{ij}_{klm} &&= \epsilon_{a_1a_2(k}\left(u^{a_1(i}v^{j)}{}_lv^{a_2}{}_{m)} + 3v^{a_1(i}{}_lv^{j)a_2}{}_{m)}\right)\\
t^{18}[0,0](u_3v_2v_2) &: (R^{(0,2)}_{18}) &&= \epsilon_{a_1a_2a_3}u^{a_1bc}v^{a_2}{}_bv^{a_3}{}_c\\
t^{20}[1,0](v_2v_2v_3) &: (R^{(0,3)}_{20})^i &&= 2v^a{}_cv^b{}_av^{ic}{}_b - 3v^i{}_av^c{}_bv^{ab}{}_c\\
t^{22}[2,0](u_2v_2v_2v_2) &: (R^{(0,3)}_{22})^{ij} &&= u^{ij}v^a{}_bv^b{}_cv^c{}_a - 3u^{a(i}v^{j)}{}_bv^b{}_cv^c{}_a + 3u^{ab}v^{(i}{}_av^{j)}{}_cv^c{}_b\\
t^{24}[0,0](u_2v_2v_2v_3) &: (R^{(0,3)}_{24}) &&= \epsilon_{a_1a_2a_3}u^{a_1b}v^{a_2}{}_bv^{a_3c}{}_dv^d{}_c\\
t^{26}[1,0](v_2v_2v_2v_3) &: (R^{(0,4)}_{26})^i &&= v^i{}_av^a{}_bv^d{}_cv^{bc}{}_d\\
t^{30}[0,0](v_2v_2v_2v_2v_2) &: (R^{(0,5)}_{30}) &&= v^a{}_bv^b{}_cv^c{}_dv^d{}_ev^e{}_a
\end{aligned}$$

$$t^{30}[3,0](v_2v_2v_2v_2v_2) \quad : (R^{(0,5)}_{30})^{ijk} = \epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)}{}_{a_2}v^b{}_cv^c{}_dv^d{}_b$$

$$\left(r^{(0,1)}_{10}\right)^i_{jk} = -2\epsilon_{a_1a_2(j}\mathrm{tr}\big(\phi^{a_1}\phi^{a_2}\phi^i\psi_{k)}\big),$$

$$r^{(0,1)}_{12} = \epsilon_{a_1a_2a_3}\big[6\mathrm{tr}(\psi_b\phi^{a_1})\mathrm{tr}\big(\phi^b\phi^{a_2}\phi^{a_3}\big) - \mathrm{tr}(\psi_b\phi^{a_1}\phi^{a_2})\mathrm{tr}\big(\phi^b\phi^{a_3}\big)\big]$$

$$-3\epsilon_{a_1a_2a_3}\left[\mathrm{tr}\left(\psi_b\phi^b\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^b\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^b\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^{a_3}\phi^b\right)\right],$$

$$\left(r_{12}^{(0,1)}\right)_{kl}^{ij}=-2\epsilon_{a_1a_2(k}\left[\mathrm{tr}\left(\psi_{l)}\phi^{(i}\phi^{j)}\phi^{a_1}\phi^{a_2}\right)+7\mathrm{tr}\left(\psi_{l)}\phi^{(i|}\phi^{a_1}\phi^{|j)}\phi^{a_2}\right)\right],$$

$$\left(r_{12}^{(0,2)}\right)_{ijk}=\frac{1}{2}\epsilon_{a_1a_2(i}\mathrm{tr}\left(\phi^{a_1}\psi_j\phi^{a_2}\psi_{k)}\right),$$

$$\left(r_{12}^{(0,2)}\right)_{j}^{i}=6\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{(a}\psi_{j)}\right)-5\mathrm{tr}\left(\phi^{[i}\psi_a\phi^{a]}\psi_j\right),$$

$$\left(r_{12}^{(0,2)}\right)_{kl}^{ij}=\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(k}\psi_{l)}\right),$$

$$\left(r_{14}^{(0,2)}\right)^{i}=3\mathrm{tr}\left(\phi^i\psi_{a_1}\phi^{a_1}\phi^{a_2}\psi_{a_2}\right)+2\mathrm{tr}\left(\phi^i\phi^{a_1}\right)\mathrm{tr}\left(\phi^{a_2}\psi_{(a_1}\psi_{a_2)}\right)$$
$$-6\mathrm{tr}\left(\phi^i\psi_{a_1}\right)\mathrm{tr}\left(\phi^{[a_1}\phi^{a_2]}\psi_{a_2}\right)-\mathrm{tr}\left(\phi^i\psi_{a_1}\psi_{a_2}\right)\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\right),$$

$$\left(r_{14}^{(0,2)}\right)_{ij}=\frac{5}{9}\epsilon_{a_1a_2a_3}\left[2\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_{(i}\phi^{a_1}\psi_{j)}\phi^{a_2}\phi^{a_3}\right)\right]$$
$$+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_j\psi_{a_3}\phi^{a_1}\phi^{a_3}\phi^{a_2}\right)+\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_3}\phi^{a_1}\phi^{a_2}\right)\right]$$
$$+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_j\phi^{a_1}\psi_{a_3}\phi^{a_3}\phi^{a_2}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_3}\psi_{a_3}\phi^{a_1}\phi^{a_2}\right)\right]$$
$$+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\psi_{a_3}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_j\phi^{a_1}\phi^{a_3}\psi_{a_3}\phi^{a_2}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\psi_{a_3}\phi^{a_2}\right)\right]$$
$$+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\psi_{a_3}\right)+\mathrm{tr}\left(\psi_j\phi^{a_1}\phi^{a_3}\phi^{a_2}\psi_{a_3}\right)+\mathrm{tr}\left(\psi_j\phi^{a_3}\phi^{a_1}\phi^{a_2}\psi_{a_3}\right)\right]$$
$$-\frac{1}{3}\epsilon_{a_1a_2(i}\left[5\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_3}\right)+2\mathrm{tr}\left(\psi_j\phi^{(a_1}\phi^{a_3)}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_2}\right)\right.$$
$$\left.-2\mathrm{tr}\left(\psi_j\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{(a_1}\phi^{a_3)}\right)\right],$$

$$\left(r_{14}^{(0,2)}\right)_{k}^{ij}=12\mathrm{tr}\left(\phi^{(i}\phi^a\phi^{j)}\psi_{(a}\psi_{k)}\right)+12\mathrm{tr}\left(\phi^{(i|}\phi^a\phi^{|j)}\psi_{(a}\psi_{k)}\right)+54\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(a}\phi^{a)}\psi_{k)}\right)-36\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(a}\phi^a$$

$$\left(r_{14}^{(0,2)}\right)_{jkl}^{i}=2\epsilon_{a_1a_2(j}\left[\mathrm{tr}\left(\phi^i\phi^{a_1}\phi^{a_2}\psi_k\psi_{l)}\right)+3\mathrm{tr}\left(\phi^i\phi^{a_1}\psi_k\phi^{a_2}\psi_{l)}\right)-2\mathrm{tr}\left(\phi^i\psi_k\phi^{a_1}\phi^{a_2}\psi_{l)}\right)\right],$$

$$\left(r_{14}^{(0,2)}\right)_{lm}^{ijk}=4\mathrm{tr}\left(\phi^{(i}\phi^j\phi^{k)}\psi_{(l}\psi_{m)}\right)+3\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(l}\phi^{k)}\psi_{m)}\right)$$

$$\left(r_{14}^{(0,3)}\right)_{jkl}^{i}=-\frac{1}{2}\mathrm{tr}\left(\phi^i\psi_{(j}\psi_k\psi_{l)}\right),$$

$$\left(r_{16}^{(0,3)}\right)_{i}=\frac{39}{4}\mathrm{tr}\left(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_1}\phi^{b_2}\}\right)+2\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\psi_{b_2}\phi^{b_2}\right)-\frac{61}{4}\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\psi_{b_2}\phi^{b_1}\right)$$
$$+\frac{97}{4}\mathrm{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\psi_{b_2}\phi^{b_2}\right)-\frac{41}{4}\mathrm{tr}\left(\psi_i\phi^{b_2}\psi_{b_1}\psi_{b_2}\phi^{b_1}\right)-5\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\phi^{b_2}\psi_{b_2}\right)$$
$$-\frac{25}{2}\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\phi^{b_1}\psi_{b_2}\right)+2\mathrm{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\phi^{b_2}\psi_{b_2}\right)-\frac{61}{4}\mathrm{tr}\left(\psi_i\phi^{b_2}\psi_{b_1}\phi^{b_1}\psi_{b_2}\right)$$
$$-\frac{11}{4}\mathrm{tr}\left(\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}\left(\psi_i\psi_{b_1}\psi_{b_2}\right)-\frac{27}{2}\mathrm{tr}\left(\psi_{b_1}\psi_{b_2}\right)\mathrm{tr}\left(\psi_i\phi^{b_1}\phi^{b_2}\right)+\frac{29}{4}\mathrm{tr}\left(\phi^{b_2}\psi_{b_2}\right)\mathrm{tr}\left(\psi_i[\psi_{b_1},\phi^{b_1}]\right),$$

$$\left(r_{16}^{(0,3)}\right)_{jk}^{i}=2\mathrm{tr}\left(\psi_{(j}\psi_{k)}\psi_b\phi^b\phi^i\right)-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\psi_b\phi^i\phi^b\right)-\mathrm{tr}\left(\psi_{(j|}\psi_b\psi_{|k)}\{\phi^b,\phi^i\}\right)-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^{(b}\psi_b\phi^{i)}\right)$$
$$+7\mathrm{tr}\left(\psi_{(j|}\{\psi_b,\phi^b\}\psi_{|k)}\phi^i\right)-11\mathrm{tr}\left(\psi_{(j|}\{\psi_b,\phi^i\}\psi_{|k)}\phi^b\right)-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^b\phi^i\psi_b\right)$$
$$+2\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^i\phi^b\psi_b\right)+3\mathrm{tr}\left(\psi_{(j|}\psi_b\right)\mathrm{tr}\left(\psi_{|k)}[\phi^b,\phi^i]\right)+6\mathrm{tr}\left(\psi_{(j}\phi^{[b}\right)\mathrm{tr}\left(\{\psi_{k)},\psi_b\}\phi^{i]}\right),$$

$$\left(r_{16}^{(0,3)}\right)_{klm}^{ij}=-\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(k}\psi_l\psi_{m)}\right)-2\mathrm{tr}\left(\phi^{(i}\psi_{(k}\phi^{j)}\psi_l\psi_{m)}\right).$$

$$\begin{aligned}\left(r_{10}^{(0,1)}\right)_{jk}^{i} &= -2\,\epsilon_{a_1a_2(j}\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\phi^i\psi_{k)}\right)\ ,\\ r_{12}^{(0,1)} &= \epsilon_{a_1a_2a_3}\left[6\mathrm{tr}\left(\psi_b\phi^{a_1}\right)\mathrm{tr}\left(\phi^b\phi^{a_2}\phi^{a_3}\right)-\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\phi^b\phi^{a_3}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&- 3\epsilon_{a_1a_2a_3}\left[\mathrm{tr}\left(\psi_b\phi^b\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^b\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^b\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^{a_3}\phi^b\right)\right] \ , \\
(r_{12}^{(0,1)})_{kl}^{ij} &= -2\epsilon_{a_1a_2(k}\left[\mathrm{tr}\left(\psi_{l)}\phi^{(i}\phi^{j)}\phi^{a_1}\phi^{a_2}\right)+7\mathrm{tr}\left(\psi_{l)}\phi^{(i|}\phi^{a_1}\phi^{|j)}\phi^{a_2}\right)\right] \ , \\
(r_{12}^{(0,2)})_{ijk} &= \frac{1}{2}\epsilon_{a_1a_2(i}\mathrm{tr}\left(\phi^{a_1}\psi_j\phi^{a_2}\psi_{k)}\right) \ , \\
(r_{12}^{(0,2)})_{j}^{i} &= 6\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{(a}\psi_{j)}\right)-5\mathrm{tr}\left(\phi^{[i}\psi_a\phi^{a]}\psi_j\right) \ , \\
(r_{12}^{(0,2)})_{kl}^{ij} &= \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(k}\psi_{l)}\right) \ , \\
(r_{14}^{(0,2)})^{i} &= 3\,\mathrm{tr}\left(\phi^i\psi_{a_1}\phi^{a_1}\phi^{a_2}\psi_{a_2}\right)+2\,\mathrm{tr}\left(\phi^i\phi^{a_1}\right)\mathrm{tr}\left(\phi^{a_2}\psi_{(a_1}\psi_{a_2)}\right) \\
&\quad -6\,\mathrm{tr}\left(\phi^i\psi_{a_1}\right)\mathrm{tr}\left(\phi^{[a_1}\phi^{a_2]}\psi_{a_2}\right)-\mathrm{tr}\left(\phi^i\psi_{a_1}\psi_{a_2}\right)\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\right) \ , \\
(r_{14}^{(0,2)})_{ij} &= \frac{5}{9}\epsilon_{a_1a_2a_3}\left[2\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_{(i}\phi^{a_1}\psi_{j)}\phi^{a_2}\phi^{a_3}\right)\right] \\
&\quad +\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_3}\phi^{a_2}\right)+\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_3}\phi^{a_1}\phi^{a_2}\right)\right] \\
&\quad +\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_3}\phi^{a_2}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_3}\psi_{a_3}\phi^{a_1}\phi^{a_2}\right)\right] \\
&\quad +\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\psi_{a_3}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_3}\psi_{a_3}\phi^{a_2}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\psi_{a_3}\phi^{a_2}\right)\right] \\
&\quad +\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\psi_{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_3}\phi^{a_2}\psi_{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\phi^{a_2}\psi_{a_3}\right)\right] \\
&\quad -\frac{1}{3}\epsilon_{a_1a_2(i}\big[5\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_3}\right)+2\mathrm{tr}\left(\psi_{j)}\phi^{(a_1}\phi^{a_3)}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_2}\right) \\
&\qquad\qquad -2\mathrm{tr}\left(\psi_{j)}\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{(a_1}\phi^{a_3)}\right)\big] \ ,
\end{aligned}
$$

$$
\begin{aligned}
(r_{14}^{(0,2)})_{k}^{ij} &= 12\mathrm{tr}\left(\phi^{(i}\phi^a\phi^{j)}\psi_{(a}\psi_{k)}\right)+12\mathrm{tr}\left(\phi^{(i|}\phi^a\phi^{|j)}\psi_{(a}\psi_{k)}\right)+54\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(a}\phi^{a)}\psi_{k)}\right)-36\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(a}\phi^a\psi_{k)}\right) \\
(r_{14}^{(0,2)})_{jkl}^{i} &= 2\epsilon_{a_1a_2(j}\left[\mathrm{tr}\left(\phi^i\phi^{a_1}\phi^{a_2}\psi_k\psi_{l)}\right)+3\mathrm{tr}\left(\phi^i\phi^{a_1}\psi_k\phi^{a_2}\psi_{l)}\right)-2\mathrm{tr}\left(\phi^i\psi_k\phi^{a_1}\phi^{a_2}\psi_{l)}\right)\right] \ , \\
(r_{14}^{(0,2)})_{lm}^{ijk} &= 4\mathrm{tr}\left(\phi^{(i}\phi^j\phi^{k)}\psi_{(l}\psi_{m)}\right)+3\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(l}\phi^{k)}\psi_{m)}\right) \\
(r_{14}^{(0,3)})_{jkl}^{i} &= -\frac{1}{2}\mathrm{tr}\left(\phi^i\psi_{(j}\psi_k\psi_{l)}\right) \ , \\
(r_{16}^{(0,3)})_{i} &= \frac{39}{4}\mathrm{tr}\left(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_1}\phi^{b_2}\}\right)+2\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\psi_{b_2}\phi^{b_2}\right)-\frac{61}{4}\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\psi_{b_2}\phi^{b_1}\right) \\
&\quad +\frac{97}{4}\mathrm{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\psi_{b_2}\phi^{b_2}\right)-\frac{41}{4}\mathrm{tr}\left(\psi_i\phi^{b_2}\psi_{b_1}\psi_{b_2}\phi^{b_1}\right)-5\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\phi^{b_2}\psi_{b_2}\right) \\
&\quad -\frac{25}{2}\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\phi^{b_1}\psi_{b_2}\right)+2\mathrm{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\phi^{b_2}\psi_{b_2}\right)-\frac{61}{4}\mathrm{tr}\left(\psi_i\phi^{b_2}\psi_{b_1}\phi^{b_1}\psi_{b_2}\right) \\
&\quad -\frac{11}{4}\mathrm{tr}\left(\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}\left(\psi_i\psi_{b_1}\psi_{b_2}\right)-\frac{27}{2}\mathrm{tr}\left(\psi_{b_1}\psi_{b_2}\right)\mathrm{tr}\left(\psi_i\phi^{b_1}\phi^{b_2}\right)+\frac{29}{4}\mathrm{tr}\left(\phi^{b_2}\psi_{b_2}\right)\mathrm{tr}\left(\psi_i[\psi_{b_1},\phi^{b_1}]\right) \ , \\
(r_{16}^{(0,3)})_{jk}^{i} &= 2\mathrm{tr}\left(\psi_{(j}\psi_{k)}\psi_b\phi^b\phi^i\right)-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\psi_b\phi^i\phi^b\right)-\mathrm{tr}\left(\psi_{(j|}\psi_b\psi_{|k)}\{\phi^b,\phi^i\}\right)-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^{(b}\psi_b\phi^{i)}\right) \\
&\quad +7\mathrm{tr}\left(\psi_{(j|}\{\psi_b,\phi^b\}\psi_{|k)}\phi^i\right)-11\mathrm{tr}\left(\psi_{(j|}\{\psi_b,\phi^i\}\psi_{|k)}\phi^b\right)-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^b\phi^i\psi_b\right) \\
&\quad +2\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^i\phi^b\psi_b\right)+3\mathrm{tr}\left(\psi_{(j|}\psi_b\right)\mathrm{tr}\left(\psi_{|k)}[\phi^b,\phi^i]\right)+6\mathrm{tr}\left(\psi_{(j}\phi^{[b}\right)\mathrm{tr}\left(\{\psi_{k)},\psi_b\}\phi^{i]}\right) \ ,
\end{aligned}
$$

$$
(r_{16}^{(0,3)})_{klm}^{ij} = -\mathrm{tr}(\phi^{(i}\phi^{j)}\psi_{(k}\psi_l\psi_{m)})-2\mathrm{tr}(\phi^{(i}\psi_{(k}\phi^{j)}\psi_l\psi_{m)})
$$

$$s_1 = \epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^c{}_d v^{|k)}{}_{a_1} v^d{}_{b_1}(R^{(0,1)}_{12})_{a_2b_2c}\ ,\qquad s_2 = \epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^{|k)}{}_d v^c{}_{a_1} v^d{}_{b_1}(R^{(0,1)}_{12})_{a_2b_2c}$$

$$s_3 = \epsilon^{a_1a_2(i}v^{j|}{}_b v^b{}_c v^c{}_{a_1}(R^{(0,1)}_{12})^{|k)}_{a_2}\ ,\qquad s_4 = \epsilon^{a_1a_2a_3}v^b{}_{a_1}v^{(i}{}_{a_2}v^j{}_{a_3}(R^{(0,1)}_{12})^{k)}_{b}\ ,$$

$$s_5 = \epsilon^{a_1a_2(i}v^j{}_b v^{k)}{}_{a_1} v^b{}_c(R^{(0,1)}_{12})^{c}_{a_2}\ ,\qquad s_6 = \epsilon^{a_1a_2(i|}v^b{}_c v^c{}_d v^d{}_{a_1}(R^{(0,1)}_{12})^{|jk)}_{a_2b}\ ,$$

$$s_7 = \epsilon^{a_1a_2a_3}v^b{}_{a_1}v^c{}_{a_2}v^{(i}{}_{a_3}(R^{(0,1)}_{12})^{jk)}_{bc}\ ,\qquad s_8 = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^b{}_d v^d{}_c(R^{(0,1)}_{12})^{|k)c}_{a_2b}\ ,$$

$$s_9 = \epsilon^{a_1a_2(i}v^{j|}{}_d v^b{}_{a_1} v^d{}_c(R^{(0,1)}_{12})^{|k)c}_{a_2b}\ ,\qquad s_{10} = \epsilon^{a_1a_2(i|}u^{bc}v^d{}_{a_1}v^{|j}{}_b(R^{(0,2)}_{14})^{k)}_{a_2cd}\ ,$$

$$s_{11} = \epsilon^{a_1a_2(i|}u^{bc}v^d{}_b v^{|j}{}_{a_1}(R^{(0,2)}_{14})^{k)}_{a_2cd}\ ,\qquad s_{12} = \epsilon^{a_1a_2(i}u^{j|b}v^c{}_{a_1}v^d{}_b(R^{(0,2)}_{14})^{|k)}_{a_2cd}\ ,$$

$$s_{13} = \epsilon^{a_1a_2(i|}u^{cd}v^b{}_{a_1}v^{|j}{}_b(R^{(0,2)}_{14})^{k)}_{a_2cd}\ ,\qquad s_{14} = \epsilon^{a_1a_2(i}u^{j|c}v^b{}_{a_1}v^d{}_b(R^{(0,2)}_{14})^{|k)}_{a_2cd}\ ,$$

$$s_{15} = \epsilon^{a_1a_2(i}u^{j|b}v^c{}_{a_1}v^{|k)}{}_d(R^{(0,2)}_{14})^{d}_{a_2bc}\ ,\qquad s_{16} = \epsilon^{a_1a_2a_3}u^{(ij|}v^c{}_{a_1}v^d{}_{a_2}(R^{(0,2)}_{14})^{|k)}_{a_3cd}\ ,$$

$$s_{17} = \epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{cd}{}_b(R^{(0,2)}_{14})^{|k)}_{a_2cd}\ ,\qquad s_{18} = \epsilon^{a_1a_2(i}v^{bc}{}_{a_1}v^{d|j}{}_b(R^{(0,2)}_{14})^{k)}_{a_2cd}\ ,$$

$$s_{19} = \epsilon^{a_1a_2(i}v^{jk)}{}_b v^{cd}{}_{a_1}(R^{(0,2)}_{14})^{b}_{a_2cd}\ ,\qquad s_{20} = \epsilon^{a_1a_2a_3}v^{b(i}{}_{a_1}v^{j|c}{}_{a_2}(R^{(0,2)}_{14})^{|k)}_{a_3bc}\ ,$$

$$s_{21} = \epsilon^{a_1a_2(i}v^j{}_b v^{k)b}{}_{a_1}(R^{(0,2)}_{16})_{a_2}\ ,\qquad s_{22} = \epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{|jk)}{}_b(R^{(0,2)}_{16})_{a_2}\ ,$$

$$s_{23} = \epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)b}{}_{a_2}(R^{(0,2)}_{16})_{b}\ ,\qquad s_{24} = \epsilon^{a_1a_2(i}v^{j|}{}_b v^{bc}{}_{a_1}(R^{(0,2)}_{16})^{|k)}_{a_2c}\ ,$$

$$s_{25} = \epsilon^{a_1a_2(i|}v^c{}_b v^{|j|b}{}_{a_1}(R^{(0,2)}_{16})^{|k)}_{a_2c}\ ,\qquad s_{26} = \epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{|j|c}{}_b(R^{(0,2)}_{16})^{|k)}_{a_2c}\ ,$$

$$s_{27} = \epsilon^{a_1a_2b}v^c{}_{a_1}v^{(ij}{}_{a_2}(R^{(0,2)}_{16})^{k)}_{bc}\ ,\qquad s_{28} = \epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)b}{}_c(R^{(0,2)}_{16})^{c}_{a_2b}\ ,$$

$$s_{29} = \epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{|jk)}{}_c(R^{(0,2)}_{16})^{c}_{a_2b}\ ,\qquad s_{30} = \epsilon^{a_1a_2a_3}v^{b_1}{}_{a_1}v^{b_2(i}{}_{a_2}(R^{(0,2)}_{16})^{jk)}_{a_3b_1b_2}\ ,$$

$$s_{31} = \epsilon^{a_1a_2a_3}v^{(i|}{}_{a_1}v^{b_1b_2}{}_{a_2}(R^{(0,2)}_{16})^{|jk)}_{a_3b_1b_2}\ ,\qquad s_{32} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{bc}{}_d(R^{(0,2)}_{16})^{d|k)}_{a_2bc}$$

$$s_{33} = \epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{c|j}{}_d(R^{(0,2)}_{16})^{k)d}_{a_2bc}\qquad s_{34} = \epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{cd}{}_{a_2}(R^{(0,2)}_{16})^{|jk)}_{bcd}$$

$$s_{35} = \epsilon^{ab(i}v^j{}_a v^{k)}{}_b R^{(0,2)}_{18}\ ,\qquad s_{36} = u^{a(i}v^j{}_a(R^{(0,3)}_{20})^{k)}\ ,$$

$$s_{37} = u^{(ij}v^{k)}{}_a(R^{(0,3)}_{20})^{a}\ ,\qquad s_{38} = v^{(ij}{}_a(R^{(0,3)}_{22})^{k)a}\ ,$$

$$s_{39} = u^{(ij}(R^{(0,4)}_{26})^{k)}\ .$$

$$s_1 = \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k}{}_{a_1}v^{l)}{}_{a_2}v^{a_5a_6}{}_{a_3}(R_{12}^{(0,1)})_{a_4a_5a_6}\ ,$$
$$s_2 = \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k}{}_{a_1}v^{l)}{}_{a_5}v^{a_5a_6}{}_{a_3}(R_{12}^{(0,1)})_{a_2a_4a_6}\ ,$$
$$s_3 = \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_2}v^{|l)a_6}{}_{a_3}(R_{12}^{(0,1)})_{a_4a_5a_6}\ ,$$
$$s_4 = \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_3}v^{|l)a_6}{}_{a_2}(R_{12}^{(0,1)})_{a_4a_5a_6}\ ,$$
$$s_5 = \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_6}v^{|l)a_6}{}_{a_3}(R_{12}^{(0,1)})_{a_2a_4a_5}\ ,$$
$$s_6 = \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_6}v^{a_5}{}_{a_1}v^{|l)a_6}{}_{a_3}(R_{12}^{(0,1)})_{a_2a_4a_5}\ ,$$
$$s_7 = \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{a_5}{}_{a_1}v^{a_6}{}_{a_2}v^{|kl)}{}_{a_3}(R_{12}^{(0,1)})_{a_4a_5a_6}\ ,$$
$$s_8 = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_2}v^{l)a_3}{}_{a_4}(R_{12}^{(0,1)})^{a_4}_{a_3}\ ,$$
$$s_9 = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_3}v^{l)a_3}{}_{a_4}(R_{12}^{(0,1)})^{a_4}_{a_2}\ ,$$
$$s_{10} = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_4}v^{l)a_3}{}_{a_2}(R_{12}^{(0,1)})^{a_4}_{a_3}\ ,$$
$$s_{11} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{|kl)}{}_{a_4}(R_{12}^{(0,1)})^{a_4}_{a_3}\ ,$$
$$s_{12} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_2}(R_{12}^{(0,1)})^{a_4}_{a_3}\ ,$$
$$s_{13} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_3}(R_{12}^{(0,1)})^{a_4}_{a_2}\ ,$$
$$s_{14} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_4}v^{a_3}{}_{a_1}v^{|kl)}{}_{a_2}(R_{12}^{(0,1)})^{a_4}_{a_3}\ ,$$
$$s_{15} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{|k|a_4}{}_{a_3}(R_{12}^{(0,1)})^{|l)}_{a_4}\ ,$$
$$s_{16} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|k|a_4}{}_{a_2}(R_{12}^{(0,1)})^{|l)}_{a_3}\ ,$$
$$s_{17} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{|jk}{}_{a_3}(R_{12}^{(0,1)})^{l)}_{a_4}\ ,$$
$$s_{18} = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_4}v^{l)a_3}{}_{a_5}(R_{12}^{(0,1)})^{a_4a_5}_{a_2a_3}\ ,$$
$$s_{19} = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k|}{}_{a_2}v^{a_3a_4}{}_{a_5}(R_{12}^{(0,1)})^{|l)a_5}_{a_3a_4}\ ,$$
$$s_{20} = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k|}{}_{a_5}v^{a_3a_4}{}_{a_2}(R_{12}^{(0,1)})^{|l)a_5}_{a_3a_4}\ ,$$
$$s_{21} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_5}(R_{12}^{(0,1)})^{a_4a_5}_{a_2a_3}\ ,$$
$$s_{22} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_4}v^{a_3}{}_{a_1}v^{|kl)}{}_{a_5}(R_{12}^{(0,1)})^{a_4a_5}_{a_2a_3}\ ,$$
$$s_{23} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_4}v^{a_3}{}_{a_5}v^{|kl)}{}_{a_1}(R_{12}^{(0,1)})^{a_4a_5}_{a_2a_3}\ ,$$
$$s_{24} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{a_4|k}{}_{a_5}(R_{12}^{(0,1)})^{l)a_5}_{a_3a_4}\ ,$$
$$s_{25} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{a_4|k}{}_{a_5}(R_{12}^{(0,1)})^{l)a_5}_{a_2a_3}\ ,$$
$$s_{26} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_5}v^{a_4|k}{}_{a_2}(R_{12}^{(0,1)})^{l)a_5}_{a_3a_4}\ ,$$
$$s_{27} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_5}v^{a_4|k}{}_{a_3}(R_{12}^{(0,1)})^{l)a_5}_{a_2a_4}\ ,$$
$$s_{28} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_5}v^{a_3}{}_{a_1}v^{a_4|k}{}_{a_2}(R_{12}^{(0,1)})^{l)a_5}_{a_3a_4}\ ,$$
$$s_{29} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{a_4a_5}{}_{a_3}(R_{12}^{(0,1)})^{kl)}_{a_4a_5}\ ,$$
$$s_{30} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{a_4a_5}{}_{a_2}(R_{12}^{(0,1)})^{kl)}_{a_3a_5}\ ,$$
$$s_{31} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{|jk}{}_{a_5}(R_{12}^{(0,1)})^{l)a_5}_{a_3a_4}\ ,$$
$$s_{32} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}v^{|jk}{}_{a_2}(R_{12}^{(0,1)})^{l)a_5}_{a_3a_4}\ ,$$
$$s_{33} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}v^{|jk}{}_{a_4}(R_{12}^{(0,1)})^{l)a_5}_{a_2a_3}\ ,$$
$$s_{34} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{a_5|j}{}_{a_3}(R_{12}^{(0,1)})^{kl)}_{a_4a_5}\ ,$$
$$s_{35} = \epsilon^{a_1a_2a_3}v^{(i}{}_{a_1}v^{j}{}_{a_2}v^{k}{}_{a_3}(R_{14}^{(0,1)})^{l)}\ ,$$
$$s_{36} = \epsilon^{a_1a_2(i|}\epsilon^{a_3a_4|j}v^{k}{}_{a_1}v^{l)}{}_{a_2}v^{a_5}{}_{a_3}(R_{14}^{(0,1)})_{a_4a_5}\ ,$$

$$s_{37} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{a_4}{}_{a_3}(R_{14}^{(0,1)})^{|kl)}_{a_4}\ ,$$
$$s_{38} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_3}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}(R_{14}^{(0,1)})^{|kl)}_{a_4}\ ,$$
$$s_{39} = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k|}{}_{a_2}v^{a_3}{}_{a_4}(R_{14}^{(0,1)})^{|l)a_4}_{a_3}\ ,$$
$$s_{40} = \epsilon^{a_1a_2(i|}\epsilon^{a_3a_4|j}v^{k|}{}_{a_1}v^{a_5}{}_{a_2}v^{a_6}{}_{a_3}(R_{14}^{(0,1)})^{|l)}_{a_4a_5a_6}\ ,$$
$$s_{41} = \epsilon^{a_1a_2(i|}\epsilon^{a_3a_4|j}v^{k|}{}_{a_1}v^{a_5}{}_{a_3}v^{a_6}{}_{a_5}(R_{14}^{(0,1)})^{|l)}_{a_2a_4a_6}\ ,$$
$$s_{42} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{a_5}{}_{a_3}(R_{14}^{(0,1)})^{|jkl)}_{a_4a_5}\ ,$$
$$s_{43} = \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_3}v^{a_5}{}_{a_4}(R_{14}^{(0,1)})^{|jkl)}_{a_2a_5}\ ,$$
$$s_{44} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{a_4}{}_{a_5}(R_{14}^{(0,1)})^{|kl)a_5}_{a_3a_4}\ ,$$
$$s_{45} = \epsilon^{a_1a_2(i}v^{j|}{}_{a_5}v^{a_3}{}_{a_1}v^{a_4}{}_{a_3}(R_{14}^{(0,1)})^{|kl)a_5}_{a_2a_4}\ ,$$
$$s_{46} = \epsilon^{a_1a_2(i}u^{jk|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}(R_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}\ ,$$
$$s_{47} = \epsilon^{a_1a_2(i}u^{jk|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_3}(R_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}\ ,$$
$$s_{48} = \epsilon^{a_1a_2(i}u^{jk|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_4}(R_{14}^{(0,2)})^{|l)}_{a_2a_3a_5}\ ,$$
$$s_{49} = \epsilon^{a_1a_2(i}u^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_2}(R_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}\ ,$$
$$s_{50} = \epsilon^{a_1a_2(i}u^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_3}(R_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}\ ,$$
$$s_{51} = \epsilon^{a_1a_2(i}u^{j|a_3a_4}v^{|k|}{}_{a_5}v^{a_5}{}_{a_1}(R_{14}^{(0,2)})^{|l)}_{a_2a_3a_4}\ ,$$
$$s_{52} = \epsilon^{a_1a_2(i|}u^{a_3a_4a_5}v^{|j}{}_{a_1}v^{k}{}_{a_2}(R_{14}^{(0,2)})^{l)}_{a_3a_4a_5}\ ,$$
$$s_{53} = \epsilon^{a_1a_2(i}u^{jk|a_3}v^{|l)}{}_{a_5}v^{a_4}{}_{a_1}(R_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}\ ,$$
$$s_{54} = \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4a_5}{}_{a_2}(R_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}\ ,$$
$$s_{55} = \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4a_5}{}_{a_3}(R_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}\ ,$$
$$s_{56} = \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_4}v^{a_4a_5}{}_{a_1}(R_{14}^{(0,2)})^{|l)}_{a_2a_3a_5}\ ,$$
$$s_{57} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_2}(R_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}\ ,$$
$$s_{58} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_3}(R_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}\ ,$$

$$s_{59} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_4}v^{a_4a_5}{}_{a_1}(R^{(0,2)}_{14})^{|l)}_{a_2a_3a_5} \ ,$$
$$s_{60} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_2}(R^{(0,2)}_{14})^{l)}_{a_3a_4a_5} \ ,$$
$$s_{61} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_3}(R^{(0,2)}_{14})^{l)}_{a_2a_4a_5} \ ,$$
$$s_{62} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_5}v^{a_5|k}{}_{a_1}(R^{(0,2)}_{14})^{l)}_{a_2a_3a_4} \ ,$$
$$s_{63} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_2}(R^{(0,2)}_{14})^{|l)}_{a_3a_4a_5} \ ,$$
$$s_{64} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_3}(R^{(0,2)}_{14})^{|l)}_{a_2a_4a_5} \ ,$$
$$s_{65} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_5}v^{k|a_5}{}_{a_1}(R^{(0,2)}_{14})^{|l)}_{a_2a_3a_4} \ ,$$
$$s_{66} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_2}(R^{(0,2)}_{14})^{l)}_{a_3a_4a_5} \ ,$$
$$s_{67} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_3}(R^{(0,2)}_{14})^{l)}_{a_2a_4a_5} \ ,$$
$$s_{68} = \epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_1}v^{a_3a_4}{}_{a_5}(R^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} \ ,$$
$$s_{69} = \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{|l)a_4}{}_{a_5}(R^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} \ ,$$
$$s_{70} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k}{}_{a_1}v^{l)a_4}{}_{a_5}(R^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} \ ,$$
$$s_{71} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{|kl)}{}_{a_5}(R^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} \ ,$$
$$s_{72} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{kl)}{}_{a_5}(R^{(0,2)}_{14})^{a_5}_{a_2a_3a_4} \ ,$$
$$s_{73} = \epsilon^{a_1a_2(i}v^{jk|}{}_{a_1}v^{a_3a_4}{}_{a_5}(R^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} \ ,$$
$$s_{74} = \epsilon^{a_1a_2(i}v^{jk|}{}_{a_5}v^{a_3a_4}{}_{a_1}(R^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} \ ,$$
$$s_{75} = \epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4|k}{}_{a_5}(R^{(0,2)}_{16})^{l)a_5}_{a_2a_3a_4} \ ,$$
$$s_{76} = \epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4a_5}{}_{a_2}(R^{(0,2)}_{16})^{|kl)}_{a_3a_4a_5} \ ,$$
$$s_{77} = \epsilon^{a_1a_2a_3}v^{(ij}{}_{a_1}v^{k|b}{}_{a_2}(R^{(0,2)}_{16})^{|l)}_{a_3b} \ ,$$
$$s_{78} = \epsilon^{a_1a_2(i}v^{jk}{}_{a_1}v^{l)c}{}_{b}(R^{(0,2)}_{16})^{b}_{a_2c} \ ,$$
$$s_{79} = \epsilon^{a_1a_2(i}v^{jk|}{}_{a_1}v^{bc}{}_{a_2}(R^{(0,2)}_{16})^{|l)}_{bc} \ ,$$
$$s_{80} = \epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{c|k}{}_{a_2}(R^{(0,2)}_{16})^{|l)}_{bc} \ ,$$
$$s_{81} = \epsilon^{a_1a_2a_3}v^{(ij}{}_{a_1}v^{kl)}{}_{a_2}(R^{(0,2)}_{16})_{a_3} \ ,$$
$$s_{82} = \epsilon^{a_1a_2(i}v^{jk}{}_{a_1}v^{l)b}{}_{a_2}(R^{(0,2)}_{16})_{b} \ ,$$
$$s_{83} = \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}(R^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} \ ,$$
$$s_{84} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4}{}_{a_5}(R^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} \ ,$$
$$s_{85} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_5}v^{a_4}{}_{a_1}(R^{(0,2)}_{16})^{|l)a_5}_{a_2a_3a_4} \ ,$$
$$s_{86} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}(R^{(0,2)}_{16})^{|kl)}_{a_3a_4a_5} \ ,$$
$$s_{87} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k}{}_{a_5}(R^{(0,2)}_{16})^{l)a_5}_{a_2a_3a_4} \ ,$$
$$s_{88} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_2}(R^{(0,2)}_{16})^{|kl)}_{a_3a_4a_5} \ ,$$
$$s_{89} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_3}(R^{(0,2)}_{16})^{|kl)}_{a_2a_4a_5} \ ,$$
$$s_{90} = \epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_1}v^{a_3}{}_{a_4}(R^{(0,2)}_{16})^{a_4}_{a_2a_3} \ ,$$
$$s_{91} = \epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_4}v^{a_3}{}_{a_1}(R^{(0,2)}_{16})^{a_4}_{a_2a_3} \ ,$$
$$s_{92} = \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}(R^{(0,2)}_{16})^{|l)}_{a_3a_4} \ ,$$
$$s_{93} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k}{}_{a_1}v^{l)}{}_{a_4}(R^{(0,2)}_{16})^{a_4}_{a_2a_3} \ ,$$
$$s_{94} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4}{}_{a_2}(R^{(0,2)}_{16})^{|l)}_{a_3a_4} \ ,$$

$$s_{95} = \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4}{}_{a_3}(R^{(0,2)}_{16})^{|l)}_{a_2a_4} \ ,$$
$$s_{96} = \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k}{}_{a_2}(R^{(0,2)}_{16})^{l)}_{a_3a_4} \ ,$$
$$s_{97} = \epsilon^{a_1a_2a_3}u^{(ij}v^{k}{}_{a_1}v^{l)}{}_{a_2}(R^{(0,2)}_{16})_{a_3} \ ,$$
$$s_{98} = \epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_1}v^{b}{}_{a_2}(R^{(0,2)}_{16})_{b} \ ,$$
$$s_{99} = \epsilon^{a_1a_2(i}u^{j|b}v^{|k}{}_{a_1}v^{l)}{}_{a_2}(R^{(0,2)}_{16})_{b} \ ,$$
$$s_{100} = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{kl)}{}_{a_2}(R^{(0,2)}_{18}) \ ,$$
$$s_{101} = u^{a(i}v^{jk}{}_{a}(R^{(0,3)}_{20})^{l)} \ ,$$
$$s_{102} = u^{(ij}v^{kl)}{}_{a}(R^{(0,3)}_{20})^{a} \ ,$$
$$s_{103} = u^{a(ij}v^{k}{}_{a}(R^{(0,3)}_{20})^{l)} \ ,$$
$$s_{104} = u^{(ijk}v^{l)}{}_{a}(R^{(0,3)}_{20})^{a} \ ,$$
$$s_{105} = u^{a(i}v^{j}{}_{a}(R^{(0,3)}_{22})^{kl)} \ ,$$
$$s_{106} = u^{(ij}v^{k}{}_{a}(R^{(0,3)}_{22})^{l)a} \ ,$$
$$s_{107} = u^{(ijk}(R^{(0,4)}_{26})^{l)} \ ,$$

$$\frac{ds_{10}^2}{R_{AdS}^2} = g_{\mu\nu}dx^{\mu}dx^{\nu} = ds_5^2 + ds^2(\mathbb{CP}^2) + (d\Psi + \Theta - A)^2$$

$$R_{AdS}^4 = 4\pi g_s N(\alpha')^2$$

$$ds^2(\mathbb{CP}^2) + (d\Psi + \Theta - A)^2 = \sum_{i=1}^{3} dl_i^2 + l_i^2(d\phi_i - A)^2$$

$$\sum_{i=1}^{3} l_i^2 = 1$$

$$\Psi = \frac{\phi_1 + \phi_2 + \phi_3}{3}$$

$$2J = d\Theta = \sum_i d\left(l_i^2\right) \wedge d\phi_i$$

$$\Theta = \sum_i l_i^2 d\phi_i - d\Psi = \sum_i \left(l_i^2 - \frac{1}{3}\right) d\phi_i$$

$$\epsilon^{(10)} = \epsilon^{(5)} \wedge \epsilon^{(4)}(\mathbb{CP}^2) \wedge (d\Psi + \Theta - A), \epsilon^{(4)}(\mathbb{CP}^2) = \frac{1}{2} J \wedge J$$

$$ds_5^2 = -\frac{r^2 W(r)}{4b(r)^2} dt^2 + \frac{dr^2}{W(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + b(r)^2(\sigma_3 + f(r)dt)^2$$

$$A = \frac{q}{r^2}\left(dt - \frac{j}{2}\sigma_3\right)$$

$$\sigma_1 = \cos\ \psi d\theta_a + \sin\ \psi \sin\ \theta_a d\phi_a$$

$$\sigma_2\ = -\sin\ \psi d\theta_a + \cos\ \psi \sin\ \theta_a d\phi_a$$

$$\sigma_3\ = d\psi + \cos\ \theta_a d\phi_a$$

$$\sum_i \frac{\sigma_i^2}{4}$$

$$\Omega\ = f(R)$$

$$\mu\ = A_t(R) - \Omega A_{\sigma_3}(R)$$

$$A = \left(\frac{q}{r^2} - \mu\right) dt - \frac{jq}{2r^2}\sigma_3$$

$$b(r)^2 = \frac{r^2}{4}\left(1 - \frac{j^2 q^2}{r^6} + \frac{2j^2 p}{r^4}\right)$$

$$f(r)\ = -\frac{j}{2b^2}\left(\frac{2p-q}{r^2} - \frac{q^2}{r^4}\right)$$

$$W(r)\ = 1 + 4b^2 - \frac{1}{r^2}(2p - 2q) + \frac{1}{r^4}(q^2 + 2pj^2)$$

$$M\ = \frac{1}{2}(3p - 3q + pj^2)$$

$$Q\ = \frac{q}{2}$$

$$J\ = \frac{1}{2} j(2p - q)$$

$$T = \frac{RW'(R)}{8\pi b(R)}$$

$$\frac{F^{(5)}}{R_{AdS_5}^4}\ = G^{(5)} +_{*_{10}} G^{(5)}$$

$$G^{(5)}\ = -4\epsilon^{(5)} + J \wedge_{*_5} F^{(2)}$$

$$dC = F^{(5)}$$

$$M - 2J - 3Q = 0$$

$$\frac{C}{R^4_{AdS_5}} = C_V + \Theta \wedge (-*_5 F + A \wedge F) + l_2^2 d(l_3^2) \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 - A \wedge (d\Psi + \Theta) \wedge J$$

$$C_V = \frac{r^4 - R^4}{8} dt \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3$$

$$\begin{aligned} ds^2 &= -\frac{Y}{R^2} dt^2 + \frac{R\rho^2}{Y} d\rho^2 + R d\Omega_3^2 \\ A_i &= \frac{r_j r_k - \Gamma r_i}{H_i} dt \\ X_i &= \frac{R}{H_i} \end{aligned}$$

$$\begin{aligned} R &= (H_1 H_2 H_3)^{\frac{1}{3}} \\ Y = R^3 + \rho^4 + \rho^2 \left(\frac{1}{3}(r_1^2 + r_2^2 + r_3^2) - \Gamma^2\right) + \frac{1}{3}\Gamma^2(r_1^2 + r_2^2 + r_3^2) \\ -2\Gamma r_1 r_2 r_3 + \left[\frac{5}{18}(r_1^2 + r_2^2 + r_3^2)^2 - \frac{1}{2}(r_1^4 + r_2^4 + r_3^4)\right] \\ H_i = \rho^2 + \frac{\mu(s_i^2 - s_j^2) + \mu(s_i^2 - s_k^2)}{3} \\ \Gamma = \sqrt{\mu}[c_1 c_2 c_3 - s_1 s_2 s_3] \\ r_i = \sqrt{\mu}(c_i s_j s_k - s_i c_j c_k) \\ c_i = \cosh\ \delta_i \\ s_i = \sinh\ \delta_i \end{aligned}$$

$$\begin{aligned} E &= \frac{\mu}{4}[\cosh\ 2\delta_1 + \cosh\ 2\delta_2 + \cosh\ 2\delta_3] \\ Q_i &= \frac{\mu}{4}\sinh\ 2\delta_i \end{aligned}$$

$$\begin{aligned} S &= \pi(H_1 H_2 H_3)^{\frac{1}{2}} \\ T &= \frac{Y'(\rho_0)}{4\pi\rho_0(H_1 H_2 H_3)^{\frac{1}{2}}} \end{aligned}$$

$$\mu_i = \frac{3\mu\sinh\ 2\delta_i}{6\rho_0^2 + \mu(2\cosh\ 2\delta_i - \cosh\ 2\delta_j - \cosh\ 2\delta_k)}$$

$$\begin{aligned} N_{\text{eff}} &= N - m \\ E^{BH} &= E^{\text{tot}} - \sum_{i=1}^{3} Q_i^D \\ Q_i^{BH} &= Q_i^{\text{tot}} - Q_i^D \\ J_i^{BH} &= J_i^{\text{tot}} \end{aligned}$$

$$S_{BH} = N_{\text{eff}}^2 s_{BH}\left(\frac{E^{BH}}{N_{\text{eff}}^2}, \frac{Q_i^{\text{BH}}}{N_{\text{eff}}^2}, \frac{J_j^{BH}}{N_{\text{eff}}^2}\right)$$

$$N_{\text{eff}} = N\left(1 - \frac{m}{N}\right)$$

$$S_{BH} = N^2 s_{BH}\left(\frac{E^{BH}}{N^2}, \frac{Q_i^{\text{BH}}}{N^2}, \frac{J_j^{BH}}{N^2}\right) + 2Nm\beta\left(\frac{E^{BH}}{N^2} - \mu_i \frac{Q_i^{\text{BH}}}{N^2} - \omega_j \frac{J_j^{\text{BH}}}{N^2} - s_{BH}T\right)$$

$$\beta = N^2 \frac{\partial s_{BH}}{\partial E^{BH}}$$

$$\beta\mu_i = -N^2 \frac{\partial s_{BH}}{\partial Q_i^{BH}}$$

$$\beta\omega_j = -N^2 \frac{\partial s_{BH}}{\partial J_j^{BH}}$$

$$\frac{\partial S_{BH}}{\partial Q_i^D} = \beta(\mu_i - 1), Q^{BH} < Q^{\text{tot}}$$

$$\mu_i\left(Q_i^{BH} = Q_i^{\text{tot}}\right) < 1$$

$$2Nm\beta\left(\frac{E^{BH}}{N^2} - \mu_i \frac{Q_i^{\text{BH}}}{N^2} - \omega_j \frac{J_j^{\text{BH}}}{N^2} - s_{BH}T\right)$$

$$e^{-\beta q_i(1-\mu_i)}$$

$$Z_{\text{Dual}}^i(\beta, \mu_i) = \sum_{n=1}^{\infty}\left(Z_{\text{excit}}^{(n)}(\beta)e^{-\beta(1-\mu)n}\right) \approx \frac{Z_{\text{excit}}^{(\infty)}(\beta)}{\beta(1-\mu_i)}$$

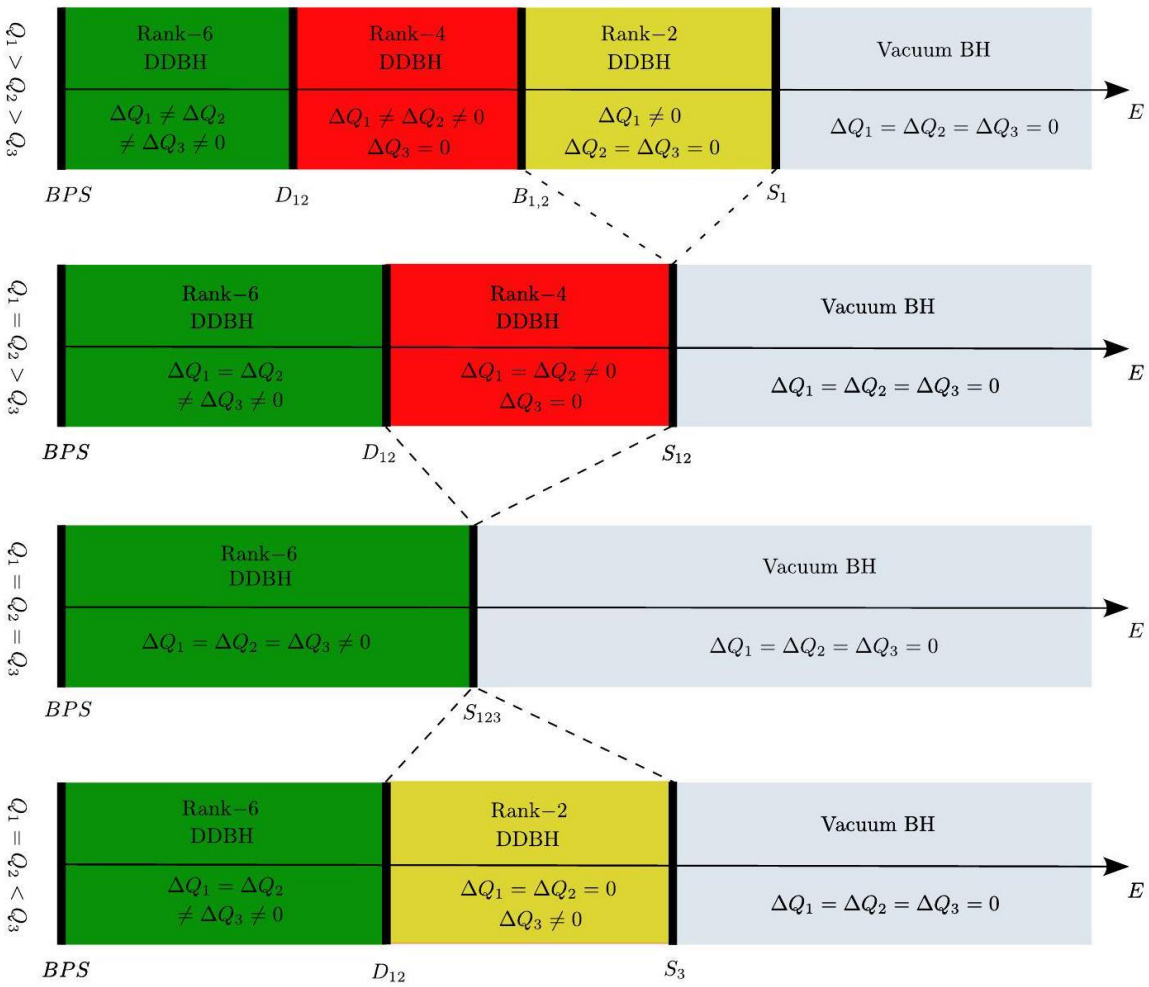
$$\ln Z_{\text{Dual}} = -\ln\left(\beta(1-\mu_i)\right)$$

$$\ln Z_{\text{Tot}} = \ln Z_{\text{BH}} - \sum_{i=1}^{3} \ln \beta(1-\mu_i)$$

$$Q_{\text{Tot}}^i(\beta, \mu_i) = Q_{BH}^i(\beta, \mu_i) + \frac{1}{\beta(1-\mu_i)}$$

$$\mu_i = 1 - \frac{\chi_i}{N^2}$$

$$\Delta E = \Delta Q_1 + \Delta Q_2 + \Delta Q_3$$



$$E = 3Q^2 + 3Q \approx 3Q^2$$

$$T = \frac{\tau}{\epsilon}, \mu = \frac{\nu}{\epsilon}$$

$$E = \frac{3(1+x^2)}{4\epsilon^4}$$

$$Q = \frac{x}{2\epsilon^3}$$

$$E = cQ^{\frac{4}{3}}$$

$$c = \frac{3}{4}(1+x^2)\left(\frac{2}{x}\right)^{4/3}$$

$$\frac{\nu}{\tau} = \frac{2\pi x}{2-x^2}$$

$$r' = \epsilon r$$

$$t' = \frac{t}{\epsilon}$$

$$x_i' = \frac{\theta_i}{\epsilon}$$

$$Q_{gg} = \frac{x}{2\epsilon^3} - \mathcal{O}\left(\frac{1}{\epsilon^2}\right)$$

$$Q_{BH} = \frac{\sqrt{1+x^2}}{2\epsilon^2} + \cdots$$
$$E_{BH} = \frac{3(1+x^2)}{4\epsilon^4} - \mathcal{O}\left(\frac{1}{\epsilon^3}\right)$$

$$r = \sqrt{\frac{Nx}{2}}\frac{1}{\epsilon^{3/2}}, \text{ i.e } r' = \sqrt{\frac{Nx}{2}}\frac{1}{\epsilon^{1/2}}$$

$$ds^2 = R^2_{AdS_5} g_{\mu\nu} dx^\mu dx^\nu$$

$$ds^2 = R^2_{AdS_5} h_{ab} dx^a dx^b$$

$$S = \frac{R^4_{AdS_5}}{(2\pi)^3 \alpha'^2 g_s}\left(-\int d^4y\sqrt{-\det h_{ab}} + \frac{1}{4!}\int \epsilon^{abcd} C_{\mu_1\mu_2\mu_3\mu_4}\frac{dx^{\mu_1}}{dy^a}\frac{dx^{\mu_2}}{dy^b}\frac{dx^{\mu_3}}{dy^c}\frac{dx^{\mu_4}}{dy^d}d^4y\right)$$
$$= \frac{N}{2\pi^2}\left(-\int d^4y\sqrt{-\det h_{ab}} + \frac{1}{4!}\int \epsilon^{abcd} C_{\mu_1\mu_2\mu_3\mu_4}\frac{dx^{\mu_1}}{dy^a}\frac{dx^{\mu_2}}{dy^b}\frac{dx^{\mu_3}}{dy^c}\frac{dx^{\mu_4}}{dy^d}d^4y\right)$$

$$R^4_{AdS_5} = 4\pi g N \alpha'^2$$

$$S = S_{DBI} + S_{WZ}$$
$$= 2N\int r^2\sqrt{b^2 + A^2_{\sigma_3}}\sqrt{-\tilde{g}_{\mu\nu}dx^\mu dx^\nu} + 8N\int\left[\left(\frac{r^4 - R^4}{8}\right)dt\right]$$

$$ds^2_7 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -\frac{r^2 W}{4b^2}dt^2 + \frac{dr^2}{W} + ds^2(\mathbb{CP}^2) + \frac{b^2}{b^2 + A^2_{\sigma_3}}\left(d\Psi + \Theta + \left(fA_{\sigma_3} - A_t\right)dt\right)^2$$

$$\Delta = \tilde{g}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$$
$$m = 2r^2\sqrt{b^2 + A^2_{\sigma_3}}$$

$$\begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Q_{ij} = \frac{i}{2}\left(z_i p_{z_j} - \bar{z}_j p_{\bar{z}_i}\right)$$

$$z_i = l_i e^{i\phi_i}$$
$$p_{z_i} = \frac{\partial L}{\partial \dot{z}_i} = -\frac{i}{z_i}\left(\frac{\partial L}{\partial \dot{\phi}_i}\right) + \frac{l_i}{z_i}\frac{\partial L}{\partial \dot{l}_i}$$
$$p_{\bar{z}_i} = \frac{\partial L}{\partial \dot{\bar{z}}_i} = \frac{i}{\bar{z}_i}\left(\frac{\partial L}{\partial \dot{\phi}_i}\right) + \frac{l_i}{\bar{z}_i}\frac{\partial L}{\partial \dot{l}_i}$$

$$Q_{ii} \equiv N q_i = \frac{\partial L}{\partial \dot{\phi}_i}$$

$$N\begin{pmatrix} q_1 & \frac{e^{i(\phi_1-\phi_2)}\left(l_2^2q_1+l_1^2q_2-2il_1^2l_2^2(i_1-i_2)\right)}{2l_1l_2} & \frac{e^{i(\phi_1-\phi_3)}\left(l_3^2q_1+ \\ -\frac{e^{-i(\phi_1-\phi_2)}\left(l_2^2q_1+l_1^2q_2-2il_1^2l_2^2(i_1-i_2)\right)}{2l_1l_2} & q_2 & \frac{e^{i(\phi_2-\phi_3)}\left(l_3^2q_2+ \\ -\frac{e^{-i(\phi_1-\phi_3)}\left(l_3^2q_1+l_1^2q_3-2il_1^2l_3^2(i_1-i_3)\right)}{2l_1l_3} & -\frac{e^{-i(\phi_2-\phi_3)}\left(l_3^2q_2+l_2^2q_3-2il_2^2l_3^2(i_2-i_3)\right)}{2l_2l_3} & \end{pmatrix}$$

$$\dot{l}_1=\dot{l}_2=0, l_3=0, \frac{l_1^2}{q_1}=-\frac{l_2^2}{q_2}$$

$$l_1=\sqrt{\frac{q_1}{q_1-q_2}}, l_2=\sqrt{\frac{q_2}{q_2-q_1}}, l_3=0$$

$$q_1= m\frac{l_1^2\dot{\phi}_1-l_1^2\left(A_t-\frac{A_tA_{\sigma_3}+b^2f-\sum_i\left(l_i^2A_{\sigma_3}\dot{\phi}_i\right)}{b^2+A_{\sigma_3}^2}A_{\sigma_3}\right)}{\sqrt{-\Delta}}$$

$$q_2= m\frac{l_2^2\dot{\phi}_2-l_2^2\left(A_t-\frac{A_tA_{\sigma_3}+b^2f-\sum_i\left(l_i^2A_{\sigma_3}\dot{\phi}_i\right)}{b^2+A_{\sigma_3}^2}A_{\sigma_3}\right)}{\sqrt{-\Delta}}$$

$$L_{\text{eff}}=L-p_i\dot{\phi}_i$$

$$L_{\text{eff}}= N\left(-\frac{1}{2b^2}\sqrt{\frac{(r^2W^2-4b^2\dot{r}^2)\left((q_1+q_2)^2A_{\sigma_3}^2+b^2(m^2+(q_1-q_2)^2)\right)}{W}}\right.$$
$$\left.+(q_1+q_2)\left(fA_{\sigma_3}-A_t\right)+r^4-R^4\right)$$

$$H=N\left(\frac{r\sqrt{W}}{2b^2}\sqrt{(q_1+q_2)^2A_{\sigma_3}^2+b^2(m^2+Wq_r^2+(q_1-q_2)^2)}+(q_1+q_2)\left(A_t-fA_{\sigma_3}\right)-r^4+R^4\right)$$

$$V_{\text{cl}}(r)=N\left(\frac{r\sqrt{W}}{2b^2}\sqrt{(q_1+q_2)^2A_{\sigma_3}^2+b^2(m^2+(q_1-q_2)^2)}+(q_1+q_2)\left(A_t-fA_{\sigma_3}\right)-r^4+R^4\right)$$

$$L_{\text{eff}}=N\left(-(q_1+q_2)A_t(r)-\sqrt{\frac{(r^6+(q_1-q_2)^2)(W(r)^2-\dot{r}^2)}{W(r)}}+r^4-R^4\right)$$

$$H=N\left((q_1+q_2)A_t+\sqrt{W(r)((q_1-q_2)^2+q_r^2W(r)+r^6)}-r^4+R^4\right)$$

$$V_{\text{cl}}(r)=N\left((q_1+q_2)A_t+\sqrt{W(r)((q_1-q_2)^2+r^6)}-r^4+R^4\right)$$

$$W(r)=\frac{(r-R)(r+R)(r^4+r^2(R^2+1)-\mu^2R^2)}{r^4}$$

$$A_t(r)=\mu\left(\frac{R^2}{r^2}-1\right)$$

$$V_{\mathrm{cl}}(r) = \frac{Nr\sqrt{W}}{2b^2}\sqrt{q_c^2 A_{\sigma_3}^2 + b^2(m^2 + q_n^2)} + q_c\left(A_t - fA_{\sigma_3}\right) - r^4 + R^4$$

$$V_{\mathrm{cl}}(R) = N\mu q_c$$

$$V_{\mathrm{cl}}(r) = N\frac{r^2}{2} + \mathcal{O}(1)$$

$$V_{\mathrm{cl}}(r) = N\left(\sqrt{(1+r^2)(q_n^2 + r^6)} - r^4\right)$$

$$V_{\mathrm{cl}}(r) = N\left(\frac{q_n^2 + r^4}{2r^2} + \left((j^2-1)p + \frac{qq_c}{r^2} + q - \frac{(q_n^2 - r^4)^2}{8r^8} + R^4\right)\right)$$

$$r_{\min} = \sqrt{|q_{\mathrm{n}}|} + \frac{qq_c}{2q_n^{3/2}}$$

$$E = V_{\mathrm{cl}}(r_{\min}) = N\left(q_n + \left((j^2-1)p + \frac{qq_c}{q_n} + q + R^4\right)\right)$$

$$\mu_c > \frac{q_n}{q_c}$$

$$E = V_{\mathrm{cl}}(r_{\min}) = N\left(q_n + \left((j^2-1)p + 2q + R^4\right)\right)$$

$$(j^2-1)p + 2q + R^4$$

$$V_{\mathrm{cl}}(r) = N\left(\frac{q_n^2 + r^4}{2r^2} + \frac{-q_n^4 + 2q_n^2 r^4 + 8\mu q_c r^6 R^2 + r^8(4R^4 - 4(\mu^2+1)R^2 - 1)}{8r^8}\right)$$

$$r_{\min} = \sqrt{q_n} + \frac{(\mu q_c R^2)}{2q_n^{3/2}}$$

$$E = V_{\mathrm{cl}}(r_{\min}) = q_n + \frac{R^2(q_n(-\mu^2 + R^2 - 1) + 2\mu q_c)}{2q_n}$$

$$q = R^2\left(1 + \frac{R^2}{2}\right)$$

$$p = 2R^2\left(1 + \frac{R^2}{2}\right)^2$$

$$j = \frac{R^2}{2l}\left(1 + \frac{R^2}{2}\right)^{-1}$$

$$V_{\mathrm{cl}}(r) = q_c + \sqrt{A^2 + B} - A$$

$$A = q_c\left(\frac{2(r^2-R^2)(2r^4-r^2R^4+R^6)}{4r^6+4r^2R^6-R^8}\right)+r^4-R^4$$

$$B = \frac{(r^2-R^2)^2(-2q_c(r^2+R^2)+2r^4-r^2R^4+R^6)^2}{4r^6+4r^2R^6-R^8}$$

$$\sqrt{A^2+B}-A \geq 0$$

$$q_c = f(r);\ f(r) = \frac{2r^4-r^2R^4+R^6}{2(r^2+R^2)}$$

$$r^2 = \frac{R^4+2q_c \pm \sqrt{(R^4+2q_c)^2-8(R^4-2q_c)R^2}}{4}$$

$$f(r) = \frac{R^2}{2}\left(4\sqrt{R^2+1}-R^2-4\right)$$

$$\frac{R^2}{2}\left(4\sqrt{R^2+1}-R^2-4\right) \leq q_c \leq \frac{R^2}{2}$$

$$h_{\mu\nu} = m(r)^2\tilde{g}_{\mu\nu}$$

$$C_t = (r^4-R^4).$$

$$\left(-\frac{1}{N^2}h^{\mu\nu}D_\mu D_\nu+1\right)\phi = 0$$

$$\left(-h^{\mu\nu}D_\mu D_\nu+N^2\right)\phi = 0$$

$$t = v-\int \frac{dr}{W(r)}$$

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -W(r)dv^2+2drdv+ds^2(\mathbb{CP}^2)+\frac{b^2}{b^2+A_{\sigma_3}^2}\left(d\Psi+\Theta+\left(fA_{\sigma_3}-A_t\right)\left(dv-\frac{dr}{W}\right)\right)^2$$

$$C_v = C_t = r^4-R^4, C_r = -\frac{C_t}{W(r)} = \frac{R^4-r^4}{W(r)}$$

$$\left(-\Theta+\left(A_t-fA_{\sigma_3}\right)dv-\frac{A_t-fA_{\sigma_3}}{W}dr\right).$$

$$-e^{-ik\Psi}\nabla^2\phi = -\frac{1}{m^2}(D_vD_r+D_rD_v)\chi-\frac{1}{m^7}D_r(m^5W(r)D^r\chi)$$
$$+\frac{k^2}{m^2}\left(\frac{b^2+A_{\sigma_3}^2}{b^2}\right)\chi-\frac{1}{m^2}\nabla_{CP^2,k}^2\chi$$

$$D_r = \partial_r + i\frac{k\left(A_t - fA_{\sigma_3}\right) - NC_t}{W}, D_v = \left(\partial_v + i\left(k\left(A_t - fA_{\sigma_3}\right) - iNC_t\right)\right.$$

$$\chi = e^{-i\omega v}\phi_{n_Z,n_{\bar{Z}}}h(r)$$

$$m^2\frac{e^{i\omega v}}{\phi_{n_Z,n_{\bar{Z}}}}(-\nabla^2 + N^2)\phi = -(D_v D_r + D_r D_v)h(r) - \frac{1}{m^5}D_r(m^5 W(r)D^r h(r)) + \left(\left(\frac{b^2 + A_{\sigma_3}^2}{b^2}\right)(n_Z - n_{\bar{Z}})^2 + 4(n_Z n_{\bar{Z}} + n_Z + n_{\bar{Z}}) + N^2 m^2\right)h(r)$$

$$D_v = -i\omega + i\left((n_Z - n_{\bar{Z}})\left(A_t(r) - fA_{\sigma_3}(r)\right) - NC_t(r)\right)$$

$$h(r) = \frac{e^{i\omega r_s}}{\sqrt{W}}g(r)$$

$$r_s = \int_R^r \frac{dr}{W(r)}$$

$$\left(-\frac{1}{2N^2}\frac{\partial^2}{\partial r^2} + V(r)\right)g(r) = 0$$

$$V(r) = \frac{(m^5 W)''}{4N^2(m^5 W)} - \frac{((m^5 W)')^2}{8N^2(m^5 W)^2} - \frac{\left(\frac{\omega}{N} + r^4 - R^4 - \left(A_t - fA_{\sigma_3}\right)\left(\frac{n_Z - n_{\bar{Z}}}{N}\right)\right)^2}{2W^2} + \frac{m^2}{2W} + \frac{1}{2W}\left(\left(\frac{b^2 + A_{\sigma_3}^2}{b^2}\right)\left(\frac{n_Z - n_{\bar{Z}}}{N}\right)^2 + 4\left(\frac{n_Z n_{\bar{Z}} + n_Z + n_{\bar{Z}}}{N^2}\right)\right)$$

$$g(r) \sim \sqrt{W(r)}e^{-i\omega r_s} \sim (r - R)^{-\frac{i\omega}{W'(R)} + \frac{1}{2}}$$

$$\frac{n_Z + n_{\bar{Z}}}{N} = q_n$$
$$\frac{n_Z - n_{\bar{Z}}}{N} = q_c$$

$$r \to \frac{r}{\epsilon}, \omega \to \frac{\omega}{\epsilon^2}\ n_Z \to \frac{n_Z}{\epsilon^2}\ n_{\bar{Z}} \to \frac{n_{\bar{Z}}}{\epsilon^2}$$

$$V(r) = \frac{r^2}{2} + \frac{(n_Z + n_{\bar{Z}})^2}{2N^2 r^2} - \frac{\omega}{N} + \mu\frac{(n_{\bar{Z}} - n_Z)}{N}$$

$$\omega = N\left(\frac{r^2}{2} + \frac{(n_Z + n_{\bar{Z}})^2}{2N^2 r^2} - \mu\frac{n_Z - n_{\bar{Z}}}{N}\right)$$

$$r_*^2 = \frac{n_Z + n_{\bar{Z}}}{N} = q_n$$

$$\omega_* = (n_Z + n_{\bar{Z}}) - \mu(n_Z - n_{\bar{Z}}) = N(q_n - \mu q_c)$$

$$V''(r_*)|_{\omega=\omega_*} = 4$$

$$V(r) = -\frac{(W'(R))^2 + 4\omega^2}{8N^2(W'(R)(r-R))^2}$$

$$\omega_\star \sim \mathcal{O}(Nq_n),$$

$$V(r) \sim -\frac{q_n^2}{(r-R)^2}$$

$$V(r) = q_n^2 \sum_{n=-2}^{\infty} A_n (r-R)^n$$

$$V(r_*) \sim q_n - \mu q_c - \frac{\omega}{N}$$

$$r_* \sim \sqrt{q_n}$$

$$\frac{\max\left(\frac{\omega}{N'}\frac{1}{N}\right)^2}{q_n^2})\omega \sim \omega_* \sim \mathcal{O}(1)$$

$$V(r) = -\frac{\frac{1}{4} + \frac{\omega_q^{*2}}{W'(R)^2}}{2N^2(r-R)^2} + \frac{b}{N^2(r-R)}$$

$$\int \sqrt{V}dr \sim q_n$$

$$b = \frac{1}{8W'(R)}\left(4\left(\frac{j^2\mu^2R^4(n_Z - n_{\bar{Z}})^2}{R^6 - j^2(q^2 - 2pR^2)} + n_Z^2 + 2n_Z n_{\bar{Z}} + 2\right) + n_{\bar{Z}}^2 + 4n_{\bar{Z}}\right) + 4m^2N^2$$
$$+W''(R) + 4\frac{W''(R)}{W'^2(R)}\omega^2\Bigg)$$

$$r_1 - R = \frac{\frac{1}{4} + \frac{\omega^2}{W'(R)^2}}{2b}$$

$$r_1 - R \sim \mathcal{O}\left(\max\left(\frac{\omega}{N}, \frac{1}{N}\right)\right)$$

$$g(r) = \frac{1}{\sqrt{N}(2V(r))^{\frac{1}{4}}}\left(a_1 e^{N\int dr\sqrt{2V(r)}} + a_2 e^{-N\int dr\sqrt{2V(r)}}\right)$$

$$\frac{V'}{NV^{\frac{3}{2}}} \ll 1$$

$$V(r) \sim q_n^2(r - r_1)$$

$$\frac{1}{Nq_n(r-r_1)^{\frac{3}{2}}}$$

$$|r-r_1| \gg \frac{1}{(Nq_n)^{\frac{2}{3}}}$$

$$|r-r_1| \ll 1$$

$$\frac{1}{(Nq_n)^{\frac{2}{3}}} \ll 1$$

$$V = \frac{\alpha^2}{(r-R)^2} + \frac{q_n^2}{(r-R)}$$

$$\alpha^2 = \max\left(\frac{\omega^2}{N^2}, \frac{1}{N^2}\right)$$

$$r_1 - R \sim \frac{\alpha^2}{q_n^2}$$

$$|r_1 - r| \ll r_1 - R \sim \frac{\alpha^2}{q_n^2}$$

$$V(r) \sim \frac{q_n^6}{\alpha^4}(r-r_1)$$

$$|r-r_1| \gg \left(\frac{\alpha^2}{Nq_n^3}\right)^{\frac{2}{3}}$$

$$\alpha \gg \frac{1}{N}$$

$$|\omega| \gg 1$$

$$\omega_* = N(q_n - \mu q_c) = -Nq_c\left(\mu - \frac{\mu q_n}{q_c}\right)$$

$$\left|\mu - \frac{q_n}{q_c}\right| \gg \frac{1}{Nq_c}$$

$$\omega = \omega_*^q + \epsilon$$

$$\omega_*^q = \omega_* + 1 + \cdots$$

$$\left(-\frac{1}{2N^2}\frac{\partial^2}{\partial r^2} + \frac{V''(r_*)}{2}(r-r_*)^2\right)g(r) = \left(\frac{\sqrt{V''(r_*)}}{2N} + \epsilon\right)g(r)$$

$$r_* = \frac{\sqrt{n_Z + n_{\bar{Z}}}}{N}V''(r_*) = 4$$

$$\left(-\frac{1}{2N^2}\frac{\partial^2}{\partial r^2}+2(r-r_*)^2\right)g(r)=\left(\frac{1}{N}+\epsilon\right)g(r)$$

$$g(r)=c_1D_{\frac{N\epsilon}{2}}\left(2\sqrt{N}(r-r_*)\right)+c_2D_{-\frac{N\epsilon}{2}-1}\left(2i\sqrt{N}(r-r_*)\right)$$

$$g(r)=c_1e^{-N(r-r_*)^2}-ic_2\frac{e^{N(r-r_*)^2}}{2\sqrt{N}(r-r_*)}$$

$$g(r)=c_1D_{\frac{N\epsilon}{2}}\left(2\sqrt{N}(r-r_*)\right)$$

$$r_*-r\gg\frac{1}{\sqrt{N}},g(r)$$

$$g(r)=c_1\left(\frac{\sqrt{\pi}e^{N(r-r_*)^2}}{\sqrt{2N}(r_*-r)\Gamma\left(-\frac{1}{2}(N\epsilon)\right)}+e^{-N(r-r_*)^2}\right)$$
$$=c_1\left(\frac{-\epsilon\sqrt{N\pi}}{2\sqrt{2}(r_*-r)}e^{N(r-r_*)^2}+e^{-N(r-r_*)^2}\right)$$

$$g(r)=\frac{1}{(2V(r))^{1/4}}\left(A\exp\left(-\int_r^{r^*}dr'\sqrt{2V(r')}\right)+B\exp\left(\int_r^{r^*}dr'\sqrt{2V(r')}\right)\right)$$

$$A=\frac{i}{\sqrt[4]{e}},B=-i\sqrt{\frac{\pi}{2}}\sqrt[4]{e}N^{3/2}\epsilon$$

$$g(r)=\frac{1}{(2V(r))^{1/4}}\left(A\exp\left(\int_{r_1}^{r}N\sqrt{2V}\right)e^{-\beta}+B\exp\left(-\int_{r_1}^{r}N\sqrt{2V}\right)e^{\beta}\right)$$

$$\beta=\int_{r_1}^{r^*}N\sqrt{2V}$$

$$g(r)\sim\frac{Ae^{-\beta}}{(2V)^{-\frac{1}{4}}}\left(\sin\left(\int_{r_1}^{r}N\sqrt{-2V}+\frac{\pi}{4}\right)+\frac{2B}{A}e^{2\beta}\cos\left(\int_{r_1}^{r}N\sqrt{-2V}+\frac{\pi}{4}\right)\right)$$

$$V(r)=-\frac{\left(\omega_*^q\right)^2}{2N^2W'(R)^2(r-R)^2}$$

$$\sqrt{-2V}=\frac{\left|\omega_*^q\right|}{N\sqrt{W'(R)}(r-R)}$$

$$g(r)= (-1)^{1/4}C(r-R)^{1/2}\left(\left(-i+\frac{2B}{A}e^{2\beta}\right)e^{i\frac{|\omega_q^*|}{\sqrt{W'(R)}}\log(r-R)}+\left(\frac{1}{2}-i\frac{B}{A}e^{2\beta}\right)e^{-i\frac{|\omega_q^*|}{\sqrt{W'(R)}}\log(r-R)}\right)$$

$$= (-1)^{1/4}C(r-R)^{1/2}\left(\left(-i+\frac{2B}{A}e^{2\beta}\right)e^{i|\omega_q^*|r_s}+\left(\frac{1}{2}-i\frac{B}{A}e^{2\beta}\right)e^{-i|\omega_q^*|r_s}\right)$$

$$C = Ae^{-\beta}\left(\frac{|\omega_*^q|}{N\sqrt{W'(R)}}\right)^{-\frac{1}{2}}$$

$$\frac{2B}{A}e^{2\beta}= i$$

$$\epsilon = -\frac{i}{N^{3/2}\sqrt{2e\pi}}\exp(-2\beta)$$

$$\omega = \omega_*^q - \frac{i}{N^{3/2}\sqrt{2e\pi}}\exp(-2\beta)$$

$$\frac{2B}{A}e^{2\beta} = -i$$

$$\epsilon = \frac{i}{N^{3/2}\sqrt{2e\pi}}\exp(-2\beta)$$

$$\frac{\omega}{N} \ll q_n.$$

$$1 \ll \omega \ll Nq_n$$

$$g(r) = 2\sqrt{2b(r-R)}f(y)$$

$$y = 2\sqrt{2b(r-R)}$$

$$y^2f''(y) + yf'(y) + \left(\frac{4(\omega_*^q)^2}{W'(R)^2} - y^2\right)f(y) = 0$$

$$f = c_1\tilde{I}_\nu(y) + c_2\tilde{K}_\nu(y)$$

$$\tilde{I}_\nu(y) = \mathrm{Re}(I_{i\nu}(y)), \tilde{K}_\nu(y) = K_{i\nu}(y)$$

$$\nu = \frac{2\omega_*^q}{W'}$$

$$f^{y\to 0}c_1\left(\frac{\sinh(\pi\nu)}{\pi\nu}\right)^{\frac{1}{2}}\cos(\nu\log(y/2)-\gamma_\nu) - c_2\left(\frac{\pi}{\nu\sinh(\pi\nu)}\right)^{\frac{1}{2}}\sin(\nu\log(y/2)-\gamma_\nu)$$

$$\frac{c_2}{c_1} = \frac{i\sinh\pi\nu}{\pi}$$

$$\tilde{I}_{\nu}(y) \approx \frac{1}{\sqrt{2\pi y}} e^{y}$$

$$\tilde{K}_{\nu}(y) \approx \sqrt{\frac{\pi}{2y}} e^{-y}$$

$$f^{y\to\infty} \frac{c_1}{\sqrt{2\pi y}} (e^{y} + i\sinh(\pi\nu) e^{-y})$$

$$g(r) = \frac{c_1\sqrt{2} b^{1/4} (r-R)^{1/4}}{\sqrt{2\pi}} \left(e^{2\sqrt{2b(r-R)}} + i\sinh\left(\frac{2\pi\omega_*^q}{W'(R)}\right) e^{-2\sqrt{2b(r-R)}} \right)$$

$$g(r) = \frac{\sqrt{N}\sqrt[4]{(r-R)}}{\sqrt[4]{2}\sqrt[4]{b}} \left(A\exp(-\beta) e^{2\sqrt{2b(r-R)}-\alpha} + B\exp(\beta) e^{\alpha-2\sqrt{2b(r-R)}} \right)$$

$$\alpha = \pi \sqrt{\frac{1}{4} + \frac{\left(\omega_*^q\right)^2}{W'(R)^2}}$$

$$\frac{B}{A} e^{2\beta} e^{2\alpha} = i\sinh\left(\frac{2\pi\omega_*^q}{W'(R)}\right)$$

$$\epsilon = -\frac{1}{N^{3/2}} \sqrt{\frac{2}{e\pi}} \left(i\sinh\left(\frac{2\pi\omega_*^q}{W'(R)}\right) \right) e^{-2\beta} e^{-2\alpha}$$

$$\omega \gg 1, \alpha \to \frac{\pi\left|\omega_*^q\right|}{W'(r)}$$

$$\epsilon = -\frac{i}{N^{3/2}\sqrt{2e\pi}} e^{-2\beta}$$

$$\epsilon = \frac{i}{N^{3/2}\sqrt{2e\pi}} e^{-2\beta}$$

$$J_{\mu} = -i\left(\phi^* D_{\mu}\phi - \phi D_{\mu}\phi^*\right)$$

$$\sqrt{g} J^{v} = 2\left(\frac{\left(A_t - f A_{\sigma_3}\right)(n_Z - n_{\bar{Z}}) + r^4 - R^4}{W} \right) |h|^2 e^{2\mathrm{Im}(\omega)v} - i(h^*\partial_r h - h\partial_r h^*) e^{2\mathrm{Im}(\omega)v}$$

$$\sqrt{g} J^{r} = -2(\mathrm{Re}(\omega))|h|^2 e^{2\mathrm{Im}(\omega)v} - iW(h^*\partial_r h - h\partial_r h^*) e^{2\mathrm{Im}(\omega)v}$$

$$h^*\partial_r h - h\partial_r h^* = (g^*\partial_r g - g\partial_r g^*) e^{-2\mathrm{Im}(\omega)r_s} + \frac{2i\mathrm{Re}(\omega)}{W} |g|^2 e^{-2\mathrm{Im}(\omega)r_s}$$

$$\sqrt{g} J^{v} = \frac{1}{W} \left(\left(\frac{2\left(A_t - f A_{\sigma_3}\right)}{W} \right) |g|^2 - i(g^*\partial_r g - g\partial_r g^*) \right) e^{2\mathrm{Im}(\omega)v} e^{-2\mathrm{Im}(\omega)r_s}$$

$$\sqrt{g} J^{r} = -i(g^*\partial_r g - g\partial_r g^*) e^{2\mathrm{Im}(\omega)v} e^{-2\mathrm{Im}(\omega)r_s}$$

$$J^r(R) = \int_r^\infty \partial_v J^v$$

$$J^r(R) = 2\omega_i \int_r^\infty J^v$$

$$J^r(R) = \begin{cases} 0, & r - r_1 \gg \dfrac{1}{\alpha} \\ 2\omega_i, & r - r_1 \ll \dfrac{1}{\alpha} \end{cases}$$

$$\sqrt{g} J^r(R) = -8N^2 \mathrm{sgn}\left(\omega_*^q\right) |A|^2 e^{-2\beta} W'(R) e^{2\omega_i v}$$

$$\omega_*^q \left(\frac{\sinh\left(\dfrac{2\pi\left|\omega_*^q\right|}{W'(R)}\right)}{\dfrac{2\left|\omega_q^*\right|}{W'(R)}} \right) \approx \pi \frac{\omega_q^*}{W'(R)}$$

$$q_c^{\mathrm{crit}}(\delta)$$

$$\omega = \omega_*^q - \frac{i}{N^{3/2}} \sqrt{\frac{2}{e\pi}} \left(\sinh\left(\frac{2\pi\omega_*^q}{W'(r)}\right) \right) e^{-2\beta} e^{-2\alpha}$$

$$\sum_{i=0}^{n_z} \sum_{j=0}^{n_z - i} \sum_{k=0}^{2i} (2n_Z + i - j - k, n_z - i + k, j)$$

$$\sum_{i=0}^{n_z} \sum_{j=0}^{n_z - i} \sum_{k=0}^{2i} (Q + 2n_Z + i - j - k, Q + n_z - i + k, Q + j)$$

$$(Q + n_Z, Q + n_Z, Q + n_Z)$$

$$\sum_{i=0}^{n_Z} (Q + n_Z + i, Q + n_Z - i, Q_1)$$

$$(Q_1 + n_Z, Q_2, Q_3)$$

$$\sum_{i=0}^{n_Z} (n_Z + i, n_Z - i, 0).$$

$$ds^2 = l_{N-1}^2 \left(ds_{(0)}^2 + ds_{(1)}^2 \right)$$
$$F^{(5)} = F_{(0)}^{(5)} + F_{(1)}^{(5)}$$

$$ds_{(0)}^2 = \left(ds_{AdS_5}^2 + ds_{S^5}^2 \right)$$

$$(l_{N-1})^4 = 4\pi g_s (N-1)(\alpha')^2$$

$$ds^2_{AdS_5} = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_3^2$$
$$ds^2_{S^5} = \sin^2\ \theta d\tilde{\Omega}_3^2 + \cos^2\ \theta d\phi^2 + d\theta^2$$

$$\frac{F^{(0)}_{(5)}}{(l_{N-1})^4} = -4\epsilon_5 + 4\tilde{\epsilon}_5$$

$$\begin{aligned} ds^2_{(1)} =& \rho_1^2 \Bigg[-\frac{(r^2+\sin^2\ \theta)^2}{2x^4} r^2 d\Omega_3^2 + \frac{(r^2+\sin^2\ \theta)^2}{2x^4} \sin^2\ \theta d\tilde{\Omega}_3^2 \\ &+ \left(\frac{(r^2+\cos^2\ \theta)^2 - 4(1+r^2)\cos^2\ \theta - 1 + 4d\sqrt{1+r^2}\cos\ \theta\cos\ (\phi - t)}{2x^4} \right)(1+r^2)dt^2 \\ &+ \left(\frac{r^4 - \sin^4\ \theta}{2x^4} \right) \frac{dr^2}{1+r^2} + \left(\frac{r^4 - \sin^4\ \theta}{2x^4} \right) d\theta^2 \\ &+ \left(\frac{(r^2+\cos^2\ \theta)^2 - 4(1+r^2)\cos^2\ \theta - 1 + 4d\sqrt{1+r^2}\cos\ \theta\cos\ (\phi - t)}{2x^4} \right)\cos^2\ \theta d\phi^2 \\ &- \frac{2d\sin\ (\phi - t)}{x^4}(dt(1+r^2) - \cos^2\ \theta d\phi) d\left(\sqrt{1+r^2}\cos\ \theta\right) \\ &+ 2d\phi dt \frac{\cos\ \theta\big(2(1+r^2)\cos\ \theta - d\sqrt{1+r^2}(1+r^2+\cos^2\ \theta)\cos\ (\phi - t)\big)}{x^4} \Bigg] + O(\rho_1^4) \end{aligned}$$

$$x^2 = \rho_0^2(r^2+\cos^2\ \theta) + d^2 - 2d\rho_0\sqrt{1+r^2}\cos\ \theta\cos\ (\phi - t)$$

$$F^{(5)}_{(1)} = 4\big(F \wedge d\Omega_3 + \tilde{F} \wedge d\tilde{\Omega}_3\big)$$

$$F_{tr} = \frac{2(d^2-1)r\sin^4\ \theta(r^2+\sin^2\ \theta)}{2Nx^6}, F_{t\phi} = \frac{d\sqrt{r^2+1}\sin^3\ \theta\sin\ 2\theta(\sin^2\ \theta + r^2)^2\sin\ (t-\phi)}{2Nx^6}$$

$$\begin{aligned} F_{t\theta} =& \frac{\sqrt{1+r^2}\sin^3\ \theta}{8Nx^6}\Big[8\sqrt{r^2+1}\cos\ \theta\big((d^2(r^2+2)+2r^2+1)\cos\ \theta + d^2\cos^2\ \theta\cos\ (2t-2\phi)\big) \\ &-4d\cos\ (t-\phi)\big((d^2+3r^2+2)\cos\ 2\theta + d^2 + 2r^4 + 7r^2 + 4\big)\Big] \end{aligned}$$

$$F_{r\theta} = \frac{dr\sin^3\ \theta\sin\ (t-\phi)\big(-(d^2+3r^2+2)\cos\ 2\theta - d^2 + 4d\sqrt{r^2+1}\cos^3\ \theta\cos\ (t-\phi) + 2r^4 + r^2\big)}{2N\sqrt{r^2+1}x^6}$$

$$F_{r\phi} = \frac{2r\sin^4\ \theta\cos\ \theta(r^2+\sin^2\ \theta)\big(d(\cos\ 2\theta + 2r^2 + 3)\cos\ (t-\phi) - 4\sqrt{r^2+1}\cos\ \theta\big)}{4N\sqrt{r^2+1}x^6}$$

$$\begin{aligned} F_{\theta\phi} =& \frac{\sin^3\ \theta\cos\ \theta}{4Nx^6}[2\cos^2\theta(2d^2(2(r^2+1)\cos^2\ (t-\phi) + 2r^2 + 1) + \sin^2\ \theta(4d^2 + 3r^4 + 2r^2 - 1) \\ &+2(2r^2+1)\sin^4\ \theta) \\ &+2d\sqrt{r^2+1}\cos\ (t-\phi)\cos\ \theta\Big(2\cos^2\ \theta\big(2\cos\ 2\theta - 5(r^2+1)\big) - \big(2(d^2-r^4) + \sin^2\ \theta(-\cos\ 2\theta + 2r^2 + 1)\big)\Big) \\ &-\frac{1}{4}(r^2+1)\big(16r^4 - 8r^2\cos\ 2\theta + (r^2+1)\cos\ 4\theta + 7r^2 - 1 + 4\cos^4\ \theta(-\cos\ 2\theta + 2r^2 + 2)\big)\Big] \end{aligned}$$

$$\tilde{F}_{tr} = \frac{r^3}{32Nx^6}[(48d^2(r^2+1) - 16r^4 + 17)\cos\ 2\theta - \cos\ 6\theta + 2(6r^2+5)\cos\ 4\theta$$
$$+2\left(-8(4d^2+3)r^4 - 6(4d^2+1)r^2 + 2d\cos\ \theta\left(\sqrt{r^2+1}\cos(t-\phi)(-8d^2 + \cos\ 4\theta - 8(2r^2+3)\cos\ 2\theta\right.\right.$$
$$\left.\left.+24(r^4+r^2) - 1) + 8d(r^2+1)\cos\ \theta\cos\ (2t-2\phi)) + 8d^2 - 16r^6 + 3)\right]$$

$$\tilde{F}_{t\theta} = -\frac{4r^4\sin\ \theta(r^2+\sin^2\ \theta)\left(4(r^2+1)\cos\ \theta - d\sqrt{r^2+1}(\cos\ 2\theta + 2r^2 + 3)\cos\ (t-\phi)\right)}{8Nx^6}$$

$$\tilde{F}_{t\phi} = -\frac{dr^4\sqrt{r^2+1}\cos\ \theta(\sin^2\ \theta + r^2)^2\sin\ (t-\phi)}{Nx^6}$$

$$\tilde{F}_{r\theta} = \frac{-dr^3\sin\ \theta\sin\ (t-\phi)}{8N\sqrt{r^2+1}x^6}\left[8d(r^2+1)\left(d - 2(r^2+1)^{1/2}\cos\ \theta\cos\ (t-\phi)\right) - \cos\ 4\theta + 4(3r^2+2)\cos\ 2\theta\right.$$
$$\left.+4r^2 + 1\right]$$

$$\tilde{F}_{r\phi} = -\frac{r^3\cos\ \theta}{8Nx^6}\left[\frac{\cos\ (t-\phi)\left(d(8d^2(r^2+1) + \cos\ 4\theta + 4(3r^2+4)\cos\ 2\theta + 4r^2 + 7)\right)}{\sqrt{r^2+1}}\right.$$
$$\left.-4\cos\ \theta\left((d^2+2)\cos\ 2\theta - 2d^2(r^2+1)\cos\ (2t-2\phi) + 3d^2\right)\right]$$

$$\tilde{F}_{\theta\phi} = \frac{2(d^2-1)r^4\sin\ (2\theta)(-\cos\ (2\theta) + 2r^2 + 1)}{8Nx^6}$$

$$-\frac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu h^\alpha_\alpha + \frac{1}{2}\nabla_\alpha\nabla_\nu h^\alpha_\mu + \frac{1}{2}\nabla_\alpha\nabla_\mu h^\alpha_\nu - h^{\alpha\beta}R_{\mu\alpha\nu\beta} + \frac{1}{2}R_{\nu\alpha}h^\alpha_\mu + \frac{1}{2}R_{\mu\alpha}h^\alpha_\nu$$
$$= \left(2\left(F^{(5)}_{(1)}\right)_{\mu\rho\sigma\delta\kappa}\left(F^{(5)}_{(0)}\right)^{\rho\sigma\delta\kappa}_{\nu} - 4h^{\alpha\rho}\left(F^{(5)}_{(0)}\right)_{\mu\rho\sigma\delta\kappa}\left(F^{(5)}_{(0)}\right)^{\sigma\delta\kappa}_{\nu\alpha}\right)$$

$$F^{(5)}_{(1)} = \star F^{(5)}_{(1)} + \frac{1}{2}h^\alpha_\alpha \star F^{(5)}_{(0)} - 5\sqrt{-g}\epsilon_{\mu_1\mu_2\cdots\mu_{10}}h^{\mu_6\nu_6}g^{\mu_7\nu_7}\cdots g^{\mu_{10}\nu_{10}}\left(F^{(5)}_{(0)}\right)_{\nu_6\cdots\nu_{10}}dx^{\mu_1}\wedge\cdots dx^{\mu_5}$$

$$ds^2 = l^2_{N-1}\left(\frac{dr^2}{d^2} - dt^2\times d^2 + (d^2-1)d\alpha_i^2 + dy_j^2 + d\phi^2\right)$$

$$\theta\hat{n}^j = y^j, j = 1\ldots 4$$

$$l_{N-1}\theta\hat{n}^j = y^j, \qquad j = 1\ldots 4$$
$$l_{N-1}\left(r - \sqrt{d^2-1}\right) = d\times y^5 \quad i = 1\ldots 3$$
$$l_{N-1}\alpha_i = \frac{x_i}{\sqrt{d^2-1}},$$
$$l_{N-1}(\phi - t) = \frac{\sqrt{d^2-1}}{d}y^6$$
$$l_{N-1}x^0 = \frac{d^2t - \phi}{\sqrt{d^2-1}}$$

$$ds^2 = dx_\mu dx^\mu + dy_i dy^i, \mu = 0\ldots 3, i = 1\ldots 6$$

$$x^2 = (d^2-1)\frac{(y_i y^i)}{l^2_{N-1}}$$

$$F_{y^i x^0 x^1 x^2 x^3} = -16\pi\alpha' g_s^2\frac{y^i}{(y^j y_j)^3}$$

$$h_{\mu\nu} = -2\pi\alpha' g_s^2 \frac{1}{\left(y_j y^j\right)^2}\eta_{\mu\nu}$$

$$h_{ij} = 2\pi\alpha' g_s^2 \frac{1}{\left(y_j y^j\right)^2}\delta_{ij}$$

$$G_{AB} = \frac{1}{2}(\partial^C\partial_A h_{BC} + \partial^C\partial_B h_{AC} - \Box\, h_{AB} - \partial_A\partial_B h - \partial^C\partial^D h_{CD}\eta_{AB} + \Box\, h\eta_{AB}) = 8\pi G_N T_{AB}\nabla_i F^{i\mu\nu\rho\sigma}$$

$$= -16\pi^4 g_s \alpha^2 \epsilon^{\mu\nu\rho\sigma}\delta^6(\vec{y})$$

$$\frac{ds^2_{(1)}}{l^2_{N-1}} \approx \left(\frac{r^2}{2}\right)\frac{dt^2}{N} + \left(\frac{1}{2r^2}\right)\frac{dr^2}{N} - \frac{r^2}{2N}d\Omega_3^2 + \frac{1}{2N}d\Omega_5^2$$

$$r = r' + \frac{r'}{2N}$$

$$\frac{ds^2_{(1)}}{l^2_{N-1}} \approx \frac{1}{2N}\left(-r^2dt^2 + \left(\frac{dr^2}{r^2}\right) + r^2 d\Omega_3^2 + d\Omega_5^2\right)$$

$$= \frac{1}{2N}\left(ds^2_{AdS_5} + d\Omega_5^2\right)$$

$$ds^2 = ds^2_{(0)} + ds^2_{(1)} = l^2_{N-1}\left(1 + \frac{1}{2N}\right)\left(ds^2_{Ads_5} + d\Omega_5^2\right) = l_N^2\left(ds^2_{Ads_5} + d\Omega_5^2\right)$$

$$\frac{l_N^2}{l^2_{N-1}} = 1 + \frac{1}{2N}$$

$$r = r' + \frac{1}{N}\left(d\cos\,\theta'\cos\left(\phi' - t'\right) - \frac{2\cos^2\,\theta\left(1 - 3d^2\cos^2\left(\phi' - t'\right)\right) + (d^2 - 1)}{2r'}\right)$$

$$\phi = \phi' - \frac{d\sin\left(\phi' - t'\right)}{r'\cos\,\theta' N}$$

$$\theta = \theta' - \frac{d\sin\,\theta'\cos\left(\phi' - t'\right)}{Nr'}$$

$$\frac{ds^2}{l_N^2} \approx ds^2_{AdS_5\times S^5} + \frac{1}{N}\Big[-\left(2\cos^2\,\theta(1 - 3d^2\cos^2\left(\phi - t\right)) + (d^2 - 1)\right)dt^2$$
$$+ \frac{4\cos^2\,\theta(1 - 2d^2\cos^2\left(\phi - t\right))}{r^2}d\phi dt + \left(\frac{\sin^2\,\theta(\cos^2\,\theta(6d^2\cos^2\left(\phi - t\right) - 1) - d^2 + \sin^2\,\theta)}{r^2}\right)d\tilde{\Omega}_3^2$$
$$+ \left(\frac{\cos^2\,\theta(6d^2\cos^2\left(\phi - t\right) - 1) - d^2}{r^4}\right)dr^2 + (\cos^2\,\theta(1 - 6d^2\cos^2\left(\phi - t\right)) + d^2 - \sin^2\,\theta)d\Omega_3^2$$
$$+ \left(\frac{2d\cos\,\theta\cos\left(\phi - t\right)}{r} + \frac{\cos^2\,\theta(6d^2\cos^2\left(\phi - t\right) - 1) - d^2}{r^2}\right)d\theta^2 - \left(\frac{d^2\sin\left(2\theta\right)\sin\left(2t - 2\phi\right)}{r^2}\right)d\theta dt\Big]$$

$$A_\mu^\zeta = -3\frac{\int\, d\Omega_5\zeta^\alpha g_{\alpha\mu}}{\int\, d\Omega_5}$$

$$A = \frac{2(d^2 - 1)}{N}\frac{dt}{r^2}$$

$$Q = N(d^2 - 1)$$

$$ds^2 = \left(-(1+r^2) + \frac{6d^2 + 6r^4 + 2r^2 - 7}{6Nr^2}\right)dt^2 + \left(\frac{r^4 - r^2 + 1}{r^6} - \frac{r^2 - 1}{3Nr^6}\right)dr^2 + \left(r^2 - \frac{-2d^2 + 6r^4 + 2r^2 + 1}{6Nr^2}\right)d\Omega_3^2$$

$$n^\mu = \left(0, \frac{1}{\sqrt{g_{rr}}}, 0,0,0\right)$$

$$\Theta_\nu^\mu = -\frac{1}{2}(\nabla^\mu n_\nu + \nabla_\nu n^\mu)$$

$$T_\nu^\mu = \frac{N^2 r^4}{4\pi^2}\left(\Theta_\nu^\mu - \Theta\gamma_\nu^\mu - 3\gamma_\nu^\mu + \frac{1}{2}G_\nu^\mu\right)$$

$$T_0^0 = \frac{(N-1)^2}{4\pi^2}\left(\frac{3}{8}\left(1 + \frac{2}{N}\right) + \frac{2(d^2-1)}{N}\right) \approx \frac{N^2}{4\pi^2}\left(\frac{3}{8} + \frac{2(d^2-1)}{N}\right)$$

$$T_j^i = -\frac{\delta_i^j}{3}\frac{(N-1)^2}{4\pi^2}\left(\frac{3}{8}\left(1 + \frac{2}{N}\right) + \frac{2(d^2-1)}{N}\right) \approx -\frac{\delta_i^j}{3}\frac{N^2}{4\pi^2}\left(\frac{3}{8} + \frac{2(d^2-1)}{N}\right)$$

$$E = \int d^3x\sqrt{\gamma}T_0^0 = \frac{N^2}{2}\left(\frac{3}{8}\left(1 + \frac{2}{N}\right) + \frac{2(d^2-1)}{N}\right)$$

$$\delta\tilde{F} = -\frac{\sin^3\theta\cos\theta}{N}d\theta \wedge d\phi$$

$$\delta F = \frac{r'^3}{N}dt \wedge dr'$$

$$F_{(5)} = l_{N-1}^4\left(1 + \frac{1}{N}\right)(-4\epsilon_5 + 4\tilde{\epsilon}_5) \approx l_N^4(-4\epsilon_5 + 4\tilde{\epsilon}_5)$$

$$F_5 = \sqrt{g'}\tilde{\epsilon}_5\chi$$

$$dx^\mu = dx'^\mu + \frac{1}{N}\frac{\partial x^\mu}{\partial x'^\nu}dx'^\nu$$

$$\chi = \sum_k (-k)(k+4)(-s^I + t^I)Y^I$$

$$s^I = -\frac{\frac{\alpha^I}{k}}{r^k}$$

$$\alpha^I = \frac{4i^k}{N(k+4)}\left(\frac{(k^2-1)}{\sqrt{2(k+1)(k+2)}}\right)(d)^k$$

$$ds^2_{(1)}=\frac{1}{d^3N}\left[\left(2(1+r^2)^{1/2}\cos\,\theta\cos\,(t-\phi)\right)\left((1+r^2)dt^2+\cos^2\,\theta d\phi^2\right)\right.$$
$$+\left(\frac{2r\cos\,\theta\sin\,(t-\phi)}{\sqrt{r^2+1}}dr-2\left((r^2+1)^{1/2}\sin\,\theta\sin\,(t-\phi)\right)d\theta\right)\left((1+r^2)dt-\cos^2\,\theta d\phi\right)$$
$$\left.-2\left(\sqrt{r^2+1}\cos\,\theta(\cos^2\,\theta+r^2+1)\cos\,(t-\phi)\right)dtd\phi\right]+\mathcal{O}\left(\frac{1}{d^4N}\right)$$

$$F_{t\theta}=\quad F_{\theta\phi}=-\frac{\sqrt{r^2+1}\sin^3\,\theta\cos^2\,\theta\cos\,(t-\phi)}{d^3N},\qquad F_{r\theta}=-\frac{r\sin^3\,\theta\cos^2\,\theta\sin\,(t-\phi)}{d^3N\sqrt{r^2+1}}$$
$$\tilde{F}_{tr}=\tilde{F}_{r\phi}=-\frac{r^3\sqrt{r^2+1}\cos\,\theta\cos\,(t-\phi)}{d^3N},\qquad \tilde{F}_{r\theta}=-\frac{r^3\sqrt{r^2+1}\sin\,\theta\sin\,(t-\phi)}{d^3N}$$

$$\phi'=\phi-\frac{\sqrt{r^2+1}\cos\,\theta\sin\,(t-\phi)}{x^3N}$$
$$t'=t-\frac{\sqrt{r^2+1}\cos\,\theta\sin\,(t-\phi)}{x^3N}$$

$$ds^2_{(1)}=\left(\frac{1}{2Nx^4}\right)\left[(r^2+\sin^2\,\theta)^2\left(r^2d\Omega_3^2+\sin^2\,\theta d\tilde{\Omega}_3^2\right)+(r^4-\sin^4\,\theta)\left(\frac{dr^2}{1+r^2}+d\theta^2\right)\right.$$
$$+\left((r^2+\cos^2\,\theta)^2-4(1+r^2)\cos^2\,\theta-1+4d\sqrt{1+r^2}\cos\,\theta\cos\,(\phi-t)\right.$$
$$\left.\left.+\frac{4\sqrt{1+r^2}\cos\,\theta}{x}\left(3d\sqrt{1+r^2}\cos\,\theta\sin^2\,(t-\phi)-x^2\cos\,(t-\phi)\right)\right)\left((1+r^2)dt^2+\cos^2\,\theta d\phi^2\right)\right.$$
$$+\left(4\left(2(1+r^2)\cos\,\theta-d\sqrt{1+r^2}(1+r^2+\cos^2\,\theta)\cos\,(\phi-t)\right)+\right.$$
$$\left.\left.+\frac{2\sqrt{1+r^2}(3+2r^2+\cos\,2\theta)}{x}\left(-3d\sqrt{1+r^2}\cos\,\theta\sin^2\,(t-\phi)+x^2\cos\,(t-\phi)\right)\right)\cos\theta d\phi dt\right.$$
$$+\frac{12}{x}\sqrt{1+r^2}\cos\,\theta\sin\,(\phi-t)(dt(1+r^2)-\cos^2\,\theta d\phi)(rdr-\sin\theta\cos\,\theta d\theta)$$
$$+\sin\,(\phi-t)(dt(1+r^2)-\cos^2\,\theta d\phi)\left(\frac{r\cos\,\theta dr}{\sqrt{1+r^2}}-\sqrt{1+r^2}\sin\,\theta d\theta\right)(4(x-d)$$
$$\left.+12\frac{d}{x}\sqrt{1+r^2}\cos\,\theta\cos\,(t-\phi)\right)\Big]$$

$$F_{tr}=\frac{2(d^2-1)r\sin^4\,\theta(r^2+\sin^2\,\theta)}{2Nx^6},F_{t\phi}=\frac{d\sqrt{r^2+1}\sin^3\,\theta\sin\,2\theta(\sin^2\,\theta+r^2)^2\sin\,(t-\phi)}{2Nx^6}$$
$$F_{t\theta}=\frac{\sin^3\,\theta}{2Nx^6}\left[2(r^2+1)(d^2(r^2+2)+2r^2+1)\cos\,\theta+2\cos^2\,\theta\sqrt{(r^2+1)}x^3\cos\,(t-\phi)\right.$$
$$+d(r^2+1)\cos^3\,\theta((2d+3x)\cos\,(2t-2\phi)-3x)-d\sqrt{r^2+1}\cos\,(t-\phi)\left((d^2+3r^2+2)\cos\,(2\theta)\right.$$
$$\left.+d^2+2r^4+7r^2+4)\right]$$
$$F_{r\theta}=\frac{r\sin^3\,\theta\sin\,(t-\phi)}{2N\sqrt{r^2+1}x^6}[d(\sin^2\,\theta+r^2)^2+6dx\cos^3\,\theta)\sqrt{(r^2+1)}\cos\,(t-\phi)$$
$$+2x\cos^2\,\theta(-dx-3r^2+x^2-3)]$$
$$F_{r\phi}=\frac{2r\sin^4\,\theta\cos\,\theta(r^2+\sin^2\,\theta)\left(d(\cos\,2\theta+2r^2+3)\cos\,(t-\phi)-4\sqrt{r^2+1}\cos\,\theta\right)}{4N\sqrt{r^2+1}x^6}$$

$$F_{\theta\phi} = \frac{\sin^3\theta\cos\theta}{8Nx^6}\Big[8\cos^2\theta\big((2r^2+1)(d^2+\sin^4\theta)+2d^2(r^2+1)\cos^2(t-\phi)-3d(r^2+1)x\sin^2(t-\phi)\big)$$
$$+\sqrt{r^2+1}\cos\theta\cos(t-\phi)(-d(8d^2-8r^4+4r^2+3)-d\cos 4\theta+4d(r^2+1)\cos(2\theta)+8x^3)$$
$$-2\big((d^2+r^4+r^2)\cos(4\theta)-d^2+4r^6+5r^4+r^2-2(r^4+r^2)\cos(2\theta)\big)$$
$$+8d\sqrt{r^2+1}\cos^3\theta\big(2\cos 2\theta-5(r^2+1)\big)\cos(t-\phi)-8(r^2+1)\cos^4\theta\big(\cos 2\theta-2(r^2+1)\big)\Big]$$

$$\tilde{F}_{tr} = \frac{r^3}{8Nx^6}\Big[-2(\cos 2\theta-2r^2-1)\big((d^2+r^4)\cos 2\theta-d^2(4r^2+1)-r^4\big)$$
$$+\cos\theta\Big(\sqrt{r^2+1}\cos(t-\phi)(d(-8d^2+24(r^4+r^2)-1)+d(\cos 4\theta-8(2r^2+3)\cos 2\theta)+8x^3)$$
$$+r^2\cos 3\theta)+\cos^2\theta(2(2d^2+(r^2+2)^2)\cos 2\theta+4d(r^2+1)(2d+3x)\cos(2t-2\phi)$$
$$+12d(d-(r^2+1)x)-2\cos 4\theta-8r^6-22r^4-13r^2-6)+4\cos^4\theta(\cos 2\theta+r^4+r^2+1)]$$
$$\tilde{F}_{t\theta} = -\frac{4r^4\sin\theta(r^2+\sin^2\theta)\big(4(r^2+1)\cos\theta-d\sqrt{r^2+1}(\cos 2\theta+2r^2+3)\cos(t-\phi)\big)}{8Nx^6}$$
$$\tilde{F}_{t\phi} = -\frac{dr^4\sqrt{r^2+1}\cos\theta(\sin^2\theta+r^2)^2\sin(t-\phi)}{Nx^6}$$
$$\tilde{F}_{r\theta} = -\frac{r^3\sin\theta\sin(t-\phi)}{8N\sqrt{r^2+1}x^6}\Big[8(r^2+1)\big(d^3-x(d^2+r^2-1)\big)+4dr^2+d$$
$$-d\big(8(r^2+1)^{3/2}\cos\theta(2d+x)\cos(t-\phi)+\cos 4\theta\big)+4\cos 2\theta(d(3r^2+2)+2(r^2+1)x)]$$
$$\tilde{F}_{r\phi} = \frac{r^3\cos\theta[2(7d^2+2)\cos\theta+2(d^2+2)\cos 3\theta+8d^2(r^2+1)\cos\theta\cos(2t-2\phi)}{8Nx^6}$$
$$-\frac{d\cos(t-\phi)(8d^2(r^2+1)+\cos 4\theta+4(3r^2+4)\cos 2\theta+4r^2+7)}{\sqrt{r^2+1}}$$
$$-24d(r^2+1)x\cos\theta\sin^2(t-\phi)+8\sqrt{r^2+1}x^3\cos(t-\phi)\Big]$$
$$\tilde{F}_{\theta\phi} = \frac{2(d^2-1)r^4\sin 2\theta(-\cos 2\theta+2r^2+1)}{8Nx^6}$$

$$\epsilon^{\text{red}} = \frac{1}{N^2}$$

$$\epsilon^{\text{irred}} = \frac{1}{N^3}$$

$$Q > Nq_c(\delta)$$

$$S = \frac{1}{16\pi(2\pi^2)^3 g_s^2 \alpha' 4}\sqrt{-g}\left(R-\frac{1}{2(5!)}F_5^2-\frac{H^2}{2(3!)}-\partial_\mu\phi\partial^\mu\phi\right)$$

$$(F_5)_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \partial_{\mu_1}C_{\mu_2\mu_3\mu_4\mu_5}+\partial_{\mu_2}C_{\mu_3\mu_4\mu_5\mu_1}+3\text{ other}$$

$$\int_{S^5} *F_5 = (2\pi)^4 N g_s \alpha'^2$$

$$S = \frac{1}{(2\pi)^3\alpha'^2 g_s}\left(-\int d^4y\sqrt{-\det h_{ab}}+\frac{1}{4!}\int \epsilon^{abcd}C_{\mu_1\mu_2\mu_3\mu_4}\frac{dx^{\mu_1}}{dy^a}\frac{dx^{\mu_2}}{dy^b}\frac{dx^{\mu_3}}{dy^c}\frac{dx^{\mu_4}}{dy^d}d^4y\right)$$

$$\frac{1}{16\pi(2\pi^2)^3 g_s^2 \alpha'^4}\nabla_\mu F^{\mu\mu_1\mu_2\mu_3\mu_4} = -\frac{N}{(2\pi)^3\alpha'^2 g_s}\delta^6(\vec{r})\epsilon^{\mu_1\dots\mu_4}$$

$$\Longrightarrow \frac{1}{16\pi^4 g_s \alpha'^2}\nabla_\mu F^{\mu\mu_1\mu_2\mu_3\mu_4} = -N\delta^6(\vec{r})\epsilon^{\mu_1\dots\mu_4}$$

$$F_{\mu_1\mu_2\mu_3\mu_4 r} = A(r)\epsilon_{\mu_1\mu_2\mu_3\mu_4 r}$$

$$\frac{1}{(2\pi)^4 g_s\alpha'^2}\frac{1}{r^5}\partial_r(r^5 F^{r\mu_1\mu_2\mu_3\mu_4}) = -N\frac{\delta(r)}{\pi^3 r^5}\epsilon^{\mu_1\dots\mu_4}$$

$$A = -N\frac{(2\pi)^4 g_s\alpha'^2}{\pi^3}, F = -A\frac{\epsilon_{\mu_1\mu_2\mu_3\mu_4 r}}{r^5}$$

$$T^{\mu\nu} = -\frac{1}{(2\pi)^3\alpha'^2 g_s}\eta^{\mu\nu}\delta^6(\vec{r})$$

$$ds^2_{(1)} = \phi(r)\left(dx^\mu dx_\mu - dy^i dy_i\right)$$

$$4\partial_i\partial^i\phi = 8\pi G_N\frac{1}{(2\pi)^3\alpha'^2 g_s}\delta^6(\vec{r})$$

$$\phi = \frac{2\pi\alpha'^2 g_s}{r^4}$$

$$\frac{1}{\alpha'^4 g_s^2}\int F^2 \sim N^2\int\frac{drr^5}{r^{10}} \sim N^2\frac{1}{r_0^4} \sim \frac{N}{g_s\alpha'^2}$$

$$T_D^{\mu\nu}(x) = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}$$

$$= \frac{-1}{(2\pi)^3\alpha'^2 g_s}\int d^4y\sqrt{-h}h^{ab}\frac{\partial x_0^\mu(y)}{\partial y^a}\frac{\partial x_0^\nu(y)}{\partial y^b}\frac{\delta^{10}\left(x^\mu - x_0^\mu(y)\right)}{\sqrt{-g(x)}}$$

$$\delta h_{ab} = \partial_a X^\mu \partial_b X^\nu \delta g_{\mu\nu}$$

$$\int\delta = \frac{\delta^{10}}{\sqrt{-g}}$$

$$n_\mu = k(x)\frac{\partial f(x)}{\partial x^\mu}$$

$$n_\mu T_D^{\mu\nu} = \frac{-1}{(2\pi)^3\alpha'^2 g_s}\int d^4y\sqrt{-h}h^{\alpha\beta}k(x)\frac{\partial f(x)}{\partial x^\mu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\frac{\delta^{10}\left(x^\mu - x_0^\mu(y)\right)}{\sqrt{-g}}$$

$$= \frac{-1}{(2\pi)^3\alpha'^2 g_s}\int d^4y\sqrt{-h}h^{\alpha\beta}k(x)\frac{\partial f(x(y))}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\frac{\delta^{10}\left(x^\mu - x_0^\mu(y)\right)}{\sqrt{-g}} = 0$$

$$\frac{-\delta\int d^4y\sqrt{-h}}{\delta x_0^\alpha(y')}=-\int d^4y\frac{\delta\sqrt{-h}}{\delta h_{ab}}\frac{\delta h_{ab}}{\delta x_0^\alpha(y')}$$

$$=-\frac{1}{2}\int d^4y\sqrt{-h}h^{ab}\frac{\delta\left(\frac{\partial x_0^\mu}{\partial y^a}\frac{\partial x_0^\nu}{\partial y^b}g_{\mu\nu}\right)}{\delta x_o^\alpha(y')}$$

$$=-\frac{\sqrt{-h}h^{ab}}{2}\frac{\partial x_0^\mu}{\partial y^a}\frac{\partial x_0^\nu}{\partial y^b}\frac{\partial g_{\mu\nu}}{\partial x_0^\alpha}+\frac{\partial}{\partial y^b}\left(\sqrt{-h}h^{ab}g_{\alpha\nu}\frac{\partial x_0^\nu}{\partial y^a}\right)$$

$$=-\frac{\sqrt{-h}h^{ab}}{2}\frac{\partial x_0^\mu}{\partial y^a}\frac{\partial x_0^\nu}{\partial y^b}\frac{\partial g_{\mu\nu}}{\partial x_0^\alpha}+\sqrt{-h}h^{ab}\left(\hat{\nabla}_b g_{\alpha\nu}\right)\left(\frac{\partial x_0^\nu}{\partial y^a}\right)+\sqrt{-h}h^{ab}(g_{\alpha\nu})\hat{\nabla}_b\left(\frac{\partial x_0^\nu}{\partial y^a}\right)$$

$$=-\frac{\sqrt{-h}h^{ab}}{2}\frac{\partial x_0^\mu}{\partial y^a}\frac{\partial x_0^\nu}{\partial y^b}\frac{\partial g_{\mu\nu}}{\partial x_0^\alpha}+\sqrt{-h}h^{ab}\frac{\partial g_{\alpha\nu}}{\partial x_0^\mu}\frac{\partial x_0^\mu}{\partial y^b}\frac{\partial x_0^\nu}{\partial y^a}+\sqrt{-h}h^{ab}g_{\alpha\nu}\hat{\nabla}_b\left(\frac{\partial x_0^\nu}{\partial y^a}\right)$$

$$=\sqrt{-h}h^{ab}g_{\alpha\nu}\left(\hat{\nabla}_b\left(\frac{\partial x_0^\nu}{\partial y^a}\right)+\frac{\partial x_0^\mu}{\partial y^a}\frac{\partial x_0^\beta}{\partial y^b}\Gamma^\nu_{\mu\beta}\right)$$

$$\partial_a\left(\sqrt{-h}h^{ab}A_b\right)=\sqrt{-h}\nabla_b A^b$$

$$\frac{1}{4!}\frac{\delta}{\delta x_0^\alpha(y')}\int d^4y\epsilon^{a_1a_2a_3a_4}C_{\mu_1\mu_2\mu_3\mu_4}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}$$

$$=\frac{1}{4!}\epsilon^{a_1a_2a_3a_4}\left(\frac{\partial C_{\mu_1\mu_2\mu_3\mu_4}}{\partial x_0^\alpha}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}-4\frac{\partial}{\partial y_{a_1}}\left(C_{\alpha\mu_2\mu_3\mu_4}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}\right)\right)$$

$$=\frac{1}{4!}\epsilon^{a_1a_2a_3a_4}\left(\frac{\partial C_{\mu_1\mu_2\mu_3\mu_4}}{\partial x_0^\alpha}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}+4\frac{\partial}{\partial y_{a_1}}\left(C_{\alpha\mu_2\mu_3\mu_4}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}\right)\right)$$

$$=\frac{1}{4!}\epsilon^{a_1a_2a_3a_4}\left(\frac{\partial C_{\mu_1\mu_2\mu_3\mu_4}}{\partial x_0^\alpha}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}-4\frac{\partial C_{\alpha\mu_2\mu_3\mu_4}}{\partial x_0^{\mu_1}}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}\right)$$

$$=\frac{1}{4!}\epsilon^{a_1a_2a_3a_4}F_{\alpha\mu_1\mu_2\mu_3\mu_4}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}$$

$$\sqrt{-h}h^{ab}g_{\alpha\nu}\left(\hat{\nabla}_b\left(\frac{\partial x_0^\nu}{\partial y^a}\right)+\frac{\partial x_0^\mu}{\partial y^a}\frac{\partial x_0^\beta}{\partial y^b}\Gamma^\nu_{\mu\beta}\right)+\frac{1}{4!}\epsilon^{a_1a_2a_3a_4}F_{\alpha\mu_1\mu_2\mu_3\mu_4}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}=0$$

$$\nabla_\mu T_D^{\mu\nu}(x)=\partial_\mu T_D^{\mu\nu}(x)+\Gamma^\nu_{\mu\alpha}T^{\mu\alpha}+\Gamma^\mu_{\mu\alpha}T^{\alpha\nu}$$

$$=\frac{-1}{(2\pi)^3\alpha'^2g_s}\left[\int d^4y\sqrt{-h}h^{ab}\frac{\partial x_0^\mu}{\partial y^a}\frac{\partial x_0^\nu}{\partial y^b}\frac{\partial}{\partial x^\mu}\left(\frac{\delta^{10}\left(x^\mu-x_0^\mu(y)\right)}{\sqrt{-g(x)}}\right)\right.$$

$$\left.+\sqrt{-h}h^{ab}\left(\frac{\partial x_0^\alpha}{\partial y^a}\frac{\partial x_0^\mu}{\partial y^b}\Gamma^\nu_{\mu\alpha}+\frac{\partial x_0^\alpha}{\partial y^a}\frac{\partial x_0^\nu}{\partial y^b}\Gamma^\mu_{\mu\alpha}\right)\left(\frac{\delta^{10}\left(x^\mu-x_0^\mu(y)\right)}{\sqrt{-g(x)}}\right)\right]$$

$$=\frac{-1}{(2\pi)^3\alpha'^2g_s}\left[\int d^4y\frac{1}{\sqrt{-g(x)}}\frac{\partial}{\partial y^a}\left(\sqrt{-h}h^{ab}\frac{\partial x_0^\nu}{\partial y^b}\right)\delta^{10}\left(x^\mu-x_0^\mu(y)\right)\right.$$

$$\left.+\sqrt{-h}h^{ab}\delta^{10}\left(x^\mu-x_0^\mu(y)\right)\left(\frac{\partial x_0^\mu}{\partial y^a}\frac{\partial x_0^\nu}{\partial y^b}\frac{\partial}{\partial x^\mu}\left(\frac{1}{\sqrt{-g(x)}}\right)+\left(\frac{\partial x_0^\alpha}{\partial y^a}\frac{\partial x_0^\mu}{\partial y^b}\Gamma^\nu_{\mu\alpha}+\frac{\partial x_0^\alpha}{\partial y^a}\frac{\partial x_0^\nu}{\partial y^b}\Gamma^\mu_{\mu\alpha}\right)\right)\right]$$

$$=\frac{-1}{(2\pi)^3\alpha'^2g_s}\frac{1}{\sqrt{-g(x)}}\left[\int d^4y\delta^{10}\left(x^\mu-x_0^\mu(y)\right)\sqrt{-h(y)}h^{ab}\left(\hat{\nabla}_a\frac{\partial x_0^\nu}{\partial y^b}+\frac{\partial x_0^\beta}{\partial y^a}\frac{\partial x_0^\mu}{\partial y^b}\Gamma^\nu_{\mu\beta}\right)\right]$$

$$\frac{\partial x_0^{\mu}}{\partial y^a} - \frac{\partial}{\partial y^a}\left(\delta^{10}\left(x^{\mu} - x_0^{\mu}(y)\right)\right.$$

$$\frac{\partial g}{\partial x^{\mu}} = 2g\Gamma^{\alpha}_{\alpha\mu}$$

$$T^{\mu\nu}_{\text{int}} = \frac{-1}{16\pi(2\pi^2)^3 g_s^2 \alpha' 45!}\left(g^{\mu\nu}F.\tilde{F} - 5F^{\mu\nu_1\nu_2\nu_3\nu_4}\tilde{F}^{\nu}_{\nu_1\nu_2\nu_3\nu_4} - 5\tilde{F}^{\mu\nu_1\nu_2\nu_3\nu_4}F^{\nu}_{\nu_1\nu_2\nu_3\nu_4}\right)$$

$$\begin{aligned}\nabla_{\mu}T^{\mu\nu}_{int}(x) &= \frac{-1}{16\pi(2\pi^2)^3 g_s^2 \alpha'_{\ 5}!}\left(g^{\mu\nu}\partial_{\mu}(F.\tilde{F}) - 5F^{\mu\nu_1\nu_2\nu_3\nu_4}\nabla_{\mu}\tilde{F}^{\nu}_{\nu_1\nu_2\nu_3\nu_4}\right.\\ &\quad \left. -5\nabla_{\mu}\tilde{F}^{\mu\nu_1\nu_2\nu_3\nu_4}F^{\nu}_{\nu_1\nu_2\nu_3\nu_4} - 5\tilde{F}^{\mu\nu_1\nu_2\nu_3\nu_4}\nabla_{\mu}F^{\nu}_{\nu_1\nu_2\nu_3\nu_4}\right)\\ &= \frac{1}{16\pi(2\pi^2)^3 g_s^2 \alpha'_{\ 4}5!}5\nabla_{\mu}\tilde{F}^{\mu\nu_1\nu_2\nu_3\nu_4}F^{\nu}_{\nu_1\nu_2\nu_3\nu_4}\\ &= \frac{-1}{(2\pi)^3\alpha'^2 g_s}\frac{1}{4!\sqrt{-g}}\int d^4y\delta^{10}\left(x^{\mu} - x_0^{\mu}(y)\right)\epsilon^{a_1a_2a_3a_4}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}F^{\nu}_{\mu_1\mu_2\mu_3\mu_4}\end{aligned}$$

$$\nabla_{\mu}F^{\mu\nu_1\nu_2\nu_3\nu_4} = 0$$

$$\frac{1}{16\pi(2\pi^2)^3 g_s^2 \alpha'^4}\nabla_{\mu}\tilde{F}^{\mu\mu_1\mu_2\mu_3\mu_4} = -\frac{1}{(2\pi)^3\alpha'^2 g_s}\int d^4y\frac{\delta^{10}\left(x^{\mu} - x_0^{\mu}(y)\right)}{\sqrt{-g}}\epsilon^{a_1a_2a_3a_4}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}$$

$$\nabla_{\mu}\left(T_D^{\mu\nu} + T_F^{\mu\nu}\right) = 0$$

$$\Longrightarrow \sqrt{-h(y)}h^{ab}\left(\hat{\nabla}_a\frac{\partial x_0^{\nu}}{\partial y^b} + \frac{\partial x_0^{\beta}}{\partial y^a}\frac{\partial x_0^{\mu}}{\partial y^b}\Gamma^{\nu}_{\mu\beta}\right) + \frac{1}{4!}\epsilon^{a_1a_2a_3a_4}\frac{dx_0^{\mu_1}}{dy^{a_1}}\frac{dx_0^{\mu_2}}{dy^{a_2}}\frac{dx_0^{\mu_3}}{dy^{a_3}}\frac{dx_0^{\mu_4}}{dy^{a_4}}F^{\nu}_{\mu_1\mu_2\mu_3\mu_4} = 0$$

$$\begin{aligned}dC_V &= -4\epsilon^{(5)}\\ d\left(\Theta\wedge(-*_5F + A\wedge F)\right) &= -2J\wedge *_5F + 2J\wedge A\wedge F\\ d(l_2^2 d(l_3^2)\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3) &= -2J\wedge J\wedge(d\Psi + \Theta) = -4*_{10}\epsilon^{(5)} - 2J\wedge J\wedge A\\ d(-A\wedge(d\Psi + \Theta)\wedge J) &= -F\wedge(d\Psi + \Theta)\wedge J + 2A\wedge J\wedge J\end{aligned}$$

$$d(-*_5F + A\wedge F) = 0$$

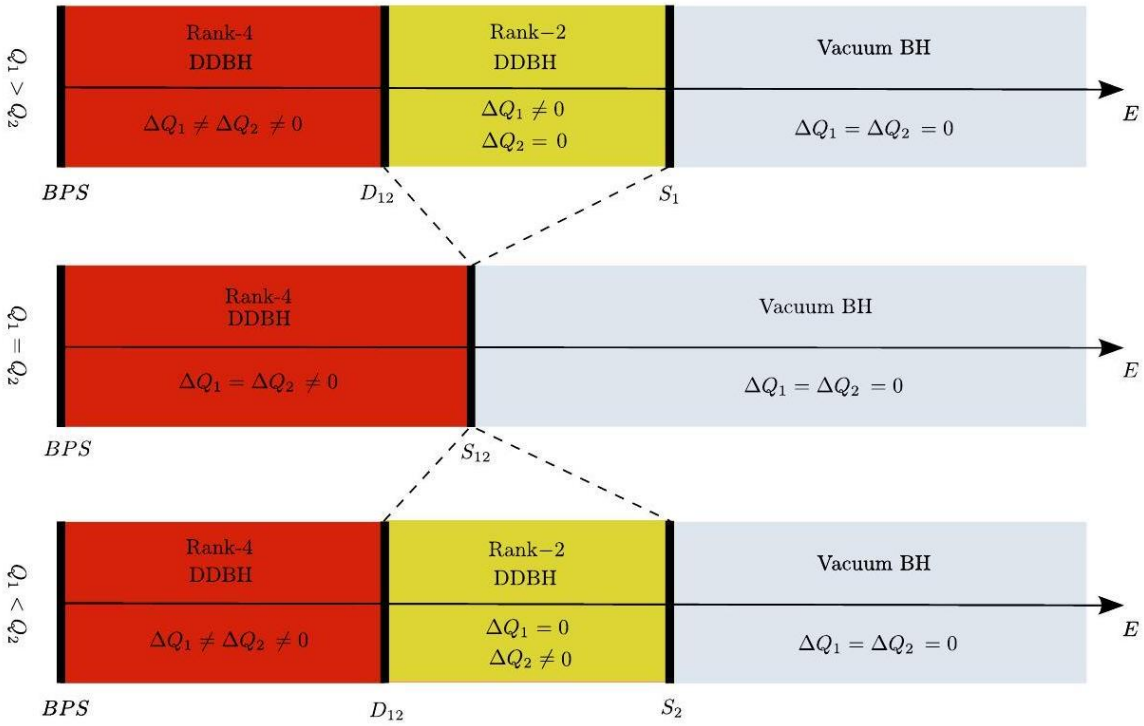
$$(-*_5F + A\wedge F) = \frac{2\pi^2 Q}{4}\sigma_1\wedge\sigma_2\wedge\sigma_3$$

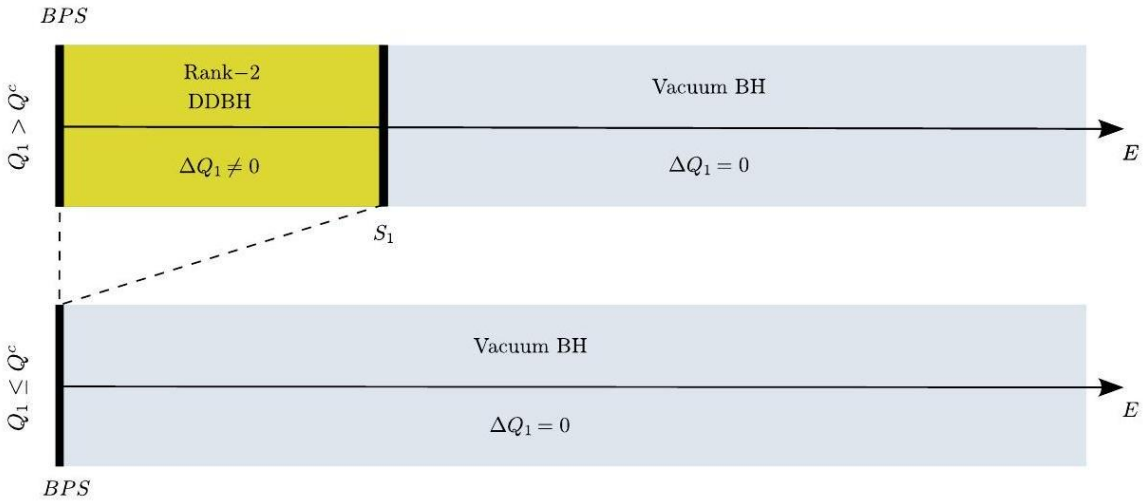
$$\begin{aligned}\frac{C}{R^4_{AdS_5^4}} &= C_V + \left(\frac{\pi^2 Q}{4}\right)\Theta\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3\\ &+ l_2^2 d(l_3^2)\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3 + A\wedge(d\Psi + \Theta)\wedge J\end{aligned}$$

$$\sigma_1\wedge\sigma_2\wedge\sigma_3 = d(\cos\,\theta_a d\psi\wedge d\phi)$$

$$\Theta\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3 = -d\left(\left(\Theta\wedge(\cos\,\theta_a d\psi\wedge d\phi)\right) + J\wedge(\cos\,\theta_a d\psi\wedge d\phi)\right.$$

$$\begin{aligned}\frac{C}{R_{AdS_5^4}} &= C_V + \left(\frac{\pi^2 Q}{4}\right)\cos\,\theta_a J\wedge d\psi\wedge d\phi\\ &+ l_2^2 d(l_3^2)\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3 + A\wedge(d\Psi + \Theta)\wedge J\end{aligned}$$





$$\frac{NV_3}{2\pi^2}V_3 = 2\pi^2\left(2r^2\sqrt{b^2+A_3^2}\right)$$

$$2Nr^2\sqrt{b^2+A_3^2}$$

$$S_{\mathrm{DBI}} = 2N\int\ r^2\sqrt{b^2+A_3^2}\sqrt{-\tilde{g}_{\mu\nu}dx^\mu dx^\nu}$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{ab}dy^a dy^b + 2g_{a\mu}dy^a dx^\mu$$

$$ds_7^2 = \left(g_{\mu\nu} - g_{\mu a} g_{\nu b} g_3^{ab}\right) dx^\mu dx^\nu = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$$

$$ds_7^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = \left(-\frac{r^2 W}{4b^2} + A_t^2 + b^2 f^2\right) dt^2 + \frac{dr^2}{W} + d\alpha^2 + \cos^2\alpha d\beta^2 + \sum_{i=1}^{3} \left(l_i^2 d\phi_i^2 - 2A_t l_i^2 d\phi_i dt\right)$$
$$- \frac{\left(dt\left(A_t A_{\sigma_3} + b^2 f\right) - A_{\sigma_3} \sum_i l_i^2 d\phi_i\right)^2}{b^2 + A_{\sigma_3}^2}$$
$$= -\frac{r^2 W}{4b^2} dt^2 + \frac{dr^2}{W} + ds^2(\mathbb{CP}^2) + \frac{b^2}{b^2 + A_{\sigma_3}^2} \left(d\Psi + \Theta + \left(f A_{\sigma_3} - A_t\right) dt\right)^2$$

$$l_i(\tau), r(\tau), \phi_{1,2,3}(\tau)$$

$$ds_{\text{ind}}^2 = \left(g_{tt} + g_{rr}\dot{r}^2 + + \sum_{i=1}^{2} \left(2 l_i \dot{l}_i\right) + \sum_{i=1}^{3} \left(g_{\phi_i \phi_i} \dot{\phi}_i^2 - 2 g_{t\phi_i} \dot{\phi}_i\right)\right) dt^2$$
$$+ \left(g_{t\sigma_3} + \sum_{i=1}^{3} g_{\sigma_3 \phi_i} \dot{\phi}_i\right) dt \sigma_3 + \sum_{i=1}^{3} g_{\sigma_i \sigma_i} \sigma_i^2$$

$$ds_{\text{ind}}^2 = \Delta dt^2 + \left(b^2 + A_{\sigma_3}^2\right) \left(\sigma_3 + \frac{\gamma}{b^2 + A_{\sigma_3}^2} dt\right)^2 + \frac{r^2}{4} \left(\sigma_1^2 + \sigma_2^2\right)$$

$$\Delta = \tilde{g}_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = -\frac{r^2 W}{4b^2} + A_t^2 + b^2 f^2 + \frac{\dot{r}^2}{W} + \sum_{i=1}^{2} \left(2 l_i \dot{l}_i\right) + \sum_{i=1}^{3} \left(l_i^2 \dot{\phi}_i^2 - 2 A_t l_i^2 \dot{\phi}_i\right) - \frac{\gamma^2}{q^2}$$

$$q^2 = b^2 + A_{\sigma_3}^2, \text{ and }, \gamma = A_t A_{\sigma_3} + b^2 f - l_i^2 A_{\sigma_3} \dot{\phi}_i$$

$$C_V = \left(\frac{r^4 - R^4}{8}\right) dt \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3$$

$$S_{WZ} = 8N \int \left(\frac{r^4 - R^4}{8}\right) dt$$

$$S = S_{DBI} + S_{WZ}$$
$$= 2N \int r^2 \sqrt{b^2 + A_3^2} \sqrt{-\tilde{g}_{\mu\nu} dx^\mu dx^\nu} + 8N \int \left[\left(\frac{r^4 - R^4}{8}\right) dt\right]$$

$$z_1 \to \cos\zeta z_1 + i \sin\zeta \bar{z}_2, z_2 \to \cos\zeta z_2 - i \sin\zeta \bar{z}_1$$

$$dz_1 \wedge d\bar{z}_1 \to \cos^2\zeta dz_1 \wedge d\bar{z}_1 - \sin^2\zeta dz_2 \wedge d\bar{z}_2 - i \sin\zeta \cos\zeta (dz_1 \wedge dz_2 + d\bar{z}_1 \wedge d\bar{z}_2)$$

$$\cos^2\zeta dz_1 \wedge d\bar{z}_1 - \sin^2\zeta dz_2 \wedge d\bar{z}_2$$

$$\begin{pmatrix} \cos^2\zeta & 0 & 0 \\ 0 & -\sin^2\zeta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$dz_1 \wedge d\bar{z}_1 \to \cos^2\,\zeta dz_1 \wedge d\bar{z}_1 - \sin^2\,\zeta dz_2 \wedge d\bar{z}_2 - i\sin\,\zeta\cos\,\zeta\left(e^{2i\alpha}dz_1 \wedge dz_2 + e^{-2i\alpha}d\bar{z}_1 \wedge d\bar{z}_2\right)$$

$$x_1 \to \cos\,\zeta x_1 + \sin\,\zeta y_2, \quad y_1 \to \cos\,\zeta y_1 + \sin\,\zeta x_2$$
$$x_2 \to \cos\,\zeta x_2 - \sin\,\zeta y_1, \quad y_2 \to \cos\,\zeta y_2 - \sin\,\zeta x_1$$

$$ds^2_{(5)} = -f_b^2(dt_b - \omega)^2 + \frac{1}{f_b}\left[\frac{\xi}{F(\xi)}d\xi^2 + \frac{F}{\xi}(d\Phi + \eta\Psi_b)^2 + \xi\left(\frac{d\eta^2}{1-\eta^2} + (1-\eta^2)d\Psi_b^2\right)\right]$$

$$\omega = \frac{1}{3f_b}\left(\frac{F'(\xi)}{2\xi} - 1\right)^{\blacksquare}$$

$$A = -f_b dt_b - \frac{3a^2 f_b}{8(1-a)^2\xi}(d\Phi + \eta\Psi_b)$$

$$t_b = t, \eta = \cos\,\theta_a, \Phi = \phi_a - 2t, \Psi_b = \psi, \xi = \frac{r^2}{4}\left(1 - \frac{R_0^2}{r^2}\right)$$

$$f_b = 1 - \frac{R_0^2}{r^2}, F = \frac{1}{16}(r^2 - R_0^2)^2(1 + 2R_0^2 + r^2), a = j$$

$$e^0 = f_b(dt - \omega), e^1 = -\frac{1}{\sqrt{f_b}}\sqrt{\frac{\xi}{F(\xi)}}d\xi, e^2 = \frac{1}{\sqrt{f_b}}\sqrt{\frac{F}{\xi}}(d\Phi + \eta d\Psi_b), e^3 = \sqrt{\frac{\xi}{f_b(1-\eta^2)}}d\eta$$

$$e^4 = \sqrt{\frac{\xi}{f_b}(1-\eta^2)}d\Psi_b, e^5 = d\alpha, e^6 = \sin\,\alpha\cos\,\alpha(\cos^2\,\beta(d\phi_1 - d\phi_2) + d\phi_2 - d\phi_3)$$

$$e^7 = \sin\,\alpha d\beta, e^8 = \sin\,\alpha\sin\,\beta\cos\,\beta d(\phi_2 - \phi_1), e^9 = \sum_i l_i^2 d\phi_i - A$$

$$\Gamma\epsilon = \epsilon$$

$$\Gamma = \frac{1}{4!}\frac{\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}}{\sqrt{-h}}\frac{\partial X^{\mu_1}}{\partial\sigma^{\alpha_1}}\frac{\partial X^{\mu_2}}{\partial\sigma^{\alpha_2}}\frac{\partial X^{\mu_3}}{\partial\sigma^{\alpha_3}}\frac{\partial X^{\mu_4}}{\partial\sigma^{\alpha_4}}e_{\mu_1}^{M_1}e_{\mu_2}^{M_2}e_{\mu_3}^{M_3}e_{\mu_4}^{M_4}\Gamma_{M_1M_2M_3M_4}$$

$$\sigma^0 = t, \sigma^i = y^i$$

$$(r, \phi_1, \phi_2, \phi_3, l_1, l_2, l_3) = (r_0, t, t, t, 1, 0, 0)$$

$$\epsilon = e^{\frac{i}{2}(\phi - 2t + \phi_1 + \phi_2 + \phi_3)}\sqrt{f}\epsilon_0$$

$$\Gamma^{09}\epsilon_0 = \epsilon_0, \Gamma^{12}\epsilon_0 = -i\epsilon_0, \Gamma^{34}\epsilon_0 = \Gamma^{56}\epsilon_0 = \Gamma^{78}\epsilon_0 = i\epsilon_0$$

$$\Gamma^{11}\epsilon = -\epsilon$$

$$\left(\Gamma^{0349} + \frac{R\sqrt{1 + R^2 + 2R_0^2}}{R^2 + R_0^2}(\Gamma^{0234} - \Gamma^{2349})\right)\epsilon_0 = \epsilon_0$$

$$(\Gamma^0 + \Gamma^9)\epsilon_0 = 0$$

$$\Gamma^{0349}\epsilon_0 = \epsilon_0$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ (a_{12})^* & a_{22} & a_{23} \\ (a_{13})^* & (a_{23})^* & a_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_{11}(i\sigma_2) & \mathrm{Re}(a_{12})(i\sigma_2) - \mathrm{Im}(a_{12})\mathbb{I}_2 & \mathrm{Re}(a_{13})(i\sigma_2) - \mathrm{Im}(a_{13})\mathbb{I}_2 \\ \mathrm{Re}(a_{12})(i\sigma_2) + \mathrm{Im}(a_{12})\mathbb{I}_2 & a_{22}(i\sigma_2) & \mathrm{Re}(a_{23})(i\sigma_2) - \mathrm{Im}(a_{23})\mathbb{I}_2 \\ \mathrm{Re}(a_{13})(i\sigma_2) + \mathrm{Im}(a_{13})\mathbb{I}_2 & \mathrm{Re}(a_{23})(i\sigma_2) + \mathrm{Im}(a_{23})\mathbb{I}_2 & a_{33}(i\sigma_2) \end{pmatrix}$$

$$\mathbf{z}' = U\mathbf{z}$$

$$V_B = \partial_{z_i} B_i{}^j z_j - \partial_{z_i^*} (B^*)_i{}^j z_j^* = B_i{}^j \left(z_i \partial_{z_j} - z^*{}_j \partial_{z_i^*} \right)$$

$$z_i = l_i e^{i\phi_i}$$

$$\partial_{z_i} = -\frac{i}{z_i}\partial_{\phi_i} + \frac{l_i}{z_i}\partial_{l_i}$$

$$B_i^j e^{i(\phi_i - \phi_j)} \left(l_i \frac{\partial}{\partial l_j} - l_j \frac{\partial}{\partial l_i} - i \frac{l_i}{l_j} \frac{\partial}{\partial \phi_j} - i \frac{l_j}{l_i} \frac{\partial}{\partial \phi_i} \right)$$

$$n_{SO_6} = \sum_{n_Z=0}^{n} (n_Z, n - n_Z)$$

$$-\nabla^2 \phi_n = n(n+4)\phi_n$$

$$ds_{S^5}^2 = ds_{CP^2}^2 + (d\Psi + \Theta)^2$$

$$\phi_n = e^{i(n_Z - n_{\bar{Z}})\Psi} \phi_{n_Z, n_{\bar{Z}}}$$

$$-\nabla_{S^5}^2 = \left(-\nabla_{CP^2,(n_Z - n_{\bar{Z}})}^2 + (n_Z - n_{\bar{Z}})^2 \right) \phi_{n_Z, n_{\bar{Z}}}$$

As $-\nabla_{S^5}^2 = (n_Z + n_{\bar{Z}})(n_Z + n_{\bar{Z}} + 4)$,

$$-\nabla_{CP^2,(n_Z - n_{\bar{Z}})}^2 \phi_{n_Z, n_{\bar{Z}}} = (n_Z + n_{\bar{Z}})(n_Z + n_{\bar{Z}} + 4) - (n_Z - n_{\bar{Z}})^2$$
$$= 4(n_Z n_{\bar{Z}} + n_Z + n_{\bar{Z}})$$

$$g(r) = \frac{1}{(2V(r))^{1/4}} \left(A \exp\left(\int_{r_1}^{r} N\sqrt{2V} \right) e^{-\beta} + B \exp\left(-\int_{r_1}^{r} N\sqrt{2V} \right) e^{\beta} \right)$$
$$= \frac{Ae^{-\beta}}{(2V(r))^{1/4}} \left(\exp\left(\int_{r_1}^{r} N\sqrt{2V} \right) + \frac{B}{A} e^{2\beta} \exp\left(-\int_{r_1}^{r} N\sqrt{2V} \right) \right)$$

$$\beta = \int_{r_1}^{r^*} N\sqrt{2V}$$

$$\left(-\frac{1}{2N^2} \frac{\partial}{\partial r} + V'(r_1)(r - r_1) \right) g(r) = 0$$

$$g(r) = c_1 \mathrm{Bi}\left(N^{2/3}V'(r_1)^{1/3}(r-r_1)\right) + c_2 \mathrm{Ai}\left(N^{2/3}V'(r_1)^{1/3}(r-r_1)\right)$$

$$g(r) \approx c_1 \frac{e^{\frac{2}{3}N\sqrt{V'(r_1)}r^{3/2}}}{\sqrt{\pi}\sqrt[12]{V'(r_1)}\sqrt[4]{r}} + c_2 \frac{e^{-\frac{2}{3}N\sqrt{V'(r_1)}r^{3/2}}}{2\sqrt{\pi}\sqrt[12]{V'(r_1)}\sqrt[4]{r}}$$

$$c_1 = \frac{\sqrt{\pi}}{V'(r_1)^{1/6}} A e^{-\beta}, c_2 = \frac{2\sqrt{\pi}}{V'(r_1)^{1/6}} B e^{\beta}$$

$$g(r) \sim \frac{A e^{-\beta}}{(2V)^{-\frac{1}{4}}}\left(\sin\left(\int_{r_1}^{r} N\sqrt{-2V} + \frac{\pi}{4}\right) + \frac{2B}{A} e^{2\beta} \cos\left(\int_{r_1}^{r} N\sqrt{-2V} + \frac{\pi}{4}\right)\right)$$

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + e^{H+G}d\Omega_3^2 + e^{H-G}d\tilde{\Omega}_3^2 \\ F_{(5)} &= 4\left(F \wedge d\Omega_3 + \tilde{F} \wedge d\tilde{\Omega}_3\right) \end{aligned}$$

$$\begin{aligned} F &= dB_t \wedge (dt + V) + B_t dV + d\hat{B} \\ \tilde{F} &= d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\hat{\tilde{B}} \end{aligned}$$

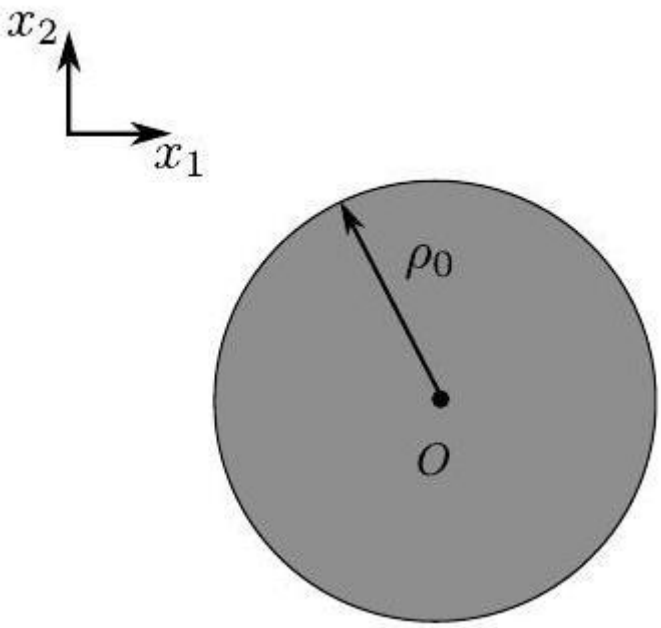
$$F = e^{3G} \star_4 \tilde{F}$$

$$\begin{aligned} ds^2 &= -h^{-2}\left(dt + V_i dx^i\right)^2 + h^2\left(dy^2 + dx^i dx^i\right) + y e^G d\Omega_3^2 + y e^{-G} d\tilde{\Omega}_3^2 \\ B_t &= -\frac{1}{4}y^2 e^{2G}, \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G} \\ d\hat{B} &= -\frac{1}{4}y^3 \star_3 \left(\frac{z+\frac{1}{2}}{y^2}\right), d\hat{\tilde{B}} = -\frac{1}{4}y^3 \star_3 \left(\frac{z-\frac{1}{2}}{y^2}\right) \end{aligned}$$

$$\begin{aligned} z &= \frac{1}{2}\tanh\ G, \quad h^{-2} = 2y\cosh\ G \\ y\partial_y V_i &= \epsilon_{ij}\partial_j z, \quad y\left(\partial_i V_j - \partial_j V_i\right) = \epsilon_{ij}\partial_y z \end{aligned}$$

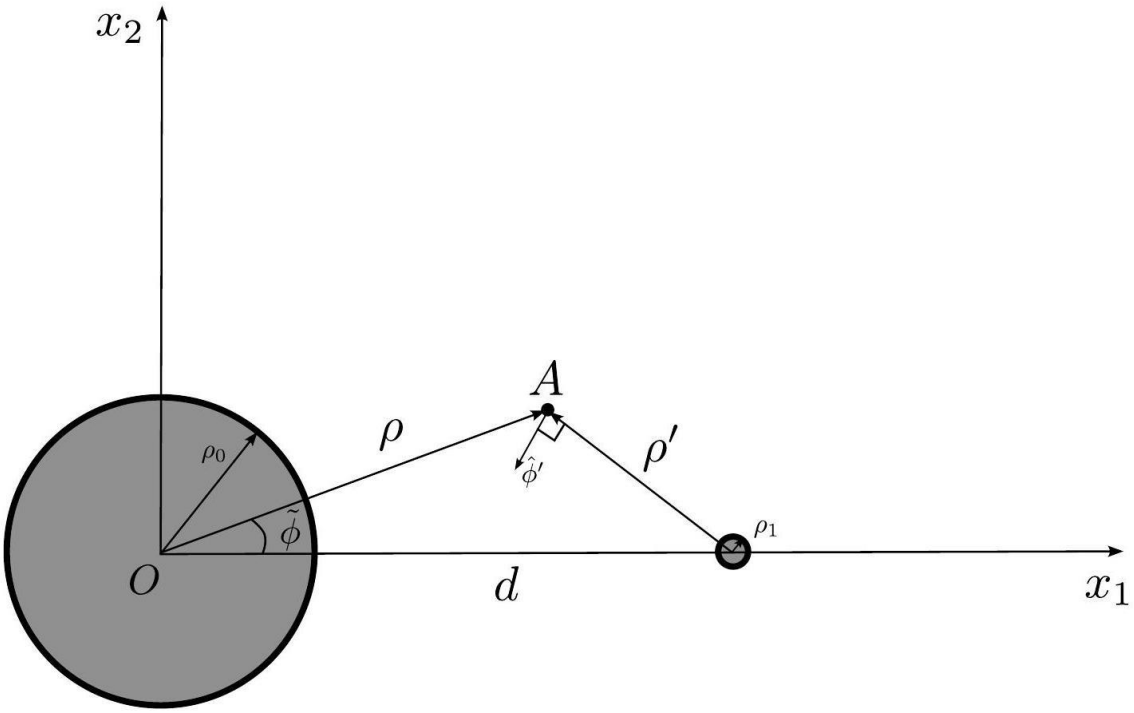
$$\begin{aligned} z(x_1, x_2, y) &= \frac{y^2}{\pi}\int_{R^2} \frac{z(x_1', x_2', 0)dx_1' dx_2'}{((\mathbf{x}-\mathbf{x}')^2 + y^2)^2} \\ V_i &= \frac{\epsilon_{ij}}{\pi}\int_{R^2} \frac{z(x_1', x_2', 0)\left(x^j - x'^j\right)dx_1' dx_2'}{((\mathbf{x}-\mathbf{x}')^2 + y^2)^2} \end{aligned}$$

$$\frac{ds^2}{l_N^2} = \rho_0\left(-(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_3^2 + d\theta^2 + \cos^2\ \theta d\phi^2 + \sin^2\ \theta d\tilde{\Omega}_3^2\right)$$



$$\rho = \rho_0\sqrt{1+r^2}\cos\,\theta$$
$$y = \rho_0 r\sin\,\theta$$
$$\tilde{\phi} = \phi - t$$

$$z = \tilde{z}_1 + \tilde{z}_2 + \frac{1}{2}$$
$$\tilde{z_1} = \frac{(\rho^2 - \rho_0^2 + y^2)}{2\sqrt{(\rho^2+\rho_0^2+y^2)^2 - 4\rho^2\rho_0^2}} - \frac{1}{2} = z_1 - \frac{1}{2}$$
$$\tilde{z_2} = \frac{(\rho'^2 - \rho_1^2 + y^2)}{2\sqrt{(\rho'^2+\rho_1^2+y^2)^2 - 4\rho'^2\rho_1^2}} - \frac{1}{2} = z_2 - \frac{1}{2} \approx -\frac{(\rho_1)^2 y^2}{(\rho'^2+y^2)^2}$$



$$V_1 = V_\phi \hat{\phi} = -\frac{1}{2}\left(\frac{\rho^2 + y^2 + \rho_0^2}{\sqrt{(\rho^2 + \rho_0^2 + y^2)^2 - 4\rho^2\rho_0^2}} - 1\right)(-\sin\,\tilde{\phi}\hat{x}_1 + \cos\,\tilde{\phi}\hat{x}_2)$$

$$V_2 = V'_\phi \hat{\phi}' = -\frac{1}{2}\left(\frac{\rho'^2 + y^2 + \rho_1^2}{\sqrt{(\rho'^2 + \rho_1^2 + y^2)^2 - 4\rho'^2\rho_1^2}} - 1\right)\frac{1}{\rho'}(-\rho\sin\,\tilde{\phi}\hat{x}_1 + (\rho\cos\,\tilde{\phi} - d)\hat{x}_2)$$

$$= V_2^\rho \hat{\rho} + V_2^\phi \hat{\phi}$$

$$\vec{\rho}' = \vec{\rho} - \vec{d} = (\rho\cos\,\tilde{\phi} - d, \rho\sin\,\tilde{\phi})$$

$$\rho' = \sqrt{\rho^2 + d^2 - 2\rho d\cos\,\tilde{\phi}}, \hat{\phi}' = \frac{(-\rho\sin\,\tilde{\phi}, \rho\cos\,\tilde{\phi} - d)}{\rho'}$$

$$V_2^\rho = \frac{d\sin\,\tilde{\phi}}{2}\left(\frac{\rho'^2 + y^2 + \rho_1^2}{\sqrt{(\rho'^2 + \rho_1^2 + y^2)^2 - 4\rho'^2\rho_1^2}} - 1\right)\frac{1}{\rho'^2} \approx \frac{(\rho_1)^2 d\sin\,\tilde{\phi}}{(\rho'^2 + y^2)^2}$$

$$V_2^\phi = -\frac{\rho^2 - d\rho\cos\,\tilde{\phi}}{2}\left(\frac{\rho'^2 + y^2 + \rho_1^2}{\sqrt{(\rho'^2 + \rho_1^2 + y^2)^2 - 4\rho'^2\rho_1^2}} - 1\right)\frac{1}{\rho'^2} \approx -\frac{(\rho_1)^2(\rho^2 - d\rho\cos\,\tilde{\phi})}{(\rho'^2 + y^2)^2}$$

$$z\ = z_1 = \frac{1}{2}\frac{r^2 - \sin^2\,\theta}{r^2 + \sin^2\,\theta}$$

$$V\ = V_1 = \frac{-\cos^2\,\theta}{r^2 + \sin^2\,\theta}\hat{\phi}$$

$$h^{-2} = \frac{2y}{\sqrt{1 - 4z^2}} = \rho_0(r^2 + \sin^2\,\theta)$$

$$\cosh\,G = \frac{1}{\sqrt{4z^2 - 1}} = \frac{r^2 + \sin^2\,\theta}{2r\sin\,\theta}$$

$$e^G = \frac{r}{\sin\,\theta}$$

$$h^{-2} = \frac{2y}{\sqrt{1 - 4(z_1 + \tilde{z}_2)^2}} \approx \rho_0(r^2 + \sin^2\,\theta)\left(1 - \frac{\rho_1^2}{2}\frac{(r^4 - \sin^4\,\theta)}{x^4}\right)$$

$$h^2 = \frac{\sqrt{1 - 4(z_1 + \tilde{z}_2)^2}}{2y} \approx \frac{1}{\rho_0(r^2 + \sin^2\,\theta)}\left(1 + \frac{\rho_1^2}{2}\frac{(r^4 - \sin^4\,\theta)}{x^4}\right)$$

$$e^G = \sqrt{\frac{1 + 2(z_1 + \tilde{z}_2)}{1 - 2(z_1 + \tilde{z}_2)}} \approx \frac{r}{\sin\,\theta}\left(1 - \frac{\rho_1^2}{2}\frac{(r^2 + \sin^2\,\theta)^2}{x^4}\right)$$

$$e^{-G} = \sqrt{\frac{1 - 2(z_1 + \tilde{z}_2)}{1 + 2(z_1 + \tilde{z}_2)}} \approx \frac{\sin\,\theta}{r}\left(1 + \frac{\rho_1^2}{2}\frac{(r^2 + \sin^2\,\theta)^2}{x^4}\right)$$

$$V^\rho \approx \rho_1^2\frac{d\sin\,(\phi - t)}{x^4}$$

$$V^\phi \approx \frac{-\cos^2\,\theta}{r^2 + \sin^2\,\theta} - \rho_1^2\frac{\sqrt{1 + r^2}\cos\,\theta(\sqrt{1 + r^2}\cos\,\theta - d\cos\,(\phi - t))}{x^4}$$

$$x^2 = r^2 + \cos^2\,\theta + d^2 - 2d\sqrt{1 + r^2}\cos\,\theta\cos\,(\phi - t)$$

$$\frac{ds^2}{l_{N-1}^2} = r^2 d\Omega_3^2\left(1-\frac{\rho_1^2}{2}\frac{(r^2+\sin^2\theta)^2}{x^4}\right)+\sin^2\theta\, d\tilde{\Omega}_3^2\left(1+\frac{\rho_1^2}{2}\frac{(r^2+\sin^2\theta)^2}{x^4}\right)$$
$$-(1+r^2)dt^2\left(1-\frac{\rho_1^2}{2}\frac{(r^2+\cos^2\theta)^2-4(1+r^2)\cos^2\theta-1+4d\sqrt{1+r^2}\cos\theta\cos(\phi-t)}{x^4}\right)$$
$$+\frac{dr^2}{1+r^2}\left(1+\frac{\rho_1^2}{2}\frac{r^4-\sin^4\theta}{x^4}\right)+d\theta^2\left(1+\frac{\rho_1^2}{2}\frac{r^4-\sin^4\theta}{x^4}\right)$$
$$+\cos^2\theta d\phi^2\left(1+\frac{\rho_1^2}{2}\frac{(r^2+\cos^2\theta)^2-4(1+r^2)\cos^2\theta-1+4d\sqrt{1+r^2}\cos\theta\cos(\phi-t)}{x^4}\right)$$
$$-\rho_1^2\frac{2d\sin(\phi-t)}{x^4}(dt(1+r^2)-\cos^2\theta d\phi)\left(\frac{r\cos\theta dr}{\sqrt{1+r^2}}-\sqrt{1+r^2}\sin\theta d\theta\right)$$
$$+2d\phi dt\rho_1^2\frac{\cos\theta\left(2(1+r^2)\cos\theta-d\sqrt{1+r^2}(1+r^2+\cos^2\theta)\cos(\phi-t)\right)}{x^4}+O(\rho_1^4)$$

$$F_{tr}=\frac{2(d^2-1)r\sin^4\theta(r^2+\sin^2\theta)}{2Nx^6}, F_{t\phi}=\frac{d\sqrt{r^2+1}\sin^3\theta\sin 2\theta(\sin^2\theta+r^2)^2\sin(t-\phi)}{2Nx^6}$$
$$F_{t\theta}=\frac{\sqrt{1+r^2}\sin^3\theta}{8Nx^6}\Big[8\sqrt{r^2+1}\cos\theta\left((d^2(r^2+2)+2r^2+1)\cos\theta+d^2\cos^2\theta\cos(2t-2\phi)\right)$$
$$-4d\cos(t-\phi)\left((d^2+3r^2+2)\cos 2\theta+d^2+2r^4+7r^2+4\right)\Big]$$
$$F_{r\theta}=\frac{dr\sin^3\theta\sin(t-\phi)\left(-(d^2+3r^2+2)\cos 2\theta-d^2+4d\sqrt{r^2+1}\cos^3\theta\cos(t-\phi)+2r^4+r^2\right)}{2N\sqrt{r^2+1}x^6}$$
$$F_{r\phi}=\frac{2r\sin^4\theta\cos\theta(r^2+\sin^2\theta)\left(d(\cos 2\theta+2r^2+3)\cos(t-\phi)-4\sqrt{r^2+1}\cos\theta\right)}{4N\sqrt{r^2+1}x^6}$$
$$F_{\theta\phi}=\frac{\sin^3\theta\cos\theta}{4Nx^6}[2\cos^2\theta(2d^2(2(r^2+1)\cos^2(t-\phi)+2r^2+1)+\sin^2\theta(4d^2+3r^4+2r^2-1)$$
$$+2(2r^2+1)\sin^4\theta)$$
$$+2d\sqrt{r^2+1}\cos(t-\phi)\cos\theta\left(2\cos^2\theta\left(2\cos 2\theta-5(r^2+1)\right)-\left(2(d^2-r^4)+\sin^2\theta(-\cos 2\theta+2r^2+1)\right)\right)$$
$$-\frac{1}{4}(r^2+1)\left(16r^4-8r^2\cos 2\theta+(r^2+1)\cos 4\theta+7r^2-1+4\cos^4\theta(-\cos 2\theta+2r^2+2)\right)\Big]$$

$$\tilde{F}_{tr}=\frac{r^3}{32Nx^6}[(48d^2(r^2+1)-16r^4+17)\cos 2\theta-\cos 6\theta+2(6r^2+5)\cos 4\theta$$
$$+2\Big(-8(4d^2+3)r^4-6(4d^2+1)r^2+2d\cos\theta\Big(\sqrt{r^2+1}\cos(t-\phi)(-8d^2+\cos 4\theta-8(2r^2+3)\cos 2\theta$$
$$+24(r^4+r^2)-1)+8d(r^2+1)\cos\theta\cos(2t-2\phi))+8d^2-16r^6+3)]$$
$$\tilde{F}_{t\theta}=-\frac{4r^4\sin\theta(r^2+\sin^2\theta)\left(4(r^2+1)\cos\theta-d\sqrt{r^2+1}(\cos 2\theta+2r^2+3)\cos(t-\phi)\right)}{8Nx^6}$$
$$\tilde{F}_{t\phi}=-\frac{dr^4\sqrt{r^2+1}\cos\theta(\sin^2\theta+r^2)^2\sin(t-\phi)}{Nx^6}$$
$$\tilde{F}_{r\theta}=\frac{-dr^3\sin\theta\sin(t-\phi)}{8N\sqrt{r^2+1}x^6}\left[8d(r^2+1)\left(d-2(r^2+1)^{1/2}\cos\theta\cos(t-\phi)\right)-\cos 4\theta+4(3r^2+2)\cos 2\theta\right.$$
$$\left.+4r^2+1\right]$$
$$\tilde{F}_{r\phi}=-\frac{r^3\cos\theta}{8Nx^6}\left[\frac{\cos(t-\phi)\left(d(8d^2(r^2+1)+\cos 4\theta+4(3r^2+4)\cos 2\theta+4r^2+7)\right)}{\sqrt{r^2+1}}\right.$$
$$\left.-4\cos\theta\left((d^2+2)\cos 2\theta-2d^2(r^2+1)\cos(2t-2\phi)+3d^2\right)\right]$$
$$\tilde{F}_{\theta\phi}=\frac{2(d^2-1)r^4\sin(2\theta)(-\cos(2\theta)+2r^2+1)}{8Nx^6}$$

$$ds^2 = d\theta^2 + \cos^2\ \theta d\phi^2 + \sin^2\ \theta d\Omega_3^2$$

$$\left(\frac{1}{\sin^3\ \theta\cos\ \theta}\frac{\partial}{\partial\theta}\left(\sin^3\ \theta\cos\ \theta\frac{\partial}{\partial\theta}\right)+\frac{\partial^2}{\partial\phi^2}\right)Y(\theta,\phi) = -k(k+4)$$

$$Y(\theta,\phi) = Y_{k,m}(\theta)e^{im\phi}$$

$$\frac{\partial^2 Y_{k,m}(\theta)}{\partial\theta^2}-\frac{m^2 Y_{k,m}(\theta)}{\cos^2\ \theta}+\left(\frac{3\cos^2\ \theta-\sin^2\ \theta}{\sin\ \theta\cos\ \theta}\right)\frac{\partial Y_{k,m}(\theta)}{\partial\theta} = -k(k+4)$$

$$Y_{k,m}(\theta)= a(i\cos\ \theta)^{-m}\ {}_2F_1\left(-\frac{k}{2}-\frac{m}{2},\frac{k}{2}-\frac{m}{2}+2;1-m;\cos^2\ \theta\right)$$
$$+b(i\cos\ \theta)^{m}\ {}_2F_1\left(\frac{m}{2}-\frac{k}{2},\frac{k}{2}+\frac{m}{2}+2;m+1;\cos^2\ (\theta)\right)$$

$$Y_{k,m}(\theta) = b(i\cos\ \theta)^{m}\ {}_2F_1\left(\frac{m}{2}-\frac{k}{2},\frac{k}{2}+\frac{m}{2}+2;m+1;\cos^2\ \theta\right)$$

$$\lim_{\theta\to 0}Y_{k,m}(\theta) = b\frac{i^m\Gamma(m+1)}{\theta^2\Gamma\left(\frac{m-k}{2}\right)\Gamma\left(\frac{1}{2}(k+m+4)\right)}$$

$$\frac{2}{\pi}\int\ d\theta d\phi\sin^3\ \theta\cos\ \theta Y(\theta,\phi)Y^*(\theta,\phi) = \delta_{k,k'}\delta_{m,m'}$$

$$Y(\theta,\phi) = b(i\cos\ \theta)^{|m|}\ {}_2F_1\left(\frac{|m|}{2}-\frac{k}{2},\frac{k}{2}+\frac{|m|}{2}+2;|m|+1;\cos^2\ \theta\right)e^{im\phi}$$

$$Y_{k,k}(\theta,\phi) = \sqrt{\frac{(k+1)(k+2)}{2}}(i\cos\ \theta)^k e^{ik\phi}$$

$$\chi= 4\left(-1-\frac{(\cos\ 2\theta-2r^2-1)}{64N\sqrt{r^2+1}x^6}\left[4\cos\ 2\theta\left(\sqrt{r^2+1}(5d^2+10r^2+6)-6d(r^2+1)\cos\ \theta\cos\ (t-\phi)\right)\right.\right.$$
$$\left.\left.+\sqrt{r^2+1}(8(5d^2+2)r^2+28d^2+8r^4+23)-24d(2r^4+5r^2+3)\cos\ \theta\cos\ (t-\phi)+\sqrt{r^2+1}\cos\ 4\theta\right]\right)$$

$$\chi(\theta,\phi)\approx 4\left(\frac{-6\cos^2\ \theta(8d^2\cos^2\ (t-\phi)+1)+8d^2+15\cos\ 2\theta+7}{16Nr^2}\right)+O\left(\frac{1}{r^3}\right)$$
$$\approx\frac{1}{r^2}\left(\frac{d^2\sqrt{6}}{2}\left(Y_{2,2}(\theta,\tilde{\phi})+Y_{2,-2}(\theta,\tilde{\phi})\right)+\sqrt{2}(d^2-1)Y_{2,0}(\theta,\tilde{\phi})\right)+O\left(\frac{1}{r^3}\right)$$

$$s^I = \frac{1}{k(k+4)}\left(\frac{2}{\pi}\int\ d\theta d\phi\sin^3\ \theta\cos\ \theta Y^*_{k,k}(\theta,\phi)\chi(\theta,\phi)\right)$$

$$z_{\pm} = \frac{\sec\ \theta\left(2d^2+\cos\ 2\theta+2r^2+1\pm\sqrt{(2d^2+\cos\ 2\theta+2r^2+1)^2-16d^2(r^2+1)\cos^2\ \theta}\right)}{4d\sqrt{r^2+1}}$$

$$s^I = -\frac{4i^k}{Nk(k+4)}\left(\int\ d\theta\left(\sqrt{2(k+1)(k+2)}(k^2-1)\sin^3\ \theta\cos^{k+1}\ \theta\right)\left(\frac{d\cos\ \theta}{r}\right)^k\right)$$

$$s^I = -\frac{4i^k}{Nk(k+4)}\left(\frac{(k^2-1)}{\sqrt{2(k+1)(k+2)}}\right)\left(\frac{d}{r}\right)^k$$

$$l_1 = \cos\,\alpha, l_2 = \sin\,\alpha\cos\,\beta, l_3 = \sin\,\alpha\sin\,\beta$$

$$Q = \frac{1}{16\pi G}\int_{S^3} * F$$

$$\partial_t - \Omega\left(\partial_\psi + \cos\,\theta\partial_\phi\right)$$

$$M_{\text{there}} = \frac{\pi}{2}M_{\text{here}}, J_{\text{there}} = \frac{\pi}{2}J_{\text{here}}, Q_{\text{there}} = 2Q_{\text{here}}$$

$$q_c < \frac{R^2}{2}\left(4\sqrt{R^2+1} - R^2 - 4\right)$$

$$l_i = \sqrt{\frac{q_i}{\sum_j\, q_j}}$$

$$-e^{-ik\psi}\nabla^2\left(B(x_i)e^{ik\psi}\right) = -H^{ij}D_iD_jB + e^{-2\sigma}k^2B$$

$$\int\, \sqrt{V}dr \sim q_n$$

$$V(r) = -\frac{\frac{R^2\omega^2}{4(\mu^2-2R^2-1)^2} + \frac{1}{4}}{2N^2(r-R)^2}$$

$$\omega = \omega_* + 1 + \mathcal{O}\left(\frac{1}{q_nN}\right)$$

$$V(r) = 2(r-r_*)^2 - \frac{1}{N} - \epsilon$$

$$r - r_1 \gg \frac{1}{N^{\frac{2}{3}}}$$

$$g(r) = 2c_1\sqrt{2b(r-R)}e^{-\frac{i\omega_*^q}{W'(r)}\log\, b(r-R)}$$

$$\tilde{\epsilon}_5 = \epsilon^{(4)}(\mathbb{CP}^2) \wedge (d\Psi + \Theta - A)$$

$$\delta F_{(5)} = \frac{l_{N-1}^4}{N}4\left(F \wedge d\Omega_3 + \tilde{F} \wedge d\tilde{\Omega}_3\right) = \frac{l_{N-1}^4}{N}(-4\epsilon_5 + 4\tilde{\epsilon}_5)$$

$$l_p^8 = g_s^2 \alpha'^4$$

$$T_F^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_F}{\delta g_{\mu\nu}}$$
$$= \frac{-1}{16\pi(2\pi^2)^3 g_s^2 \alpha'^4 5!} \left(\frac{1}{2} g^{\mu\nu} F_5^2 - 5 F^{\mu\nu_1\nu_2\nu_3\nu_4} F^{\nu}_{\nu_1\nu_2\nu_3\nu_4} \right)$$

$$\int \vec{E}_1 \cdot \vec{E}_2 = \int \vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2$$

$$-\int \phi_2 \vec{\nabla}^2 \phi_1 + \int_B \phi_2 \vec{\nabla}\phi_1$$

$$ds_3^2 = \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + \left(b^2 + A_{\sigma_3}^2\right)\sigma_3^2 \otimes \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{4}$$

$$l_1 = \cos\,\alpha, l_2 = \sin\,\alpha\cos\,\beta, l_3 = \sin\,\alpha\sin\,\beta$$

$$\langle A_2 \mid A_1 \rangle = \mathrm{Tr}\left(A_2^\dagger A_1\right)$$

$$u_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} i\sigma_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$u_{12,r} = \frac{1}{2} \begin{pmatrix} 0 & -\mathbb{I}_2 & 0 \\ \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, u_{12,c} = \frac{1}{2} \begin{pmatrix} 0 & i\sigma_2 & 0 \\ i\sigma_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{Tr}\left(OB^\dagger\right)$$

$$\frac{\partial M_{ext}(\vec{Q}, \epsilon)}{\partial \epsilon} = \lim_{M \to M_{ext}(\vec{Q}, \epsilon)} -T \left(\frac{\partial S(M, \vec{Q}, \epsilon)}{\partial \epsilon} \right)_{M, \vec{Q}}$$

$$\Delta \mathcal{I} = \mathcal{K} \int d^4x \sqrt{-g} \epsilon \Lambda$$

$$\mathcal{I}_{\text{Total}} = \mathcal{I} + \Delta \mathcal{I}.$$

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \epsilon \Delta g_{\mu\nu} = \hat{g}_{\mu\nu} + \epsilon \mathrm{h}_{\mu\nu}$$

$$G(T, \vec{\mu}) = M(T, \vec{\mu}) - TS - \vec{\mu} \cdot \vec{Q}$$

$$G(T, \vec{\mu}) \to G(T, \vec{\mu}) + \epsilon \Delta G(T, \vec{\mu}).$$

$$S(T, \vec{\mu}, \epsilon) = -\left(\frac{\partial G}{\partial T}\right)_{\vec{\mu}, \epsilon} \; ; \; Q_i(T, \vec{\mu}, \epsilon) = -\left(\frac{\partial G}{\partial \mu_i}\right)_{T, \mu_{j \neq i}, \epsilon}$$

$$\lim_{T \to 0} T \left(\frac{\partial S(T, \vec{Q}, \epsilon)}{\partial \epsilon} \right)_{T, \vec{Q}} = 0$$

$$M_{\text{ext}}(\vec{Q}, \epsilon) \equiv \lim_{T \to 0} M(T, \vec{Q}, \epsilon)$$

$$\left(\frac{\partial M}{\partial \epsilon}\right)_{T,\vec{Q}} = \left(\frac{\partial}{\partial \epsilon}(G + TS + \vec{\mu}\cdot\vec{Q})\right)_{T,\vec{Q}}$$
$$= \left(\frac{\partial G}{\partial \vec{\mu}}\right)_{T,\epsilon}\cdot\left(\frac{\partial \vec{\mu}}{\partial \epsilon}\right)_{T,\vec{Q}} + \left(\frac{\partial G}{\partial \epsilon}\right)_{T,\vec{\mu}} + T\left(\frac{\partial S}{\partial \epsilon}\right)_{T,\vec{Q}} + \vec{Q}\cdot\left(\frac{\partial \vec{\mu}}{\partial \epsilon}\right)_{T,\vec{Q}}$$
$$= \left(\frac{\partial G}{\partial \epsilon}\right)_{T,\vec{\mu}} + T\left(\frac{\partial S}{\partial \epsilon}\right)_{T,\vec{Q}}.$$

$$\lim_{T\to 0}\left(\frac{\partial M}{\partial \epsilon}\right)_{T,\vec{Q}} = \lim_{T\to 0}\left(\frac{\partial G}{\partial \epsilon}\right)_{T,\vec{\mu}}$$

$$-T\left(\frac{\partial S}{\partial \epsilon}\right)_{M,\vec{Q}} = \left(\frac{\partial G}{\partial \epsilon}\right)_{T,\vec{\mu}}.$$

$$\Delta M_{ext}(\vec{Q}) \approx -T_0(M,\vec{Q})\Delta S(M,\vec{Q})\big|_{M\approx M^0_{ext}(\vec{Q})}$$

$$\mathcal{I}_{\text{AdS}} = \frac{1}{2}\int d^4x\sqrt{-g}(R - 2\Lambda + \mathcal{I}_{\text{interaction}})$$

$$\mathcal{I}_{\text{PM}} = \frac{1}{2}\int_M d^4x\sqrt{-g}[R - 2\Lambda - \mathcal{F}^\gamma]$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \gamma(\mathcal{F})^{\gamma-1}\left(F_{\mu\lambda}F^{\lambda}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) = 0$$
$$\frac{1}{\sqrt{-g}}\nabla_\nu\left(\sqrt{-g}(\mathcal{F})^{\gamma-1}F^{\mu\nu}\right) = 0$$

$$ds^2 = -f_{\text{PM}}(r)dt^2 + \frac{dr^2}{f_{\text{PM}}(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

$$f_{\text{PM}}(r) = 1 + \frac{r^2}{\ell^2} - \frac{2M}{r} - \frac{(2\gamma-1)^{2-2\gamma}q^{2\gamma}}{(2^{1-\gamma}(2\gamma-3)^{1-2\gamma})r^{\frac{2}{2\gamma-1}}}.$$

$$A = -qr^{(2\gamma-3)/(2\gamma-1)}dt\,;\; F = dA,$$

$$Q = 2^{\gamma-1}\frac{(3-2\gamma)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}}q^{2\gamma-1}.$$

$$T = \frac{f'_{\text{PM}}(r_+)}{4\pi} = \frac{1}{4\pi}\left[\frac{2r_+}{\ell^2} + \frac{2M}{r_+^2} + 2^\gamma\left(\frac{2}{1-2\gamma}+1\right)^{2\gamma-1}\frac{q^{2\gamma}}{r_+^{\frac{2}{2\gamma-1}+1}}\right].$$

$$M = -\frac{1}{2}r_+\left(-\frac{r_+^2}{l^2} + 2^{\gamma-1}\frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}}\frac{q^{2\gamma}}{r_+^{\frac{2}{2\gamma-1}}} - 1\right)$$

$$\Phi = \frac{q}{r_+^{(3-2\gamma)/(2\gamma-1)}}\;;\; S = \pi r_+^2$$

$$P = -\frac{1}{8\pi}\Lambda = \frac{3}{8\pi\ell^2}\ ;\ V = \frac{4\pi r_+^3}{3}$$

$$M = 2TS + \frac{\Phi Q}{\gamma(2\gamma - 1)} - VP$$

$$dM = TdS + \Phi dQ + VdP$$

$$\mathcal{I}_{\text{PYM}} = \frac{1}{2}\int\ d^4x\sqrt{-g}\left[R - 2\Lambda - \mathcal{F}^{\xi}\right]$$

$$\mathcal{F} = \text{Tr}\left(F^{(a)}_{\lambda\sigma}F^{(a)\lambda\sigma}\right)\ ;\ \text{Tr}(.) = \sum_{a=1}^{3}(.)$$

$$F^{(a)}_{\lambda\sigma} = \partial_\lambda A^{(a)}_\sigma - \partial_\sigma A^{(a)}_\lambda + \frac{1}{2\zeta}\Xi^{(a)}_{(b)(c)}A^{(b)}_\lambda A^{(c)}_\sigma$$

$$F^{(a)} = dA^{(a)} + \frac{1}{2\sigma}\Xi^{(a)}_{(b)(c)}A^{(b)} \wedge A^{(c)}.$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{1}{2}\left(\delta^\mu_\nu\mathcal{F}^{\xi} - 4\xi\text{Tr}\left[F^{(a)}_{\nu\lambda}F^{(a)\mu\lambda}\right]\mathcal{F}^{\xi-1}\right),$$

$$d\left({}^*F^{(a)}\mathcal{F}^{\xi-1}\right) + \frac{1}{\zeta}\Xi^{(a)}_{(b)(c)}\mathcal{F}^{\xi-1}A^{(b)} \wedge^* F^{(c)} = 0$$

$$f_{\text{PYM}}(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2} + \frac{(2Q^2)^{\xi}}{2(4\xi - 3)r^{4\xi-2}}.$$

$$M = \frac{1}{4}r_h\left(\frac{2r_h^2}{\ell^2} + \frac{2^{\xi}r_h^{2-4\xi}}{4\xi - 3}Q^{2\xi} + 2\right)$$

$$T = \frac{1}{4\pi}\left[\frac{2r_h}{l^2} + \frac{2M}{r_h^2} + \frac{2^{\xi}(1 - 2\xi)r_h^{1-4\xi}}{4\xi - 3}Q^{2\xi}\right].$$

$$\Phi_Q = \frac{\xi(2Q^2)^{\xi}}{2(4\xi - 3)Q}r_h^{3-4\xi}\ ;\ S = \pi r_h^2$$

$$M = 2TS + \frac{2\xi - 1}{\xi}\Phi_Q Q - 2VP$$

$$dM = TdS + \Phi_Q dQ + VdP$$

$$\mathcal{I}_{\text{MPYM}} = \frac{1}{2}\int\ d^4x\sqrt{-g}\left[(R - 2\Lambda) - F_{\mu\nu}F^{\mu\nu} - \mathcal{F}^{\xi}\right],$$

$$f_{\text{MPYM}}(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2} + \frac{Q_M}{r^2} + \frac{(2Q_{YM})^{\xi}}{2(4\xi - 3)r^{4\xi-2}},$$

$$M=\frac{1}{4r_h}\left[2Q_M^2+\frac{2r_h^4}{\ell^2}+\frac{2^\xi (Q_{YM})^{2\xi} r_h^{4-4\xi}}{4\xi-3}+2r_h^2\right].$$

$$T=\frac{1}{4\pi}\left[-\frac{2Q_M^2}{r_h^3}+\frac{2r_h}{\ell^2}+\frac{2M}{r_h^2}+\frac{2^\xi (1-2\xi)(Q_{YM})^{2\xi} r_h^{1-4\xi}}{4\xi-3}\right].$$

$$\Phi_{Q_M}=\frac{Q_M}{r_h} \text{ and } \Phi_{Q_{YM}}=\frac{\xi(2Q^2)^\xi}{2(4\xi-3)Q} r_h^{3-4\xi}.$$

$$M=2TS+\Phi_{Q_M}Q_M+\frac{2\xi-1}{\xi}\Phi_{Q_{YM}}Q_{YM}-2VP.$$

$$dM=TdS+\Phi_{Q_M}dQ_M+\Phi_{Q_{YM}}dQ_{YM}+VdP$$

$$ds^2_{\text{Perturbed}}=-\frac{r^2}{\ell^2}dt^2-\frac{r^2}{\ell^2 f_{\text{PM}}(r)}dr^2.$$

$$M_{\text{ext}}(\vec{Q},\epsilon)=\frac{\sqrt{S}}{2\sqrt{\pi}}\left(1+\frac{S(\epsilon+1)}{\pi\ell^2}-2^{\gamma-1}\pi^{\frac{1}{2\gamma-1}}\frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}}\frac{q^{2\gamma}}{S^{\frac{1}{2\gamma-1}}}\right).$$

$$T(\vec{Q},\epsilon)=\frac{\epsilon+1}{8\pi^{3/2}\ell^2\sqrt{S}}\left[\frac{\pi\ell^2}{\epsilon+1}\left(2-2^\gamma\pi^{\frac{1}{2\gamma-1}}\frac{(2\gamma-3)^{2\gamma}}{(2\gamma-1)^{2\gamma-1}}\frac{q^{2\gamma}}{S^{\frac{1}{2\gamma-1}}}\right)+6S\right].$$

$$\epsilon=-\frac{\ell^2}{2S^{3/2}}\left[\frac{2S^{3/2}}{\ell^2}-2^\gamma\pi^{\frac{1}{2\gamma-1}+1}\frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}}\frac{q^{2\gamma}}{S^{\frac{1}{2\gamma-1}-\frac{1}{2}}}+2\pi\sqrt{S}-4M\pi^{3/2}\right]$$

$$-T\left(\frac{\partial S}{\partial\epsilon}\right)_{M,Q}=\frac{S^2\left[\pi\ell^2\left(2-2^\gamma\pi^{\frac{1}{2\gamma-1}}\frac{(2\gamma-3)^{2\gamma}}{(2\gamma-1)^{2\gamma-1}}\frac{q^{2\gamma}}{S^{2\gamma-1}}\right)+6S(\epsilon+1)\right]}{8\pi^{5/2}\ell^4\left[-3\sqrt{\pi}M-2^\gamma\pi^{\frac{1}{2\gamma-1}}\left(\frac{2}{1-2\gamma}+1\right)^{2\gamma-1}\gamma\frac{q^{2\gamma}}{S^{2\gamma-1}+\frac{1}{2}}+\sqrt{S}\right]}$$

$$-T\left(\frac{\partial S}{\partial\epsilon}\right)_{M,\vec{Q}}=\left(\frac{\partial M_{\text{ext}}}{\partial\epsilon}\right)_{\vec{Q}}=\frac{r_+^3}{2\ell^2}.$$

$$\epsilon(P)=-\frac{1}{16PS^{3/2}}\left[16PS^{3/2}-32^\gamma\pi^{\frac{1}{2\gamma-1}}\frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}}\frac{q^{2\gamma}}{S^{\frac{1}{2\gamma-1}-\frac{1}{2}}}+6\sqrt{S}-12M\sqrt{\pi}\right]$$

$$-V\left(\frac{\partial P}{\partial\epsilon}\right)=\frac{64P^2S^3}{3\sqrt{\pi}\left(-12\sqrt{\pi}M-32^\gamma\pi^{\frac{1}{2\gamma-1}}\frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}}\frac{q^{2\gamma}}{S^{2\gamma-1}-\frac{1}{2}}+6\sqrt{S}\right)}$$

$$-V\left(\frac{\partial P}{\partial \epsilon}\right)=\left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}}=\frac{r_+^3}{2\ell^2}$$

$$ds^2_{\text{Perturbed}}=-\frac{r^2}{\ell^2}dt^2-\frac{r^2}{\ell^2 f_{\text{PYM}}(r)}dr^2.$$

$$M_{\text{ext}}(\vec{Q},\epsilon)=\frac{1}{4\pi^{3/2}}\left[\frac{2\sqrt{S}(\pi l^2+S\epsilon+S)}{l^2}+\frac{2^{\xi}\pi^{2\xi}(Q^2)^{\xi}S^{\frac{3}{2}-2\xi}}{4\xi-3}\right],$$

$$T(\vec{Q},\epsilon)=\frac{S^{-2\xi-\frac{1}{2}}}{8\pi^{3/2}\ell^2}\left[2(\epsilon+1)(\pi\ell^2+3S)S^{2\xi}-2^{\xi}\pi^{2\xi}\ell^2S(Q^2)^{\xi}\right].$$

$$\epsilon=\ell^2\left(\frac{2\pi^{3/2}M}{S^{3/2}}+\frac{2^{\xi-1}\pi^{2\xi}(Q^2)^{\xi}S^{-2\xi}}{3-4\xi}-\frac{\pi}{S}\right)-1.$$

$$-T\left(\frac{\partial S}{\partial \epsilon}\right)_{M,Q}=\frac{(4\xi-3)S^2\left(2(\epsilon+1)(\pi l^2+3S)S^{2\xi}-2^{\xi}\pi^{2\xi}l^2S(Q^2)^{\xi}\right)}{8\pi^{3/2}l^4\left(-3\pi^{3/2}(4\xi-3)MS^{2\xi}+2^{\xi}\pi^{2\xi}\xi S^{3/2}(Q^2)^{\xi}+\pi(4\xi-3)S^{2\xi+\frac{1}{2}}\right)}.$$

$$-T\left(\frac{\partial S}{\partial \epsilon}\right)_{M,\vec{Q}}=\left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}}=\frac{1}{2\ell^2}\left(\frac{S}{\pi}\right)^{3/2}.$$

$$\epsilon(P)=\frac{3}{16P}\left[\frac{4\sqrt{\pi}M}{S^{3/2}}+\frac{2^{\xi}\pi^{2\xi-1}(Q^2)^{\xi}S^{-2\xi}}{3-4\xi}-\frac{2}{S}\right]-1$$

$$-V\left(\frac{\partial P}{\partial \epsilon}\right)=\frac{64P^2S^{3/2}}{9\sqrt{\pi}}\left[-\frac{4\sqrt{\pi}M}{S^{3/2}}+\frac{2^{\xi}\pi^{2\xi-1}(Q^2)^{\xi}S^{-2\xi}}{4\xi-3}+\frac{2}{S}\right]^{-1}.$$

$$-V\left(\frac{\partial P}{\partial \epsilon}\right)=\left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}}=\frac{1}{2\ell^2}\left(\frac{S}{\pi}\right)^{3/2}.$$

$$-\Phi_Q\left(\frac{\partial Q}{\partial \epsilon}\right)=\left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}}=\frac{1}{2\ell^2}\left(\frac{S}{\pi}\right)^{3/2}.$$

$$ds^2_{\text{Perturbed}}=-\frac{r^2}{\ell^2}dt^2+\frac{r^2}{\ell^2 f_{\text{MPYM}}(r)}dr^2$$

$$M_{\text{ext}}(\vec{Q},\epsilon)=\frac{\sqrt{S}}{2\sqrt{\pi}}\left[\frac{1}{2\pi}S\left(\frac{2(\epsilon+1)}{\ell^2}+\frac{2^{\xi}\pi^{2\xi}S^{-2\xi}\left(Q^2_{\text{YM}}\right)^{\xi}}{4\xi-3}\right)+\frac{\pi Q_M^2}{S}+1\right]$$

$$T(\vec{Q},\epsilon)=\frac{S^{-2\xi-\frac{3}{2}}\left(2S^{2\xi}(S(\pi\ell^2+3S(\epsilon+1))-\pi^2\ell^2Q_M^2)-2^{\xi}\pi^{2\xi}\ell^2S^2\left(Q^2_{\text{YM}}\right)^{\xi}\right)}{8\pi^{3/2}\ell^2}.$$

$$\epsilon=\frac{1}{2}\ell^2\left[\frac{2^{\xi}\pi^{2\xi}S^{-2\xi}\left(Q^2_{\text{YM}}\right)^{\xi}}{3-4\xi}-\frac{2\pi\left(\pi Q_M^2-2\sqrt{\pi}M\sqrt{S}+S\right)}{S^2}\right]-1.$$

$$-T\left(\frac{\partial S}{\partial \epsilon}\right)_{M,Q}=\frac{(\epsilon+1)S^{\frac{3}{2}-2\xi}\left(\frac{l^2\left(-2\pi S^{2\xi}(\pi Q_M^2-S)-2^{\xi}\pi^{2\xi}S^2\left(Q_{\mathrm{YM}}^2\right)^{\xi}\right)}{\epsilon+1}+6S^{2\xi+2}\right)}{8\pi^{3/2}l^4\left(\pi\left(2\pi Q_M^2-3\sqrt{\pi}M\sqrt{S}+S\right)+\frac{2\xi\pi^{2\xi}\xi S^{2-2\xi}(Q_{\mathrm{YM}}^2)^{\xi}}{4\xi-3}\right)}.$$

$$-T\left(\frac{\partial S}{\partial \epsilon}\right)_{M,\vec{Q}}=\left(\frac{\partial M_{\mathrm{ext}}}{\partial \epsilon}\right)_{\vec{Q}}=\frac{1}{2\ell^2}\left(\frac{S}{\pi}\right)^{3/2}.$$

$$\epsilon(P)=\frac{3}{16\pi P}\left[\frac{2^{\xi}\pi^{2\xi}(Q_{YM}^2)^{\xi}S^{-2\xi}}{3-4\xi}-\frac{2\pi\left(\pi Q_M^2-2\sqrt{\pi}M\sqrt{S}+S\right)}{S^2}\right]-1.$$

$$-V\left(\frac{\partial P}{\partial \epsilon}\right)=-\frac{64\sqrt{\pi}P^2S^{3/2}}{9\left(\frac{2^{\xi}\pi^{2\xi}S^{-2\xi}(Q_{\mathrm{YM}}^2)^{\xi}}{3-4\xi}-\frac{2\pi\left(\pi Q_M^2-2\sqrt{\pi}M\sqrt{S}+S\right)}{S^2}\right)}.$$

$$-V\left(\frac{\partial P}{\partial \epsilon}\right)=\left(\frac{\partial M_{\mathrm{ext}}}{\partial \epsilon}\right)_{\vec{Q}}=\frac{1}{2\ell^2}\left(\frac{S}{\pi}\right)^{3/2}.$$

$$-\Phi_{Q_M}\left(\frac{\partial Q_M}{\partial \epsilon}\right)=\left(\frac{\partial M_{\mathrm{ext}}}{\partial \epsilon}\right)_{\vec{Q}}=\frac{1}{2\ell^2}\left(\frac{S}{\pi}\right)^{3/2}$$

$$-\Phi_{Q_{YM}}\left(\frac{\partial Q_{YM}}{\partial \epsilon}\right)=\left(\frac{\partial M_{\mathrm{ext}}}{\partial \epsilon}\right)_{\vec{Q}}=\frac{1}{2\ell^2}\left(\frac{S}{\pi}\right)^{3/2}$$

$$dM=TdS+\vec{\mu}\cdot d\vec{Q}$$

$$L_0-\frac{k_R}{4}=\frac{\epsilon+j}{2},\tilde{L}_0-\frac{k_L}{4}=\frac{\epsilon-j}{2}.$$

$$6k=c=\frac{3R}{2G_3}.$$

$$T=\frac{\hbar c^3}{8\pi G_N k_B M},$$

$$S=2\pi\sqrt{\frac{1}{2}k_R(E+J)-\frac{1}{4}Q_R^2}+2\pi\sqrt{\frac{1}{2}k_L(E-J)-\frac{1}{4}Q_L^2}.$$

$$TdS=dE-\mu dJ-\omega_L dQ_L-\omega_R dQ_R$$

$$\beta=\frac{\partial S}{\partial E}=\frac{1}{2}\sqrt{\frac{2\pi^2 k_L}{E-J-\frac{Q_L^2}{2k_L}}}+\frac{1}{2}\sqrt{\frac{2\pi^2 k_R}{E+J-\frac{Q_R^2}{2k_R}}}$$

$$\mu=-\frac{1}{\beta}\frac{\partial S}{\partial J}=\frac{-\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}+\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}}{\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}+\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}}$$

$$\omega_R=-\frac{1}{\beta}\frac{\partial S}{\partial Q_R}=\frac{\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}}{\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}+\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}}\cdot\frac{Q_R}{k_R}$$

$$\omega_L=-\frac{1}{\beta}\frac{\partial S}{\partial Q_L}=\frac{\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}}{\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}+\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}}\cdot\frac{Q_L}{k_L}. \quad (2.5)$$

$$E_R\equiv E+J=\frac{2k_R}{\beta^2(1-\mu)^2}(\pi^2+\beta^2\omega_R^2)$$

$$E_L\equiv E-J=\frac{2k_L}{\beta^2(1+\mu)^2}(\pi^2+\beta^2\omega_L^2)$$

$$Q_R=\frac{2k_R}{1-\mu}\omega_R$$

$$Q_L=\frac{2k_L}{1+\mu}\omega_L$$

$$Z(\beta,\mu,\omega_R,\omega_L)=\mathrm{Tr}\left[e^{-\beta(E-\mu J-\omega_R Q_R-\omega_L Q_L)}\right]$$

$$\log Z=S-\beta(E-\mu J-\omega_R Q_R-\omega_L Q_L)$$
$$=\frac{k_R}{\beta(1-\mu)}(\pi^2+\beta^2\omega_R^2)+\frac{k_L}{\beta(1+\mu)}(\pi^2+\beta^2\omega_L^2).$$

$$E-\mu J-\omega_R Q_R-\omega_L Q_L=-\frac{\partial\log Z}{\partial\beta}$$
$$=\frac{k_R}{\beta^2(1-\mu)}(\pi^2-\beta^2\omega_R^2)+\frac{k_L}{\beta^2(1+\mu)}(\pi^2-\beta^2\omega_L^2)$$

$$J=\frac{1}{\beta}\frac{\partial\log Z}{\partial\mu}$$
$$=\frac{k_R}{\beta^2(1-\mu)^2}(\pi^2+\beta^2\omega_R^2)-\frac{k_L}{\beta^2(1+\mu)^2}(\pi^2+\beta^2\omega_L^2)$$

$$Q_{L,R}=\frac{1}{\beta}\frac{\partial\log Z}{\partial\omega_{L,R}}=\frac{2k_{L,R}}{1\pm\mu}\omega_{L,R}$$

$$E - J - \frac{1}{2k_L} Q_L^2 \geq 0$$
$$E + J - \frac{1}{2k_R} Q_R^2 \geq 0$$

$$\epsilon - j + \frac{1}{2} k_L \geq q_L$$

$$E - J + \frac{1}{2} k_L \geq Q_L$$

$$Q_L = k_L$$

$$E = \frac{k_R}{\beta^2(1-\mu)^2}(\pi^2 + \beta^2\omega_R^2) + \frac{k_L}{\beta^2(1+\mu)^2}(\pi^2 + \beta^2\omega_L^2),$$
$$J = \frac{k_R}{\beta^2(1-\mu)^2}(\pi^2 + \beta^2\omega_R^2) - \frac{k_L}{\beta^2(1+\mu)^2}(\pi^2 + \beta^2\omega_L^2),$$
$$Q_{L,R} = \frac{2k_{L,R}}{1 \pm \mu}\omega_{L,R}$$

$$E + J - \frac{Q_R^2}{2k_R} = \frac{2k_R\pi^2}{\beta^2(1-\mu)^2} \geq 0$$
$$E - J - \frac{Q_L^2}{2k_L} = \frac{2k_L\pi^2}{\beta^2(1+\mu)^2} \geq 0$$

Extremal limit: $\begin{cases} \beta \to \infty, & \\ \mu \to 1 & \text{with } \tilde{\mu} \equiv \beta(\mu - 1) \text{ finite}, \\ \omega_R \to 0 & \text{with } \tilde{\omega}_R \equiv \beta\omega_R \text{ finite}, \\ \omega_L \text{ finite.} & \end{cases}$

$$E = \frac{k_R}{\tilde{\mu}^2}(\pi^2 + \tilde{\omega}_R^2) + \frac{k_L}{4}\omega_L^2,$$
$$J = \frac{k_R}{\tilde{\mu}^2}(\pi^2 + \tilde{\omega}_R^2) - \frac{k_L}{4}\omega_L^2,$$
$$Q_R = -\frac{2k_R}{\tilde{\mu}}\tilde{\omega}_R,$$
$$Q_L = k_L\omega_L.$$

$$E_{\text{ext}} = J + \frac{1}{2k_L} Q_L^2$$

$$S_{\text{ext}} = -\frac{2k_R\pi^2}{\tilde{\mu}} = 2\pi\sqrt{\frac{1}{2}k_R(E_{\text{ext}} + J) - \frac{1}{4}Q_R^2}$$
$$= 2\pi\sqrt{k_R J + \frac{k_R}{4k_L}Q_L^2 - \frac{1}{4}Q_R^2}$$

$$E - J - Q_L + \frac{1}{2}k_L \geq 0$$

$$E - J - Q_L + \frac{1}{2}k_L = \frac{2k_L\pi^2}{\beta^2(1+\mu)^2} + \frac{k_L}{2}\left(1 - \frac{2\omega_L}{1+\mu}\right)^2$$

$$\varphi \equiv 1 + \mu - 2\omega_L = 0$$

$$T = \beta^{-1} \to 0$$

$$\tilde{\mu} = \beta(\mu - 1), \tilde{\omega}_R = \beta\omega_R, \tilde{\omega}_L = \beta(\omega_L - 1)$$

$$\begin{aligned} Z(\beta, \mu, \omega_R, \omega_L) &= \mathrm{Tr}\left[e^{-\beta(E - \mu J - \omega_R Q_R - \omega_L Q_L)}\right] \\ &= e^{\frac{1}{2}\beta k_L}\mathrm{Tr}\left[e^{-\beta\left(E - J - Q_L + \frac{k_L}{2}\right) + \tilde{\mu}J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L}\right] \end{aligned}$$

$$E_{\mathrm{SUSY}} = -\frac{1}{2}k_L,$$

$$\begin{aligned} Z_{\mathrm{BPS}}(\beta, \tilde{\mu}, \tilde{\omega}_R, \tilde{\omega}_L) &= \lim_{\beta\to\infty} e^{\frac{1}{2}\beta k_L}\mathrm{Tr}\left[e^{-\beta\left(E - J - Q_L + \frac{k_L}{2}\right) + \tilde{\mu}J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L}\right] \\ &= \left(\lim_{\beta\to\infty} e^{\frac{1}{2}\beta k_L}\right)\mathrm{Tr}_{\mathrm{BPS}}\left[e^{\tilde{\mu}J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L}\right] \end{aligned}$$

$$\log\, Z_{\mathrm{BPS}} = \frac{1}{2}\beta k_L - \frac{k_R}{\tilde{\mu}}(\pi^2 + \tilde{\omega}_R^2) + k_L\left(\tilde{\omega}_L - \frac{1}{4}\tilde{\mu}\right).$$

$$\begin{aligned} E &= \frac{k_R}{\tilde{\mu}^2}(\pi^2 + \tilde{\omega}_R^2) + \frac{k_L}{4} \\ J &= \frac{k_R}{\tilde{\mu}^2}(\pi^2 + \tilde{\omega}_R^2) - \frac{k_L}{4} \\ Q_R &= -\frac{2k_R}{\tilde{\mu}}\tilde{\omega}_R \end{aligned}$$

$$Q_L = k_L$$

$$S_{\mathrm{BPS}} = 2\pi\sqrt{k_R\left(J + \frac{1}{4}k_L\right) - \frac{1}{4}Q_R^2}$$

$$TdS = dE - \mu dJ - \omega_R dQ_R - \omega_L dQ_L$$

$$TdS = -\epsilon + \mu j + \omega_R q_R + \omega_L q_L$$

$$\omega_R \leq 1, \omega_L \leq 1$$

$$0 \geq \omega_R - \frac{1-\mu}{2} \Leftrightarrow Q_R \leq k_R$$

$$0 \geq \omega_L - \frac{1+\mu}{2} \Leftrightarrow Q_L \leq k_L$$

$$E - J + \frac{1}{2}k_L \geq Q_L, E + J + \frac{1}{2}k_R \geq Q_R$$

$$S_{\mathrm{bh}} = 2\pi\sqrt{\frac{1}{2}k_R E_{R,\mathrm{bh}} - \frac{1}{4}Q_{R,\mathrm{bh}}^2} + 2\pi\sqrt{\frac{1}{2}k_L E_{L,\mathrm{bh}} - \frac{1}{4}Q_{L,\mathrm{bh}}^2}.$$

$$\begin{aligned} S_{R,\mathrm{bh}} &= 2\pi\sqrt{\frac{1}{2}k_R E_{R,\mathrm{bh}} - \frac{1}{4}Q_{R,\mathrm{bh}}^2} \\ &= 2\pi\sqrt{\frac{1}{2}k_R\left(E_{R,\mathrm{tot}} - E_{R,\mathrm{gp}}\right) - \frac{1}{4}\left(Q_{R,\mathrm{tot}} - E_{R,\mathrm{gp}}\right)^2} \\ &= \pi\sqrt{2k_L\left(E_{R,\mathrm{tot}} - Q_{R,\mathrm{tot}} + \frac{k_R}{2}\right) - \left(Q_{R,\mathrm{tot}} - k_R - E_{R,\mathrm{gp}}\right)^2} \end{aligned}$$

$$E_{R,\mathrm{gp}} = \begin{cases} Q_{R,\mathrm{tot}} - k_R \Rightarrow S_{R,\mathrm{bh}} = \begin{cases} 2\pi\sqrt{\frac{1}{2}k_L\left(E_{R,\mathrm{tot}} - Q_{R,\mathrm{tot}} + \frac{k_R}{2}\right)} & \left(Q_{R,\mathrm{tot}} > k_R\right) \\ 2\pi\sqrt{\frac{1}{2}k_R E_{R,\mathrm{tot}} - \frac{1}{4}Q_{R,\mathrm{tot}}^2} & \left(Q_{R,\mathrm{tot}} \le k_R\right) \end{cases} \\ 0 \end{cases}$$

$$E = \frac{\pi}{4G_5}\frac{m\left(2(1-a^2)+2(1-b^2)-(1-a^2)(1-b^2)\right)+2qab\left((1-a^2)+(1-b^2)\right)}{(1-a^2)^2(1-b^2)^2}$$

$$Q = \frac{\pi}{4G_5}\frac{q}{(1-a^2)(1-b^2)}$$

$$J_1 = \frac{\pi}{4G_5}\frac{2ma + qb(1+a^2g^2)}{(1-a^2)^2(1-b^2)}$$

$$J_2 = \frac{\pi}{4G_5}\frac{2mb + qa(1+b^2g^2)}{(1-a^2)(1-b^2)^2}$$

$$2m = \frac{(r_+^2+a^2)(r_+^2+b^2)(1+g^2r_+^2)+q^2+2abq}{r_+^2}.$$

$$S = 2\pi\cdot\frac{\pi}{4G_5}\frac{(r_+^2+a^2)(r_+^2+b^2)+abq}{(1-a^2)(1-b^2)r_+}.$$

$$TdS = dE - \Phi dQ - \Omega_1 dJ_1 - \Omega_2 dJ_2.$$

$$T = \left(\frac{\partial S}{\partial E}\right)^{-1} = \frac{r_+^4[1+(2r_+^2+a^2+b^2)]-(ab+q)^2}{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]}$$

$$E - (3Q + J_1 + J_2) \ge 0$$

$$E^* - 3Q^* - J_1^* - J_2^* = 0$$

$$E - (3Q + J_1 + J_2) = \frac{\pi}{4G_5}\frac{3+(a+b)-ab}{(1-a)(1+a)^2(1-b)(1+b)^2}[m - q(1+a+b)],$$

$$q^* = \frac{m^*}{1+a+b}$$

$$m - q(1+a+b) = \frac{r_+^2(q-q^*)^2 + \left(((1+a+b)^2 + r_+^2)(r_+^2 - r^{*2}) - (1+a+b)(q-q^*)\right)^2}{2r_+^2((1+a+b)^2 + r_+^2)}$$

$$q^* = (a+b)(1+a)(1+b), r^* \equiv r_+ = \sqrt{a+b+ab}.$$

$$\begin{aligned} E^* &= \frac{\pi}{4G_5}\frac{(3(a+b) - (a^3+b^3) - ab(a+b)^2)}{(1-a)^2(1-b)^2}, \\ Q^* &= \frac{\pi}{4G_5}\frac{a+b}{(1-a)(1-b)}, \\ J_1^* &= \frac{\pi}{4G_5}\frac{(a+b)(2a+b+ab)}{(1-a)^2(1-b)}, \\ J_2^* &= \frac{\pi}{4G_5}\frac{(a+b)(a+2b+ab)}{(1-a)(1-b)^2}. \end{aligned}$$

$$T = \frac{[1+3(a+b)+(a^2+b^2+3ab)](r_+^2 - r^{*2}) - (1+a+b)(q-q^*)}{\pi r^* q^*}.$$

$$\begin{aligned} &\left(Q_1Q_2Q_3 + \frac{\pi}{4G_5}J_1J_2\right) \\ =&\left(Q_1+Q_2+Q_3+\frac{\pi}{4G_5}\right)\left(Q_1Q_2+Q_2Q_3+Q_3Q_1 - \frac{\pi}{4G_5}(J_1+J_2)\right). \end{aligned}$$

$$S^* = 2\pi \cdot \frac{\pi}{4G_5}\frac{a+b}{(1-a)(1-b)}\sqrt{a+b+ab}$$

$$S^* = 2\pi\sqrt{Q_1^*Q_2^* + Q_2^*Q_3^* + Q_3^*Q_1^* - \frac{\pi}{4G_5}(J_1^* + J_2^*)}.$$

$$\begin{aligned} J_1 &= \frac{j+\bar{J}}{2}, & J_2 &= \frac{j-\bar{J}}{2}, \\ Q_1 &= R_2 + \frac{R_1+R_3}{2}, & Q_2 &= \frac{R_1+R_3}{2}, & Q_3 &= \frac{R_1-R_3}{2}. \end{aligned}$$

$$Z(\beta, \Delta_I, \omega_i) \equiv \mathrm{Tr}\left[e^{-\beta E}e^{\Delta_I Q_I + \omega_i J_i}\right]$$

$$2\{Q, Q^\dagger\} = E - (Q_1 + Q_2 + Q_3 + J_1 + J_2)$$

$$E \geq Q_1 + Q_2 + Q_3 + J_1 + J_2$$

$$E = Q_1 + Q_2 + Q_3 + J_1 + J_2$$

$$Z(\beta, \Delta_I, \omega_i) = \mathrm{Tr}\left[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)}e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}\right]$$

$$\begin{aligned} Z_{\mathrm{BPS}}(\tilde{\Delta}_I, \tilde{\omega}_i) &= \lim_{\beta\to\infty}\mathrm{Tr}\left[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)}e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}\right] \\ &= \mathrm{Tr}_{\mathrm{BPS}}\left[e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}\right] \end{aligned}$$

$$e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}} = -1$$

$$\mathcal{I}\left(\tilde{\Delta}_I,\tilde{\omega}_i\right)= \mathrm{Tr}\left[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)}e^{\tilde{\Delta}_I Q_I+\tilde{\omega}_i J_i}\right]$$
$$= \mathrm{Tr}_{\mathrm{BPS}}\left[e^{\tilde{\Delta}_I Q_I+\tilde{\omega}_i J_i}\right],\ \text{where } e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}}=-1.$$

$$\mathcal{I}\left(\tilde{\Delta}_I,\tilde{\omega}_i\right)= \mathrm{Tr}\left[(-1)^F e^{\tilde{\Delta}_I Q_I+\tilde{\omega}_i J_i}\right],\ \text{where } e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}}=1$$

$$A_\mu \sim A_{\alpha\dot{\beta}}, \qquad (\mu=1,2,3,4, \alpha=\pm, \dot{\beta}=\dot{\pm})$$
$$\Phi_{ij}\left(=-\Phi_{ji}\right), \bar{\Phi}^{ij} \sim \frac{1}{2}\epsilon^{ijkl}\Phi_{kl}, \quad (i,j,k,l=1,2,3,4)$$
$$: \Psi_{i\alpha}, \bar{\Psi}^i_{\dot{\alpha}}$$

$$\partial_{+\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}=0 \Leftrightarrow \partial_{+[\dot{\alpha}}\bar{\lambda}_{\dot{\beta}]}=0$$

$$J_1=\frac{j+\bar{j}}{2}, \quad J_2=\frac{j-\bar{j}}{2},$$
$$Q_1=R_2+\frac{R_1+R_3}{2}, \quad Q_2=\frac{R_1+R_3}{2}, \quad Q_3=\frac{R_1-R_3}{2},$$

$$x_B \equiv e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}}$$

$$1+x_B+x_B^2+\cdots=\frac{1}{1-x_B}$$
$$=\exp\left[-\log\left(1-x_B\right)\right]$$
$$=\exp\left[\sum_{n=1}^{\infty}\frac{x_B^n}{n}\right]$$
$$\equiv \mathrm{PE}[x_B]$$

$$1-x_F=\exp\left[\log\left(1-x_F\right)\right]$$
$$=\exp\left[-\sum_{n=1}^{\infty}\frac{x_F^n}{n}\right]$$
$$\equiv \mathrm{PE}[-x_F]$$

$$\mathcal{I}_{\mathrm{Fock}}=\mathrm{PE}\left[\sum_{\text{dark particle letters}} x_B-\sum_{\text{white particle letters}} x_F\right]$$

$$x^2=e^{\tilde{\Delta}_1}, y^2=e^{\tilde{\Delta}_2}, z^2=e^{\tilde{\Delta}_3}, p^2=e^{\tilde{\omega}_1}, q^2=e^{\tilde{\omega}_2}$$

$$\mathcal{I}_{\mathrm{Fock}}\left(x,y,z,p,q;\zeta_a\right)=\mathrm{Tr}_{\mathrm{Fock}}\left[(-1)^F x^{2Q_1}y^{2Q_2}z^{2Q_3}p^{2J_1}q^{2J_2}\prod_a e^{i\alpha_a\zeta_a}\right]$$

$$\mathcal{I}_{\mathrm{Fock}}\left(x,y,z,p,q;\zeta_a\right)=\mathrm{PE}\left[f(x,y,z,p,q)\cdot\chi^{\mathrm{adj.}}\left(\alpha_a\right)\right]$$

$$f(x,y,z,p,q)=\frac{x^2+y^2+z^2-xyzpq\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{p^2}+\frac{1}{q^2}-1\right)+p^2q^2}{(1-p^2)(1-q^2)}$$
$$=1-\frac{(1-x^2)(1-y^2)(1-z^2)}{(1-p^2)(1-q^2)}$$

$$\chi^{adj}\cdot(\alpha_a)=\sum_{a,b=1}^{N}e^{i(\alpha_a-\alpha_b)}$$

$$\mathcal{I}(x,y,z,p,q)=\oint d\mu[\alpha_a]\mathcal{I}_{\text{Fock}}\,(x,y,z,p,q;\alpha_a)$$

$$\oint d\mu[\alpha_a]=\frac{1}{N!}\cdot\int_0^{2\pi}\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\cdot\prod_{a,b=1}^{N}\left(1-e^{i(\alpha_a-\alpha_b)}\right)$$
$$=\frac{1}{N!}\cdot\int_0^{2\pi}\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\cdot\text{PE}\left[-\sum_{a,b=1}^{N}e^{i(\alpha_a-\alpha_b)}\right]$$

$$\mathcal{I}(x,y,z,p,q)\;=\frac{1}{N!}\int_0^{2\pi}\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\cdot\text{PE}\left[-(1-f(x,y,z,p,q))\cdot\sum_{a,b=1}^{N}e^{i(\alpha_a-\alpha_b)}\right]$$
$$=\frac{1}{N!}\int_0^{2\pi}\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\cdot\exp\left[-\sum_{n=1}^{\infty}\sum_{a,b=1}^{N}\frac{1}{n}\left(1-f(x^n,y^n,z^n,p^n,q^n)\right)\cdot e^{in(\alpha_a-\alpha_b)}\right]$$

$$\int_0^{2\pi}d\theta\rho(\alpha)=1$$

$$\mathcal{I}(x,y,z,p,q)=\frac{1}{N!}\int\;[d\rho]e^{-S[\rho(\alpha)]},$$

$$S[\rho(\alpha)]=N^2\sum_{n=1}^{\infty}\int_0^{2\pi}d\alpha_1d\alpha_2\rho(\alpha_1)\rho(\alpha_2)e^{in(\alpha_1-\alpha_2)}\cdot\frac{1}{n}\left(1-f(x^n,y^n,z^n,p^n,q^n)\right)$$
$$=N^2\sum_{n=1}^{\infty}\frac{1}{n}\left(1-f(x^n,y^n,z^n,p^n,q^n)\right)\cdot|\rho_n|^2$$

$$\rho_n\equiv\int_0^{2\pi}d\alpha\rho(\alpha)e^{in\alpha}$$

$$1-f(x^n,y^n,z^n,p^n,q^n)=\frac{(1-x^{2n})(1-y^{2n})(1-z^{2n})}{(1-p^{2n})(1-q^{2n})}>0,$$

$$\mathcal{I}(x,y,z,p,q)=e^{\mathcal{O}(N^0)}$$

$$\frac{\pi}{4G_5}=\frac{N^2}{2}$$

$$e^{\frac{\bar{\Delta}_1+\bar{\Delta}_2+\bar{\Delta}_3-\bar{\omega}_1-\bar{\omega}_2}{2}}=-1$$

$$\mathcal{I}(x)=\text{Tr}[(-1)^Fx^{2Q_1+2Q_2+2Q_3+3J_1+3J_2}]\equiv\text{Tr}[(-1)^Fx^{\mathcal{J}}]$$

$$\mathcal{J}\equiv2Q_1+2Q_2+2Q_3+3J_1+3J_2$$

$$x^2=e^{\Delta_1}=e^{\Delta_2}=e^{\Delta_3}$$

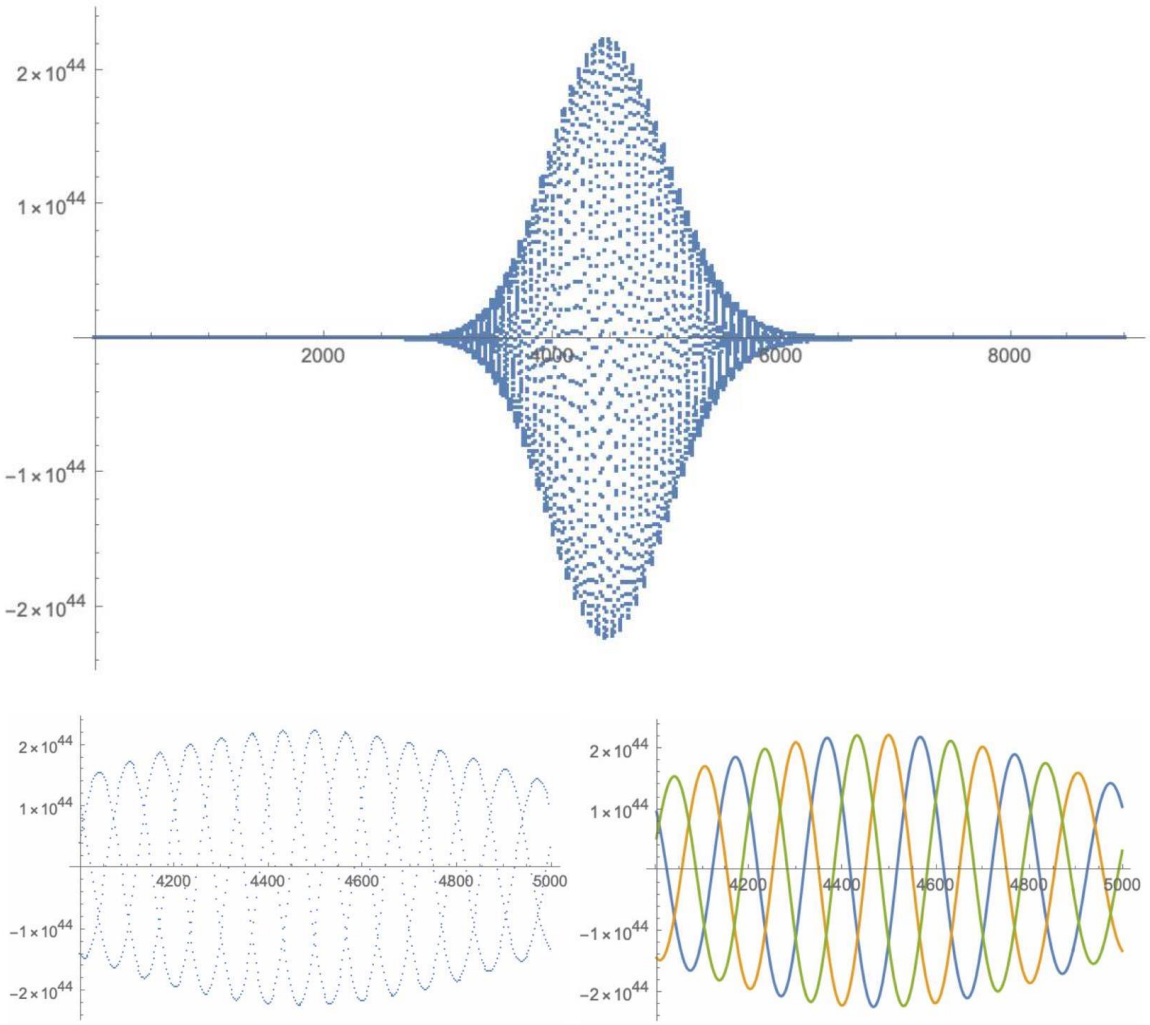
$$x^3 = e^{\omega_1} = e^{\omega_2}$$

$$e^{\frac{\bar{\Delta}_1+\bar{\Delta}_2+\bar{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}} = 1$$

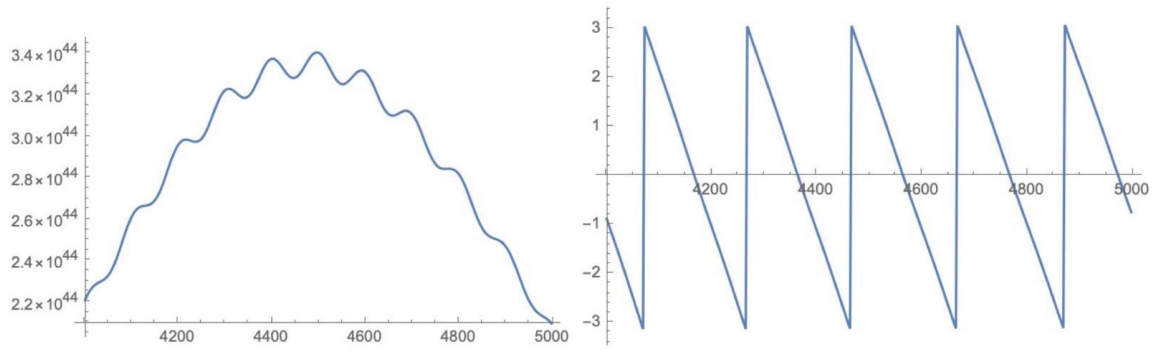
$$\mathcal{J}(x) = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\left(1-f(x^n)\right)\right],$$

$$f(x)=\frac{3x^2-2x^3-3x^4+2x^6}{(1-x^3)^2}$$
$$=1-\frac{(1-x^2)^3}{(1-x^3)^2}$$

$$\mathcal{J}(x) = \sum_{j=0}^{\infty}\Omega_{\mathcal{J}}x^{\mathcal{J}}$$



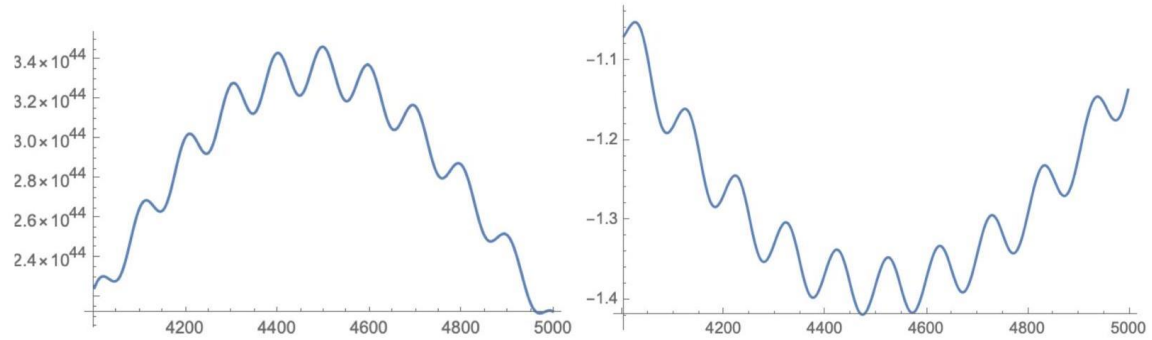
$$\Omega_{\mathcal{J}}x^{\mathcal{J}} + \Omega_{\mathcal{J}+1}x^{\mathcal{J}+1} + \Omega_{\mathcal{J}+2}x^{\mathcal{J}+2}$$



$$\Omega_{\mathcal{J}} x^{\mathcal{J}} + \Omega_{\mathcal{J}+1} x^{\mathcal{J}+1} + \Omega_{\mathcal{J}+2} x^{\mathcal{J}+2}$$

$$\Omega_{\mathcal{J}} x^{\mathcal{J}} + \Omega_{\mathcal{J}+1} x^{\mathcal{J}+1} + \Omega_{\mathcal{J}+2} x^{\mathcal{J}+2}, \text{ where } x = 0.95 \cdot \exp\left(\frac{2\pi i}{3}\right),$$

$$x = 0.95 \cdot \exp\, i\left(\frac{2\pi}{3} + 0.0313458\right).$$



$$\Omega_{\mathcal{J}} x^{\mathcal{J}} + \Omega_{\mathcal{J}+1} x^{\mathcal{J}+1} + \Omega_{\mathcal{J}+2} x^{\mathcal{J}+2} \text{ where } x = 0.95 \cdot e^{i\left(\frac{2\pi}{3}+0.0313458\right)}$$

$$\mathcal{J}\left(0.95 \cdot e^{i\left(\frac{2\pi}{3}+0.0313458\right)}\right) = 1.33 \times 10^{47}$$

$$\mathcal{J}(0.95) = 2.1 \times 10^{31}.$$

$$\log \mathcal{J} = \frac{8(5a-3c)}{27\omega^2}(-\pi i + \omega)^3 + \frac{8\pi^2(a-c)}{3\omega^2}(-\pi i + \omega),$$

$$\mathrm{Re}\omega = \sqrt{3} \cdot \mathrm{Im}\omega \;\leftrightarrow\; -\log\,|x| = \sqrt{3} \cdot \left(\arg\,(x) - \frac{2\pi}{3}\right)$$

$$\mathrm{Re}(\log\,\mathcal{J}) = \frac{2\pi^2(3c-2a)}{9\sqrt{3}(\mathrm{Im}\omega)^2}$$

$$\begin{aligned} Z(\beta,\mu,\omega_R,\omega_L) &= \mathrm{Tr}\left[e^{-\beta(E-\mu J-\omega_R Q_R-\omega_L Q_L)}\right] \\ &= e^{\frac{1}{2}\beta k_L}\mathrm{Tr}\left[e^{-\beta\left(E-J-Q_L+\frac{k_L}{2}\right)+\tilde{\mu}J+\tilde{\omega}_R Q_R+\tilde{\omega}_L Q_L}\right] \end{aligned}$$

$$Z_{\mathrm{BPS}}(\beta,\tilde{\mu},\tilde{\omega}_R,\tilde{\omega}_L) = \left(\lim_{\beta\to\infty} e^{\frac{1}{2}\beta k_L}\right)\mathrm{Tr}_{\mathrm{BPS}}\left[e^{\tilde{\mu}J+\tilde{\omega}_R Q_R+\tilde{\omega}_L Q_L}\right]$$

$$\tilde{\mu} - 2\tilde{\omega}_L = 2\pi i$$

$$\begin{aligned}\mathcal{I} &\equiv e^{\beta E_{\text{SUSY}}} Z\big|_{\tilde{\omega}_L=\frac{\tilde{\mu}}{2}-i\pi} \\ &= \text{Tr}_{\text{BPS}}\left[e^{\tilde{\mu}J+\tilde{\omega}_R Q_R+\tilde{\omega}_L Q_L}\right]\Big|_{\tilde{\omega}_L=\frac{\tilde{\mu}}{2}-i\pi},\end{aligned}$$

$$e^{-\beta E_{\text{SUSY}}}\big|_{\beta\to\infty}$$

$$\begin{aligned}\log \mathcal{I} &= -\frac{k_L}{2}\beta + \frac{k_R}{\beta(1-\mu)}(\pi^2+\beta^2\omega_R^2) + \frac{k_L}{\beta(1+\mu)}(\pi^2+\beta^2\omega_L^2) \\ &= -\frac{k_L}{2}\beta - \frac{k_R}{\tilde{\mu}}(\pi^2+\tilde{\omega}_R^2) + \frac{k_L}{\tilde{\mu}+2\beta}(\pi^2+(\tilde{\omega}_L+\beta)^2) \\ &= -\frac{k_R}{\tilde{\mu}}(\pi^2+\tilde{\omega}_R^2) + \frac{k_L}{4}(\tilde{\mu}-4\pi i) \\ &= -\frac{k_R}{\tilde{\mu}}(\pi^2+\tilde{\omega}_R^2) + \frac{k_L}{\tilde{\mu}}(\pi^2+\tilde{\omega}_L^2)\end{aligned}$$

$$\log \mathcal{I} = k\frac{\tilde{\omega}_1\tilde{\omega}_2}{\tilde{\mu}},$$

$$\tilde{\omega}_{L,R} = \frac{1}{2}(\tilde{\omega}_1 \pm \tilde{\omega}_2)$$

$$S(\tilde{\mu},\tilde{\omega}_R,\tilde{\omega}_L) \equiv \log \mathcal{I} - \tilde{\omega}_L Q_L - \tilde{\omega}_R Q_R - \tilde{\mu}J,$$

$$\begin{aligned}S &= S(\tilde{\mu},\tilde{\omega}_R,\tilde{\omega}_L) - \Lambda(\tilde{\mu}-2\tilde{\omega}_L-2\pi i) \\ &= \frac{k_L(\tilde{\omega}_L^2+\pi^2)-k_R(\tilde{\omega}_R^2+\pi^2)}{\tilde{\mu}} - \tilde{\omega}_L Q_L - \tilde{\omega}_R Q_R - \tilde{\mu}J - \Lambda(\tilde{\mu}-2\tilde{\omega}_L-2\pi i)\end{aligned}$$

$$0 = \left(\tilde{\omega}_L\partial_{\tilde{\omega}_L} + \tilde{\omega}_R\partial_{\tilde{\omega}_R} + \tilde{\mu}\partial_{\tilde{\mu}}\right)S = S - 2\pi i\Lambda + \frac{2\pi^2(k_R-k_L)}{\tilde{\mu}},$$

$$S = 2\pi i\Lambda - \frac{2\pi^2(k_R-k_L)}{\tilde{\mu}}.$$

$$\begin{aligned}\partial_{\tilde{\omega}_L} S &= k_L\frac{2\tilde{\omega}_L}{\tilde{\mu}} + (2\Lambda - Q_L) = 0, \\ \partial_{\tilde{\omega}_R} S &= -k_R\frac{2\tilde{\omega}_R}{\tilde{\mu}} - Q_R = 0, \\ \partial_{\tilde{\mu}} S &= -\frac{k_L(\tilde{\omega}_L^2+\pi^2)-k_R(\tilde{\omega}_R^2+\pi^2)}{\tilde{\mu}^2} - (\Lambda + J) = 0.\end{aligned}$$

$$k_L\frac{\tilde{\mu}-2\pi i}{\tilde{\mu}} = Q_L - 2\Lambda \Rightarrow \frac{\pi i k_L}{\tilde{\mu}} = \Lambda - \frac{1}{2}(Q_L - k_L).$$

$$S = 2\pi i\left[\Lambda + \frac{i\pi}{\tilde{\mu}}(k_R-k_L)\right] = 2\pi i\left[\frac{k_R}{k_L}\Lambda - \frac{1}{2k_L}(k_R-k_L)(J_L-k_L)\right] \equiv 2\pi i\Lambda_{\text{eff}}$$

$$\Lambda_{\text{eff}} = \frac{k_R}{k_L}\Lambda - \frac{1}{2k_L}(k_R-k_L)(Q_L-k_L).$$

$$-\frac{1}{k_L}\left(\Lambda-\frac{1}{2}Q_L\right)^2+\frac{1}{4k_R}Q_R^2-(\Lambda+J)-\frac{1}{k_L^2}(k_R-k_L)\left(\Lambda-\frac{1}{2}(Q_L-k_L)\right)^2=0,$$

$$\Lambda_{\text{eff}}^2-(Q_L-k_L)\Lambda_{\text{eff}}+\frac{1}{4}(Q_L-k_L)^2+k_R\left(J+\frac{Q_L}{2}-\frac{k_L}{4}\right)-\frac{1}{4}Q_R^2=0.$$

$$S=2\pi i\Lambda_{\text{eff}}=2\pi\sqrt{k_R\left(J+\frac{Q_L}{2}-\frac{k_L}{4}\right)-\frac{Q_R^2}{4}}+\pi i(Q_L-k_L).$$

$$Q_L=k_L,$$

$$S_{\text{BPS}}=2\pi\sqrt{k_R\left(J+\frac{1}{4}k_L\right)-\frac{1}{4}Q_R^2},$$

$$\mathcal{I}(\tilde{\Delta}_I,\tilde{\omega}_i)=\text{Tr}\left[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)}e^{\tilde{\Delta}_IQ_I+\tilde{\omega}_iJ_i}\right]$$
$$=\text{Tr}_{\text{BPS}}\left[e^{\tilde{\Delta}_IQ_I+\tilde{\omega}_iJ_i}\right](3.61)$$

$$\tilde{\Delta}_I\equiv\Delta_I-\beta\otimes\tilde{\omega}_i\equiv\omega_i-\beta\iiint e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}}=-1$$

$$\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2=2\pi i$$

$$\log\mathcal{I}=-\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}$$

$$S(\tilde{\Delta}_I,\tilde{\omega}_i)\equiv\log\mathcal{I}-\sum_I\tilde{\Delta}_IQ_I-\sum_i\tilde{\omega}_iJ_i$$

$$S=S(\tilde{\Delta}_I,\tilde{\omega}_i)-\Lambda\left(\sum_I\tilde{\Delta}_I-\sum_i\tilde{\omega}_i-2\pi i\right)$$
$$=-\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}-\sum_I\tilde{\Delta}_IQ_I-\sum_i\tilde{\omega}_iJ_i-\Lambda\left(\sum_I\tilde{\Delta}_I-\sum_i\tilde{\omega}_i-2\pi i\right)$$

$$0=\left(\sum_I\tilde{\Delta}_I\partial_{\tilde{\Delta}_I}+\sum_i\tilde{\omega}_i\partial_{\tilde{\omega}_i}\right)S=S-2\pi i\Lambda$$

$$S=2\pi i\Lambda$$

$$-\tilde{\Delta}_I\partial_{\tilde{\Delta}_I}S=\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}+\tilde{\Delta}_I(Q_I+\Lambda)=0$$
$$\tilde{\omega}_i\partial_{\tilde{\omega}_i}S=\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}+\tilde{\omega}_i(-J_i+\Lambda)=0$$

$$\tilde{\Delta}_I = -\frac{1}{Q_I+\Lambda}\cdot\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}$$
$$\tilde{\omega}_i = -\frac{1}{-J_i+\Lambda}\cdot\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}$$

$$\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2} = \frac{N^2}{2}\cdot\left(-\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}\right)^{3-2}\cdot\frac{(-J_1+\Lambda)(-J_2+\Lambda)}{(Q_1+\Lambda)(Q_2+\Lambda)(Q_3+\Lambda)}$$
$$\Rightarrow\ 0 = (Q_1+\Lambda)(Q_2+\Lambda)(Q_3+\Lambda)+\frac{N^2}{2}(-J_1+\Lambda)(-J_2+\Lambda)$$

$$\Lambda^3+\left(Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{N^2}{2}(J_1+J_2)\right)\Lambda\ =0$$
$$\left(Q_1+Q_2+Q_3+\frac{N^2}{2}\right)\Lambda^2+\left(Q_1Q_2Q_3+\frac{N^2}{2}J_1J_2\right)\ =0$$

$$Q_1Q_2Q_3+\frac{N^2}{2}J_1J_2$$
$$=\left(Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{N^2}{2}(J_1+J_2)\right)\left(Q_1+Q_2+Q_3+\frac{N^2}{2}\right),$$

$$S = 2\pi\sqrt{Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{N^2}{2}(J_1+J_2)}$$

$$N^2 = \frac{\pi}{2G_5}$$

$$E \geq J+Q_L-\frac{k_L}{2}$$

$$E \geq J-\frac{Q_L^2}{2k_L},$$

$$Q_L = k_L$$

$$\mathrm{ch}_{h,q_L}(q,y) = q^h F^{NS}\sum_{m=-\infty}^{\infty}\left(y^{2(k_L+1)m+q_L+1}-y^{-2(k_L+1)m-q_L-1}\right)\frac{q^{(k_L+1)m^2+(q_L+1)m}}{y-y^{-1}}$$

$$\chi_{q_L}(q,y) = q^{\frac{q_L}{2}}F^{NS}\sum_{m=-\infty}^{\infty}\left(\frac{y^{2(k_L+1)m+q_L+1}}{\left(1+yq^{m+\frac{1}{2}}\right)^2}-\frac{y^{-2(k_L+1)m-q_L-1}}{\left(1+y^{-1}q^{m+\frac{1}{2}}\right)^2}\right)\frac{q^{(k_L+1)m^2+(q_L+1)m}}{y-y^{-1}}$$

$$F^{NS} = \prod_{n\geq 1}\frac{\left(1+yq^{n-\frac{1}{2}}\right)^2\left(1+y^{-1}q^{n-\frac{1}{2}}\right)^2}{(1-y^2q^n)(1-q^n)^2(1-y^{-2}q^n)},$$

$$\chi_{q_L}(q,y)= \chi_{q_L}(q,q^{-1}y^{-1})\cdot q^{k_L}y^{2k_L}$$
$$\mathrm{ch}_{h,q_L}(q,y) = \mathrm{ch}_{h,q_L}(q,q^{-1}y^{-1})\cdot q^{k_L}y^{2k_L}.$$

$$q^h(q^{-1}y^{-1})^{q_L}\cdot q^k y^{2k_L} = q^{h+k_L-q_L}y^{2k_L-q_L}.$$

$$h-\frac{1}{2}q_L = (h+k_L-q_L)-\frac{1}{2}(2k_L-q_L).$$

$$\langle Q_L\rangle = y\frac{\partial \log Z}{\partial y}\Big|_{y=q^{-\frac{1}{2}}} = k_L,$$

$$\chi_{h,q_L-1}(q,y)+2\chi_{h,q_L}(q,y)+\chi_{h,q_L+1}(q,y) = \mathrm{ch}_{h,q_L}(q,y)\big|_{h=\frac{1}{2}q_L}$$

$$\text{BPS: } \{Q,Q^\dagger\} = E-\sum_i J_i-\sum_I Q_I = 0$$

$$\begin{aligned} Z &\equiv \mathrm{Tr}_{\mathrm{BPS}}\left[e^{-\beta\{Q,Q^\dagger\}}e^{\sum_i \omega_i J_i+\sum_I \Delta_I Q_I}\right] \\ &= \mathrm{Tr}_{\mathrm{BPS}}\left[e^{\sum_i \omega_i J_i+\sum_I \Delta_I Q_I}\right] \end{aligned}$$

$$1+x_B+x_B^2+\cdots = \frac{1}{1-x_B}$$

$$Z = \frac{\prod_{j=1}^{N_F}(1+x_{F,j})}{\prod_{i=1}^{N_B}(1-x_{B,i})}.$$

$$Z = e^{Z_{\mathrm{sp}}} = \exp\left(\sum_{i=1}^{N_B}x_{B,i}+\sum_{j=1}^{N_F}x_{F,j}\right).$$

$$\begin{aligned} Q_J &= \frac{\mathrm{Tr}_{\mathrm{BPS}}\left[Q_J\cdot e^{\sum_i \omega_i J_i+\sum_I \Delta_I Q_I}\right]}{\mathrm{Tr}_{\mathrm{BPS}}\left[e^{\sum_i \omega_i J_i+\sum_I \Delta_I Q_I}\right]} = \frac{\partial}{\partial \Delta_J}\log Z, \\ J_j &= \frac{\mathrm{Tr}_{\mathrm{BPS}}\left[J_j\cdot e^{\sum_i \omega_i J_i+\sum_I \Delta_I Q_I}\right]}{\mathrm{Tr}_{\mathrm{BPS}}\left[e^{\sum_i \omega_i J_i+\sum_I \Delta_I Q_I}\right]} = \frac{\partial}{\partial \omega_j}\log Z. \end{aligned}$$

$$E \geq J_1+J_2+Q_1+Q_2+Q_3$$

$$\begin{aligned} &\left(Q_1Q_2Q_3+\frac{N^2}{2}J_1J_2\right) \\ =&\left(Q_1+Q_2+Q_3+\frac{N^2}{2}\right)\left(Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{N^2}{2}(J_1+J_2)\right). \end{aligned}$$

$$\frac{1}{2}N^2 = \frac{\pi\ell_5^3}{4G_5}$$

$$\left(Q^3+\frac{N^2}{2}J^2\right)-\left(3Q+\frac{N^2}{2}\right)(3Q^2-N^2J) = 0$$

$$[j;\bar{J}]_E^{[R_1,R_2,R_3]}$$

$$J_1 = \frac{j+\bar{J}}{2}, \qquad J_2 = \frac{j-\bar{J}}{2},$$
$$Q_1 = R_2 + \frac{R_1+R_3}{2}, \quad Q_2 = \frac{R_1+R_3}{2}, \quad Q_3 = \frac{R_1-R_3}{2}$$

$$\begin{aligned} 2\{Q,Q^\dagger\} &= E - \left(j + \frac{3}{2}R_1 + R_2 + \frac{1}{2}R_3\right) \\ &= E - (Q_1 + Q_2 + Q_3 + J_1 + J_2) \geq 0 \end{aligned}$$

$$e^{\omega_1} = p^2, e^{\omega_2} = q^2, e^{\Delta_1} = x^2, e^{\Delta_2} = y^2, e^{\Delta_3} = z^2$$

$$Z_{\text{sp}} \equiv \text{Tr}_{\text{BPS}}[p^{2J_1} q^{2J_2} x^{2Q_1} y^{2Q_2} z^{2Q_3}]$$

$$x^2 + y^2 + z^2 + xyzpq\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{p^2} + \frac{1}{q^2} - 1\right) + p^2q^2.$$

$$Z_{\text{sp}} = \frac{x^2 + y^2 + z^2 + xyzpq\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{p^2} + \frac{1}{q^2} - 1\right) + p^2q^2}{(1-p^2)(1-q^2)}$$

$$Z \equiv \exp\left[Z_{\text{sp}}\right]$$

$$2Q_1 = x\frac{\partial}{\partial x}\log\, Z, 2J_1 = p\frac{\partial}{\partial p}\log\, Z$$

$$Q_1 = \frac{2x^2 + xyzpq\left(-\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{p^2} + \frac{1}{q^2} - 1\right)}{2(1-p^2)(1-q^2)}$$

$$J_1 = \frac{2p^2(q^2 + x^2 + y^2 + z^2) + xyzpq(1+p^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} - \frac{1}{p^2} + \frac{1}{q^2} - 1\right) + 4xyzpq}{2(1-p^2)^2(1-q^2)}$$

$$(E, J_1, J_2, Q_1, Q_2, Q_3) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$\frac{xyz}{pq} = 1,$$

$$(Q_1Q_2Q_3 + J_1J_2) - (Q_1 + Q_2 + Q_3 + 1)(Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - J_1 - J_2) = 0$$

$$\frac{1}{2}N^2 = 1 \leftrightarrow \frac{\pi\ell_5^3}{4G_5} = 1$$

$$Q = \frac{2x^2 + 2x^3 + xp^2 - x^3p^2}{2(1-p^2)^2}$$
$$J = \frac{3x(1+x)^2p^2 + (2 + 3x - x^3)p^4}{2(1-p^2)^3}$$

$$E \geq J + \frac{1}{2}(Q_1 + Q_2 + Q_3 + Q_4)$$

$$(\mathbb{Q}_3)^2-(\mathbb{Q}_1)(\mathbb{Q}_2)(\mathbb{Q}_3)+(\mathbb{Q}_1)^2(\mathbb{Q}_4)=0,$$

$$(\mathbb{Q}_1)\equiv Q_1+Q_2+Q_3+Q_4$$

$$(\mathbb{Q}_2)\equiv Q_1Q_2+Q_1Q_3+Q_1Q_4+Q_2Q_3+Q_2Q_4+Q_3Q_4+\frac{2N^3}{9}$$

$$(\mathbb{Q}_3)\equiv Q_1Q_2Q_3+Q_1Q_2Q_4+Q_1Q_3Q_4+Q_2Q_3Q_4-\frac{4N^3}{9}J$$

$$(\mathbb{Q}_4)\equiv Q_1Q_2Q_3Q_4+\frac{2N^3}{9}J^2.$$

$$N^{\frac{3}{2}}=\frac{3}{2\sqrt{2}G_4},$$

$$Q_1Q_2(Q_1+Q_2)^2-(Q_1+Q_2)\cdot\frac{2N^3}{9}J-\frac{2N^3}{9}J^2=0.$$

$$4Q^4-2Q\cdot\frac{2N^3}{9}J-\frac{2N^3}{9}J^2=0.$$

$$[j]_E^{[R_1,R_2,R_3,R_4]}$$

$$J=\frac{j}{2}$$

$$Q_1=R_3+R_2+\frac{R_1+R_4}{2},Q_2=R_2+\frac{R_1+R_4}{2},Q_3=\frac{R_1+R_4}{2},Q_4=\frac{R_1-R_4}{2}.$$

$$E\geq\frac{1}{2}j+R_1+R_2+\frac{1}{2}R_3+\frac{1}{2}R_4$$
$$=J+\frac{1}{2}(Q_1+Q_2+Q_3+Q_4)$$

$$Z_{\mathrm{sp}}\equiv \mathrm{Tr}_{\mathrm{BPS}}[p^{2J}x^{2Q_1}y^{2Q_2}z^{2Q_3}w^{2Q_4}]$$
$$=\frac{x^2+y^2+z^2+w^2+pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)}{1-p^2}$$

$$Q_1=\frac{pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)+2x^2-\frac{2pxyzw}{x^2}}{2(1-p^2)}$$

$$J=\frac{pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)}{2(1-p^2)}+\frac{p^2\left(x^2+y^2+z^2+w^2+pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)\right)}{(1-p^2)^2}$$

$$\frac{xyzw}{p}=1$$

$$(\mathbb{Q}_3)_{\frac{9}{2}}^2-(\mathbb{Q}_1)_{\frac{9}{2}}(\mathbb{Q}_2)_{\frac{9}{2}}(\mathbb{Q}_3)_{\frac{9}{2}}+(\mathbb{Q}_1)_{\frac{9}{2}}^2(\mathbb{Q}_4)_{\frac{9}{2}}=0$$

$$(\mathbb{Q}_1)_{\frac{9}{2}} \equiv Q_1 + Q_2 + Q_3 + Q_4$$
$$(\mathbb{Q}_2)_{\frac{9}{2}} \equiv Q_1Q_2 + Q_1Q_3 + Q_1Q_4 + Q_2Q_3 + Q_2Q_4 + Q_3Q_4 + 1$$
$$(\mathbb{Q}_3)_{\frac{9}{2}} \equiv Q_1Q_2Q_3 + Q_1Q_2Q_4 + Q_1Q_3Q_4 + Q_2Q_3Q_4 - 2J$$
$$(\mathbb{Q}_4)_{\frac{9}{2}} \equiv Q_1Q_2Q_3Q_4 + J^2$$

$$\frac{\sqrt{2}}{3}N^{\frac{3}{2}} = 1 \leftrightarrow G_4 = \frac{1}{2}$$

$$Q_1 = Q_3 = \frac{x^2}{1-p}$$
$$Q_2 = Q_4 = \frac{y^2}{1-p}$$
$$J = \frac{p(x^2+y^2)}{(1-p)^2}$$

$$Q_1Q_2(Q_1+Q_2)^2 - (Q_1+Q_2)J - J^2 = 0.$$

$$Q \equiv Q_{1,2,3,4} = \frac{x^2}{1-p}$$
$$J = \frac{2px^2}{(1-p)^2}$$

$$4Q^4 - 2QJ - J^2 = 0$$

$$E \geq J_1 + J_2 + J_3 + Q_1 + Q_2$$

$$\frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2) + \frac{N^3}{3}(J_1 + J_2 + J_3) - \frac{Q_1Q_2(Q_1+Q_2) - \frac{N^3}{3}(J_1J_2 + J_2J_3 + J_3J_1)}{Q_1 + Q_2 - \frac{N^3}{3}}$$
$$= \sqrt{\left(\frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2) + \frac{N^3}{3}(J_1 + J_2 + J_3)\right)^2 - \left(Q_1^2Q_2^2 + \frac{2N^3}{3}J_1J_2J_3\right)}$$

$$N^3 = \frac{3\pi^2}{16G_7}$$

$$\left(Q^4 + \frac{2N^3}{3}J^3\right)\left(Q - \frac{N^3}{6}\right)^2$$
$$= 2(3Q^2 + N^3J)\left(Q^3 - \frac{N^3}{2}J^2\right)\left(Q - \frac{N^3}{6}\right) - \left(Q^3 - \frac{N^3}{2}J^2\right)^2.$$

$$J_1 = j_1 + \frac{j_2 + j_3}{2}\ , \qquad J_2 = \frac{j_2 + j_3}{2}\ , \qquad J_3 = \frac{-j_2 + j_3}{2}\ ,$$
$$Q_1 = R_1 + \frac{R_2}{2}\ , \qquad Q_2 = \frac{R_2}{2}\ .$$

$$\begin{aligned} E &\geq j_1 + \frac{1}{2}j_2 + \frac{3}{2}j_3 + 2R_1 + 2R_2 \\ &= J_1 + J_2 + J_3 + 2Q_1 + 2Q_2 \end{aligned}$$

$$\begin{aligned} Z_{\text{sp}} &\equiv \text{Tr}_{\text{BPS}}[p^{2J_1} q^{2J_2} r^{2J_3} x^{2Q_1} y^{2Q_2}] \\ &= \frac{x^2 + y^2 + xypqr\left(\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} - 1\right)}{(1-p^2)(1-q^2)(1-r^2)} \end{aligned}$$

$$Q_1 = \frac{2x^2 + \text{xypqr}\left(\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} - 1\right)}{2(1-p^2)(1-q^2)(1-r^2)}$$

$$J_1 = \frac{2p^2(x^2+y^2) + \text{xypqr}(1+p^2)\left(-\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} - 1\right) + 4xypqr}{2(1-p^2)^2(1-q^2)(1-r^2)}$$

$$\frac{1}{2}(J_1 + J_2 + J_3) + \frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2) - \frac{Q_1Q_2(Q_1+Q_2) - \frac{1}{2}(J_1J_2 + J_2J_3 + J_3J_1)}{Q_1 + Q_2 - \frac{1}{2}}$$
$$= \sqrt{\left(\frac{1}{2}(J_1 + J_2 + J_3) + \frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2)\right)^2 - (J_1J_2J_3 + Q_1^2Q_2^2)}$$

$$\frac{2}{3}N^3 = 1 \leftrightarrow \frac{\pi^2}{8G_7} = 1.$$

$$Q = \frac{2x^2 + 3x^2p - x^2p^3}{2(1-p^2)^3},$$
$$J = \frac{x^2p + 4x^2p^2 + 4x^2p^3 - x^2p^5}{2(1-p^2)^4}.$$

$A_\mu \sim A_{\alpha\dot{\beta}}$, $\qquad (\mu = 1,2,3,4, \alpha = \pm, \dot{\beta} = \dot{\pm})$

$\Phi_{ij}\left(= -\Phi_{ji}\right), \bar{\Phi}^{ij} \sim \frac{1}{2}\epsilon^{ijkl}\Phi_{kl}$, $\qquad (i,j,k,l = 1,2,3,4)$

$\Psi_{i\alpha}, \bar{\Psi}^i_{\dot{\alpha}}$.

$$A_{\alpha\dot{\beta}}, \lambda_\alpha = \Psi_{4\alpha}, \bar{\lambda}_{\dot{\alpha}} = \bar{\Psi}^4_{\dot{\alpha}}$$
$$\phi_m = \Phi_{4m}, \bar{\phi}^m = \bar{\Phi}^{4m}, \psi_{m\alpha} = -i\Psi_{m\alpha}, \bar{\psi}^m_{\dot{\alpha}} = i\bar{\Psi}^m_{\dot{\alpha}}$$

$$\left\{Q^i_\alpha, S^\beta_j\right\} = \frac{1}{2}E\delta^i_j\delta^\beta_\alpha + R^i{}_j\delta^\beta_\alpha + J_\alpha{}^\beta\delta^i_j$$

$$2\{Q, Q^\dagger\} = E - (Q_1 + Q_2 + Q_3 + J_1 + J_2).$$

$$[Q, O\} = 0, \left[Q^{\dagger}, O\right\} = 0$$

$$E = Q_1 + Q_2 + Q_3 + J_1 + J_2$$

$$\phi^m, \psi_m, f, \lambda_{\dot{\alpha}}, \partial_{\dot{\alpha}}$$

$$\partial_{\dot{\alpha}}\lambda^{\dot{\alpha}} = 0 \Leftrightarrow \partial_{[\dot{\alpha}}\lambda_{\dot{\beta}]} = 0$$

$$\partial_{(\dot{\alpha}_1}\cdots\partial_{\dot{\alpha}_n)}\phi^m, \quad \partial_{(\dot{\alpha}_1}\cdots\partial_{\dot{\alpha}_n)}\psi_m, \partial_{(\dot{\alpha}_1}\cdots\partial_{\dot{\alpha}_n)}f$$
$$\partial_{(\dot{\alpha}_1}\cdots\partial_{\dot{\alpha}_{n-1}}\lambda_{\dot{\alpha}_n)}$$

$$g_{\mathrm{YM}}^2, H(g_{\mathrm{YM}}) = \sum_{L=0}^{\infty} g_{\mathrm{YM}}^{2L} H_{(L)}$$

$$\left\{Q(g_{\mathrm{YM}}), Q^{\dagger}(g_{\mathrm{YM}})\right\} = H(g_{\mathrm{YM}}) - \sum_I R_I - \sum_i J_i = \sum_{L=1}^{\infty} g_{\mathrm{YM}}^{2L} H_{(L)}.$$

$$Q\phi^m = 0, Q\lambda_{\dot{\alpha}} = 0, Q\psi_m = -\frac{i}{2}\epsilon_{mnp}[\phi^n, \phi^p]$$
$$Qf = -i[\phi^m, \psi_m], [Q, D_{\dot{\alpha}}] = -i[\lambda_{\dot{\alpha}}\}$$

$$\phi^m, \psi_m, f$$

$$Q\colon [Q, \phi^m] = 0, \{Q, \psi_m\} = -i\epsilon_{mnp}[\phi^m, \phi^n], [Q, f] = -i[\psi_m, \phi^m]$$

$$Z(\Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{\Delta_I Q_I + \omega_i J_i}\right], \text{ where } e^{\frac{\Delta_1+\Delta_2+\Delta_3-\omega_1-\omega_2}{2}} = 1$$

$$e^{\Delta_1} = e^{\Delta_2} = e^{\Delta_3} = t^2, e^{\omega_1} = e^{\omega_2} = t^3$$
$$\Rightarrow Z(t) = \mathrm{Tr}\left[(-1)^F t^{2(Q_1+Q_2+Q_3)+3(J_1+J_2)}\right] \equiv \mathrm{Tr}[(-1)^F x^{\mathcal{J}}]$$

$$\mathcal{J} \equiv 2Q_1 + 2Q_2 + 2Q_3 + 3J_1 + 3J_2$$

$$\omega = \frac{\Delta_1 + \Delta_2 + \Delta_3}{2}$$

$$Z_{\mathrm{BMN}}(\Delta_I) = \mathrm{Tr}_{\mathrm{BMN}}\left[(-1)^F e^{\Delta_I(Q_I+J)}\right]$$

$$q_I \equiv Q_I + J$$

$$Z_{\mathrm{BMN}}(t) = \mathrm{Tr}_{\mathrm{BMN}}\left[(-1)^F t^{2(Q_1+Q_2+Q_3)+6J}\right]$$

$$S(j, N) = N^2 f\left(\frac{j}{N^2}\right)$$

$$Z_{\mathrm{BMN}}(\Delta_I) = \frac{1}{N!}\int_0^{2\pi}\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\frac{\prod_{a\neq b}\left(1-e^{i\alpha_{ab}}\right)\prod_{a,b=1}^{N}\prod_{I<J}\left(1-e^{\Delta_I+\Delta_J}e^{i\alpha_{ab}}\right)}{\prod_{a,b=1}^{N}\left[\left(1-e^{\Delta_1+\Delta_2+\Delta_3}e^{i\alpha_{ab}}\right)\prod_{I=1}^{3}\left(1-e^{\Delta_I}e^{i\alpha_{ab}}\right)\right]}$$
$$\times\frac{\left(1-e^{\Delta_1+\Delta_2+\Delta_3}\right)\prod_{I=1}^{3}\left(1-e^{\Delta_I}\right)}{\prod_{I<J}\left(1-e^{\Delta_I+\Delta_J}\right)}$$

$$\left(e^{\Delta_1}, e^{\Delta_2}, e^{\Delta_3}\right) = t^2(x, y^{-1}, x^{-1}y)$$

$$\rho(\alpha) = \frac{3}{4\pi^3}(\pi^2-\alpha^2).$$

$$\log Z = \frac{3N^2}{\pi^2}\Delta_1\Delta_2\Delta_3.$$

$$S_{\mathrm{BMN}}(q_I;\Delta_I) = \log Z - \sum_I q_I\Delta_I,$$

$$S_{\mathrm{BMN}}(q_I) = 2\pi\sqrt{\frac{q_1q_2q_3}{3N^2}}$$

$$q_I = N^2\epsilon_I \sim N^2\sqrt{\epsilon_1\epsilon_2\epsilon_3} \propto N^2$$

$$S(q_I) = 2\pi\sqrt{\frac{2q_1q_2q_3}{N^2}}$$

$$u^{i_1i_2\cdots i_n} = \mathrm{tr}\left(\phi^{(i_1}\phi^{i_2}\cdots\phi^{i_n)}\right)$$

$$(u_n)^{i_1\cdots i_n} = \mathrm{tr}\left(\phi^{(i_1}\cdots\phi^{i_n)}\right)$$
$$(v_n)^{i_1\cdots i_{n-1}}{}_j = \mathrm{tr}\left(\phi^{(i_1}\cdots\phi^{i_{n-1})}\psi_j\right) - \mathrm{trace}$$
$$(w_n)^{i_1\cdots i_{n-1}} = \mathrm{tr}\left(\phi^{(i_1\cdots i_{n-1})}f + \frac{1}{2}\epsilon^{jk(i_p}\sum_{p=1}^{n-1}\phi^{i_1}\cdots\phi^{i_{p-1}}\psi_j\phi^{i_{p+1}}\cdots\phi^{i_{n-1})}\psi_k\right).$$

$$R_{ij} \equiv \epsilon_{ikm}\epsilon_{jln}(u_2)^{kl}(u_2)^{mn} = Q\left[-i\epsilon_{a_1a_2(i}\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\right)\right]$$

$$\mathrm{tr}(X^2)\mathrm{tr}(Y^2) - [\mathrm{tr}(XY)]^2 \sim 0, \mathrm{tr}(XY)\mathrm{tr}(XZ) - \mathrm{tr}(X^2)\mathrm{tr}(YZ) \sim 0,$$

$$\sum_a f_a(\{u_n, v_n, w_n\})R_a(\{u_n, v_n, w_n\}).$$

$(-1)^F E'$	J'	R'_1	R'_2	construction
n	0	n	0	$\|n\rangle$
$-(n+\frac{1}{2})$	$\frac{1}{2}$	$n-1$	0	$\overline{Q}_{m\dot{\alpha}}\|n\rangle$
$n+1$	0	$n-2$	0	$\overline{Q}_{m\dot{+}}\overline{Q}_{n\dot{-}}\|n\rangle$
$-(n+1)$	0	$n-1$	1	$Q^m_+\|n\rangle$
$n+\frac{3}{2}$	$\frac{1}{2}$	$n-2$	1	$Q^m_+\overline{Q}_{n\dot{\alpha}}\|n\rangle$
$-(n+2)$	0	$n-3$	1	$Q^m_+\overline{Q}_{n\dot{+}}\overline{Q}_{p\dot{-}}\|n\rangle$
$n+2$	0	$n-1$	0	$Q^m_+Q^n_+\|n\rangle$
$-(n+\frac{5}{2})$	$\frac{1}{2}$	$n-2$	0	$Q^m_+Q^n_+\overline{Q}_{p\dot{\alpha}}\|n\rangle$
$n+3$	0	$n-3$	0	$Q^m_+Q^n_+\overline{Q}_{p\dot{+}}\overline{Q}_{q\dot{-}}\|n\rangle$

$$u^{ik}R_{jk}(u_2)-\frac{1}{3}\delta^i_j u^{kl}R_{kl}(u_2)=0,$$

$$u^{1i}R_{2i}=u^{11}[u^{23}u^{13}-u^{12}u^{33}]+u^{12}[u^{33}u^{11}-(u^{13})^2]+u^{13}[u^{12}u^{13}-u^{11}u^{23}]=0.$$

$$G_i(g_i,\lambda_I)\equiv g_i-g_i(\lambda_I)=0$$

$$\sum_i q_i(g_i,\lambda_I)G_i(g_i,\lambda_I)$$

$$p(g_i,\lambda_I)=\sum_a q_a(g_i,\lambda_I)G_a(g_i,\lambda_I)+r(g_i,\lambda_I)$$

$$f=\begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & -f_1-f_2 \end{pmatrix}$$

$$f=\begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & f_3-f_1 & 0 \\ 0 & 0 & 0 & -f_2-f_3 \end{pmatrix}$$

$$f=\mathrm{diag}(f_1,f_2,f_3,-f_1-f_2-f_3)$$

6: $x^2, y^2, z^2, xy, yz, zx$
8: $\psi_1\cdot(y,z), \psi_2\cdot(z,x), \psi_3\cdot(x,y), \psi_1 x-\psi_2 y, \psi_2 y-\psi_3 z$
3: $xf-\frac{1}{2}\psi_2\psi_3, yf-\frac{1}{2}\psi_3\psi_1, zf-\frac{1}{2}\psi_1\psi_2$

$$V_k = \text{span}\{\mathfrak{v} \times (xf)^a (yf)^b (zf)^c \mid \mathfrak{v} \in \mathfrak{V}, a+b+c \le k\}$$

$$\tilde{V}_k = \text{span}\left\{\mathfrak{v} \times \left(xf - \frac{1}{2}\psi_2\psi_3\right)^a \left(yf - \frac{1}{2}\psi_3\psi_1\right)^b \left(zf - \frac{1}{2}\psi_1\psi_2\right)^c \middle| \mathfrak{v} \in \mathfrak{V}, a+b+c \le k\right\}$$

$$v = \sum_{i=1}^{n} r_i (xf)^{a_i} (yf)^{b_i} (zf)^{c_i}$$

$$\tilde{v} = \sum_{i=1}^{n} r_i \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i}$$

$$\begin{aligned}\tilde{v} = -\frac{1}{2}\sum_i^n r_i [& a_i\psi_2\psi_3 \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i-1} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i} \\ & + b_i\psi_3\psi_1 \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i-1} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i} \\ & + c_i\psi_1\psi_2 \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i-1}\Big].\end{aligned}$$

$$r_1 = \alpha_1\psi_1 y, r_2 = \alpha_2\psi_1 z, \cdots, r_8 = \alpha_8(\psi_2 y - \psi_3 z)$$

$$G_0 = \{x^a y^b z^c f^d \mid a,b,c,d \in \mathbb{Z}^{\ge 0}, a+b+c \ge d, a+b+c+d = 0(\text{mod}2)\}.$$

$$Z_0(x,y,z,f) = \sum_{g \in G_0} g$$

$$\frac{1}{(1-x)(1-y)(1-z)(1-f)}$$

$$\frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{f}{1-f}.$$

$$\begin{aligned}Z_0 &= \left[\frac{1}{(1-x)(1-y)(1-z)(1-f)} - \frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{f}{1-f}\right]_{\text{even}} \\ &= \left[\frac{1 - f(xy+yz+zx-xyz) + f^2xyz}{(1-x)(1-y)(1-z)(1-xf)(1-yf)(1-zf)}\right]_{\text{even}} \\ &= \frac{1 + \chi_2 + f(\chi_3 - \chi_1\chi_2) + f^2(\chi_3^2 + \chi_1\chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)}\end{aligned}$$

$$\begin{aligned}\chi_1 &= x+y+z \\ \chi_2 &= xy+yz+zx \\ \chi_3 &= xyz\end{aligned}$$

$$\{x^a y^b z^c f^d \psi_1 \mid a,b,c,d \in \mathbb{Z}^{\ge 0}, b+c \ge 1, a+b+c-1 \ge d, a+b+c+d = 1(\text{mod}2)\}.$$

$$G_{\psi_1} = \{x^a y^b z^c f^d \psi_1 \mid a,b,c,d \in \mathbb{Z}^{\ge 0}, a+b+c-1 \ge d, a+b+c+d = 1(\text{mod}2)\} \setminus \{x\psi_1\}.$$

$$G_1 = G_{\psi_1} \cup G_{\psi_2} \cup G_{\psi_3} \cup \{x\psi_1 - y\psi_2, y\psi_2 - z\psi_3\}.$$

$$x \to [x], y \to [y], z \to [z], f \to [f], \psi_1 \to \frac{[\psi]}{[x]}, \psi_2 \to \frac{[\psi]}{[y]}, \psi_3 \to \frac{[\psi]}{[z]}.$$

$$\frac{1}{(1-x)(1-y)(1-z)(1-f)}\cdot\frac{\psi}{x}$$

$$(x,y,z,f)\to(-x,-y,-z,-f)x\psi_1\to\psi$$

$$\begin{aligned}Z_{\psi_1}&=\left[\frac{1}{(1-x)(1-y)(1-z)(1-f)}-\frac{1}{(1-xf)(1-yf)(1-zf)}\cdot\frac{1}{1-f}\right]_{\text{odd}}\cdot\frac{\psi}{x}-\psi\\&=\left[\frac{x+y+z-(xy+yz+zx)(1+f)+xyz(1+f+f^2)}{(1-x)(1-y)(1-z)(1-xf)(1-yf)(1-zf)}\right]_{\text{odd}}\cdot\frac{\psi}{x}-\psi\\&=\frac{\chi_1+\chi_3-f(\chi_2+\chi_2^2-\chi_1\chi_3-\chi_3^2)+f^2\chi_3(1+\chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)}\cdot\frac{\psi}{x}-\psi\end{aligned}$$

$$Z_1=\frac{\chi_1+\chi_3-f(\chi_2+\chi_2^2-\chi_1\chi_3-\chi_3^2)+f^2\chi_3(1+\chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)}\cdot\frac{\chi_2}{\chi_3}\cdot\psi-\psi$$

$$\{x^ay^bz^cf^d\psi_1\psi_2\mid a,b,c,d\in\mathbb{Z}^{\geq0},a+c\geq1,a+b+c-2\geq d\\ a+b+c+d=0(\text{mod}2)\}\setminus\{x^2\psi_1\psi_2\}.$$

$$\{x^ay^bz^cf^d\psi_1\psi_2\mid a,b,c,d\in\mathbb{Z}^{\geq0},b+c\geq1,a+b+c-2\geq d\\ a+b+c+d=0(\text{mod}2)\}\setminus\{y^2\psi_1\psi_2\}.$$

$$G_{\psi_1\psi_2}=\{x^ay^bz^cf^d\psi_1\psi_2\mid a,b,c,d\in\mathbb{Z}^{\geq0},a+b+c-2\geq d,a+b+c+d=0(\text{mod}2)\}$$

$$\begin{aligned}&(x\psi_1-y\psi_2)(x\psi_1-y\psi_2)\sim xy\psi_1\psi_2\\&(y\psi_2-z\psi_3)(y\psi_2-z\psi_3)\sim yz\psi_2\psi_3\\&(x\psi_1-y\psi_2)(y\psi_2-z\psi_3)\sim xy\psi_1\psi_2+yz\psi_2\psi_3+zx\psi_3\psi_1\end{aligned}$$

$$G_2=G_{\psi_1\psi_2}\cup G_{\psi_2\psi_3}\cup G_{\psi_3\psi_1}$$

$$Z_{\psi_1\psi_2}=\frac{\chi_1^2-\chi_2-\chi_2^2+2\chi_1\chi_3+\chi_3^2+f(\chi_3-\chi_1\chi_2)+f^2\chi_3(\chi_1+\chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)}\cdot\frac{\psi^2}{xy}$$

$$\begin{aligned}Z_2&=Z_{\psi_1\psi_2}+Z_{\psi_2\psi_3}+Z_{\psi_3\psi_1}\\&=\frac{\chi_1^2-\chi_2-\chi_2^2+2\chi_1\chi_3+\chi_3^2+f(\chi_3-\chi_1\chi_2)+f^2\chi_3(\chi_1+\chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)}\cdot\frac{\chi_1}{\chi_3}\cdot\psi^2\end{aligned}$$

$$\{x^ay^bz^cf^d\psi_1\psi_2\psi_3\mid a,b,c,d\in\mathbb{Z}^{\geq0},a+b\geq1,a+b+c-3\geq d,a+b+c+d=1(\text{mod}2)\}.$$

$$G_{\psi_1\psi_2\psi_3}=\{x^ay^bz^cf^d\psi_1\psi_2\psi_3\mid a,b,c,d\in\mathbb{Z}^{\geq0},a+b+c-3\geq d,a+b+c+d=1(\text{mod}2)\}.$$

$$Z_3=\left[\frac{\begin{matrix}-1+\chi_1^2-2\chi_2-\chi_2^2+2\chi_1\chi_3+\chi_3^2+f(\chi_1+\chi_3)\\-f^2(\chi_2+\chi_2^2-\chi_1\chi_3-\chi_3^2)+f^3\chi_3(1+\chi_2)\end{matrix}}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)}+1\right]\cdot\frac{\psi^3}{f\chi_3}.$$

$$(x,y,z)=\left(e^{\Delta_1},e^{\Delta_2},e^{\Delta_3}\right)$$

$$x,y,z\to t^2,f\to t^6,\psi\to-t^6.$$

$$Z_{\text{grav}} = \frac{1+3t^4-8t^6-6t^{10}+10t^{12}+9t^{14}-9t^{16}+16t^{18}}{(1-t^4)^3(1-t^8)^3}.$$

$$\left(e^{\Delta_1}=e^{\Delta_2}=e^{\Delta_3}\equiv t^2\right)$$

$$Z = [1+3t^2+12t^4+20t^6+42t^8+48t^{10}+75t^{12}+66t^{14}+81t^{16} +55t^{18}+54t^{20}+27t^{22}+19t^{24}+6t^{26}+3t^{28}]\frac{(1-t^2)^3}{(1-t^{12})(1-t^8)^3}$$

$$Z - Z_{\text{grav}} = \left[-\frac{e^{4(\Delta_1+\Delta_2+\Delta_3)}}{1-e^{2(\Delta_1+\Delta_2+\Delta_3)}}\right]\cdot\left[\prod_{I=1}^{3}\left(1-e^{\Delta_I}\right)\right]\cdot\left[\prod_{I=1}^{3}\frac{1}{1-e^{\Delta_I}e^{\Delta_1+\Delta_2+\Delta_3}}\right]$$

$$Z - Z_{\text{grav}} = -\frac{t^{24}}{1-t^{12}}\cdot\frac{(1-t^2)^3}{(1-t^8)^3}$$

$$\begin{aligned} Z - Z_{\text{grav}} = &\Big[-t^{24}-\chi_{(1,3)}t^{32}-\left(\chi_{(1,\overline{3})}+\chi_{(3,6)}\right)t^{34}-\chi_{(2,3)}t^{35}+\left(\chi_{(3,1)}+\chi_{(3,8)}\right)t^{36} \\ &-\left(\chi_{(2,\overline{3})}+\chi_{(4,6)}\right)t^{37}+\chi_{(5,3)}t^{38}+\left(\chi_{(2,1)}+2\chi_{(4,1)}+\chi_{(4,8)}\right)t^{39} \\ &-\left(2\chi_{(1,6)}+\chi_{(3,\overline{3})}+\chi_{(5,\overline{3})}+\chi_{(5,6)}\right)t^{40}\Big]\chi_{\text{D}}+\mathcal{O}(t^{41}) \end{aligned}$$

$$\chi_{(2J'+1,\text{R})} \equiv \chi_{J'}^{SU(2)_R}(p)\chi_{\text{R}}^{SU(3)}(x,y)$$

$$\chi_D \equiv \frac{(1-t^2z_1)\left(1-\frac{t^2}{z_2}\right)\left(1-\frac{t^2z_2}{z_1}\right)\left(1-\frac{tp}{z_1}\right)\left(1-\frac{t}{pz_1}\right)(1-tz_2p)\left(1-\frac{tz_2}{p}\right)\left(1-\frac{tz_1p}{z_2}\right)\left(1-\frac{tz_1}{z_2p}\right)}{(1-t^3p)\left(1-\frac{t^3}{p}\right)}$$

$$t^6 = e^{\Delta_1+\Delta_2+\Delta_3} = e^{\omega_1+\omega_2}, z_1 = e^{\frac{2\Delta_1-\Delta_2-\Delta_3}{3}}, z_2^{-1} = e^{\frac{-\Delta_1+2\Delta_2-\Delta_3}{3}}, p = e^{\frac{\omega_1-\omega_2}{2}}$$

$$Z - Z_{\text{grav}} = Z_{\text{core}}(\Delta_I)\cdot\prod_{I=1}^{3}\frac{1}{1-e^{\Delta_I}e^{\Delta_1+\Delta_2+\Delta_3}}\cdot\prod_{I<J}\left(1-e^{\Delta_I+\Delta_J}\right).$$

$$Z_{\text{core}}(\Delta_I) \equiv f(t,x,y) \text{ with } e^{\Delta_1} = t^2x, e^{\Delta_2} = t^2y^{-1}, e^{\Delta_3} = t^2x^{-1}y$$

$$f(t,x,y) = \sum_{\mathcal{J}=0}^{54}\sum_{\mathbf{R}_j}(-1)^{F(\mathbf{R}_\mathcal{J})}\chi_{\mathbf{R}_\mathcal{J}}(x,y)t^{\mathcal{J}}+\mathcal{O}(t^{56}),$$

$$\prod_{I<J}\left(1-e^{\Delta_1+\Delta_2}\right)$$

$$Z - Z_{\text{grav}} = \left[-\chi_{[2,0]}(x,y)t^{28}-\chi_{[3,0]}(x,y)t^{30}+\mathcal{O}(t^{32})\right]\cdot\prod_{I<J}\left(1-e^{\Delta_I+\Delta_J}\right)$$

$$-\frac{t^{24}}{1-t^{12}} = -t^{24}-t^{36}-t^{48}-t^{60}-\cdots$$

$$Q\phi^m = 0, Q\psi_m = \frac{1}{2}\epsilon_{mnp}\phi^n \times \phi^p, Qf = \phi^m \times \psi_m$$

$$E = \frac{19}{2}, Q_1 = Q_2 = Q_3 = \frac{3}{2}, J_1 = J_2 = \frac{5}{2}$$

$$\begin{aligned} O_0' =& (X\cdot\psi_1 - Y\cdot\psi_2)(X\cdot\psi_3)(\psi_2\cdot\psi_1\times\psi_1) + (Y\cdot\psi_2 - Z\cdot\psi_3)(Y\cdot\psi_1)(\psi_3\cdot\psi_2\times\psi_2) \\ &+(Z\cdot\psi_3 - X\cdot\psi_1)(Z\cdot\psi_2)(\psi_1\cdot\psi_3\times\psi_3) \end{aligned}$$

$$\begin{aligned} O_0'' =& (X\cdot\psi_1 - Y\cdot\psi_2)(Y\cdot\psi_3)(\psi_1\cdot\psi_2\times\psi_2) + (Y\cdot\psi_2 - Z\cdot\psi_3)(Z\cdot\psi_1)(\psi_2\cdot\psi_3\times\psi_3) \\ &+(Z\cdot\psi_3 - X\cdot\psi_1)(X\cdot\psi_2)(\psi_3\cdot\psi_1\times\psi_1) \end{aligned}$$

$$O_0' - O_0'' = -2Q[(\psi_1\cdot\psi_2)(\psi_2\cdot\psi_3)(\psi_3\cdot\psi_1)]$$
$$O_0 \equiv -5(O_0' + O_0'') = \epsilon^{p_1p_2p_3}v_{p_1}^m v_{p_2}^n(\psi_m\cdot\psi_n\times\psi_{p_3})$$

$$v_n^m \equiv (\phi^m\cdot\psi_n) - \frac{1}{3}\delta_n^m(\phi^p\cdot\psi_p)$$

$$O_0 = \epsilon^{p_1p_2p_3}(\phi^m\cdot\psi_{p_1})(\phi^n\cdot\psi_{p_2})(\psi_m\cdot\psi_n\times\psi_{p_3}).$$

$$\psi_m\cdot\psi_n\times\psi_{p_3}$$

$$Q(\psi_m\cdot\psi_n\times\psi_p) = \frac{3}{2}\epsilon_{(m|qr}(\phi^q\times\phi^r)\cdot(\psi_{|n}\times\psi_{p)}) = 3\epsilon_{(m|qr}(\phi^q\cdot\psi_{|n})(\phi^r\cdot\psi_{p)}) = 3\epsilon_{(m|qr}v_{|n}^q v_{p)}^r$$

$$\phi^2\psi^5 = Q_f[f\phi\psi^4] + Q[\psi^6].$$

$$\phi_{(i}^{(m}\phi_{j)}^{n)} : (\mathbf{6},\mathbf{5}+\mathbf{1}) \in SU(3)\times SU(2)$$
$$\psi_{p_1(i}\psi_{p_2|j)}(\psi_m\cdot\psi_n\times\psi_{p_3}) : (\mathbf{3},\mathbf{5}+\mathbf{1})\otimes(\overline{\mathbf{10}},\mathbf{1})$$

$$(f\cdot\phi^m)(\psi_{[n_1}\cdot\psi_{n_2]})(\psi_{[p_1}\cdot\psi_{p_2]}), (\phi^m\cdot\psi_{n_1})(f\cdot\psi_{n_2})(\psi_{[p_1}\cdot\psi_{p_2]}).$$

$$Q[f\cdot\phi^m] = (\phi^n\times\psi_n)\cdot\phi^m = (\phi^m\times\phi^n)\cdot\psi_n$$

$$(\mathbf{8}\oplus\mathbf{1})\otimes\mathbf{3}\otimes\overline{\mathbf{3}} \to \mathbf{27}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}\oplus\mathbf{1}$$

$$(\phi^m\cdot\psi_m)\epsilon^{npq}(f\cdot\psi_n)(\psi_p\cdot\psi_q), (\phi^m\cdot\psi_n)(f\cdot\psi_m)\epsilon^{npq}(\psi_p\cdot\psi_q)$$

$$\psi_{(m}^{(i}(\psi_{r)}\times\psi_n)^{j)}\epsilon^{npq}(\psi_p\cdot\psi_q), \psi_n^{(i}(\psi_r\times\psi_m)^{j)}\epsilon^{npq}(\psi_p\cdot\psi_q)$$

$$\epsilon^{pqr}\psi_p^{(i}\psi_q^{j)}(\psi_m\cdot\psi_n\times\psi_r) = A\psi_{(m}^{(i}(\psi_{n)}\times\psi_r)^{j)}\epsilon^{pqr}(\psi_p\cdot\psi_q) + B\psi_r^{(i}(\psi_n\times\psi_m)^{j)}\epsilon^{pqr}(\psi_p\cdot\psi_q)$$

$$\begin{aligned} &Q_+^2Q_+^1O_0' = \\ &-(Y\cdot f+\psi_3\cdot\psi_1)^2\psi_3\cdot(\psi_2\times\psi_2) - (X\cdot f+\psi_2\cdot\psi_3)(Z\cdot f+\psi_1\cdot\psi_2)\psi_1\cdot(\psi_3\times\psi_3) \\ &-(X\cdot f+\psi_2\cdot\psi_3)(X\cdot\psi_3)f\cdot(\psi_1\times\psi_1) + 2(Y\cdot\psi_2 - Z\cdot\psi_3)(Y\cdot f+\psi_3\cdot\psi_1)\psi_3\cdot(\psi_2\times f) \\ &-2(Y\cdot f+\psi_3\cdot\psi_1)(X\cdot\psi_3)\psi_2\cdot(\psi_1\times f) - (Z\cdot\psi_3 - X\cdot\psi_1)(Z\cdot f+\psi_1\cdot\psi_2)f\cdot(\psi_3\times\psi_3) \end{aligned}$$

$$\begin{aligned} &Q_+^1Q_+^2Q_+^3O_0' = Q_+^1Q_+^2Q_+^3O_0'' = \\ &(X\cdot f+\psi_2\cdot\psi_3)^2 f\cdot(\psi_1\times\psi_1) + 2(X\cdot f+\psi_2\cdot\psi_3)(Y\cdot f+\psi_3\cdot\psi_1)f\cdot(\psi_1\times\psi_2) \\ &+(1,2,3\to2,3,1) + (1,2,3\to3,1,2) = G^mG^nf\cdot(\psi_m\times\psi_n) \end{aligned}$$

$$G^m \equiv \phi^m \cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p$$

$$Q_+^2 Q_+^1[(Z \cdot f + \psi_1 \cdot \psi_2)O_0'] = (Z \cdot f + \psi_1 \cdot \psi_2)Q_+^2 Q_+^1 O_0'$$

$$\begin{aligned} O_1 = {} & (f \cdot f)\epsilon^{c_1c_2c_3}(\phi^a \cdot \psi_{c_1})(\phi^b \cdot \psi_{c_2})(\psi_a \cdot \psi_b \times \psi_{c_3}) \\ & +\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(f \cdot \psi_{b_1})(\phi^a \cdot \psi_{c_1})(\psi_{b_2} \cdot \psi_{c_2})(\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\ & -\frac{1}{72}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1})(\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2})(\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3}) \end{aligned}$$

$$O_1 = (f \cdot f)O_0 + f \cdot \xi + \chi$$

$$\begin{aligned} \vec{\xi} &= \epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}\vec{\psi}_{b_1}(\phi^a \cdot \psi_{c_1})(\psi_{b_2} \cdot \psi_{c_2})(\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\ \chi &= -\frac{1}{72}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1})(\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2})(\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3}) \\ &= -120\psi_1^1\psi_1^2\psi_1^3\psi_2^1\psi_2^2\psi_2^3\psi_3^1\psi_3^2\psi_3^3 \end{aligned}$$

$$2(\vec{\phi}^m \times \vec{\psi}_m)O_0 + Q_\psi\vec{\xi} = 0\,, \vec{\phi}^m \cdot (\overrightarrow{\psi_m} \times \vec{\xi}) + Q_\psi\chi = 0$$

$$\vec{\xi} = -\frac{1}{2}\epsilon^{mnp}\vec{\psi}_m\psi_n \cdot \frac{\partial}{\partial\phi^p}O_0$$

$$\begin{aligned} & 4(\vec{\phi}^m \times \vec{\psi}_m)O_0 \\ = & \left[(\vec{\phi}^a \times \vec{\phi}^b)\psi_a \cdot \frac{\partial}{\partial\phi^b} + \vec{\psi}_a(\phi^a \times \phi^b) \cdot \frac{\partial}{\partial\phi^b} - \vec{\psi}_a(\psi_b \times \phi^a) \cdot \frac{\partial}{\partial\psi_b} + \vec{\psi}_b(\psi_a \times \phi^a) \cdot \frac{\partial}{\partial\psi_b}\right]O_0. \end{aligned}$$

$$(\ \phi^b \times \tfrac{\partial}{\partial\phi^b} + \psi_b \times \tfrac{\partial}{\partial\psi_b}\ )O_0$$

$$\phi^m \cdot \frac{\partial}{\partial\phi^m}O_0 = n_{\mathrm{B}}O_0, \psi_m \cdot \frac{\partial}{\partial\psi_m}O_0 = n_{\mathrm{F}}O_0\ (n_{\mathrm{B}}, n_{\mathrm{F}}) = \mathfrak{G}$$

$$\vec{\varepsilon}_a^b\left[\phi^a \cdot \frac{\partial}{\partial\phi^b} - \psi_b \cdot \frac{\partial}{\partial\psi_a}\right]O_0 = 0\ (\vec{\varepsilon}_a^a = 0).$$

$$\left[\phi^a \cdot \frac{\partial}{\partial\phi^b} - \psi_b \cdot \frac{\partial}{\partial\psi_a}\right]O_0 = \frac{1}{3}(n_{\mathrm{B}} - n_{\mathrm{F}})\delta_b^a O_0.$$

$$\left[\phi_i^a \cdot \frac{\partial}{\partial\phi_j^a} - \psi_a^j \cdot \frac{\partial}{\partial\psi_a^i}\right]O_0 = \frac{1}{3}(n_{\mathrm{B}} - n_{\mathrm{F}})\delta_i^j O_0.$$

$$(\vec{\psi}_a \times \vec{\phi}^b)\left(\phi^a \cdot \frac{\partial}{\partial\phi^b}\right)O_0 = \left(4 - \frac{n_{\mathrm{F}} + 2n_{\mathrm{B}}}{3}\right)(\vec{\phi}^m \times \vec{\psi}_m)O_0 = (\vec{\phi}^m \times \vec{\psi}_m)O_0$$

$$\epsilon_{\mathrm{man}}\,\epsilon_{ijk}\frac{\partial}{\partial\psi_n^k}\chi$$

$$\phi^m \cdot (\psi_m \times \xi) = -\frac{1}{2}\phi_i^m\phi_j^a\epsilon_{\mathrm{man}}\epsilon_{ijk}\frac{\partial\chi}{\partial\psi_n^k} = -\frac{1}{2}\epsilon_{\mathrm{man}}(\phi^m \times \phi^a) \cdot \frac{\partial\chi}{\partial\psi_n} = -Q_\psi\chi,$$

$$O_0 \text{ is } (Q_+^m Q_+^n O_0)\left(\phi^p \cdot f + \tfrac{1}{2}\epsilon^{pqr}\psi_q \cdot \psi_r\right)$$

$$O_n \equiv (f\cdot f)^n O_0 + n(f\cdot f)^{n-1} f\cdot\xi + \frac{2n^2+n}{3}(f\cdot f)^{n-1}\chi$$

$$QO_n = (f\cdot f)^{n-1}\left[\vec{f}\cdot\left(2n(\vec{\phi}^m\times\vec{\psi}_m)O_0 + nQ_\psi\vec{\xi}\right) + n(\vec{\phi}^m\times\vec{\psi}_m)\cdot\vec{\xi} + \frac{2n^2+n}{3}Q_\psi\chi\right]$$
$$+2(n^2-n)(f\cdot f)^{n-2}\vec{f}\cdot(\vec{\phi}^m\times\vec{\psi}_m)(f\cdot\xi) + \frac{2n(n-1)(2n+1)}{3}(f\cdot f)^{n-2}\vec{f}\cdot(\vec{\phi}^m\times\vec{\psi}_m)\chi$$

$$QO_n = \frac{2(n^2-n)}{3}(f\cdot f)^{n-2}[-(f\cdot f)(\phi^m\times\psi_m)\cdot\xi + 3(f\times\phi^m)\cdot\psi_m(f\cdot\xi)]$$

$$\epsilon_{\text{man}}\delta_{ij}\frac{\partial}{\partial\psi_n^k}\chi\,,\epsilon_{\text{man}}\delta_{k(i}\frac{\partial}{\partial\psi_n^{j)}}\chi$$

$$\epsilon_{\text{man}}\left[\delta_{ij}\frac{\partial}{\partial\psi_n^k} - \delta_{k(i}\frac{\partial}{\partial\psi_n^{j)}}\right]\chi$$

$$\epsilon_{\text{man}}\left[(f\cdot\phi^a)(f\times\phi^m)\cdot\frac{\partial}{\partial\psi_n} - \frac{1}{2}[(f\times\phi^m)\cdot\phi^a]f\cdot\frac{\partial}{\partial\psi_n}\right]\chi$$
$$= \frac{1}{2}\epsilon_{\text{man}}\left[[f\times(f\times(\phi^m\times\phi^a))]\cdot\frac{\partial}{\partial\psi_n} - [(f\times\phi^m)\cdot\phi^a]f\cdot\frac{\partial}{\partial\psi_n}\right]\chi$$
$$= -\frac{1}{2}\epsilon_{\text{man}}(f\cdot f)(\phi^m\times\phi^a)\cdot\frac{\partial}{\partial\psi_n}\chi = -(f\cdot f)Q_\psi\chi = (f\cdot f)(\phi^m\times\psi_m)\cdot\xi$$

$$O_n = (f\cdot f)^n\epsilon^{c_1c_2c_3}(\phi^a\cdot\psi_{c_1})(\phi^b\cdot\psi_{c_2})(\psi_a\cdot\psi_b\times\psi_{c_3})$$
$$+n(f\cdot f)^{n-1}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(f\cdot\psi_{b_1})(\phi^a\cdot\psi_{c_1})(\psi_{b_2}\cdot\psi_{c_2})(\psi_a\cdot\psi_{b_3}\times\psi_{c_3})$$
$$-\left(\frac{n}{72}+\frac{n^2-n}{108}\right)(f\cdot f)^{n-1}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}$$
$$\cdot(\psi_{a_1}\cdot\psi_{b_1}\times\psi_{c_1})(\psi_{a_2}\cdot\psi_{b_2}\times\psi_{c_2})(\psi_{a_3}\cdot\psi_{b_3}\times\psi_{c_3})$$

$$2p+8q = 12(n-n'), p = 0,1,2,3, q\geq 0$$

$$(p,q,n-n') = (2,1,1),(0,3,2),(2,4,3),(0,6,4),(2,7,5),(0,9,6),\cdots$$

$$\phi^m\cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n\psi_p$$

$$\mathcal{O}(f^{2n-2}\phi^0\psi^9)(Q_+^aQ_+^bO_{n'})(\phi\cdot f+\psi\cdot\psi)^q \boxtimes f^{2n-2}\phi^0\psi^9$$

$$f^{2n'}\phi^2\psi^5\to f^{2n'}\psi^7, f^{2n'-1}\phi\psi^7\to f^{2n'}\psi^7, f^{2n'-2}\psi^9\to f^{2n'}\psi^7.$$

$$O_n \overset{\Cup}{\sim} \epsilon_{abc}(Q_+^aQ_+^bO_{n-1})\left(\phi^c\cdot f + \frac{1}{2}\epsilon^{cde}\psi_d\cdot\psi_e\right)$$

$$O_n = (f\cdot f)^nO_0 + \frac{20n}{3}(f\cdot f)^{n-1}\sum_{\text{cyclic}}(f\cdot\psi_3)(\psi_3\cdot\psi_2)(X\cdot\psi_2)(\psi_1\cdot\psi_1\times\psi_1)$$
$$-\frac{10}{3}\left(\frac{n}{6}+\frac{n^2-n}{9}\right)(f\cdot f)^{n-1}(\psi_1\cdot\psi_1\times\psi_1)(\psi_2\cdot\psi_2\times\psi_2)(\psi_3\cdot\psi_3\times\psi_3)$$

$$Q_1^+Q_2^+Q_3^+O_0$$
$$= -10(X\cdot f+\psi_2\cdot\psi_3)^2 f\cdot(\psi_1\times\psi_1)$$
$$-20(X\cdot f+\psi_2\cdot\psi_3)(Y\cdot f+\psi_3\cdot\psi_1)f\cdot(\psi_1\times\psi_2)+\ \mathfrak{C}_{\text{cyclic}}$$
$$\to -20(\psi_2\cdot\psi_3)^2(f\cdot\psi_1\times\psi_1)+\ \mathfrak{C}_{\text{cyclic}}$$
$$Q_1^+Q_2^+Q_3^+(f\cdot\psi_3)(\psi_3\cdot\psi_2)(X\cdot\psi_2)(\psi_1\cdot\psi_1\times\psi_1)+\mathfrak{C}_{\text{cyclic}}$$
$$\to -3(f\cdot f)(\psi_2\cdot\psi_3)^2(f\cdot\psi_1\times\psi_1)+6(f\cdot\psi_2)(f\cdot\psi_3)(\psi_2\cdot\psi_3)(f\cdot\psi_1\times\psi_1)+\mathfrak{C}_{\text{cyclic}}$$
$$Q_1^+Q_2^+Q_3^+(\psi_1\cdot\psi_1\times\psi_1)(\psi_2\cdot\psi_2\times\psi_2)(\psi_3\cdot\psi_3\times\psi_3)$$
$$= -27(f\cdot\psi_1\times\psi_1)(f\cdot\psi_2\times\psi_2)(f\cdot\psi_3\times\psi_3)$$
$$= 18\big((f\cdot f)(\psi_2\cdot\psi_3)^2(f\cdot\psi_1\times\psi_1)\big)-2(f\cdot\psi_2)(f\cdot\psi_3)(\psi_2\cdot\psi_3)(f\cdot\psi_1\times\psi_1)+\mathfrak{C}_{\text{cyclic}}\big)$$

$$\chi_{24}\equiv -t^{24}\chi_D(t,x,y,p)$$

$$\begin{aligned}Z-Z_{\text{grav}}-\chi_{24}=&\Big[-\chi_{(1,3)}t^{32}-\Big(\chi_{(1,\overline{3})}+\chi_{(3,6)}\Big)t^{34}-\chi_{(2,3)}t^{35}+\big(\chi_{(3,1)}+\chi_{(3,8)}\big)t^{36}\\&-\Big(\chi_{(2,\overline{3})}+\chi_{(4,6)}\Big)t^{37}+\chi_{(5,3)}t^{38}+\big(\chi_{(2,1)}+2\chi_{(4,1)}+\chi_{(4,8)}\big)t^{39}\\&-\Big(2\chi_{(1,6)}+\chi_{(3,\overline{3})}+\chi_{(5,\overline{3})}+\chi_{(5,6)}\Big)t^{40}\Big]\chi_D+\mathcal{O}(t^{41}).\end{aligned}$$

$$O_0\cdot\big(\phi^{(m}\cdot\phi^{n)}\big),O_0\cdot(\phi^m\cdot\lambda_{\dot\alpha}),O_0\cdot(\lambda_{\dot+}\cdot\lambda_{\dot-})$$
$$O_0\cdot\Big(\phi^m\cdot\psi_n-\frac{1}{3}\delta_n^m\phi^p\cdot\psi_p\Big),O_0\cdot\Big(\lambda_{\dot\alpha}\cdot\psi_m-\frac{1}{2}\epsilon_{mnp}\phi^n\cdot D_{\dot\alpha}\phi^p\Big)$$
$$O_0\cdot\partial_{\dot\alpha}\big(\phi^{(m}\cdot\phi^{n)}\big)$$

$$\bar{Q}O_0\times\Big(\phi^m\cdot\phi^n,\phi^m\cdot\lambda_{\dot\alpha},\lambda_{\dot+}\cdot\lambda_{\dot-},\phi^m\cdot\psi_n-\frac{1}{3}\delta_n^m\phi^p\cdot\psi_p\Big)$$
$$(Q,\overline{QQ})O_0\times(\phi^m\cdot\phi^n,\phi^m\cdot\lambda_{\dot\alpha})$$
$$(Q\bar{Q},\overline{QQQ},\partial)O_0\times(\phi^m\cdot\phi^n)$$

$$O_0\cdot(\phi^m\cdot\phi^n),O_0\cdot(\phi^m\cdot\lambda_{\dot\alpha}),O_0\cdot\Big(\phi^m\cdot\psi_n-\frac{1}{3}\delta_n^m\phi^p\cdot\psi_p\Big)$$

$$\begin{aligned}O_0\cdot\big(\phi^{(m}\cdot\phi^{n)}\big)=-\frac{1}{14}Q\big[&20\epsilon^{rs(m}\big(\phi^{n)}\cdot\psi_p\big)(\phi^p\cdot\psi_r)\big(\phi^q\cdot\psi_q\big)(f\cdot\psi_s)\\&-20\epsilon^{prs}\big(\phi^{(m}\cdot\psi_p\big)\big(\phi^{n)}\cdot\psi_r\big)\big(\phi^q\cdot\psi_q\big)(f\cdot\psi_s)\\&+30\epsilon^{prs}\big(\phi^{(m}\cdot\psi_p\big)\big(\phi^{n)}\cdot\psi_r\big)(\phi^q\cdot\psi_s)\big(f\cdot\psi_q\big)\\&-7\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}\big(\phi^{n)}\cdot\psi_p\big)\big(\phi^q\cdot\psi_q\big)\big(\psi_{a_1}\cdot\psi_{a_2}\big)\big(\psi_{b_1}\cdot\psi_{b_2}\big)\\&+18\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}\big(\phi^{n)}\cdot\psi_q\big)\big(\phi^q\cdot\psi_p\big)\big(\psi_{a_1}\cdot\psi_{a_2}\big)\big(\psi_{b_1}\cdot\psi_{b_2}\big)\big].\end{aligned}$$

$$O_0\cdot(\phi^m\cdot\lambda_{\dot\alpha})$$

$$\begin{aligned}O_0\cdot(\phi^m\cdot\lambda_{\dot\alpha})=\frac{1}{8}Q\big[&40\epsilon^{mnp}\big(f\cdot\psi_q\big)(\lambda_{\dot\alpha}\cdot\psi_r)(\phi^q\cdot\psi_n)\big(\phi^r\cdot\psi_p\big)\\&-4\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}(\lambda_{\dot\alpha}\cdot\psi_n)\big(\phi^p\cdot\psi_p\big)\big(\psi_{a_1}\cdot\psi_{a_2}\big)\big(\psi_{b_1}\cdot\psi_{b_2}\big)\\&+6\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}\big(\lambda_{\dot\alpha}\cdot\psi_p\big)(\phi^p\cdot\psi_n)\big(\psi_{a_1}\cdot\psi_{a_2}\big)\big(\psi_{b_1}\cdot\psi_{b_2}\big)\\&+\epsilon^{na_1a_2}\epsilon^{pb_1b_2}(\lambda_{\dot\alpha}\cdot\psi_n)\big(\phi^m\cdot\psi_p\big)\big(\psi_{a_1}\cdot\psi_{a_2}\big)\big(\psi_{b_1}\cdot\psi_{b_2}\big)\big].\end{aligned}$$

$$O_0 \cdot \left(\phi^m \cdot \psi_n - \frac{1}{3}\delta_n^m \phi^p \cdot \psi_p\right)$$

$$O_0 \cdot \left(\phi^m \cdot \psi_n - \frac{1}{3}\delta_n^m \phi^p \cdot \psi_p\right)$$

$$= \frac{1}{4} Q\left[\epsilon_{npq}\epsilon^{ra_1a_2}\epsilon^{qb_1b_2}\epsilon^{mc_1c_2}(\phi^p \cdot \psi_r)(\psi_{a_1} \cdot \psi_{a_2})(\psi_{b_1} \cdot \psi_{b_2})(\psi_{c_1} \cdot \psi_{c_2})\right]$$

$$O_0 \cdot \left(\phi^m \cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p\right)$$

$$f \cdot \lambda_{\dot{\alpha}} + \tfrac{2}{3}\psi_m \cdot D_{\dot{\alpha}}\phi^m - \tfrac{1}{3}\phi^m \cdot D_{\dot{\alpha}}\psi_m \text{ at } \mathcal{O}(t^9)$$

$$\phi^m \cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p$$

$$\phi^m \cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p$$

$$O_0 \cdot \partial_{\dot{\alpha}}\left(\lambda_{\dot{\beta}} \cdot \lambda^{\dot{\beta}}\right), O_0 \cdot \left(f \cdot \lambda_{\dot{\alpha}} + \frac{2}{3}\psi_m \cdot D_{\dot{\alpha}}\phi^m - \frac{1}{3}\phi^m \cdot D_{\dot{\alpha}}\psi_m\right)$$

$$Z - Z_{\text{grav}} - \chi_{24} - \chi_{32} \text{ is } -\left(\chi_{(1,\overline{3})} + \chi_{(3,6)}\right)t^{34}$$

$$O_0 \cdot \partial^{\dot{\alpha}}\left(\lambda_{\dot{\alpha}} \cdot \psi_m - \frac{1}{2}\epsilon_{mnp}\phi^n \cdot D_{\dot{\alpha}}\phi^p\right), O_0 \cdot \partial_{\dot{\alpha}}\partial_{\dot{\beta}}(\phi^m \cdot \phi^n)$$

$$O_0 \cdot \partial_{\dot{\alpha}}\left(f \cdot \phi^m + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p\right)$$

$$O_0 \cdot \partial_{\dot{\alpha}}\left(f \cdot \lambda_{\dot{\beta}} + \frac{2}{3}\psi_m \cdot D_{\dot{\beta}}\phi^m - \frac{1}{3}\phi^m \cdot D_{\dot{\beta}}\psi_m\right)$$
$$O_0 \cdot \partial_{\dot{\alpha}}\partial_{\dot{\beta}}\left(\phi^m \cdot \psi_n - \frac{1}{3}\delta_n^m \phi^p \cdot \psi_p\right)$$

$$O_0 \cdot \partial_{\dot{\alpha}}\partial^{\dot{\beta}}\left(\lambda_{\dot{\beta}} \cdot \psi_m - \frac{1}{2}\epsilon_{mnp}\phi^n \cdot D_{\dot{\beta}}\phi^p\right), O_0 \cdot \partial_{\dot{\alpha}}\partial_{\dot{\beta}}\partial_{\dot{\gamma}}(\phi^m \cdot \phi^n).$$

$$O_0 \cdot \partial_{(\dot{\alpha}}\partial_{\dot{\beta}}\partial_{\dot{\gamma}}(\lambda_{\dot{\delta})} \cdot \phi^m)$$

$$O_0 \cdot \partial_{\dot{\alpha}}\partial_{\dot{\beta}}\partial^{\dot{\gamma}}(\lambda_{\dot{\gamma}} \cdot \phi^m), O_0 \cdot \partial_{\dot{\alpha}}\partial_{\dot{\beta}}\left(f \cdot \phi^m + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p\right)$$

$$O_0\partial_{\dot{\alpha}}\partial_{\dot{\beta}}\partial_{\dot{\gamma}}(\lambda_{\dot{\delta}} \cdot \lambda^{\dot{\delta}})$$
$$O_0\partial_{\dot{\alpha}}\partial^{\dot{\beta}}\left(f \cdot \lambda_{\dot{\beta}} + \frac{2}{3}\psi_m \cdot D_{\dot{\beta}}\phi^m - \frac{1}{3}\phi^m \cdot D_{\dot{\beta}}\psi_m\right)$$
$$O_0\partial_{(\dot{\alpha}}\partial_{\dot{\beta}}(f \cdot \lambda_{\dot{\gamma}}) + \frac{2}{3}\psi_m \cdot D_{\dot{\gamma}}\phi^m - \frac{1}{3}\phi^m \cdot D_{\dot{\gamma}}\psi_m\right)$$
$$O_0\partial_{\dot{\alpha}}\partial_{\dot{\beta}}\partial_{\dot{\gamma}}\left(\phi^m \cdot \psi_n - \frac{1}{3}\delta_n^m \phi^p \cdot \psi_p\right)$$

$$O_0 \equiv \epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b \text{tr}(\psi_{(c}\psi_m\psi_{n)})$$

$$Q\mathrm{tr}\left(\psi_{(c}\psi_m\psi_{n)}\right) \propto \epsilon_{ab(c}(v_2)^a{}_m(v_2)^b{}_{n)} \equiv R(v_2)_{cmn}$$

$$QO_0 \propto \epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b R(v_2)_{cmn} = 0.$$

$$\mathcal{J} \equiv 2(Q_1+Q_2+Q_3)+6J = 24$$

$$R \equiv \frac{Q_1+Q_2+Q_3}{3}$$

$$SU(3) \subset SU(4)_R$$

$$(R,J) = \left(\frac{n}{2}, \frac{8-n}{2}\right)$$

$$u^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right), u^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j}\phi^{k)}\right)$$
$$v^i_j \equiv \mathrm{tr}\left(\phi^i\psi_j\right) - \frac{1}{3}\delta^i_j\mathrm{tr}\left(\phi^a\psi_a\right), v^{ij}{}_k \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_k\right) - \frac{1}{4}\delta^i_k\mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_a\right) - \frac{1}{4}\delta^j_k\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_a\right)$$
$$w^i \equiv \mathrm{tr}\left(f\phi^i + \frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right), w^{ij} \equiv \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right)$$

$$\begin{aligned} O^{(2,1)} \equiv & 65u^{ij}\left(r^{(2,1)}_{20}\right)_{ij} - 39w^{ij}\left(r^{(1,1)}_{14}\right)_{ij} + 5w^i\left(r^{(1,1)}_{16}\right)_i \\ & + 312v^{jk}_i\left(r^{(1,2)}_{16}\right)^i_{jk} + 26v^j_i\left(r^{(1,2)}_{18}\right)^i_j + 6w^i\left(r^{(0,3)}_{16}\right)_i \end{aligned}$$

$$O^{(1,2)}_1 \equiv -3v^{(j}{}_i w^{k)}\left(r^{(0,1)}_{10}\right)^i_{jk} - 3u^{(ij}w^{k)}\left(r^{(0,2)}_{12}\right)_{ijk} + \epsilon_{a_1a_2i}u^{a_1j}w^{a_2}\left(r^{(0,2)}_{12}\right)^i_j$$

$$\begin{aligned} O^{(1,2)}_2 \equiv & -9u^{a(i}v^{j)}{}_a\left(r^{(1,1)}_{14}\right)_{ij} + 10\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(r^{(1,1)}_{14}\right)^{ij}_k \\ & + 30v^{(j}{}_i w^{k)}\left(r^{(0,1)}_{10}\right)^i_{jk} + 60u^{(jk}v^{l)}{}_i\left(r^{(0,3)}_{14}\right)^i_{jkl}, \end{aligned}$$

$$\begin{aligned} O^{(1,2)}_3 \equiv & -3u^{a(i}v^{j)}{}_a\left(r^{(1,1)}_{14}\right)_{ij} + 6\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(r^{(1,1)}_{14}\right)^{ij}_k + 4u^{ijk}\left(r^{(1,2)}_{18}\right)_{ijk} + 14v^{(j}{}_i w^{k)}\left(r^{(0,1)}_{10}\right)^i_{jk} \\ & - 6w^{ij}\left(r^{(0,2)}_{14}\right)_{ij} - 12\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)}{}_{a_2}\left(r^{(0,2)}_{12}\right)_{ijk} - 4v^j{}_a v^a{}_i\left(r^{(0,2)}_{12}\right)^i_j \end{aligned}$$

$$\begin{aligned} O^{(1,2)}_4 \equiv & -3u^{a(i}v^{j)}{}_a\left(r^{(1,1)}_{14}\right)_{ij} + 14\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(r^{(1,1)}_{14}\right)^{ij}_k + 8v^{jk}{}_i\left(r^{(1,1)}_{16}\right)^i_{jk} + 42v^{(j}{}_i w^{k)}\left(r^{(0,1)}_{10}\right)^i_{jk} \\ & + 12u^{(ij}w^{k)}\left(r^{(0,2)}_{12}\right)_{ijk} - 24w^{ij}\left(r^{(0,2)}_{14}\right)_{ij} - 36\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)}{}_{a_2}\left(r^{(0,2)}_{12}\right)_{ijk} - 8v^{jk}{}_i\left(r^{(0,3)}_{16}\right)^i_{jk}. \end{aligned}$$

$$O_1^{(1,1)} \equiv \epsilon_{a_1a_2i} u^{a_1(j} w^{k)a_2} \left(r_{10}^{(0,1)}\right)^i_{jk}$$

$$O_2^{(1,1)} \equiv \epsilon_{a_1a_2i} u^{a_1jk} w^{a_2} \left(r_{10}^{(0,1)}\right)^i_{jk}$$

$$O_3^{(1,1)} \equiv \epsilon_{a_1a_2i}\epsilon_{b_1b_2j} u^{a_1b_1} u^{a_2b_2k} \left(r_{14}^{(1,1)}\right)^{ij}_k + 5v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)}\right)^i_{jk} - 2v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)}\right)^i_{jk}$$

$$O_1^{(0,3)} = -\epsilon_{ia_1a_2}\left(4u^{a_1b} v^{ja_2}{}_b + 3u^{ja_1b} v^{a_2}{}_b\right)\left(r_{12}^{(0,2)}\right)^i_j = \frac{1}{2} iQ\left(\left(r_{12}^{(0,2)}\right)^i_j \left(r_{12}^{(0,2)}\right)^j_i\right)$$

$$O_2^{(0,3)} = -\epsilon_{a_1a_2(i}\left(u^{a_1(k} v^{l)a_2}{}_{j)} + u^{kla_1} v^{a_2}{}_{j)}\right)\left(r_{12}^{(0,2)}\right)^{ij}_{kl} = \frac{1}{2} iQ\left(\left(r_{12}^{(0,2)}\right)^{kl}_{ij} \left(r_{12}^{(0,2)}\right)^{ij}_{kl}\right)$$

$$O_3^{(0,3)} \equiv -u^{a(i} v^{jk)}{}_a \left(r_{12}^{(0,2)}\right)_{ijk},$$

$$O_4^{(0,3)} \equiv -\epsilon_{a_1a_2i} u^{a_1b} v^{a_2}{}_b \left(r_{14}^{(0,2)}\right)^i,$$

$$O_5^{(0,3)} \equiv 6v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)}\right)^i_{jk} + 6u^{a(ij} v^{k)}{}_a \left(r_{12}^{(0,2)}\right)_{ijk} + \epsilon_{a_1a_2i} u^{a_1bj} v^{a_2}{}_b \left(r_{12}^{(0,2)}\right)^i_j,$$

$$O_6^{(0,3)} \equiv 24v^{(j}{}_a v^{k)}{}_i \left(r_{10}^{(0,1)}\right)^i_{jk} + 6u^{a(i} v^{j)}{}_a \left(r_{14}^{(0,2)}\right)_{ij} - \epsilon_{a_1a_2(i} u^{a_1k} v^{a_2}{}_{j)} \left(r_{14}^{(0,2)}\right)^{ij}_k,$$

$$O_7^{(0,3)} \equiv v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)}\right)^i_{jk} - 10v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)}\right)^i_{jk} + 6u^{a(ij} v^{k)} \left(r_{12}^{(0,2)}\right)_{ijk} + 10\epsilon_{a_1a_2(i} u^{a_1kl} v^{a_2)} \left(r_{12}^{(0,2)}\right)^{ij}_{kl},$$

$$O_8^{(0,3)} \equiv 5v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)}\right)^i_{jk} - 2v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)}\right)^i_{jk} + 9u^{a(ij} v^{k)}{}_a \left(r_{12}^{(0,2)}\right)_{ijk} + 6\epsilon_{a_1a_2i} u^{a_1(j} u^{kl)a_2} \left(r_{14}^{(0,3)}\right)^i_{jkl},$$

$$O_9^{(0,3)} \equiv 6v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)}\right)^i_{jk} + 12v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)}\right)^i_{jk} + 18u^{a(ij} v^{k)} \left(r_{12}^{(0,2)}\right)_{ijk} - \epsilon_{a_1a_2(i} u^{a_1k} v^{a_2}{}_{j)} \left(r_{14}^{(0,2)}\right)^{ij}_k,$$

$$O_{10}^{(0,3)} \equiv 38v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)}\right)^i_{jk} + 4v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)}\right)^i_{jk} + 24u^{a(ij} v^{k)}{}_a \left(r_{12}^{(0,2)}\right)_{ijk} + 5u^{(jk} v^{l)}{}_i \left(r_{14}^{(0,2)}\right)^i_{jkl}.$$

$$\begin{aligned}
O \equiv\ & -6O_6^{(0,3)} \\
=\ & 288v^j{}_a v^{ka}{}_i \epsilon_{c_1c_2(j} \mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^i\psi_{k)}\right) - 72v^a{}_b v^{bk}{}_a \epsilon_{c_1c_2(k} \mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^d\psi_{d)}\right) \\
& +36\epsilon_{a_1a_2} u^{a_1k} v^{a_2}{}_j \left[2\mathrm{tr}\left(\phi^{(i}\phi^c\phi^{j)}\psi_{(c}\psi_{k)}\right) + 2\mathrm{tr}\left(\phi^{(i|}\phi^c\phi^{|j)}\psi_{(c}\psi_{k)}\right)\right. \\
& \left. +9\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(c}\phi^{c)}\psi_{k)}\right) - 6\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(c}\phi^c\psi_{k)}\right)\right] \\
& -9\epsilon_{a_1a_2j} u^{a_1b} v^{a_2}{}_b \left[2\mathrm{tr}\left(\phi^{(j}\phi^c\phi^{d)}\psi_{(c}\psi_{d)}\right) + 2\mathrm{tr}\left(\phi^{(j|}\phi^c\phi^{|d)}\psi_{(c}\psi_{d)}\right)\right. \\
& \left. +9\mathrm{tr}\left(\phi^{(j}\phi^d\psi_{(c}\phi^{c)}\psi_{d)}\right) - 6\mathrm{tr}\left(\phi^{(j}\phi^{d)}\psi_{(c}\phi^c\psi_{d)}\right)\right] \\
& -20u^{ai} v^j{}_a \epsilon_{b_1b_2b_3} \left[2\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^{b_3}\right) + \mathrm{tr}\left(\psi_{(i}\phi^{b_1}\psi_{j)}\phi^{b_2}\phi^{b_3}\right)\right] \\
& -36u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \left[\mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^c\phi^{b_1}\phi^{b_2}\right)\right] \\
& -36u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \left[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_j\phi^c\psi_c\phi^{b_1}\phi^{b_2}\right)\right] \\
& -36u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \left[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\psi_c\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^c\psi_c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\phi^{b_1}\psi_c\phi^{b_2}\right)\right] \\
& -36u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \left[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^c\psi_c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^c\phi^{b_2}\psi_c\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\phi^{b_1}\phi^{b_2}\psi_c\right)\right] \\
& +12u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \left[5\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}\left(\psi_c\phi^c\right) + 2\mathrm{tr}\left(\psi_j\phi^{(b_1}\phi^{c)}\right)\mathrm{tr}\left(\psi_c\phi^{b_2}\right)\right. \\
& \left. -2\mathrm{tr}\left(\psi_{j)}\phi^{b_2}\right)\mathrm{tr}\left(\psi_c\phi^{(b_1}\phi^{c)}\right)\right].
\end{aligned}$$

$$O_1^{(0,2)} \equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} + 2\epsilon_{a_1a_2}u^{a_1b}u^{a_2(j}v^{k)}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk}$$

$$O_2^{(0,2)} \equiv -6\epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} - \epsilon_{a_1a_2i}u^{a_1(k}v^{l)a_2}{}_{j)}\left(r_{12}^{(0,1)}\right)^{ij}_{kl}$$

$$O_3^{(0,2)} \equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} - \epsilon_{a_1a_2i}u^{a_1kl}v^{a_2}{}_{j)}\left(r_{12}^{(0,1)}\right)^{ij}_{kl}$$

$$O_4^{(0,2)} \equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} + \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}\left(r_{12}^{(0,2)}\right)^{ij}_{kl}$$

$$O_5^{(0,2)} \equiv -4\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} - 24\epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk}$$
$$-\epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2k}\left(r_{14}^{(0,2)}\right)^{ij}_{k}$$

$$O_6^{(0,2)} \equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} + 12\epsilon_{a_1a_2i}u^{a_1b\left(jv^{k)a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk}+3\epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}\left(r_{14}^{(0,2)}\right)^i_{jkl}\right.}.$$

$$O^{(0,1)} \equiv 36\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{a_3jk}\left(r_{10}^{(0,1)}\right)^i_{jk} + 5\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}u^{a_3b_3}r_{12}^{(0,1)}$$
$$-6\epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}\left(r_{12}^{(0,1)}\right)^{ij}_{kl}$$

$$\begin{aligned}
&\mathrm{tr}(\phi^1\phi^1\psi_1\phi^1\psi_1\psi_1\phi^1\psi_1) + \mathrm{tr}(\phi^2\phi^2\psi_2\phi^2\psi_2\psi_2\phi^2\psi_2) + \mathrm{tr}(\phi^3\phi^3\psi_3\phi^3\psi_3\psi_3\phi^3\psi_3)\\
&\mathrm{tr}(\phi^1\phi^1\phi^2\psi_2)\mathrm{tr}(\psi_1\psi_2)\mathrm{tr}(\phi^2\psi_1) + \mathrm{tr}(\phi^2\phi^2\phi^3\psi_3)\mathrm{tr}(\psi_2\psi_3)\mathrm{tr}(\phi^3\psi_2)\\
&+\mathrm{tr}(\phi^3\phi^3\phi^1\psi_1)\mathrm{tr}(\psi_3\psi_1)\mathrm{tr}(\phi^1\psi_3) + \mathrm{tr}(\phi^3\phi^3\phi^2\psi_2)\mathrm{tr}(\psi_3\psi_2)\mathrm{tr}(\phi^2\psi_3)\\
&+\mathrm{tr}(\phi^1\phi^1\phi^3\psi_3)\mathrm{tr}(\psi_1\psi_3)\mathrm{tr}(\phi^3\psi_1) + \mathrm{tr}(\phi^2\phi^2\phi^1\psi_1)\mathrm{tr}(\psi_2\psi_1)\mathrm{tr}(\phi^1\psi_2)\\
&\mathrm{tr}(\phi^2\phi^2\psi_1\psi_2\phi^3)\mathrm{tr}(f\phi^1\phi^1) + \mathrm{tr}(\phi^3\phi^3\psi_2\psi_3\phi^1)\mathrm{tr}(f\phi^2\phi^2)\\
&+\mathrm{tr}(\phi^1\phi^1\psi_3\psi_1\phi^2)\mathrm{tr}(f\phi^3\phi^3) - \mathrm{tr}(\phi^3\phi^3\psi_1\psi_3\phi^2)\mathrm{tr}(f\phi^1\phi^1)\\
&-\mathrm{tr}(\phi^1\phi^1\psi_2\psi_1\phi^3)\mathrm{tr}(f\phi^2\phi^2) - \mathrm{tr}(\phi^2\phi^2\psi_3\psi_2\phi^1)\mathrm{tr}(f\phi^3\phi^3)
\end{aligned}$$

$$f = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & -f_1 - f_5 \end{pmatrix},$$

$$u^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right), u^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^j\phi^{k)}\right)$$

$$v_j^i \equiv \mathrm{tr}\left(\phi^i\psi_j\right) - \frac{1}{3}\delta_j^i\mathrm{tr}(\phi^a\psi_a), v_k^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_k\right) - \frac{1}{4}\delta_k^i\mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_a\right) - \frac{1}{4}\delta_k^j\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_a\right)$$

$$w^i \equiv \mathrm{tr}\left(f\phi^i + \frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right), w^{ij} \equiv \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right)$$

$$Q\phi^m = 0, Q\psi_m = -\frac{i}{2}\epsilon_{mnp}[\phi^n, \phi^p], Qf = -i[\phi^m, \psi_m]$$

$$[n,0]: O^{i_1i_2i_3\cdots i_n} \to O^{i_1i_2i_3\cdots i_n}, [0,n]: O_{i_1i_2i_3\cdots i_n} \to O_{i_1i_2i_3\cdots i_n}$$

$$[1,1]: O_j^i \to O_j^i - \frac{1}{3}\delta_j^i O_a^a$$

$$[2,1]: O_k^{ij} \to O_k^{ij} - \frac{1}{2}\delta_k^{(i}O_a^{j)a}, [1,2]: O_{jk}^i \to O_{jk}^i - \frac{1}{2}\delta_{(j}^i O_{k)a}^a$$

$$[3,1]: O_l^{ijk} \to O_l^{ijk} - \frac{3}{5}\delta_l^{(i}O_a^{jk)a}, [1,3]: O_{jkl}^i \to O_{jkl}^i - \frac{3}{5}\delta_{(j}^i O_{kl)a}^a$$

$$[2,2]: O_{kl}^{ij} \to O_{kl}^{ij} - \frac{4}{5}\delta_{(k}^{(i}O_{l)a}^{j)a} + \frac{1}{10}\delta_{(k}^{(i}\delta_{l)}^{j)}O_{a_1a_2}^{a_1a_2}$$

$$t^{10}[1,2](u_2u_3):\left(R_{10}^{(0,0)}\right)_{jk}^{i}=\epsilon_{a_1a_2(j}\epsilon_{k)b_1b_2}u^{a_1b_1}u^{ia_2b_2}$$

$$t^{12}[0,0](u_2u_2u_2):R_{12}^{(0,0)}=\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}u^{a_3b_3}$$

$$t^{12}[2,2](u_2u_2u_2,u_3u_3):\left(R_{12}^{(0,0)}\right)_{kl}^{ij}=\epsilon_{a_1a_2(k}\epsilon_{l)b_1b_2}\left(u^{a_1b_1}u^{a_2b_2}u^{ij}+6u^{a_1b_1(i}u^{j)a_2b_2}\right)$$

$$t^{12}[0,3](u_2v_3):\left(R_{12}^{(0,1)}\right)_{ijk}=\epsilon_{(i|a_1a_2}\epsilon_{|j|b_1b_2}u^{a_1b_1}v^{a_2b_2}{}_{|k)}$$

$$t^{12}[1,1](u_2v_3,u_3v_2):\left(R_{12}^{(0,1)}\right)_{j}^{i}=\epsilon_{ja_1a_2}\left(4u^{a_1b}v^{ia_2}{}_{b}+3u^{ia_1b}v^{a_2}{}_{b}\right)$$

$$t^{12}[2,2](u_2v_3,u_3v_2):\left(R_{12}^{(0,1)}\right)_{kl}^{ij}=\epsilon_{a_1a_2(k}\left(u^{a_1(i}v^{j)a_2}{}_{l)}+u^{ija_1}v^{a_2}{}_{l)}\right)$$

$$t^{14}[1,0](u_2u_2v_2):\left(R_{14}^{(0,1)}\right)^{i}=\epsilon_{a_1a_2a_3}u^{ia_1}u^{ba_2}v^{a_3}{}_{b}$$

$$t^{14}[0,2](u_2u_2v_2,u_3v_3):\left(R_{14}^{(0,1)}\right)_{ij}=\epsilon_{a_1a_2(i|}\left(\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}v^{b_3}{}_{|j)}-2\epsilon_{|j)b_1b_2}u^{a_1b_1c}v^{a_2b_2}{}_{c}\right)$$

$$t^{14}[2,1](u_2u_2v_2,u_3v_3):\left(R_{14}^{(0,1)}\right)_{k}^{ij}=\epsilon_{ka_1a_2}\left(3u^{(a_1b}u^{ij)}v^{a_2}{}_{b}+4u^{a_1b}u^{a_2(i}v^{j)}{}_{b}+24u^{a_1b(i}v^{j)a_2}{}_{b}\right)$$

$$t^{14}[1,3](u_2u_2v_2,u_3v_3):\left(R_{14}^{(0,1)}\right)_{jkl}^{i}=\epsilon_{(j|a_1a_2}\epsilon_{|k|b_1b_2}\left(u^{a_1b_1}u^{a_2b_2}v^{i}{}_{|l)}+6u^{ia_1b_1}v^{a_2b_2}{}_{|l)}\right)$$

$$t^{14}[3,2](u_2u_2v_2,u_3v_3):\left(R_{14}^{(0,1)}\right)_{lm}^{ijk}=\epsilon_{a_1a_2(l}\left(u^{(a_1i}u^{jk)}v^{a_2}{}_{m)}+6u^{a_1(ij}v^{k)a_2}{}_{m)}\right)$$

$$t^{14}[1,3](v_2v_3):\left(R_{14}^{(0,2)}\right)_{jkl}^{i}=\epsilon_{a_1a_2(j}v^{a_1}{}_{k}v^{ia_2}{}_{l)}$$

$$t^{16}[0,1](u_2v_2v_2,v_3v_3):\left(R_{16}^{(0,2)}\right)_{i}=\epsilon_{ia_1a_2}\left(12u^{bc}v^{a_1}{}_{b}v^{a_2}{}_{c}+13u^{a_1b}v^{a_2}{}_{c}v^{c}{}_{b}+12v^{a_1b}{}_{c}v^{a_2c}{}_{b}\right)$$

$$t^{16}[1,2](u_2v_2v_2,v_3v_3):\left(R_{16}^{(0,2)}\right)_{jk}^{i}=\epsilon_{a_1a_2(j}\left(3u^{ib}v^{a_1}{}_{k)}v^{a_2}{}_{b}-7u^{ia_1}v^{b}{}_{k)}v^{a_2}{}_{b}\right.$$
$$\left.+6u^{a_1b}v^{i}{}_{k}v^{a_2}{}_{b}+24v^{a_1b}{}_{k}v^{ia_2}{}_{b}\right)$$

$$t^{16}[2,3](u_2v_2v_2,v_3v_3):\left(R_{16}^{(0,2)}\right)_{klm}^{ij}=\epsilon_{a_1a_2(k}\left(u^{a_1(i}{}_{j}v^{j}{}_{l}v^{a_2}{}_{m)}+3v^{a_1(i}{}_{l}v^{j)a_2}{}_{m)}\right)$$

$$t^{18}[0,0](u_3v_2v_2):R_{18}^{(0,2)}=\epsilon_{a_1a_2a_3}u^{a_1bc}v^{a_2}{}_{b}v^{a_3}{}_{c}$$

$$t^{20}[1,0](v_2v_2v_3):\left(R_{20}^{(0,3)}\right)^{i}=2v^{a}{}_{c}v^{b}{}_{a}v^{ic}{}_{b}-3v^{i}{}_{a}v^{c}{}_{b}v^{ab}{}_{c}$$

$$t^{22}[2,0](u_2v_2v_2v_2):\left(R_{22}^{(0,3)}\right)^{ij}=u^{ij}v^{a}{}_{b}v^{b}{}_{c}v^{c}{}_{a}-3u^{a(i}v^{j)}{}_{b}v^{b}{}_{c}v^{c}{}_{a}+3u^{ab}v^{(i}{}_{a}v^{j)}{}_{c}v^{c}{}_{b}$$

$$t^{24}[0,0](u_2v_2v_2v_3):R_{24}^{(0,3)}=\epsilon_{a_1a_2a_3}u^{a_1b}v^{a_2}{}_{b}v^{a_3c}{}_{d}v^{d}{}_{c}$$

$$t^{26}[1,0](v_2v_2v_2v_3):\left(R_{26}^{(0,4)}\right)^{i}=v^{i}{}_{a}v^{a}{}_{b}v^{d}{}_{c}v^{bc}{}_{d}$$

$$t^{30}[0,0](v_2v_2v_2v_2v_2):R_{30}^{(0,5)}=v^{a}{}_{b}v^{b}{}_{c}v^{c}{}_{d}v^{d}{}_{e}v^{e}{}_{a}$$

$$t^{30}[3,0](v_2v_2v_2v_2v_2):\left(R_{30}^{(0,5)}\right)^{ijk}=\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k)}{}_{a_2}v^{b}{}_{c}v^{c}{}_{d}v^{d}{}_{b}.$$

$t^{14}[0,2](v_2v_3,u_2w_3)$: $\left(R_{14}^{(1,0)}\right)_{ij}=\epsilon_{a_1a_2(i}\left(8v^{a_1b}{}_{j)}v^{a_2}{}_{b}+5\epsilon_{j)b_1b_2}u^{a_1b_1}w^{a_2b_2}\right)$

$t^{14}[2,1](v_2v_3,u_2w_3,u_3w_2)$:

$\left(R_{14}^{(1,0)}\right)_{k}^{ij}=2v^{i}{}_{a}v^{j)a}{}_{k}-5v^{a}{}_{k}v^{ij}{}_{a}+3\epsilon_{ka_1a_2}u^{a_1(i}w^{j)a_2}+3\epsilon_{ka_1a_2}u^{ija_1}w^{a_2}$

$t^{16}[0,1](v_3v_3,u_2v_2v_2,u_2u_2w_2)$:

$\left(R_{16}^{(1,0)}\right)_{i}=\epsilon_{ia_1a_2}\left(48v^{a_1b_1}{}_{b_2}v^{a_2b_2}{}_{b_1}+9u^{b_1b_2}v^{a_1}{}_{b_1}v^{a_2}{}_{b_2}-13\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}w^{b_3}\right)$

$t^{16}[1,2](v_3v_3,u_2v_2v_2,u_3w_3,u_2u_2w_2)$:

$\left(R_{16}^{(1,0)}\right)_{jk}^{i}=\epsilon_{a_1a_2(j|}\left(24v^{ia_1}{}_{b}v^{ba_2}{}_{|k)}+2u^{ia_1}v^{a_2}{}_{b}v^{b}{}_{|k)}-6u^{a_1b}v^{a_2}{}_{b}v^{i}{}_{|k)}\right.$

$\left.+6\epsilon_{|k)b_1b_2}u^{ia_1b_1}w^{a_2b_2}+\epsilon_{|k)b_1b_2}u^{a_1b_1}u^{a_2b_2}w^{i}\right)$

$t^{16}[3,1](v_3v_3,u_2v_2v_2,u_3w_3,u_2u_2w_2):$

$$(R_{16}^{(1,0)})_{l}^{ijk}=24v^{(ij}{}_{a}v^{k)a}{}_{l}+7u^{(ij}v^{k)}{}_{a}v^{a}{}_{l}-6u^{a(i}v^{j}{}_{a}v^{k)}{}_{l}$$
$$+18\epsilon_{la_1a_2}u^{a_1(ij}w^{k)a_2}+3\epsilon_{la_1a_2}u^{(ij}u^{k)a_1}w^{a_2}$$

$t^{16}[1,2](v_2w_3,v_3w_2):(R_{16}^{(1,1)})_{jk}^{i}=\epsilon_{a_1a_2(j}\left(v^{a_1}{}_{k)}w^{a_2i}+v^{a_1i}{}_{k)}w^{a_2}\right)$

$t^{18}[0,0](v_2v_2v_2,u_2v_2w_2):R_{18}^{(1,1)}=v^{a_1}{}_{a_2}v^{a_2}{}_{a_3}v^{a_3}{}_{a_1}-3\epsilon_{a_1a_2a_3}u^{a_1b}v^{a_2}{}_{b}w^{a_3}$

$t^{18}[1,1](v_2v_2v_2,v_3w_3,u_2v_2w_2):$

$$(R_{18}^{(1,1)})_{j}^{i}=9v^{i}{}_{a_1}v^{a_1}{}_{a_2}v^{a_2}{}_{j}-24\epsilon_{ja_1a_2}v^{ia_1}{}_{b}w^{ba_2}$$
$$-13\epsilon_{ja_1a_2}u^{ia_1}v^{a_2}{}_{b}w^{b}-16\epsilon_{ja_1a_2}u^{ib}v^{a_1}{}_{b}w^{a_2}+5\epsilon_{ja_1a_2}u^{a_1b}v^{i}{}_{b}w^{a_2}$$

$t^{18}[0,3](v_2v_2v_2,v_3w_3,u_2v_2w_2):$

$$(R_{18}^{(1,1)})_{ijk}=\epsilon_{a_1a_2(i|}\left(3v^{a_1}{}_{|j|}v^{a_2}{}_{b}v^{b}{}_{|k)}-3\epsilon_{b_1b_2|j}v^{a_1b_1}{}_{k)}w^{a_2b_2}-\epsilon_{b_1b_2|j}u^{a_1b_1}v^{a_2}{}_{k)}w^{b_2}\right)$$

$t^{18}[2,2](v_2v_2v_2,v_3w_3,u_2v_2w_2):$

$$(R_{18}^{(1,1)})_{ij}^{kl}=2v^{(i}{}_{a}v^{j)}{}_{(k}v^{a}{}_{l)}-6\epsilon_{a_1a_2(k}v^{a_1(i}{}_{l)}w^{j)a_2}-\epsilon_{a_1a_2(k}u^{ij}v^{a_1}{}_{l)}w^{a_2}$$

$t^{20}[0,2](v_2v_2w_2,u_2w_2w_2,w_3w_3):$

$$(R_{20}^{(2,0)})_{ij}=2\epsilon_{a_1a_2(i|}v^{a_1}{}_{b}v^{b}{}_{|j)}w^{a_2}-3\epsilon_{a_1a_2a_3}v^{a_1}{}_{i}v^{a_2}{}_{j}w^{a_3}$$
$$+\epsilon_{ia_1a_2}\epsilon_{jb_1b_2}u^{a_1b_1}w^{a_2}w^{b_2}+3\epsilon_{ia_1a_2}\epsilon_{jb_1b_2}w^{a_1b_1}w^{a_2b_2}\ .$$

$$iQr_{j}^{(n_f,n_\psi)}=R_{j}^{(n_f,n_\psi-1)}$$

$$\left(r_{10}^{(0,1)}\right)_{jk}^{i} = -2\epsilon_{a_1a_2(j}\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\phi^{i}\psi_{k)}\right),$$

$$r_{12}^{(0,1)} = \epsilon_{a_1a_2a_3}\left[6\mathrm{tr}(\psi_b\phi^{a_1})\mathrm{tr}\left(\phi^{b}\phi^{a_2}\phi^{a_3}\right) - \mathrm{tr}(\psi_b\phi^{a_1}\phi^{a_2})\mathrm{tr}\left(\phi^{b}\phi^{a_3}\right)\right]$$
$$-3\epsilon_{a_1a_2a_3}\left[\mathrm{tr}\left(\psi_b\phi^{b}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{b}\phi^{a_2}\phi^{a_3}\right)\right.$$
$$\left.+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^{b}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^{a_3}\phi^{b}\right)\right],$$

$$\left(r_{12}^{(0,1)}\right)_{kl}^{ij} = -2\epsilon_{a_1a_2(k}\left[\mathrm{tr}\left(\psi_{l)}\phi^{(i}\phi^{j)}\phi^{a_1}\phi^{a_2}\right) + 7\mathrm{tr}\left(\psi_{l)}\phi^{(i|}\phi^{a_1}\phi^{|j)}\phi^{a_2}\right)\right],$$

$$\left(r_{12}^{(0,2)}\right)_{ijk} = \frac{1}{2}\epsilon_{a_1a_2(i}\mathrm{tr}\left(\phi^{a_1}\psi_j\phi^{a_2}\psi_{k)}\right),$$

$$\left(r_{12}^{(0,2)}\right)_{j}^{i} = 6\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{(a}\psi_{j)}\right) - 5\mathrm{tr}\left(\phi^{[i}\psi_a\phi^{a]}\psi_j\right),$$

$$\left(r_{12}^{(0,2)}\right)_{kl}^{ij} = \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(k}\psi_{l)}\right),$$

$$\left(r_{14}^{(0,2)}\right)^{i} = 3\mathrm{tr}\left(\phi^{i}\psi_{a_1}\phi^{a_1}\phi^{a_2}\psi_{a_2}\right) + 2\mathrm{tr}\left(\phi^{i}\phi^{a_1}\right)\mathrm{tr}\left(\phi^{a_2}\psi_{(a_1}\psi_{a_2)}\right)$$
$$-6\mathrm{tr}\left(\phi^{i}\psi_{a_1}\right)\mathrm{tr}\left(\phi^{[a_1}\phi^{a_2]}\psi_{a_2}\right) - \mathrm{tr}\left(\phi^{i}\psi_{a_1}\psi_{a_2}\right)\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\right),$$

$$\left(r_{14}^{(0,2)}\right)_{ij} = \frac{5}{9}\epsilon_{a_1a_2a_3}\left[2\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{(i}\phi^{a_1}\psi_{j)}\phi^{a_2}\phi^{a_3}\right)\right]$$
$$+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_j\psi_{a_3}\phi^{a_1}\phi^{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_3}\phi^{a_1}\phi^{a_2}\right)\right]$$
$$+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_j\right)^{a_1}\psi_{a_3}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_j\phi^{a_1}\psi_{a_3}\phi^{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_j\phi^{a_3}\psi_{a_3}\phi^{a_1}\phi^{a_2}\right)\right]$$
$$+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\psi_{a_3}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_j\phi^{a_1}\phi^{a_3}\psi_{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_j\phi^{a_3}\phi^{a_1}\psi_{a_3}\phi^{a_2}\right)\right]$$
$$+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_j\right)\phi^{a_1}\phi^{a_2}\phi^{a_3}\psi_{a_3}\right) + \mathrm{tr}\left(\psi_j\phi^{a_1}\phi^{a_3}\phi^{a_2}\psi_{a_3}\right) + \mathrm{tr}\left(\psi_j\phi^{a_3}\phi^{a_1}\phi^{a_2}\psi_{a_3}\right)\right]$$
$$-\frac{1}{3}\epsilon_{a_1a_2(i}\left[5\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_3}\right) + 2\mathrm{tr}\left(\psi_j\phi^{(a_1}\phi^{a_3)}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_2}\right)\right.$$
$$\left.-2\mathrm{tr}\left(\psi_j\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{(a_1}\phi^{a_3)}\right)\right],$$

$$\left(r_{14}^{(0,2)}\right)_{k}^{ij} = 12\mathrm{tr}\left(\phi^{(i}\phi^{a}\phi^{j)}\psi_{(a}\psi_{k)}\right) + 12\mathrm{tr}\left(\phi^{(i|}\phi^{a}\phi^{j)}\psi_{(a}\psi_{k)}\right)$$
$$+54\mathrm{tr}\left(\phi^{(i}\phi^{j}\psi_{(a}\phi^{a)}\psi_{k)}\right) - 36\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(a}\phi^{a}\psi_{k)}\right),$$

$$\left(r_{14}^{(0,2)}\right)_{jkl}^{i} = 2\epsilon_{a_1a_2(j}\left[\mathrm{tr}\left(\phi^{i}\phi^{a_1}\phi^{a_2}\psi_k\psi_{l)}\right) + 3\mathrm{tr}\left(\phi^{i}\phi^{a_1}\psi_k\phi^{a_2}\psi_{l)}\right) - 2\mathrm{tr}\left(\phi^{i}\psi_k\phi^{a_1}\phi^{a_2}\psi_{l)}\right)\right],$$

$$\left(r_{14}^{(0,3)}\right)_{jkl}^{i} = -\frac{1}{2}\mathrm{tr}\left(\phi^{i}\psi_{(j}\psi_k\psi_{l)}\right),$$

$$\left(r_{16}^{(0,3)}\right)_{i} = \frac{39}{4}\mathrm{tr}\left(\psi_i\{\psi_{b_1}\psi_{b_2}, \phi^{b_1}\phi^{b_2}\}\right) + 2\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\psi_{b_2}\phi^{b_2}\right) - \frac{61}{4}\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\psi_{b_2}\phi^{b_1}\right)$$
$$+\frac{97}{4}\mathrm{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\psi_{b_2}\phi^{b_2}\right) - \frac{41}{4}\mathrm{tr}\left(\psi_i\phi^{b_2}\psi_{b_1}\psi_{b_2}\phi^{b_1}\right) - 5\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\phi^{b_2}\psi_{b_2}\right)$$
$$-\frac{25}{2}\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\phi^{b_1}\psi_{b_2}\right) + 2\mathrm{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\phi^{b_2}\psi_{b_2}\right) - \frac{61}{4}\mathrm{tr}\left(\psi_i\phi^{b_2}\psi_{b_1}\phi^{b_1}\psi_{b_2}\right)$$
$$-\frac{11}{4}\mathrm{tr}\left(\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}\left(\psi_i\psi_{b_1}\psi_{b_2}\right) - \frac{27}{2}\mathrm{tr}\left(\psi_{b_1}\psi_{b_2}\right)\mathrm{tr}\left(\psi_i\phi^{b_1}\phi^{b_2}\right)$$

$$+\frac{29}{4}\mathrm{tr}\left(\phi^{b_2}\psi_{b_2}\right)\mathrm{tr}\left(\psi_i[\psi_{b_1},\phi^{b_1}]\right)$$

$$\left(r_{16}^{(0,3)}\right)_{jk}^{i} = 2\mathrm{tr}\left(\psi_{(j}\psi_{k)}\psi_b\phi^b\phi^i\right) - 4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\psi_b\phi^i\phi^b\right) - \mathrm{tr}\left(\psi_{(j|}\psi_b\psi_{|k)}\{\phi^b,\phi^i\}\right)$$
$$-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^{(b}\psi_b\phi^{i)}\right) + 7\mathrm{tr}\left(\psi_{(j|}\{\psi_b,\phi^b\}\psi_{|k)}\phi^i\right) - 11\mathrm{tr}\left(\psi_{(j|}\{\psi_b,\phi^i\}\psi_{|k)}\phi^b\right)$$
$$-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^b\phi^i\psi_b\right) + 2\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^i\phi^b\psi_b\right)$$
$$+3\mathrm{tr}\left(\psi_{(j|}\psi_b\right)\mathrm{tr}\left(\psi_{|k)}[\phi^b,\phi^i]\right) + 6\mathrm{tr}\left(\psi_{(j}\phi^{[b}\right)\mathrm{tr}\left(\{\psi_{k)},\psi_b\}\phi^{i]}\right)$$

$$\left(r_{14}^{(1,1)}\right)_{ij} = 5\epsilon_{a_1a_2(i}\mathrm{tr}\left(f\phi^{a_1}\psi_{j)}\phi^{a_2}\right) + \mathrm{tr}\left(\phi^a\{\psi_a,\psi_{(i}\psi_{j)}\}\right) - 4\mathrm{tr}\left(\phi^a\psi_{(i|}\psi_a\psi_{|j)}\right),$$
$$\left(r_{14}^{(1,1)}\right)_{k}^{ij} = 3\mathrm{tr}\left(f\phi^{(i}\phi^{j)}\psi_k\right) - 3\mathrm{tr}\left(f\psi_k\phi^{(i}\phi^{j)}\right)$$
$$+\epsilon^{a_1a_2(i}\mathrm{tr}\left(\phi^{j)}\psi_k\psi_{a_1}\psi_{a_2}\right) - \epsilon^{a_1a_2(i}\mathrm{tr}\left(\phi^{j)}\psi_{a_1}\psi_{a_2}\psi_k\right),$$
$$\left(r_{16}^{(1,1)}\right)_{i} = 13\epsilon_{a_1a_2a_3}\mathrm{tr}(f\psi_i)\mathrm{tr}(\phi^{a_1}\phi^{a_2}\phi^{a_3}) + \frac{10}{3}\epsilon_{a_1a_2a_3}\mathrm{tr}(f\phi^{a_1})\mathrm{tr}(\psi_i\phi^{a_2}\phi^{a_3})$$
$$+\frac{10}{3}\epsilon_{a_1a_2a_3}\mathrm{tr}\left(f\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\psi_i\phi^{a_3}\right) + 46\epsilon_{ia_1a_2}\mathrm{tr}\left(f\phi^b\right)\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\right)$$
$$-7\epsilon_{ia_1a_2}\mathrm{tr}\left(f\phi^{a_1}\right)\mathrm{tr}\left(\psi_b\phi^{a_2}\phi^b\right) - 7\epsilon_{ia_1a_2}\mathrm{tr}\left(f\phi^b\phi^{a_1}\right)\mathrm{tr}\left(\psi_b\phi^{a_2}\right)$$
$$+6\epsilon_{ia_1a_2}\mathrm{tr}\left(f\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\psi_b\phi^b\right) - \frac{115}{3}\epsilon_{a_1a_2a_3}\mathrm{tr}\left(f\psi_i\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)$$
$$-\frac{95}{3}\epsilon_{a_1a_2a_3}\mathrm{tr}\left(f\phi^{a_1}\psi_i\phi^{a_2}\phi^{a_3}\right) + 5\epsilon_{a_1a_2a_3}\mathrm{tr}\left(f\phi^{a_1}\phi^{a_2}\psi_i\phi^{a_3}\right)$$
$$+36\epsilon_{ia_1a_2}\mathrm{tr}\left(f\psi_b\phi^{a_1}\phi^{a_2}\phi^b\right) - 43\epsilon_{ia_1a_2}\mathrm{tr}\left(f\psi_b\phi^{a_1}\phi^b\phi^{a_2}\right)$$
$$+39\epsilon_{ia_1a_2}\mathrm{tr}\left(f\phi^{a_1}\psi_b\phi^{a_2}\phi^b\right) - 68\epsilon_{ia_1a_2}\mathrm{tr}\left(f\phi^{a_1}\phi^{a_2}\psi_b\phi^b\right)$$
$$+39\epsilon_{ia_1a_2}\mathrm{tr}\left(f\phi^{a_1}\phi^b\psi_b\phi^{a_2}\right) + 13\mathrm{tr}\left(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_1}\phi^{b_2}\}\right)$$
$$-31\mathrm{tr}\left(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_2}\phi^{b_1}\}\right) + 14\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\psi_{b_2}\phi^{b_2}\right)$$
$$-22\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\phi^{b_1}\psi_{b_2}\right) + 14\mathrm{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\phi^{b_2}\psi_{b_2}\right),$$
$$\left(r_{16}^{(1,1)}\right)_{jk}^{i} = \epsilon_{a_1a_2(j}\left[-4\mathrm{tr}\left(f\phi^i\right)\mathrm{tr}\left(\psi_{k)}\phi^{a_1}\phi^{a_2}\right) - \mathrm{tr}\left(\phi^i\phi^{a_2}\right)\mathrm{tr}\left(f[\psi_{k)},\phi^{a_1}]\right)\right]$$
$$+\epsilon_{a_1a_2(j}\left[3\mathrm{tr}\left(f\phi^{a_1}\{\psi_{k)},\phi^i\}\phi^{a_2}\right) + 5\mathrm{tr}\left(f\{\psi_{k)},\phi^{a_1}\phi^i\phi^{a_2}\}\right)$$
$$-4\mathrm{tr}\left(f\psi_{k)}\phi^i\phi^{a_1}\phi^{a_2}\right) - 4\mathrm{tr}\left(f\phi^{a_1}\phi^{a_2}\phi^i\psi_{k)}\right)\Big)\Big]$$
$$+2\mathrm{tr}\left(\psi_{(j}\psi_{k)}\psi_b[\phi^b,\phi^i]\right) - 3\mathrm{tr}\left(\psi_{(j|}\psi_b\psi_{|k)}\{\phi^b,\phi^i\}\right) + 6\mathrm{tr}\left(\psi_{(j|}\{\psi_b,\phi^b\}\psi_{|k)}\phi^i\right)$$
$$-9\mathrm{tr}\left(\psi_{(j|}\{\psi_b,\phi^i\}\psi_{|k)}\phi^b\right) - 2\mathrm{tr}\left(\psi_{(j}\psi_{k)}[\phi^b,\phi^i]\psi_b\right)$$
$$+\mathrm{tr}\left(\psi_{(j|}\psi_b\right)\mathrm{tr}\left(\psi_{|k)}[\phi^b,\phi^i]\right) + \mathrm{tr}\left(\psi_{(j|}\phi^b\right)\mathrm{tr}\left(\{\psi_{|k)},\psi_b\}\phi^i\right),$$
$$\left(r_{16}^{(1,2)}\right)_{jk}^{i} = -\frac{1}{2}\mathrm{tr}\left(f\phi^i\psi_{(j}\psi_{k)}\right) - \frac{1}{2}\mathrm{tr}\left(f\psi_{(j}\phi^i\psi_{k)}\right)$$
$$-\frac{1}{2}\mathrm{tr}\left(f\psi_{(j}\psi_{k)}\phi^i\right) - \frac{1}{4}\epsilon^{ia_1a_2}\mathrm{tr}\left(\psi_{a_1}\psi_{a_2}\psi_{(j}\psi_{k)}\right),$$

$$\left(r_{18}^{(1,2)}\right)_{j}^{i} = -4\mathrm{tr}(f\phi^{i}\phi^{a})\mathrm{tr}(\psi_{j}\psi_{a}) - 5\mathrm{tr}(f\phi^{a}\phi^{i})\mathrm{tr}(\psi_{j}\psi_{a}) - \frac{53}{2}\mathrm{tr}(f\phi^{i}\psi_{j})\mathrm{tr}(\phi^{a}\psi_{a})$$
$$+7\mathrm{tr}(f\phi^{i}\psi_{a})\mathrm{tr}(\phi^{a}\psi_{j}) + \frac{15}{2}\mathrm{tr}(f\phi^{a}\psi_{j})\mathrm{tr}(\phi^{i}\psi_{a}) + 12\mathrm{tr}(f\phi^{a}\psi_{a})\mathrm{tr}(\phi^{i}\psi_{j})$$
$$+2\mathrm{tr}(f\psi_{j}\phi^{i})\mathrm{tr}(\phi^{a}\psi_{a}) - 13\mathrm{tr}(f\psi_{a}\phi^{i})\mathrm{tr}(\phi^{a}\psi_{j}) + 4\mathrm{tr}(f\psi_{j}\phi^{a})\mathrm{tr}(\phi^{i}\psi_{a})$$
$$+6\mathrm{tr}(f\psi_{j}\psi_{a})\mathrm{tr}(\phi^{i}\phi^{a}) + \frac{13}{2}\mathrm{tr}(f\psi_{a}\psi_{j})\mathrm{tr}(\phi^{i}\phi^{a}) - 4\mathrm{tr}(f\phi^{i})\mathrm{tr}(\phi^{a}\psi_{j}\psi_{a})$$
$$+14\mathrm{tr}(f\phi^{i})\mathrm{tr}(\phi^{a}\psi_{a}\psi_{j}) - 8\mathrm{tr}(f\phi^{a})\mathrm{tr}(\phi^{i}\psi_{j}\psi_{a}) - 8\mathrm{tr}(f\phi^{a})\mathrm{tr}(\phi^{i}\psi_{a}\psi_{j})$$
$$-4\mathrm{tr}(f\psi_{j})\mathrm{tr}(\psi_{a}\phi^{i}\phi^{a}) - 9\mathrm{tr}(f\psi_{a})\mathrm{tr}(\psi_{j}\phi^{i}\phi^{a}) + 6\mathrm{tr}(f\psi_{a})\mathrm{tr}(\psi_{j}\phi^{a}\phi^{i})$$
$$+3\mathrm{tr}(f\phi^{i}\phi^{a}\psi_{j}\psi_{a}) - \frac{31}{2}\mathrm{tr}(f\phi^{i}\phi^{a}\psi_{a}\psi_{j}) + 3\mathrm{tr}(f\phi^{a}\phi^{i}\psi_{j}\psi_{a})$$
$$+\frac{5}{2}\mathrm{tr}(f\phi^{a}\phi^{i}\psi_{a}\psi_{j}) + 12\mathrm{tr}(f\phi^{i}\psi_{j}\phi^{a}\psi_{a}) - \frac{13}{2}\mathrm{tr}(f\phi^{i}\psi_{a}\phi^{a}\psi_{j})$$
$$-6\mathrm{tr}(f\phi^{a}\psi_{j}\phi^{i}\psi_{a}) - \frac{13}{2}\mathrm{tr}(f\phi^{a}\psi_{a}\phi^{i}\psi_{j}) + 18\mathrm{tr}(f\phi^{i}\psi_{j}\psi_{a}\phi^{a})$$
$$-12\mathrm{tr}(f\psi_{j}\phi^{i}\phi^{a}\psi_{a}) + \frac{17}{2}\mathrm{tr}(f\psi_{a}\phi^{i}\phi^{a}\psi_{j}) - \frac{43}{2}\mathrm{tr}(f\psi_{a}\phi^{a}\phi^{i}\psi_{j})$$
$$+\frac{1}{3}\epsilon^{a_1a_2a_3}\mathrm{tr}(\phi^{i}\psi_{j})\mathrm{tr}(\psi_{a_1}\psi_{a_2}\psi_{a_3}) - 2\epsilon^{a_1a_2i}\mathrm{tr}(\phi^{b}\psi_{a_1})\mathrm{tr}(\psi_{b}\psi_{j}\psi_{a_2})$$
$$-10\epsilon^{a_1a_2a_3}\mathrm{tr}(\phi^{i}\psi_{j}\psi_{a_1}\psi_{a_2}\psi_{a_3}) + 8\epsilon^{a_1a_2a_3}\mathrm{tr}(\phi^{i}\psi_{a_1}\psi_{j}\psi_{a_2}\psi_{a_3})$$
$$-2\epsilon^{a_1a_2a_3}\mathrm{tr}(\phi^{i}\psi_{a_1}\psi_{a_2}\psi_{j}\psi_{a_3}),$$
$$\left(r_{18}^{(1,2)}\right)_{ijk} = -\epsilon_{a_1a_2(i}\left[\mathrm{tr}(f\phi^{a_1})\mathrm{tr}(\phi^{a_2}\psi_{j}\psi_{k)}) - \frac{3}{2}\mathrm{tr}(f\psi_{j})\mathrm{tr}(\psi_{k)})\phi^{a_1}\phi^{a_2}\right)$$
$$+3\mathrm{tr}(f\phi^{a_1}\psi_{j}\phi^{a_2}\psi_{k)}) - 3\mathrm{tr}(f\psi_{j}\phi^{a_1}\psi_{k)}\phi^{a_2})]$$
$$-\frac{1}{2}\mathrm{tr}(\phi^{a}\psi_{a})\mathrm{tr}(\psi_{(i}\psi_{j}\psi_{k)}) + \frac{3}{2}\mathrm{tr}(\phi^{a}\psi_{(i)})\mathrm{tr}(\psi_{a}\psi_{|j}\psi_{k)})$$
$$+\frac{1}{2}\mathrm{tr}(\phi^{a}\psi_{(i}\psi_{j|})\mathrm{tr}(\psi_{a}\psi_{|k)}) + \frac{3}{2}\mathrm{tr}(\phi^{a}\psi_{(i|}\psi_{a}\psi_{|j}\psi_{k)}) - \frac{3}{2}\mathrm{tr}(\phi^{a}\psi_{(i}\psi_{j|}\psi_{a}\psi_{|k)}),$$
$$\left(r_{20}^{(2,1)}\right)_{ij} = -\epsilon_{a_1a_2(i}\left[\mathrm{tr}(ff)\mathrm{tr}(\phi^{a_1}\phi^{a_2}\psi_{j}) + \frac{1}{2}\mathrm{tr}(f\psi_{j})\right)\mathrm{tr}(f\phi^{a_1}\phi^{a_2})$$
$$+2\mathrm{tr}(f\phi^{a_1})\mathrm{tr}(f[\phi^{a_2},\psi_{j}])]$$
$$+\epsilon_{a_1a_2(i}[4\mathrm{tr}(ff\phi^{a_1}\phi^{a_2}\psi_{j}) - \mathrm{tr}(f\phi^{a_1}\phi^{a_2}f\psi_{j})] + 2\mathrm{tr}(f\phi^{a}\psi_{(i)})\mathrm{tr}(\psi_{j)}\psi_{a})$$
$$-4\mathrm{tr}(f\psi_{(i}\phi^{a})\mathrm{tr}(\psi_{j)}\psi_{a}) - \frac{1}{2}\mathrm{tr}(f\psi_{(i)})(\phi^{a}\psi_{j)})\psi_{a}) - \frac{5}{2}\mathrm{tr}(f\psi_{(i|})(\phi^{a}\psi_{a}\psi_{|j)})$$
$$+2\mathrm{tr}(f\psi_{a})(\phi^{a}\psi_{(i}\psi_{j)}) - 4\mathrm{tr}(f\psi_{(i|}\psi_{a})(\phi^{a}\psi_{|j)}) + 2\mathrm{tr}(f\phi^{a}\psi_{(i}[\psi_{j)},\psi_{a}])$$
$$+4\mathrm{tr}(f\phi^{a}\psi_{a}\psi_{(i}\psi_{j)}) + 4\mathrm{tr}(f\psi_{(i}\phi^{a}\psi_{j)}\psi_{a}) - 3\mathrm{tr}(f\psi_{(i|}\phi^{a}\psi_{a}\psi_{|j)})$$
$$-2\mathrm{tr}(f\psi_{a}\phi^{a}\psi_{(i}\psi_{j)}) - \mathrm{tr}(f\psi_{(i}\psi_{a}\phi^{a}\psi_{|j)}) + 4\mathrm{tr}(f\psi_{a}\psi_{(i}\phi^{a}\psi_{j)})$$

$$+\frac{2}{5}\epsilon^{a_1a_2a_3}[2\mathrm{tr}(\psi_{a_1}\psi_{a_2})\mathrm{tr}(\psi_{a_3}\psi_{(i}\psi_{j)}) - 3\mathrm{tr}(\psi_{(i|}\psi_{a_1}\psi_{|j)}\psi_{a_2}\psi_{a_3})]$$

$$s_1^{(2,0)} = u^{ij}\left(R_{20}^{(2,0)}\right)_{ij}, \quad s_2^{(2,0)} = w^{ij}\left(R_{14}^{(1,0)}\right)_{ij}, \quad s_3^{(2,0)} = w^{i}\left(R_{16}^{(1,0)}\right)_{i}$$
$$s_1^{(1,2)} = v_i^{jk}{}_{i}\left(R_{16}^{(1,1)}\right)_{jk}^{i}, \quad s_2^{(1,2)} = v_i^{j}{}_{\left(R_{18}^{(1,1)}\right)_{j}^{i}}, \quad s_3^{(1,2)} = w^{i}\left(R_{16}^{(0,2)}\right)_{i}.$$

$$iQO^{(2,1)} \equiv 65s_1^{(2,0)} - 39s_2^{(2,0)} + 5s_3^{(2,0)} - 312s_1^{(1,2)} - 26s_2^{(1,2)} + 6s_3^{(1,2)} = 0$$

$$s_1^{(1,1)} = u^{a(i}v^{j)}{}_a\left(R_{14}^{(1,0)}\right)_{ij}, s_2^{(1,1)} = \epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(R_{14}^{(1,0)}\right)_k^{ij}, s_3^{(1,1)} = v^{jk}{}_i\left(R_{16}^{(1,0)}\right)_{jk}^{i},$$
$$s_4^{(1,1)} = u^{ijk}\left(R_{18}^{(1,1)}\right)_{ijk}, s_5^{(1,1)} = v^{(j}{}_i w^{k)}\left(R_{10}^{(0,0)}\right)_{jk}^{i}, s_6^{(1,1)} = u^{(ij}w^{k)}\left(R_{12}^{(0,1)}\right)_{ijk},$$
$$s_7^{(1,1)} = \epsilon_{a_1a_2i}u^{a_1j}w^{a_2}\left(R_{12}^{(0,1)}\right)_j^{i}, s_8^{(1,1)} = w^{ij}\left(R_{14}^{(0,1)}\right)_{ij},$$
$$s_1^{(0,3)} = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k)}{}_{a_2}\left(R_{12}^{(0,1)}\right)_{ijk}, s_2^{(0,3)} = v^j{}_a v^a{}_i\left(R_{12}^{(0,1)}\right)_j^{i},$$
$$s_3^{(0,3)} = u^{(jk}v^{k)}{}_i\left(R_{14}^{(0,2)}\right)_{jkl}^{i}, s_4^{(0,3)} = v^{jk}{}_i\left(R_{16}^{(0,2)}\right)_{jk}^{i}.$$

$$iQO_1^{(1,2)} \equiv 3s_5^{(1,1)} - 3s_6^{(1,1)} + s_7^{(1,1)} = 0$$
$$iQO_2^{(1,2)} \equiv 9s_1^{(1,1)} - 10s_2^{(1,1)} - 30s_5^{(1,1)} - 60s_3^{(0,3)} = 0$$
$$iQO_3^{(1,2)} \equiv 3s_1^{(1,1)} - 6s_2^{(1,1)} + 4s_4^{(1,1)} - 14s_5^{(1,1)} - 6s_8^{(1,1)} - 12s_1^{(0,3)} - 4s_2^{(0,3)} = 0$$
$$iQO_4^{(1,2)} \equiv 3s_1^{(1,1)} - 14s_2^{(1,1)} - 8s_3^{(1,1)} - 42s_5^{(1,1)} + 12s_6^{(1,1)} - 24s_8^{(1,1)} - 36s_1^{(0,3)} + 8s_4^{(0,3)} = 0$$

$$s_1^{(1,0)} = \epsilon_{a_1a_2i}\epsilon_{b_1b_2j}u^{a_1b_1}u^{a_2b_2k}\left(R_{14}^{(1,0)}\right)_k^{ij},$$
$$s_2^{(1,0)} = \epsilon_{a_1a_2i}u^{a_1(j}w^{k)a_2}\left(R_{10}^{(0,0)}\right)_{jk}^{i},$$
$$s_3^{(1,0)} = \epsilon_{a_1a_2i}u^{a_1jk}w^{a_2}\left(R_{10}^{(0,0)}\right)_{jk}^{i},$$
$$s_1^{(0,2)} = v^a{}_i v^{jk}{}_a\left(R_{10}^{(0,0)}\right)_{jk}^{i},$$
$$s_2^{(0,2)} = v^{(j}{}_a v^{k)a}{}_i\left(R_{10}^{(0,0)}\right)_{jk}^{i},$$
$$s_3^{(0,2)} = u^{a(i}v^{jk)}{}_a\left(R_{12}^{(0,1)}\right)_{ijk},$$
$$s_4^{(0,2)} = u^{a(ij}v^{k)}{}_a\left(R_{12}^{(0,1)}\right)_{ijk},$$
$$s_5^{(0,2)} = \epsilon_{a_1a_2i}u^{a_1b}v^{a_2j}{}_b\left(R_{12}^{(0,1)}\right)_j^{i},$$
$$s_6^{(0,2)} = \epsilon_{a_1a_2i}u^{a_1bj}v^{a_2}{}_b\left(R_{12}^{(0,1)}\right)_j^{i},$$
$$s_7^{(0,2)} = \epsilon_{a_1a_2i}u^{a_1(k}v^{l)a_2}{}_{j)}\left(R_{12}^{(0,1)}\right)_{kl}^{ij},$$
$$s_8^{(0,2)} = \epsilon_{a_1a_2i}u^{a_1kl}v^{a_2}{}_{j)}\left(R_{12}^{(0,1)}\right)_{kl}^{ij},$$
$$s_9^{(0,2)} = \epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}\left(R_{14}^{(0,2)}\right)_{jkl}^{i},$$
$$s_{10}^{(0,2)} = \epsilon_{a_1a_2i}u^{a_1b}v^{a_2}{}_b\left(R_{14}^{(0,1)}\right)^{i},$$
$$s_{11}^{(0,2)} = u^{a(i}v^{j)}{}_a\left(R_{14}^{(0,1)}\right)_{ij},$$
$$s_{12}^{(0,2)} = \epsilon_{a_1a_2i}u^{a_1k}v^{a_2}{}_{j)}\left(R_{14}^{(0,1)}\right)_k^{ij},$$

$$s_{13}^{(0,2)} = u^{(jk}v^{l)}{}_i\left(R_{14}^{(0,1)}\right)_{jkl}^{i}.$$

$$iQO_1^{(1,1)} \equiv s_2^{(1,0)} = 0$$
$$iQO_2^{(1,1)} \equiv s_3^{(1,0)} = 0$$
$$iQO_3^{(1,1)} \equiv s_1^{(1,0)} + 5s_1^{(0,2)} - 2s_2^{(0,2)} = 0$$
$$iQO_1^{(0,3)} \equiv 4s_5^{(0,2)} + 3s_6^{(0,2)} = \left(R_{12}^{(0,1)}\right)_j^i \left(R_{12}^{(0,1)}\right)_i^j = iQ\left[\frac{1}{2}iQ\left(\left(r_{12}^{(0,2)}\right)_j^i \left(r_{12}^{(0,2)}\right)_i^j\right)\right] = 0$$
$$iQO_2^{(0,3)} \equiv s_7^{(0,2)} + s_8^{(0,2)} = \left(R_{12}^{(0,1)}\right)_{kl}^{ij} \left(R_{12}^{(0,1)}\right)_{ij}^{kl} = iQ\left[\frac{1}{2}iQ\left(\left(r_{12}^{(0,2)}\right)_{kl}^{ij} \left(r_{12}^{(0,2)}\right)_{ij}^{kl}\right)\right] = 0$$
$$iQO_3^{(0,3)} \equiv s_3^{(0,2)} = 0$$
$$iQO_4^{(0,3)} \equiv s_{10}^{(0,2)} = 0$$
$$iQO_5^{(0,3)} \equiv 6s_1^{(0,2)} - 6s_4^{(0,2)} - s_6^{(0,2)} = 0$$
$$iQO_6^{(0,3)} \equiv 24s_2^{(0,2)} - 6s_{11}^{(0,2)} + s_{12}^{(0,2)} = 0$$
$$iQO_7^{(0,3)} \equiv s_1^{(0,2)} - 10s_2^{(0,2)} - 6s_4^{(0,2)} - 10s_8^{(0,2)} = 0$$
$$iQO_8^{(0,3)} \equiv 5s_1^{(0,2)} - 2s_2^{(0,2)} - 9s_4^{(0,2)} + 6s_9^{(0,2)} = 0$$
$$iQO_9^{(0,3)} \equiv 6s_1^{(0,2)} + 12s_2^{(0,2)} - 18s_4^{(0,2)} + s_{12}^{(0,2)} = 0$$
$$iQO_{10}^{(0,3)} \equiv 38s_1^{(0,2)} + 4s_2^{(0,2)} - 24s_4^{(0,2)} - 5s_{13}^{(0,2)} = 0$$

$$s_1^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(R_{10}^{(0,0)}\right)_{jk}^i, s_2^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1b}u^{a_2(j}v^{k)}{}_b\left(R_{10}^{(0,0)}\right)_{jk}^i,$$
$$s_3^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b\left(R_{10}^{(0,0)}\right)_{jk}^i, s_4^{(0,1)} = \epsilon_{a_1a_2(i}u^{a_1(k}v^{l)a_2}{}_{j)}\left(R_{12}^{(0,0)}\right)_{kl}^{ij},$$
$$s_5^{(0,1)} = \epsilon_{a_1a_2(i}u^{a_1kl}v^{a_2}{}_{j)}\left(R_{12}^{(0,0)}\right)_{kl}^{ij}, s_6^{(0,1)} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}\left(R_{12}^{(0,1)}\right)_{kl}^{ij},$$
$$s_7^{(0,1)} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2k}\left(R_{14}^{(0,1)}\right)_k^{ij}, s_8^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}\left(R_{14}^{(0,1)}\right)_{jkl}^i.$$

$$iQO_1^{(0,2)} \equiv s_1^{(0,1)} - 2s_2^{(0,1)} = 0$$
$$iQO_2^{(0,2)} \equiv 6s_3^{(0,1)} + s_4^{(0,1)} = 0$$
$$iQO_3^{(0,2)} \equiv s_1^{(0,1)} + s_5^{(0,1)} = 0$$
$$iQO_4^{(0,2)} \equiv s_1^{(0,1)} + s_6^{(0,1)} = 0$$
$$iQO_5^{(0,2)} \equiv 4s_1^{(0,1)} + 24s_3^{(0,1)} - s_7^{(0,1)} = 0$$
$$iQO_6^{(0,2)} \equiv s_1^{(0,1)} - 12s_3^{(0,1)} + 3s_8^{(0,1)} = 0$$

$$s_1^{(0,0)} = \epsilon_{a_1a_2a_3}\epsilon_{b_1b_2i}u^{a_1b_1}u^{a_2b_2}u^{a_3jk}\left(R_{10}^{(0,0)}\right)_{jk}^i,$$
$$s_2^{(0,0)} = R_{12}^{(0,0)}R_{12}^{(0,0)}$$
$$s_3^{(0,0)} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}\left(R_{12}^{(0,0)}\right)_{kl}^{ij},$$
$$s_4^{(0,0)} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1(k}u^{l)a_2b_2}\left(R_{12}^{(0,0)}\right)_{kl}^{ij}.$$

$$iQO^{(0,1)} \equiv 36s_1^{(0,0)} + 5s_2^{(0,0)} - 6s_3^{(0,0)} = 0$$

$$r_c \ll \min(\mathcal{R}, \mathcal{L}).$$

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}_D}\left(\star\,(R-2\Lambda) - \frac{1}{2}d\phi \wedge\star\, d\phi - \frac{1}{2}e^{a_q\phi}F_{q+2}\wedge\star\, F_{q+2}\right)$$

$$ds^2_{AdS_D} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{D-2}, f(r) = 1 + \frac{r^2}{L^2}$$

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}_{10}}\left(\star\, R - \frac{1}{2}d\phi \wedge\star\, d\phi - \frac{1}{2}e^{-\phi}H_3\wedge\star\, H_3 - \frac{1}{4}\sum_q\, e^{a_q\phi}F_{q+2}\wedge\star\, F_{q+2}\right)$$

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}_{11}}\left(\star\, R - \frac{1}{2}F_4\wedge\star\, F_4\right) - \frac{1}{6}\int\, A_3\wedge F_4\wedge\star\, F_4$$

$$ds^2 = ds^2_{AdS_l} + ds^2_{S^m}$$

$$ds^2_{AdS_l} = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{l-2}$$
$$ds^2_{S^m} = L^2(d\zeta^2 + \cos^2\,\zeta d\phi^2 + \sin^2\,\zeta d\Omega^2_{m-2})$$

$$f(r) \equiv 1 + \frac{r^2}{\tilde{L}^2}$$

$$\tilde{L} = 2L/(l-3), d\Omega^2_{l-2}\otimes d\Omega^2_{m-2}$$

$$A_{l-1} = -\frac{r^{l-1}}{\tilde{L}}dt\wedge d\Omega_{l-2}$$
$$A_{m-1} = \beta_m(L\sin\,\zeta)^{m-1}d\phi\wedge d\Omega_{m-2}$$

$$\beta_m = (-1)^{D-m-1}$$

$$\tilde{T}^{\mu\nu}(x) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\tilde{\delta}(x)T^{\mu\nu}(\sigma)$$

$$\tilde{\delta}(x) = \frac{\sqrt{-\gamma}}{\sqrt{-g}}\delta^{(D)}(x - X(\sigma))$$

$$\tilde{\mathcal{J}}_{q+1}(x) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\tilde{\delta}(x)\mathcal{J}_{q+1}(\sigma)$$

$$\nabla_\mu\tilde{T}^{\mu\nu} = \frac{1}{(q+1)!}F^{\nu\alpha_1\ldots\alpha_{q+1}}\tilde{\mathcal{J}}_{\alpha_1\ldots\alpha_{q+1}}$$
$$\nabla_\mu\tilde{\mathcal{J}}^{\mu\alpha_1\ldots\alpha_q} = 0$$

$$D_a T^{ab} = 0, D_a\mathcal{J}^{ab_1\ldots b_q} = 0$$

$$T^{ab}K_{ab}{}^{\mu} = \perp_\nu{}^{\mu}F^{\nu a_1\ldots a_{q+1}}\mathcal{J}_{a_1\ldots a_{q+1}},$$

$$K^{\mu}_{ab} = \perp^{\mu}_{\nu}\left(\partial_a\partial_b X^\nu + \Gamma^\nu_{\alpha\beta}\partial_a X^\alpha\partial_b X^\beta\right),$$

$$SO(p) \to SO\left(p - \sum_{l}^{m} q_l\right) \times SO(q_l) \times \ldots \times SO(q_l)$$

$$\mathcal{J}_{q+1} = \mathcal{Q}_q \mathcal{V}_{q+1}$$

$$\mathcal{C}_{q+1} \subset \mathcal{W}_{p+1}$$

$$\mathcal{V}_{q+1} = u \wedge v^{(1)} \wedge \ldots \wedge v^{(q)}$$

$$\partial_a \mathcal{Q}_p = 0$$

$$D_a\left(\mathcal{Q}_q u^a\right) + \mathcal{Q}_q u^a \dot{v}_a = 0,$$
$$D_a\left(\mathcal{Q}_q v^{a,i}\right) + \mathcal{Q}_q v^{a,i} \dot{v}_a - \mathcal{Q}_q v^{a,i} \dot{u}_a = 0,$$

$$\dot{v}_a \equiv v^{b,j} D_b v_{a,j}$$

$$P_\perp - P_\parallel = \Phi_q \mathcal{Q}_q$$

$$\hat{h}_{ab}^{(q)} = -u_a u_b + \left(1 - \delta_{q,0}\right) v_a^i v_{b,i},$$

$$\hat{h}_{ab}^{(0)} = -u_a u_b, \hat{h}_{ab}^{(1)} = -u_a u_b + v_a v_b$$

$$\hat{h}_{ab}^{(p)} = \gamma_{ab}$$

$$\hat{\perp}_b^a \left(v_{[i}^c D_c v_{j]}^b\right) = \hat{\perp}_b^a \left(v_i^c D_c u^b - u^c D_c v_i^b\right) = 0.$$

$$T_{ab} = \left(\varepsilon + P_\parallel\right) u_a u_b + \left(P_\parallel - P_\perp\right) \hat{h}_{ab}^{(q)} + P_\perp \gamma_{ab},$$

$$\varepsilon + P_\perp = \mathcal{T} s + \Phi_q \mathcal{Q}_q$$

$$d\varepsilon = \mathcal{T} ds + \Phi_q d\mathcal{Q}_q, dP_\perp = s d\mathcal{T} + \mathcal{Q}_q d\Phi_q, dP_\parallel = s d\mathcal{T} - \mathcal{Q}_q d\Phi_q$$

$$D_a(s u^a) = 0,$$
$$\left(\hat{h}^{ab} + u^a u^b\right)\left(\dot{u}_b + D_b \ln \mathcal{T}\right) = 0,$$
$$\mathcal{T} s \, \hat{\perp}^{ab} \left(\dot{u}_b + D_b \ln \mathcal{T}\right) - \Phi_q \mathcal{Q}_q \left(\hat{K}_{(q)}^a - \hat{\perp}^{ab} D_b \ln \Phi_q\right) = 0,$$

$$u^a \partial_a f|_{\partial w_{p+1}} = 0,$$

$$\prod_{j=1}^{q} \mathcal{Q}_q u^{[a} v_j^{b_j]} \partial_a f \Big|_{\partial \mathcal{W}_{p+1}} = 0$$

$$\mathcal{Q}_q v_j^a \partial_a f\big|_{\partial \mathcal{W}_{p+1}} = 0$$

$$P_{\perp}|_{\partial \mathcal{W}_{p+1}} = 0$$

$$r_0|_{\partial \mathcal{W}\ p+1} = 0,$$

$$k = \xi + \Omega\chi$$

$$u^a \partial_a = \frac{k^a \partial_a}{|k|}, |k| = \sqrt{-\gamma_{ab} k^a k^b}$$

$$v_i^a \partial_a = \mathbb{M}_i{}^j \zeta_j^a \partial_a, \mathbb{M}_i{}^k \mathbb{M}_j{}^l \zeta_k^a \zeta_{a,l} = \delta_{ij}, \zeta_j^a \partial_a \equiv \left(\psi_j^a \partial_a - \frac{k_b \psi_j^b}{k^2} k^a \partial_a \right)$$

$$\left(\hat{h}^{ab} + u^a u^b \right) D_b \ln\left(|k| \mathcal{T} \right) = 0,$$
$$\mathcal{T} s\ \hat{\perp}^{ab}\ D_b \ln\left(|k| \mathcal{T} \right) - \Phi_q \mathcal{Q}_q\ \hat{\perp}^{ab} \left(\dot{v}_b - D_b \ln\left(|k| \Phi_q \right) \right) = 0,$$

$$\mathcal{T}(\sigma) = \frac{T}{|k|},$$

$$\Phi_q(\sigma) = \frac{\phi_q(\sigma)}{\left| \hat{h}^{(q)} \right|^{1/2}}$$

$$\left| \hat{h}^{(q)} \right|^{1/2} = |k||\mathbb{M}|$$

$$\Phi_H^{(q)} = \int_{\mathcal{C}_q} d^q \sigma \left| \hat{h}^{(q)} \right|^{1/2} \Phi_q(\sigma)$$

$$v^{a,i} \dot{v}_a = v^{a,i} \dot{u}_a = u^a \dot{v}_a = 0$$

$$v_i^a D_a \mathcal{Q} = u^a D_a \mathcal{Q} = 0$$

$$-\xi^2 \left| \mathcal{W}_{p+1} = R_0^2(\sigma), \chi^2 \right| \mathcal{W}_{p+1} = R^2(\sigma),$$

$$n^a \partial_a = \frac{1}{R_0} \xi^a \partial_a$$

$$n^a u_a = -\cosh\ \eta$$

$$\mathcal{Q}_k = \int_{\mathcal{B}_p} dV_p (T^{\mu\nu} + \mathcal{V}^{\mu\nu}) n_\mu k_\nu \Big|_{x^\mu = X^\mu}$$

$$\mathcal{V}^{\mu\nu} = \frac{1}{q!} A^\nu_{\alpha_1 \ldots \alpha_q} \mathcal{J}^{\mu \alpha_1 \ldots \alpha_q}$$

$$M = \int_{\mathcal{B}_p} dV_p (T^{\mu\nu} + \mathcal{V}^{\mu\nu}) n_\mu \xi_\nu \Big|_{x^\mu = X^\mu}, J = -\int_{\mathcal{B}_p} dV_p (T^{\mu\nu} + \mathcal{V}^{\mu\nu}) n_\mu \chi_\nu \Big|_{x^\mu = X^\mu}$$

$$\omega_q = \frac{-\mathcal{V}_{q+1}\cdot n}{\sqrt{-\hat{h}^{(q)}_{ab}n^a n^b}},$$

$$Q_q = -\int_{\mathcal{B}_{p-q}} dV_{p-q}\mathcal{J}_{q+1}\cdot(n\wedge\omega_q) = \int_{\mathcal{B}_{p-q}} dV_{p-q}\sqrt{-\hat{h}^{(q)}_{ab}n^a n^b}\mathcal{Q}_q$$

$$\Phi_H^{(q)} = \int_{\mathcal{C}_q} dV_q \frac{R_0}{\sqrt{-\hat{h}^{(q)}_{ab}n^a n^b}}\Phi_q$$

$$I = -\int^{\star}_{\mathcal{W}_{p+1}} {}_{(p+1)}\mathcal{G}$$

$$\mathcal{G} = \varepsilon - \mathcal{T}s - \sum_q^p \Phi_q\mathcal{Q}_q$$

$$\bar{\Phi}_H^{(p)} = R_0\int_{\mathcal{B}_p} {}^{\star}(p)\Phi_p - \int_{\mathcal{B}_p}\mathbb{P}[A_{q+1}]$$

$$\tilde{I} = -\int_{\mathcal{B}_p} {}^{\star}(p)R_0\left(\mathcal{G}+\sum_{q\neq p}\Phi_q\mathcal{Q}_q\right) - \bar{\Phi}_H^{(p)}\mathcal{Q}_p = \int_{\mathcal{B}_p} {}^{\star}{}_{(p)}\left(R_0 P_{\|} - \mathbb{P}[A_{q+1}]\right)$$

$$I = \int_{\mathcal{W}_{p+1}} \left({}^{\star}(p+1)P_{\|} + Q_p\mathbb{P}[A_{p+1}]\right)$$

$$\tilde{I}_E = M - \Omega J - TS,$$

$$S = -\int_{\mathcal{B}_p} dV_p s u^a n_a$$

$$\varepsilon = \frac{\Omega_{n+1}}{16\pi G}r_0^n(1+n+nN\sinh^2\alpha)$$

$$\mathcal{T} = \frac{n}{4\pi r_0\cosh^N\alpha}, s = \frac{\Omega_{n+1}}{4G}r_0^{n+1}\cosh^N\alpha$$

$$\mathcal{Q}_q = \frac{\Omega_{n+1}}{16\pi G}n\sqrt{N}r_0^n\sinh\alpha\cosh\alpha, \Phi_q = \sqrt{N}\tanh\alpha$$

$$P_{\|} = -\frac{\Omega_{n+1}}{16\pi G}r_0^n(1+nN\sinh^2\alpha), P_{\perp} = -\frac{\Omega_{n+1}}{16\pi G}r_0^n$$

$$a_q^2 = \frac{4}{N} - \frac{2(p+1)n}{D-2} \Longrightarrow N \leq 2\left(\frac{n+p+1}{n(p+1)}\right)$$

$$\mathcal{G} = \frac{\Omega_{n+1}}{16\pi G}r_0^n$$

$$\varepsilon = -(n+1)P_{\perp} + \Phi_q \mathcal{Q}_q$$
$$\varepsilon = -(n+1)P - n\Phi_p \mathcal{Q}_p$$

$$T_{ab} = \mathcal{T}s\left(u_a u_b - \frac{1}{n}\gamma_{ab}\right) - \Phi_q \mathcal{Q}_q \hat{h}_{ab}^{(q)}$$

$$r_0 = \frac{n|k|}{4\pi T}\left(1 - \frac{\Phi_H^2}{c^2 N\left|\hat{h}^{(q)}\right|}\right)^{N/2}, \tanh\,\alpha = \frac{\Phi_H}{\sqrt{N}c\left|\hat{h}^{(q)}\right|^{1/2}}$$

$$c\left|\hat{h}^{(q)}\right|^{1/2} = \begin{cases} |k| \text{ for } q = 0 \\ \Omega_q |k||\mathbb{M}| \text{ for } 0 < q < p \\ \Omega_p |\gamma| \text{ for } q = p \end{cases}$$

$$(D-3)M - (D-2)(TS + \Omega J) - (D-q-3)\Phi_H^{(q)} Q_q = \mathcal{T}_{tot}$$

$$\mathcal{T}_{\text{tot}} = -\int_{\mathcal{B}_p} dV_p R_0 \left(\gamma^{ab} + n^a n^b\right) T_{ab}$$

$$F = M - TS - \Omega J - \Phi_H^{(q)} Q_q$$

$$T_{ab} = -\sqrt{N}\mathcal{Q}_q \hat{h}_{ab}^{(q)}$$

$$-\sqrt{N}\mathcal{Q}_p \hat{h}_{ab}^{(p)} = P\gamma_{ab}$$

$$M = (q+1)\Phi_H^{(q)} Q_q - \mathcal{T}_{tot}, \Omega J = q\Phi_H^{(q)} Q_q - \mathcal{T}_{tot}.$$

$$M = \Phi_H^{(q)} Q_q + \Omega J$$

$$\sqrt{\mathcal{T}}su^a = \mathcal{K}^{1/2} l^a$$

$$T_{ab} = \mathcal{K} l_a l_b - \sqrt{N}\mathcal{Q}_q \hat{h}_{ab}^{(q)}$$

$$\mathcal{T}_{tot} = -\int_{\mathcal{B}_p} dV_p R_0 \left(\mathcal{K} - q\sqrt{N}\mathcal{Q}_q\right)$$
$$M = \int_{\mathcal{B}_p} dV_p R_0 \left(\mathcal{K} + \sqrt{N}\mathcal{Q}_q\right)$$
$$J = \int_{\mathcal{B}_p} dV_p \mathcal{R}\mathcal{K}, \mathcal{R} \equiv l_a \chi^a$$

$$M = (q+1)\Phi_H^{(q)} Q_q - \mathcal{T}_{tot} = \frac{1}{q}\left(V_p R_0 (q+1)\mathcal{K} + \mathcal{T}_{tot}\right)$$
$$\frac{J}{\mathcal{R}} = \frac{1}{R_0}\left(q\Phi_H^{(q)} Q_q - \mathcal{T}_{tot}\right) = V_p \mathcal{K}$$

$$M = V_p R_0 \sqrt{N} Q_p + R_0 \frac{J}{\mathcal{R}},$$

$$\sigma^0 = t = \tau, r = R, \sigma_i = \phi_i, \sigma_j = \mu_i$$

$$\gamma_{ab}d\sigma^a d\sigma^b = -f(R)d\tau^2 + R^2\sum_{i=1}^{k+1}\left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right), \sum_{i=1}^{k+1}\mu_i^2 = 1$$

$$k^a\partial_a = \partial_\tau + \Omega\partial_\Phi, \partial_\Phi \equiv \sum_{i=1}^{k+1}\partial_{\phi_i}$$

$$\zeta = \frac{1}{|k|^2}(\Omega^2 R^2\partial_\tau + f(R)\Omega\partial_\Phi)$$

$$\tilde{I}_E = \frac{\Omega_{n+1}V_p}{16\pi G}\sqrt{f(R)}\left(\frac{n}{4\pi T}\right)^n (f(R) - \Omega^2R^2)^{n/2}\left(1 - \frac{\Phi_H^2}{c^2N\left|\hat{h}^{(q)}\right|}\right)^{nN/2}$$

$$\Omega^2 = \frac{f(R)}{R^2}\frac{p\mathcal{T}s + qn\Phi_q\mathcal{Q}_q + \frac{R^2}{L^2}\Xi_q}{(n+p)\mathcal{T}s + \left(\delta_{q,0}+q\right)n\Phi_q\mathcal{Q}_q + \frac{R^2}{L^2}\Xi_q},$$

$$\Xi_q = \mathcal{T}s(1+n+p) + n(1+q)\Phi_q\mathcal{Q}_q$$

$$M = V_pR_0\left(\cosh^2\eta\left(\mathcal{T}s + \delta_{q,0}\Phi_q\mathcal{Q}_q\right) + \frac{\mathcal{T}s}{n} + \left(1-\delta_{q,0}\right)\Phi_q Q_q\right)$$
$$J = V_pR\left(\mathcal{T}s + \delta_{q,0}\Phi_q\mathcal{Q}_q\right)\cosh\eta\sinh\eta$$
$$S = V_p s\cosh\eta, T = \mathcal{T}R_0\mathrm{sech}\eta$$

$$\Phi_H^{(0)} = R_0\mathrm{sech}\eta\Phi_0, Q_0 = V_p\cosh\eta\mathcal{Q}_0$$
$$\Phi_H^{(1)} = 2\pi RR_0\Phi_1, Q_1 = \frac{V_p}{2\pi R}\mathcal{Q}_1$$
$$\Phi_H^{(p)} = V_pR_0\Phi_p, Q_p = \mathcal{Q}_p$$

$$\mathcal{T}_{tot} = -\frac{V_p}{n}\sqrt{f(R)}\frac{R^2}{L^2}\Xi_q$$

$$\Omega^2 = \frac{f'(R)}{R^2}\frac{p\mathcal{T}s + qn\Phi_q\mathcal{Q}_q - \frac{R^2}{L^2}\Xi_q}{(n+p)\mathcal{T}s + \left(\delta_{q,0}+q\right)n\Phi_q\mathcal{Q}_q - \frac{R^2}{L^2}\Xi_q}, f'(R) = 1 - \frac{R^2}{L^2}.$$

$$\left(\frac{(n+p)\mathcal{T}s + \left(\delta_{q,0}+q\right)n\Phi_q\mathcal{Q}_q}{\Xi_q}\right)^{1/2} < \frac{R}{L} \leq 1 \vee \frac{R}{L} \leq \left(\frac{p\mathcal{T}s + qn\Phi_q\mathcal{Q}_q}{\Xi_q}\right)^{1/2}$$

$$\mathcal{T}_{tot} = \frac{V_p}{n}\sqrt{f(R)}\frac{R^2}{L^2}\Xi_q,$$

$$\Omega^2 = \pm\frac{1}{L^2}$$

$$M = \frac{J}{L} + \sqrt{Nf(R)}Q_0$$

$$l = \frac{1}{\sqrt{f(R)}}\partial_\tau + \frac{1}{R}\partial_\Phi$$

$$R = \frac{L(q+1)J}{\sqrt{L^2(qM - \mathcal{T}_{tot})^2 - (q+1)^2 J^2}}$$

$$Q_q = \frac{(M + \mathcal{T}_{tot})(L^2(qM - \mathcal{T}_{tot})^2 - (q+1)^2 J^2)^{\frac{2q+1}{2}}}{\sqrt{N}\Omega_q (L(q+1))^{q+1} J^q (qM - \mathcal{T}_{tot})}$$

$$M = V_q \sqrt{N f(R)} Q_q + \sqrt{f(R)}\frac{J}{R}$$

$$R = \frac{L(q+1)J}{\sqrt{L^2(qM - \mathcal{T}_{tot})^2 + (q+1)^2 J^2}}$$

$$\lim_{\alpha\to\infty}\left(\lim_{\Omega\to 0}\Omega^2\right) \Longrightarrow \frac{R^2}{L^2} = \frac{p}{1+p} \vee \frac{R^2}{L^2} = 1$$

$$\mathcal{B}_p = \prod_{a=1}^{b} S^{pa}$$

$$p_a = 2k_a + 1$$

$$\sum_{a=1}^{b} p_a = p$$

$$\prod_a \mathbb{S}^{p_a} \times \mathbb{S}^{n+1}$$

$$\gamma_{ab} d\sigma^a d\sigma^b = -f(R)d\tau + \sum_{a=1}^{b}\sum_{i=1}^{k+1} R_a^2\left(d\mu_{a,i}^2 + \mu_{a,i}^2 d\phi_{a,i}^2\right), R^2 \equiv \sum_{a=1}^{b} R_a^2, \sum_{i=1}^{k_a+1} \mu_i^2 = 1$$

$$k = \partial_\tau + \sum_{a=1}^{b} \Omega_a \partial_{\Phi_a}, \partial_{\Phi_a} \equiv \sum_{i=1}^{k+1} \partial_{\phi_{a,i}}$$

$$\psi^\mu \partial_\mu = \sum_a^m \Omega_a \partial_{\Phi_a}$$

$$\zeta = \frac{1}{|k|^2}\sum_{a=1}^{b}\left(\Omega_a^2 R_a^2 \partial_\tau + f(R)\Omega_a \partial_{\Phi_a}\right)$$

$$\tilde{I}_E = \frac{\Omega_{n+1} V_p}{16\pi G}\left(\frac{n}{4\pi T}\right)^n \left(f(R) - \sum_{a=1}^{b}\Omega_a^2 R_a^2\right)^{n/2}\left(1 - \frac{\Phi_H^2}{c^2 N\left|\hat{h}^{(q)}\right|}\right)^{nN/2},$$

$$V_p = \prod_{a=1}^{m} V_{p_a} \text{ and } c^2\left|\hat{h}^{(q)}\right|$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2}\frac{p_a\mathcal{T}s + q_a n\Phi_q\mathcal{Q}_q + \frac{R_a^2}{L^2}\Xi_q}{(n+p)\mathcal{T}s + \left(\delta_{q,0}+q\right)n\Phi_q\mathcal{Q}_q + \frac{R^2}{L^2}\Xi_q}, q_a = (0, p_a),$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2}\frac{\left(p_a\mathcal{T}s + \frac{R_a^2}{L^2}\Xi_1\right)\left(p\mathcal{T}s + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{L^2}\Xi_1\right)}{\left(p\mathcal{T}s + \frac{R^2}{L^2}\Xi_1\right)\left((n+p)\mathcal{T}s + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{L^2}\Xi_1\right)}, q = 1.$$

$$\tanh\,\eta = \frac{\sum_{a=1}^{b}\Omega_a R_a}{\sqrt{f(R)}}$$

$$J_a = V_p\frac{\Omega_a R_a^2}{|k|}\mathcal{T}s\cosh\,\eta$$

$$\frac{R_a}{L} \leq \left(\frac{p_a\mathcal{T}s + q_a n\Phi_q\mathcal{Q}_q}{\Xi_q}\right)^{1/2} \vee$$

$$\frac{R_a}{L} \geq \left(\frac{p_a\mathcal{T}s + q_a n\Phi_q\mathcal{Q}_q}{\Xi_q}\right)^{1/2} \wedge \left(\frac{(n+p)\mathcal{T}s + \left(\delta_{q,0}+q\right)n\Phi_q\mathcal{Q}_q}{\Xi_q}\right)^{1/2} < \frac{R}{L} < 1$$

$$\frac{R_a}{L} \leq \left(\frac{p_a\mathcal{T}s}{\Xi_1}\right)^{1/2} \vee \frac{R_a}{L} \geq \left(\frac{p_a\mathcal{T}s}{\Xi_1}\right)^{1/2} \wedge$$

$$\left(\frac{R}{L} \leq \left(\frac{p\mathcal{T}s + n\Phi_1\mathcal{Q}_1}{\Xi_1}\right)^{1/2} \vee \left(\frac{(n+p)\mathcal{T}s}{\Xi_1}\right)^{1/2} < \frac{R}{L} < 1\right)$$

$$M = \sum_{a}^{b}\Omega_a J_a + \Phi_H^{(q)}Q_q$$

$$\Longrightarrow M = \sqrt{f(R)}\frac{J}{L} + \sqrt{Nf(R)}Q_0$$

$$J \equiv \sum_{a}^{b} J_a$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2}\frac{p_a + (1+p)\frac{R_a^2}{L^2}}{p + (1+p)\frac{R^2}{L^2}} \equiv \frac{f(R)}{R_a^2}\rho_a$$

$$\sum_{a=1}^{b}\Omega_a^2 R_a^2 = f(R)$$

$$l = \frac{1}{\sqrt{f(R)}}\partial_t + \sum_{a=1}^{b}\left(\frac{\rho_a}{R_a^2}\right)^{1/2}\partial_{\Phi_a}$$

$$l_a \chi^a = \sum_{a=1}^{b} R_a \sqrt{\rho_a}$$

$$J_a = V_p \mathcal{K} \sum_{a=1}^{b} R_a \sqrt{\rho_a}$$

$$M = \frac{q+1}{q}\left(\sum_{a=1}^{b} \frac{\sqrt{f(R)} J_a}{R_a \sqrt{\rho_a}} + \frac{\mathcal{T}_{tot}}{q+1}\right)$$

$$\lim_{\alpha \to \infty}\left(\lim_{\Omega \to 0} \Omega^2\right) \Longrightarrow \frac{R_a^2}{L^2} = \frac{p_a}{1+p} \vee \frac{R^2}{L^2} = 1$$

$$\gamma_{ab} d\sigma^a d\sigma^b = -f(r) d\tau^2 + f(r)^{-1} dr^2 + r^2 d\phi^2$$

$$k^a \partial_a = \partial_\tau + \Omega \partial_\phi$$

$$\zeta^a \partial_a = \frac{1}{|k|^2}\left(\Omega r^2 \partial_\tau + f(r) \partial_\phi\right)$$

$$r_0(r) = \frac{n}{4\pi T}\left(f(r) - \Omega^2 r^2\right)^{\frac{1-N}{2}}\left(f(r) - \Omega^2 r^2 - \frac{\Phi_H^2}{N}\right)^{\frac{N}{2}}$$

$$\tanh\, \alpha(r) = \frac{\Phi_H}{\sqrt{N}\sqrt{f(r) - \Omega^2 r^2}}$$

$$r = r_{\max} = \sqrt{\frac{1 - \frac{\Phi_H^2}{N}}{\Omega^2 - \frac{1}{L^2}}},$$

$$\tilde{I}_E = \frac{\Omega_{(n+1)}}{8G}\left(\frac{n}{4\pi T}\right)^n \int_0^{r_{\max}} dr r \left(f(r) - \Omega^2 r^2\right)^{\frac{n-nN}{2}}\left(f(r) - \Omega^2 r^2 - \frac{\Phi_H^2}{N}\right)^{\frac{nN}{2}}$$
$$= \frac{\Omega_{(n+1)}}{8G}\left(\frac{n}{4\pi T}\right)^n \frac{x^{2+nN}}{(2+nN)\left(\Omega^2 - \frac{1}{L^2}\right)}\, {}_2F_1\left(1, \frac{1}{2}(N-1)n; \frac{Nn}{2} + 2; x^2\right)$$

$$x \equiv \left(1 - \frac{\Phi_H^2}{N}\right)^{1/2}$$

$$r_0(r) = \frac{n}{4\pi T}\left(f(r) - \Omega^2 r^2\right)^{\frac{1}{2}}\left(1 - \frac{\Phi_H^2}{N f(r)(2\pi r)^2}\right)^{\frac{N}{2}}$$

$$\tanh\, \alpha(r) = \frac{\Phi_H}{2\pi\sqrt{N}\sqrt{f(r)} r}$$

$$r_{\min} = \frac{L}{\sqrt{2}}\sqrt{\sqrt{1+\frac{\Phi_H^2}{NL^2\pi^2}}-1} \le r \le r_{\max} = \sqrt{\frac{1}{\Omega^2-\frac{1}{L^2}}},$$

$$\tilde{I}_E = \frac{\Omega_{(n+1)}}{8G}\left(\frac{n}{4\pi T}\right)^n \int_{r_{\min}}^{r_{\max}} drr(f(r)-\Omega^2 r^2)^{\frac{n}{2}}\left(1-\frac{\Phi_H^2}{Nf(r)(2\pi r)^2}\right)^{\frac{nN}{2}}$$

$$\Phi_H \to \sqrt{N} \text{ as } \alpha \to \infty$$

$$r_{\max}\sqrt{\Omega^2 - 1/L^2} \to 0$$

$$\Phi_H \le \frac{2\pi\Omega\sqrt{N}}{\Omega^2 - 1/L^2}$$

$$r_{\max}\sqrt{\Omega^2 + 1/L^2} \to 0$$

$$r_{\max} = L\sqrt{1-\Phi_H^2/N}$$

$$r_{\min} = \frac{L}{\sqrt{2}}\sqrt{1-\sqrt{1-\frac{\Phi_H^2}{NL^2\pi^2}}} \le r \le r'_{\max} = \frac{L}{\sqrt{2}}\sqrt{1+\sqrt{1-\frac{\Phi_H^2}{NL^2\pi^2}}},$$

$$T_{ab} = \mathcal{T}s u_a u_b - \mathcal{G}\gamma_{ab} - \sum_q^p \Phi_q \mathcal{Q}_q \hat{h}_{ab}^{(q)},$$

$$\varepsilon = \frac{n+1}{n}\mathcal{T}s + \sum_q \Phi_q \mathcal{Q}_q \Rightarrow \mathcal{G} = \frac{1}{n}\mathcal{T}s$$

$$T_{ab} = \mathcal{T}s\left(u_a u_b + \frac{1}{n}\gamma_{ab}\right) - \sum_q \Phi_q \mathcal{Q}_q \hat{h}_{ab}^{(q)},$$

$$(D-3)M - (D-2)(TS+\Omega J) - \sum_q^p (D-q-3)\Phi_H^{(q)} Q_q = \mathcal{T}_{tot},$$

$$\mathcal{T}_{\text{tot}} = -\int_{\mathcal{B}_p} dV_p \left(R_0 T + (D-2)\mathcal{L}_p + (T^{\mu\nu} + \mathcal{V}^{\mu\nu})n_\mu \xi_\nu\big|_{x^\mu = X^\mu}\right)$$

$$\mathcal{L}_p \equiv \mathcal{Q}_p \mathbb{P}[A_{l-1}]$$

$$\mathcal{T}s\left(\dot{u}^\mu - \frac{1}{n}K^\mu\right) - \sum_q \Phi_q \mathcal{Q}_q \hat{K}_{(q)}^\mu = \mathcal{F}^\mu$$

$$\mathcal{F}^\mu = \perp_\nu {}^\mu F^{\nu a_1 \dots a_{q+1}} \mathcal{J}_{a_1 \dots a_{q+1}}/(q+1)!$$

$$\hat{K}^{\mu}_{(0)} = -\perp^{\mu}_{\nu} \dot{u}^{\nu}, \hat{K}^{\mu}_{(q)} = -\perp^{\mu}_{\nu} (\dot{u}^{\nu} - \dot{v}^{\nu}), \hat{K}^{\mu}_{(p)} = K^{\mu},$$

$$I = -\int_{\mathcal{W}_{p+1}} \mathcal{Q}_p \left(\star_{(p+1)} \mathbf{1} - \mathbb{P}[A_{l-1}] \right)$$

$$M = \Omega J + \Phi_H^{(p)} Q_p (1 - \tilde{A}), \tilde{A} \equiv \frac{A_{\sigma^0 \dots \sigma^p}}{R_0 R^p}$$

$$R_0 V_p Q_p = \Phi_H^{(p)} Q_p$$

$$\begin{aligned} M &= \Phi_H Q_p ((n + p + 1)\tilde{A} + p + 1) - \mathcal{T}_{tot} \\ \Omega J &= \Phi_H Q_p ((p + n)\tilde{A} + p) - \mathcal{T}_{tot} \end{aligned}$$

$$M = \Omega J + \sum_{q}^{p} \Phi_H^{(q)} Q_q - \Phi_H^{(p)} Q_p \tilde{A}$$

$$T_{ab} = \mathcal{K} l_a l_b - \sum_{q} \Phi_q \mathcal{Q}_q \hat{h}^{(q)}_{ab}$$

$$\mathcal{T}_{tot} = -\int_{\mathcal{B}_p} dV_p R_0 \left(\mathcal{K} - \sum_{q} q \Phi_q \mathcal{Q}_q + \tilde{\mathcal{V}}_{\mathcal{T}} \right)$$

$$M = \int_{\mathcal{B}_p} dV_p R_0 \left(\mathcal{K} + \sum_{q} \Phi_q \mathcal{Q}_q + \tilde{\mathcal{V}}_M \right)$$

$$J = \int_{\mathcal{B}_p} dV_p \left(\mathcal{R}\mathcal{K} + \tilde{\mathcal{V}}_J \right)$$

$$\tilde{\mathcal{V}}_M = \mathcal{V}^{\mu\nu} n_\mu \xi_\nu / R_0$$

$$\tilde{\mathcal{V}}_{\mathcal{T}} = \tilde{\mathcal{V}}_M + (D - 2)\mathcal{L}_p$$

$$M = V_p R_0 \left(\tilde{\mathcal{V}}_M - \frac{\tilde{\mathcal{V}}_J}{\mathcal{R}} + \sum_{q}^{p} \Phi_q \mathcal{Q}_q \right) + R_0 \frac{J}{\mathcal{R}}$$

$$\sigma^0 = t = \tau, r = R, \sigma_i = \theta_i, \sigma_j = \mu_i, \zeta = \rho_i = 0$$

$$\begin{aligned} \dot{u}^r &= \cosh^2 \eta \left(\frac{R}{\tilde{L}^2} - \Omega^2 R \right), \dot{v}^r = \frac{R}{\tilde{L}^2} \sinh^2 \eta - \frac{f(R)}{R} \cosh^2 \eta \\ K^r &= -\left(\frac{R}{\tilde{L}^2} + \frac{p}{R} f(R) \right), \mathcal{F}^r_{p+2} = \frac{l - 1}{\tilde{L}} \sqrt{f(R)} \mathcal{Q}_p \end{aligned}$$

$$\Omega^2 = \frac{f(R)}{R^2}\frac{p\left(\mathcal{T}s + n\Phi_p\mathcal{Q}_p\right) + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p} - \frac{Rn(l-1)}{\tilde{L}}\sqrt{f(R)}\mathcal{Q}_p}{(n+p)\mathcal{T}s + pn\Phi_p\mathcal{Q}_p + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p} - \frac{Rn(l-1)}{\tilde{L}}\sqrt{f(R)}\mathcal{Q}_p},$$

$$\Xi_{1,p} \equiv \mathcal{T}s(1+n+p) + n(1+p)\Phi_p\mathcal{Q}_p + 2n\Phi_1\mathcal{Q}_1$$

$$\mathcal{V}^{\mu\nu}n_\mu\chi_\nu\big|_{x^\mu = X^\mu} = 0$$

$$M = V_p\left(R_0\left(\cosh^2\,\eta\left(\mathcal{T}s + \delta_{\bar{q},0}\Phi_{\bar{q}}\mathcal{Q}_{\bar{q}}\right) + \frac{\mathcal{T}s}{n} + \Phi_p\mathcal{Q}_p + \left(1-\delta_{\bar{q},0}\right)\Phi_{\bar{q}}\mathcal{Q}_{\bar{q}}\right) - \frac{R\mathcal{Q}_p}{\tilde{L}}\right)$$

$$\mathcal{T}_{\text{tot}} = -\frac{V_p}{n}\left(R_0\frac{R^2}{L^2}\Xi_{\bar{q},p} + n\mathcal{Q}_p\frac{R}{L}(R_0^2(l-1)+1)\right)$$

$$\sigma^0 = t = \tau, r = \theta_i = 0, \sigma_i = \rho_i, \sigma_j = \mu_i, \zeta = \zeta_0$$

$$\gamma_{ab}d\sigma^a d\sigma^b = -d\tau^2 + R^2 d\Omega_{m-2}$$

$$\mathcal{V}^{\mu\nu}n_\mu\xi_\nu = 0 \;\text{ since } A^t\cdot\xi_t = 0$$
$$\mathcal{V}^{\mu\nu}n_\mu\chi_\nu = 0 \;\text{ since } A^{\chi_i}\cdot J^t = 0$$

$$R_0 \equiv \sqrt{-\xi^2}\Big|_{\mathcal{W}_{p+1}} = 1$$

$$\Omega^2 = \frac{1}{R^2}\frac{p\left(\mathcal{T}s + n\Phi_p\mathcal{Q}_p\right) + n\Phi_1\mathcal{Q}_1}{(n+p)\mathcal{T}s + pn\Phi_p\mathcal{Q}_p + n\Phi_1\mathcal{Q}_1},$$

$$d\Omega_{m-2}^2 = d\delta_1^2 + \cos^2\,\delta_1 d\delta_2^2 + \sin^2\,\delta_1 d\Omega_{m-4}^2$$

$$\sigma^0 = t = \tau, r = R_1, \sigma_1 = \theta_1, \theta_{2,3} = \pi/2, \zeta = \zeta_0, \sigma_2 = \phi, \delta_1 = \delta_0, \sigma_3 = \delta_2, \rho_1 = 0$$

$$\sigma^0 = t = \tau, r = R_1, \sigma_i = \theta_i, \sigma_j = \mu_i, \zeta = \zeta_0, \sigma_5 = \phi, \delta_1 = \delta_0, \sigma_6 = \delta_2, \rho_1 = 0$$

$$d\Omega_5^2 = d\omega_1^2 + \cos^2\,\omega_1 d\omega_2^2 + \sin^2\,\omega_1 d\Omega_3^2$$

$$\gamma_{ab}d\sigma^a d\sigma^b = -f(R_1)d\tau^2 + R_1^2 d\Omega_{p_a}^2 + R_2^2 d\phi^2 + R_3^2 d\delta_2^2$$

$$k^a\partial_a = \partial_\tau + \Omega_1\partial_\Theta + \Omega_2\partial_\phi + \Omega_3\partial_{\delta_2}, \zeta^a\partial_a = \frac{1}{|k|^2}\left(\sum_{a=1}^{3}\Omega_a^2R_a^2\partial_\tau + f(R_1)\left(\Omega_1\partial_\Theta + \Omega_2\partial_\phi + \Omega_3\partial_{\delta_2}\right)\right),$$

$$\dot{u}^r = \frac{f(R_1)}{|k|^2}\left(\frac{R_1}{L^2} - \Omega_1^2R_1\right), \dot{u}^\zeta = \frac{R_2}{L^2|k|^2}\left(\Omega_2^2c_\zeta + \frac{\Omega_3^2R_3^2}{c_\zeta}\right), \dot{u}^{\delta_1} = \frac{\Omega_3^2R_3c_{\delta_1}}{|k|^2},$$

$$\dot{v}^r = \frac{1}{|k|^2}\left(\frac{R_1}{L^2}\sum_{a=1}^{2}\Omega_a^2R_a^2 - \frac{\Omega_1^2R_1f(R_1)}{\sum_{a=1}^{2}\Omega_a^2R_a^2}\right), \dot{v}^\zeta = \frac{f(R_1)}{\sum_{a=1}^{3}\Omega_a^2R_a^2}\dot{u}^\zeta, \dot{v}^{\delta_1} = \frac{f(R_1)\dot{u}^{\delta_1}}{\sum_{a=1}^{3}\Omega_a^2R_a^2},$$

$$K^r = -\left(\frac{R_1}{L^2} + \frac{p_a}{R_1}f(R_1)\right), K^\zeta = \frac{c_\zeta}{R_2} + \frac{R_3^2}{R_2L^2c_\zeta}, K^{\delta_1} = \frac{c_{\delta_1}}{R_3},$$

$$c_{\delta_1} = \sqrt{L^2 - R_2^2 - R_3^2}/c_\zeta^2$$

$$c_\zeta = \sqrt{L^2 - R_2^2}$$

$$\Omega_1^2 = \frac{f(R_1)}{R_1^2}\frac{\left(p_a(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + n\Phi_1\mathcal{Q}_1 + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s + n(p\Phi_p\mathcal{Q}_p + \Phi_1\mathcal{Q}_1) + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\Omega_a^2 = \frac{f(R_1)}{R_a^2}\frac{(\mathcal{T}s + n\Phi_p\mathcal{Q}_p)\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + n\Phi_1\mathcal{Q}_1 + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p(\mathcal{T}s + \Phi_p\mathcal{Q}_p) + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s + p\Phi_p\mathcal{Q}_p + \Phi_1\mathcal{Q}_1 + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\sigma^0 = t = \tau, r = R, \omega_1 = \omega_0, \sigma_1 = \omega_2, \sigma_2 = \theta, \zeta = \zeta_0, \phi = \phi_0, \sigma_3 = \rho_1, \rho_{2,3} = \pi/2$$

$$\sigma^0 = t = \tau, r = R, \omega_1 = \omega_0, \sigma_1 = \omega_2, \sigma_2 = \theta, \zeta = \zeta_0, \phi = \phi_0, \sigma_i = \rho_i, \sigma_j = \mu_i,$$

$$\gamma_{ab}d\sigma^a d\sigma^b = -f(R)d\tau^2 + R_1^2 d\theta^2 + R_2^2 d\omega^2 + R_3^2 d\Omega_{p_a}^2,$$

$$k^a\partial_a = \partial_\tau + \Omega_1\partial_{\theta_1} + \Omega_2\partial_{\omega_2} + \Omega_3\partial_{\mathcal{P}}, \zeta^a\partial_a = \frac{1}{|k|^2}\left(\sum_{a=1}^{3}\Omega_a^2 R_a^2\partial_\tau + f(R)\left(\Omega_1\partial_{\omega_2} + \Omega_2\partial_\phi + \Omega_3\partial_{\mathcal{P}}\right)\right),$$

$$\dot{u}^r = \frac{f(R)}{|k|^2}\left(\frac{R}{L^2} - \frac{\sum_a^2 \Omega_a^2 R_a^2}{R}\right), \dot{u}^{\omega_1} = \frac{R_1 R_2}{R^2|k|^2}(\Omega_1^2 - \Omega_2^2), \dot{u}^\zeta = -\frac{\Omega_3^2 R_3\sqrt{L^2 - R_3^2}}{L^2|k|^2}$$

$$\dot{v}^r = \frac{1}{|k|^2}\left(\frac{R}{L^2}\sum_a^3 \Omega_a^2 R_a^2 - \frac{\sum_a^2 \Omega_a^2 R_a f(R)}{\sum_a^3 \Omega_a^2 R_a^2}\right), \dot{v}^{\omega_1} = \frac{f(R)\dot{u}^{\omega_1}}{\sum_a^2 \Omega_a^2 R_a^2}, \dot{v}^\zeta = \frac{f(R)\dot{u}^\zeta}{\sum_a^2 \Omega_a^2 R_a^2}$$

$$K^r = -\left(\frac{R_1}{L^2} + 2\frac{f(R)}{R_1}\right), K^{\omega_1} = \frac{1}{R^2}\left(\frac{R_2}{R_1} - \frac{R_1}{R_2}\right), K^\zeta = \frac{p_a\dot{u}^\zeta}{\Omega_3^2 R_3^2}$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2}\frac{\left(\mathcal{T}s + n\Phi_p\mathcal{Q}_p + \frac{R_a^2}{\tilde{L}^2}\Xi_{1,p}\right)\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s + n(p\Phi_p\mathcal{Q}_p + \Phi_1\mathcal{Q}_1) + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\Omega_3^2 = \frac{f(R)}{R_3^2}\frac{\left(p_a(\mathcal{T}s + n\Phi_p\mathcal{Q}_p)\right)\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p(\mathcal{T}s + \Phi_p\mathcal{Q}_p) + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s + p\Phi_p\mathcal{Q}_p + \Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\sigma^0 = t = \tau, r = R_1, \sigma_1 = \theta_1, \theta_2 = \pi/2, \zeta = \zeta_0, \sigma_2 = \phi, \delta_1 = \delta_0, \sigma_i = \rho_i, \sigma_j = \mu_i$$

$$\gamma_{ab}d\sigma^a d\sigma^b = -f(R_1)d\tau^2 + R_1^2 d\theta_1^2 + R_2^2 d\phi^2 + R_3^2 d\Omega_3^2$$

$$k^a\partial_a = \partial_\tau + \Omega_1\partial_{\theta_1} + \Omega_2\partial_\phi + \Omega_3\partial_{\mathcal{P}}, \zeta^a\partial_a = \frac{1}{|k|^2}\left(\sum_{a=1}^{3}\Omega_a^2 R_a^2\partial_\tau + f(R)\left(\Omega_1\partial_{\theta_1} + \Omega_2\partial_\phi + \Omega_3\partial_{\mathcal{P}}\right)\right)$$

$$K^{\zeta}=\frac{c_{\zeta}}{R_2}+\frac{3R_3^2}{R_2L^2c_{\zeta}}$$

$$\Omega_1^2=\frac{f(R_1)}{R_1^2}\frac{\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p+\frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+n\Phi_1\mathcal{Q}_1+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+\frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s+n\left(p\Phi_p\mathcal{Q}_p+\Phi_1\mathcal{Q}_1\right)+\frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\Omega_2^2=\frac{f(R_1)}{R_2^2}\frac{\left(\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)\right)\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+n\Phi_1\mathcal{Q}_1+\frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+\frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s+n\left(p\Phi_p\mathcal{Q}_p+\Phi_1\mathcal{Q}_1\right)+\frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\Omega_3^2=\frac{f(R_1)}{R_3^2}\frac{\left(3\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)\right)\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+n\Phi_1\mathcal{Q}_1+\frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p\left(\mathcal{T}s+\Phi_p\mathcal{Q}_p\right)+\frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s+p\Phi_p\mathcal{Q}_p+\Phi_1\mathcal{Q}_1+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\sigma^0=t=\tau,r=R_1,\omega_1=\omega_0,\sigma_1=\omega_2,\sigma_{2,3}=\theta_{2,3},\sigma_{4,5}=\mu_{2,3},\zeta=\zeta_0,\sigma_6=\phi$$
$$\delta_1=\pi/2$$

$$\gamma_{ab}d\sigma^a d\sigma^b=-f(R)d\tau^2+R_1^2d\omega_2^2+R_2^2d\Omega_3^2+R_3^2d\phi^2$$

$$k^a\partial_a=\partial_\tau+\Omega_1\partial_{\omega_2}+\Omega_2\partial_\Theta+\Omega_3\partial_\phi,\zeta^a\partial_a=\frac{1}{|k|^2}\left(\sum_{a=1}^{3}\Omega_a^2R_a^2\partial_\tau+f(R)\left(\Omega_1\partial_{\omega_2}+\Omega_2\partial_\Theta+\Omega_3\partial_\phi\right)\right),$$

$$K^{\omega_1}=\frac{1}{R^2}\left(\frac{R_2}{R_1}-3\frac{R_1}{R_2}\right)$$

$$\Omega_1^2=\frac{f(R)}{R_1^2}\frac{\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p+\frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+n\Phi_1\mathcal{Q}_1+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s+n\left(p\Phi_p\mathcal{Q}_p+\Phi_1\mathcal{Q}_1\right)+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\Omega_2^2=\frac{f(R)}{R_2^2}\frac{\left(3\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+\frac{R_2^2}{\tilde{L}^2}\Xi_{1,p}\right)\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+n\Phi_1\mathcal{Q}_1+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s+n\left(p\Phi_p\mathcal{Q}_p+\Phi_1\mathcal{Q}_1\right)+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\Omega_3^2=\frac{f(R)}{R_3^2}\frac{\left(\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)\right)\left(p\left(\mathcal{T}s+n\Phi_p\mathcal{Q}_p\right)+n\Phi_1\mathcal{Q}_1+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p\left(\mathcal{T}s+\Phi_p\mathcal{Q}_p\right)+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s+p\Phi_p\mathcal{Q}_p+\Phi_1\mathcal{Q}_1+\frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\gamma_{ab}d\sigma^a d\sigma^b=-f(R_1)d\tau^2+R_2^2d\phi^2+R_3^2d\delta_2^2$$

$$M\ =V_pR_0\left(\mathcal{T}s\cosh^2\eta+\frac{\mathcal{T}s}{n}+\Phi_1\mathcal{Q}_1+\Phi_p\mathcal{Q}_p\right)$$

$$J_a\ =V_p\frac{\Omega_aR_a^2}{|k|}\mathcal{T}s\cosh\eta$$

$$\mathcal{T}_{\text{tot}}\ =-\frac{V_pR_0}{n}\frac{R^2}{L^2}\Xi_{1,p}$$

$$\Omega^2=\frac{f(R)}{R^2}\frac{p(1+nsh_\alpha^2)+\frac{R^2}{\tilde{L}^2}(1+n+p+n(1+p)sh_\alpha^2)-\frac{Rn(l-1)}{\tilde{L}}\sqrt{f(R)}sh_\alpha ch_\alpha}{n+p(1+nsh_\alpha^2)+\frac{R^2}{\tilde{L}^2}(1+n+p+n(1+p)sh_\alpha^2)-\frac{Rn(l-1)}{\tilde{L}}\sqrt{f(R)}sh_\alpha ch_\alpha}.$$

$$r_c \ll r_{\text{int}}, r_c \ll r_{\text{ext}}, r_c \ll \mathcal{L}$$

$$r_{\text{int}} = \left|\mathcal{R}_\gamma\right|^{-1/2}$$

$$r_{\text{ext}} = \left|K^\mu K_\mu\right|^{-1/2}$$

$$\frac{r_0}{R} \ll (\cosh\,\alpha)^{-2/n} \text{ and } \frac{r_0}{L} \ll (\cosh\,\alpha)^{-2/n}.$$

$$r_c \sim N_{l-2}^{\frac{1}{l-1}} T_{l-2}^{-\frac{1}{m-1}}$$

$$1 \ll N_{D_p} \ll N, 1 \ll N_2 \ll N^2, 1 \ll N_5^2 \ll N$$

$$\prod_a^b \mathbb{S}^{p_a} \times \mathbb{S}^{n+1}$$

$$u^b\partial_b = \frac{1}{\sqrt{f(R)}}\partial_\tau, v_a^b\partial_b = \frac{1}{R_a}\partial_{\Phi_a}$$

$$\tilde{I}_E = \frac{\Omega_{n+1}V_p}{16\pi G}\left(\frac{n}{4\pi T}\right)^n f'(R)^{\frac{n+1}{2}}\left(1-\frac{\Phi_H^2}{4\pi^4 N V_q^2}\right)^{nN/2}$$

$$c\left|\hat{h}^{(q)}\right|^{1/2} = V_q = (2\pi)^q \prod_a^q R_a\sqrt{f'(R)}$$

$$\frac{R}{L} = \left(\frac{p+qnN\sinh^2\,\alpha}{1+n+p+(q+1)nN\sinh^2\,\alpha}\right)^{1/2}$$

$$M=\frac{\Omega_{(n+1)}x^{nN}}{8\pi G\left(\Omega^2-\frac{1}{L^2}\right)^2}\left(\frac{n}{4\pi T}\right)^n\left[\left(\Omega^2-\frac{1}{L^2}\right)\frac{\Phi_H^2}{N}\,{}_2F_1\left(1,1+\frac{n}{2}(N-1);1+\frac{nN}{2};x^2\right)\right.$$
$$+x^2\left(\frac{1}{L^2}+\frac{\frac{-1-2n}{L^2}+(1+n)\Omega^2-\frac{n(N-1)}{L^2}\frac{\Phi_H^2}{N}}{2+nN}\,{}_2F_1\left(1,1+\frac{n}{2}(N-1);2+\frac{nN}{2},x^2\right)\right)$$
$$\left.+\frac{2\left(\frac{n+1}{L^2}-\Omega^2\right)x^4}{(2+nN)(4+nN)}\,{}_2F_1\left(2,1+\frac{n}{2};3+\frac{nN}{2};x^2\right)\right]$$
$$J=\frac{\Omega_{(n+1)}\Omega}{4G\left(\Omega^2-\frac{1}{L^2}\right)^2}\left(\frac{n}{4\pi T}\right)^n\left[\frac{\Phi_H^2x^{nN+2}}{N(nN+2)}\,{}_2F_1\left(2,1+\frac{n}{2}(N-1);\frac{nN}{2}+2;x^2\right)\right.$$
$$\left.+\frac{nx^{nN+4}}{(nN+2)(nN+4)}\,{}_2F_1\left(2,1+\frac{n}{2}(N-1);\frac{nN}{2}+3;x^2\right)\right],$$
$$S=\frac{n\Omega_{(n+1)}}{8GT}\left(\frac{n}{4\pi T}\right)^n\frac{x^{2+nN}}{(2+nN)\left(\Omega^2-\frac{1}{L^2}\right)}\,{}_2F_1\left(1,\frac{1}{2}(N-1)n;\frac{Nn}{2}+2;x^2\right)$$
$$Q=\frac{\Phi_H\Omega_{(n+1)}}{8GN}\left(\frac{n}{4\pi T}\right)^n\frac{x^{nN}}{\left(\Omega^2-\frac{1}{L^2}\right)}\,{}_2F_1\left(1,\frac{1}{2}(N-1)n;\frac{Nn}{2}+1;x^2\right),$$
$$\mathcal{T}_{\text{tot}}=\frac{\Omega_{(n+1)}x^{nN+2}}{8G\left(\Omega^2-\frac{1}{L^2}\right)^2}\left[\left(\Omega^2-\frac{1}{L^2}\right)\frac{\Phi_H^2}{N}\,{}_2F_1\left(1,1+\frac{n}{2}(N-1);1+\frac{nN}{2};x^2\right)\right.$$
$$+\frac{\frac{4+n}{L^2}-(2+n)\Omega^2}{(2+n)}+\frac{n}{(2+n)(2+nN)}\left((2+n)\left(\Omega^2-\frac{1}{L^2}\right)\right.$$
$$\left.\left.+\left(\frac{4+n-2N}{L^2}-\Omega^2(2+n)\right)\frac{\Phi_H^2}{N}\right)\times\,{}_2F_1\left(1,1+\frac{n}{2}(N-1);2+\frac{nN}{2};x^2\right)\right]$$

$$x\equiv\left(1-\frac{\Phi_H^2}{N}\right)^{1/2}$$

$$M = \frac{\Omega_{(n+1)}}{8G}\left(\frac{n}{4\pi T}\right)^n \int_{r_{\min}}^{r_{\max}} dr I_r \left(f(r) - \Omega^2 r^2 + n + n\frac{\Phi_H^2}{4\pi^2 r^2}\frac{f(r) - \Omega^2 r^2}{f(r) - \frac{\Phi_H^2}{N(2\pi r)^2}} \right)$$

$$J = \frac{n\Omega_{(n+1)}\Omega}{8G}\left(\frac{n}{4\pi T}\right)^n \int_{r_{\min}}^{r_{\max}} dr r^2 I_r$$

$$S = \frac{n\Omega_{(n+1)}}{8GT}\left(\frac{n}{4\pi T}\right)^n \int_{r_{\min}}^{r_{\max}} dr I_r$$

$$Q = \frac{n\Omega_{(n+1)}\Phi_H}{16\pi G}\left(\frac{n}{4\pi T}\right)^n \int_{r_{\min}}^{r_{\max}} dr r^{-2} I_r$$

$$\mathcal{T}_{\text{tot}} = -\frac{\Omega_(n+1)}{16\pi G}\int_{r_{\min}}^{r_{\max}} dr I_r [(n+p)\Omega^2 r^2 - p f(r)$$

$$+nN\left(-f(r) + 2\Omega^2 r^2 - f(r)\Omega^2 r^2 \frac{\frac{\Phi_H^2}{N(2\pi r)^2}}{f(r) - \frac{\Phi_H^2}{N(2\pi r)^2}} \right)\Bigg]$$

$$I_r = r(f(r) - \Omega^2 r^2)^{\frac{n}{2}}\left(1 - \frac{\Phi_H^2}{N f(r)(2\pi r)^2}\right)^{\frac{nN}{2}}$$

$$\sinh\,\alpha_i \equiv \sin\,\theta \sinh\,\alpha, \sinh\,\alpha_p \equiv \cos\,\theta \sinh\,\alpha$$

$$s_i \equiv \sinh\,\alpha_i, c \equiv \cosh\,\alpha$$

$$\Omega^2 = \frac{f(R)}{R^2}\frac{p\left(1 + (7-p)s_p^2\right) + (7-p)s_1^2 + \frac{R^2}{L^2}\Xi_{1,p} - 4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}{7 + p(7-p)s_p^2 + (7-p)s_1^2 + \frac{R^2}{L^2}\Xi_{1,p} - 4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}$$

$$M = \frac{\Omega_{8-p}V_p}{16\pi G} r_0^{7-p}\left(f(R)^{3/2}\Xi_{1,p} - (7-p)\frac{R}{L}(4f(R)+1)s_p c \right)$$

$$J = \frac{\Omega_{8-p}V_p R}{16\pi G} r_0^{7-p}\sqrt{p\left(1 + (7-p)s_p^2\right) + (7-p)s_1^2 + \frac{R^2}{L^2}\Xi_{1,p} - 4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c} \times$$

$$\times \sqrt{7 + p(7-p)s_p^2 + (7-p)s_1^2 + \frac{R^2}{L^2}\Xi_{1,p} - 4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c},$$

$$S = \frac{\Omega_{8-p}V_p c}{4G\sqrt{7-p}} r^{8-p}\sqrt{7 + p(7-p)s_p^2 + (7-p)s_1^2 + \frac{R^2}{L^2}\Xi_{1,p} - 4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}$$

$$T = \frac{1}{4\pi r_0 c}\sqrt{\frac{f(R)(7-p)^3}{7 + p(7-p)s_p^2 + (7-p)s_1^2 + \frac{R^2}{L^2}\Xi_{1,p} - 4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}}$$

$$\mathcal{T}_{\text{tot}} = -\frac{\Omega_{8-p}V_p}{16\pi G}\left(\sqrt{f(R)}\frac{R^2}{L^2}\Xi_{1,p} + (7-p)\frac{R}{L}(4f(R)+1)s_p c\right)$$

$$\Xi_{1,p} = 8 + (7-p)\left((1+p)s_p^2 + 2s_1^2\right)$$

$$\sin\,\theta = 0, \tilde{L} = \frac{L}{2}, n = 3$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2}\frac{1 + 2s_p^2 + \frac{R_a^2}{L^2}\Xi_5}{7 + 2\left(5s_p^2 + s_1^2\right) + \frac{R^2}{L^2}\Xi_5}$$

$$\Omega_3^2 = \frac{f(R)}{R_3^2}\frac{3\left(1 + 2s_p^2\right)}{7 + 2\left(5s_p^2 + s_1^2\right) + \frac{R^2}{L^2}\Xi_5}$$

$$M = \frac{\Omega_3 V_5}{16\pi G} r_0^2 \left(f(R)^{3/2}\Xi_{1,5}\right)$$

$$J_a = \frac{\Omega_3 V_5 R_a}{16\pi G} r_0^2 \sqrt{1 + 2s_p^2 + \frac{R_a^2}{L^2}\Xi_5}\sqrt{7 + 2\left(5s_p^2 + s_1^2\right) + \frac{R^2}{L^2}\Xi_5}$$

$$J_3 = \frac{\Omega_3 V_5 R_3}{16\pi G} r_0^2 \sqrt{3\left(1 + 2s_p^2\right)}\sqrt{7 + 2\left(5s_p^2 + s_1^2\right) + \frac{R^2}{L^2}\Xi_5}$$

$$S = \frac{\Omega_3 V_5 c r_0^3}{4G\sqrt{f(R) - \sum_a^3\ \Omega_2^2 R_a^2}}$$

$$T = \frac{1}{4\pi r_0 c}\sqrt{f(R) - \sum_a^3 \Omega_2^2 R_a^2}$$

$$\mathcal{T}_{tot} = -\frac{\Omega_3 V_3}{16\pi G}\sqrt{f(R)\frac{R^2}{L^2}\Xi_5}$$

$$\Omega_a^2 = \frac{f(R_1)}{R_a^2}\frac{1 + 6s_2^2}{7 + 2s_p^2 + \frac{R_1^2}{\tilde{L}^2}\Xi_2}$$

$$M = \frac{\Omega_7 V_2}{16\pi G} r_0^7 \left(f(R)^{3/2}\Xi_2\right)$$

$$J_a = \frac{\Omega_7 V_2 R_a}{16\pi G} r_0^7 \sqrt{1 + 6s_2^2}\sqrt{7 + 2s_p^2 + \frac{R_1^2}{\tilde{L}^2}\Xi_2}$$

$$S = \frac{\Omega_7 V_2 c r_0^7}{4G\sqrt{f(R) - \sum_a^2\ \Omega_2^2 R_a^2}}$$

$$T = \frac{1}{4\pi r_0 c}\sqrt{f(R) - \sum_a^2 \Omega_2^2 R_a^2}$$

$$\mathcal{T}_{tot} = -\frac{\Omega_7 V_2}{16\pi G}\sqrt{f(R)}\frac{R^2}{L^2}\Xi_2$$

$$d\varepsilon = \mathcal{T} ds + \sum_q^{p-1} \Phi_q d\mathcal{Q}_q$$

$$\mathcal{Q}_2 u^{[a} v^b w^{c]} \partial_a f\big|_{\partial W_{p+1}} = 0$$

$$Q_2 v^a \partial_a f|_{\partial w_{p+1}} = 0$$

$$v^{ia} D_{(a} v_{b)i} = \dot{v}_b = \delta^{ij} v_i^a \zeta_{ka} D_b \mathbb{M}_j{}^k$$

$$v_i^a \zeta_{ka} = (\mathbb{M}^{-1})_k{}^j \delta_{ij}$$

$$\dot{v}_b = (\mathbb{M}^{-1})_k{}^j D_b \mathbb{M}_j{}^k = D_b \ln |\mathbb{M}|$$

$$I = \frac{1}{16\pi} \int d^D x \sqrt{-\mathfrak{g}} \left[R - 2\Lambda + \frac{g}{D-4} \mathcal{G} + F_{\mu\alpha}^{(a)} F^{(a)\mu\alpha} \right] + \int_\Sigma \mu \sqrt{-\gamma} d\tau d\sigma$$

$$\Lambda = \frac{-(D-1)(D-2)}{2l^2}$$

$$\mathcal{G} = R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} - i \left[A_\mu^{(a)}, A_\nu^{(a)} \right]$$

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{2,\kappa}^2, d\Omega_{2,\kappa}^2 = \frac{dx^2}{1-\kappa x^2} + x^2 d\varphi^2$$

$$f(r) = 1 - r^2 \psi(r)$$

$$\nabla_\mu F^{(a)\mu\nu} = 0$$

$$\mathbf{A}^{(a)} = \frac{i}{2} h(r) dt \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$h(r) = C_2 + Q \int^r \frac{1}{u^{D-2}} \frac{1}{u^{D-2}} du$$

$$h'(r) = \frac{Q}{r^{D-2}}$$

$$I = \frac{\Omega_{D-2}(D-2)}{16\pi}$$
$$\int dt dr \left[r^{D-1} \psi (1 + g(D-3)\psi) + \frac{r^{D-1}}{l^2} + \frac{2Q^2 r^{3-D}}{(D-3)(D-2)} - \frac{2ar}{(D-2)} \right]'$$

$$\psi + g(D-3)\psi^2 = \frac{16\psi M}{(D-2) r^{D-1} \Omega_{D-2}} - \frac{1}{l^2} - \frac{2Q^2 r^{4-2D}}{(D-3)(D-2)} - \frac{2a}{(D-2) r^{D-2}},$$

$$f(r) = 1 - \frac{r^2}{2g(D-3)} [-1 \pm$$
$$\sqrt{1 - 4(D-3)g \left(\frac{1}{l^2} + \frac{2Q^2 r^{4-2D}}{(D-3)(D-2)} - \frac{16\pi M r^{1-D}}{(D-2)\Omega_{D-2}} + \frac{2a r^{2-D}}{D-2} \right)}]$$

$$M = \frac{(D-2)\Omega_{D-2}}{16\pi} m$$

$$Q^2 = \frac{(D-2)(D-3)}{2} q^2$$

$$\Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}$$

$$\frac{(D-3)g}{l^2} = \lambda_{gb}$$

$$\frac{2a}{D-2} = A,$$

$$f(r) = 1 + \frac{r^2}{2\lambda_{gb} l^2}\left[1 \mp \sqrt{1 - 4\lambda_{gb}\left(1 + \frac{q^2 l^2}{r^{2D-4}} - \frac{m l^2}{r^{D-1}} + \frac{A l^2}{r^{D-2}}\right)}\right].$$

$$f(r) = 1 + \frac{r^2}{2g}\left(1 - \sqrt{1 - 4g\left(-\frac{\Lambda}{3} + \frac{Q^2}{r^4} - \frac{2m}{r^3} + \frac{a}{r^2}\right)}\right)$$

$$m = -\frac{r_+^3 \Lambda}{6} + \frac{a r_+}{2} + \frac{Q^2}{2r_+} + \frac{r_+}{2} + \frac{g}{2r_+},$$

$$K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

$$K = \frac{4}{r^4}(f(r) - 1)^2 + \frac{4f'(r)^2}{r^2} + f''(r)^2$$

$$K = \frac{(1-\sqrt{\chi})^2}{g^2} + \frac{4\left(\frac{r(1-\sqrt{\chi})}{g} + \frac{2r^2 \mathcal{Y}}{\sqrt{\chi}}\right)^2}{r^2}$$
$$+ \left(\frac{1-\sqrt{\chi}}{g} + \frac{8r\mathcal{Y}}{\sqrt{\chi}} + \frac{8r^2 \mathcal{Y}^2 g}{\chi^{\frac{3}{2}}} - \frac{2r\mathcal{C}}{\sqrt{\chi}}\right)^2$$

$$\chi = 1 - 4g\left(-\frac{\Lambda}{3} + \frac{Q^2}{r^4} - \frac{2m}{r^3} + \frac{a}{r^2}\right),$$

$$\mathcal{Y} = -\frac{2Q^2}{r^5} + \frac{3m}{r^4} - \frac{a}{r^3}$$

$$\mathcal{C} = -\frac{10Q^2}{r^5} + \frac{12m}{r^4} - \frac{3a}{r^3}$$

$$H = -\frac{r_h^3 \Lambda}{6} + \frac{r_h a}{2} + \frac{g}{2r_h} + \frac{r_h}{2} + \frac{Q^2}{2r_h} =$$
$$\frac{4r_h^3 \pi P}{3} + \frac{r_h a}{2} + \frac{g}{2r_h} + \frac{r_h}{2} + \frac{Q^2}{2r_h}$$

$$T = \frac{\mathcal{K}(r_h)}{2\pi} = \frac{1}{4\pi}\left(\frac{\partial f(r)}{\partial r}\right)_{r=r_h} = \frac{-r_h^4\Lambda + (a+1)r_h^2 - Q^2 - g}{4\pi r_h(r_h^2+2g)} = \frac{8r_h^4\pi P + (a+1)r_h^2 - Q^2 - g}{4\pi r_h(r_h^2+2g)}$$

$$S = \int_0^{r_h} \frac{1}{T}\frac{\partial H}{\partial r_h} dr_h$$

$$S = \pi r_h^2 + 4\pi g \ln(r_h)$$

$$dH(S,P,Q,a,g) = TdS + VdP + \phi dQ + \mathcal{A}da + \mathcal{B}dg$$

$$V = \left(\frac{\partial H}{\partial P}\right)_{S,Q,a,g} = \frac{4\pi}{3}r_h^3$$

$$\phi = \left(\frac{\partial H}{\partial Q}\right)_{S,P,a,g} = \frac{Q}{r_h}$$

$$\mathcal{A} = \left(\frac{\partial H}{\partial a}\right)_{S,P,Q,g} = \frac{1}{2}r_h$$

$$\mathcal{B} = \left(\frac{\partial H}{\partial g}\right)_{S,P,Q,a} = \frac{1}{2r_h}$$

$$H = 2TS - 2PV + Q\phi + a\mathcal{A} + g\mathcal{B}$$

$$P = \frac{T}{2r_h} + \frac{gT}{r_h^3} - \frac{a+1}{8\pi r_h^2} + \frac{Q^2}{8\pi r_h^4} + \frac{g}{8\pi r_h^4}$$

$$\left.\frac{\partial P}{\partial r_h}\right|_{T=T_c, r_h=r_c} = 0, \left.\frac{\partial^2 P}{\partial^2 r_h}\right|_{T=T_c, r_h=r_c} = 0$$

$$r_c = \sqrt{\frac{3Q^2 + 3ag + \Upsilon + 6g}{a+1}}$$

$$P_c = \frac{(Q^2ag + 3a^2g^2 + 3\Upsilon Q^2 + 4\Upsilon ag + Q^2g + 7ag^2 + \Upsilon^2 + 7\Upsilon g + 4g^2)(a+1)^2}{8\pi(3Q^2 + 3ag + \Upsilon + 6g)^2(3Q^2 + 9ag + \Upsilon + 12g)}$$

$$T_c = \frac{(Q^2 + 3ag + \Upsilon + 4g)(a+1)}{2\pi\sqrt{\frac{3Q^2 + 3ag + \Upsilon + 6g}{a+1}}(3Q^2 + 9ag + \Upsilon + 12g)}$$

$$\Upsilon = \sqrt{9Q^4 + 30Q^2ag + 9a^2g^2 + 48Q^2g + 48ag^2 + 48g^2}$$

$$Z = \frac{PV}{T}.$$

$$C_P = \left(\frac{\partial H}{\partial r_h}\right)\left(\frac{\partial r_h}{\partial T}\right) =$$

$$\frac{2\left(r_h^2 + 2g\right)^2\left(8r_h^4\pi P + r_h^2 a - Q^2 + r_h^2 - g\right)\pi}{8\pi P r_h^6 + 48\pi g P r_h^4 - a r_h^4 + 3Q^2 r_h^2 + 2agr_h^2 - r_h^4 + 2Q^2 g + 5r_h^2 g + 2g^2}$$

$$\delta = |r_c - r_d| \simeq 0.01$$

$$G = \frac{\left(-8r_h^4\pi P - (a+1)r_h^2 + Q^2 + g\right)\left(4\ln(r_h)g + r_h^2\right)}{4r_h(r_h^2 + 2g)} +$$
$$\frac{4r_h^3\pi P}{3} + \frac{r_h a}{2} + \frac{g}{2r_h} + \frac{r_h}{2} + \frac{Q^2}{2r_h}$$

$$C_v = \left(T\frac{\partial S}{\partial T}\right)_v \propto |t|^{-\alpha},$$
$$\eta = v_l - v_s \propto |t|^{\beta},$$
$$\kappa_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T \propto |t|^{-\gamma}$$

$$|P - P_c| \propto |v - v_c|^{-\delta}.$$

$$p = \frac{T_c}{v_c p_c}\frac{\tau}{\nu} + \frac{8gT_c}{v_c^3 p_c}\frac{\tau}{\nu^3} - \frac{a+1}{2\pi v_c^2 p_c}\frac{1}{\nu^2} + \frac{2Q^2 + 2g}{\pi v_c^4 p_c}\frac{1}{\nu^4}.$$

$$p = 1 + \Theta t + \Phi t\omega + \Omega\omega^3 + \cdots$$

$$\Theta = \frac{T_c(v_c^2 + 8g)}{v_c^3 p_c}$$
$$\Phi = -\frac{T_c(v_c^2 + 24g)}{v_c^3 p_c}$$

$$\Omega = -\frac{\pi T_c v_c^3 + 80\pi g T_c v_c - 2a v_c^2 + 40Q^2 - 2v_c^2 + 40g}{\pi v_c^4 p_c}$$

$$1 + \Theta t + \Phi t\omega_l + \Omega\omega_l^3 = 1 + \Theta t + \Phi t\omega_s + \Omega\omega_s^3$$

$$\int_{\omega_s}^{\omega_l} \omega\frac{dp}{d\omega}d\omega = 0$$

$$\Phi t\left(\omega_l^2 - \omega_s^2\right) = -\frac{3}{2}\Omega\left(\omega_l^4 - \omega_s^4\right)$$

$$\omega_l = -\omega_s = \sqrt{-\frac{\Phi t}{\Omega}}$$

$$\eta = v_l - v_s = v_c(\omega_l - \omega_s) = 2v_c\omega_l \propto \sqrt{-t}$$

$$\kappa_T = -\frac{1}{v_c(1+\omega)}\frac{\partial v}{\partial \omega}\left(\frac{\partial \omega}{\partial P}\right)_T \propto \left(-\frac{1}{\frac{\partial p}{\partial \omega}}\right)_{\omega=0} = -\frac{t^{-1}}{\Phi}$$

$$p = 1 + \Omega\omega^3$$

$$\alpha + 2\beta + \gamma = 2, \alpha + \beta(\delta+1) = 2, (2-\alpha) = \delta(1-\alpha) - 1$$
$$\gamma = \beta(\delta-1), \gamma(\delta+1) = (\delta-1)(2-\alpha)$$

$$\frac{dP}{dT} = \frac{T\Delta S}{\Delta V} = \frac{(\Delta H)_P}{\Delta V}$$

$$\mu = \left(\frac{\partial T}{\partial P}\right)_H$$

$$\mu = \frac{1}{C_P}\left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right]$$

$$T_i = V\left(\frac{\partial T}{\partial V}\right)_P.$$

$$I_{\mathrm{QT}} = \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{-g}\left(R - 2\Lambda + \sum_n^{n_{\max}} \alpha_n \mathcal{Z}_n\right)$$

$$\Lambda = -(D-1)(D-2)/(2L^2)$$

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + \frac{r^2}{L^2}\,\mathrm{d}\Sigma_{D-2,k}^2,\ \mathrm{d}\Sigma_{D-2,k}^2 = \begin{cases} L^2\,\mathrm{d}\Omega_{D-2}^2, & \text{for } k=1 \\ \mathrm{d}\mathbf{x}_{D-2}^2, & \text{for } k=0 \\ L^2\,\mathrm{d}\Xi_{D-2}^2, & \text{for } k=-1 \end{cases}$$

$$\sum_{n=0}^{n_{\max}} \alpha_n r^{D-2n-3}(k - f(r))^n = m$$

$$f(r) \propto \frac{1}{r^{\frac{D-2n_{\max}-1}{n_{\max}}}} + \cdots$$

$$h(\psi) = \frac{1}{L^2} + \psi + \sum_{n=2}^{\infty}\frac{D-2n}{D-2}\alpha_n\psi^n = \frac{m}{r^{D-1}},\ \text{with } \psi := \frac{k - f(r)}{r^2}$$

$$h_{\mathrm{H}}(\psi) = \frac{1}{L^2} + \frac{\psi}{1-\alpha\psi}, \qquad \text{if } \alpha_n = \frac{D-2}{D-2n}\alpha^{n-1}$$
$$h_{\mathrm{D}}(\psi) = \frac{1}{L^2} - \frac{\log(1-\alpha\psi)}{\alpha}, \quad \text{if } \alpha_n = \frac{D-2}{n(D-2n)}\alpha^{n-1}$$

$$h_{\mathrm{B}}(\psi) = \frac{1}{L^2} + \frac{\psi}{\sqrt{1-\alpha^2\psi^2}},\ \text{if } \alpha_n = \frac{(1-(-1)^n)(D-2)\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}(D-2n)\Gamma\left(\frac{D+1}{2}\right)}\alpha^{n-1}.$$

$$f_{\mathrm{H}}(r) = k - \frac{r^{D+1} - L^2 m r^2}{\alpha r^{D-1} - L^2(r^{D-1} + \alpha m)}$$

$$f_{\mathrm{D}}(r) = k + \frac{r^2}{\alpha}\left(\mathrm{e}^{\alpha\left(L^{-2} - m r^{1-D}\right)} - 1\right)$$

$$f_{\mathrm{B}}(r) = k + \frac{r^2(r^{D-1} - L^2 m)}{\sqrt{L^4 r^{2D-2} + \alpha^2 (r^{D-1} - L^2 m)^2}}$$

$$M = \frac{(D-2) r_+^{D-1} \Sigma_{D-2,k}}{16\pi G} h(\psi_+)$$

$$T = \frac{1}{4\pi r_+}\left(\frac{(D-1) r_+^2 h(\psi_+)}{h'(\psi_+)} - 2k\right)$$

$$S = -\frac{(D-2)\Sigma_{D-2,k} k^{D/2-1}}{8G} \int \frac{h'(\psi_+)}{\psi_+^{D/2}}\, \mathrm{d}\psi_+$$

$$\mathrm{d}M = T\, \mathrm{d}S$$

$$f_{\mathrm{H}}(r) \overset{m\to 0}{=} k + \frac{r^2}{L^2 - \alpha} = k + \frac{r^2}{L_\star^2}$$

$$\frac{m}{L_\star^{D-3}} = \frac{y_+^D (1-\beta)^{\frac{D-3}{2}} (y_+^2 + k(1-\beta))}{y_+^2 - k\beta}$$

$$\beta = \alpha / L^2 = \alpha / (\alpha + L_\star^2)$$

$$y_+ = r_+ / L = r_+ \sqrt{1-\beta} / L_\star$$

$$\frac{m_{\mathrm{ext}}}{L_\star^{D-3}} = \frac{k^{\frac{D-1}{2}}}{2^{\frac{D+3}{2}} \beta (1-\beta)^{\frac{D-3}{2}} (D-1)^{\frac{D-1}{2}}} \left(D - 3 + 4\beta + \sqrt{(D-3)^2 + 8(D-1)\beta}\right)$$

$$\times \left(3 - D + 2(D-1)\beta + \sqrt{(D-3)^2 + 8(D-1)\beta}\right)^{\frac{D-1}{2}},$$

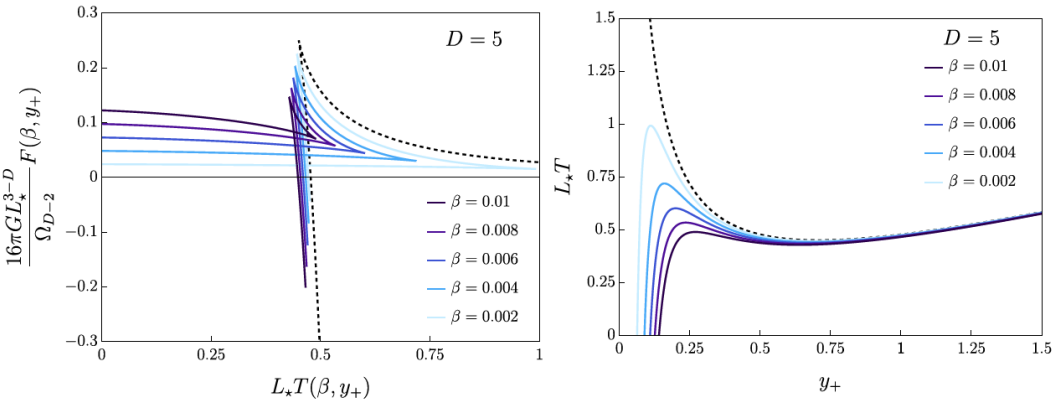
$$L_\star T = \frac{\sqrt{1-\beta}\left(k(D-3) y_+^2 + (D-1)\left(y_+^4 - k\beta(k(1-\beta) + 2y_+^2)\right)\right)}{4\pi y_+^3}.$$

$$\frac{M}{L_\star^{D-3}} = \frac{(D-2)\Sigma_{D-2,k} y_+^{D-1}}{16\pi G (1-\beta)^{\frac{D-3}{2}}} \frac{(k(\beta-1) - y_+^2)}{(k\beta - y_+^2)}$$

$$\frac{S}{L_\star^{D-2}} = \frac{\Sigma_{D-2,k} y_+^{D-2}}{4G(1-\beta)^{\frac{D-2}{2}}}\, {}_2F_1\left(2, 1 - \frac{D}{2}, 2 - \frac{D}{2}, \frac{k\beta}{y_+^2}\right).$$

$$\frac{F}{L_\star^{D-3}} = \frac{y_+^{D-5}\Sigma_{D-2}^k}{16\pi G(k\beta - y_+^2)(1-\beta)^{\frac{D-3}{2}}}((D-2)y_+^4(k(\beta-1)-y_+^2)-(k\beta-y_+^2)$$

$$\times\left(k(D-3)y_+^2+(D-1)\left(y_+^4+k\beta(k(\beta-1)-2y_+^2)\right)\right)\,{}_2F_1\left(2,1-\frac{D}{2},2-\frac{D}{2},\frac{k\beta}{y_+^2}\right)\Bigg)$$



$$f_{\mathrm{D}}(r) \overset{m\to 0}{=} k+\frac{r^2}{\alpha}\left(\mathrm{e}^{\alpha/L^2}-1\right)=1+\frac{r^2}{L_\star^2},$$

$$L_\star^2=\alpha/\left(\mathrm{e}^{\alpha/L^2}-1\right)$$

$$\frac{m}{L_\star^{D-3}}=\left(\frac{e^\beta-1}{\beta}\right)^{\frac{D-3}{2}}\left[1-\frac{1}{\beta}\log\left(1-k\frac{\beta}{y_+^2}\right)\right]y_+^{D-1}$$

$$L_\star T=\sqrt{\frac{\beta}{e^\beta-1}}\frac{(D-1)y_+^2+k[(D-1)\beta-2]+(D-1)\left(k-\frac{y_+^2}{\beta}\right)\log\left(1-k\frac{\beta}{y_+^2}\right)}{4\pi y_+}.$$

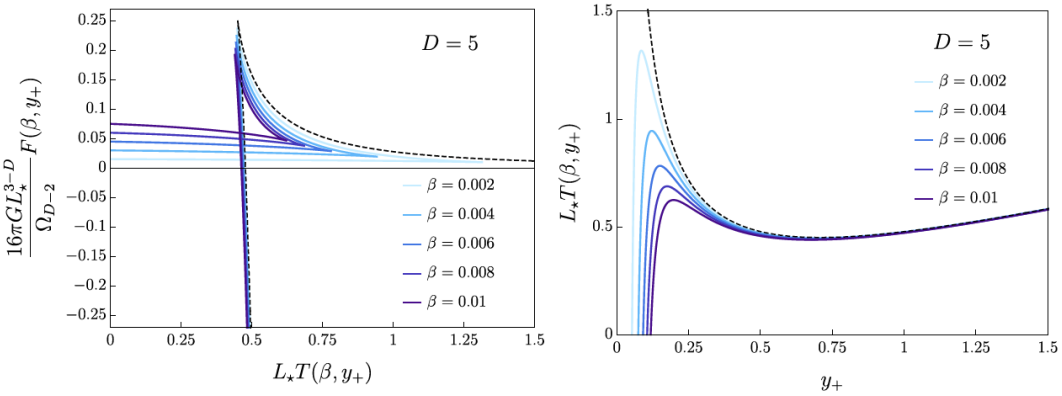
$$(D-1)y_{+,\mathrm{ext}}^2+k[(D-1)\beta-2]+(D-1)\left(k-\frac{y_{+,\mathrm{ext}}^2}{\beta}\right)\log\left(1-k\frac{\beta}{y_{+,\mathrm{ext}}^2}\right)=0$$

$$\frac{M}{L_\star^{D-3}}=\frac{(D-2)\Sigma_{D-2,k}y_+^{D-1}}{16\pi G}\left(\frac{e^\beta-1}{\beta}\right)^{\frac{D-3}{2}}\left[1-\frac{1}{\beta}\log\left(1-k\frac{\beta}{y_+^2}\right)\right]$$
$$=\frac{(D-2)\Sigma_{D-2,k}}{16\pi G}m$$

$$\frac{S}{L_\star^{D-2}}=\frac{\Sigma_{D-2,k}y_+^{D-2}}{4G}\left(\frac{e^\beta-1}{\beta}\right)^{\frac{D-2}{2}}\,{}_2F_1\left(1,1-\frac{D}{2},2-\frac{D}{2},\frac{k\beta}{y_+^2}\right)$$

$$\frac{F}{L_\star^{D-3}}=\frac{\Sigma_{D-2,k}y_+^{D-1}}{16\pi G}\left(\frac{e^\beta-1}{\beta}\right)^{\frac{D-3}{2}}\left\{(D-2)\left[1-\frac{1}{\beta}\log\left(1-k\frac{\beta}{y_+^2}\right)\right]\right.$$

$$-\left[(D-1)y_+^2+k[(D-1)\beta-2]+(D-1)\left(k-\frac{y_+^2}{\beta}\right)\ln\left(1-k\frac{\beta}{y_+^2}\right)\right]$$
$$\times\ {}_2F_1\left(1,1-\frac{D}{2},2-\frac{D}{2},\frac{k\beta}{y_+^2}\right)\Bigg\}$$



$$f_B(r) \overset{m\to 0}{=} 1+\frac{r^2}{\sqrt{L^4+\alpha^2}}=1+\frac{r^2}{L_\star^2},$$

$$L_\star^2=\sqrt{L^4+\alpha^2}$$

$$\frac{m}{L_\star^{D-3}}=\frac{y_+^{D-1}}{(1+\beta)^{\frac{D-3}{4}}}\left(1+\frac{k}{\sqrt{y_+^4-k^2\beta^2}}\right)$$

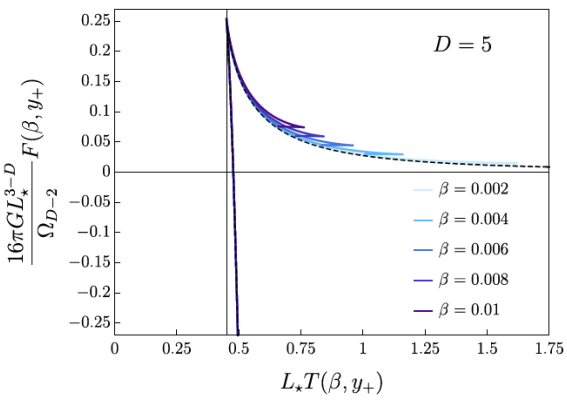
$$L_\star T=(1+\beta^2)^{1/4}\frac{(D-1)\left((y_+^4-k^2\beta^2)^{3/2}-k^3\beta^2\right)+(D-3)ky_+^4}{4\pi y_+^5}$$

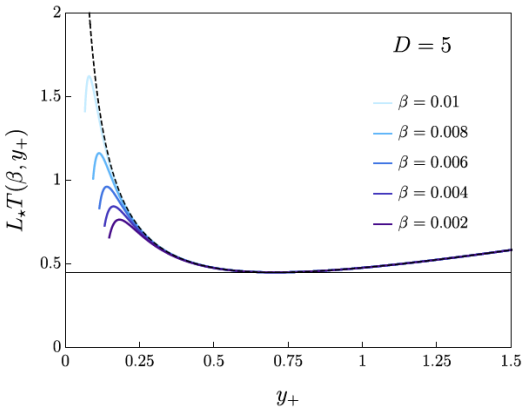
$$(D-1)\left((y_+^4-k^2\beta^2)^{3/2}-k^3\beta^2\right)+(D-3)ky_+^4=0$$

$$\frac{M}{L_\star^{D-3}}=\frac{(D-2)\Sigma_{D-2,k}y_+^{D-1}}{16\pi G}\frac{1}{(1+\beta)^{\frac{D-3}{4}}}\left(1+\frac{k}{\sqrt{y_+^4-k^2\beta^2}}\right)$$
$$=\frac{(D-2)\Sigma_{D-2,k}}{16\pi G}m$$

$$\frac{S}{L_\star^{D-2}}=\frac{\Sigma_{D-2,k}y_+^{D-2}}{4G}\frac{1}{(1+\beta)^{\frac{D-2}{4}}}2F_1\left(\frac{3}{2},\frac{1}{2}-\frac{D}{4},\frac{3}{2}-\frac{D}{4},\frac{k^2\beta^2}{y_+^4}\right)$$

$$\frac{F}{L_\star^{D-3}}=\frac{\Sigma_{D-2,k}y_+^{D-7}}{16\pi G}\frac{1}{(1+\beta^2)^{\frac{D-3}{4}}}\Bigg\{(D-2)y_+^6\left(1+\frac{k}{\sqrt{y_+^4-k^2\beta^2}}\right)-$$
$$\left[(D-1)\left((y_+^4-k^2\beta^2)^{3/2}-k^3\beta^2\right)+(D-3)ky_+^4\right]\times$$
$${}_2F_1\left(\frac{3}{2},\frac{1}{2}-\frac{D}{4},\frac{3}{2}-\frac{D}{4},\frac{k^2\beta^2}{y_+^4}\right)\Bigg\}$$





$$\Box\,\Phi = 0,$$

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\Omega_{D-2}^2$$

$$\Phi = \mathrm{e}^{-\mathrm{i}\omega v} R(r) Y(\Omega_{D-2}),$$

$$\nabla_{\Omega_{D-2}} Y = -l(l+D-3)Y$$

$$fR'' + (Dr^{-1}f - 2\mathrm{i}\omega - 2r^{-1}f + f')R' + (-\mathrm{i}\omega r^{-1}D + 2\mathrm{i}\omega r^{-1} - l(l+D-3)r^{-2})R(r) = 0$$

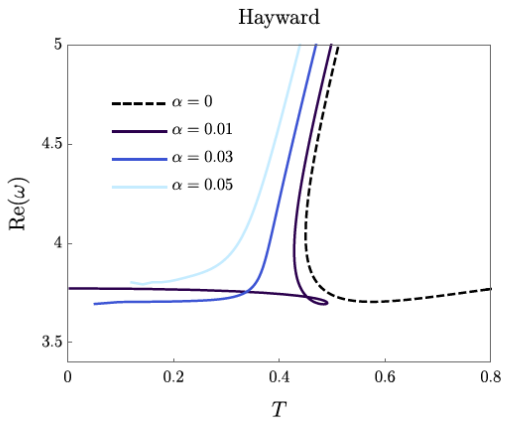
$$R(r) = C_1\big(1 + \mathcal{O}(r - r_+)\big) + C_2(r - r_+)^{2\mathrm{i}\omega/4\pi T}\big(1 + \mathcal{O}(r - r_+)\big)$$

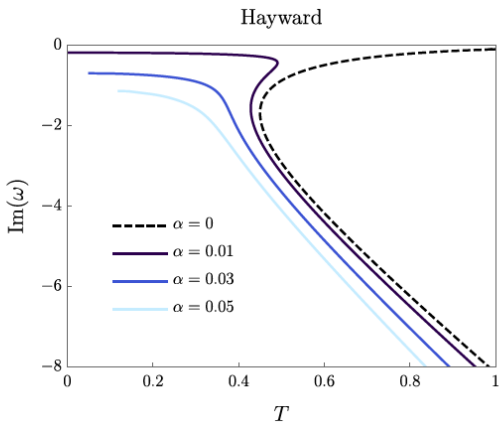
$$T = f'(r_+)/4\pi$$

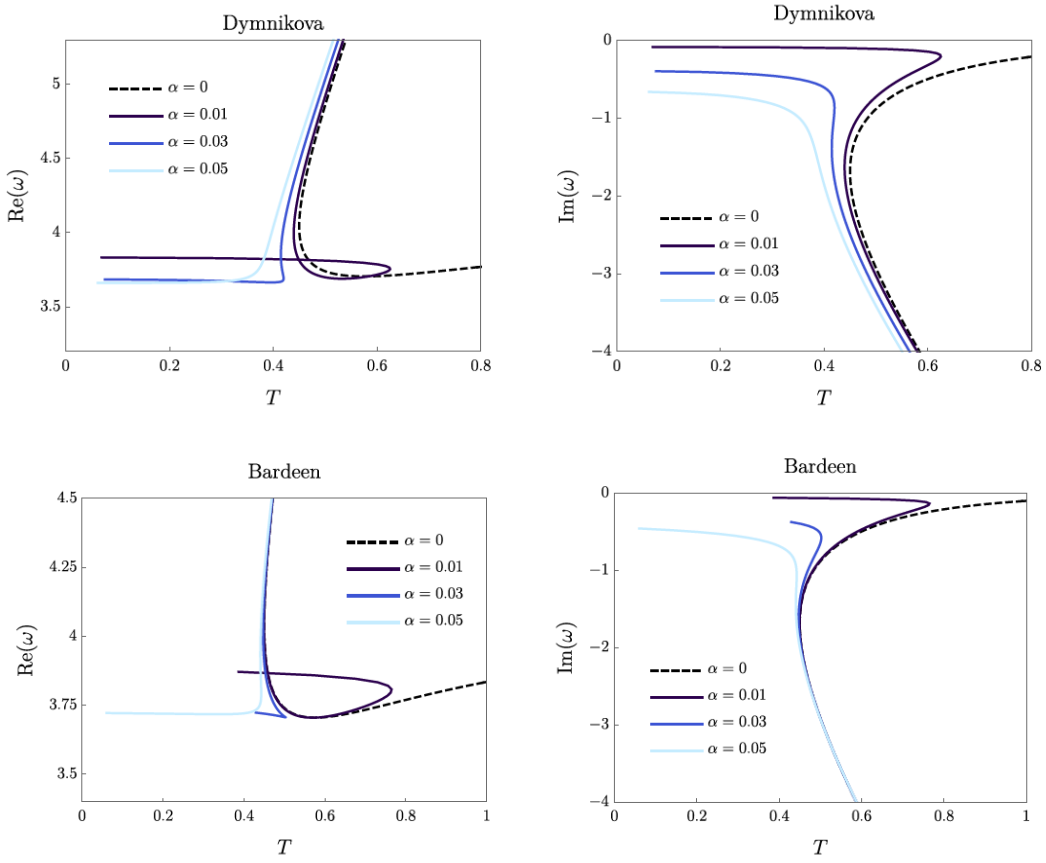
$$\Phi \sim \exp\left[-\mathrm{i}\omega\big(t + (4\pi T)^{-1}\log(r - r_+)\big)\right]$$

$$R(r) = \frac{D_1}{r^{\Delta_+}}\big(1 + \mathcal{O}(r^{-1})\big) + \frac{D_2}{r^{\Delta_-}}\big(1 + \mathcal{O}(r^{-1})\big)$$

$$\Phi \sim r^{-\Delta_+} \sim r^{-4}, \Delta_+$$







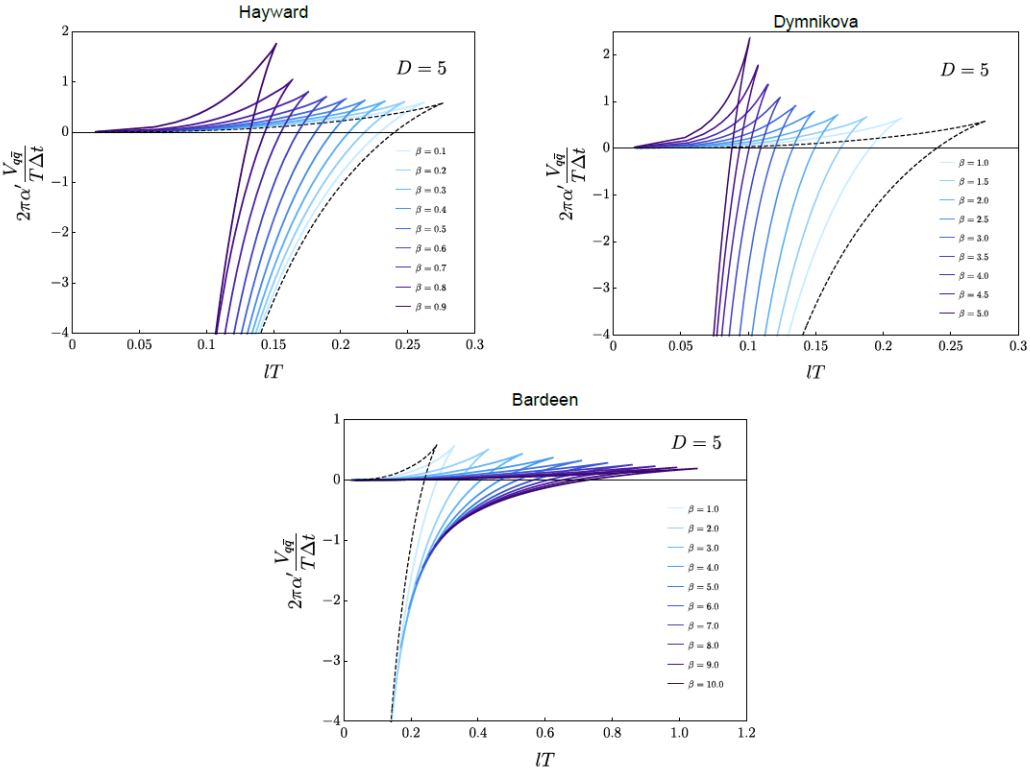
$$\langle W(\mathcal{C})\rangle \sim e^{-\Delta t V_{q\bar{q}}},$$

$$\langle W(\mathcal{C})\rangle \sim e^{-I_{\mathrm{NG}}}.$$

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\,\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

$$\frac{I_{\mathrm{reg}}(r_0)}{2\Delta t} = \int_{r_0}^{\infty} dr \left[\frac{f(r)^{1/2} r}{\sqrt{f(r)r^2 - f(r_0)r_0^2}} - 1\right] - \int_{r_{\mathrm{min}}}^{r_0} dr$$

$$l(r_0) = 2\int_{r_0}^{\infty} \frac{r_0}{r}\left(\frac{f(r_0)}{f(r)}\right)^{1/2} \frac{dr}{\sqrt{f(r)r^2 - f(r_0)r_0^2}}$$



$$I_{\text{hyperspace}_{\mathcal{A}_5\times S^5}} = \frac{1}{16\pi G}\int \mathrm{d}^5x\sqrt{|g|}\left[R + \frac{12}{L^2} + \frac{\zeta(3)}{8}\alpha'^3 W^4\right]$$

$$W^4 = \left(W_{abcd}W^{ebcf} + \frac{1}{2}W_{adbc}W^{efbc}\right)W^{ag}{}_{he}W_{fg}{}^{hd}$$

$$g_{ab} \to g_{ab} - \frac{\zeta(3)}{8}\alpha'^3\left[\frac{1}{L^4}\left(\hat{C}^{(1)}_{ab} - \frac{g_{ab}}{3}\hat{C}^{(1)}\right) + \frac{1}{L^2}\left(\hat{C}^{(2)}_{ab} - \frac{g_{ab}}{3}\hat{C}^{(2)}\right) + \hat{C}^{(3)}_{ab} - \frac{g_{ab}}{3}\hat{C}^{(3)}\right]$$

$$\begin{aligned}\hat{C}^{(3)}_{ab} = &-\frac{6935}{1584}R^{cdef}R_c{}^g{}_{ea}R_{dgfb} - \frac{18625}{19008}R^{cdef}R_{cd}{}^g{}_{ef}R_{gb} + \frac{7}{44}R_a{}^c{}_b{}^dR_{efgc}R^{efg}{}_d \\ &+\frac{2053}{1584}g_{ab}R_g{}^c{}_h{}^dR_c{}^e{}_d{}^fR_e{}^g{}_f{}^h + \frac{1241}{19008}g_{ab}R_{gh}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{gh} + \frac{6035}{3168}R^{cd}R^e{}_a{}^f{}_cR_{ebfd} \\ &+\frac{24215}{6336}R^{cd}R^e{}_a{}^f{}_bR_{ecfd} - \frac{1291}{12672}R^{cd}R^{ef}{}_{ac}R_{efbd} - \frac{767}{1584}R_{ab}R_{cdef}R^{cdef} \\ &+\frac{2155}{12672}g_{ab}R_{ghcd}R^{ghc}{}_eR^{de} - \frac{521}{1408}RR_{ghca}R^{ghc} + \frac{1679}{12672}g_{ab}RR_{ghcd}R^{ghcd} \\ &-\frac{25}{44}R^{gh}R_{gahd}R^d_b - \frac{3569}{3168}RR_{gacb}R^{gc} + \frac{7}{704}g_{ab}R^3\end{aligned}$$

$$\hat{C}^{(2)}_{ab} = \frac{15}{44}R_{acde}R_b{}^{cde} - \frac{285}{176}R_{cd}R_a{}^c{}_b{}^d + \frac{15}{44}RR_{ab}, \hat{C}^{(1)}_{ab} = \frac{45}{44}R_{ab}$$

$$\begin{aligned}\tilde{I} = &\frac{1}{16\pi G}\int \mathrm{d}^5x\sqrt{|g|}\left(\frac{12}{L^2} + R + \frac{15\zeta(3)}{88}\alpha'^3\left[\frac{3}{L^6}R + \frac{1}{L^4}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right)\right.\right. \\ &\left.\left.+\frac{1}{L^2}Q^{(3)}(\mathcal{R}^3) + Q^{(4)}(\mathcal{R}^4)\right]\right)\end{aligned}$$

$$Q^{(3)}(\mathcal{R}^3) = \frac{11}{15}\left(4\hat{C}^{(3)} + R^{ab}\hat{C}^{(2)}_{ab}\right)$$
$$Q^{(4)}(\mathcal{R}^4) = \frac{11}{15}\left(W^4 + R^{ab}\hat{C}^{(3)}_{ab}\right)$$

$$\frac{r^2}{L^2} + (k-f) + \frac{15\zeta(3)\alpha^3}{88}\left(\frac{3(k-f)}{L^6} + \frac{2(k-f)^2}{r^2L^4} - \frac{4(k-f)^3}{3r^4L^2} - \frac{(k-f)^4}{r^6}\right) = \frac{m}{r^2}$$

$$S = -\frac{(D-2)\Sigma_{D-2,k}}{8G}\left(\frac{h(\psi_+)}{\psi_+^{D/2}} - \int h(\psi_+)\mathrm{d}\left(\frac{1}{\psi_+^{D/2}}\right)\right)$$

$$t = 1/\psi^{D/2} = r_+^D$$

$$h' = -\frac{D}{2}\frac{\dot{h}}{\psi^{D/2+1}} = -\frac{D}{2}\dot{h}t^{1+\frac{2}{D}}$$

$$M = \frac{(D-2)\Sigma_{D-2,k}}{16\pi G}t^{1-\frac{1}{D}}h(t)$$
$$T = \frac{1}{4\pi}\frac{1}{t^{\frac{1}{D}}}\left(-\frac{2(D-1)}{D}\frac{h(t)}{\dot{h}(t)t} - 2\right)$$
$$S = -\frac{(D-2)\Sigma_{D-2,k}}{8G}\left(h(t)t - \int h(t)\mathrm{d}t\right)$$

$$M = \frac{(D-2)\Sigma_{D-2,k}}{16\pi G}t^{1-\frac{1}{D}}\dot{H}(t)$$
$$T = -\frac{1}{2\pi}\frac{1}{t^{\frac{1}{D}}}\left(\frac{(D-1)}{D}\frac{\dot{H}(t)}{t\ddot{H}(t)} + 1\right)$$
$$S = -\frac{(D-2)\Sigma_{D-2,k}}{8G}(\dot{H}(t)t - H(t))$$

$$Dt^2\ddot{H}(t)\dot{H}(t) + (D-2)((D-1)\dot{H}(t)(\dot{H}(t)t - H(t)) - Dt\ddot{H}(t)H(t)) = 0$$

$$\mathrm{d}M = T\,\mathrm{d}S + V\,\mathrm{d}P + \mu_\alpha \mathrm{d}\alpha$$

$$P = -\frac{\Lambda}{8\pi G} = \frac{(D-1)(D-2)}{16\pi GL^2}$$
$$V = \frac{\Sigma_{D-2,k}}{(D-1)}r_+^{D-1}$$
$$\mu_\alpha = \frac{(D-2)\Sigma_{D-2,k}}{16\pi G}\left[r_+^{D-1}\frac{\partial h(\psi_+)}{\partial\alpha}\right|_{L,r_+}^{+}$$
$$\left.\frac{k^{D/2-1}}{2\pi r_+}\left(\frac{(D-1)r_+^2h(\psi_+)}{h'(\psi_+)} - 2k\right)\int \frac{\mathrm{d}\psi_+}{\psi_+^{D/2}}\frac{\partial h'(\psi_+)}{\partial\alpha}\right|_{L,r_+}\Bigg]$$

$$(D-3)M = (D-2)TS - 2VP + 2\mu_\alpha\alpha$$

$$I = \int \sqrt{-g} d^D x \left(R - 2\Lambda + \sum_{n=2}^{\infty} \alpha_n \mathcal{Z}_n \right)$$
$$= \int \sqrt{-g} d^D x \left(R - 2\Lambda + F(R, \mathcal{Z}_2, \ldots, \mathcal{Z}_5) \right)$$

$$y_{+,\text{ext}} = \frac{\sqrt{k\left(3 - 2\beta + D(-1 + 2\beta) + \sqrt{(D-3)^2 + 8(D-1)\beta}\right)}}{\sqrt{2(D-1)}}, \text{ such that } T = 0$$

$$S = \frac{1}{2\kappa_4^2} \int \mathcal{R} \star 1 + \frac{1}{2} \mathrm{Re}\mathcal{N}_{AB} F^A \wedge F^B + \frac{1}{2} \mathrm{Im}\mathcal{N}_{AB} F^A \wedge \star F^B$$
$$- \frac{1}{\kappa_4^2} \int G_{a\bar{b}} dz^a \wedge \star d\bar{z}^b + h_{pq} d\xi^p \wedge \star d\xi^q$$

$$K = -\log i\left(\bar{X}^A \mathcal{F}_A - X^A \overline{\mathcal{F}}_A\right)$$

$$\mathcal{N}_{AB} = \overline{\mathcal{F}}_{AB} + 2i \frac{(\mathrm{Im}\mathcal{F})_{AC} X^C (\mathrm{Im}\mathcal{F})_{BD} X^D}{X^C (\mathrm{Im}\mathcal{F})_{CD} X^D},$$

$$\mathcal{L}_{\text{h.d.}} \supset -\frac{i}{2} \int \mathrm{d}^4\theta \sum_{g \geq 1} \mathcal{F}_g(\mathcal{X}^A) \left(\mathcal{W}^{ij} \mathcal{W}_{ij}\right)^g + \text{ h.c.}$$

$$\mathcal{W}_{\mu\nu}^{ij} = W_{\mu\nu}^{ij,-} - \mathcal{R}_{\mu\nu\rho\sigma}^{-} \theta^i \sigma^{\rho\sigma} \theta^j + \cdots$$
$$\mathcal{X}^A = X^A + \frac{1}{2} \epsilon_{ij} \theta^i \sigma^{\mu\nu} \theta^j \left(F_{\mu\nu}^{A,-} - ie^{K/2} \bar{X}^A W_{\mu\nu}^{-}\right) + \cdots$$

$$W_{\mu\nu}^{-} = 2ie^{K/2} \mathrm{Im}\mathcal{N}_{AB} X^A F_{\mu\nu}^{B,-}, W_{\mu\nu}^{ij,-} = \frac{\epsilon^{ij}}{2} W_{\mu\nu}^{-}$$

$$\mathcal{L}_{\text{h.d.}} \supset -\frac{i}{2} \sum_{g \geq 1} \mathcal{F}_g(X^A) \mathcal{R}_{-}^2 W_{-}^{2g-2} + \text{ h.c.}$$

$$\sum_{g \geq 0} \delta\mathcal{F}_g^{(\text{hyp})}(X^A) W_{-}^{2g-2} = -\frac{1}{4} \int_{i0^+}^{i\infty} \frac{\mathrm{d}\tau}{\tau} \frac{1}{\sin^2 \frac{\tau W_{-} e^{K/2} \bar{Z}_p}{2}} e^{-\tau |Z_p|^2}$$

$$Y^A = CX^A, \Upsilon = C^2 W^2, \text{ with } C^2 = e^{\mathcal{K}} \overline{\mathcal{Z}}^2,$$

$$|\mathcal{Z}|^2 = e^{\mathcal{K}} |p^A F_A(X, W^2) - q_A X^A|^2$$
$$e^{-\mathcal{K}} = i\bar{X}^A F_A(X, W^2) - iX^A \bar{F}_A(\bar{X}, \bar{W}^2)$$

$$F_A(X, W^2) = \partial_{X^A} F(X, W^2)$$

$$F(X, W^2) = \sum_{g=0}^{\infty} F_g(X^A) W^{2g}$$

$$F(Y,\Upsilon) := C^2 F(X,W^2) = \sum_{g=0}^{\infty} F_g(Y^A)\Upsilon^g$$

$$2F(\Upsilon,Y^A) = 2\Upsilon F_\Upsilon + Y^A F_A$$

$$F_g(Y^A) = (-1)^g 2^{-6g}\mathcal{F}_g(Y^A)$$

$$\mathcal{Z} = \overline{\mathcal{Z}}^{-1}(p^A F_A(Y,\Upsilon) - q_A Y^A),$$

$$ds^2 = -e^{2U(y)}dt^2 + e^{-2U(y)}(dy^2 + y^2 d\Omega_2^2), \text{ with } e^{-2U(y)} = \frac{|\mathcal{Z}|^2\kappa_4^2}{8\pi y^2}$$

$$ip^A = Y^A - \bar{Y}^A$$
$$iq_A = F_A(Y,\Upsilon) - \bar{F}_A(\bar{Y},\bar{\Upsilon})$$

$$\mathcal{S}_{\mathrm{BH}} = \pi[|\mathcal{Z}|^2 + 4\mathrm{Im}(\Upsilon\partial_\Upsilon F(Y,\Upsilon))],$$

$$\mathcal{S}_{\mathrm{BH}} = 4\pi\left[\mathrm{Im}F - \frac{1}{2}q_A \mathrm{Re}Y^A\right],$$

$$F(Y,\Upsilon) = \frac{D_{abc}Y^aY^bY^c}{Y^0} + d_a\frac{Y^a}{Y^0}\Upsilon + G(Y^0,\Upsilon)$$

$$\mathcal{S}_{\mathrm{BH}} = \frac{\pi}{3p^0}\sqrt{\frac{4}{3}\left(\tilde{\Delta}_a\tilde{x}^a\right)^2 - 9(p^0p^A\tilde{q}_A - 2D_{abc}p^ap^bp^c)^2}$$
$$+4\pi\left[\mathrm{Im}G - \mathrm{Re}Y^0\mathrm{Im}G_0 - \frac{1}{\sqrt{3}}\frac{(\mathrm{Re}Y^0)^2}{|Y^0|^3}\tilde{x}^a d_a\Upsilon\right]$$

$$\mathcal{F}_0(X^A) = -\frac{1}{6}\mathcal{K}_{abc}\frac{X^aX^bX^c}{X^0} + K_{ab}^{(1)}X^aX^b + K_a^{(2)}X^0X^a + K^{(3)}(X^0)^2$$
$$-\frac{(X^0)^2}{(2\pi i)^3}\sum_{k>0} n_k^{(0)}\sum_{m\geq 1}\frac{1}{m^3}e^{2\pi i m k_a z^a}$$

$$\mathcal{K}_{abc} = \omega_a\cdot\omega_b\cdot\omega_c, K_a^{(2)} = \frac{1}{24}c_2(TX_3)\cdot\omega_a, K^{(3)} = \frac{i\zeta(3)}{2(2\pi)^3}\chi_E(X_3)$$

$$\chi_E(X_3) = 2h^{1,1}(X_3) - 2h^{2,1}(X_3)$$

$$\boldsymbol{k} = k_a\gamma^a \in H_2^+(X_3,\mathbb{Z})$$

$$\mathcal{F}_1(X^A) = \frac{1}{24}\int_{X_3} J_{\mathbb{C}}\wedge c_2(TX_3) + \mathcal{O}(\log z^a) = \frac{1}{24}c_{2,a}z^a + \mathcal{O}\left(e^{2\pi i z^a}\right)$$

$$J_{\mathbb{C}} = z^a\omega_a = (b^a + it^a)\omega_a$$

$$\mathcal{F}_{g>1}(X^A) \supset \frac{\chi(X_3)}{2}(-1)^{g-1}2(2g-1)\frac{\zeta(2g)\zeta(3-2g)}{(2\pi)^{2g}}(X^0)^{2-2g},$$

$$F(Y,\Upsilon)=\frac{D_{abc}Y^aY^bY^c}{Y^0}+d_a\frac{Y^a}{Y^0}\Upsilon+G(Y^0,\Upsilon)+\mathcal{O}\left(e^{2\pi iz^a}\right)$$

$$G(Y^0,\Upsilon)=-\frac{i}{2(2\pi)^3}\chi_E(X_3)(Y^0)^2\sum_{g=0,2,3,\ldots}c_{g-1}^3\alpha^{2g}+\cdots$$

$$c_{g-1}^3=(-1)^{g-1}2(2g-1)\frac{\zeta(2g)\zeta(3-2g)}{(2\pi)^{2g}},\alpha^2=-\frac{1}{64}\frac{\Upsilon}{(Y^0)^2}$$

$$\begin{gathered}q_a=2\mathrm{Im}F_a\\q_0=2\mathrm{Im}F_0\\p^a=2\mathrm{Im}Y^a\\p^0=2\mathrm{Im}Y^0\end{gathered}$$

$$D_{abc}\left(\mathrm{Re}Y^b-\frac{\mathrm{Re}Y^0}{p^0}p^b\right)\left(\mathrm{Re}Y^c-\frac{\mathrm{Re}Y^0}{p^0}p^c\right)=\Delta_a$$

$$\Delta_a=|Y^0|^2\left(-\frac{\tilde{q}_a}{3p^0}+\frac{1}{(p^0)^2}D_{abc}p^bp^c\right),\tilde{q}_a\equiv q_a+p^0\frac{d_a\Upsilon}{|Y^0|^2}$$

$$\mathrm{Re}Y^a=x^a+\frac{\mathrm{Re}Y^0}{p^0}p^a$$

$$\tilde{q}_0=2\mathrm{Im}\left[-\frac{1}{(Y^0)^2}D_{abc}Y^aY^bY^c\right]$$

$$\tilde{q}_0\equiv q_0+2\mathrm{Im}\left[\frac{1}{(Y^0)^2}d_aY^a\Upsilon\right]-2\mathrm{Im}G_0$$

$$\left(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c\right)|Y^0|^4=2(p^0)^3\mathrm{Re}Y^0x^a\Delta_a$$

$$\begin{aligned}\frac{\mathcal{S}_{\mathrm{BH}}}{4\pi}=&-2\mathrm{Im}\left(\frac{1}{Y^0}\right)\Delta_a\mathrm{Re}Y^a+\mathrm{Im}G-\frac{1}{2}q_0\mathrm{Re}Y^0\\&+\frac{1}{2}\mathrm{Re}\left(\frac{1}{Y^0}\right)p^a\left(\frac{q_a}{6\mathrm{Im}\left(\frac{1}{Y^0}\right)}+\frac{2}{3}d_a\Upsilon\right)\end{aligned}$$

$$\frac{\mathcal{S}_{\mathrm{BH}}}{4\pi}=\frac{1}{8\mathrm{Re}Y^0}\left(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c\right)+\mathrm{Im}G+\frac{\mathrm{Re}Y^0}{2p^0}p^A\delta q_A$$

$$x^a=\tilde{x}^a\frac{|Y^0|}{\sqrt{3}p^0},\tilde{\Delta}_a=D_{abc}\tilde{x}^b\tilde{x}^c=\Delta_a\frac{3(p^0)^2}{|Y^0|^2},$$

$$\left(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c\right)|Y^0|=\frac{2}{3\sqrt{3}}\mathrm{Re}Y^0\tilde{x}^a\tilde{\Delta}_a$$

$$|Y^0|^2=\frac{(p^0)^2}{3}\frac{\left(\tilde{\Delta}_a\tilde{x}^a\right)^2}{\frac{4}{3}\left(\tilde{\Delta}_a\tilde{x}^a\right)^2-9(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c)^2}$$

$$\mathcal{S}_{\rm BH}=\frac{\pi}{3p^0}\sqrt{\frac{4}{3}\left(\tilde{\Delta}_a\tilde{x}^a\right)^2-9(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c)^2}$$
$$+4\pi\left[{\rm Im}G-{\rm Re}Y^0{\rm Im}G_0-\frac{1}{\sqrt{3}}\frac{({\rm Re}Y^0)^2}{|Y^0|^3}\tilde{x}^ad_a\Upsilon\right]$$

$$\mathcal{S}_{\rm BH}=\frac{\pi}{3p^0}\sqrt{\frac{4}{3}\left(\tilde{\Delta}_a\tilde{x}^a\right)^2-9(p^0p^Aq_A-2D_{abc}p^ap^bp^c)^2},$$

$$\mathcal{S}_{\rm BH}=-\frac{4\pi}{3}Y^a\left(q_a+\frac{4d_a\Upsilon}{p^0}\right)+4\pi{\rm Im}G,$$

$$d_a=-\frac{1}{24}\frac{1}{64}c_a$$

$$\tilde{x}^a=-\sqrt{3}p^a+p^0\left[\frac{\sqrt{3}}{6}D^{ab}q_b\right]+(p^0)^2\frac{\sqrt{3}}{2}\left[\frac{p^a}{4({\rm Re}Y^0)^2}-\frac{1}{({\rm Re}Y^0)^3}\Gamma^a\right]+\cdots$$
$$\tilde{\Delta}_a=3D_{abc}p^bp^c-p^0q_a-(p^0)^2\frac{d_a\Upsilon}{({\rm Re}Y^0)^2}+\cdots$$
$$\tilde{q}_a=q_a+p^0\frac{d_a\Upsilon}{({\rm Re}Y^0)^2}+\cdots$$
$$\tilde{q}_0=q_0-2{\rm Im}G_0+\frac{p^ad_a\Upsilon}{({\rm Re}Y^0)^2}+\cdots$$

$$D_{abc}p^ap^b\Gamma^c={\rm Re}Y^0\left[-\frac{1}{36}({\rm Re}Y^0)^2+\frac{1}{4}D_{abc}p^ap^bp^c-\frac{1}{3}p^ad_a\Upsilon\right]+\mathcal{O}(p^0)$$

$$\frac{\mathcal{S}_{\rm BH}}{4\pi}=\frac{1}{2}\sqrt{D_{abc}p^ap^bp^c\left[\hat{q}_0-2{\rm Im}G_0+\frac{p^ad_a\Upsilon}{(Y^0)^2}\right]}+{\rm Im}G-Y^0{\rm Im}G_0+\frac{p^ad_a\Upsilon}{Y^0}$$

$$\hat{q}_0=q_0+\frac{1}{12}D^{ab}q_aq_b$$

$$(Y^0)^2=\frac{\frac{1}{4}D_{abc}p^ap^bp^c-d_ap^a\Upsilon}{\hat{q}_0+i(G_0-\bar{G}_0)}$$

$$\frac{\mathcal{S}_{\rm BH}}{4\pi}=-\frac{1}{4}\frac{D_{abc}p^ap^bp^c}{Y^0}+{\rm Im}G-Y^0{\rm Im}G_0+\frac{p^ad_a\Upsilon}{Y^0}$$
$$=-\hat{q}_0Y^0+{\rm Im}G+Y^0{\rm Im}G_0$$

$$F(Y^A)=D_{abc}\frac{Y^aY^bY^c}{Y^0}+K_a^{(2)}Y^0Y^a$$

$$F(Y^A,\Upsilon)=\frac{D_{abc}Y^aY^bY^c}{Y^0}+d_a\frac{Y^a}{Y^0}\Upsilon+G(Y^0,\Upsilon)$$

$$|\mathcal{Z}|^2=-\frac{D_{abc}p^ap^bp^c-2d_ap^a\Upsilon}{Y^0}-2Y^0{\rm Im}G_0$$

$$\frac{\mathcal{S}_{\mathrm{BH}}^{\mathrm{shift}}}{\pi}=2\sqrt{D_{abc}p^{a}p^{b}p^{c}\left[\hat{q}_{0}-2\mathrm{Im}G_{0}+\frac{p^{a}d_{a}\Upsilon}{(Y^{0})^{2}}\right]}=-\frac{D_{abc}p^{a}p^{b}p^{c}}{Y^{0}}\neq|\mathcal{Z}|^{2}$$

$$\frac{\mathcal{S}_{\mathrm{BH}}}{\pi}=\left[-\frac{D_{abc}p^{a}p^{b}p^{c}}{Y^{0}}+2\frac{p^{a}d_{a}\Upsilon}{Y^{0}}-2Y^{0}\mathrm{Im}G_{0}\right]+\left[4\mathrm{Im}G-2Y^{0}\mathrm{Im}G_{0}+2\frac{p^{a}d_{a}\Upsilon}{Y^{0}}\right]$$

$$F_{\mu\nu}^{A,-}=\frac{1}{2}\left(F_{\mu\nu}^{A}+i\star F_{\mu\nu}^{A}\right),G_{A}^{-}=\overline{\mathcal{N}}_{AB}F^{B,-}$$

$$S\supset\frac{1}{4\kappa_{4}^{2}}\int F^{A,-}\wedge G_{A}^{-}+\mathrm{h.c.}$$

$$\begin{pmatrix}F^{A,-}\\G_{A}^{-}\end{pmatrix}\rightarrow\begin{pmatrix}\mathcal{A}&\mathcal{B}\\\mathcal{C}&\mathcal{D}\end{pmatrix}\begin{pmatrix}F^{A,-}\\G_{A}^{-}\end{pmatrix}$$

$$\mathcal{A}^{\mathrm{T}}\mathcal{D}-\mathcal{C}^{\mathrm{T}}\mathcal{B}=\mathbb{1}_{n_{V}+1},\mathcal{A}^{\mathrm{T}}\mathcal{C}-\mathcal{C}^{\mathrm{T}}\mathcal{A}=\mathcal{B}^{\mathrm{T}}\mathcal{D}-\mathcal{D}^{\mathrm{T}}\mathcal{B}=0$$

$$\mathcal{N}_{AB}\rightarrow(\mathcal{D}_{A}{}^{C}\mathcal{N}_{CD}+\mathcal{C}_{AD})[(\mathcal{C}+\mathcal{B}\mathcal{N})^{-1}]_{B}^{D}$$

$$\begin{pmatrix}X^{A}\\\mathcal{F}_{A}\end{pmatrix}\rightarrow\begin{pmatrix}X^{A\prime}\\\mathcal{F}_{A}'\end{pmatrix}=\begin{pmatrix}\mathcal{A}&\mathcal{B}\\\mathcal{C}&\mathcal{D}\end{pmatrix}\begin{pmatrix}X^{A}\\\mathcal{F}_{A}\end{pmatrix}$$

$$F'(X^{A\prime})=F(X^{A})-\frac{1}{2}X^{A}\mathcal{F}_{A}+\frac{1}{2}X^{A}\left(\mathcal{C}^{\mathrm{T}}\mathcal{B}+\mathcal{A}^{\mathrm{T}}\mathcal{D}\right)_{A}{}^{B}\mathcal{F}_{B}$$
$$+\frac{1}{2}X^{A}\left(\mathcal{C}^{\mathrm{T}}\mathcal{A}\right)_{AB}X^{B}+\frac{1}{2}\mathcal{F}_{A}\left(\mathcal{D}^{\mathrm{T}}\mathcal{B}\right)^{AB}\mathcal{F}_{B}$$

$$F'(Y^{A\prime})=F(Y^{A})-\frac{1}{2}Y^{A}F_{A}+\frac{1}{2}Y^{A}\left(\mathcal{C}^{\mathrm{T}}\mathcal{B}+\mathcal{A}^{\mathrm{T}}\mathcal{D}\right)_{A}^{B}F_{B}$$
$$+\frac{1}{2}Y^{A}\left(\mathcal{C}^{\mathrm{T}}\mathcal{A}\right)_{AB}Y^{B}+\frac{1}{2}F_{A}\left(\mathcal{D}^{\mathrm{T}}\mathcal{B}\right)^{AB}F_{B}$$

$$Y^{0\prime}=Y^{0},Y^{a\prime}=Y^{a}+\frac{\gamma^{a}}{p^{0}}Y^{0}$$

$$F_{0}'=F_{0}-\frac{\gamma^{a}F_{a}}{p^{0}}-3D_{abc}\frac{\gamma^{a}\gamma^{b}Y^{c}}{(p^{0})^{2}}-D_{abc}\frac{\gamma^{a}\gamma^{b}\gamma^{c}Y^{0}}{(p^{0})^{3}}$$
$$F_{a}'=F_{a}+6D_{abc}\frac{Y^{b}\gamma^{c}}{p^{0}}+3D_{abc}\frac{\gamma^{b}\gamma^{c}Y^{0}}{(p^{0})^{2}}$$

$$p^{0\prime}=p^{0},p^{a\prime}=p^{a}+\gamma^{a},q_{a}'=q_{a}+6D_{abc}\frac{p^{b}\gamma^{c}}{p^{0}}+3D_{abc}\frac{\gamma^{b}\gamma^{c}}{p^{0}}$$
$$q_{0}'=q_{0}-\frac{\gamma^{a}q_{a}}{p^{0}}-3D_{abc}\frac{\gamma^{a}\gamma^{b}p^{c}}{(p^{0})^{2}}-D_{abc}\frac{\gamma^{a}\gamma^{b}\gamma^{c}}{(p^{0})^{2}}$$

$$p^{A}F_{A}\rightarrow p^{A}F_{A}+\Delta,q_{A}Y^{A}\rightarrow q_{A}Y^{A}+\Delta,\mathrm{Im}(\Upsilon\partial_{\Upsilon}F)\rightarrow\mathrm{Im}(\Upsilon\partial_{\Upsilon}F)$$

$$\Delta=6D_{abc}\frac{p^{a}Y^{b}\gamma^{c}}{p^{0}}+3D_{abc}\frac{\gamma^{a}Y^{b}\gamma^{c}}{p^{0}}+3D_{abc}\frac{p^{a}\gamma^{b}\gamma^{c}Y^{0}}{(p^{0})^{2}}+2D_{abc}\frac{\gamma^{a}\gamma^{b}\gamma^{c}Y^{0}}{(p^{0})^{2}},$$

$$\mathcal{S}'_{\mathrm{BH}}[p^{0\prime},p^{a\prime},q'_a,q'_0]=\mathcal{S}_{\mathrm{BH}}[p^{0\prime},p^{a\prime},q'_a,q'_0]=\mathcal{S}_{\mathrm{BH}}[p^0,p^a,q_a,q_0],$$

$$\begin{pmatrix}Y^{0\prime}\\Y^{a\prime}\\F'_0\\F'_a\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0\\ \dfrac{\gamma^a}{p^0} & \delta^a_b & 0 & 0\\ -D_{abc}\dfrac{\gamma^a\gamma^b\gamma^c}{(p^0)^3} & -3D_{bcd}\dfrac{\gamma^c\gamma^d}{(p^0)^2} & 1 & -\dfrac{\gamma^a}{p^0}\\ 3D_{abc}\dfrac{\gamma^\gamma\gamma^c}{(p^0)^2} & 6D_{abc}\dfrac{\gamma^c}{(p^0)^2} & 0 & \delta^b_a\end{pmatrix}\begin{pmatrix}Y^0\\Y^b\\F_0\\F_b\end{pmatrix}$$

$$\mathrm{M}^{\mathrm{T}}\eta\mathrm{M}=\eta\equiv\begin{pmatrix}0 & \mathbb{1}_{h^{1,1}+1}\\ -\mathbb{1}_{h^{1,1}+1} & 0\end{pmatrix}$$

$$|\mathcal{Z}|^2=\mathbf{q}^{\mathrm{T}}\cdot\eta\cdot\mathcal{C}\mathbf{\Pi},\mathrm{Im}(\Upsilon\partial_\Upsilon F(Y,\Upsilon))$$

$$F'(Y^{A\prime},\Upsilon')=F(Y^A,\Upsilon)+3\frac{\gamma^aD_{abc}Y^bY^c}{p^0}+3\frac{\gamma^a\gamma^bD_{abc}Y^0Y^c}{(p^0)^2}+\frac{D_{abc}\gamma^a\gamma^b\gamma^c(Y^0)^2}{(p^0)^3}$$
$$=\frac{D_{abc}Y^{a\prime}Y^{b\prime}Y^{c\prime}}{Y^{0\prime}}+d_a\frac{Y^{a\prime}}{Y^{0\prime}}\Upsilon'+G(Y^{0\prime},\Upsilon')-d_a\frac{\gamma^a}{p^0}\Upsilon'=F(Y^{A\prime})$$

$$q_a=-\frac{1}{p^0}3D_{abc}\left(2\mathrm{Re}Y^b\right)(2\mathrm{Re}Y^c)+\frac{1}{p^0}3D_{abc}p^bp^c-\frac{4}{p^0}d_a\Upsilon$$
$$q_0=\frac{1}{(p^0)^2}3D_{abc}p^a\left(2\mathrm{Re}Y^b\right)(2\mathrm{Re}Y^c)-\frac{1}{(p^0)^2}D_{abc}p^ap^bp^c+\frac{4}{(p^0)^2}d_ap^a\Upsilon$$

$$q_0=-\frac{p^aq_a}{p^0}+\frac{2}{(p^0)^2}D_{abc}p^ap^bp^c$$

$$p^{0\prime}=p^0,p^{a\prime}=\gamma^a,q'_a=q_a+3D_{abc}\frac{\gamma^b\gamma^c}{p^0},q'_0=-\frac{\gamma^aq_a}{p^0}-D_{abc}\frac{\gamma^a\gamma^b\gamma^c}{(p^0)^2}$$

$$q'_0=-\frac{p^{a\prime}q'_a}{p^{0\prime}}+\frac{2}{(p^{0\prime})^2}D_{abc}p^{a\prime}p^{b\prime}p^{c\prime}$$

$$p^{0\prime}=p^0,q'_a=q_a-3D_{abc}\frac{p^bp^c}{p^0}$$

$$D_{abc}x^ax^b=\Delta_a=-|Y^0|^2\frac{\tilde{q}_a}{3p^0},\tilde{q}_a\equiv q_a+p^0\frac{d_a\Upsilon}{|Y^0|^2},$$

$$q_0-2\mathrm{Im}G_0=\left[\frac{2}{3p^0}|Y^0|^2q_ax^a-\frac{4}{3}d_ax^a\Upsilon\right]\mathrm{Im}((Y^0)^{-2})=\left[-\frac{2q_ax^a}{3|Y^0|^2}+\frac{4p^0d_ax^a\Upsilon}{3|Y^0|^4}\right]\mathrm{Re}Y^0$$

$$\frac{\mathcal{S}_{\mathrm{BH}}}{4\pi}=2\mathrm{Im}\left(\frac{1}{Y^0}\right)\left[\frac{1}{3p^0}|Y^0|^2q_ax^a+\frac{1}{3}d_ax^a\Upsilon\right]+\mathrm{Im}G-\frac{1}{2}q_0\mathrm{Re}Y^0$$
$$=-\frac{\tilde{q}_ax^a}{3}+\mathrm{Im}G-\frac{1}{2}q_0\mathrm{Re}Y^0$$

$$p^0 = p^{0\prime}, q_a = q'_a - 3D_{abc}\frac{p^{b\prime}p^{c\prime}}{p^{0\prime}}, q_0 = q'_0 + \frac{p^{a\prime}q'_a}{p^{0\prime}} - 2D_{abc}\frac{p^{a\prime}p^{b\prime}p^{c\prime}}{(p^{0\prime})^2}$$

$$\tilde{q}_a = \tilde{q}'_a - 3D_{abc}\frac{p^{b\prime}p^{c\prime}}{p^{0\prime}} = -3p^{0\prime}\frac{\Delta'_a}{|Y^{0\prime}|^2}, x^a = x^{a\prime} = \mathrm{Re}Y^{a\prime} - \frac{\mathrm{Re}Y^{0\prime}}{p^{0\prime}}p^{a\prime}, Y^0 = Y^{0\prime}$$

$$\frac{\mathcal{S}_{\mathrm{BH}}}{4\pi} = p^{0\prime}\frac{\Delta'_a}{|Y^{0\prime}|^2}\mathrm{Re}Y^{a\prime} + \mathrm{Im}G' - \frac{1}{2}q'_0\mathrm{Re}Y^{0\prime} + \frac{1}{2}\frac{\mathrm{Re}Y^{0\prime}}{|Y^{0\prime}|^2}p^{a\prime}\left(-\frac{q'_a|Y^{0\prime}|^2}{6p^{0\prime}} + \frac{2}{3}d_a\Upsilon'\right)$$

$$G(Y^0,\Upsilon) = \frac{i}{4(2\pi)^3}\chi_E(X_3)(Y^0)^2\alpha^2\oint \frac{ds}{s}\frac{1}{1-e^{-2\pi i s}}\frac{1}{\sinh^2\left(\frac{\pi\alpha s}{2}\right)}$$

$$s = \frac{2\pi n}{|\alpha|}e^{i\left(\frac{\pi}{2}-\theta_\alpha\right)}, \text{ with } n \in \mathbb{Z}$$

$$G^{(p)}(Y^0,\Upsilon) = -\frac{i}{2(2\pi)^3}\chi_E(X_3)(Y^0)^2\alpha^2\sum_{n=1}^{\infty} n\log\left(1-e^{-\alpha n}\right)$$

$$G^{(np)}(Y^0,\Upsilon=-64) = \frac{\chi_E}{8\pi^2}Y^0\sum_{n,k=1}^{\infty}\frac{n}{k}e^{-4\pi^2 knY^0}\left(1+\frac{1}{4\pi^2 knY^0}\right)$$
$$= \frac{\chi_E}{8\pi^2}Y^0\sum_{n}^{\infty}\left(n\mathrm{Li}_1\left(e^{-4\pi^2 nY^0}\right) + \frac{1}{4\pi^2 Y^0}\mathrm{Li}_2\left(e^{-4\pi^2 nY^0}\right)\right)$$

$$G_0^{(np)}(Y^0,\Upsilon=-64) = -\frac{\chi_E}{2}Y^0\sum_{n,k=1}^{\infty}n^2e^{-4\pi^2 knY^0} = \frac{\chi_E}{2}Y^0\sum_{n=1}^{\infty}n^2\frac{1}{1-e^{4\pi^2 nY^0}}$$
$$= -\frac{\chi_E}{2}Y^0\sum_{n=1}^{\infty}n^2\mathrm{Li}_0\left(e^{-4\pi^2 nY^0}\right)$$

$$\frac{\mathcal{S}_{\mathrm{BH}}}{4\pi} = -\frac{\tilde{q}_a x^a}{3} + \mathrm{Im}G - \frac{1}{2}q_0\mathrm{Re}Y^0,$$

$$\mathrm{Im}G^{(np)} = \frac{\chi_E}{8\pi^2}\sum_{n,k=1}^{\infty}\frac{n}{k}e^{-4\pi^2 kn\mathrm{Re}Y^0}\left[\frac{p^0}{2}\cos\left(2\pi^2 knp^0\right) - \left(\mathrm{Re}Y^0 + \frac{1}{4\pi^2 kn}\right)\sin\left(2\pi^2 knp^0\right)\right]$$

$$\mathrm{Im}G_0^{(np)} = -\frac{\chi_E}{2}\sum_{n,k=1}^{\infty}n^2e^{-4\pi^2 kn\mathrm{Re}Y^0}\left[\frac{p^0}{2}\cos\left(2\pi^2 knp^0\right) - \mathrm{Re}Y^0\sin\left(2\pi^2 knp^0\right)\right],$$

$$ds^2 = -f(r)^2dt^2 + f(r)^{-2}dr^2 + r^2d\Omega_2^2, \text{ with } f(r) = 1 - \frac{r_h}{r}.$$

$$f(y) = \left(1+\frac{r_h}{y}\right)^{-1}$$

$$ds^2 = -\frac{y^2}{r_h^2}dt^2 + \frac{r_h^2}{y^2}dy^2 + r_h^2d\Omega_2^2$$

$$ds^2 = \frac{r_h^2}{\rho^2}(-dt^2 + d\rho^2 + \rho^2 d\Omega_2^2)$$

$$-(X^0) - (X^2)^2 + (X^1) = -r_h^2 := -R_{\text{AdS}}^2$$

$$X^0 = R_{\text{AdS}} \cosh(\xi)\sin(\tau/R_{\text{AdS}}), X^2 = R_{\text{AdS}} \cosh(\xi)\cos(\tau/R_{\text{AdS}})$$

$$X^1 = R_{\text{AdS}} \sinh(\xi).$$

$$ds^2 = -\cosh^2(\xi)d\tau^2 + R_{\text{AdS}}^2(d\xi^2 + d\Omega_2^2)$$

$$\psi = \sin^{-1}\left(\frac{1}{\cosh\ \xi}\right), \tilde{\tau} = \frac{\tau}{R_{\text{AdS}}},\ \text{with}\ \psi \in [0,\pi], \tilde{\tau} \in (-\infty,\infty)$$

$$ds^2 = \frac{R_{\text{AdS}}^2}{\sin^2(\psi)}(-d\tilde{\tau}^2 + d\psi^2) + R_{\text{AdS}}^2 d\Omega_2^2$$

$$r_h^2 = R_{\text{AdS}}^2 = |C|^2 e^{-K} = |Z_{\text{BH}}|^2.$$

$$m = |Z|, Z = e^{K/2}(q_A X^A - p^A \mathcal{F}_A),$$

$$S_{wl} = -2|Z|R_{\text{AdS}} \int_\gamma d\sigma \sqrt{\rho^{-2}(\dot{t}^2 - \dot{\rho}^2) - \dot{\theta}^2 - \sin^2\ \theta\dot{\phi}^2} + \int_\Sigma q_A F^A - p^A G_A,$$

$$p^{A\prime} = \frac{1}{4\pi}\int_{\mathbf{S}^2} F^A, q'_A = \frac{1}{4\pi}\int_{\mathbf{S}^2} G_A, G_A^- = \overline{\mathcal{N}}_{AB}F^{B,-},$$

$$F^A = R_{\text{AdS}}^{-2}\left(p^{A\prime}\omega_{\text{S}^2} + 2\text{Re}CX^A\omega_{\text{AdS}_2}\right), G_A = R_{\text{AdS}}^{-2}\left(q'_A\omega_{\text{S}^2} + 2\text{Re}C\mathcal{F}_A\omega_{\text{AdS}_2}\right),$$

$$CX^A = \text{Re}CX^A + \frac{i}{2}p^{A\prime}, C\mathcal{F}_A = \text{Re}C\mathcal{F}_A + \frac{i}{2}q'_A,$$

$$\omega_{\text{S}^2} = R_{\text{AdS}}^2 \sin\ \theta d\theta \wedge d\phi, \omega_{\text{AdS}_2} = \frac{R_{\text{AdS}}^2}{\rho^2}dt \wedge d\rho$$

$$\star_4\, \omega_{\mathbf{S}^2} = -\omega_{\text{AdS}_2}, \star_4\, \omega_{\text{AdS}_2} = \omega_{\mathbf{S}^2},$$

$$R_{\text{AdS}}^2 F^{A,-} = \overline{CX}^A\left(\omega_{\text{AdS}_2} + i\omega_{\mathbf{S}^2}\right), R_{\text{AdS}}^2 G_A^- = \overline{C\mathcal{F}}_A\left(\omega_{\text{AdS}_2} + i\omega_{\mathbf{S}^2}\right)$$

$$\begin{aligned} W^- &= e^{K/2}\mathcal{F}_A F^{A,-} - e^{K/2}X^A G_A^- = \frac{e^{K/2}}{R_{\text{AdS}}^2}\left[\mathcal{F}_A\overline{CX}^A - X^A\overline{C\mathcal{F}}_A\right]\left(\omega_{\text{AdS}_2} + i\omega_{\mathbf{S}^2}\right) \\ &= -ie^{-K/2}\bar{C}R_{\text{AdS}}^{-2}\left(\omega_{\text{AdS}_2} + i\omega_{\mathbf{S}^2}\right) = Z_{\text{BH}}R_{\text{AdS}}^{-2}\left(\omega_{\mathbf{S}^2} - i\omega_{\text{AdS}_2}\right) \end{aligned}$$

$$ie^{K/2}\bar{X}^A W^- = \overline{CX}^A R_{\mathrm{AdS}}^{-2}\left(\omega_{\mathrm{AdS}_2} + i\omega_{\mathbf{S}^2}\right) = F^{A,-}$$

$$S_{wl} \supset \int_\Sigma q_A F^A - p^A G_A = q_e \int_\gamma \frac{dt}{\rho} + q_m \int_\gamma \cos\,\theta d\phi$$

$$q_e = 2e^{-K/2}\mathrm{Re}CZ, q_m = -2e^{-K/2}\mathrm{Im}CZ = p^A q'_A - q_A p^{A\prime}$$

$$\tilde{m} := 2|Z|R_{\mathrm{AdS}} = 2e^{-K/2}|CZ|,$$

$$q_e^2 + q_m^2 = \tilde{m}^2 \Longrightarrow |q_e| = 2e^{-K/2}|\mathrm{Re}CZ| \leq 2e^{-K/2}|CZ| = \tilde{m},$$

$$(2mR_{\mathrm{AdS}})^2 \geq \left|2e^{-K/2}CZ\right|^2 = q_e^2 + q_m^2$$

$$S_{wl} = -\tilde{m}\int_\gamma d\sigma\sqrt{\rho^{-2}(\dot{t}^2 - \dot{\rho}^2) - \dot{\theta}^2 - \sin^2\,\theta\dot{\phi}^2} + q_e\int_\gamma \frac{dt}{\rho} + q_m\int_\gamma \cos\,\theta d\phi$$

$$S_{wl} = \frac{1}{2}\int_\gamma d\sigma\left[h^{-1}\left(\rho^{-2}(-\dot{t}^2 + \dot{\rho}^2) + \dot{\theta}^2 + \sin^2\,\theta\dot{\phi}^2\right) - h\tilde{m}^2\right] + \int_\gamma d\sigma\left(q_e\rho^{-1}\dot{t} + q_m\cos\,\theta\dot{\phi}\right)$$

$$\rho^{-2}(-\dot{t}^2 + \dot{\rho}^2) + \dot{\theta}^2 + \sin^2\,\theta\dot{\phi}^2 + h^2\tilde{m}^2 = 0$$

$$H = \frac{\rho^2}{2}\left[p_\rho^2 - \left(p_t - \frac{q_e}{\rho}\right)^2\right] + \frac{1}{2}p_\theta^2 + \frac{1}{2}\csc^2\,\theta\left(p_\phi - q_m\cos\,\theta\right)^2 \overset{!}{=} -\frac{\tilde{m}^2}{2}.$$

$$p_\rho = \frac{\dot{\rho}}{\rho^2}, p_t = -\frac{\dot{t}}{\rho^2} + \frac{q_e}{\rho}, p_\theta = \dot{\theta}, p_\phi = \sin^2\,\theta\dot{\phi} + q_m\cos\,\theta$$

$$\frac{dp_\phi}{d\sigma} = \frac{dp_t}{d\sigma} = 0$$

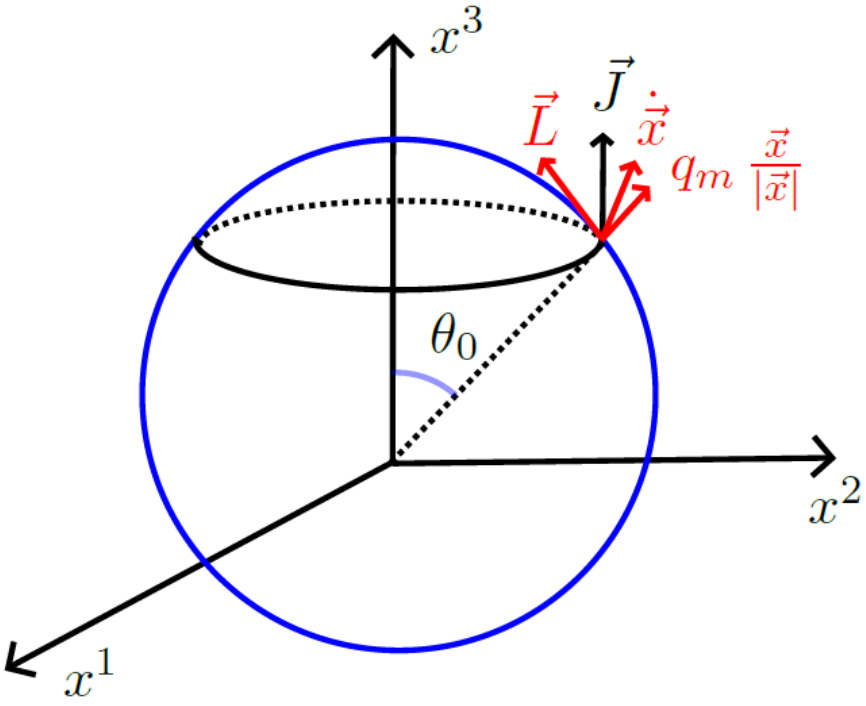
$$j = p_\phi = \sin^2\,\theta\dot{\phi} + q_m\cos\,\theta, E = -p_t = \frac{\dot{t}}{\rho^2} - \frac{q_e}{\rho}.$$

$$\dot{p}_\rho = \frac{d}{d\sigma}\left(\frac{\dot{\rho}}{\rho^2}\right) = E\frac{\dot{t}}{\rho} - \frac{\dot{\rho}^2}{\rho^3}$$
$$\dot{p}_\theta = \ddot{\theta} = \sin\,\theta\left(\cos\,\theta\dot{\phi}^2 - q_m\dot{\phi}\right) = \dot{\phi}\tan\,\theta(\dot{\phi} - j).$$

$$p_\rho^2 + \frac{p_\theta^2}{\rho^2} + V(\rho,\theta) = 0, V(\rho,\theta) = \frac{\tilde{m}^2}{\rho^2} - \left(E + \frac{q_e}{\rho}\right)^2 + \frac{1}{\rho^2\sin^2\,\theta}(j - q_m\cos\,\theta)^2.$$

$$\dot{p}_\theta = 0 \Longrightarrow \cos\,\theta_0 = \frac{q_m}{j},\ \text{with}\ j = \dot{\phi}$$

$$|\boldsymbol{L}| = \ell = \sin\,\theta_0\dot{\phi} = \sqrt{j^2 - q_m^2}$$



$$p_\rho^2 + V(\rho) = 0, V(\rho) = \frac{\tilde{m}^2 + \ell^2}{\rho^2} - \left(E + \frac{q_e}{\rho}\right)^2$$

$$\tilde{m}^2 \to m_{\text{eff}}^2 = \tilde{m}^2 + \ell^2$$

$$p_\rho^2 + V(\rho) = 0, V(\rho) = \frac{\tilde{m}^2}{\rho^2} - \left(E + \frac{q_e}{\rho}\right)^2$$

$$p_\theta = \pm i\left(\cot\,\theta p_\phi - q_m \csc\,\theta\right) = 0 \Leftrightarrow \cos\,\theta_0 = \frac{q_m}{j}$$

$$p_\rho^2 + V_{\text{ext}}(\rho) = 0, V_{\text{ext}}(\rho) = \frac{\tilde{m}^2 + \ell^2}{\rho^2} - \left(E + \frac{\tilde{m}}{\rho}\right)^2.$$

$$p_\rho^2 + V_{\text{mag}}(\rho) = 0, V_{\text{mag}}(\rho) = \frac{j^2}{\rho^2} - E^2,$$

$$\ell = \sqrt{j^2 - \tilde{m}^2}$$

$$V_{\text{mag}}\,(\rho) = \frac{\tilde{m}^2}{\rho^2} - E^2$$

$$H = \frac{1}{2}\rho^2\left[p_\rho^2 - \left(p_t - \frac{q_e}{\rho}\right)^2\right] + \frac{1}{2}p_\theta^2 + \frac{1}{2}\csc^2\,\theta\left(p_\phi - q_m \cos\,\theta\right)^2.$$

$$\{q^j, p_k\}_{\text{PB}} = \delta_k^j$$

$$\{A(q,p),B(q,p)\}_{\mathrm{PB}}=\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i}-\frac{\partial B}{\partial q^i}\frac{\partial A}{\partial p_i}$$

$$\dot{q}^i=\frac{\partial H}{\partial p_i},\dot{p}_i=-\frac{\partial H}{\partial q^i}$$

$$\frac{dA(q,p)}{d\sigma}=\{A(q,p),H(q,p)\}_{\mathrm{PB}}$$

$$\begin{aligned}K_+&=p_t\\K_-&=(t^2+\rho^2)p_t+2t\rho p_\rho-2q_e\rho\\K_0&=tp_t+\rho p_\rho\end{aligned}$$

$$\begin{aligned}J_1&=-\sin\,\phi p_\theta-\cot\,\theta\cos\,\phi p_\phi+q_m\csc\,\theta\cos\,\phi\\J_2&=\cos\,\phi p_\theta-\cot\,\theta\sin\,\phi p_\phi+q_m\csc\,\theta\sin\,\phi\\J_3&=p_\phi\end{aligned}$$

$$\begin{aligned}\{J_i,J_j\}_{\mathrm{PB}}&=\epsilon_{ijk}J_k\\\{K_+,K_-\}_{\mathrm{PB}}&=-2K_0,\{K_0,K_\pm\}_{\mathrm{PB}}=\pm K_\pm\\\{J_i,K_j\}_{\mathrm{PB}}&=0\end{aligned}$$

$$2H=\delta^{ik}J_iJ_k+K_0^2-\frac{1}{2}(K_+K_-+K_-K_+)-q_e^2-q_m^2=C_2^{\mathbb{S}^2}+C_2^{\mathrm{AdS}_2}-\tilde{m}^2$$

$$H\overset{!}{=}-\frac{1}{2}\tilde{m}^2\Leftrightarrow C_2^{\mathbb{S}^2}+C_2^{\mathrm{AdS}_2}=0$$

$$C_2^{\mathbb{S}^2}+C_2^{\mathrm{AdS}_2}=-\Delta^2$$

$$\Delta^2=\tilde{m}^2-q_e^2-q_m^2>0$$

$$\left(\rho-\frac{q_e}{K_+}\right)^2-\left(t-\frac{K_0}{K_+}\right)^2=\frac{\tilde{m}^2+\ell^2}{K_+^2}.$$

$$\mathrm{sgn}(q_e/K_+)=\mathrm{sgn}(q_ep_t)$$

$$S_E=\frac{1}{2}\int\,d\sigma\left[\left(\frac{1}{h\rho^2}(\dot{t}^2+\dot{\rho}^2)\right)+hm_{\mathrm{eff}}^2\right]-q_e\int\,d\sigma\frac{\dot{\tau}}{\rho}$$

$$m_{\mathrm{eff}}\,=\sqrt{\tilde{m}^2+\ell(\ell+1)}$$

$$H_E=\frac{\rho^2}{2}\left[p_\rho^2+\left(p_\tau-\frac{q_e}{\rho}\right)^2\right]-\frac{m_{\mathrm{eff}}^2}{2}\overset{!}{=}0$$

$$p_\rho^2-V(\rho)=0,V(\rho)=\frac{m_{\mathrm{eff}}^2}{\rho^2}-\left(p_\tau-\frac{q_e}{\rho}\right)^2$$

$$V(\rho)=0\Leftrightarrow\rho_\pm=\frac{q_e\pm m_{\mathrm{eff}}}{p_\tau}$$

$$\mathcal{L}_E = p_i \dot{q}^i - H_E = p_i \dot{q}^i$$

$$S_E = \oint p_i dq^i$$

$$S_E = 2\int_{\rho_-}^{\rho_+} d\rho p_\rho = 2\int_{\rho_-}^{\rho_+} d\rho \sqrt{V(\rho)}$$

$$S_E = 2m_{\text{eff}} \int_{-1}^{1} dy \frac{\sqrt{1-y^2}}{y + q_e/m_{\text{eff}}}$$

$$S_E = 2\pi\left(q_e - \sqrt{q_e^2 - \tilde{m}^2 - \ell(\ell+1)}\right)$$

$$\frac{d\tau}{d\rho} = \pm \frac{q_e}{m_{\text{eff}}} \frac{\frac{p_\tau}{q_e} - 1}{\sqrt{1 - \frac{q_e^2}{m_{\text{eff}}^2}\left(1 - \frac{p_\tau}{q_e}\rho\right)^2}}$$

$$\dot{\tau} = q_e \rho \left(\frac{p_\tau}{q_e}\rho - 1\right), \dot{\rho} = \pm m_{\text{eff}} \rho \sqrt{1 - \frac{q_e^2}{m_{\text{eff}}^2}\left(1 - \frac{p_\tau}{q_e}\rho\right)^2}.$$

$$\tau = \tau_0 \mp m_{\text{eff}}^2 \frac{\sqrt{1 - \frac{q_e^2}{m_{\text{eff}}^2}\left(1 - \frac{p_\tau}{q_e}\rho\right)^2}}{p_\tau q_e} \Longrightarrow (\tau - \tau_0)^2 + \left(\rho - \frac{q_e}{p_\tau}\right)^2 = \frac{m_{\text{eff}}^2}{p_\tau^2},$$

$$S_{\text{eff}}^{2\,\text{d}}[A] = -\int_0^\infty \frac{dh}{h} \int \mathcal{D}\tau(h)\mathcal{D}\rho(h) e^{-S_{\text{E}}[x^i, A]}$$

$$S_{\text{eff}}^{2\,\text{d}}[A] \sim i \int d^2x \sqrt{g} \sum_{k=1}^{\infty} \sum_{\ell=0}^{\ell_{\max}} f(q_e^2 - \tilde{m}^2 - \ell(\ell+1)) e^{-2\pi k\left(q_e - \sqrt{q_e^2 - \tilde{m}^2 - \ell(\ell+1)}\right)} + \cdots$$

$$e^{-K/2} C Z_{\text{D0}} = nCX^0 = \frac{1}{2}\left(q_e^{(n)} - i q_m^{(n)}\right)$$

$$\text{Im} G^{(np)} \supset -\frac{\chi_E}{16\pi^2} \sum_{n,k=1}^{\infty} \frac{q_m^{(n)}}{k} \text{Re}\left[e^{-2\pi^2 k\left(q_e^{(n)} - i q_m^{(n)}\right)}\right]$$

$$S_{wl} = -\tilde{m} \int_\gamma d\sigma \sqrt{\csc^2\,\psi\left(\dot{\tilde{\tau}}^2 - \dot{\psi}^2\right) - \dot{\theta}^2 - \sin^2\,\theta \dot{\phi}^2} + \int_\Sigma q_A F^A - p^A G_A$$

$$\omega_{\text{AdS}_2} = \frac{R_{\text{AdS}}^2}{\sin^2\,\psi} d\tilde{\tau} \wedge d\psi$$

$$S_{wl} \supset \int_{\Sigma} q_A F^A - p^A G_A = \int_{\gamma} d\sigma \left(q_e \frac{\dot{\tilde{\tau}}}{\tan\ \psi} + q_m \cos\ \theta \dot{\phi} \right)$$

$$S_{wl} = \frac{1}{2} \int_{\gamma} d\sigma \left[h^{-1} \left(\csc^2\ \psi \left(-\dot{\tilde{\tau}}^2 + \dot{\psi}^2 \right) + \dot{\theta}^2 + \sin^2\ \theta \dot{\phi}^2 \right) - h\tilde{m}^2 \right] + \int_{\gamma} d\sigma \left(q_e \frac{\dot{\tilde{\tau}}}{\tan\ \psi} + q_m \cos\ \theta \dot{\phi} \right)$$

$$H = \frac{\sin^2\ \psi}{2} \left[p_{\psi}^2 - \left(p_{\tilde{\tau}} - \frac{q_e}{\tan\ \psi} \right)^2 \right] + \frac{1}{2} p_{\theta}^2 + \frac{1}{2} \csc^2\ \theta \left(p_{\phi} - q_m \cos\ \theta \right)^2 \stackrel{!}{=} - \frac{\tilde{m}^2}{2},$$

$$p_{\psi} = \frac{\dot{\psi}}{\sin^2\ \psi}, p_{\tilde{\tau}} = - \frac{\dot{\tilde{\tau}}}{\sin^2\ \psi} + \frac{q_e}{\tan\ \psi}, p_{\theta} = \dot{\theta}, p_{\phi} = \sin^2\ \theta \dot{\phi} + q_m \cos\ \theta$$

$$j = p_{\phi}, \tilde{E} = - p_{\tilde{\tau}},$$

$$\begin{aligned} \dot{p}_{\psi}\ &= \frac{d}{d\sigma} \left(\frac{\dot{\psi}}{\sin^2\ \psi} \right) = \frac{\dot{\tilde{\tau}}}{\cos\ \psi \sin\ \psi} (\tilde{E} - \dot{\tilde{\tau}}) - \frac{\dot{\psi}^2}{\sin^3\ \psi} \cos\ \psi \\ \dot{p}_{\theta}\ &= \ddot{\theta} = \sin\ \theta \left(\cos\ \theta \dot{\phi}^2 - q_m \dot{\phi} \right) = \dot{\phi} \tan\ \theta (\dot{\phi} - j) \end{aligned}$$

$$p_{\psi}^2 + V(\psi) = 0, V(\psi) = \frac{\tilde{m}^2 + \ell^2}{\sin^2\ \psi} - \left(\tilde{E} + \frac{q_e}{\tan\ \psi} \right)^2$$

$$\begin{aligned} K_0\ &= \cos\ \psi \cos\ \tilde{\tau} p_{\psi} + \cos\ \psi \sin\ \tilde{\tau} p_{\tilde{\tau}} + q_e \sin\ \psi \sin\ \tilde{\tau} \\ K_{\pm}\ &= \mp \sin\ \psi \sin\ \tilde{\tau} p_{\psi} + (1 \pm \cos\ \psi \cos\ \tilde{\tau}) p_{\tilde{\tau}} \pm q_e \sin\ \psi \cos\ \tilde{\tau} \end{aligned}$$

$$\{K_+, K_-\}_{\mathrm{PB}} = -2K_0, \{K_0, K_{\pm}\}_{\mathrm{PB}} = \pm K_{\pm}$$

$$(K_+ - K_-) \cos\ \tilde{\tau} + 2K_0 \sin\ \tilde{\tau} = 2q_e \sin\ \psi + (K_+ + K_-) \cos\ \psi$$

$$\frac{1}{2} (K_+ K_- + K_- K_+) - K_0^2 = \Delta^2 + j^2,$$

$$\Delta^2 = \tilde{m}^2 - q_e^2 - q_m^2$$

$$\tanh\ \xi = \frac{\mathrm{Re} CZ}{|CZ|} = \frac{q_e}{\tilde{m}},$$

$$K_+ = K_-, K_0 = 0, \ell = 0$$

$$K_+^2 = q_m^2$$

$$\tan\ \psi = - \frac{q_m}{q_e} = \tan\ [\arg\ (CZ)]$$

$$\sin\ \psi = \sqrt{1 - \tanh^2\ \xi} = \sin\ [\arg\ (CZ)].$$

$$\psi = \arg\ (CZ), \theta, \phi = \lambda,$$

$$K_+^2 = j^2 = \ell^2 + q_m^2.$$

$$\tan\,\psi = -\frac{j}{q_e},$$

$$S_{wl} = -\tilde{m}\int\, d\tilde{\tau}\left[\sqrt{\csc^2\,\psi - \csc^2\,\psi\left(\frac{d\psi}{d\tilde{\tau}}\right)^2 - \left(\frac{d\theta}{d\tilde{\tau}}\right)^2 - \sin^2\,\theta\left(\frac{d\phi}{d\tilde{\tau}}\right)^2} + \frac{q_e}{\tan\,\psi} + q_m\cos\,\theta\frac{d\phi}{d\tilde{\tau}}\right].$$

$$\mathcal{H} = \mathrm{P}_i\frac{d}{d\tilde{\tau}}\mathrm{q}^i - \mathcal{L} = \csc\,\psi\sqrt{\tilde{m}^2 + \sin^2\,\psi\mathrm{P}_\psi^2 + \mathrm{P}_\theta^2 + \csc^2\,\theta\left(\mathrm{P}_\phi - q_m\cos\,\theta\right)^2} - \frac{q_e}{\tan\,\psi},$$

$$\mathrm{P}_\psi = \frac{\tilde{m}}{\sqrt{h}}\csc^2\,\psi\frac{d\psi}{d\tilde{\tau}}, \mathrm{P}_\theta = \frac{\tilde{m}}{\sqrt{h}}\frac{d\theta}{d\tilde{\tau}}, \mathrm{P}_\psi = \frac{\tilde{m}}{\sqrt{h}}\sin^2\,\theta\frac{d\phi}{d\tilde{\tau}} + q_m\cos\,\theta$$

$$h = g_{\mu\nu}\frac{dx^\mu}{d\tilde{\tau}}\frac{dx^\nu}{d\tilde{\tau}} = \frac{\tilde{m}^2\csc^2\,\psi}{\tilde{m}^2 + \sin^2\,\psi\mathrm{P}_\psi^2 + \mathrm{P}_\theta^2 + \csc^2\,\theta\left(\mathrm{P}_\phi - q_m\cos\,\theta\right)^2}$$

$$\begin{aligned}\mathrm{J}_1 &= -\sin\,\phi\mathrm{P}_\theta - \cot\,\theta\cos\,\phi\mathrm{P}_\phi + q_m\csc\,\theta\cos\,\phi\\ \mathrm{J}_2 &= \cos\,\phi\mathrm{P}_\theta - \cot\,\theta\sin\,\phi\mathrm{P}_\phi + q_m\csc\,\theta\sin\,\phi\\ \mathrm{J}_3 &= \mathrm{P}_\phi\end{aligned}$$

$$\mathcal{H} = \csc\,\psi\sqrt{q_e^2 + \sin^2\,\psi\mathrm{P}_\psi^2 + \mathrm{J}^2} - \frac{q_e}{\tan\,\psi}.$$

$$\cos\,\psi = q_e/\sqrt{q_e^2 + \mathrm{J}^2}$$

$$\mathcal{H} \geq \sqrt{\mathrm{J}^2}$$

$$\mathrm{Z}_{1-\,\mathrm{loop}}\,[A] = -\int_0^\infty\frac{dh}{h}\int\,\mathcal{D}\tau(h)\mathcal{D}\rho(h)e^{-S_\mathrm{E}[x^i,A]}$$

$$\tau(\sigma) = \tau_0 - R_0\sin\,\theta(\sigma), \rho(\sigma) = \rho_0 \mp R_0\cos\,\theta(\sigma),$$

$$\dot{\theta} = \pm p_\tau\rho \Longrightarrow \theta(\sigma) = 2\tan^{-1}\left(\sqrt{\frac{1 \mp m_{\mathrm{eff}}/q_e}{1 \pm m_{\mathrm{eff}}/q_e}}\tan\left(\frac{\sigma}{2}\sqrt{q_e^2 - m_{\mathrm{eff}}^2} + c_1\right)\right)$$

$$\tau(\sigma) = \tau_0 - R\sin\,\theta(\sigma), \rho(\sigma) = \rho_0 - R\cos\,\theta(\sigma),$$

$$\theta(\sigma) = 2\tan^{-1}\left(\sqrt{\frac{\rho_0/R - 1}{\rho_0/R + 1}}\tan\left(\frac{q_e\sigma}{2\rho_0}\sqrt{\rho_0^2 - R^2}\right)\right)$$

$$R = \frac{m_{\mathrm{eff}}}{q_e}\rho_0, \rho_0 = \frac{q_e}{p_\tau}$$

$$2\pi\rho_0/\left(q_e\sqrt{\rho_0^2-R^2}\right)$$

$$\mathrm{Z_{mini}}[A]=-\int\frac{d\tau_0 d\rho_0}{\rho_0^2}\int\frac{RdR}{\rho_0^2}\int_0^\infty\frac{dh}{h}e^{-S_\mathrm{E}}$$

$$S_E=\int_0^T d\sigma\left[\frac{(q_eR)^2}{2h\rho_0^2}+\frac{hm_{\mathrm{eff}}^2}{2}-q_e^2\frac{\rho-\rho_0}{\rho_0}\right]$$
$$=\pi\left[\frac{q_eR^2}{h\rho_0\sqrt{\rho_0^2-R^2}}+\frac{hm_{\mathrm{eff}}^2\rho_0}{q_e\sqrt{\rho_0^2-R^2}}\right]-\frac{q_e^2R}{\rho_0}\int_0^T d\sigma\cos\,\theta(\sigma)$$

$$\dot\tau^2+\dot\rho^2=R^2\dot\theta^2=(q_eR)^2\frac{\rho^2}{\rho_0^2}$$
$$\dot\tau=\dot\theta(\rho-\rho_0)=\frac{q_e\rho}{\rho_0}(\rho-\rho_0)$$

$$\int_0^T d\sigma\cos\,\theta(\sigma)=\frac{2\pi\rho_0}{q_eR}\left(\frac{\rho_0}{\sqrt{\rho_0^2-R^2}}-1\right)$$

$$S_\mathrm{E}=\pi\left[\frac{q_eR^2}{h\rho_0\sqrt{\rho_0^2-R^2}}+\frac{hm_{\mathrm{eff}}^2\rho_0}{q_e\sqrt{\rho_0^2-R^2}}-2q_e\left(\frac{\rho_0}{\sqrt{\rho_0^2-R^2}}-1\right)\right]$$

$$\frac{\partial S_\mathrm{E}}{\partial h}=0\Longrightarrow h_{\min}=\frac{q_e}{m_{\mathrm{eff}}}\frac{R}{\rho_0},$$

$$S_\mathrm{E}=2\pi\left[\frac{m_{\mathrm{eff}}}{\sqrt{\rho_0^2/R^2-1}}-q_e\frac{\rho_0/R}{\sqrt{\rho_0^2/R^2-1}}+q_e\right]$$

$$R_{\min}=\frac{m_{\mathrm{eff}}}{q_e}\rho_0\Longrightarrow S_{\mathrm{E,min}}=2\pi\left(q_e-\sqrt{q_e^2-m_{\mathrm{eff}}^2}\right)$$

$$\delta S_\mathrm{E}=\pi\left[-\frac{q_eR}{h^2\rho_0\sqrt{\rho_0^2/R^2-1}}+\frac{m_{\mathrm{eff}}^2\rho_0/R}{q_e\sqrt{\rho_0^2/R^2-1}}\right]\delta h$$
$$+\pi\left[\frac{q_e}{h\rho_0\sqrt{\rho_0^2/R^2-1}}+\frac{q_e\rho_0}{R^2(\rho_0^2/R^2-1)^{3/2}}\left(h^{-1}-2+h\frac{m_{\mathrm{eff}}^2}{q_e^2}\right)\right]\delta R$$

$$h_{\min}=\frac{m_{\mathrm{eff}}}{q_e}\rho_0,\frac{R_{\min}}{\rho_0}=\sqrt{2-2h+\frac{h^2m_{\mathrm{eff}}^2}{q_e^2}}.$$

$$\delta^2S_\mathrm{E}=\frac{\delta^2S_\mathrm{E}}{\delta h^2}(\delta h)^2+2\frac{\delta^2S_\mathrm{E}}{\delta h\delta R}\delta h\delta R+\frac{\delta^2S_\mathrm{E}}{\delta R^2}(\delta R)^2$$

$$\frac{\delta^2 S_{\mathrm{E}}}{\delta h^2}=\frac{2\pi q_e R}{h^3\rho_0\sqrt{\rho_0^2/R^2-1}}\triangleq\frac{2\pi m_{\mathrm{eff}}}{\sqrt{q_e^2/m_{\mathrm{eff}}^2-1}}$$

$$\frac{\delta^2 S_{\mathrm{E}}}{\delta h\delta R}=-\frac{\pi q_e}{h^2\rho_0\sqrt{\rho_0^2/R^2-1}}-\frac{\pi q_e\rho_0}{h^2R^2(\rho_0^2/R^2-1)^{3/2}}+\frac{\pi h m_{\mathrm{eff}}^2 R}{q_e\rho_0^2(1-R^2/\rho_0^2)^{3/2}}\overset{\star}{=}-\frac{2\pi q_e}{\rho_0\sqrt{q_e^2/m_{\mathrm{eff}}^2-1}}$$

$$\frac{\delta^2 S_{\mathrm{E}}}{\delta R^2}=-\frac{2\pi q_e\rho_0\left((2h)^{-1}-2+h\frac{m_{\mathrm{eff}}^2}{q_e^2}\right)}{R^3(\rho_0^2/R^2-1)^{3/2}}+\frac{3\pi q_e\rho_0^3\left(h^{-1}-2+h\frac{m_{\mathrm{eff}}^2}{q_e^2}\right)}{R^5(\rho_0^2/R^2-1)^{5/2}}\overset{\dagger}{=}-\frac{2\pi q_e^2}{m_{\mathrm{eff}}\rho_0^2\left(q_e^2/m_{\mathrm{eff}}^2-1\right)^{3/2}}$$

$$\mathrm{Z}_{\mathrm{mini}}[A]=-\sum_{\text{saddles}}\int\frac{d\tau_0 d\rho_0}{\rho_0^2}\frac{R_{\min}}{h_{\min}\rho_0^2}\mathcal{A}_{1-\text{ loop}}\,e^{-S_{\mathrm{E,\,min}}}+(\mathfrak{H})$$

$$=i\int\frac{d\tau_0 d\rho_0}{\rho_0^2}\sum_{k=1}^{\infty}\sum_{\ell=0}^{\ell_{\max}}\frac{(q_e^2-\tilde{m}^2-\ell(\ell+1))}{2\pi k q_e^3}e^{-2\pi k\left(q_e-\sqrt{q_e^2-\tilde{m}^2-\ell(\ell+1)}\right)}+(\mathfrak{H})$$

$$\mathcal{A}_{1-\text{ loop}}=\left(\det\frac{\delta^2 S_{\mathrm{E}}}{\delta x^i\delta x^j}\right)^{-1/2}=\frac{m_{\mathrm{eff}}\rho_0\left(q_e^2/m_{\mathrm{eff}}^2-1\right)}{2\pi i q_e^2}$$

$$s=\frac{\tilde{\Delta}_a\tilde{x}^a}{\sqrt{\left(\tilde{\Delta}_a\tilde{x}^a\right)^2}}$$

$$X^0=\frac{y}{r_h}t=\frac{r_h}{\rho}t, X^2-X^1=y=\frac{r_h^2}{\rho}, X^2+X^1=\frac{r_h^2}{y}-\frac{yt^2}{r_h^2}=\rho-\frac{t^2}{\rho}$$

$$\tilde{\sigma}=\tilde{\rho}_+ +\tilde{\rho}_-, \tilde{t}=\tilde{\rho}_+ -\tilde{\rho}_-, \text{ with } \tilde{\rho}_\pm=\tan^{-1}(\rho\pm t),$$

$$J_\pm=J_1\pm iJ_2=\pm ie^{\pm i\phi}\left[p_\theta\pm i\left(\cot\theta p_\phi-q_m\csc\theta\right)\right]$$

$$d\rho/dt=\sqrt{1-\frac{\tilde{m}^2}{(E\rho+q_e)^2}}$$

$$ds^2=-\left(1-\frac{2m}{r}\right)dt^2+\frac{dr^2}{1-2m/r}+r^2d\Omega^2$$

$$t\to t+F(r);\ dt\to dt+f(r)dr$$

$$ds^2=-\left(1-\frac{2m}{r}\right)(dt+f(r)dr)^2+\frac{dr^2}{1-2m/r}+r^2d\Omega^2$$

$$ds^2=-\left(1-\frac{2m}{r}\right)dt^2-2(1-2m/r)f(r)drdt+\left[\frac{1}{1-2m/r}-(1-2m/r)f(r)^2\right]dr^2+r^2d\Omega^2$$

$$f(r) = \frac{h(r)}{1 - 2m/r}$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 - 2h(r)drdt + \left[\frac{1 - h(r)^2}{1 - 2m/r}\right]dr^2 + r^2 d\Omega^2$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 \mp 2\sqrt{2m/r}drdt + dr^2 + r^2 d\Omega^2$$

$$-dt^2 + (dr \mp \sqrt{2m/r}dt)^2 = 0$$

$$\frac{dr}{dt} = \pm 1 \pm \sqrt{2m/r}$$

$$\frac{dr}{dt} = \pm 1 - \sqrt{2m/r}$$

$$\frac{dr}{dt} = \pm 1 + \sqrt{2m/r}$$

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 + \frac{2m}{r}(dt \pm dr)^2$$

$$-dt^2 + dr^2 + (2m/r)(dt \pm dr)^2 = 0$$

$$\frac{dr}{dt} = \mp 1, \text{ or } \frac{dr}{dt} = \pm 1 \mp \frac{4m}{r + 2m},$$

$$\frac{dr}{dt} = -1 \text{ (ingoing), or } \frac{dr}{dt} = 1 - \frac{4m}{r + 2m} \text{ ("outgoing"),}$$

$$\frac{dr}{dt} = 1 \text{ (outgoing); or } \frac{dr}{dt} = -1 + \frac{4m}{r + 2m} \text{ ("ingoing"),}$$

$$ds^2 = -(1 - 2m/r)dt^2 \mp 2drdt + r^2 d\Omega^2$$

$$ds^2 = -(1 - 2m/r)dv^2 + 2dvdr + r^2 d\Omega^2$$

$$[-(1 - 2m/r)dv + 2dr]dv = 0$$

$$\frac{dr}{dv} = -\infty; \frac{dr}{dv} = \frac{1 - 2m/r}{2}$$

$$ds^2 = -(1 - 2m/r)du^2 - 2dudr + r^2 d\Omega^2$$

$$[-(1 - 2m/r)du - 2dr]du = 0$$

$$\frac{dr}{du} = +\infty; \frac{dr}{du} = -\frac{1 - 2m/r}{2}$$

$$F(r) = \int f(r)dr = \int \frac{h(r)}{1 - 2m/r}dr$$

$$F_{PG}(r) = \pm \int \frac{\sqrt{2m/r}}{1-2m/r} dr = \pm 2\sqrt{2mr} \pm 2m\ln\left(\frac{1-\sqrt{2m/r}}{1+\sqrt{2m/r}}\right)$$

$$F_{KS}(r) = \pm \int \frac{2m/r}{1-2m/r} dr = \pm 2m\ln(r-2m)$$

$$F_{EF}(r) = \pm \int \frac{1}{1-2m/r} dr = \pm r \pm 2m\ln(r-2m)$$

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 - 2h(r)drdt + \left[\frac{1-h(r)^2}{1-2m/r}\right]dr^2 + r^2 d\Omega^2$$

$$-\left(1-\frac{2m}{r}\right)dt^2 - 2h(r)drdt + \left[\frac{1-h(r)^2}{1-2m/r}\right]dr^2 = 0$$

$$-\left(1-\frac{2m}{r}\right)^2 - 2\left(1-\frac{2m}{r}\right)h(r)\dot{r} + [1-h(r)^2]\dot{r}^2 = 0$$

$$\dot{r} = \mp\frac{1-2m/r}{1 \pm h(r)}.$$

$$\dot{r}_H = \pm\frac{1}{2mh'(2m)}$$

$$\Delta F = F(\infty) - F(0) = F(r_>) - F(r_<) = \int_{r_<}^{r_>} \frac{h(r)}{1-2m/r} dr = \int_{r_<}^{r_>} \frac{rh(r)}{r-2m} dr$$

$$h(r) \to s(t)h(r).$$

$$ds^2 = -(1-2m/r)dt^2 - 2s(t)h(r)drdt + \left[\frac{1-s(t)^2h(r)^2}{1-2m/r}\right]dr^2 + r^2 d\Omega^2$$

$$G_{tt} = 0;\ G_{rr} = -\frac{2\dot{s}(t)h(r)}{r(1-2m/r)}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{d^2[s^2(t)]/dt^2 h(r)^2}{2(1-2m/r)} + h'(r)\dot{s}(t) - \frac{(1-m/r)h(r)\dot{s}(t)}{r(1-2m/r)}$$

$$R = -\frac{d^2[s^2(t)]/dt^2 h(r)^2}{(1-2m/r)} - 2h'(r)\dot{s}(t) + \frac{2(2-3m/r)h(r)\dot{s}(t)}{r(1-2m/r)}.$$

$$\sqrt{-g} = r^2 \sin\theta$$

$$S = \int \sqrt{-g}Rd^4x = \int \sqrt{-g}Rd^4x = 4\pi \int r^2 Rdtdr$$

$$\int_{-\infty}^{+\infty} \left(\frac{d^2[s^2(t)]}{dt^2}\right)dt = \left[\frac{d[s^2(t)]}{dt}\right]_{-\infty}^{+\infty} = 0 - 0 = 0$$

$$\int_{-\infty}^{+\infty} \left(\frac{ds(t)}{dt}\right)dt = [s(t)]_{-\infty}^{+\infty} = \pm 1 - (\mp 1) = \pm 2$$

$$S = \pm 4\pi \int \; r^2 \left[4h'(r) + \frac{4(2-3m/r)h(r)}{r(1-2m/r)} \right] dr$$

$$\int_{-\infty}^{+\infty} r^2 h'(r) dr = [r^2 h(r)]_{-\infty}^{\pm\infty} - \int_{-\infty}^{+\infty} 2rh(r) dr = -\int_{-\infty}^{+\infty} 2rh(r) dr$$

$$S = \pm 4\pi \int \; r \left[-8h(r) + \frac{4(2-3m/r)h(r)}{(1-2m/r)} \right] dr$$

$$S = \pm 16\pi m \int_{r_<}^{r_>} \frac{h(r)}{(1-2m/r)} dr = \pm 16\pi m \int_{r_<}^{r_>} \frac{rh(r)}{(r-2m)} dr$$

$$S = \pm 16\pi m \Delta F$$

$$S = \; \pm 16\pi m \int_{r_<}^{r_>} \frac{h(r)}{(1-2m/r)} dr = \pm 16\pi m \int_{2m-\Delta}^{2m+\Delta} \frac{(2m/r)B(|r-2m|)}{(1-2m/r)} dr$$
$$= \pm 32\pi m^2 \int_{2m-\Delta}^{2m+\Delta} \frac{B(|r-2m|)}{(r-2m)} dr = \pm 32\pi m^2 \int_{-\Delta}^{+\Delta} \frac{B(|z|)}{z} dz$$

$$S \propto \lim_{\epsilon \to 0} \left(\int_{-\Delta}^{-\epsilon} \frac{B(|z|)}{z} dz + \int_{\epsilon}^{\Delta} \frac{B(|z|)}{z} dz \right)$$

$$S \propto \lim_{\epsilon \to 0} \left(\int_{\epsilon}^{\Delta} \frac{B(|z|)}{z} dz - \int_{\epsilon}^{\Delta} \frac{B(|z|)}{z} dz \right) = 0$$

$$-(1-2m/r)dt^2 - 2s(t)h(r)drdt + \left[\frac{1-s(t)^2 h(r)^2}{1-2m/r} \right] dr^2 = 0$$

$$-(1-2m/r)^2 - 2s(t)h(r)(1-2m/r)\dot{r} + [1-s(t)^2 h(r)^2]\dot{r}^2 = 0$$

$$\frac{dr}{dt} = \pm \frac{(1-2m/r)}{[1 \mp s(t)h(r)]}$$

$$k^a \propto (1, \dot{r}; 0,0) = \left(1, \pm \frac{(1-2m/r)}{[1 \mp s(t)h(r)]}; 0,0 \right)$$

$$G_{ab} k^a k^b \propto G_{rr} \left(\frac{(1-2m/r)^2}{[1 \mp s(t)h(r)]^2} \right) = -\frac{2\dot{s}(t)h(r)(1-2m/r)}{[1 \mp s(t)h(r)]^2}$$

$$G_{ab} k^a k^b \propto -\dot{s}(t)h(r)(1-2m/r)$$

$$g_{ab} T^{ab} = -(\rho - 3p) \leq 0$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 - 2h(r)drdt + \left[\frac{1-h(r)^2}{1-2m/r} \right] dr^2 + r^2 d\Omega^2$$

$$\dot{r}_H = \pm \frac{1}{2mh'(2m)}$$

$$ds^2 = -(1-2m/r)dt^2 - 2s(t)h(r)drdt + \left[\frac{1-s(t)^2h(r)^2}{1-2m/r}\right]dr^2 + r^2d\Omega^2$$

$$\dot{r}_H = \pm\frac{1}{2mh'(2m)}$$

$$S = 16\pi m\int_{r_<}^{r_>}\frac{h(r)}{(1-2m/r)}dr$$

$$g^{ab} = \Sigma^{-1}\begin{pmatrix}A_3-B_3 & A_4-B_4\\ A_4-B_4 & A_5-B_5\end{pmatrix}, g^{ij} = -\Sigma^{-1}\begin{pmatrix}A_2 & 0\\ 0 & B_2\end{pmatrix}$$

$$\Sigma = A_1 + B_1.$$

$$\nabla_{(\alpha}K_{\mu\nu)} = 0$$

$$\Sigma K^{ab} = \begin{pmatrix}A_1B_3+A_3B_1 & A_1B_4+A_4B_1\\ A_1B_4+A_4B_1 & A_1B_5+A_5B_1\end{pmatrix}, \Sigma K^{ij} = \begin{pmatrix}-A_2B_1 & 0\\ 0 & A_1B_2\end{pmatrix}$$

$$A_1 \leftrightarrow -B_1, A_i \leftrightarrow B_i, (i = 2..4), g^{rr} \leftrightarrow -g^{yy}, K^{rr} \leftrightarrow -K^{yy}.$$

$$g_{ab} = \frac{\Sigma}{\mathcal{P}}\begin{pmatrix}A_5-B_5 & -A_4+B_4\\ -A_4+B_4 & A_3-B_3\end{pmatrix}, g_{ij} = -\Sigma\begin{pmatrix}A_2^{-1} & 0\\ 0 & B_2^{-1}\end{pmatrix}$$

$$\mathcal{P} = (A_3-B_3)(A_5-B_5) - (A_4-B_4)^2.$$

$$A_3 - B_3 > 0, A_5 - B_5 < 0, \mathcal{P} < 0.$$

$$A_4 \equiv 0, B_4 \equiv 0.$$

$$r \to \tilde{r}(r), y \to \tilde{y}(y),$$

$$A_2A_5 = a^2 = \text{ const }, B_2B_5 = b^2 = \Re.$$

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu\bar{m}^\nu - \bar{m}^\mu m^\nu.$$

$$l = 1/\sqrt{2\Sigma}\left(\sqrt{A_3}\partial_t + C\partial_\varphi - \sqrt{A_2}\partial_r\right), \quad m = 1/\sqrt{2\Sigma}\left(\sqrt{B_3}\partial_t + D\partial_\varphi - i\sqrt{B_2}\partial_y\right),$$
$$n = 1/\sqrt{2\Sigma}\left(\sqrt{A_3}\partial_t + C\partial_\varphi + \sqrt{A_2}\partial_r\right), \quad \bar{m} = 1/\sqrt{2\Sigma}\left(\sqrt{B_3}\partial_t + D\partial_\varphi + i\sqrt{B_2}\partial_y\right).$$

$$(A_3-B_3)C = \sqrt{A_3}(A_4-B_4) + \sqrt{B_3}\sqrt{-\mathcal{P}}, (A_3-B_3)D = \sqrt{B_3}(A_4-B_4) + \sqrt{A_3}\sqrt{-\mathcal{P}}.$$

$$A_4 = \sqrt{A_3A_5}, B_4 = \sqrt{B_3B_5}.$$

$$\sqrt{-\mathcal{P}} = \sqrt{A_3B_5} - \sqrt{A_5B_3}, \Rightarrow C = \sqrt{A_5}, D = \sqrt{B_5}.$$

$$l = 1/\sqrt{2\Sigma}\left(\sqrt{A_3}\partial_t + \sqrt{A_5}\partial_\varphi - \sqrt{A_2}\partial_r\right), \quad m = 1/\sqrt{2\Sigma}\left(\sqrt{B_3}\partial_t + \sqrt{B_5}\partial_\varphi - i\sqrt{B_2}\partial_y\right),$$
$$n = 1/\sqrt{2\Sigma}\left(\sqrt{A_3}\partial_t + \sqrt{A_5}\partial_\varphi + \sqrt{A_2}\partial_r\right), \quad \bar{m} = 1/\sqrt{2\Sigma}\left(\sqrt{B_3}\partial_t + \sqrt{B_5}\partial_\varphi + i\sqrt{B_2}\partial_y\right).$$

$$A_5 \equiv 0, B_3 \equiv 0, A_3 \neq 0, B_5 \neq 0$$

$$D = l^\mu\nabla_\mu, \Delta = n^\mu\nabla_\mu, \delta = m^\mu\nabla_\mu, \bar{\delta} = \bar{m}^\mu\nabla_\mu$$

$$Dl^{\mu} = (\epsilon + \bar{\epsilon})l^{\mu} - \bar{\kappa}m^{\mu} - \kappa\bar{m}^{\mu}$$
$$\Delta n^{\mu} = -(\gamma + \bar{\gamma})n^{\mu} + \nu m^{\mu} + \bar{\nu}\bar{m}^{\mu}$$

$$\sigma = -m^{\mu}\delta l_{\mu}, \bar{\lambda} = m^{\mu}\delta n_{\mu}$$

$$\kappa = \nu = 0, \sigma = \lambda = 0$$

$$\mu = \rho, \tau = \pi, \epsilon = \gamma, \alpha = \beta.$$

$$\Psi_0 = -C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}m^{\delta}$$
$$\Psi_1 = -C_{\alpha\beta\gamma\delta}l^{\alpha}n^{\beta}l^{\gamma}m^{\delta}$$
$$\Psi_2 = -C_{\alpha\beta\gamma\delta}\left(l^{\alpha}n^{\beta}l^{\gamma}n^{\delta} - l^{\alpha}n^{\beta}m^{\gamma}\bar{m}^{\delta}\right)/2$$
$$\Psi_3 = -C_{\alpha\beta\gamma\delta}n^{\alpha}l^{\beta}n^{\gamma}m^{\delta}$$
$$\Psi_4 = -C_{\alpha\beta\gamma\delta}n^{\alpha}\bar{m}^{\beta}n^{\gamma}\bar{m}^{\delta}$$

$$\Psi_0 = 0 = \Psi_4$$

$$\Psi_1 = \Psi_3$$

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2, J = \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}$$

$$I^3 = 27J^2$$

$$\Psi_1^2 = k\Psi_2^2, \text{ with } k = 9/16, \text{ or } 0.$$

$$l^{\mu} \to f^{-1}(r, y)l^{\mu}, n^{\mu} \to f(r, y)n^{\mu}, f = \sqrt{\Sigma/A_2}$$

$$\mathbf{Q} = \begin{pmatrix} \Psi_2 - \frac{1}{2}(\Psi_0 + \Psi_4) & \frac{1}{2}i(\Psi_4 - \Psi_0) & \Psi_1 - \Psi_3 \\ \frac{1}{2}i(\Psi_4 - \Psi_0) & \Psi_2 + \frac{1}{2}(\Psi_0 + \Psi_4) & i(\Psi_1 + \Psi_3) \\ \Psi_1 - \Psi_3 & i(\Psi_1 + \Psi_2) & -2\Psi_2 \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} \Psi_2 & 0 & 0 \\ 0 & \Psi_2 & 2i\Psi_1 \\ 0 & 2i\Psi_1 & -2\Psi_2 \end{pmatrix}.$$

$$K_{ln} = B_1, K_{m\bar{m}} = A_1.$$

$$\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\nu}}g^{\mu\nu} = \mu^2$$

$$S = -Et + L\varphi + S_r(r) + S_y(y),$$

$$A_2p_r^2 + U_r = 0, \qquad B_2p_y^2 + U_y = 0,$$
$$U_r = \mathcal{C} - \varepsilon_a^2 + A_1\mu^2, \quad U_y = -\mathcal{C} + \varepsilon_b^2 + B_1\mu^2,$$
$$\varepsilon_a = E\sqrt{A_3} - L\sqrt{A_5}, \quad \varepsilon_b = E\sqrt{B_3} - L\sqrt{B_5},$$

$$\mathcal{C} = p_{\mu}p_{\nu}K^{\mu\nu} = \Sigma^{-1}\left[A_1\left(\varepsilon_b^2 + B_2p_y^2\right) + B_1(\varepsilon_a^2 - A_2p_r^2)\right].$$

$$U_r = 0 = U_r'$$

$$\mu^2 \Rightarrow \mathcal{M}^2(x)$$

$$\mathcal{M}^2(x) = \frac{\mathcal{M}_r^2(r) + \mathcal{M}_y^2(y)}{A_1 + B_1}.$$

$$\Box\,\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = -\mu^2\phi$$

$$\sqrt{-g} = \frac{\Sigma^2}{\sqrt{A_2B_2}\sqrt{-\mathcal{P}}}$$

$$\sqrt{-g} = \frac{\Sigma^2}{\sqrt{A_2B_2}\left(\sqrt{A_3B_5} - \sqrt{A_5B_3}\right)}$$

$$\sqrt{-g} = \Sigma$$

$$\Sigma = \sqrt{A_2B_2}\left(\sqrt{A_3B_5} - \sqrt{B_3A_5}\right)$$

$$A_1 + B_1 = bA_{23} - aB_{23}$$

$$A_{23} = \sqrt{A_2A_3}, B_{23} = \sqrt{B_2B_3},$$

$$A_1' = bA_{23}', B_1' = aB_{23}'$$

$$\phi(x^\mu) = \mathrm{e}^{-i\omega t + im\varphi}R(r)Y(y),$$

$$\frac{((A_2)'R)'}{R} + \frac{((B_2)'Y)'}{Y} + U(r) - V(y) = 0,$$

$$U(r) = \left(\omega\sqrt{A_3} - m\sqrt{A_5}\right)^2 - \mu^2A_1,$$
$$V(y) = \left(\omega\sqrt{B_3} - m\sqrt{B_5}\right)^2 + \mu^2B_1.$$

$$\hat{K} = \nabla_\mu K^{\mu\nu}\nabla_\nu$$

$$[\Box, \hat{K}]\phi = \frac{4}{3}\nabla_\alpha\left(K_\sigma\,{}^{[\alpha}R^{\beta]\sigma}\right)\nabla_\beta\phi$$

$$K_\sigma\,{}^{[\alpha}R^{\beta]\sigma} = 2(K_{ln} + K_{m\bar{m}})\left(n^\beta(\bar{m}^\alpha\Phi_{01} + m^\alpha\Phi_{10}) - n^\alpha\left(\bar{m}^\beta\Phi_{01} + m^\beta\Phi_{10}\right)\right.$$
$$\left.+\left(l^\beta\bar{m}^\alpha - l^\alpha\bar{m}^\beta\right)\Phi_{12} + \left(l^\beta m^\alpha - l^\alpha m^\beta\right)\Phi_{21}\right)$$

$$\Phi_{01} = R_{\mu\nu}l^\mu m^\nu/2 = \overline{\Phi_{10}}, \Phi_{12} = R_{\mu\nu}n^\mu m^\nu/2 = \overline{\Phi_{21}}$$

$$aA_{23}'' - bB_{23}'' = 0$$

$$a^2A_1'' - b^2B_1'' = 0$$

$$\begin{aligned}\Box=&(D+\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho})\Delta+(\Delta+\mu+\bar{\mu}-\gamma-\bar{\gamma})D\\&-(\delta-\tau-\bar{\alpha}+\beta+\bar{\pi})\bar{\delta}-(\bar{\delta}-\alpha-\bar{\tau}+\pi+\bar{\beta})\delta\\\hat{K}=&(D+\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho})K_{ln}\Delta+(\Delta+\mu+\bar{\mu}-\gamma-\bar{\gamma})K_{ln}D\\&+(\delta-\tau-\bar{\alpha}+\beta+\bar{\pi})K_{m\bar{m}}\bar{\delta}+(\bar{\delta}-\alpha-\bar{\tau}+\pi+\bar{\beta})K_{m\bar{m}}\delta\end{aligned}$$

$$\begin{aligned}D_n^{\pm}&=\sqrt{A_3/A_2}\partial_t\pm\partial_r+\sqrt{A_5/A_2}\partial_\varphi\pm n\partial_r(\ln\,A_2)\\L_s^{\pm}&=\sqrt{B_3/B_2}\partial_t\pm i\partial_y+\sqrt{B_5/B_2}\partial_\varphi\pm is\partial_y(\ln\,B_2)\end{aligned}$$

$$D_n^{\pm}A_2{}^k=A_2{}^kD_{n+k}^{\pm},L_s^{\pm}B_2{}^l=B_2{}^lL_{s+l}^{\pm}$$

$$D=\sqrt{A_2/2\Sigma}D_0^-,\Delta=\sqrt{A_2/2\Sigma}D_0^+,\delta=\sqrt{B_2/2\Sigma}L_0^-,\bar{\delta}=\sqrt{B_2/2\Sigma}L_0^+$$

$$\begin{aligned}\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho}&=\sqrt{A_2/8\Sigma^3}\left(2\Sigma\partial_r\ln\left[\sqrt{A_3B_5}-\sqrt{A_5B_3}\right]-3\partial_r\Sigma\right)\\\bar{\pi}-\tau+\beta-\bar{\alpha}&=-i\sqrt{B_2/8\Sigma^3}\left(2\Sigma\partial_y\ln\left[\sqrt{A_3B_5}-\sqrt{A_5B_3}\right]-3\partial_y\Sigma\right)\end{aligned}$$

$$\begin{aligned}\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho}&=\sqrt{A_2}\partial_r(2\Sigma)^{-1/2}-(2\Sigma)^{-1/2}\partial_r\sqrt{A_2}\\\bar{\pi}-\tau+\beta-\bar{\alpha}&=i\sqrt{B_2}\partial_y(2\Sigma)^{-1/2}-i(2\Sigma)^{-1/2}\partial_r\sqrt{B_2}\end{aligned}$$

$$\Box=\frac{1}{2\Sigma}[A_2(D_1^-D_0^++D_1^+D_0^-)-B_2(L_1^-L_0^++L_1^+L_0^-)],$$

$$\hat{K}=\frac{1}{2\Sigma}[A_2B_1(D_1^-D_0^++D_1^+D_0^-)+B_2A_1(L_1^-L_0^++L_1^+L_0^-)]=\frac{B_2}{2}(L_1^-L_0^++L_1^+L_0^-)+B_1\,\Box.$$

$$ds^2=\frac{A_2B_2}{\Sigma}\left(\sqrt{B_5}dt-\sqrt{B_3}d\varphi\right)^2-\frac{A_2B_2}{\Sigma}\left(\sqrt{A_5}dt-\sqrt{A_3}d\varphi\right)^2-\frac{\Sigma}{A_2}dr^2-\frac{\Sigma}{B_2}dy^2$$

$$\Sigma=\sqrt{A_2B_2}\left(\sqrt{A_3B_5}-\sqrt{A_5B_3}\right).$$

$$S=\int\,d^4x\left[\left(R-\frac{1}{2}f_{AB}\partial_\mu\Psi^A\partial^\mu\Psi^B-\frac{1}{2}K_{IJ}F_{\mu\nu}^IF^{J\mu\nu}\right)\sqrt{-g}-\frac{1}{2}H_{IJ}F_{\mu\nu}^IF_{\lambda\tau}^J\epsilon^{\mu\nu\lambda\tau}\right]$$

$$R_{\mu\nu}=\frac{1}{2}f_{AB}\Psi_{,\mu}^A\Psi_{,\nu}^B-K_{IJ}\left(F_{\mu\lambda}^IF_\nu^{J\lambda}+\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}^IF^{J\alpha\beta}\right)$$

$$\nabla_{(\alpha}Y_{\mu)\nu}=0$$

$$Y_\mu{}^\alpha Y_{\alpha\nu}=K_{\mu\nu}$$

$$Y_{ln}^2=K_{ln},Y_{m\bar{m}}^2=-K_{m\bar{m}}$$

$$Y_{ln}=p\sqrt{B_1},Y_{m\bar{m}}=iq\sqrt{A_1},p=\pm1,q=\pm1$$

$$\begin{aligned}&(\tau+\bar{\pi})Y_{ln}-(\tau-\bar{\pi})Y_{m\bar{m}}=0,\quad&&(\rho+\bar{\rho})Y_{ln}-(\rho-\bar{\rho})Y_{m\bar{m}}=0\\&DY_{m\bar{m}}-\rho(Y_{ln}-Y_{m\bar{m}})=0,\quad&&\delta Y_{ln}-\bar{\pi}(Y_{ln}+Y_{m\bar{m}})=0\\&\Delta Y_{m\bar{m}}+\mu(Y_{ln}-Y_{m\bar{m}})=0,\quad&&\delta Y_{ln}+\tau(Y_{ln}-Y_{m\bar{m}})=0\end{aligned}$$

$$\mu=\rho=\frac{b}{2\Sigma}\sqrt{A_2/2\Sigma}(A_{23}'-iB_{23}'),\tau=\pi=\frac{a}{2\Sigma}\sqrt{B_2/2\Sigma}(A_{23}'-iB_{23}')$$

$$pA'_{23}\sqrt{B_1} = qB'_{23}\sqrt{A_1}$$

$$pA'_{23} = 2\sqrt{A_1}, qB'_{23} = 2\sqrt{B_1}$$

$$\left(\sqrt{A_1}\right)' = b/p, \left(\sqrt{B_1}\right)' = -a/q$$

$$A_1 + B_1 = bA_{23} - aB_{23}$$

$$A'_1 = 2b\sqrt{A_1}, \quad B'_1 = 2a\sqrt{B_1}$$
$$A'_{23} = 2\sqrt{A_1}, \quad B'_{23} = -2\sqrt{B_1}$$

$$A_1 = (br + c_1)^2, \qquad B_1 = (ay + d_1)^2$$
$$A_{23} = br^2 + 2c_1 r + c_2, \quad B_{23} = -(ay^2 + 2d_1 y + d_2)$$

$$bc_2 + ad_2 = c_1^2 + d_1^2$$

$$Y_{ln} = \sqrt{B_1}, Y_{m\bar{m}} = -i\sqrt{A_1}$$

$$8\Sigma^3\Psi_{1,3} = -\sqrt{A_2 B_2}\{\Sigma(aA''_{23} - bB''_{23}) - ab({A'_{23}}^2 + {B'_{23}}^2)\},$$

$$\nabla_{(\alpha}\mathcal{K}_{\mu\nu)} = \Omega_{(\alpha}g_{\mu\nu)}, 6\Omega_\alpha = (2\nabla_\sigma \mathcal{K}_\alpha{}^\sigma + \nabla_\alpha \mathcal{K}), \mathcal{K} = g^{\mu\nu}\mathcal{K}_{\mu\nu}$$

$$(D + \rho + \bar{\rho})(\mathcal{K}_{ln} + \mathcal{K}_{m\bar{m}}) = 0, \quad (\delta + \tau - \bar{\pi})(\mathcal{K}_{ln} + \mathcal{K}_{m\bar{m}}) = 0$$
$$(\Delta - \mu - \bar{\mu})(\mathcal{K}_{ln} + \mathcal{K}_{m\bar{m}}) = 0, \quad (\bar{\delta} + \bar{\tau} - \pi)(\mathcal{K}_{ln} + \mathcal{K}_{m\bar{m}}) = 0.$$

$$\mathcal{K}_{ln} = \Sigma - S(r, y), \mathcal{K}_{m\bar{m}} = S(r, y)$$

$$K^{\mu\nu} = -A_1 g^{\mu\nu} - A_2 \delta^\mu_r \delta^\nu_r + \tilde{K}^{\mu\nu}_r$$

$$\tilde{K}^{\mu\nu}_r = A_3 \delta^\mu_t \delta^\nu_t + 2A_4 \delta^{(\mu}_t \delta^{\nu)}_\varphi + A_5 \delta^\mu_\varphi \delta^\nu_\varphi$$

$$K^{\mu\nu} = B_1 g^{\mu\nu} + B_2 \delta^\mu_y \delta^\nu_y + \tilde{K}^{\mu\nu}_y$$

$$\tilde{K}^{\mu\nu}_y = B_3 \delta^\mu_t \delta^\nu_t + 2B_4 \delta^{(\mu}_t \delta^{\nu)}_\varphi + B_5 \delta^\mu_\varphi \delta^\nu_\varphi$$

$$A_2(r_h) = 0, A_2 > 0 \text{ for } r > r_h$$

$$n_\mu = -\sqrt{\Sigma/A_2}\delta^r_\mu$$

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \chi_{\mu\nu} = h^\alpha_\mu h^\beta_\nu \nabla_\alpha n_\beta = h^\alpha_\mu h^\beta_\nu \left(n_{\beta,\alpha} - \Gamma^\lambda_{\alpha\beta} n_\lambda\right)$$

$$\chi_{\mu\nu}dx^\mu dx^\nu = \frac{n_r}{2} g^{rr}\left(g_{ab,r}dx^a dx^b + g_{yy,r}dy^2\right)$$

$$\chi_{\mu\nu} = \frac{\chi^\alpha_\alpha}{3} h_{\mu\nu}, \mu, \nu = t, \varphi, y$$

$$(\ln\ g_{tt})_{,r} = \left(\ln\left|g_{\varphi\varphi}\right|\right)_{,r} = \left(\ln\left|g_{yy}\right|\right)_{,r}$$

$$(\ln\ A_2)' = 2(\ln\ A_{23})', \text{ or simply } A'_3 = 0,$$

$$A_2 = r^2 - 2Mr + Q^2, A_{23} = r^2$$

$$r^2 - 3Mr + 2Q^2 = 0 \Rightarrow r = \frac{3}{2}\left(M \pm \sqrt{M^2 - 8Q^2/9}\right)$$

$$h_{\mu\nu}dx^\mu dx^\nu = \frac{A_2}{\Sigma}(bdt - B_{23}d\varphi)^2 - \frac{B_2}{\Sigma}(adt - A_{23}d\varphi)^2 - \frac{\Sigma}{B_2}dy^2$$

$$\chi_{\mu\nu}dx^\mu dx^\nu = \frac{1}{2}\sqrt{\frac{A_2}{\Sigma}}\partial_r h_{\mu\nu}dx^\mu dx^\nu$$

$$\frac{1}{2}h_{yy}\sqrt{|g^{rr}|}\partial_r p^2 = \chi_{yy}(p^2 - \mu^2), p^2 = g^{ab}p_a p_b, p_a = (-E, L)$$

$$\left(\sqrt{A_3}E - \sqrt{A_5}L\right)'\left(\sqrt{A_3}E - \sqrt{A_5}L\right) = \frac{1}{2}\mu^2 A_1'$$

$$ds^2 = \Delta_r/\Sigma(dt - a\sin^2\theta d\varphi)^2 - 1/\Sigma\sin^2\theta(adt - (r^2 + a^2)d\varphi)^2 - \Sigma/\Delta_r dr^2 - \Sigma d\theta^2$$
$$\Delta_r = r^2 - 2Mr + a^2 + Q^2, \Sigma = r^2 + a^2\cos\theta$$

$$\begin{aligned} A_1 &= r^2, & B_1 &= a^2y^2 \\ A_3 &= (r^2 + a^2)^2/\Delta_r, & B_3 &= a^2(1 - y^2) \\ A_5 &= a^2/A_2 = a^2/\Delta_r, & B_5 &= 1/B_2 = 1/(1 - y^2) \end{aligned}$$

$$A_4^2 = A_3A_5, B_4^2 = B_3B_5, b = 1, A_1 + B_1 = \sqrt{A_2B_2}\left(\sqrt{A_3B_5} - \sqrt{A_5B_3}\right).$$

$$\Psi_2 = \frac{Q^2 - M(r - iay)}{(r - iay)^3(r + iay)}, \Phi_{11} = \frac{Q^2}{2(r^2 + a^2)^2}$$

$$(E(r^2 + a^2) - aL)[Er(\Delta_r - Mr + Q^2) + aL(r - m) + a^2EM] - \mu^2 r\Delta_r^2 = 0,$$

$$ds^2 = \frac{\Delta_r - a^2\sin^2\theta}{\Sigma}(dt - wd\varphi)^2 - \Sigma\left(\frac{dr^2}{\Delta_r} + d\theta^2 + \frac{\Delta_r\sin^2\theta}{\Delta_r - a^2\sin^2\theta}d\varphi^2\right)$$
$$\Delta_r = (r - r_-)(r - 2M) + a^2 - (N - N_-)^2$$
$$\Sigma = r(r - r_-) + (a\cos\theta + N)^2 - N_-^2$$
$$w = \frac{2}{a^2\sin^2\theta - \Delta_r}\left[N\Delta_r\cos\theta + a\sin^2\theta\left(M(r - r_-) + N(N - N_-)\right)\right]$$
$$r_- = \frac{M|Q - iP|^2}{|M + iN|^2}, N_- = \frac{N|Q - iP|^2}{2|M + iN|^2}$$

$$\begin{aligned} A_1 &= r(r - r_-), & B_1 &= (ay + N)^2 - N_-^2 \\ A_3 &= (r(r - r_-) + a^2 + N^2 - N_-^2)^2/\Delta_r, & B_3 &= [a(1 - y^2) - 2Ny]^2/1 - y^2 \\ A_5 &= a^2/A_2 = a^2/\Delta_r, & B_5 &= 1/B_2 = 1/1 - y^2 \\ A_4^2 &= A_3A_5, B_4^2 = B_3B_5, a = a, b = 1, & A_1 + B_1 &= \sqrt{A_2B_2}\left(\sqrt{A_3B_5} - \sqrt{A_5B_3}\right), \end{aligned}$$

$$\Psi_1 = \Psi_3 = \frac{a(4N_-^2 + r_-^2)\sin\,\theta\sqrt{\Delta_r}}{8\Sigma^3}$$

$$\begin{aligned}
12\Sigma^3\Psi_2 =& -12(M - iN)(r + i(a\cos\,\theta + N))^3 + 6NN_-(2(r + i(N + a\cos\,\theta)) - r_-)^2 \\
&+ r_-^3(8M - r) + 8NN_-^3 - 4N_-^4 - 6ar_-\cos\,\theta(5M - 3iN)(a\cos\,\theta + 2N - 2ir) \\
&+ 2r_-(M(15(r + iN)^2 + 7N_-^2) + 18N^2r + 9iN^3 + 3iN(N_-^2 - 3r^2) - 2N_-^2r) \\
&+ 4N_-^2(2a^2 - a\cos\,\theta(3i(M + iN) + a\cos\,\theta) - 5Mr + 2N^2 - 3iN(M + r)) \\
&+ 4N_-^2r^2 + r_-^2(2a^2 - 24iMN - 7N^2 + 2NN_- - N_-^2) \\
&- r_-^2(a\cos\,\theta(6i(4M - iN) + a\cos\,\theta) + 26Mr - 6iNr - r^2)
\end{aligned}$$

$$\begin{aligned}
16\Sigma^3\Phi_{11} =& a^2\cos^2\,(\theta)(8Mr_- + 16NN_- - 4N_-^2 - r_-^2) + 16MN_-^2r_- - 24MN_-^2r \\
&+ 8aN\cos\,\theta(2Mr_- + 4NN_- - 4N_-^2 - r_-^2) + N^2(8Mr_- - 28N_-^2 - 7r_-^2) \\
&+ 8Mr_-r^2 - 14Mr_-^2r + 16N^3N_- + 2NN_-(4N_-^2 + 3r_-^2 + 8r^2 - 8r_-r) \\
&+ N_-^2r_-^2 - 4N_-^2r^2 + 4N_-^2r_-r + 4N_-^4 - r_-^2r^2 + r_-^3r + 6Mr_-^3
\end{aligned}$$

$$\Lambda = \frac{R}{24} = -\frac{(4N_-^2 + r_-^2)(a^2\sin^2\,\theta + \Delta_r)}{48\Sigma^3}$$

$$\begin{aligned}
&E(r(r - r_-) + N^2 - N_-^2)[(2r - r_-)(r^2 - 2rr_- + a^2 + N^2 - 2NN_- + 3N_-^2) - \\
&-2M(3r^2 - rr_- + 2r_-^2 + N^2 - N_-^2)] - 4a^2ELM + a^4E^2(2r + 2M - r_-) + \\
&+a^2[2E^2(2(2r - r_-)(N_-^2 - NN_-) - 2M((r - r_-)^2) - N^2 + N_-^2) - L^2(2r - r_- - 2M)] + \\
&+4aEL[M(r - r_-) - N^2 + N_-^2 + N_-(N - N_-)(2r - r_-)] - \mu^2\Delta^2(2r - r_-) = 0.
\end{aligned}$$

$$w = \frac{2}{\Delta_r - a^2\sin^2\,\theta}\left(N\Delta_r\cos\,\theta + a\sin^2\,\theta\left[m(r - (m + |\Upsilon|)) + \frac{1}{2}(|\mathcal{M}|^2 - |\Upsilon|^2)\right]\right)$$

$$\Delta_r = r[r - 2(m + |\Upsilon|)] + a^2 + (m + |\Upsilon|)^2$$

$$\Sigma = r(r - 2|\Upsilon|) + (a\cos\,\theta + N)^2$$

$$\Upsilon = -\frac{2}{\mathcal{M}}\sum_n\,\left[\bar{\Gamma}_{(n)}\right]^2$$

$$\Gamma_{(n)} = \frac{1}{2}\left(Q_{(n)} + iP_{(n)}\right), n = 1,2$$

$$|\mathcal{M}|^2 + |\Upsilon|^2 - 4\sum_n\,\left|\Gamma_{(n)}\right|^2 = 0$$

$$\begin{aligned}
&A_1 = r(r - 2|\Upsilon|), && B_1 = N + ay \\
&A_3 = (r(r - 2|\Upsilon|) + N^2 + a^2)^2/\Delta_r, && B_3 = (a(1 - y^2) - 2Ny)^2/1 - y^2 \\
&A_5 = a^2/A_2 = a^2/\Delta_r, && B_5 = 1/B_2 = 1/(1 - y^2)
\end{aligned}$$

$$\Psi_1 = \Psi_3 = \frac{a\sin\theta|\Upsilon|^2\sqrt{\Delta_r}}{2\Sigma^3}$$

$$\begin{aligned}3\Sigma^3\Psi_2 =& 3(m-iN)(m-r-ia\cos\theta)(r+i(N+a\cos\theta))^2 + \\ &+3(m-iN)(3(r+ia\cos\theta)-2m+iN)(r+i(N+a\cos\theta))|\Upsilon| - \\ &-2|\Upsilon|^2(3N^2-2(m^2+r^2)+7mr+6iN(m-r)+a^2\cos 2\theta) - \\ &-2|\Upsilon|^3(4(r-m)+3i(N+a\cos\theta))+3ia|\Upsilon|^2\cos\theta(2(m-iN)-r)+4|\Upsilon|^4\end{aligned}$$

$$4\Sigma^3\Phi_{11} = 3|\Upsilon|^2(a^2\cos^2\theta + (m-r+|\Upsilon|)^2) +$$

$$+2\Sigma(m^2 - a\cos\theta(2N+a\cos\theta) - (r-|\Upsilon|)^2 + \Sigma)$$

$$\Phi_{00} =$$

$$\Lambda = \Phi_{22} = \frac{|\Upsilon|^2\Delta_r}{2\Sigma^3}, \Phi_{02} = -\frac{a^2|\Upsilon|^2\sin^2\theta}{2\Sigma^3}$$

$$\Psi_1 = \Psi_3 = \frac{a\sin\theta|\Upsilon|^2\sqrt{\Delta_r}}{2\Sigma^3},$$

$$\begin{aligned}3\Sigma^3\Psi_2 =& 3(m-iN)(m-r-ia\cos\theta)(r+i(N+a\cos\theta))^2+ \\ &+3(m-iN)(3(r+ia\cos\theta)-2m+iN)(r+i(N+a\cos\theta))|\Upsilon|- \\ &-2|\Upsilon|^2(3N^2-2(m^2+r^2)+7mr+6iN(m-r)+a^2\cos 2\theta)- \\ &-2|\Upsilon|^3(4(r-m)+3i(N+a\cos\theta))+3ia|\Upsilon|^2\cos\theta(2(m-iN)-r)+4|\Upsilon|^4,\end{aligned}$$

$$\begin{aligned}4\Sigma^3\Phi_{11} =& 3|\Upsilon|^2(a^2\cos^2\theta+(m-r+|\Upsilon|)^2)+ \\ &+2\Sigma(m^2-a\cos\theta(2N+a\cos\theta)-(r-|\Upsilon|)^2+\Sigma),\end{aligned}$$

$$\Phi_{00} = \Phi_{22} = \frac{|\Upsilon|^2\Delta_r}{2\Sigma^3}, \qquad \Phi_{02} = -\frac{a^2|\Upsilon|^2\sin^2\theta}{2\Sigma^3},$$

$$\Lambda = -\frac{|\Upsilon|(a^2(1+\sin^2\theta)+(m-r+|\Upsilon|)^2)}{12\Sigma^3}.$$

$$E(r(r-2|\Upsilon|)+N^2+a^2)[\{E(\Delta_r+|\Upsilon|^2-M^2-N^2)+aL\}(r-M-|\Upsilon|)+2a^2EM]$$
$$-\mu^2\Delta_r^2(r-|\Upsilon|)=0$$

$$\Delta_r = r^2 - 2Mr + a^2$$

$$w = -\frac{2Ma\omega\sin^2\theta}{\Delta_r - a^2\sin^2\theta}, \omega = \left((\Pi_c - \Pi_s)r + 2M\Pi_s\right)$$

$$\Sigma^2 = \prod_{I=0}^{4}(r+2Ms_I^2) + a^4\cos^4\theta +$$

$$+2a^2\cos^2\theta\left(r^2 + Mr\sum_{I=0}^{3}s_I^2 + 4M^2(\Pi_c-\Pi_s)\Pi_s - 2M^2\sum_{I<J<K}^{3}s_I^2s_J^2s_K^2\right)$$

$$\Pi_s = \prod_{i=1}^{4}s_I = \prod_{I=1}^{4}\sinh\delta_I, \Pi_c = \prod_{I=1}^{4}\sqrt{1+s_I^2} = \prod_{I=1}^{4}\cosh\delta_I, s_I^2 = \sinh^2\delta_I$$

$$\Sigma_{2ch} = r^2 + a^2\cos^2\theta + 2Mr(\mathcal{S}_1^2+\mathcal{S}_2^2) + 4M^2\mathcal{S}_1^2\mathcal{S}_2^2 = r^2 - 2Mr + a^2\cos^2\theta + 2Mw_{2ch}$$

$$A_1 = r^2 - 2Mr + 2Mw_{2ch}, \qquad B_1 = a^2y^2$$
$$A_3 = (r^2 - 2Mr + a^2 + 2Mw_{2ch})^2/\Delta_r, \qquad B_3 = a^2(1-y^2)$$
$$A_5 = a^2/A_2 = a^2/\Delta_r, \qquad B_5 = 1/B_2 = 1/(1-y^2)$$

$$\Psi_1 = \Psi_3 = \frac{aM^2\sin\,\theta(S_1^2 - S_2^2)^2\sqrt{\Delta_r}}{2\Sigma_{2ch}^3}$$

$$\frac{3\Sigma_{2ch}^3}{M}\Psi_2 = -3(r + ia\cos\,\theta)^3 - 3(r + ia\cos\,\theta)^2(M + r + ia\cos\,\theta)(\mathcal{S}_1^2 + \mathcal{S}_2^2)$$
$$-2M(r^2 + 2Mr - a^2 + 3iar\cos\,\theta + a^2y^2)(\mathcal{S}_1^4 + \mathcal{S}_2^4)$$
$$+12M^2\mathcal{S}_1^2\mathcal{S}_2^2(M - r - ia\cos\,\theta)(\mathcal{S}_1^2 + \mathcal{S}_2^2)$$
$$+4M\mathcal{S}_1^2\mathcal{S}_2^2(5a^2\cos^2\,\theta - a^2 + 4Mr - 5r^2 + 3iaM\cos\,\theta - 9iar\cos\,\theta)$$
$$\Phi_{00} = \Phi_{22} = \frac{M^2\Delta_r(\mathcal{S}_1^2 - \mathcal{S}_2^2)^2}{2\Sigma_{2ch}^3}, \Phi_{02} = \overline{\Phi_{20}} = -\frac{a^2M^2\sin^2\,\theta(\mathcal{S}_1^2 - \mathcal{S}_2^2)^2}{2\Sigma_{2ch}^3}$$

$$\frac{4\Sigma_{2ch}^3}{M^2}\Phi_{11} = 4(r^2 + a^2\cos^2\,\theta)(\mathcal{S}_1^2 + \mathcal{S}_2^2) + (2Mr + 3r^2 + 3a^2\cos^2\,\theta)(\mathcal{S}_1^4 + \mathcal{S}_2^4)$$
$$+16M(r + M)\mathcal{S}_1^2\mathcal{S}_2^2(\mathcal{S}_1^2 + \mathcal{S}_2^2) + 32M^2\mathcal{S}_1^4\mathcal{S}_2^4 + 2\mathcal{S}_1^2\mathcal{S}_2^2(r^2 + 14Mr + a^2\cos^2\,\theta)$$
$$\Lambda = \frac{1}{24R} = -\frac{M^2(\mathcal{S}_1^2 - \mathcal{S}_2^2)^2(\Delta_r + a^2\sin^2\,\theta)}{12\Sigma_{2ch}^3}$$

$$E^2\left[\Delta^2(r - M + 2M\Sigma_{s^2}) - 4M^2r(Mr - a^2)\Sigma_{s^2}^2 - 16M^4(r - M)\Pi_{s^2}^2 - 8M^3(r^2 - a^2)\Pi_{s^2}\Sigma_{s^2}\right]$$
$$+2aELM[4M(r - M)\Pi_{s^2} - (r^2 - a^2)\Sigma_{s^2}] - a^2L(r - M) - \mu^2\Delta^2(r - M + M\Sigma_{s^2}) = 0,$$

$$\Sigma_{s^2} = \mathcal{S}_1^2 + \mathcal{S}_2^2, \Pi_{s^2} = \mathcal{S}_1^2\mathcal{S}_2^2$$

$$\mu = \rho = -\sqrt{A_2}\frac{\partial}{\partial_r}\frac{1}{\sqrt{2\Sigma}} - \frac{iB_5\sqrt{B_2}}{2\sqrt{2\Sigma}}\frac{\partial_y\sqrt{B_3/B_5}}{\sqrt{A_3B_5} - \sqrt{A_5B_3}}$$
$$\tau = \pi = -i\sqrt{B_2}\frac{\partial}{\partial_y}\frac{1}{\sqrt{2\Sigma}} + \frac{A_5\sqrt{A_2}}{2\sqrt{2\Sigma}}\frac{\partial_r\sqrt{A_3/A_5}}{\sqrt{A_3B_5} - \sqrt{A_5B_3}}$$
$$\epsilon = \gamma = \frac{1}{2}\rho + \frac{\sqrt{A_2}}{2\sqrt{2}\Sigma^{3/2}}\left(\Sigma\partial_r\ln\left[\sqrt{A_3B_5} - \sqrt{A_5B_3}\right] - \partial_r\Sigma\right)$$
$$\alpha = \beta = \frac{1}{2}\tau + i\frac{\sqrt{B_2}}{2\sqrt{2}\Sigma^{3/2}}\left(\Sigma\partial_y\ln\left[\sqrt{A_3B_5} - \sqrt{A_5B_3}\right] - \partial_y\Sigma\right)$$

$$\Phi_{00} = \Phi_{22} = \frac{bA_2}{8\Sigma^3}(bA'^{\,2}_{23} + bB'^{\,2}_{23} - 2\Sigma A''_{23}),$$

$$\Phi_{01} = \overline{\Phi_{10}} = \Phi_{12} = \overline{\Phi_{21}} = -\frac{\sqrt{A_2 B_2}}{8\Sigma^3}(aA''_{23} - bB''_{23}),$$

$$\Phi_{02} = \overline{\Phi_{20}} = -\frac{aB_2}{8\Sigma^3}(aA'^{\,2}_{23} + aB'^{\,2}_{23} + 2\Sigma B''_{23}),$$

$$64\Sigma^3\Phi_{11} = \Sigma^2\{A_2[(\ln A_5)^2_{,r} - 4(\ln A_5)_{,rr}] - B_2[(\ln B_5)^2_{,r} - 4(\ln B_5)_{,rr}]\}$$
$$+3\left\{4b^2A_2B'^{\,2}_{23} - 4a^2B_2A'^{\,2}_{23} + A_2[\Sigma(\ln A_5)_{,r} + 2bA'_{23}]^2 - B_2[\Sigma(\ln B_5)_{,y} - 2aB'_{23}]^2\right\}$$
$$+4\Sigma\{A_2((\ln A_5)_{,r})[\Sigma(\ln A_5)_{,r} + 2bA'_{23}] - A_2[\Sigma(\ln A_5)_{,r} + 2bA'_{23}], r\}$$
$$-4\Sigma\{B_2((\ln B_5)_{,y}[\Sigma(\ln B_5)_{,y} - 2aB'_{23}] - B_2[\Sigma(\ln B_5)_{,y} - 2aB'_{23}], y\},$$
$$192\Sigma^3\Lambda = 8\Sigma^3 R = 3\Sigma^2\{A_2[3(\ln A_5)^2_{,r} - 4(\ln A_5)_{,rr}] + B_2[3(\ln B_5)^2_{,y} - 4(\ln B_5)_{,yy}]\}$$
$$+4A_2[\Sigma(\ln A_5)_{,r} + 2bA'_{23}], r + 4B_2[\Sigma(\ln B_5)_{,y} - 2aB'_{23}], y - 4a^2B_2A'^{\,2}_{23}$$
$$-A_2[\Sigma(\ln A_5)_{,r} + 2bA'_{23}]^2 - B_2[\Sigma(\ln B_5)_{,y} - 2aB'_{23}]^2 - 4b^2A_2B'^{\,2}_{23}$$
$$-12\Sigma^3\Psi_2 = \Sigma\{A_2[\Sigma(\ln A_5)_{,r} + 2bA'_{23}], r + B_2[\Sigma(\ln B_5)_{,y} - 2aB'_{23}], y\} + 2a^2B_2A'^{\,2}_{23}$$
$$-A_2[\Sigma(\ln A_5)_{,r} + 2bA'_{23}]^2 - B_2[\Sigma(\ln B_5)_{,y} - 2aB'_{23}]^2 + 2b^2A_2B'^{\,2}_{23}$$
$$+3i\{aB_2A'_{23}[\Sigma(\ln B_5)_{,y} - 2aB'_{23}] + bA_2B'_{23}[\Sigma(\ln A_5)_{,r} + 2bA'_{23}]\}$$

$$S = \frac{1}{16\pi G}\int d^5xe\left[R + 12g^2 - \frac{1}{4}F^2 - \frac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda\right]$$

$$E = \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2\pi qabg^2(\Xi_a + \Xi_b)}{4G\Xi_a^2\Xi_b^2}, Q = \frac{\sqrt{3}\pi q}{4G\Xi_a\Xi_b}$$
$$J_1 = \frac{\pi[2am + qb(1 + a^2g^2)]}{4G\Xi_a^2\Xi_b}, J_2 = \frac{\pi[2bm + qa(1 + b^2g^2)]}{4G\Xi_b^2\Xi_a}$$

$$\frac{\pi\ell_5^3}{4G} = \frac{1}{2}N^2$$

$$\Delta_r(r) = \frac{(r^2 + a^2)(r^2 + b^2)(1 + r^2) + q^2 + 2abq}{r^2} - 2m = 0,$$

$$T \equiv \beta^{-1} = \frac{r_+^4[(1 + g^2(2r_+^2 + a^2 + b^2)] - (ab + q)^2}{2\pi r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}$$
$$\Omega_1 = \frac{a(r_+^2 + b^2)(1 + g^2r_+^2) + bq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}, \Omega_2 = \frac{b(r_+^2 + a^2)(1 + g^2r_+^2) + aq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}$$
$$\Phi = \frac{\sqrt{3}qr_+^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}$$

$$\mathcal{S} = \frac{\text{Area}}{4G} = \frac{\pi^2[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{2G\Xi_a\Xi_b r_+}$$

$$dE = Td\mathcal{S} + \Omega_1 dJ_1 + \Omega_2 dJ_2 + \Phi dQ$$

$$I = \beta E - \mathcal{S} - \beta\Omega_1 J_1 - \beta\Omega_2 J_2 - \beta\Phi Q,$$

$$I = \frac{\pi\beta}{4G\Xi_a\Xi_b}\left[m - g^2(r_+^2 + a^2)(r_+^2 + b^2) - \frac{q^2r_+^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}\right]$$

$$E=\frac{\partial I}{\partial \beta}, J_1=-\frac{1}{\beta}\frac{\partial I}{\partial \Omega_1}, J_2=-\frac{1}{\beta}\frac{\partial I}{\partial \Omega_2}, Q=-\frac{1}{\beta}\frac{\partial I}{\partial \Phi}$$

$$q=\frac{m}{1+ag+bg}$$

$$q=\frac{1}{g}(a+b)(1+ag)(1+bg)$$

$$r_*=\sqrt{\frac{1}{g}(a+b+abg)}$$

$$\beta\to\infty, \Omega_1\to\Omega_1^*=g, \Omega_2\to\Omega_2^*=g, \Phi\to\Phi^*=\sqrt{3}$$

$$\{\mathcal{Q},\overline{\mathcal{Q}}\}\propto E-gJ_1-gJ_2-\sqrt{3}Q.$$

$$E^*-gJ_1^*-gJ_2^*-\sqrt{3}Q^*=0$$

$$G(T,\Omega_a,\Omega_b,\Phi)=M-TS-\Phi Q-\Omega_a J_a-\Omega_b J_b$$

$$F=E-T\mathcal{S}-\Phi Q-2\Omega J$$

$$m=\frac{(r_+^2+a^2)^2(1+r_+^2)+q^2+2a^2q}{2r_+^2}$$

$$\begin{aligned}F(r_+,T,\Phi,\Omega)=&\left(\frac{\pi}{8(a^2-1)^3Gr_+^2}\right)\left(-\left(a^6(r_+^2+4\pi r_+T+1)\right)-3q^2+r_+^4(-3r_+^2+4\pi r_+T-3)\right.\\&+4a^5(r_+^2+1)\Omega+4a^3\Omega((q+2)r_+^2+2q+2r_+^4)+4a\Omega(q^2+qr_+^2+r_+^6+r_+^4)\\&+2\sqrt{3}qr_+^2\Phi-a^2\left(q^2+2q\left(r_+\left(\sqrt{3}r_+\Phi+4r_+-2\pi T\right)+3\right)\right.\\&\left.+r_+^2(r_+^4+4\pi(r_+^2-2)r_+T+7r_+^2+6)\right)\\&\left.-a^4(4\pi r_+T(q+2r_+^2-1)+2q+2r_+^4+5r_+^2+3)\right)\end{aligned}$$

$$q=\frac{\Phi(r_+^2+a^2)^2}{\sqrt{3}r_+^2-a^2\Phi}, 1-\Omega=\frac{1-a}{r_+^2+a^2}\left[r_+^2-r_*^2+a(1+a)\left(1-\frac{\Phi}{\sqrt{3}}\right)\right]$$

$$F(r_+,T,\Phi)=\frac{1}{4}r_+^2(3+3r_+^2-4\pi r_+T-\Phi^2)$$

$$T=\frac{6r_+^2-\Phi^2+3}{6\pi r_+},$$

$$F(r_+,\Phi)=-\frac{1}{12}r_+^2(3r_+^2+\Phi^2-3),$$

$$F=0,$$
$$\frac{\partial F}{\partial r_+}=0,$$

$$r_+=\frac{\sqrt{3-\Phi^2}}{\sqrt{3}}, T_{\mathrm{HP}}=\frac{\sqrt{3}\sqrt{3-\Phi^2}}{2\pi}.$$

$$a = \frac{1 + r_+^2 - f_\Omega}{2\Omega}, \text{ where } f_\Omega = \sqrt{(2 - 4\Omega^2)r_+^2 + r_+^4 + 1}.$$

$$F = \left(\frac{1}{8r_+^2(\Omega^2 - 1)^2}\right)(-4\pi r_+^3 T(f_\Omega - 2\Omega^2) + r_+^4(3f_\Omega - 10\Omega^2 + 3)$$
$$+r_+^2\left((6 - 4f_\Omega)\Omega^2 - 1\right) + 4\pi(f_\Omega - 1)r_+ T + f_\Omega - 4\pi r_+^5 T + 3r_+^6 - 1)$$

$$T = \frac{f_\Omega + r_+^2}{2\pi r_+}$$

$$T_{\text{HP}} = \frac{2\sqrt{1 - \Omega^2} + 1}{2\pi},$$

$$F = \frac{1}{4}r_+^2(3r_+^2 - 4\pi r_+ T + 3), T = \frac{(2r_+^2 + 1)}{2\pi r_+}$$

$$S = \frac{1}{16\pi G}\int d^5x e\left\{\tilde{c}_0 R + 12\tilde{c}_1 g^2 - \frac{\tilde{c}_2}{4}F^2 - \frac{\tilde{c}_3}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda\right.$$
$$\left.+\lambda_1\alpha\left[\mathcal{X}_{\text{GB}} - \frac{1}{2}C_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{1}{8}F^4 - \frac{1}{2\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}R_{\mu\nu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}A_\lambda\right]\right\}$$

$$\mathcal{X}_{\text{GB}} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{3}\left(R_{\mu[\rho}g_{\sigma]\nu} + R_{\nu[\sigma}g_{\rho]\mu}\right) + \frac{1}{6}Rg_{\mu[\rho}g_{\sigma]\nu}$$

$$\tilde{c}_i = 1 + \alpha g^2\delta\tilde{c}_i$$

$$\delta\tilde{c}_0 = 4\lambda_2, \delta\tilde{c}_1 = -10\lambda_1 + 4\lambda_2, \delta\tilde{c}_2 = 4\lambda_1 + 4\lambda_2, \delta\tilde{c}_3 = -12\lambda_1 + 4\lambda_2.$$

$$Q = \frac{\sqrt{3}\pi q}{4G(1 - a^2)^2}(1 + 4\lambda_2\alpha) + \lambda_1\alpha\Delta Q(q, a, r_+)$$
$$J = \frac{a\pi(1 + 4\lambda_2\alpha)}{4(1 - a^2)^3 G r_+^2}[(a^2 + q)^2 + (a^4 + q + a^2(2 + q))r_+^2 + (1 + 2a^2)r_+^4 + r_+^6]$$
$$+\lambda_1\alpha\Delta J(q, a, r_+)$$
$$\mathcal{S} = \frac{\pi^2[r_+^4 + a^4 + a^2(q + 2r_+^2)]}{2Gr_+(1 - a^2)^2}(1 + 4\lambda_2\alpha) + \lambda_1\alpha\Delta\mathcal{S}(q, a, r_+)$$

$$F(r_+, T, \Phi, \Omega, \alpha, \lambda_1, \lambda_2) = E - T\mathcal{S} - \Phi Q - 2\Omega J$$

$$F(r_+, T, \Phi, \alpha, \lambda_1, \lambda_2) = F_0(r_+, T, \Phi) + 4\pi\alpha r_+^3 T\left(-\lambda_2 - \frac{3\lambda_1(9r_+^2 + \Phi^2 + 9)}{6r_+^2 + \Phi^2 - 3}\right)$$
$$+\frac{\alpha}{6(6r_+^2 + \Phi^2 - 3)}[6\lambda_2 r_+^2(-3r_+^2(\Phi^2 - 3) + 18r_+^4 - (\Phi^2 - 3)^2)$$
$$\left.+\lambda_1\left(r_+^4(729 - 99\Phi^2) - 12r_+^2(\Phi^4 + 6\Phi^2 - 27) + 486r_+^6 + (\Phi^2 - 3)^2(\Phi^2 + 9)\right)\right]$$

$$M = 3Q + 2J$$

$$M - 3Q - 2J = \frac{1}{2}N^2 \frac{3 + 2a - a^2}{(1-a)^2(1+a)^4}(m - q(1+2a)),$$

$$W(\Phi', \Omega') = \frac{G}{T} = -S - \Phi' Q - \Omega' J$$

$$\Phi' = (\Phi - \Phi^*)/T, \Omega' = (\Omega - \Omega^*)/T$$

$$\Phi' = -\frac{2}{\tau} + \varphi',$$
$$\Omega' = -\frac{1}{\tau}.$$

$$W(a) = \frac{2a\left(\frac{\pi(-2a^2 - 5a + 1)}{\sqrt{\frac{2}{a} + 1}} - \frac{1}{2}(3-a)a\varphi'\right)}{(1-a)^2(5a+1)}$$

$$\tau_{\mathrm{H}} = \frac{5a + 1}{(1-a)\left(\frac{3\pi}{\sqrt{\frac{2}{a} + 1}} + \frac{\varphi'}{2}\right)}$$

$$F(u) \approx \sqrt{2}\pi u^3 - \frac{33\pi u^5}{2\sqrt{2}} + \mathcal{O}(u^7)$$

$$\tau_{\mathrm{H}} \approx \frac{25u}{6\sqrt{2}\pi} + \frac{\sqrt{2}}{3\pi u} + \mathcal{O}(u^3)$$

$$F(u, \tau) = \frac{\pi}{2\sqrt{2}}(c_3 u^3 - c_4 \tau u^4 + c_5 u^5)$$

$$F(u) = \frac{1}{12}u^3\left(3\sqrt{2}\pi c_3 - 2c_4\right) + \frac{1}{24}u^5\left(6\sqrt{2}\pi c_5 - 25c_4\right).$$

$$c_5 = -\frac{198\sqrt{2}\pi - 25c_4}{6\sqrt{2}\pi}, c_3 = \frac{\sqrt{2}\left(c_4 + 6\sqrt{2}\pi\right)}{3\pi}$$

$$F(u, \tau) = -\frac{\pi c_4 \tau u^4}{2\sqrt{2}} + \frac{1}{24}\left(25c_4 - 198\sqrt{2}\pi\right)u^5 + \frac{1}{6}\left(c_4 + 6\sqrt{2}\pi\right)u^3$$

$$\partial_u F(u, \tau) = 0$$

$$\frac{u(25u^2 + 4)c_4'(u) + (125u^2 + 12)c_4(u) + 18\sqrt{2}\pi(4 - 55u^2)}{6\sqrt{2}\pi u(uc_4'(u) + 4c_4(u))} = \frac{25u}{6\sqrt{2}\pi} + \frac{\sqrt{2}}{3\pi u},$$

$$c_4(u) = 18\sqrt{2}\pi - 135(\sqrt{2}\pi)u^2 + O(u^4)$$

$$F(u,\tau) = 4\sqrt{2}\pi u^3 - 9(\pi^2\tau)u^4 - 12(\sqrt{2}\pi)u^5 + \frac{135}{2}\pi^2\tau u^6 + O(u^7)$$

$$F(r_+,T) = G(r_+) + g(r_+)\left(1 - \frac{T}{T_H}\right).$$

$$F'(r_+,T) = \left[G'(r_+) + g(r_+)\left(1 - \frac{T}{T_H(r_+)}\right)' + g'(r_+)\left(1 - \frac{T}{T_H}\right)\right] = 0$$

$$G = \frac{r_+^2(1-r_+^2)}{4}$$

$$T_H = \frac{1+2r_+^2}{2\pi r_+}$$

$$F(r_+,T) = \frac{r_+^2(1-r_+^2)}{4} + g(r_+)\left[1 - \left(\frac{2\pi r_+}{1+2r_+^2}\right)T\right]$$

$$g(r_+) = r^4 + \frac{r^2}{2}$$

$$F(r_+,T) = \frac{1}{4}r^2(3r^2 - 4\pi rT + 3)$$

$$F(a,\tau,\phi') = W(a) + g(a,\varphi')\left(1 - \frac{\tau}{\tau_H}\right)$$

$$g(a,\varphi') = \frac{a(a+1)^2\left(\pi\sqrt{\frac{a+2}{a}}(17a^2+20a-1)\varphi + 6\pi^2(11a^2+8a-1) + (a+2)^2\varphi^2\right)}{(a-1)^2(5a+1)\left(\pi\sqrt{\frac{a+2}{a}}(11a^2+8a-1) + (a+2)^2\varphi\right)}.$$

$$F = \frac{a}{(a-1)^2(5a+1)^2}\left(2(5a+1)\left(\frac{\pi(-2a^2-5a+1)}{\sqrt{\frac{a+2}{a}}} + \frac{1}{2}(a-3)a\varphi'\right) - \frac{(a+1)^2\left((a+2)\varphi' + 6\pi a\sqrt{\frac{a+2}{a}}\right)\left((1-a)\tau\left(\frac{3\pi}{\sqrt{\frac{a+2}{a}}} + \frac{\varphi'}{2}\right) - 5a - 1\right)}{a+2}\right)$$

$$F(a,\tau) = W(a) + 2\left(\frac{J^* + Q^*}{\tau_H}\right)\left(1 - \frac{\tau}{\tau_H}\right)$$

$$W(a) = -S^* + 2\left(\frac{J^* + Q^*}{\tau_H}\right)$$

$$M^* = N^2\frac{a(3-a^2-2a^3)}{(1-a)^4}, Q^* = N^2\frac{a}{(1-a)^2}$$
$$J^* = N^2\frac{a^2(a+3)}{(1-a)^3}, S^* = N^2\frac{2\pi a\sqrt{a(a+2)}}{(1-a)^2}$$

$$\left(Q^{*3}+\frac{1}{2}N^2J^{*2}\right)-\left(3Q^*+\frac{1}{2}N^2\right)(3Q^{*2}-N^2J^*)=0$$

$$F(a,\tau)=4\sqrt{2}\pi a^{3/2}-12\sqrt{2}\pi a^{5/2}-9\pi^2a^2\tau+\varphi'\left(\frac{87\pi a^{5/2}\tau}{2\sqrt{2}}-3\sqrt{2}\pi a^{3/2}\tau-4a^2+a\right)$$
$$+\mathcal{O}\left(a^{5/2}\right)$$

$$F=\mathcal{E}-\tau\mathcal{S}$$

$$\mathcal{S}=-\frac{\partial F(a,\tau)}{\partial\tau}=2\left(\frac{J^*+Q^*}{\tau_H^2}\right)=\frac{18\pi^2a^2(a+1)^2}{(5a+1)^2(2-a-a^2)},$$

$$\mathcal{E}=-S^*+4\left(\frac{J^*+Q^*}{\tau_H}\right)=\frac{2\pi a^{3/2}(a^2+a+4)}{(a-1)^2\sqrt{a+2}(5a+1)}$$

$$Q^* = \frac{\pi a}{2Gg(1-ag)^2}\left[1+4\lambda_2\alpha g^2+4\lambda_1\alpha g^2\frac{1+8ag+36a^2g^2+44a^3g^3+19a^4g^4}{ag(-1+8ag+11a^2g^2)}\right],$$
$$J^* = \frac{\pi a^2(3+ag)}{2Gg(1-ag)^3}\left[1+4\lambda_2\alpha g^2+24\lambda_1\alpha g^2\frac{1+9ag+29a^2g^2+25a^3g^3+8a^4g^4}{ag(3+ag)(-1+8ag+11a^2g^2)}\right],$$
$$\mathcal{S}^* = \frac{\pi^2a\sqrt{ag(ag+2)}}{g^2G(1-ag)^2}\left[1+4\lambda_2\alpha g^2+48\lambda_1\alpha g^2\frac{2a^2g^2+5ag+2}{11a^2g^2+8ag-1}\right].$$

$$F(a,\tau,\alpha,\lambda_1,\lambda_2)=F_0(a,\tau)+\frac{\left(8\sqrt{a}\pi\alpha\left(f_{\lambda_1}+f_{\lambda_2}\right)\right)}{(a-1)^2(a+2)^{3/2}(5a+1)^2(a(11a+8)-1)},$$
$$f_{\lambda_1}=3\lambda_1\left(3\pi\sqrt{a(a+2)}(29a^6+96a^5+57a^4-100a^3-69a^2-12a-1)\tau\right.$$
$$\left.+90a^7+508a^6+1158a^5+1272a^4+622a^3+192a^2+42a+4\right),$$
$$f_{\lambda_2}=\lambda_2\left(9\pi\sqrt{a^3(a+2)}(11a^3-3a^2-9a+1)(a+1)^2\tau\right.$$
$$\left.+a(a+2)(5a+1)(a^2+a+4)(a(11a+8)-1)\right),$$

$$G(r_+,\alpha,\lambda_1,\lambda_2)=-\frac{1}{4}(r_+^2-1)r_+^2(4\alpha\lambda_2+1)-\frac{9}{2}\alpha\lambda_1(r_+^2+1)^2,$$

$$W=-\frac{\pi r_+\left((r_+^2-1)r_+^2(4\alpha\lambda_2+1)+18\alpha\lambda_1(r_+^2+1)^2\right)}{4r_+^2+2}$$

$$W\sim-9r_+(\pi\alpha\lambda_1)+\frac{1}{2}\pi r^3(4\alpha\lambda_2+1)+O(r_+^4)$$

$$W\sim-T^{-1}\ \text{as}\ T\to\infty$$

$$W\sim-\frac{1}{4}\pi r_+^3(18\alpha\lambda_1+4\alpha\lambda_2+1)-\frac{3}{8}\pi r_+(18\alpha\lambda_1-4\alpha\lambda_2-1)+\cdots$$

$$W \sim -T^3 \text{ as } T \to \infty$$

$$W(a,\tau,\alpha) = W(a,\tau) - \frac{8\pi\sqrt{\frac{a}{a+2}}\alpha(3(a^3+5a^2+5a+1)\lambda_1 + a(2a^2+5a-1)\lambda_2)}{(a-1)^2(5a+1)}$$

$$W(a,\alpha) = -12\sqrt{2}\pi\alpha\sqrt{a}\lambda_1 - \sqrt{2}\pi a^{3/2}(21\alpha\lambda_1 - 4\alpha\lambda_2 - 1) + O\left(a^{5/2}\right)$$

$$W \sim -\tau^{-1} \text{ as } \tau \to \infty,$$

$$W(a,\alpha) = -\frac{4\pi(4\alpha(3\lambda_1+\lambda_2)+1)}{\sqrt{3}(1-a)} - \frac{2\pi(4\alpha(6\lambda_1+\lambda_2)+1)}{\sqrt{3}(1-a)^2} - \frac{\pi(4\alpha(6\lambda_1+5\lambda_2)+5)}{3\sqrt{3}} + \cdots,$$

$$W \sim -\tau^2 \text{ as } \tau \to \infty,$$

$$W(Q,\Phi,T) = \frac{Q\left(6Q - \Phi\left(4\pi\sqrt{2}T\sqrt{\frac{Q}{\Phi}} + \Phi^2 - 3\right)\right)}{2\Phi^2}$$

$$T = \frac{\sqrt{\frac{Q}{\Phi}}(12Q - \Phi^3 + 3\Phi)}{6\sqrt{2}\pi Q}$$

$$T = \frac{\sqrt{\frac{Q}{\Phi}}(6Q - \Phi^3 + 3\Phi)}{4\sqrt{2}\pi Q}$$

$$Q_c = -\frac{1}{6}\Phi(\Phi^2 - 3)$$

$$T_c = \frac{\sqrt{9 - 3\Phi^2}}{2\pi}$$

CONCLUSIONES.

Los agujeros negros cuánticos se forman en gravedad cuántica o en su defecto en supergravedad cuántica, aunque es inevitable en el segundo caso. La singularidad de un agujero negro cuántico, es en sí, la masa compacta de la partícula colapsada o colisionada, cuyo punto concéntrico es extremadamente denso, capaz de devorar materia y energía, transformándola en materia y energía oscuras, es decir, materia y energía inertes atravesadas por gravedad en condiciones perturbativas o entrópicas. El agujero cuántico de gusano, es distinto al agujero negro cuántico, es decir, no se forma por el colapso o colisión de una partícula oscura o de una partícula blanca, sino por la distorsión del espacio – tiempo cuántico, en condiciones de gravedad exógena, esto es, cuando cualquiera de las partículas antes referidas, interactúan con el gravitón o su supercompañera, el supergravitón o gravitino, formándose así, lo que se

conoce también como puente de Einstein – Rosen, lo que explicaría la superposición y entrelazamiento cuánticos desde la perspectiva de la relatividad general. El radio de un agujero negro cuántico, es infinito en proporción a su disco de acreción. El horizonte de eventos de un agujero negro cuántico, inicialmente es débil, más su acaparamiento es mayor en tanto la absorción de materia y energía se vuelve más agresiva. El agujero negro cuántico, sea en gravedad o supergravedad cuánticas, produce materia y energía oscuras, es decir, tejido cuánticamente muerto, sin interacción más que la gravedad en sí misma, sin que sea posible la formación de otro fenómeno subatómico que requiera energía o materia. Un agujero negro cuántico, no puede crecer más allá del límite del campo en el que se ha formado, además que, es capaz de producir radiación bajo parámetros termodinámicos hostiles. Finalmente, es preciso señalar que un agujero blanco cuántico, se forma por el colapso o colisión de una partícula blanca en sentido estricto, más no por una partícula oscura, aunque puede formarse en condiciones de antimateria, lo que causa una supercurvatura y en consecuencia, la formación de un agujero blanco cuántico cuya principal característica es, expulsar materia y energía, sin capacidad de absorción o atracción, por lo que, todo escapa a su horizonte de eventos. La singularidad de un agujero blanco cuántico, es el centro de masa y energía de la partícula colapsada o colisionada, cuya densidad es extremadamente ligera, la misma que expulsa paquetes de energía de manera infinita al igual que repele materia aunque la produce. Por tanto, la gravedad no interactúa en un agujero blanco cuántico, aunque éstos pueden formarse a partir de la distorsión del espacio – tiempo cuántico. El agujero blanco cuántico, es vital para la creación de multidimensiones cuánticas y por ende, de hiperespacios paralelos al espacio – tiempo cuántico de origen, pues se constituye como un punto de salida de un puente de Einstein – Rosen propiamente dicho, arrojando energía y materia, bien sea, por atrapamiento desde el otro extremo del puente, por un agujero negro cuántico, aunque el referido puente, no necesariamente se forma a partir de un agujero negro cuántico, es decir, que su opuesto puede producir un puente ER de manera directa, cuando una partícula blanca o estrella colapsa o colisiona, en cuyo caso, la formación de materia y energía y la repulsión de la misma, se origina desde la singularidad del agujero blanco cuántico, aunque en condiciones extremas y para estos efectos, el agujero blanco cuántico pueda apalancarse de un agujero negro cuántico para efectos de desplegar su fenomenología, es decir, se forma de manera ulterior al agujero blanco cuántico. En todos los casos, el puente ER, es el ducto de comunicación entre ambos agujeros, pero esta

configuración no es absoluta, pues un agujero blanco cuántico o su opuesto, pueden formarse a partir de una partícula blanca u oscura, según sea el caso, expulsando o en su defecto, absorbiendo masa y energía, que no se transporta a través del puente ER, sino que interactúa con la singularidad del agujero cuántico de que se trate, sea formando gravedad o reduciéndola a cero.

ACLARACIONES FINALES:

Algunas aclaraciones finales a tener en consideración y aplicar, a propósito de la Teoría Cuántica de Campos Relativistas o Curvos (TCCR) propuesta por este autor:

1. En todos los casos, este símbolo † será reemplazado por este símbolo†o por este símbolo †, equivaliendo lo mismo.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
†	†
	†

2. En todos los casos, este símbolo ‡, será reemplazado por este símbolo‡o por este símbolo ‡.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
‡	‡
	‡

3. En todos los casos, se añadirá y por ende, se calculará la magnitud§que equivale a un campo de Yang – Mills y por ende, a la teoría de Yang – Mills en sentido amplio, en relación a la Teoría Cuántica de Campos Relativistas propuesta por este autor.

4. Este símbolo • podrá usarse como exponente u operador, según sea el caso.

Las aclaraciones antes referidas aplican tanto a este trabajo como a todos los trabajos previos y posteriores publicados por este autor, según corresponda.

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