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RESUMEN.

En este trabajo, compuesto por diez volúmenes, abordaremos aspectos esenciales de la Teoría Cuántica de Campos Relativistas (TCCR), con propósitos de optimización de los cálculos expuestos en trabajos anteriores pero sobre todo, posicionar la referida teoría, como una alternativa de solución al problema del milenio de Yang – Mills y la brecha de masa. La idea esencial es la misma, todo espacio – tiempo cuántico, es decir, todo campo cuántico, es curvo y esa deformación ocurre por la gravedad y supergravedad cuánticas, según sea el caso, que provocan las partículas oscuras o estrella, al momento de interactuar con un campo gravitónico o supergravitónico, según corresponda, o en relación a la criticidad de su centro de masa y/o energía, lo que afecta su espín, velocidad y momento angular y por ende, sus trayectorias orbitales. Por tanto, la TCCR, no es un intento por cuantizar la gravedad, sino por introducir la gravedad, como principio de mínima acción de un sistema cuántico y de sus estados fundamentales.

Las métricas siguen siendo las mismas, es decir, que para un campo cuántico curvo o geométricamente deformado, la densidad lagrangiana/hamiltoniana equivale a:

$$\mathcal{LH}_{curvature} = \frac{\langle \int \hat{e}^{i\hbar t}\sqrt{\bar{\bar{g}}^{\mu\nu}} \otimes \overleftrightarrow{m\psi}\bar{\psi} - \partial^2 \triangle' \rangle' \langle \otimes_{\mathfrak{R}}^{\mathfrak{G}} \left| d^4x/\partial\mathcal{R} \right\rangle' \int \left\| \widetilde{\frac{\partial\phi'_{\sigma\rho}}{\partial\phi^{\dagger}_{\sigma\rho}}} \right\| -}{\left\langle \frac{\partial\phi^{\star}_{\sigma\rho}}{\partial^{\mathfrak{H}}_{\mu\nu}} \right| \partial\uparrow/\partial t \setminus \partial\downarrow/\partial t \partial^2 \square \left| {}^{\blacksquare^{\circlearrowleft}}_{\blacksquare_{\circlearrowright}} \partial^2\varphi/\partial\psi \blacksquare \right\rangle \Lambda_\nu^\mu \sum_{\substack{0\leq \imath \leq m \\ 0<\jmath<n}} P(\imath,\jmath) \prod_{k=1}^n A_k \bigcup_{n=1}^m (X_n \cap Y_n) \bigcup_{n=1}^m (X_n \cap Y_n) \circledcirc \Lambda_\nu^\mu \odot \Gamma_\nu^\mu},$$

respecto de una partícula pesada ρ, sea oscura o blanca (partícula estrella), según corresponda, a propósito de la criticidad de su masa y/o energía $\langle 0 | \sum_\delta \partial\mathfrak{m}/\partial\mathfrak{e} \rangle$ o de su interacción con un gravitón o un gravitino, según corresponda, en coordenadas $\langle \rho^\mu \rho^\nu \rho^\sigma \rho^\varrho \rangle$, esto último, lo que ocurre por permeabilización del campo gravitónico o supergravitónico en $\blacksquare = \int \langle \partial\mathfrak{G}/\partial\mathfrak{S}\mathfrak{G} \rangle$, lo que corresponde al espacio – cuántico deformado en $\mathfrak{C}_{\mathfrak{H}\mathfrak{R}} = \langle \sum_{\boxdot}^{\sigma\rho} \mathcal{R}_\nu^{\mu\dagger} | \otimes \mathcal{H}_\mu^{\nu\star} \rangle$ lo que en dimensiones $\mathbb{R}^\eta$, representa, gravedad o supergravedad cuánticas por curvatura o supercurvatura del espacio - tiempo cuántico multidimensional.

Palabras Clave: Supergravedad cuántica, gravedad cuántica, partícula oscura, partícula estrella, hiperpartículas, suprapartículas, teoría cuántica de campos relativistas, problema del milenio de Yang – Mills y la brecha de masa, partículas ligeras, curvatura, supercurvatura, multidimensiones, agujeros cuánticos.

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ABSTACT.

In this work, composed of ten volumes, we will address essential aspects of the Quantum Theory of Relativistic Fields (TCCR), with the purpose of optimizing the calculations exposed in previous works but above all, positioning the aforementioned theory as an alternative solution to the Yang-Mills millennium problem and the mass gap. The essential idea is the same, all quantum space-time, that is, every quantum field, is curved and that deformation occurs due to quantum gravity and supergravity, as the case may be, caused by dark particles or stars, when interacting with a gravitonic or supergravitonic field, as appropriate, or in relation to the criticality of its center of mass and/or energy which affects their spin, velocity and angular momentum and therefore, their orbital trajectories. Therefore, the TCCR is not an attempt to quantize gravity, but to introduce gravity, as the principle of least action of a quantum system and its fundamental states.

The metrics remain the same, i.e., for a curved or geometrically warped quantum field, the Lagrangian/Hamiltonian density is equal to:

$$\mathcal{LH}_{curvature} = \frac{\langle \int \hat{e}^{i\hbar t}\sqrt{\bar{\bar{g}}^{\mu\nu}} \otimes \overleftrightarrow{m\psi}\bar{\psi} - \partial^2 \triangle' \rangle' \langle \otimes_{\mathfrak{R}}^{\mathfrak{G}} \left| d^4x/\partial\mathcal{R} \right\rangle' \int \left\| \widetilde{\frac{\partial\phi'_{\sigma\rho}}{\partial\phi^{\dagger}_{\sigma\rho}}} \right\| -}{\left\langle \frac{\partial\phi^{\star}_{\sigma\rho}}{\partial^{\mathfrak{H}}_{\mu\nu}} \right| \partial\uparrow/\partial t \setminus \partial\downarrow/\partial t\partial^2\square \left| {\blacksquare^{\circlearrowleft}}_{\blacksquare_{\circlearrowright}} \partial^2\varphi/\partial\psi\blacksquare \right\rangle \Lambda^{\mu}_{\nu} \sum_{\substack{0\le\imath\le m\\ 0<\jmath<n}} P(\imath,\jmath) \prod_{k=1}^{n} A_k \bigcup_{n=1}^{m}(X_n\cap Y_n)\bigcup_{n=1}^{m}(X_n\cap Y_n) \circledcirc \Lambda^{\mu}_{\nu} \odot \Gamma^{\mu}_{\nu}}$$

with respect to a heavy particle ρ, whether dark or white (star particle), as appropriate, regarding the criticality of its mass and/or energy $\langle 0|\sum_{\delta}\partial\mathfrak{m}/\partial\mathfrak{e}\rangle$ or its interaction with a graviton or a gravitin, as appropriate, in coordinates $\langle \rho^{\mu}\rho^{\nu}\rho^{\sigma}\rho^{\varrho}\rangle$, the latter, which occurs by permeabilization of the gravitonic or supergravitonic field in $\blacksquare = \int\langle \partial\mathfrak{G}/\partial\mathfrak{S}\mathfrak{G}\rangle$, what corresponds to the space – quantum deformed in $\mathfrak{C}_{\mathfrak{H}\mathfrak{R}} = \langle \sum_{\boxdot}^{\sigma\rho} \mathcal{R}^{\mu\dagger}_{\nu} \middle| \otimes \mathcal{H}^{\nu\star}_{\mu}\rangle$ the which in dimensions $\mathbb{R}^{\eta}$, represents, quantum gravity or supergravity by curvature or supercurvature of multidimensional quantum space-time.

Keywords: Quantum supergravity, quantum gravity, dark particle, star particle, quantum theory of relativistic fields.

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INTRODUCCIÓN

En este punto, es indispensable establecer las bases teóricas que conforman la Teoría Cuántica de Campos Relativistas (TCCR) y que se encuentran desarrolladas en trabajos previos. Por tanto, estos son los puntos más relevantes.

1. Todo campo cuántico, es curvo por acción inmediata de una partícula cuya masa y/o energía alcanzan el mayor grado de criticidad. En este caso, la gravedad es endógena o implícita, es decir, una cualidad propia de la partícula interactuante.

2. Siguiendo lo dicho, en el numeral que antecede, las partículas se dividen en:

2.1. Partículas Supermasivas (Tipo IA): Son aquellas, cuyo centro de masa/energía en unidades de Planck dados en $\mathcal{M}_p = \sqrt{\frac{\hbar c}{\mathfrak{G}}} \approx 2{,}18 \times 10^{-8}\ kg$ (masa) y $E_p = \frac{\hbar}{t_P}$, $E_p = m_P c^2$, $E_p = \sqrt{\frac{\hbar c^5}{G}} \approx 1.956 \times 10^9\ \mathrm{J} \approx 1.22 \times 10^{19}\mathrm{GeV} \approx 0.5433\mathrm{MWh}\sqrt{\frac{\hbar c^5}{8\pi G}} \approx 0.390 \times 10^9\ \mathrm{J} \approx 2.43 \times 10^{18}\mathrm{GeV}$ (energía $\approx 10^{-120}$), alcanza el mayor grado de criticidad, deformando el espacio – tiempo cuántico, lo que afecta el estado fundamental de los orbitales (espín, momentum, velocidad, trayectorias, etc), desplegados por las partículas repercutidas. Esta partícula también se la denomina "partícula oscura", en la medida en que, su centro de energía/masa, es oscuro. Principal candidata para explicar la materia oscura, en la medida en que, la gravedad converge en su centro, absorbiendo energía y materia.

2.2. Partículas Blancas (Tipo IB): Son aquellas, cuyo centro de masa/ energía en unidades de Planck, alcanza el mayor grado de criticidad, deformando el espacio – tiempo cuántico, lo que afecta el estado fundamental de los orbitales (espín, momentum, velocidad, trayectorias, etc), desplegados por las partículas repercutidas. Esta partícula también se la denomina "partícula estrella", en la medida en que, su centro de masa/energía es extremadamente denso, superando la masa, temperatura y energía de Planck, en $\mathcal{M}_p = \sqrt{\frac{\hbar c}{\mathfrak{G}}} \approx 2{,}18 \times 10^{-8}\ kg$ (masa), $\mathcal{M}_p = \sqrt{\frac{\hbar c^5}{\mathfrak{G} k^2}}$ $T_P \approx 1.416784(16) \times 10^{32}\mathrm{K}$ (temperatura) y $E_p = \frac{\hbar}{t_P}$, $E_p = m_P c^2$, $E_p = \sqrt{\frac{\hbar c^5}{G}} \approx 1.956 \times 10^9\ \mathrm{J} \approx 1.22 \times 10^{19}\mathrm{GeV} \approx 0.5433\mathrm{MWh}\sqrt{\frac{\hbar c^5}{8\pi G}} \approx 0.390 \times 10^9\ \mathrm{J} \approx 2.43 \times 10^{18}\mathrm{GeV}$. También se la denomina "partícula estrella".

2.3. Hiperpartículas (Tipo IIA): Son aquellas, cuyo centro de masa/energía es extremadamente bajo, en unidades de Planck, más sin embargo, son capaces de igualar o superar la velocidad de la luz.

2.4. Suprapartículas (Tipo IIB): Son aquellas, cuyo centro de masa/energía es el equivalente al de una partícula oscura o blanca, más sin embargo, éstas, a diferencia de las referidas en los numerales 2.1 y 2.2, ésta igual o supera la velocidad de la luz.

3. Agujero negro cuántico: Fenómeno que ocurre en un espacio cuántico de Sitter, esto es, cuando una partícula oscura colisiona con otra o en su defecto, cuando una partícula blanca colisiona con otra o cuando una partícula blanca y una partícula oscura colisionan entre sí. Los agujeros negros cuánticos, también se forman por el colapso (por compresión gravitacional) o por la aniquilación (por interacción) de una partícula oscura o de una partícula blanca. Lo primero, ocurre cuando se atraen mutuamente por gravedad en tanto que lo segundo, ocurre cuando su centro de masa/energía alcanza el mayor grado de criticidad posible. En el centro del agujero negro cuántico, se encuentra la masa de la partícula aniquilada o comprimida, la que comporta condiciones gravitatorias extremas. Ahí es donde radica la singularidad de un agujero negro cuántico. La información que ingresa al agujero negro cuántico, no se destruye, muy al contrario, se transforma en materia y energía, las mismas que son repulsadas por el agujero negro cuántico blanco que se encuentra en el otro extremo del agujero cuántico de gusano. Por tanto, la materia y energía atrapada por el agujero negro cuántico, se convierte en materia y energía oscuras interferidas por gravedad extrema.

4. Agujero cuántico de gusano: Túnel cuántico por el cual, se conectan un agujero negro cuántico y un agujero blanco cuántico. A través de este túnel, por teletransportación cuántica, la información es procesada y convertida en materia y energía, todo esto, en un espacio de Sitter.

5. Agujero blanco cuántico: Fenómeno que ocurre en un espacio cuántico de Sitter, volviéndose la región de salida o repulsión de materia y energía, a propósito de lo que devora el agujero negro cuántico y de lo que procesa el canal cuántico de gusano. Lo que repulsa el agujero blanco cuántico, es materia y energía procesadas.

6. Espacio – tiempo cuántico: Entiéndase por espacio – tiempo cuántico, al campo en sí mismo, cuya Longitud de Planck, es superior a $\ell_{\mathcal{P}} = \sqrt{\frac{\hbar \mathcal{G}}{c^3}} \approx 1{,}616199(97) \times 10^{-35}\, metros$**.** La métrica es la curvatura escalar de Ricci, así: $\mathcal{R} = \sum_{\alpha,\beta=0}^{3} g^{\alpha\beta}\mathcal{R}_{\alpha\beta} \approx o\left(\mathcal{L}_p^{-2}\right) \approx 3{,}828 \cdot 10^{69} m^{-2}$. Ahora bien, el espacio – tiempo cuántico puede ser, bien de Sitter (dS) o bien, anti de Sitter (AdS). En el primero, se forma la curvatura cuántica y sus subniveles, subespacios o subcapas, en tanto que en el segundo, se forman los agujeros cuánticos y las multidimensiones.

7. Todo campo cuántico, es curvo por acción inmediata de la gravedad, esto a propósito de la existencia (Modelo – Higgs):

7.1. De un campo gravitónico, es decir, cuando una partícula cualquiera, interactúa con un gravitón, lo que supone la permeabilidad del campo cuántico, por un campo gravitónico que transfiere gravedad al campo primario, curvándolo.

7.2. De un campo supergravitónico, es decir, cuando una partícula cualquiera, interactúa con un gravitino o supergravitón, lo que supone la permeabilidad del campo cuántico, por un campo gravitónico que transfiere gravedad al campo primario, deformándolo.

7.3. Lo referido en este numeral se denomina gravedad exógena.

8. La gravedad cuántica, sea endógena o exógena comporta la curvatura del espacio – tiempo cuántico, en tanto que, la supergravedad cuántica, sea endógena o exógena, comporta la deformación (supercurvatura) del espacio – tiempo cuántico, formándose pliegues multidimensionales (en alta configuración – membranas dimensionales) en rango superior a $\mathbb{R}^4 - AdS$. Cabe indicar que las membranas dimensionales, se dividen en TIPO I y TIPO II respectivamente, la primera a propósito de la curvatura del campo en gravedad cuántica y la segunda, la deformación del campo en supergravedad cuántica, todo esto, lo cual también depende de la naturaleza de la gravedad que interfiere, es decir, si es exógena o endógena, lo que llamaríamos membranas dimensionales tipo IA, IB, IIA y IIB respectivamente, las cuales, pueden contener dimensiones y subdimensiones infinitas, en relación a las interacciones de la partícula que provoca de la deformación del espacio – tiempo cuántico. Esto es lo que llamamos supersimetrías de gauge en dimensiones altas a $\mathbb{R}^4$, es decir, cuando estamos ante membranas dimensionales tipo IA, IB y IIB, según sea el caso en tanto que, las membranas dimensionales del tipo IIA, contienen dimensiones infinitas en $\mathbb{R}^4 - dS$.

9. Cuando una partícula colisiona con otra y se aniquilan o cuando la partícula pesada colapsa por compresión, la extinción provoca ondas cuánticas que se desplazan en longitud sobre el campo cuántico deformado el mismo que, es superfluido.

10. El puente ER, en esta teoría, explica la superposición y el entrelazamiento cuánticos en sentido estricto, en un espacio AdS.

11. Los enunciados antes referidos, aplican a la antimateria, es decir, a la región de antipartículas.

12. La brecha de masa, provoca la curvatura del espacio – tiempo cuántico pero no lo deforma por completo, pues este fenómeno, no ocurre con una partícula deformante, sino en partículas ligeras como las hiperpartículas, esto en la medida en que, no registran estado de vacío.

13. Adicionalmente, es importante, establecer las siguientes reglas:

13.1. La gravedad cuántica relativista, ocurre concretamente en un espacio cuántico de Sitter, en el que se pueden formar subdimensiones o subespacios dentro del límite de $\mathbb{R}^4$.

13.2. La supergravedad cuántica relativista, ocurre concretamente en un espacio cuántico anti de Sitter, en el que se pueden formar hiperespacios o dimensiones más altas, superiores a $\mathbb{R}^4$.

13.3. Las partículas propuestas, viajan en gravedad cuántica más, interactúan en supergravedad cuántica por permeabilización.

13.4. Cualquier partícula, de las aquí propuestas, se puede convertir en otra, por aniquilación, siguiendo los diagramas de Feynman.

13.5. Las dimensiones en alta configuración así como las de ensamble, son infinitas.

13.6. La materia y energía oscuras, están formadas esencialmente por partículas aniquiladas o colapsadas por gravedad. En consecuencia, es la criticidad de la masa la que las vuelve compatibles.

13.7. Los agujeros cuánticos, absorben partículas ligeras y pesadas, sin distinción, lo que explica la expansión del universo por acción gravitacional en la materia.

13.8. Las partículas aquí propuestas, son susceptibles de enganche, como ocurre con un diquark.

13.9. En esta teoría, se incorpora el concepto de cuerda, pero en un espacio cuántico anti de Sitter.

13.10. Las partículas pesadas, cuando se desplazan de un punto a otro en forma infinita hasta su aniquilación o colapso, lo hacen por medio de gravedad, deformando, en el caso de las partículas blancas y las hiperpartículas, un espacio de Sitter, creando capas dimensionales en límite de $\mathbb{R}^4$ en tanto que, la partícula oscura, crea capas dimensiones en alta configuración a $\mathbb{R}^4$ en un espacio anti de Sitter.

13.11. La hiperpartícula es la única en este modelo, que no tiene masa, es por ello que puede viajar a la velocidad de la luz.

13.12. La suprapartícula es por excepción, un caso de mutación por aniquilación, en la medida en que, pese a tratarse de una partícula pesada, con un centro de masa/energía extremadamente crítico y denso, es capaz de viajar a la velocidad de la luz. La suprapartícula solamente existe por aniquilación en entre dos o más partículas pesadas, quedando excluidas las partículas ligeras. Adicionalmente, la suprapartícula, tiene la capacidad de desplazarse entre dimensiones dS y AdS, lo que esta teoría denomina dimensiones en $\mathbb{R}^\eta$. En consecuencia, las dimensiones por gravedad y supergravedad, pueden intersecarse por gravedad. En este punto, es pertinente para efectos de ejemplificar, citar el diagrama de Penrose expandido al infinito.

13.13. Los campos de las partículas ligeras, son deformados por acción a distancia, debido a las interacciones de una partícula pesada, esto es, por gravedad.

13.14. Solamente las partículas pesadas pueden deformar el campo propio y de las partículas ligeras, por acción de la gravedad que se desprende de su centro de masa/energía extremo. En consecuencia, la gravedad endógena, se materializa por impermeabilización del campo de Braut – Englert – Higgs respecto de la partícula pesada. El bosón de Higgs es el que transfiere la masa, a las partículas pesadas, aniquilándose con éstas.

13.15. La gravedad exógena, se vuelve posible, por permeabilización de un campo cuántico arbitrario, lo que, como ha quedado explicado en esta teoría, funciona como un mecanismo de Higgs.

13.16. El colapso de una partícula pesada, ocurre por la expansión de su centro de masa/energía, debido a la gravedad interferente, ditalación que es comprimida en contrario, por los límites del campo de la partícula de que se trate, lo que provoca, la deformación del plano cuántico e incluso la formación de agujeros cuánticos, según la criticidad de los valores de masa/energía involucrados.

13.17. La fusión de campos cuánticos, es posible, por acción de la gravedad entre ambos, lo que vuelve posible, su aniquilación.

13.18. Las ondas en un plano cuántico, no solamente se forman por la aniquilación o colpaso de una partícula pesada, sino también, cuando viaja de un punto a otro.

13.19. Las partículas ligeras, crean gravedad mínima a propósito de su centro de masa/energía, la cual sin embargo, es imperceptible aunque superior a cero, pues, contribuye a la aniquilación con otro campo más pesado.

13.20. La gravedad endógena, se debe a que, el campo de Higgs, y por ende, el bosón de Higgs, no solamente transfiere masa a las partículas pesadas y ligeras, con excepción de la hiperpartícula, sino que también, le dota de gravedad, a propósito de la masa transferida.

13.21. Esta teoría es estrictamente de gauge.

RESULTADOS Y DISCUSIÓN

Suponemos que, en un mapa cuántico de Einstein – Hilbert, una partícula deformante $\alpha\beta\gamma\delta$ se desplaza en el espacio cuántico, en el que interactúa, deformando el plano por gravedad, y por ende, creando, bien dimensiones altas en $\mathbb{R}^4 - AdS$ por supercurvatura (supergravedad cuántica) o bien, dimensiones en $\mathbb{R}^4 - ds$ por curvatura, esto es, en condiciones de gravedad. Para estos efectos, una partícula deformante debe colapsar por compresión gravitacional, aniquilarse cuando interactúa con otras más inestables o con otra partícula pesada, o por permeabilidad del campo gravitónico o supergravitónico en el espacio cuántico curvo, esto último, lo cual ocurre, cuando una partícula pesada interactúa con el gravitón o el gravitino (supergravitino), según sea el caso. Por tanto, la gravedad actúa a nivel cuántico, sea por aniquilación, compresión, ésta última gravitacional o por permeabilización. Suponemos en simultáneo, que una vez, causada la aniquilación o compresión por gravedad, de una partícula pesada o cuando ocurre la permeabilización, se produce, bien la curvatura cuántica, cuya métrica es el tensor es de Riemann – Ricci – Einstein, incluyendo el flujo de la simetría, o en su defecto, la supercurvatura de Weyl, cuya métrica es la de Chern-Simons-Nambu-Goto para supergravedad. La primera, produce subcampos que son subdimensiones de un mismo plano de Sitter (dS), en tanto que la segunda, produce campos en dualidad holográfica, que son dimensiones altas al plano cuatridimensional en un espacio anti de Sitter (AdS). En este sentido, el campo pasa a ser no homeomorfo, difeomorfo e isométrico, afectando los orbitales de las partículas cuyo centro de masa/energía es inferior en unidades de Planck (partículas ligeras) en relación a la partícula que deforma el plano. La interacción y/o aniquilación de estas partículas deformantes, provoca un agujero negro cuántico (con excepción de las interacciones dadas por las hiperpartículas tipo IIA), formado por materia y energía oscuras, cuya naturaleza es fermiónica/bosónica, esto a propósito de que, la partícula aniquilada o comprimida, engendra materia y energía oscuras, lo que no ocurre en escenarios de permeabilización gravitónica más sí, en escenarios de permeabilización supergravitónica. El agujero cuántico de salida, es blanco, por ende, repulsivo de materia y energía transformada por la gravedad, a través del tracto Einstein – Rosen. Cuando la materia y la energía son transformadas en oscuras, por la gravedad, éstas se comprimen hasta un punto de no retorno/densidad supermasiva, causando dos especies de singularidad inherentes al agujero negro cuántico, siendo éstas, primaria y secundaria, la primera en la que la gravedad es extrema y deforma la materia y la energía, fundiéndose con el núcleo del agujero negro cuántico (que contiene la partícula muerta) y la segunda, en la que la gravedad transforma la materia y la energía, desplazándola a través del tracto Einstein – Rosen y expulsándola a través de un agujero blanco cuántico. Esto es lo que ocurre en escenarios de entrelazamiento y túneles cuánticos supermasivos en los que, la partícula deformante genera gravedad extrema. Llámese también, gravedad absoluta. Queda claro entonces, que el sistema cuántico de agujeros, no se produce en condiciones de gravedad relativa, esto es, cuando ocurre únicamente la curvatura cuántica por gravedad moderada, lo que sucede por ejemplo, con las interacciones dadas por las hiperpartículas tipo IIA o en el caso de la brecha de masa de las partículas ligeras respecto del estado de vacío.

Dicho lo anterior, es que, propongo una posible alternativa de solución al problema del milenio de Yang – Mills y la brecha de masa, a partir de la Teoría Cuántica de Campos Relativistas, la cual se constituye además, como un intento por reconciliar la relatividad general y la mecánica cuántica.

A partir de aquí, sugerimos los cálculos de instantones (para regular la brecha de masa y la densidad de energía por carga), osciladores, propagadores, operadores, mapas, coordenadas vectoriales, orbitales, correladores, propulsores, tensores de stress por curvatura, torsión, escalares, spinors, potenciadores, simetrías y supersimetrías de calibre abelianas y no abelianas en relación a las partículas pesadas y sus interacciones con el espacio cuántico deformado, en tanto que respecto de éste último, los cálculos están vinculados a su geometría e hipergeometría (análisis cohomológico), incluyendo los agujeros cuánticos, no sin antes aclarar, que las demostraciones matemáticas contenidas en trabajos anteriores, son interdependientes a éste manuscrito y sus diez volúmenes.

Aclarado lo anterior, pasamos a precisar que el Modelo aquí referido, se divide en:

1. Supergravedad cuántica en SYM (Super Yang – Mills).

2. Gravedad cuántica en YM (Yang – Mills).

3. Agujeros cuánticos en YM (Yang – Mills).

4. Modelo de Unificación.

Las métricas usadas son, entre otras:

- Espacios de Einstein – Hilbert.
- Métrica de Chern – Simons.
- Métrica de Kaluza – Klein.
- Métrica de Nambu – Goto.
- Métrica de Feynman – Wheeler.
- Métrica de Born – Oppenheimer.
- Métrica de Hartree – Fock.
- Métrica de Yang – Mills.
- Métrica de Kerr – Newman.
- Espacios de Sitter y anti de Sitter.
- Espacios de Riemann – Perelman – Poincaré.
- Tensores y flujo de Ricci.
- Métrica de Green.
- Métrica de Goldstone.
- Métrica de Brout – Englert – Higgs.
- Métrica de Schwinger – Dyson.
- Métrica de Yukawa.
- Métrica de Von Neumann
- Métrica de Friedman.

$$\bar{Q}_A^{A'}\mathcal{R}^{(k)}_{n_1,\ldots,n_m}=\frac{1}{4}\Gamma_{\text{cusp}}\sum_r\left[\iint\ \left[d^{2|3}\mathcal{Z}_{n_r+1}\right]_A^{A'}\left[\mathcal{R}^{(k+1)}_{n_1,\ldots,n_r+1,\ldots,n_m}-\mathcal{R}^{(k)}_{n_1,\ldots,n_m}\mathcal{R}^{(1,0)}_{n_r+1}\right]+\text{cyc}_r\right]$$

$$S=\frac{1}{g^2_{\text{YM}}}\int\ d^4x\left[-\frac{1}{2}\text{tr}F^{\mu\nu}F_{\mu\nu}+\cdots\right]$$

$$g^2=\frac{g^2_{\text{YM}}N}{16\pi^2}.$$

$$\mathcal{L}(C)=\frac{1}{N}\text{tr}\mathcal{P}\exp\ i\oiiint_C dx^\mu A_\mu$$

$$W_n=\langle\mathcal{L}(C)\rangle=\left[\prod_{i=1}^n\ D_i\right]F_nR_n$$

$$W_{n_1,\ldots,n_m}=\langle\mathcal{L}(C_1)\ldots\mathcal{L}(C_m)\rangle=\prod_{r=1}^m\left[\left[\prod_{i=1_r}^{n_r}\ D_{i_r}\right]F_{n_r}\right]R_{n_1,\ldots,n_m}$$

$$W^{\text{conn}}_{n_1,n_2}=\prod_{r=1}^2\left[\left[\prod_{i_r=1}^{n_r}\ D_{i_r}\right]F_{n_r}\right]R^{\text{conn}}_{n_1,n_2}$$

$$S_1(\mathcal{A})=\frac{i}{2\pi}\alpha\int\ D^{3|4}\mathcal{Z}\wedge\text{tr}\left(\mathcal{A}\wedge\bar{\partial}\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

$$S_2(\mathcal{A})=g^2_{\text{YM}}\alpha^2\int\ d^{4|8}X\log\ \det(\bar{\partial}+\mathcal{A})_X$$

$$S_1(\mathcal{A})=\frac{iC_F}{4\pi^3}\int\ D^{3|4}\mathcal{Z}\wedge\text{tr}\left(\mathcal{A}\wedge\bar{\partial}\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

$$S_2(\mathcal{A})=\frac{4g^2C_F^2}{\pi^2N}\int\ d^{4||8}X\log\ \det(\bar{\partial}+\mathcal{A})_X$$

$$S_2(\mathcal{A})=-\frac{4g^2C_F^2}{\pi^2N}\int\ d^{4|8}X\sum_{r=2}^{\infty}\frac{1}{r}\text{tr}\left(-\bar{\partial}_X^{-1}\mathcal{A}\right)^r$$

$$\left(\bar{\partial}_X^{-1}\omega\right)(s)=\int_X G(s,s')\wedge\omega(s')$$

$$G(s,s')=-\frac{1}{2\pi i}\frac{ds'}{(s-s')}$$

$$\text{tr}\left(\bar{\partial}^{-1}\mathcal{A}\right)^r=\text{tr}\left\{\int_{X^r}G(s_r,s_1)\wedge\mathcal{A}\big(Z(s_1)\big)\ldots G(s_{n-1},s_r)\wedge\mathcal{A}\big(Z(s_r)\big)\right\}$$

$$\mathcal{L}(C) = \frac{1}{N}\mathrm{tr}\mathcal{P}\prod_{i=1}^{n}\sum_{l_i=0}^{\infty}\left(-\bar{\partial}_i^{-1}a(Z_i(0))\right)^{l_i}$$

$$\mathcal{L}(C) = \frac{1}{N}\mathrm{tr}\mathcal{P}\prod_{i=1}^{n}\sum_{l_i=0}^{\infty}\left(-\bar{\partial}_i^{-1}\mathcal{A}(Z_i(0))\right)^{l_i}$$

$$\begin{aligned}\left\langle \mathcal{A}_a(Z_i(s))\mathcal{A}_b(Z_j(t))\right\rangle^{\mathrm{CS}} &= -\frac{4\pi^2}{C_F}\delta_{ab}\Delta_*(Z_i(s),Z_j(t)) \\ &= -\frac{4\pi^2}{C_F}\delta_{ab}\bar{\delta}^{2|4}(Z_*,Z_i(s),Z_j(t)) \\ &= -\frac{4\pi^2}{C_F}\delta_{ab}\int \frac{D^2c}{c_1c_2c_3}\bar{\delta}^{4|4}(c_1Z_* + c_2Z_i(s) + c_3Z_j(t))\end{aligned}$$

$$\mathcal{W}^{\mathrm{CS}}_{n_1,\dots,n_m} = \mathcal{W}^{(0),\mathrm{CS}}_{n_1,\dots,n_m} + \mathcal{W}^{(1),\mathrm{CS}}_{n_1,\dots,n_m} + \cdots$$

$$\langle \mathcal{L}(C_1)\mathcal{L}(C_2)\dots\mathcal{L}(C_m)\rangle = \int [d\mathcal{A}]e^{-(S_1+S_2)}\mathcal{L}(C_1)\mathcal{L}(C_2)\dots\mathcal{L}(C_m)$$

$$\langle \mathcal{L}(C_1)\mathcal{L}(C_2)\dots\mathcal{L}(C_m)\rangle = \int [d\mathcal{A}]e^{-S_1}\mathcal{L}(C)\left[1 + \frac{g_{\mathrm{YM}}^2C_F^2}{4\pi^4}\int d^{4|8}X\sum_{r=2}^{\infty}\frac{1}{r}\mathrm{tr}\left(-\bar{\partial}_X^{-1}\mathcal{A}\right)^r + O\left(g_{\mathrm{YM}}^4\right)\right]$$

$$\frac{4g^2C_F^2}{\pi^2N}\sum_{r=2}^{\infty}\frac{1}{r}\int d^{4|8}X\left\langle \mathcal{L}(C_1)\mathcal{L}(C_2)\dots\mathcal{L}(C_m)\mathrm{tr}\left(-\bar{\partial}_X^{-1}\mathcal{A}\right)^r\right\rangle^{\mathrm{CS}} + O(g^4)$$

$$\frac{4g^2C_F^2}{N}\int \frac{d^{4|8}x_{AB}}{\pi^2}\left(\prod_{p=1}^{n}\int ds_{x,p}\right)\frac{1}{(s_{x,1}-s_{x,2})(s_{x,2}-s_{x,3})\dots(s_{x,n}-s_{x,1})}$$

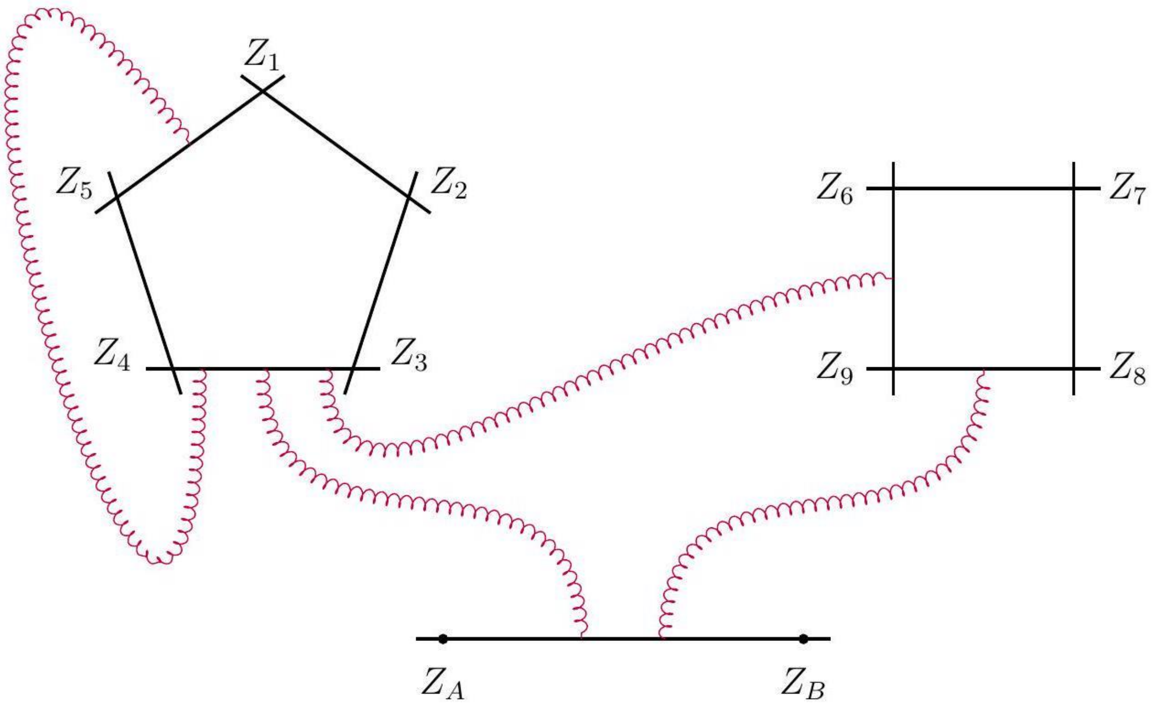
$$\int \frac{ds_{i,1}}{s_{i,1}}\int \frac{ds_{i,2}}{s_{i,2}-s_{i,1}}\int \frac{ds_{i,3}}{s_{i,3}-s_{i,2}}\dots\int \frac{ds_{i,n}}{s_{i,n}-s_{i,n-1}}$$

$$\frac{1}{C_F}\int \frac{D^4a_{(i,j)}}{a_{(i,j),1}a_{(i,j),2}a_{(i,j),3}}\bar{\delta}^{4|8}\left(a_{(i,j),1}\mathcal{Z}_* + a_{(i,j),2}s_{i,m_1}\mathcal{Z}_{i-1} + a_{(i,j),2}\mathcal{Z}_i + a_{(i,j),3}s_{j,m_2}\mathcal{Z}_{j-1} + a_{(i,j),3}\mathcal{Z}_j\right)$$

$$\begin{aligned}\frac{4g^2}{N^3C_F^2}\mathrm{tr}(t_at_bt_ct_c)\mathrm{tr}(t_at_d)\mathrm{tr}(t_bt_d)\int &\frac{d^{4|8}x_{AB}}{\pi^2}\frac{D^4a}{a_1a_2a_3}\frac{D^4b}{b_1b_2b_3}\frac{D^4c}{c_1c_2c_3}\frac{D^4d}{d_1d_2d_3} \\ &\times\frac{d\rho_1d\rho_2}{(\rho_1-\rho_2)(\rho_2-\rho_1)}\frac{ds_1}{s_1}\frac{ds_2}{s_2-s_1}\frac{ds_3}{s_3-s_2}\frac{du}{u}\frac{dv}{v}\frac{dw}{w} \\ &\times\bar{\delta}^{4|4}(a_1\mathcal{Z}_* + a_2s_1\mathcal{Z}_3 + a_2\mathcal{Z}_4 + a_3u\mathcal{Z}_5 + a_3\mathcal{Z}_1) \\ &\times\bar{\delta}^{4|4}(b_1\mathcal{Z}_* + b_2s_2\mathcal{Z}_3 + b_2\mathcal{Z}_4 + b_3\rho_2\mathcal{Z}_A + b_3\mathcal{Z}_B) \\ &\times\bar{\delta}^{4|4}(c_1\mathcal{Z}_* + c_2s_3\mathcal{Z}_3 + c_2\mathcal{Z}_4 + vc_3\mathcal{Z}_9 + c_3\mathcal{Z}_6) \\ &\times\bar{\delta}^{4|4}(d_1\mathcal{Z}_* + d_2\rho_1\mathcal{Z}_A + d_2\mathcal{Z}_B + d_3w\mathcal{Z}_8 + d_3\mathcal{Z}_9)\end{aligned}$$

$$\frac{g^2}{N^2}\int \frac{d^{4|8}}{\pi^2}[*,5,1,3,4][*,3,\hat{4}_1,A,\hat{B}_9][*,3,\hat{4}_B,9,6][*,A,\hat{B}_4,8,9]$$

$$\mathcal{Z}_{i,j} = (i-1i)\cap(j-1j*) = \mathcal{Z}_{i-1}\langle ij-1j*\rangle - \mathcal{Z}_i\langle i-1j-1j*\rangle$$



$$\mathcal{R}_{n_1,\ldots,n_m}^{(k,1),\text{conn}} = \mathcal{W}_{n_1,\ldots,n_m}^{(k,1),\text{conn}} - \mathcal{W}_{n_1,\ldots,n_m}^{(k,0),\text{conn}} \sum_{r=1}^{m} \mathcal{W}_{n_r}^{(0,1)}$$

$$\mathcal{Z}_2 = \mathcal{Z}_1 + \epsilon\left(\mathcal{Z}_{r_1} + a\mathcal{Z}_{r_2} + b\mathcal{Z}_{r_3}\right)$$
$$\mathcal{Z}_3 = \mathcal{Z}_1 + \epsilon\left(\mathcal{Z}_{r_1} + c\mathcal{Z}_{r_2} + d\mathcal{Z}_{r_3}\right)$$

$$\chi_{(12)(23)(34)(45),1}^{1m} = \int \frac{d^4x_{AB}}{\pi^2} \frac{\langle AB(123)\cap(345)\rangle\langle X24\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle ABX\rangle}$$

$$\chi_{(12)(23)(34)(45),2}^{1m} = \int \frac{d^4x_{AB}}{\pi^2} \frac{\langle AB24\rangle\langle X(123)\cap(345)\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle ABX\rangle}$$

$$\chi_{(12)(23)(34)(41),1}^{0m} = \int \frac{d^4x_{AB}}{\pi^2} \frac{\langle 1234\rangle\langle X24\rangle\langle AB13\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle\langle ABX\rangle}$$

$$\chi_{(12)(23)(34)(41),2}^{0m} = \int \frac{d^4x_{AB}}{\pi^2} \frac{\langle 1234\rangle\langle X13\rangle\langle AB24\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle\langle ABX\rangle}$$

$$-3\text{Li}_2(1) - \frac{1}{2}\log\left(\frac{\langle 12X\rangle\langle 34X\rangle}{\langle 23X\rangle\langle 14X\rangle}\right)^2$$

$$\mathcal{I}_a = \int \frac{d^4x_{AB}}{\pi^2} \frac{\langle a-1aa+1a+2\rangle\langle Xa+1a\rangle}{\langle ABa-1a\rangle\langle ABa+1a\rangle\langle ABa+1a+2\rangle\langle ABX\rangle}$$

$$P_{ij} = \int \frac{d^4x_{AB}}{\pi^2} \frac{\langle AB(i-1ii+1)\cap(j-1jj+1)\rangle\langle ABX\rangle}{\langle i-1iAB\rangle\langle ii+1AB\rangle\langle j-1jAB\rangle\langle jj+1AB\rangle\langle ABX\rangle}$$

$$-\sum_{i<j} P_{ij}$$

$$-\sum_{i,j} P_{ij}$$

$$P_{23} = \frac{\langle 1234\rangle\langle 23X\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle ABX\rangle} = -\mathcal{I}_1$$

$$[a,b,c,d,e] = \frac{\bar{\delta}^{0|4}(\langle bcde\rangle\chi_a + cyc)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}$$

$$f(Z)\prod_{i=1}^{k+2} \bar{\delta}^{4|4}\left(\sum_{j=1}^{5} r_{i,j}(Z)\mathcal{Z}_{i,j}\right)$$

$$\mathcal{R}^{(k,1),\mathrm{c}}_{n_1,\ldots,n_m} = \int d^4x_{AB}\sum_P \left(C^k_{P,1}\chi_{P,1}(x_{AB}) + C^k_{P,2}\chi_{P,2}(x_{AB})\right)$$

$$\mathcal{R}^{(k,1),\mathrm{c}}_{n_1,\ldots,n_m} = \sum_P \left(C^k_{P,1} + C^k_{P,2}\right)\bar{\chi}_P = \sum_P C^k_P\bar{\chi}_P,$$

$$\bar{\chi}_P = \int d^4x_{AB}\chi_{P,1}(x_{AB}) = \int d^4x_{AB}\chi_{P,2}(x_{AB})$$

$$\mathcal{W}^{(1,0)}_I = \sum_{(i<j)\in I} [*, i-1, i, j-1, j]$$

$$\mathcal{W}^{(1,0),\mathrm{c}}_{I,J} = \sum_{i\in I}\sum_{j\in J} [*, i-1, i, j-1, j]$$

$$\mathcal{W}^{(2,0),\mathrm{c}}_{I,J} = \frac{1}{2}\left(\sum_{i\in I}\sum_{j\in J} [*, i-1, i, j-1, j]\right)^2 = \frac{1}{2}\left(\mathcal{W}^{(1,0),\mathrm{c}}_{I,J}\right)^2$$

$$C_P = \mathcal{W}^{(2,0),\mathrm{c}}_{\{1,2,3,4\},\{5,6,7,8\}} - \mathcal{W}^{(2,0),\mathrm{c}}_{\{1,2,3\},\{5,6,7,8\}} - \mathcal{W}^{(2,0),\mathrm{c}}_{\{2,3,4\},\{5,6,7,8\}} - \mathcal{W}^{(2,0),\mathrm{c}}_{\{3,4,1\},\{5,6,7,8\}} - \mathcal{W}^{(2,0),\mathrm{c}}_{\{4,1,2\},\{5,6,7,8\}}$$

$$\bar{\chi}_P = -3\mathrm{Li}_2(1) - \frac{1}{2}\left[\log\left[\frac{\langle 12X\rangle\langle 34X\rangle}{\langle 14X\rangle\langle 23X\rangle}\right]\right]^2$$

$$C_P = F + 2\left[\frac{\langle 2356\rangle\langle 6712\rangle}{\langle 2367\rangle\langle 5612\rangle} - 1\right][(12)\cap(765),2,3,5,6][(56)\cap(321),6,7,1,2] + \mathcal{W}^{(2,0),\mathrm{c}}_{\{1,2,3,4\},\{5,6,7,8\}}$$

$$\bar{\chi}_P = \mathrm{Li}_2\left[1 - \frac{\langle 12X\rangle\langle 2367\rangle}{\langle 23X\rangle\langle 1267\rangle}\right] + \mathrm{Li}_2\left[1 - \frac{\langle 2356\rangle\langle 1267\rangle}{\langle 2367\rangle\langle 1256\rangle}\right] + \mathrm{Li}_2\left[1 - \frac{\langle 1256\rangle\langle 67X\rangle}{\langle 56X\rangle\langle 1267\rangle}\right]$$
$$-\mathrm{Li}_2\left[1 - \frac{\langle 12X\rangle\langle 2356\rangle}{\langle 23X\rangle\langle 1256\rangle}\right] - \mathrm{Li}_2\left[1 - \frac{\langle 2356\rangle\langle 67X\rangle}{\langle 2367\rangle\langle 56X\rangle}\right] + \log\left[\frac{\langle 12X\rangle\langle 2367\rangle}{\langle 23X\langle 1267\rangle}\right]\log\left[\frac{\langle 1256\rangle\langle 67X\rangle}{\langle 56X\rangle\langle 1267\rangle}\right]$$

$$
\begin{aligned}
F = &-[2,3,4,6,7][1,2,4,6,(67)\cap(234)] - [2,3,4,7,8][1,2,4,7,(78)\cap(234)]\\
&-[2,3,4,6,7][2,3,5,6,(34)\cap(267)] - [2,3,4,7,8][2,3,5,6,(34)\cap(278)]\\
&+[2,3,4,5,8][2,3,5,6,(34)\cap(285)] + [2,3,4,6,7][2,3,5,8,(34)\cap(267)]\\
&+[2,3,4,7,8][2,3,5,8,(34)\cap(278)] - [2,3,4,6,7][2,3,7,8,(34)\cap(267)]\\
&-[1,2,4,6,7][2,4,5,6,(41)\cap(267)] - [1,2,4,7,8][2,4,5,6,(41)\cap(278)]\\
&+[1,2,4,5,8][2,4,5,6,(41)\cap(285)] + [1,2,4,6,7][2,4,5,8,(41)\cap(267)]\\
&+[1,2,4,7,8][2,4,5,8,(41)\cap(278)] - [1,2,4,6,7][2,4,7,8,(41)\cap(267)]\\
&+[2,5,6,7,8][1,2,4,6,(67)\cap(285)] - [2,5,6,7,8][1,2,4,7,(78)\cap(256)]\\
&+[2,5,6,7,8][2,3,4,6,(67)\cap(285)] - [2,5,6,7,8][2,3,4,7,(78)\cap(256)]\\
&-[2,3,4,5,6][2,5,7,8,(56)\cap(234)] - [1,2,4,5,6][2,5,7,8,(56)\cap(241)]\\
&+[2,3,4,5,8][2,6,7,8,(85)\cap(234)] + [1,2,4,5,8][2,6,7,8,(85)\cap(241)]\\
&+[2,3,4,5,8][1,2,4,8,(85)\cap(234)] - [2,3,4,5,6][1,2,4,5,(56)\cap(234)]\\
&+[1,2,4,5,8][2,3,4,5,6] - [1,2,4,6,7][2,3,4,5,6] - [1,2,4,7,8][2,3,4,5,6]\\
&+[1,2,4,6,7][2,3,4,5,8] + [1,2,4,7,8][2,3,4,5,8] - [1,2,4,6,7][2,3,4,7,8]\\
&+[1,2,4,5,6][2,5,6,7,8] - [1,2,4,6,7][2,5,6,7,8] + [2,3,4,5,6][2,5,6,7,8]\\
&-[2,3,4,6,7][2,5,6,7,8]
\end{aligned}
$$

$$C_P\bar{\chi}_P + (\{1,2,3,4\} \leftrightarrow \{5,6,7,8\}) + \mathrm{cyc}_1 + \mathrm{cyc}_2$$

$$
\begin{aligned}
C_P = [2,3,4,5,6]\Big[&\mathcal{W}^{(1,0)}_{\{2,3,(43)\cap(562),(56)\cap(432),6,7,8,5,(56)\cap(432)\}} - \mathcal{W}^{(1,0)}_{\{(43)\cap(562),4,1,2,(56)\cap(432),6,7,8,5,(56)\cap(432)\}}\\
&+\mathcal{W}^{(1,0)}_{\{(12)\cap(653),2,3,(65)\cap(123),6,7,8,5,(65)\cap(1,2,3)\}} - \mathcal{W}^{(1,0)}_{\{3,4,1,(12)\cap(653),(65)\cap(123),6,7,8,5,(65)\cap(123)\}}\Big]\\
\bar{\chi}_P = &\mathrm{Li}_2\left[1-\frac{\langle 12X\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 23X\rangle}\right] - \mathrm{Li}_2\left[1-\frac{\langle 23X\rangle\langle 3456\rangle}{\langle 2356\rangle\langle 34X\rangle}\right]\\
&-\frac{1}{2}\log\left[\frac{\langle 12X\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 23X\rangle}\right]\log\left[\frac{\langle 1256\rangle\langle 34X\rangle}{\langle 1234\rangle\langle 56X\rangle}\right] + \frac{1}{2}\log\left[\frac{\langle 23X\rangle\langle 3456\rangle}{\langle 2356\rangle\langle 34X\rangle}\right]\log\left[\frac{\langle 1234\rangle\langle 56X\rangle}{\langle 1256\rangle\langle 34X\rangle}\right]
\end{aligned}
$$

$$C_P\bar{\chi}_P + \{1,2,3,4\} \leftrightarrow \{5,6,7,8\} + \mathrm{cyc}_1 + \mathrm{cyc}_2,$$

$$
\begin{aligned}
C_P = &[2,5,6,7,8]\left(\mathcal{W}^{(1,0)}_{\{2,3,4,1,2,(65)\cap(872),6,7,8,(78)\cap(652)\}} - \mathcal{W}^{(1,0)}_{\{2,3,4,1,2,(78)\cap(872),8,5,(65)\cap(872)\}}\right)\\
&+2\left[\frac{\langle 2356\rangle\langle 7812\rangle}{\langle 2378\rangle\langle 5612\rangle}-1\right][(12)\cap(87(65)\cap(321)),2,3,5,6][(65)\cap(321),7,8,1,2]\\
\bar{\chi}_P = &\mathrm{Li}_2\left[1-\frac{\langle 2356\rangle\langle 1278\rangle}{\langle 2378\rangle\langle 1256\rangle}\right] - \mathrm{Li}_2\left[1-\frac{\langle 12X\rangle\langle 2356\rangle}{\langle 23X\rangle\langle 1256\rangle}\right] - \mathrm{Li}_2\left[1-\frac{\langle 23X\rangle\langle 1278\rangle}{\langle 12X\rangle\langle 2378\rangle}\right]\\
&\frac{1}{2}\log\left[\frac{\langle 12X\rangle\langle 2356\rangle}{\langle 23X\rangle\langle 1256\rangle}\right]\log\left[\frac{\langle 23X\rangle\langle 5678\rangle}{\langle 2378\rangle\langle 56X\rangle}\right] + \frac{1}{2}\log\left[\frac{\langle 23X\rangle\langle 1278\rangle}{\langle 12X\rangle\langle 2378\rangle}\right]\log\left[\frac{\langle 12X\rangle\langle 5678\rangle}{\langle 1256\rangle\langle 78X\rangle}\right].
\end{aligned}
$$

$$C_P\bar{\chi}_P + \mathrm{cyc}_1 + \mathrm{cyc}_2$$

$$
\begin{aligned}
C_P = &\,x\phi_1[\delta_1,1,2,3,4][\beta_1,5,6,7,8] + \bar{x}\phi_2[\alpha_2,2,3,5,6][\gamma_2,7,8,1,2]\\
\bar{\chi}_P = &\,\mathrm{Li}_2\left[\frac{1}{2}(1-u+v+\Delta)\right] + \mathrm{Li}_2\left[\frac{1}{2}(1+u-v+\Delta)\right] - \mathrm{Li}_2(1)\\
&+\frac{1}{2}\log(u)\log(v) - \log\left[\frac{1}{2}(1-u+v+\Delta)\right]\log\left[\frac{1}{2}(1+u-v+\Delta)\right] \quad (4.44)
\end{aligned}
$$

$$u = x\bar{x} = \frac{\langle 1234\rangle\langle 5678\rangle}{\langle 1256\rangle\langle 3478\rangle}, \qquad v = (1-x)(1-\bar{x}) = \frac{\langle 3456\rangle\langle 1278\rangle}{\langle 1256\rangle\langle 3478\rangle},$$

$$x = \frac{1}{2}(\Delta - u + v - 1), \qquad \bar{x} = \frac{1}{2}(-\Delta - u + v - 1),$$

$$\Delta = x - \bar{x} = \sqrt{(1-u-v)^2 - 4uv}.$$

$$\mathcal{Z}_{\alpha_2} = \mathcal{Z}_2 + \frac{\langle 234(56)\cap(781)\rangle + \langle 134(56)\cap(782)\rangle + \langle 1256\rangle\langle 3478\rangle\Delta}{2\langle 34(56)\cap(781)1\rangle}\mathcal{Z}_1$$

$$\mathcal{Z}_{\gamma_2} = (56)\cap(78\alpha_2),$$

$$\mathcal{Z}_{\beta_1} = \mathcal{Z}_3 + \frac{\langle 321(78)\cap(654)\rangle + \langle 421(78)\cap(653)\rangle + \langle 1256\rangle\langle 3478\rangle\Delta}{2\langle 21(78)\cap(654)4\rangle}\mathcal{Z}_4$$

$$\mathcal{Z}_{\delta_1} = (78)\cap(56\beta_1)$$

$$\phi_1 = \left(1 - \frac{\langle \beta_1 812\rangle\langle \delta_1 456\rangle}{\langle \beta_1 856\rangle\langle \delta_1 412\rangle}\right)^{-1} \quad \phi_2 = \left(1 - \frac{\langle \alpha_2 678\rangle\langle \gamma_2 234\rangle}{\langle \alpha_2 634\rangle\langle \gamma_2 278\rangle}\right)^{-1}$$

$$\bar{Q}_A^{A'}\mathcal{R}_n^{(k)} = \frac{1}{4}\Gamma_{\text{cusp}} \int \left[d^{2|3}\mathcal{Z}_{n+1}\right]_A^{A'} \left[\mathcal{R}_{n+1}^{(k+1)} - \mathcal{R}_n^{(k)}\mathcal{R}_{n+1}^{(1,0)}\right] + \text{ cyc}$$

$$\bar{Q}_A^{A'} = \sum_i \chi_i^{A'} \frac{\partial}{\partial Z_i^A}$$

$$\int \left[d^{2|3}\mathcal{Z}_{n+1}\right]_A^{A'} = C(n-1n1)_A \oiiint_{\epsilon=0} \epsilon d\epsilon \int_0^\infty d\tau (d^3\chi_{n+1})^{A'}$$

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon\mathcal{Z}_{n-1} + C\epsilon\tau\mathcal{Z}_1 + C'\epsilon^2\mathcal{Z}_2$$

$$C = \frac{\langle n-1n23}{\langle n123\rangle} \quad C' = \frac{\langle n-2n-1n1}{\langle n-2n-121\rangle}$$

$$\frac{1}{4}\Gamma_{\text{cusp}} = \frac{2C_F}{N}g^2$$

$$\bar{Q}_A^{A'}\mathcal{R}_{n+1}^{(1,1)} = \frac{2C_F}{N}\int \left[d^{2|3}\mathcal{Z}_{n+1}\right]_A^{A'} \left[\mathcal{R}_{n+1}^{(2,0)} - \mathcal{R}_n^{(1,0)}\mathcal{R}_{n+1}^{(1,0)}\right] + \text{ cyc.}$$

$$c_n = -\frac{1}{2NC_F} = -\frac{1}{N^2-1}$$

$$\mathcal{R}_{n+1}^{(2,0)} = c_p\mathcal{D}_p + c_n\mathcal{D}_n$$

$$\mathcal{D}_n = \mathcal{D}_a - \mathcal{D}_p$$

$$\mathcal{R}_{n+1}^{(2,0)} = (c_p - c_n)\mathcal{D}_p + c_n\mathcal{D}_a$$
$$= \frac{N^2}{N^2-1}\mathcal{D}_p - \frac{1}{N^2-1}\mathcal{D}_a$$

$$\frac{N^2}{N^2-1}\mathcal{D}_p - \frac{1}{N^2-1}\mathcal{D}_a - \mathcal{R}_n^{(1,0)}\mathcal{R}_{n+1}^{(1,0)}$$

$$\frac{N^2}{N^2-1}\left(\mathcal{D}_p - \mathcal{R}_n^{(1,0)}\mathcal{R}_{n+1}^{(1,0))}\right) - \frac{1}{N^2-1}\left(\mathcal{D}_a - \mathcal{R}_n^{(1,0)}\mathcal{R}_{n+1}^{(1,0)}\right)$$

$$\int \; [d^{2|3}\mathcal{Z}_{n+1}]_A^{A'}\left[\mathcal{D}_p - \mathcal{R}_n^{(1,0)}\mathcal{R}_{n+1}^{(1,0)}\right] = \bar{Q}_A^{A'}\mathcal{R}_{n+1}^{(1,1),\text{ abelian curvature}}$$

$$\int \; [d^{2|3}\mathcal{Z}_{n+1}]_A^{A'}\left[\mathcal{D}_a - \mathcal{R}_n^{(1,0)}\mathcal{R}_{n+1}^{(1,0)}\right] = \bar{Q}_A^{A'}\mathcal{R}_{n+1}^{(1,1),\text{ abelian curvature}} = 0$$

$$\bar{Q}_A^{A'}\mathcal{R}_{n+1}^{(1,1)} = \frac{2C_F}{N}\frac{N^2}{N^2-1}\bar{Q}_A^{A'}\mathcal{R}_{n+1}^{(1,1),\text{abelian curvature}} = \bar{Q}_A^{A'}\mathcal{R}_{n+1}^{(1,1),\text{abelian curvature}}$$

$$\bar{Q}_A^{A'}\mathcal{R}_{n+1}^{(1,1)} = \bar{Q}_A^{A'}\mathcal{R}_{n+1}^{(1,1),\text{ abelian curvature}}$$

$$\mathcal{R}_{n+1}^{(1,1)} = \mathcal{R}_{n+1}^{(1,1),\text{ abelian curvature}}$$

$$\mathcal{W}_{n+1}^{(1,1)} - \mathcal{W}_{n+1}^{(1,0)}\mathcal{W}_{n+1}^{(0,1)} = \mathcal{W}_{n+1}^{(1,1),\text{ abelian curvature}} - \frac{N^2}{N^2-1}\mathcal{W}_{n+1}^{(1,0)}\mathcal{W}_{n+1}^{(0,1)}$$

$$\mathcal{W}_{n+1}^{(1,1)} + \frac{1}{N^2-1}\mathcal{W}_{n+1}^{(1,0)}\mathcal{W}_{n+1}^{(0,1)} = \mathcal{W}_{n+1}^{(1,1),\text{ abelian curvature}}$$

$$\frac{N^2-2}{N^2} - \left(-\frac{2}{N^2}\right) = 1$$

$$\frac{1}{N^2-1}\mathcal{W}_{n+1}^{(1,0)}\mathcal{W}_{n+1}^{(0,1)}$$

$$\frac{1}{N^2-1}\times\left(1-\frac{1}{N^2}\right) = \frac{1}{N^2}$$

$$\bar{Q}_A^{A'}\mathcal{R}_{n_1,\dots,n_m} = \frac{1}{4}\Gamma_{\text{cusp}}\sum_{r=1}^{m}\left[\int \; [d^{2|3}\mathcal{Z}_{X_r}]_A^{A'}\left[\mathcal{R}_{n_1,\dots,n_r+1,\dots,n_m} - \mathcal{R}_{n_1,\dots,n_m}\mathcal{R}_{n_r+1}^{(1,0)}\right] + \text{cyc}_r\right]$$

$$\mathcal{Z}_{X_r} \to \mathcal{Z}_{n_r}^r - \epsilon\mathcal{Z}_{n_r-1}^r + C\epsilon\tau\mathcal{Z}_1^r + C'\epsilon^2\mathcal{Z}_2^r$$

$$C = \frac{\langle n_r-1, n_r, 2^r, 3^r\rangle}{\langle n_r, 1^r, 2^r, 3^r\rangle}, C' = \frac{\langle n_r-2, n_r-1, n_r, 1^r\rangle}{\langle n_r-2, n_r-1, 2^r, 1^r\rangle}$$

$$\int \; [d^{2|3}\mathcal{Z}_{X_r}]_A^{A'} = C(n_r-1, n_r, 1^r)_A \oiiint_{\epsilon=0} \epsilon d\epsilon \int_0^\infty d\tau\left(d^3\chi_{X_r}\right)^{A'}$$

$$\bar{Q}_A^{A'}\mathcal{R}_{n_1,n_2} = \frac{1}{4}\Gamma_{\text{cusp}}\left[\int \left[d^{2|3}\mathcal{Z}_{X_1}\right]_A^{A'}\left[\mathcal{R}_{n_1+1,n_2} - \mathcal{R}_{n_1,n_2}\mathcal{R}_{n_1+1}^{(1,0)}\right] + \text{cyc}_1 + \int \left[d^{2|3}\mathcal{Z}_{X_2}\right]_A^{A'}\left[\mathcal{R}_{n_1,n_2+2} - \mathcal{R}_{n_1,n_2}\mathcal{R}_{n_2+1}^{(1,0)}\right] + \text{cyc}_2\right]$$

$$\frac{1}{4}\Gamma_{\text{cusp}} = \frac{2C_F}{N}g^2 + O(g^4)\ \mathcal{R}_{n_1,n_2}^{(k)} = \mathcal{R}_{m,n}^{(k,0)} + g^2\mathcal{R}_{n_1,n_2}^{(k,1)} + g^4\mathcal{R}_{n_1,n_2}^{(k,2)} + O(g^6)$$

$$g^2 = \frac{g_{YM}^2 N}{16\pi^2}$$

$$\bar{Q}_A^{A'}\mathcal{R}_{n_1,n_2}^{(k,1)} = \int \left[d^{2|3}\mathcal{Z}_{X_1}\right]_A^{A'}\left[\mathcal{R}_{n_1+1,n_2}^{(k+1,0)} - \mathcal{R}_{n_1,n_2}^{(k,0)}\mathcal{R}_{n_1+1}^{(1,0)}\right] + \text{cyc}_1 + \int \left[d^{2|3}\mathcal{Z}_{X_2}\right]_A^{A'}\left[\mathcal{R}_{n_1,n_2+1}^{(k+1,0)} - \mathcal{R}_{n_1,n_2}^{(k,0)}\mathcal{R}_{n_2+1}^{(1,0)}\right] + \text{cyc}_2$$

$$\bar{Q}_A^{A'}\mathcal{R}_{n_1,n_2}^{(k,1),\text{c}} = \int \left[d^{2|3}\mathcal{Z}_{X_1}\right]_A^{A'}\left[\mathcal{R}_{n_1+1,n_2}^{(k+1,0),\text{c}} - \mathcal{R}_{n_1,n_2}^{(k,0),\text{c}}\mathcal{R}_{n_1+1}^{(1,0)}\right] + \text{cyc}_1 + \int \left[d^{2|3}\mathcal{Z}_{X_2}\right]_A^{A'}\left[\mathcal{R}_{n_1,n_2+1}^{(k+1,0),\text{c}} - \mathcal{R}_{n_1,n_2}^{(k,0),\text{c}}\mathcal{R}_{n_2+1}^{(1,0)}\right] + \text{cyc}_2$$

$$\begin{aligned}&\bar{Q}_A^{A'}\mathcal{R}_{n_1,n_2}^{(k,1),\text{c}}\\ &= \int \left[d^{2|3}Z_{X_1}\right]_A^{A'}\Big[\big\langle\mathcal{L}[1,\dots,n_1,X_1]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k+1,0),\text{c}}\\ &-\big\langle\mathcal{L}[1,\dots,n_1]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k,0),\text{c}}\langle\mathcal{L}[1,\dots,n_1,X_1]\rangle^{(1,0)}\Big] + \text{cyc}_1\\ &+\int \left[d^{2|3}Z_{X_2}\right]_A^{A'}\Big[\big\langle\mathcal{L}[1,\dots,n_1]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2,X_2]\big\rangle^{(k+1,0),\text{c}}\\ &-\big\langle\mathcal{L}[1,\dots,n_1]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k,0),\text{c}}\big\langle\mathcal{L}[\tilde{1},\dots,\tilde{n}_2,X_2]\big\rangle^{(1,0)}\Big] + \text{cyc}_2\end{aligned}$$

$$Z_{X_1} = Z_{n_1} - \epsilon Z_{n_1-1} + C\epsilon\tau Z_1 + C'\epsilon^2 Z_2$$

$$C = \frac{\langle n_1 - 1 n_1 2 3\rangle}{\langle n_1 1 2 3\rangle}$$

$$C' = \frac{\langle n_1 - 2 n_1 - 1 n_1 1\rangle}{\langle n_1 - 2 n_1 - 1 2 1\rangle}$$

$$\mathcal{R}_{n_1,n_2}^{(k,1),\text{c}} = \sum_P C_P^k \bar{\chi}_P$$

$$\bar{Q}_A^{A'}\mathcal{R}_{n_1,n_2}^{(k,1),\text{c}} = \sum_P C_P^k \bar{Q}_A^{A'}\bar{\chi}_P$$

$$\bar{Q}_A^{A'}\text{Li}_2(f) = -\log(1-f)\bar{Q}_A^{A'}\log(f)$$

$$\bar{Q}_A^{A'}\log(f)\log(g) = \log(f)\bar{Q}_A^{A'}\log(g) + \log(g)\bar{Q}_A^{A'}\log(f).$$

$$\big\langle\mathcal{L}[1,\dots,n_1,X_1]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k+1,0),\text{c}}\langle\mathcal{L}[1,\dots,n_1,X_1]\rangle^{(1,0)}$$

$$\langle\mathcal{L}[1,\dots,n_1,X_1]\rangle^{(1,0)}=\sum_{i=1}^{n_1-1}\ \sum_{j=i+2,j\neq i-1,i,i+1}^{X_1}[*,i-1,i,j-1,j]$$

$$\sum_{i=2}^{n-2}[n-1,i-1,i,n,X_1]$$

$$\begin{aligned}\left\langle\mathcal{L}[1,\dots,n_1,X_1]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\right\rangle^{(k+1,0),c} =&\left\langle\mathcal{L}[1,\dots,n_1]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\right\rangle^{(k+1,0),c}\\ &+\sum_{i=2}^{n_1-2}[n_1-1,n_1,X_1,i-1,i]T_i\\ &+\sum_{\tilde{i}=\tilde{1}}^{\tilde{n}_2}[n_1-1,n_1,X_1,\tilde{i}-1,\tilde{i}]\tilde{T}_i\end{aligned}$$

$$T_i=\sum_{k'=0}^{k}\Big[\left\langle\mathcal{L}[X_1,1,\dots,i-1,I_i]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\right\rangle^{(k',0),c}\left\langle\mathcal{L}[I_i,i,\dots,n_1-1,\hat{n}_{1,i}]\right\rangle^{(k-k',0)}$$
$$+\langle\mathcal{L}[X_1,1,\dots,i-1,I_i]\rangle^{(k',0)}\left\langle\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\mathcal{L}[I_i,i,\dots,n_1-1,\hat{n}_{1,i}]\right\rangle^{(k-k',0),c}\Big]$$
$$\tilde{T}_i=\left\langle\mathcal{L}[X_1,1,\dots,\hat{n}_{1,\tilde{i}},I_{\tilde{i}},\tilde{i},\dots,\tilde{i}-1,I_{\tilde{i}}]\right\rangle^{(k,0)}$$
$$-\alpha\sum_{k'=0}^{k}\left\langle\mathcal{L}[X_1,1,\dots,\hat{n}_{1,\tilde{i}}]\right\rangle^{(k',0)}\left\langle\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\right\rangle^{(k-k',0)}$$

$$\begin{aligned}I_i&=(i-1i)\cap(n_1-1n_1X_1),\quad Z_{\hat{n}_{1,i}}=(n_1-1n_1)\cap(i-1iX_1)\\ I_{\tilde{i}}&=(\tilde{i}-1\tilde{i})\cap(n_1-1n_1X_1),\quad Z_{\hat{n}_{1,\tilde{i}}}=(n_1-1n_1)\cap(\tilde{i}-1\tilde{i}X_1)\end{aligned}$$

$$\left\langle\mathcal{L}[X_1,1,\dots,n_1-1,\hat{n}_{1,\tilde{i}}]\right\rangle=\left\langle\mathcal{L}[1,\dots,n_1-1,\hat{n}_{1,\tilde{i}}]\right\rangle$$
$$+\sum_{j=2}^{n_1-2}[n_1-1,\hat{n}_{1,\tilde{i}},X_1,j-1,j]\left\langle\mathcal{L}[X_1,1,\dots,j-1,I'_j]\right\rangle\left\langle\mathcal{L}\left[I'_j,j,\dots,n_1-1,\widehat{(\hat{n}_{1,\tilde{i}})}\right]\right\rangle$$

$$I'_j=(j-1j)\cap\left(n_1-1\hat{n}_{1,\tilde{i}}X_1\right)\text{ and }\widehat{(\hat{n}_{1,\tilde{i}})}_j=\left(n_1-1\hat{n}_{1,\tilde{i}}\right)\cap(j-1jX_1)$$

$$\left\langle\mathcal{L}[X_1,1,\dots,\hat{n}_{1,\tilde{i}},I_{\tilde{i}},\tilde{i},\dots,\tilde{i}-1,I_{\tilde{i}}]\right\rangle^{(k,0)}$$

$$\begin{aligned}&I_2|_{\epsilon\to0}=Z_1\equiv\bar{I}_2\\ &I_i|_{\epsilon\to0}=(i-1i)\cap(n_1-1n_11)\equiv\bar{I}_i, && i=3,\dots,n_1-2\\ &I_{\tilde{i}}|_{\epsilon\to0}=(\tilde{i}-1\tilde{i})\cap(n_1-1n_11)\equiv\bar{I}_{\tilde{i}}, && \tilde{i}=\tilde{1},\dots,\tilde{n}_2\\ &Z_{\hat{n}_{1,i}}\big|_{\epsilon\to0}=Z_{n_1}, && i=1,\dots,n_1\\ &Z_{\hat{n}_{1,\tilde{i}}}\big|_{\epsilon\to0}=Z_{n_1}, && \tilde{i}=\tilde{1},\dots,\tilde{n}_2\\ &Z_X|_{\epsilon\to0}=Z_{n_1}\end{aligned}$$

$[X_1,i,j,n_1-1,n_1],[X_1,1,i,j,n_1],[X_1,1,i,n_1-1,n_1],[i,j,k,n_1,X_1]$, where $i,j,k\neq n_1-1,1$

$$\int \left[d^{2|3}Z_{X_1}\right]_A^{A'}[X_1,a,b,c,d]=0$$

$$\Bigg[\sum_{i=3}^{n_1-2}\int \left[d^{2|3}Z_{X_1}\right]_A^{A'}[i-1,i,n_1-1,n_1,X_1]$$
$$\times\Bigg[\sum_{k'=2}^{k}\Big(\big\langle\mathcal{L}[n_1,1,\dots,i-1,\bar{I}_i]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k',0),c}\langle\mathcal{L}[\bar{I}_i,i,\dots,n_1-1,n_1]\rangle^{(k-k',0)}$$
$$+\langle\mathcal{L}[n_1,1,\dots,i-1,\bar{I}_i]\rangle^{(k'-k,0)}\big\langle\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\mathcal{L}[\bar{I}_i,i,\dots,n_1-1,n_1]\big\rangle^{(k',0),c}\Big)$$
$$-\big\langle\mathcal{L}[1,\dots,n_1]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k,0),c}\Bigg]$$
$$+\sum_{\tilde{\imath}=\tilde{1}}^{\tilde{n}_2}\int \left[d^{2|3}Z_{X_1}\right]_A^{A'}[\tilde{\imath}-1,\tilde{\imath},n_1-1,n_1,X_1]$$
$$\times\Bigg[\langle\mathcal{L}[n_1,1,\dots,n_1,\bar{I}_{\tilde{\imath}},\tilde{\imath},\dots,\tilde{\imath}-1,\bar{I}_{\tilde{\imath}}]\rangle^{(k,0)}-\sum_{k'=0}^{k}\langle\mathcal{L}[1,\dots,n_1]\rangle^{(k',0)}\big\langle\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k-k',0)}\Bigg]$$
$$+\mathrm{cyc}_1\Bigg]+\mathrm{loop}_1\leftrightarrow\mathrm{loop}_2$$

$$\langle\mathcal{L}[1,\dots,n]\rangle^{(0,0)}=1$$
$$\langle\mathcal{L}[1,\dots,n]\mathcal{L}[\tilde{1},\dots,\tilde{m}]\rangle^{(0,0),c}=0$$
$$\langle\mathcal{L}[1,\dots,n]\mathcal{L}[\tilde{1},\dots,\tilde{m}]\rangle^{(1,0),c}=0$$

$$\int \left[d^{2|3}Z_{X_1}\right]_A^{A'}[i,j,n_1-1,n_1,X_1]=\int_{\tau=0}^{\tau=\infty}\mathrm{dlog}\,\frac{\langle Yij\rangle}{\langle Yn_1-2n_1-1\rangle}\bar{Q}_A^{A'}\log\frac{\langle\bar{n}_1 j\rangle}{\langle\bar{n}_1 i\rangle}$$

$$Y=n\wedge B, Z_B=Z_{n_1-1}-C\tau Z_1,(\bar{n}_1)=(n_1-1n_11)$$

$$\int_{\tau=0}^{\tau=\infty}\mathrm{dlog}\,\langle Yij\rangle\bar{Q}\log\frac{\langle\bar{n}_1 j\rangle}{\langle\bar{n}_1 i\rangle}=\log\langle Yij\rangle\bar{Q}\log\frac{\langle\bar{n}_1 j\rangle}{\langle\bar{n}_1 i\rangle}\Bigg|_{\tau=0}^{\tau=\infty}=\log\frac{\langle n_1 1ij\rangle}{\langle n_1-1n_1ij\rangle}\bar{Q}\log\frac{\langle\bar{n}_1 j\rangle}{\langle\bar{n}_1 i\rangle}$$

$$\bar{Q}_A^{A'}\mathcal{R}_{n_1,n_2}^{(k,1),c}=\Bigg[\sum_{i=3}^{n_1-2}\log\frac{\langle n_1 1i-1i\rangle}{\langle n_1-1n_1i-1i\rangle}\bar{Q}_A^{A'}\log\frac{\langle\bar{n}_1 i\rangle}{\langle\bar{n}_1 i-1\rangle}$$
$$\times\Bigg[\sum_{k'=2}^{k}\Big(\big\langle\mathcal{L}[n_1,1,\dots,i-1,\bar{I}_i]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k',0),c}\langle\mathcal{L}[\bar{I}_i,i,\dots,n_1-1,n_1]\rangle^{(k-k',0)}$$
$$+\langle\mathcal{L}[n_1,1,\dots,i-1,\bar{I}_i]\rangle^{(k'-k,0)}\big\langle\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\mathcal{L}[\bar{I}_i,i,\dots,n_1-1,n_1]\big\rangle^{(k',0),c}\Big)$$
$$-\big\langle\mathcal{L}[1,\dots,n_1]\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k,0),c}\Bigg]$$
$$+\sum_{\tilde{\imath}=\tilde{1}}^{\tilde{n}_2}\log\frac{\langle n_1 1\tilde{\imath}-1\tilde{\imath}\rangle}{\langle n_1-1n_1\tilde{\imath}-1\tilde{\imath}\rangle}\bar{Q}_A^{A'}\log\frac{\langle\bar{n}_1\tilde{\imath}\rangle}{\langle\bar{n}_1\tilde{\imath}-1\rangle}$$
$$\times\Bigg[\langle\mathcal{L}[n_1,1,\dots,n_1,\bar{I}_{\tilde{\imath}},\tilde{\imath},\dots,\tilde{\imath}-1,\bar{I}_{\tilde{\imath}}]\rangle^{(k,0)}$$
$$-\sum_{k'=0}^{k}\langle\mathcal{L}[1,\dots,n_1]\rangle^{(k',0)}\big\langle\mathcal{L}[\tilde{1},\dots,\tilde{n}_2]\big\rangle^{(k-k',0)}\Bigg]+\mathrm{cyc}_1\Bigg]+\mathrm{loop}_1\leftrightarrow\mathrm{loop}_2$$

$$\frac{1}{\pi^2}\frac{\langle ijkl\rangle\langle Xjl\rangle\langle ABik\rangle}{\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABli\rangle\langle ABX\rangle}$$

$$\frac{1}{\pi^2}\frac{\langle ijkl\rangle\langle Xik\rangle\langle ABjl\rangle}{\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABli\rangle\langle ABX\rangle}$$

$$-3\mathrm{Li}_2(1)-\frac{1}{2}\log\left(\frac{\langle ijX\rangle\langle klX\rangle}{\langle jkX\rangle\langle ilX\rangle}\right)^2$$

$$\frac{1}{\pi^2}\frac{\langle AB(ijk)\cap(klm)\rangle\langle Xjl\rangle}{\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABlm\rangle\langle ABX\rangle}$$

$$\frac{1}{\pi^2}\frac{\langle X(ijk)\cap(klm)\rangle\langle ABjl\rangle}{\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABlm\rangle\langle ABX\rangle}$$

$$-\mathrm{Li}_2(1)+\mathrm{Li}_2\left(1-\frac{\langle ijX\rangle\langle jklm\rangle}{\langle jkX\rangle\langle ijlm\rangle}\right)+\mathrm{Li}_2\left(1-\frac{\langle ijkl\rangle\langle lmX\rangle}{\langle klX\rangle\langle ijlm\rangle}\right)+\log\left(\frac{\langle ijX\rangle\langle jklm\rangle}{\langle jkX\rangle\langle ijlm\rangle}\right)\log\left(\frac{\langle ijkl\rangle\langle lmX\rangle}{\langle klX\rangle\langle ijlm\rangle}\right)$$

$$\frac{1}{\pi^2}\frac{\langle AB(ijk)\cap(lmn)\rangle\langle Xjm\rangle}{\langle ABij\rangle\langle ABjk\rangle\langle ABlm\rangle\langle ABmn\rangle\langle ABX\rangle}$$

$$\frac{1}{\pi^2}\frac{\langle X(ijk)\cap(lmn)\rangle\langle ABjm\rangle}{\langle ABij\rangle\langle ABjk\rangle\langle ABlm\rangle\langle ABmn\rangle\langle ABX\rangle}$$

$$-\mathrm{Li}_2\left(1-\frac{\langle ijX\rangle\langle jklm\rangle}{\langle jkX\rangle\langle ijlm\rangle}\right)+\mathrm{Li}_2\left(1-\frac{\langle ijX\rangle\langle jkmn\rangle}{\langle jkX\rangle\langle ijmn\rangle}\right)+\mathrm{Li}_2\left(1-\frac{\langle jklm\rangle\langle ijmn\rangle}{\langle jkmn\rangle\langle ijlm\rangle}\right)$$
$$-\mathrm{Li}_2\left(1-\frac{\langle jklm\rangle\langle mnX\rangle}{\langle jkmn\rangle\langle lmX\rangle}\right)+\mathrm{Li}_2\left(1-\frac{\langle ijlm\rangle\langle mnX\rangle}{\langle lmX\rangle\langle ijmn\rangle}\right)+\log\left(\frac{\langle ijX\rangle\langle jkmn\rangle}{\langle jkX\rangle\langle ijmn\rangle}\right)\log\left(\frac{\langle ijlm\rangle\langle mnX\rangle}{\langle lmX\rangle\langle ijmn\rangle}\right)$$

$$\frac{1}{\pi^2}\frac{\langle X(jkl)\cap(mn(ij)\cap(ABk))\rangle-\langle X(jmn)\cap(kl(ij)\cap(ABk))\rangle}{2\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABmn\rangle\langle ABX\rangle}$$

$$\frac{1}{\pi^2}\frac{\langle X(kmn)\cap(ij(kl)\cap(ABj))\rangle-\langle X(ijk)\cap(mn(kl)\cap(ABj))}{2\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABmn\rangle\langle ABX\rangle}$$

$$-\mathrm{Li}_2\left(1-\frac{\langle jkX\rangle\langle klmn\rangle}{\langle jkmn\rangle\langle klX\rangle}\right)+\mathrm{Li}_2\left(1-\frac{\langle ijX\rangle\langle jkmn\rangle}{\langle jkX\rangle\langle ijmn\rangle}\right)-\frac{1}{2}\log\left(\frac{\langle ijX\rangle\langle jkmn\rangle}{\langle jkX\rangle\langle ijmn\rangle}\right)\log\left(\frac{\langle klX\rangle\langle ijmn\rangle}{\langle ijkl\rangle\langle mnX\rangle}\right)$$
$$-\frac{1}{2}\log\left(\frac{\langle jkX\rangle\langle klmn\rangle}{\langle jkmn\rangle\langle klX\rangle}\right)\log\left(\frac{\langle ijkl\rangle\langle mnX\rangle}{\langle klX\rangle\langle ijmn\rangle}\right)$$

$$\frac{1}{\pi^2}\frac{\langle X(jlm)\cap(no(ijk)\cap(AB))-\langle X(jno)\cap(lm(ijk)\cap(AB))\rangle}{2\langle ABij\rangle\langle ABjk\rangle\langle ABlm\rangle\langle ABno\rangle\langle ABX\rangle}$$

$$\frac{1}{\pi^2}\frac{\langle X(lm)\cap(ijk)(no)\cap(ABj)\rangle-\langle X(no)\cap(ijk)(lm)\cap(ABj)\rangle}{2\langle ABij\rangle\langle ABjk\rangle\langle ABlm\rangle\langle ABno\rangle\langle ABX\rangle}$$

$$-\mathrm{Li}_2\left(1-\frac{\langle ijX\rangle\langle jklm\rangle}{\langle jkX\rangle\langle ijlm\rangle}\right)-\mathrm{Li}_2\left(1-\frac{\langle jkX\rangle\langle ijno\rangle}{\langle ijX\rangle\langle jkno\rangle}\right)+\mathrm{Li}_2\left(1-\frac{\langle jklm\rangle\langle ijno\rangle}{\langle jkno\rangle\langle ijlm\rangle}\right)$$
$$+\frac{1}{2}\log\left(\frac{\langle ijX\rangle\langle jklm\rangle}{\langle jkX\rangle\langle ijlm\rangle}\right)\log\left(\frac{\langle jkX\rangle\langle lmno\rangle}{\langle jkno\rangle\langle lmX\rangle}\right)+\frac{1}{2}\log\left(\frac{\langle jkX\rangle\langle ijno\rangle}{\langle ijX\rangle\langle jkno\rangle}\right)\log\left(\frac{\langle ijX\rangle\langle lmno\rangle}{\langle ijlm\rangle\langle noX\rangle}\right)$$

$$\frac{1}{\pi^2}\frac{\frac{1}{2}\langle ijmn\rangle\langle klop\rangle\langle ABX\rangle\Delta-\frac{1}{12}\epsilon^{\alpha\beta\gamma\delta}\langle ABL_\alpha\cap(L_\beta X_1)L_\gamma\cap(L_\delta X_2)\rangle}{\langle ABij\rangle\langle ABkl\rangle\langle ABmn\rangle\langle ABop\rangle\langle ABX\rangle}$$

$$(L_1,L_2,L_3,L_4)=((ij),(kl),(mn),(op)).$$

$$\frac{1}{\pi^2}\frac{\frac{1}{2}\langle ijmn\rangle\langle klop\rangle\langle ABX\rangle\Delta+\frac{1}{12}\epsilon^{\alpha\beta\gamma\delta}\langle ABL_\alpha\cap(L_\beta X_1)L_\gamma\cap(L_\delta X_2)\rangle}{\langle ABij\rangle\langle ABkl\rangle\langle ABmn\rangle\langle ABop\rangle\langle ABX\rangle}$$

$$\begin{aligned}&\mathrm{Li}_2\left(\frac{1}{2}(1-u+v+\Delta)\right)+\mathrm{Li}_2\left(\frac{1}{2}(1+u-v+\Delta)\right)-\mathrm{Li}_2(1)\\&+\frac{1}{2}\log(u)\log(v)-\log\left(\frac{1}{2}(1-u+v+\Delta)\right)\log\left(\frac{1}{2}(1+u-v+\Delta)\right)\end{aligned}$$

$$\Delta=\sqrt{(1-u-v)^2-4uv}$$

$$u=\frac{\langle ijkl\rangle\langle mnop\rangle}{\langle ijmn\rangle\langle klop\rangle}$$

$$v=\frac{\langle \mathrm{klmn}\rangle\langle ijop\rangle}{\langle ijmn\rangle\langle klop\rangle}$$

$$\bar{Q}_A^{A'}\mathcal{R}_n^{(k,1)}=\frac{2C_F}{N}\int\ [d^{2|3}\mathcal{Z}_X]_A^{A'}\left[\mathcal{R}_{n+1}^{(k+1,0)}-\mathcal{R}_n^{(k,0)}\mathcal{R}_{n+1}^{(1,0)}\right]+\text{ cyc.}$$

$$\begin{aligned}&\bar{Q}_A^{A'}\mathcal{R}_n^{(k,1)}\\&=\frac{2C_F}{N}\int\ [d^{2|3}\mathcal{Z}_X]_A^{A'}\left[\langle\mathcal{L}[1,\dots,n,X]\rangle^{(k+1,0)}-\langle\mathcal{L}[1,\dots,n]\rangle^{(k,0)}\langle\mathcal{L}[1,\dots,n,X]\rangle^{(1,0)}\right]+\text{cyc}\end{aligned}$$

$$Z_X=Z_n-\epsilon Z_{n-1}+C\epsilon\tau Z_1+C'\epsilon^2 Z_2$$

$$\langle\mathcal{L}[1,\dots,n_1,X]\rangle^{(1,0)}=\sum_{i=1}^{n_1-1}\ \sum_{j=i+2,j\neq i-1,i,i+1}^{X}\ [*,i-1,i,j-1,j]$$

$$\sum_{i=2}^{n-2}\ [n-1,i-1,i,n,X]$$

$$\begin{aligned}\langle\mathcal{L}[1,\dots,n,X]\rangle^{(k+1,0)}&=\langle\mathcal{L}[1,\dots,n]\rangle^{(k+1,0)}\\+\sum_{i=2}^{n-2}\ [n-1,n,X,i-1,i]\Big(&\langle\mathcal{L}[X,1,\dots,i-1,I_i]\mathcal{L}[I_i,i,\dots,n-1,\hat{n}_i]\rangle^{(k,0)}\\&-\frac{\alpha}{N^2}\langle\mathcal{L}[X,1,\dots,i-1,I_i,i,\dots,n-1,\hat{n}_i,I_i]\rangle^{(k,0)}\Big)\end{aligned}$$

$$I_i=(i-1i)\cap(n-1nX),Z_{\hat{n}_i}=(n-1n)\cap(i-1iX)$$

$$\begin{aligned}&I_2|_{\epsilon\to0}=Z_1\equiv\bar{I}_2\\&I_i|_{\epsilon\to0}=(i-1i)\cap(n-1n1)\equiv\bar{I}_i,\quad i=3,\dots,n-2\\&Z_{\hat{n}_i}\big|_{\epsilon\to0}=Z_n,\qquad\qquad\qquad\quad i=1,\dots,n\end{aligned}$$

$$C = \frac{\langle n{-}1\ n\ 2\ 3\rangle}{\langle n\ 1\ 2\ 3\rangle}$$

$$C' = \frac{\langle n{-}2\ n{-}1\ n\ 1\rangle}{\langle n{-}2\ n{-}1\ 2\ 1\rangle}.$$

$$\begin{aligned}&\bar{Q}_A^{A'}\mathcal{R}_n^{(k,1)}\\&= \frac{2C_F}{N}\sum_{i=3}^{n-2} \log\frac{\langle n1i-1i\rangle}{\langle n-1ni-1i\rangle}\bar{Q}\log\frac{\langle n-1n1i\rangle}{\langle n-1n1i-1\rangle}\\&\times\Big(\langle\mathcal{L}[n,1,\dots,i-1,\bar{I}_i]\mathcal{L}[\bar{I}_i,i,\dots,n]\rangle^{(k,0)}-\frac{\alpha}{N^2}\langle\mathcal{L}[n,1,\dots,i-1,\bar{I}_i,i,\dots,n,\bar{I}_i]\rangle^{(k,0)}\\&+\left(1-\frac{1}{N^2}\right)\langle\mathcal{L}[1,\dots,n]\rangle^{(k,0)}\Big)+\mathrm{cyc}\end{aligned}$$

$$\left\langle[\mathcal{O}_{r_1}\mathcal{O}_{r_2}](z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\right\rangle$$

$$\left\langle\mathcal{O}_{r_1}(z_0)\mathcal{O}_{r_2}(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\right\rangle$$

$$\left\langle[\mathcal{O}_{r_1}\mathcal{O}_{r_2}]\mathcal{O}_s\mathcal{O}_p\mathcal{O}_q\right\rangle = \underline{O\left(\frac{1}{N^1}\right)}\ \text{disconnected}\ + \underline{O\left(\frac{1}{N^3}\right)+\cdots}\ \text{connected}$$

$$\left\langle\mathcal{O}_r^2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\right\rangle;\ \left\langle[\mathcal{O}_2\mathcal{O}_4]\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\right\rangle;\ \left\langle\mathcal{O}_2^2\mathcal{O}_s\mathcal{O}_p\mathcal{O}_{p+s-2}\right\rangle$$

$$\mathcal{M}_{[2^2],p_2,p_3,p_4}\ \forall p_2,p_3,p_4;\ \ \mathcal{M}_{[r^2],2,p,p}\ \forall r,p$$

$$\phi(z) = Y^I\phi_I(\vec{X});\ z\equiv(\vec{X},\vec{Y})$$

$$T_p(z) := \mathrm{Tr}(\phi(z)^p)$$

$$T_{p_1,p_2,\dots,p_n}(z) = T_{p_1}(z)T_{p_2}(z)\dots T_{p_n}(z)\ ,\Delta = \sum_i p_i$$

$$\mathcal{O}_p = T_p + \sum_{\underline{q}\vdash p} c_{\underline{q}}(p,N)T_{\underline{q}},$$

$$\langle\phi_b^a(z_1)\phi_m^n(z_2)\rangle = \left(\delta_m^a\delta_m^b - \frac{1}{N}\delta_b^a\delta_m^n\right)g_{12};\ g_{12} = \frac{\vec{Y}_{12}^2}{\vec{X}_{12}^2};\ \vec{Y}_{12}^2 \equiv \vec{Y}_1\cdot\vec{Y}_2$$

$$\langle\mathcal{O}_\Delta(z_1)\mathcal{O}_\Delta(z_2)\rangle = g_{12}^\Delta f_{O,\Delta}(N);\ |\mathcal{O}_\Delta|^2 \equiv f_{O,\Delta}(N)$$

$$\left\langle\mathcal{O}_{p_1}(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\right\rangle = \mathcal{P}_{\vec{p}}[g_{ij}]\mathcal{C}_{\vec{p}}(U,V,\hat{\sigma},\hat{\tau})$$

$$U = \frac{\vec{X}_{12}^2\vec{X}_{34}^2}{\vec{X}_{13}^2\vec{X}_{24}^2};\ V = \frac{\vec{X}_{14}^2\vec{X}_{23}^2}{\vec{X}_{13}^2\vec{X}_{24}^2};\ \hat{\sigma} = \frac{\vec{Y}_{13}^2\vec{Y}_{24}^2}{\vec{Y}_{12}^2\vec{X}_{34}^2};\ \hat{\tau} = \frac{\vec{Y}_{14}^2\vec{Y}_{23}^2}{\vec{Y}_{12}^2\vec{Y}_{34}^2}$$

$$\mathcal{C}_{\vec{p}}(U,V,\hat{\sigma},\hat{\tau}) = \mathcal{G}_{\vec{p}}^{\text{free}}(U,V,\hat{\sigma},\hat{\tau}) + \mathcal{I}(U,V,\hat{\sigma},\hat{\tau})\mathcal{H}_{\vec{p}}(U,V,\hat{\sigma},\hat{\tau})$$

$$\mathcal{I}(U,V,\hat{\sigma},\hat{\tau}) = V + \hat{\sigma}^2 UV + \hat{\tau}^2 U + \hat{\sigma}V(-1-U+V) + \hat{\tau}(1-U-V) + \hat{\sigma}\hat{\tau}U(-1+U-V).$$

$$\kappa = -2 - \Sigma + \sum_{ij=12,13,14} \min\left(\frac{p_i+p_j}{2}, \Sigma - \frac{p_i+p_j}{2}\right),$$

$$\Sigma = \frac{p_1+p_2+p_3+p_4}{2}$$

$$U = x_1x_2, V = (1-x_1)(1-x_2)\,;\ \hat{\sigma} = \frac{1}{y_1y_2}, \hat{\tau} = \frac{(1-y_1)(1-y_2)}{y_1y_2}$$

$$\mathcal{I}(x_1,x_2,y_1,y_2) = \frac{1}{(y_1y_2)^2}\prod_{i=1,2}(x_i - y_j)$$

$$\left[\left(\partial_{x_i} + \partial_{y_j}\right)\mathcal{C}\right]_{x_i=y_j} = 0$$

$$\mathcal{D}_{pq,\tau,l,[aba]} = \mathcal{O}_p \partial^l \,\Box^{\frac{1}{2}(\tau-p-q)}\, \mathcal{O}_q\ (2 \le p \le q)$$

$$R_{\vec{\tau}} := \left\{(p,q): \begin{matrix} p = i+a+2+r \\ q = i+a+2+b-r \end{matrix} \text{ for } \begin{matrix} i = 0,\dots,(t-2) \\ r = 0,\dots,(\mu-1) \end{matrix}\right\}$$

$$t \equiv \frac{(\tau-b)}{2} - a, \mu \equiv \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a+l \text{ even} \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a+l \text{ odd} \end{cases}$$

$$\mathcal{D}_{pq,[aba]} = \mathcal{O}_p\partial^l\mathcal{O}_q, 2a+b+2 = p+q\ (2 \le p \le q)$$

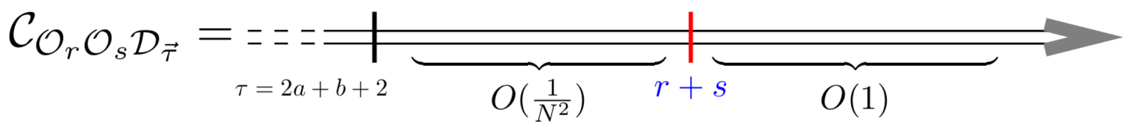
$$\bigoplus_{\tau=4,6,\dots} R_{\tau,l,[000]} = \{\underbrace{\{\mathcal{O}_2\partial^l\mathcal{O}_2\}}_{\tau=4}, \underbrace{\{\mathcal{O}_2\partial^l\,\Box\,\mathcal{O}_2, \mathcal{O}_3\partial^l\mathcal{O}_3\}}_{\tau=6}, \dots\}$$

$$\bigoplus_{\tau=5,7,\dots} R_{\tau,l,[010]} = \{\underbrace{\{\mathcal{O}_2\partial^l\mathcal{O}_3\}}_{\tau=5}, \underbrace{\{\mathcal{O}_2\partial^l\,\Box\,\mathcal{O}_3, \mathcal{O}_3\partial^l\mathcal{O}_4\}}_{\tau=7}, \dots\}$$

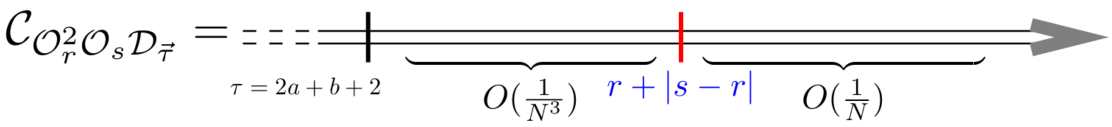
$$\bigoplus_{\tau=6,8,\dots} R_{\tau,l,[101]} = \{\underbrace{\{\mathcal{O}_3\partial^l\mathcal{O}_3\}}_{\tau=6}, \underbrace{\{\mathcal{O}_3\partial^l\,\Box\,\mathcal{O}_3, \mathcal{O}_4\partial^l\mathcal{O}_4\}}_{\tau=8} \dots\}$$

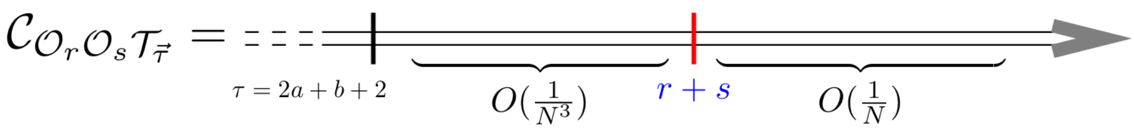
$$\bigoplus_{\tau=6,8,\dots} R_{\tau,l,[020]} = \{\underbrace{\{\mathcal{O}_2\partial^l\mathcal{O}_4, \mathcal{O}_3\partial^l\mathcal{O}_3\}}_{\tau=6}, \underbrace{\{\mathcal{O}_2\partial^l\,\Box\,\mathcal{O}_4, \mathcal{O}_3\partial^l\mathcal{O}_5, \mathcal{O}_3\partial^l\,\Box\,\mathcal{O}_3, \mathcal{O}_4\partial^l\mathcal{O}_4\}}_{\tau=8}, \dots\}$$

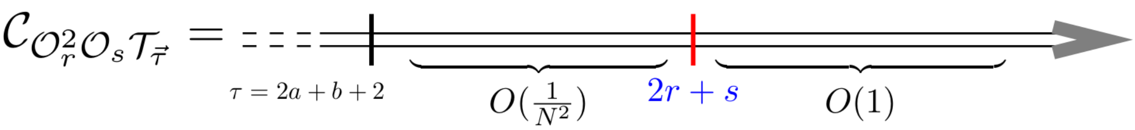
$$\mathcal{C}_{\mathcal{O}_r\mathcal{O}_s\mathcal{D}_{\vec{\tau}}} = \mathcal{C}^{(0)}_{\mathcal{O}_r\mathcal{O}_s\mathcal{D}_{\vec{\tau}}} + \frac{1}{N}\mathcal{C}^{\left(\frac{1}{2}\right)}_{\mathcal{O}_r\mathcal{O}_s\mathcal{D}_{\vec{\tau}}} + \frac{1}{N^2}\mathcal{C}^{(1)}_{\mathcal{O}_r\mathcal{O}_s\mathcal{D}_{\vec{\tau}}} + \cdots,$$



$$\mathcal{C}_{[\mathcal{O}_{r_1}\mathcal{O}_{r_2}]\mathcal{O}_s\mathcal{D}_{\vec{\tau}}}$$







$$\langle[\mathcal{O}_{r_1}\mathcal{O}_{r_2}]\mathcal{O}_s\mathcal{O}_p\mathcal{O}_q\rangle = O\left(\frac{1}{N}\right)\times \text{ free } + O\left(\frac{1}{N^3}\right)\times(\text{ free } + \text{ dynamical })+\cdots.$$

$$\frac{1}{N}\mathcal{C}^{\left(\frac{1}{2}\right)}_{[r_1r_2]s;\mathcal{D}_{\tau,l}}\times\begin{cases}\frac{1}{N^2}\eta_{\mathcal{D}}\mathcal{C}^{(0)}_{pq;\mathcal{D}}\log(U) & \text{if } p+q\leq\tau<r_1+r_2+s\\ \frac{1}{N^2}\mathcal{C}^{(1)}_{pq;\mathcal{D}} & \text{if } \tau_{\text{unitary}}\leq\tau<p+q\end{cases}$$

$$\langle\mathcal{O}_3^2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle,\langle[\mathcal{O}_2\mathcal{O}_4]\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle,\langle\mathcal{O}_r^2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle$$

$$\langle\mathcal{O}_2^2(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_3(z_4)\rangle\,;\,\langle\mathcal{O}_2^2(z_1)\mathcal{O}_3(z_2)\mathcal{O}_2(z_3)\mathcal{O}_3(z_4)\rangle$$

$$|\mathcal{O}_3|^2=\frac{3a^2}{N}\left(1-\frac{3}{a}\right)\,;\,|\mathcal{O}_2|^2=2a\,;\,\left|\mathcal{O}_2^2\right|^2=2|\mathcal{O}_2|^4\left(1+\frac{2}{a}\right)$$

$$\mathcal{P}_{[2^2]233}=|\mathcal{O}_3|^2|\mathcal{O}_2|\left|\mathcal{O}_2^2\right|g_{12}^2g_{14}g_{13}g_{34}^2\,;\,\mathcal{P}_{[2^2]323}=|\mathcal{O}_3|^2|\mathcal{O}_2|\left|\mathcal{O}_2^2\right|g_{12}^3g_{14}g_{34}^2$$

$$|\mathcal{O}_3|^2|\mathcal{O}_2|\left|\mathcal{O}_2^2\right|=12N^6(1+\cdots)$$

$$\mathcal{G}_{[2^2]233}=\frac{6}{\sqrt{a+2}}\left[1+\frac{2}{a}+\frac{4}{a}\left[U\hat{\sigma}+\frac{U\hat{\tau}}{V}\right]+\frac{2}{a}\frac{U^2\hat{\sigma}\hat{\tau}}{V}\right]$$

$$\mathcal{G}_{[2^2]323} = \frac{6}{\sqrt{a+2}}\left[U^2\hat{\sigma}^2 + \frac{2}{a}\left[\frac{U\hat{\tau}}{V} + 2U\hat{\sigma}\right] + \frac{2}{a}\left[U^2\hat{\sigma}^2 + 2\frac{U^2\hat{\sigma}\hat{\tau}}{V}\right]\right].$$

$$\mathcal{D}_{5,l} = \mathcal{O}_2\partial^l\mathcal{O}_3$$

$$\eta_{\mathcal{D}_{5,l}} = -\frac{80}{(l+1)(l+4)}, l = 0,2,4,\ldots, \eta_{\mathcal{D}_{5,l}} = -\frac{80}{(l+4)(l+7)}, l = 1,3,5,\ldots.$$

$$\mathcal{C}_{[2^2]3;\mathcal{D}_{5,l}} \propto O\left(\frac{1}{N}\right) \;;\; \mathcal{C}_{23;\mathcal{D}_5} \propto O(1)$$

$$\mathcal{H}_{[2^2]323}\Big|_{\frac{1}{N^3}} = \log(U)\sum_l \mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]3;\mathcal{D}_{5,l}}\eta_{\mathcal{D}_{5,l}}\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}}H^{4323}_{5,l,[010]} + \cdots$$

$$\mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]3;\mathcal{D}_{5,l}}\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}}\Big|_{\frac{1}{N}} = 6M^{4323}_{k=0,\gamma=5,[l+2,2]} = \frac{6}{10}(l+1)(l+7)\frac{(l+3)!\,(l+4)!}{(2l+7)!}$$

$$\mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]3;\mathcal{D}_{5,l}}\eta_{\mathcal{D}_{5,l}}\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}}\Big|_{\frac{\log(U)}{N^3}} = -\frac{12\times 4(l+7)(l+3)!^2}{(2l+7)!}\frac{1+(-1)^l}{2} - \frac{12\times 4(l+1)(l+3)!^2}{(2l+7)!}\frac{1-(-1)^l}{2}$$

$$\sum_l \mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]3;\mathcal{D}_{5,l}}\eta_{\mathcal{D}_{5,l}}\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}}H^{4323}_{5,l,[010]}\Big|_{\frac{\log(U)}{N^3}} = U^2\left[-\sum_{l\geq 0}\frac{144(l+1)(l+2)}{(l+4)(l+5)(l+6)}x_1^l + \cdots\right]$$

$$-\sum_{l\geq 0}\frac{144(l+1)(l+2)}{(l+4)(l+5)(l+6)}x_1^l = \frac{48(30-21x_1+x_1^2)}{x_1^5} + \frac{144(10-12x_1+3x_1^2)\log(1-x_1)}{x_1^6}.$$

$$\mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]3;\mathcal{D}_{5,l}}\eta_{\mathcal{D}_{5,l}}\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}} = \left[\mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]3;\mathcal{D}_{5,l}}\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}}\right]_{(3)}\left[\frac{1}{\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}}\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}}}\right]_{(2)}\left[\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}}\eta_{\mathcal{D}_{5,l}}\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}}\right]_{(1)}$$

$$\mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]3;\mathcal{D}_{5,l}}\eta_{\mathcal{D}_{5,l}}\mathcal{C}^{(0)}_{23;\mathcal{D}_{5,l}} = \frac{\left(\mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2]323}\Big|_{[010],\tau=5,l}\right)\times\left(\mathcal{H}^{(1)}_{2323}\Big|_{\log(U),[010],\tau=5,l}\right)}{\left(\mathcal{G}^{(0)}_{2323}\Big|_{[010],\tau=5,l}\right)}$$

$$\begin{aligned}\mathcal{H}_{[2^2]323}\Big|_{\frac{\log^0(U)}{N^3}} &= -\sum_l 12\times\left(2M^{4323}_{k=0,\gamma=3,[2+l]} + M^{4323}_{k=1,\gamma=3,[2+l]}\right)H^{4323}_{3,l,[010]} + \text{ higher twists}\\ &= -\left[U\sum_l\frac{12(l+1)(l+6)}{(l+3)(l+4)}x_1^l + O(U^2)\right]\end{aligned}$$

$$\mathcal{H}_{[2^2]323}\Big|_{\frac{\log^0(U)}{N^3}} = -x_1x_2\left[\frac{12(6-9x_1+2x_1^2)}{(-1+x_1)x_1^3} + \frac{12\times 6(-1+x_1)\log(1-x_1)}{x_1^4}\right] + O((x_1x_2)^2)$$

$$\left.\begin{pmatrix}\langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle & \langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_3\mathcal{O}_3\rangle\\ \langle\mathcal{O}_2^2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle & \langle\mathcal{O}_2^2\mathcal{O}_2\mathcal{O}_3\mathcal{O}_3\rangle\end{pmatrix}\right|_{[020],\tau=4}$$

$$\mathcal{S}_{[2^2]233,[020],\tau=4,l}\Big|_{\frac{1}{N^3}} \equiv C^{\left(\frac{1}{2}\right)}_{[2^2]2,\mathcal{O}_2\partial^l\mathcal{O}_2} C^{(1)}_{33,\mathcal{O}_2\partial^l\mathcal{O}_2}$$

$$\mathcal{S}_{[2^2]233,[020],\tau=4,l}\Big|_{\frac{1}{N^3}} = \frac{\left(\mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2]222}\Big|_{[0,2,0],\tau=4,l}\right) \times \left(\mathcal{G}^{(1)}_{2233}\Big|_{[0,2,0],\tau=4,l}\right)}{\left(\mathcal{G}^{(0)}_{2222}\Big|_{[0,2,0],\tau=4,l}\right)}$$

$$\mathcal{P}_{[2^2]222} = 8a^2\sqrt{a+2}\,g_{12}^2 g_{13} g_{14} g_{34}, \mathcal{G}_{[2^2]222} = \frac{4\sqrt{a+2}}{a}\left[1 + U\hat{\sigma} + \frac{U\hat{\tau}}{V}\right]$$

$$\mathcal{P}_{22pp} = 2a\left|\mathcal{O}_p\right|^{2p} g_{12}^2 g_{34}^{2p}, \mathcal{G}_{22pp} = 1 + \delta_{p,2}\left(U\hat{\sigma}^2 + \frac{U^2\hat{\tau}^2}{V^2}\right) + \frac{2p}{a}\left(U\hat{\sigma} + \frac{U\hat{\tau}}{V} + (p-1)\frac{U^2\hat{\sigma}\hat{\tau}}{V}\right)$$

$$\mathcal{S}_{[2^2]233,[020],\tau=4,l}\Big|_{\frac{1}{N^3}} = \frac{\left(4M^{4222}_{0,4,[l+2]} + 4M^{4222}_{1,4,[l+2]}\right) \times \left(12M^{2233}_{1,4,[l+2]}\right)}{\left(M^{2222}_{0,4,[l+2]} + M^{2222}_{2,4,[l+2]}\right)} = \frac{12 \times 8(l+2)!\,(l+3)!}{(2l+6)!}\frac{1+(-1)^l}{2}.$$

$$A_{[2^2]233,[020],\tau=4,l}\Big|_{\frac{1}{N^3}} = 6 \times 4\left(M^{4233}_{0,4,[l+2]} + M^{4233}_{1,4,[l+2]}\right) = \frac{12 \times 4(l+3)!\,(l+4)!}{(2l+6)!}\frac{1+(-1)^l}{2}$$

$$\mathcal{H}_{[2^2]233}\Big|_{\frac{\log^0(U)}{N^3}} = \sum_l \left(-A_{[2^2]233} + \mathcal{S}_{[2^2]233}\right)\Big|_{\frac{1}{N^3}} H_{[020],4,l} + \cdots = -U\left[\sum_l \frac{24(l+1)}{(l+3)} x_1^l + O(U^2)\right]$$

$$\mathcal{H}_{[2^2]233}\Big|_{\frac{\log^0(U)}{N^3}} = -x_1x_2\left[\frac{24(2-x_1)}{(1-x_1)x_1^2} + \frac{48\log(1-x_1)}{x_1^3}\right] + O((x_1x_2)^2)$$

$$\mathcal{H}^{ansatz}_{[2^2]323} = \frac{N_2\mathcal{P}^{(1)}(x_1,x_2)}{(x_1-x_2)^{d-1}} + \frac{N_{1,U}\log(U) + N_{1,V}\log(V)}{V(x_1-x_2)^{d-2}} + \frac{N_0}{V(x_1-x_2)^{d-2}}$$

$$\frac{U^2}{V^2}\mathcal{H}^{ansatz}_{[2^2]323}(1-x_1, 1-x_2) = \mathcal{H}^{ansatz}_{[2^2]323}(x_1,x_2)$$

$$\mathcal{H}_{[2^2]233}\Big|_{\frac{1}{N^3}} = U^2(-36 - 12U\partial_U) \circ \bar{D}_{2422}$$

$$\mathcal{H}_{[2^2]233}\Big|_{\frac{1}{N^3}} = 24\oint_{-i\infty}^{+\infty}\frac{dsdt}{(2\pi i)^2}U^{s+2}V^t\Gamma[-s]^2\Gamma[-t]^2\Gamma[-u]^2\frac{\frac{1}{2}(3+s)}{(1+s)(1+t)(1+u)},$$

$$\mathcal{P}_{[2^2]2pp} = \left|\mathcal{O}_p\right|^2|\mathcal{O}_2|\left|\mathcal{O}_2^2\right| g_{12}^2 g_{14} g_{13} g_{34}^{p-1}\ ;\ \mathcal{P}_{[2^2]p2p} = \left|\mathcal{O}_p\right|^2|\mathcal{O}_2|\left|\mathcal{O}_2^2\right| g_{12}^3 g_{14} g_{24}^{p-3} g_{34}^2,$$

$$\left|\mathcal{O}_p\right|^2|\mathcal{O}_2|\left|\mathcal{O}_2^2\right| = 4pN^{p+3}(1+\cdots)$$

$$\mathcal{G}_{[2^2]2pp} = \frac{2p}{\sqrt{a+2}}\left[1 + \frac{2}{a} + \frac{2}{a}\left[(p-1)U\hat{\sigma} + (p-1)\frac{U\hat{\tau}}{V} + \frac{(p-2)(p-1)}{2}\frac{U^2\hat{\sigma}\hat{\tau}}{V}\right]\right],$$

$$\mathcal{G}_{[2^2]p2p} = \frac{2p}{\sqrt{a+2}}\left[U^2\hat{\sigma}^2 + \frac{2}{a}\left[\frac{(p-2)(p-1)}{2}\frac{U\hat{\tau}}{V} + (p-1)U\hat{\sigma} + U^2\hat{\sigma}^2 + (p-1)\frac{U^2\hat{\sigma}\hat{\tau}}{V}\right]\right]$$

$\log^1(U)$	$\tau = p+2$	$\mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]p;[2p]}\eta_{[2p]}\mathcal{C}^{(0)}_{2p;[2p]}$
$\log^0(U)$	$\tau = p$	$\mathcal{S}^{\left(\frac{3}{2}\right)}_{[2^2]p2p} - A^{\left(\frac{3}{2}\right)}_{[2^2]p2p}$

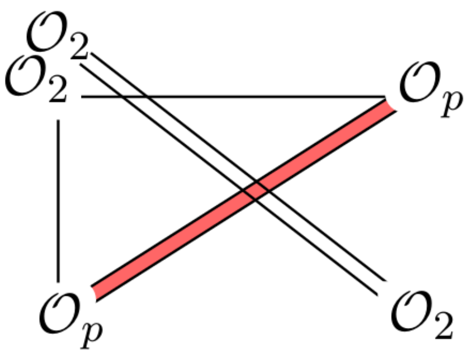
$$\mathcal{C}_{[2^2]p;\mathcal{T}_{p+2}}\frac{\eta}{N^2}\mathcal{C}_{2p\mathcal{T}_{p+2}}\log^1(U)\,;\;\mathcal{C}_{[2^2]p;\mathcal{T}_p}\mathcal{C}_{2p;\mathcal{T}_p}\log^0(U)$$

$$\mathcal{C}_{2p;\mathcal{T}_{p+2}} \propto O\left(\frac{1}{N}\right)\,;\;\mathcal{C}_{[2^2]p\mathcal{T}_{p+2}} \propto O\left(\frac{1}{N^2}\right)$$

$$\mathcal{C}_{[2^2]p;\mathcal{T}_{p+2}}\frac{\eta}{N^2}\mathcal{C}_{2p\mathcal{T}_{p+2}}\log^1(U) \propto O\left(\frac{1}{N^5}\right)$$

$$\mathcal{C}_{[2^2]p\mathcal{T}_p} \propto O\left(\frac{1}{N^2}\right)\,;\;\mathcal{C}_{2p;\mathcal{T}_p} \propto O\left(\frac{1}{N^3}\right)$$

$$\sum_{R_{p+2,[0,p-2,0]}}\mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]p;\mathcal{D}}\eta_{\mathcal{D}}\mathcal{C}^{(0)}_{2p;\mathcal{D}} = \sum_{ij}\frac{\left(\mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2]pij}\Big|_{[0,p-2,0],\tau=p+2,l}\right)\times\left(\mathcal{H}^{(1)}_{ij2p}\Big|_{\log(U),[0,p-2,0],\tau=p+2,l}\right)}{\left(\mathcal{G}^{(0)}_{ijij}\Big|_{[0,p-2,0],\tau=p+2,l}\right)}$$



$$\sum_{R_{p+2,[0,p-2,0]}}\mathcal{C}^{\left(\frac{1}{2}\right)}_{[2^2]p;\mathcal{D}}\eta_{\mathcal{D}}\mathcal{C}^{(0)}_{2p;\mathcal{D}} = \frac{\left(\mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2]p2p}\Big|_{[0,p-2,0],\tau=p+2,l}\right)\times\left(\mathcal{H}^{(1)}_{2p2p}\Big|_{\log(U),[0,p-2,0],\tau=p+2,l}\right)}{\left(\mathcal{G}^{(0)}_{2p2p}\Big|_{[0,p-2,0],\tau=p+2,l}\right)}$$

$$\mathcal{G}^{(0)}_{2p2p}\Big|_{[0,p-2,0],\tau=p+2,l}=M^{2p2p}_{0,p+2,[2+l,2]}=\frac{p^2(p-1)}{(p+2)!}\frac{(l+1)(l+p+1)!\,(l+p+4)!}{(l+p+2)(l+p+3)(2l+p+4)!}.$$

$$\mathcal{H}^{(1)}_{2p2p}\Big|_{\log(U),[0,p-2,0],\tau=p+2,l}=\frac{(l+3)!\,(l+p+1)!}{(2l+p+4)!}\begin{cases}(-1)^p c_p(-l-p-5)-4(p-1)p^2\frac{1+(-1)^p}{2} & l \text{ even}\\ c_p(l) & l \text{ odd}\end{cases}$$

$$\begin{aligned}c_3(l)\ &=-12(l+1)\\ c_4(l)\ &=-8(l+1)(l+8)\\ c_5(l)\ &=-\frac{10}{3}(l+1)(60+14l+l^2)\\ c_6(l)\ &=-(l+1)(l+10)(48+11l+l^2)\\ &\ \vdots\end{aligned}$$

$$\mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2]p2p}\Bigg|_{[0,p-2,0],\tau=p+2,l}=(2p)\times M^{4,p,2,p}_{0,p+2,[2+l,2]}=\frac{2p^2(p-1)^2(p-2)}{(p+2)!}\frac{(l+1)(l+p+4)(l+p+1)!^2}{(l+p+1)(2l+p+4)!}$$

$$2|\mathcal{O}_2|^2 2p|\mathcal{O}_p|^2\simeq\left(|\mathcal{O}_p|^2|\mathcal{O}_2||\mathcal{O}_2^2|\right)\binom{2p}{N}$$

$$\mathcal{H}_{2^2p2p}\Big|_{\frac{\log(U)}{N^3}}=-4p\times(p-2)(p-1)^2p\left[U^2\sum_l\frac{(l+1)(l+2)}{(l+p+1)_3}x_1^l+\cdots\right]$$

$$\Phi(z,s,a)=\sum_{k=0}^{\infty}\frac{z^k}{(k+a)^s}$$

$$\sum_l\frac{(l+1)(l+2)}{(l+p+1)_3}x_1^l=\frac{p-1}{2x_1}-\frac{p+2}{2x_1^2}+\left[\frac{(p+1)(p+2)}{2x_1^2}-\frac{(p+1)p}{x_1}+\frac{(p-1)p}{2}\right]\Phi(x_1,1,p+1)$$

$$\Phi(x_1,1,p+1)=\frac{1}{x_1^p}\left(-\frac{\log(1-x_1)}{x_1}-\sum_{n=0}^{p-1}\frac{x_1^n}{n+1}\right)$$

$$\begin{pmatrix}\langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}_a\mathcal{O}_b\rangle & V_{2p}^T\\ V_{[2^2]p} & \langle\mathcal{O}_2^2\mathcal{O}_p\mathcal{O}_2\mathcal{O}_p\rangle\end{pmatrix},$$

$$V_{2p}=\left(\langle\mathcal{O}_2\mathcal{O}_{p-2}\mathcal{O}_2\mathcal{O}_p\rangle,\langle\mathcal{O}_3\mathcal{O}_{p-3}\mathcal{O}_2\mathcal{O}_p\rangle,\dots\right),$$

$$V_{[2^2]p}=\left(\langle\mathcal{O}_2^2\mathcal{O}_p\mathcal{O}_2\mathcal{O}_{p-2}\rangle,\langle\mathcal{O}_2^2\mathcal{O}_p\mathcal{O}_3\mathcal{O}_{p-3}\rangle,\dots\right)$$

$$\mathcal{S}_{[2^2]p2p}=0$$

$$\mathcal{H}_{[2^2]p2p}\Big|_{\frac{\log^0(U)}{N^3}}=2p\times(p-2)(p-1)\left[-U\sum_{l=0}^{\infty}\frac{(l+1)(l+2p)}{(l+p)(l+p+1)}x_1^l+\cdots\right],$$

$$\sum_{l=0}^{\infty}\frac{(l+1)(l+2p)}{(l+p)(l+p+1)}x_1^l=-\frac{(p-1)p(x_1-1)}{x_1}\Phi(x_1,1,p)-\frac{1+p(x_1-1)}{(x_1-1)x_1}.$$

$$\mathcal{S}_{[2^2]2pp,[020],\tau=4,l}\Big|_{\frac{1}{N^3}} = \frac{\left(\mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2]222}\Big|_{[0,2,0],\tau=4,l}\right)\times\left(\mathcal{G}^{(1)}_{22pp}\Big|_{[0,2,0],\tau=4,l}\right)}{\left(\mathcal{G}^{(0)}_{2222}\Big|_{[0,2,0],\tau=4,l}\right)}.$$

$$\mathcal{S}_{[2^2]2pp,[020],\tau=4,l}\Big|_{\frac{1}{N^3}} = \frac{4\left[M^{4222}_{0,4,[l+2]}+M^{4222}_{1,4,[l+2]}\right]\times\left[2p(p-1)M^{22pp}_{1,4,[l+2]}\right]}{\left(M^{2222}_{0,4,[l+2]}+M^{2222}_{2,4,[l+2]}\right)} = \frac{16p(p-1)(l+2)!\,(l+3)!}{(2l+6)!}\frac{1+(-1)^l}{2}.$$

$$A_{[2^2]2pp,[020],4,l}\Big|_{\frac{1}{N^3}} \equiv 4p(p-1)\left(M^{42pp}_{0,4,[l+2]}+M^{42pp}_{1,4,[l+2]}\right) = \frac{4p\times 2(p-1)(l+3)!\,(l+4)!}{(2l+6)!}\frac{1+(-1)^l}{2}$$

$$\mathcal{H}_{[2^2]2pp}\Big|_{\frac{\log^0(U)}{N^3}} = \sum_l \left(-A_{[2^2]2pp}+\mathcal{S}_{[2^2]2pp}\right)H_{[020],4,l}+\cdots.$$

$$\mathcal{H}_{[2^2]2pp}\Big|_{\frac{\log^0(U)}{N^3}} = -x_2\left[\frac{4p(p-1)(2-x_1)}{(1-x_1)x_1}+\frac{8p(p-1)\log(1-x_1)}{x_1^2}+\cdots\right]$$

$$\mathcal{H}^{\text{ansatz}}_{[2^2]p2p} = \frac{N_2\mathcal{P}^{(1)}(x_1,x_2)}{(x_1-x_2)^{d-1}}+\frac{N_{1,U}\log(U)+N_{1,V}\log(V)}{V(x_1-x_2)^{d-2}}+\frac{N_0}{V(x_1-x_2)^{d-2}}$$

$$\frac{U^2}{V^2}\mathcal{H}^{ansatz}_{[2^2]p2p}(1-x_1,1-x_2) = \mathcal{H}^{ansatz}_{[2^2]p2p}(x_1,x_2)$$

$$\mathcal{H}_{[2^2]2pp}\Big|_{\frac{1}{N^3}} = 4p\frac{(p-1)}{(p-3)!}\oiint_{-i\infty}^{+\infty}\frac{dsdt}{(2\pi i)^2}U^{s+2}V^t\frac{\Gamma[-s+p-3]^2\Gamma[-t]^2\Gamma[-u]^2}{(-s)_{p-3}}\frac{\frac{1}{2}(3+s)}{(1+s)(1+t)(1+u)},$$

$$\mathcal{H}_{[2^2]2pp}\Big|_{\frac{1}{N^3}} = 2p\frac{(p-1)}{(p-3)!}U^{p-1}\partial_U^{p-3}(-3-U\partial_U)\bar{D}_{2422}$$

$$\sum_{\mathcal{T}} C^{(0)}_{[2^2]2\mathcal{T}}\times C^{\left(\frac{3}{2}\right)}_{pp\mathcal{T}}+\sum_{\mathcal{D}} C^{\left(\frac{1}{2}\right)}_{[2^2]2\mathcal{D}}\times C^{(1)}_{pp\mathcal{D}}$$

$$\sum_{\mathcal{T}} C^{(0)}_{[2^2]2\mathcal{T}}\eta_{\mathcal{T}}C^{\left(\frac{1}{2}\right)}_{pp\mathcal{T}}+\sum_{\mathcal{D}} C^{\left(\frac{1}{2}\right)}_{[2^2]2\mathcal{D}}\eta_{\mathcal{D}}C^{(0)}_{pp\mathcal{D}}$$

$$\mathcal{H}_{[2^2]sp(p+s-2)}\ ;\ \mathcal{H}_{[3^2]2pp},\mathcal{H}_{[24]2pp}\ ;\ \mathcal{H}_{[r^2]2pp}$$

$$d = (r_1+r_2)+s+p+q-2$$

$\log^0(U)$	$\tau = p+2$	$[0,p,0]$ $[1,p-2,1]$ $[0,p-2,0]$	non trivial
$\log^0(U)$	$\tau = p$	$[0,p-2,0]$	$-A_{[2^2]p3(p+1)}$

$\log^0(U)$	$\tau = 5$ $[0,1,0]$	$[0,3,0]$ $[1,1,1]$	non trivial
$\log^0(U)$	$\tau = 3$	$[0,1,0]$	$-A_{[2^2]3p(p+1)}$

$\log^1(U)$	$\tau = p+3$	$[0,p-3,0]$ $[1,p-3,1]$ $[0,p-1,0]$	0 0 $\mathcal{G}^{(\frac{1}{2})}_{[2^2](p+1)2p+1}\mathcal{H}^{(1)}_{2(p+1)3p}/\mathcal{G}^{(0)}_{2(p+1)2(p+1)}$
$\log^0(U)$	$\tau = p+1$	$[0,p-3,0]$ $[1,p-3,1](3.81)$ $[0,p-1,0](3.81)$	$-A_{[2^2](p+1)3p}$
$\log^0(U)$	$\tau = p-1$	$[\,0,p-3,0\,]$	0

$$\mathcal{P}_{[2^2](p+1)3p} = \sqrt{24p(p+1)}N^{4+p}g_{12}^4 g_{24}^{p-3} g_{34}^3$$

$$\mathcal{G}^{(\frac{3}{2})}_{[2^2](p+1)3p} = \sqrt{24p(p+1)}\left[U^2\hat{\sigma}^2 + 2(p-1)\frac{U^2\hat{\sigma}\hat{\tau}}{V} + \frac{(p-1)(p-2)}{2}\frac{U^2\hat{\tau}^2}{V^2} + \frac{2U^3\hat{\sigma}^2\hat{\tau}}{V} + (p-1)\frac{\hat{\sigma}\hat{\tau}^2}{V^2}\right].$$

$$\mathcal{H}_{[2^2]3p(p+1)}(U,V,\hat{\sigma},\hat{\tau}) = U\hat{\sigma}\mathcal{H}_{[2^2]p3(p+1)}\left(\frac{1}{U},\frac{V}{U},\frac{1}{\hat{\sigma}},\frac{\hat{\tau}}{\hat{\sigma}}\right)$$

$$\mathcal{H}_{[2^2](p+1)3p}(U,V,\hat{\sigma},\hat{\tau}) = \frac{U^3}{V^3}\hat{\tau}\mathcal{H}_{[2^2]p3(p+1)}\left(V,U,\frac{\hat{\sigma}}{\hat{\tau}},\frac{1}{\hat{\tau}}\right)$$

$$\mathcal{P}_{[2^2]p3p+1} = \sqrt{24p(p+1)}N^{p+4}g_{12}^3 g_{14} g_{24}^{p-3} g_{34}^3$$

$$\mathcal{H}_{[2^2]p3(p+1)}\Big|_{\frac{1}{N^3}} = \sqrt{24p(p+1)} \oiint \frac{dsdt}{(2\pi i)^2} U^{s+3}V^t \frac{\Gamma[-s]^2\Gamma[-t]^2\Gamma[-u+p-3]^2}{(-u)_{p-3}} \frac{1}{(1+s)(1+t)(1+u)}$$

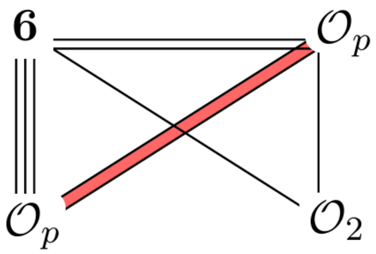
$$\times \left[\frac{1}{2(p-3)!}\frac{(p-1)t+p(s+2)}{(2+s)} + \frac{\hat{\sigma}}{2(p-2)!}\frac{2t+p(u+2)}{(2+u)} - \frac{\hat{\tau}}{2(p-3)!}\right]$$

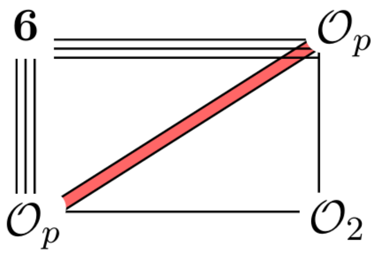
$\langle[\mathbf{3}^2], \mathbf{2}, \mathbf{p}, \mathbf{p}\rangle$ and $\langle[\mathbf{24}], \mathbf{2}, \mathbf{p}, \mathbf{p}\rangle$

$$\mathcal{H}_{[3^2]2pp}\ ;\ \mathcal{H}_{[24]2pp}$$

$$\mathcal{H}_{6p2p}\Big|_{\frac{\log^0(U)}{N^3}} = \sum_l \left(\mathcal{S}_{6p2p}^{\left(\frac{3}{2}\right)} - A_{6p2p}^{\left(\frac{3}{2}\right)}\right) H_{[0,p-2,0],\tau=p,l} + \cdots,$$

$$\mathcal{H}_{\mathbf{6}p2p}\Big|_{\frac{\log^0(U)}{N^3}} = \sum_l \left(-A_{\mathbf{6}p2p}^{\left(\frac{3}{2}\right)}\right) H_{[0,p-2,0],\tau=p,l} + \cdots.$$





$$\frac{A_{[3^2]p2p}}{\sqrt{(18)(p)(2)(p)}}\Bigg|_{\frac{1}{N^3}} = \frac{A_{[24]p2p}}{\sqrt{(8)(p)(2)(p)}}\Bigg|_{\frac{1}{N^3}} = 3(p-2)M_{0,p,[l+2]}^{6p2p} + (p-2)(p-3)M_{1,p,[l+2]}^{6p2p},$$

$$\sum_{R_{p+2,[0,p-2,0]}} C_{\mathbf{6}p;\mathcal{D}}^{\left(\frac{1}{2}\right)} \eta_{\mathcal{D}} C_{2p;\mathcal{D}}^{(0)} = \sum_{ij} \frac{\left(\mathcal{G}_{\mathbf{6}pij}^{\left(\frac{1}{2}\right)}\Big|_{[0,p-2,0],\tau=p+2,l}\right) \times \left(\mathcal{H}_{i,j,2,p}^{(1)}\Big|_{[0,p-2,0],\tau=p+2,l}\right)}{\left(\mathcal{G}_{i,j,i,j}^{(0)}\Big|_{[0,p-2,0],\tau=p+2,l}\right)}$$

$$\mathcal{G}_{[3^2]p3p-1}^{\left(\frac{1}{2}\right)} \neq 0$$

$$\sum_{R_{p+2,[0,p-2,0]}} C_{[3^2]p;\mathcal{D}}^{\left(\frac{1}{2}\right)} \eta_{\mathcal{D}} C_{2p;\mathcal{D}}^{(0)} = \frac{\left(\mathcal{G}_{[3^2]p3p-1}^{\left(\frac{1}{2}\right)}\Big|_{[0,p-2,0],\tau=p+2,l}\right) \times \left(\mathcal{H}_{3,p-1,2,p}^{(1)}\Big|_{[0,p-2,0],\tau=p+2,l}\right)}{\left(\mathcal{G}_{3,p-1,3,p-1}^{(0)}\Big|_{[0,p-2,0],\tau=p+2,l}\right)}.$$

$$\sum_{R_{p+2,[0,p-2,0]}} C_{[24]p;\mathcal{D}}^{\left(\frac{1}{2}\right)} \eta_{\mathcal{D}} C_{2p;\mathcal{D}}^{(0)} = \frac{\left(\mathcal{G}_{[24]p2p}^{\left(\frac{1}{2}\right)}\Big|_{[0,p-2,0],\tau=p+2,l}\right) \times \left(\mathcal{H}_{2,p,2,p}^{(1)}\Big|_{[0,p-2,0],\tau=p+2,l}\right)}{\left(\mathcal{G}_{2p2p}^{(0)}\Big|_{[0,p-2,0],\tau=p+2,l}\right)}$$

$$\sum_{R_{p+4,[0,p-2,0]}} \mathcal{C}^{\left(\frac{1}{2}\right)}_{\mathbf{6}p;\mathcal{D}} \eta_{\mathcal{D}} \mathcal{C}^{(0)}_{2p;\mathcal{D}} = \sum_{ij} \frac{\left(\mathcal{G}^{\left(\frac{1}{2}\right)}_{\mathbf{6}pij}\Big|_{[0,p-2,0],\tau=p+4,l}\right) \times \left(\mathcal{H}^{(1)}_{ij2p}\Big|_{[0,p-2,0],\tau=p+4,l}\right)}{\left(\mathcal{G}^{(0)}_{i,j,i,j}\Big|_{[0,p-2,0],\tau=p+2,l}\right)}.$$

$$\mathcal{P}_{[3^2]2pp} = \sqrt{36p^2N^{2p+8}} g_{12}^2 g_{13}^2 g_{14}^2 g_{34}^{p-2} \, ; \, \mathcal{P}_{[24]2pp} = \sqrt{16p^2N^{2p+8}} g_{12}^2 g_{13}^2 g_{14}^2 g_{34}^{p-2}$$

$$\mathcal{H}_{[24]2pp}\Big|_{\frac{1}{N^3}} = (4p)\frac{(p-2)}{(p-4)!} \oiint_{-i\infty}^{+\infty} \frac{dsdt}{(2\pi i)^2} U^{s+2}V^t \frac{\Gamma[-s+p-4]^2\Gamma[-t]^2\Gamma[-u]^2}{(-s)_{p-4}} \frac{\frac{1}{6}(4+s)(2tu+s+3)}{(s+1)(t+1)(u+1)}$$

$$\mathcal{H}_{[3^2]2pp}\Big|_{\frac{1}{N^3}} = (6p)\frac{(p-2)}{(p-4)!} \oiint_{-i\infty}^{+\infty} \frac{dsdt}{(2\pi i)^2} U^{s+2}V^t \frac{\Gamma[-s+p-4]^2\Gamma[-t]^2\Gamma[-u]^2}{(-s)_{p-4}} \frac{\frac{1}{6}(2(s+4)^2+(s+1)(tu+1))}{(s+1)(t+1)(u+1)}$$

$$\mathcal{P}_{[r^2]2pp} = (2rp)N^{r+p+1} g_{12}^2 g_{13}^{r-1} g_{14}^{r-1} g_{34}^{1+p-r}$$

$$\mathcal{H}_{[r^2]2pp}\Big|_{\frac{1}{N^3}} = (2rp)\frac{(p+1-r)}{(p-1-r)!} \oiint_{-i\infty}^{+\infty} \frac{dsdt}{(2\pi i)^2} U^{s+2}V^t \Gamma[-s]\Gamma[-s+p-r-1]\Gamma[-t]^2\Gamma[-u]^2 \frac{N_r(s,t)}{(s+1)(t+1)(u+1)}$$

$$N_2 = (3+s)\left(\frac{1}{2}\right)$$

$$N_3 = (3+s)\left(\frac{s}{3}+\frac{11}{6}\right) + \frac{tu}{6}(s+1)$$

$$N_4 = (3+s)\left(\frac{s^2}{8}+\frac{35s}{24}+\frac{13}{3}\right) + \left[\frac{t^2u^2}{40} + tu\left(\frac{s}{10}+\frac{71}{120}\right)\right](s+1)$$

$$N_5 = (3+s)\left(\frac{s^2}{30}+\frac{5s^2}{8}+\frac{157s}{40}+\frac{25}{3}\right) + \left[\frac{t^3u^3}{504} + t^2u^2\left(\frac{11}{126}+\frac{s}{72}\right) + tu\left(\frac{43s^2}{1260}+\frac{547s}{168}+\frac{233}{168}\right)\right](s+1)$$

$$\cdots$$

$$N_r = \frac{r(r-1)}{2} + (s+1)n_r(s,t)$$

$$n_3 = \frac{1}{6}(-1-t)(-1-u) + \left[3-\frac{1}{2}(-s)\right]$$

$$n_4 = \frac{1}{40}(-1-t)_2(-1-u)_2 + \left[\frac{2}{3}-\frac{1}{8}(-s)\right](-1-t)(-1-u) + \left[9-3(-s)+\frac{1}{4}(-s)_2\right]$$

$$\vdots$$

$$n_r = \sum_{I=0}^{r-2}\sum_{J=0}^{I} \frac{r(r-1)}{2} \frac{(-1)^{I+J}\frac{\Gamma[2+I]}{\Gamma[1+J]\Gamma[r-1-I]\Gamma[r+1-J]}}{((r-1-J)+(r-2-I))\Gamma[1+I-J]} (-s)_{I-J}(-1-t)_{r-2-I}(-1-u)_{r-2-I}$$

$$\sum_{j=1}^{4} \delta_{ij} = p_i + 2 \, ; \sum_{j=1}^{4} d_{ij} = p_i - 2$$

$$\sum_{1234} \equiv \sum_{[\tilde{s},\tilde{t}]} \prod_{i<j} \frac{\vec{Y}_{ij}^{2d_{ij}}}{\Gamma[d_{ij}+1]} \oiint_{-i\infty}^{i\infty} \frac{dsdt}{(2\pi i)^2} \prod_{i<j} \frac{\Gamma[\delta_{ij}]}{\vec{X}_{ij}^{2\delta_{ij}}}$$

$$\langle \mathcal{O}_{p_1}(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\rangle = \text{ free } + \mathcal{R}_{1234}\left(N^{-2+\frac{1}{2}\sum_i p_i} p_1 p_2 p_3 p_4 \sum_{1234} \mathcal{M}_{\vec{p}}(\delta_{ij}, d_{ij})\right),$$

$$\mathcal{R}_{1234} = \vec{X}_{13}^4 \vec{X}_{24}^4 \vec{Y}_{13}^4 \vec{Y}_{24}^4 \prod_{i,j=1,2} (x_i - y_j)$$

$$\mathcal{R}_{1234} = \vec{X}_{13}^4 \vec{X}_{24}^4 \vec{Y}_{12}^4 \vec{Y}_{34}^4 \mathcal{I}(U, V, \hat{\sigma}, \hat{\tau})$$

$$\boldsymbol{\rho}_{ij} = \delta_{ij} - d_{ij}\,;\ \sum_i \boldsymbol{\rho}_{ij} = +4$$

$$\mathcal{M}_{\vec{p}}(\delta_{ij}, d_{ij}) - \mathbf{s} \equiv \boldsymbol{\rho}_{12} = \boldsymbol{\rho}_{34}\,;\ -\mathbf{t} \equiv \boldsymbol{\rho}_{14} = \boldsymbol{\rho}_{23}\,;\ -\mathbf{u} \equiv \boldsymbol{\rho}_{13} = \boldsymbol{\rho}_{24}$$

$$\mathcal{M}_{\vec{p}}(\delta_{ij}, d_{ij}) = \mathcal{M}_{\vec{p}}^{\text{tree-level}}(\delta_{ij}, d_{ij}) + \frac{1}{N^2}\mathcal{M}_{\vec{p}}^{1-\text{ loop}}(\delta_{ij}, d_{ij}) + \cdots$$

$$\mathcal{M}_{\vec{p}}^{\text{tree-level}}(\delta_{ij}, d_{ij}) = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)}$$

$$\langle \mathcal{O}_2^2(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\rangle = free + \mathcal{R}_{1234}\left(N^{-1+\frac{1}{2}\sum_{i=2}^4 p_i}(4)p_2p_3p_4 \sum_{1234} \mathcal{M}_{[2^2]p_2p_3p_4}(\delta_{ij}, d_{ij})\right)$$

$$\mathcal{M}^{1}_{[2^2]p_2p_3(p_2+p_3-2)} = +\frac{p_2-1}{(1+\mathbf{s})(1+\mathbf{t})(1+\mathbf{u})} + \frac{p_3-2}{(\mathbf{s})(1+\mathbf{t})(1+\mathbf{u})} + \frac{3-p_2-p_3}{(\mathbf{s})(\mathbf{t})(1+\mathbf{u})}$$

$$\mathcal{M}^{\hat{\tau}}_{[2^2]p_2p_3(p_2+p_3-2)} = -\frac{2}{(1+\mathbf{s})(\mathbf{t})(1+\mathbf{u})} + \frac{2}{(1+\mathbf{s})(-1+\mathbf{t})(1+\mathbf{u})}$$

$$\mathcal{M}^{\hat{\sigma}}_{[2^2]p_2p_3(p_2+p_3-2)} = \mathcal{M}^{1}_{[2^2]p_3p_2(p_2+p_3-2)}(\mathbf{s} \leftrightarrow \mathbf{u})$$

$$(1+\mathbf{s})(\mathbf{t})(1+\mathbf{u}),\ (1+\mathbf{s})(-1+\mathbf{t})(1+\mathbf{u}),\ (\mathbf{s})(1+\mathbf{t})(\mathbf{u})$$

$$\mathcal{M}_{[2^2]p_2p_3p_4} = \left(\mathcal{M}_{[2^2]su} + \text{ crossing }\right) + \mathcal{M}_{[2^2]stu}$$

$$\mathcal{M}_{[2^2]su} = \frac{p_{1,0,1}(d_{ij}, \vec{p})}{(1+\mathbf{s})(\mathbf{t})(1+\mathbf{u})} + \frac{p_{1,-1,1}(d_{ij}, \vec{p})}{(1+\mathbf{s})(-1+\mathbf{t})(1+\mathbf{u})} + \frac{p_{0,1,0}(d_{ij}, \vec{p})}{(\mathbf{s})(1+\mathbf{t})(\mathbf{u})},\ \mathcal{M}_{[2^2]stu} = \frac{p_{1,1,1}(d_{ij}, \vec{p})}{(1+\mathbf{s})(1+\mathbf{t})(1+\mathbf{u})}$$

$$f_{ij;mn} = \frac{p_i + p_j - p_m - p_n}{2} + d_{ij} + d_{mn}, \quad \begin{array}{l} f_s := f_{12;34} \\ f_t := f_{14;23} \\ f_u := f_{13,24} \end{array}$$

$$\mathcal{M}_{[2^2]su} = \frac{\frac{1}{4}\left((p_4-p_2-p_3)(f_t-3)+(f_t-4)\right)f_t}{(1+\mathbf{s})(\mathbf{t})(1+\mathbf{u})} + \frac{\frac{1}{8}(p_2+p_3-p_4)(f_t-2)f_t}{(1+\mathbf{s})(-1+\mathbf{t})(1+\mathbf{u})} + \frac{\frac{1}{8}(4-p_2-p_3-p_4)f_sf_u}{(\mathbf{s})(1+\mathbf{t})(\mathbf{u})}$$

$$\mathcal{M}_{[2^2]stu} = \frac{4 - \frac{1}{4}(f_s^2 + f_t^2 + f_u^2) - \frac{1}{8}\left((4 - p_2 + p_3 - p_4)f_s f_t + (4 - p_2 - p_3 + p_4)f_s f_u + (4 + p_2 - p_3 - p_4)f_t f_u\right)}{(1+\mathbf{s})(1+\mathbf{t})(1+\mathbf{u})}$$

$$\Lambda \to \infty \text{ of } \mathcal{M}_{[2^2]p_2p_3p_4}(\Lambda\delta_{ij}, d_{ij}, p_i)$$

$$\mathcal{M}_{[2^2]su}(\Lambda\delta_{ij}, d_{ij}, p_i)\Big|_{\frac{1}{\Lambda^3}} = \frac{1}{\underline{stu}}\left(\frac{1}{8}(2 + p_4 - p_2 - p_3)f_t^2 + \frac{1}{8}(4 - p_2 - p_3 - p_4)f_s f_u - \frac{1}{2}(2 + p_4 - p_2 - p_3)f_t\right)$$

$$\mathcal{M}_{[2^2]p_2p_3p_4}(\Lambda\delta_{ij}, d_{ij}, p_i)\Big|_{\frac{1}{\Lambda^3}} = \frac{1}{\underline{stu}}(f_s + f_t + f_u - 4)\left(-\frac{1}{3} + \frac{1}{8}(p_2 - p_3 - p_4)f_s + \text{ crossing}\right) = 0$$

$$\lim_{\Lambda\to\infty} \mathcal{M}_{[2^2]p_2p_3p_4}(\Lambda\delta_{ij}, d_{ij}, p_i)\Big|_{\frac{1}{\Lambda^4}} = \frac{\frac{1}{8}(p_2 + p_3 + p_4 - 4)\left(-\underline{s}^2(4 - 6f_s + f_s^2) + 2(-f_s + f_t f_u)\underline{tu} + \text{ crossing}\right)}{\underline{s}^2\underline{t}^2\underline{u}^2} - \frac{\left(\underline{s}^2(-2 + f_s) + \frac{1}{2}f_s\left(d_{12} + d_{34} + 2 - \frac{1}{2}f_s\right)\underline{tu} + \text{ crossing}\right)}{\underline{s}^2\underline{t}^2\underline{u}^2}$$

$$\mathcal{M}_{[2^2]p_2p_3p_4}(\Lambda\delta_{ij}, \Lambda d_{ij}, \Lambda p_i) \approx \underbrace{(f_{\underline{s}} + f_{\underline{t}} + f_{\underline{u}})}_{=0}(\dots) + \frac{1}{\Lambda}\left(-\frac{(p_2 + p_3 + p_4)}{8}\frac{(\underline{\mathbf{s}}f_{\underline{t}} - \underline{\mathbf{t}}f_{\underline{s}})^2}{(\underline{\mathbf{s}})^2(\underline{\mathbf{t}})^2(\underline{\mathbf{u}})^2}\right) + O\left(\frac{1}{\Lambda^2}\right),$$

$$\langle \mathcal{O}_r^2(z_1)\mathcal{O}_2(z_2)\mathcal{O}_p(z_3)\mathcal{O}_p(z_4)\rangle = \mathit{free} + \mathcal{R}_{1234}\left(N^{-2+r+p}\, r^2(2)p^2 \oint_{1234} \mathcal{M}_{[r^2]2pp}(\delta_{ij}, d_{ij})\right)$$

$$\mathcal{M}_{[r^2]2pp} = \sum_{I,J\geq 0} \frac{\Gamma[\delta_{12} + I - J]\Gamma[\delta_{14} - (2+I)]\Gamma[\delta_{13} - (2+I)]}{\Gamma[\delta_{12}]\Gamma[\delta_{14}]\Gamma[\delta_{13}]} \frac{\Gamma[1 + d_{14}]\Gamma[1 + d_{13}]}{\Gamma[1 + d_{14} - (1+I)]\Gamma[1 + d_{13} - (1+I)]}\mathcal{M}_{[r^2]2pp}^{(I,J)}$$

$$\mathcal{M}_{[r^2]2pp}^{(I,J)} = (p + 1 - r)r(r-1)\frac{(-1)^{I+J}\frac{\Gamma[2+I]\Gamma[r-1-I]}{\Gamma[1+J]\Gamma[r+1-J]}}{((r-1-J) + (r-2-I))\Gamma[1+I-J]}$$

$$\mathcal{M}_{[r^2]2pp}(\Lambda\delta_{ij}, d_{ij}, p_i)\Big|_{\frac{1}{\Lambda^3}} = 0,\ \ \mathcal{M}_{[r^2]2pp}(\Lambda\delta_{ij}, d_{ij}, p_i)\Big|_{\frac{1}{\Lambda^4}} = \frac{1}{(2r-3)(r-2)!^2\,\underline{t}^2\underline{u}^2}$$

$$\frac{(-s)_{I-J}}{(1+s)(r-I-3-t)_{I+2}(r-I-3-u)_{I+2}}.$$

$$\mathcal{M}_{[r^2]2pp} = \frac{(p+1-r)(r-1)}{(1+\mathbf{s})(1+\mathbf{t})(1+\mathbf{u})} + \sum_{m,n=0}^{r-2}\frac{(-1)^{m+n+1}(p+1-r)(r-1)!^2}{(m+n+1)_2(r-2-m)!\,(r-2-n)!\,m!\,n!}\frac{2}{(1+\mathbf{s})(-m+\mathbf{t})(-n+\mathbf{u})}$$

$$\mathcal{M}_{[r^2]2pp}(\Lambda\delta_{ij})\Big|_{\frac{1}{\Lambda^3}} = \frac{(p+1-r)(r-1)}{\underline{stu}} + \frac{2}{\underline{stu}}\sum_{m,n}\frac{(-1)^{m+n+1}(p+1-r)(r-1)!^2}{(m+n+1)_2(r-2-m)!\,(r-2-n)!\,m!\,n!} = 0$$

$$\sum_{m,n} -\frac{(p+1-r)(r-1)}{2}$$

$$\lim_{z_0\to z_1}\langle \mathcal{O}_{p_0}(z_0)\mathcal{O}_{p_1}(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\rangle = \langle [\mathcal{O}_{p_0}\mathcal{O}_{p_1}](z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\rangle$$

$$\mathcal{O}_{p_0}(\vec{X}_0,\vec{Y}_0)\mathcal{O}_{p_1}(\vec{X}_1,\vec{Y}_1) = \sum_{\gamma=|p_0-p_1|O_{\gamma,\underline{\lambda}}}^{p_0+p_1} g_{01}^{\frac{p_0+p_1}{2}-\frac{\gamma}{2}} C_{p_0p_1}^{O_{\gamma,\underline{\lambda}}}\mathbb{D}^{\gamma,\underline{\lambda}}(\vec{X}_{01},\partial_1)O_{\gamma,\underline{\lambda}}(\vec{X}_1)$$

$$O_{\gamma,\underline{\lambda}} \sim \text{span}\left[\phi^{\frac{p_0-p_1+\gamma}{2}}(\partial^{|\underline{\lambda}|})\phi^{\frac{p_1-p_0+\gamma}{2}}+\cdots\right]$$

$$\lim_{z_0\to z_1}\langle \mathcal{O}_{p_0}(z_0)\mathcal{O}_{p_1}(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\rangle \equiv \lim_{\vec{X}_0\to\vec{X}_1}\left(\lim_{\vec{Y}_0\to\vec{Y}_1}\langle \mathcal{O}_{p_0}(z_0)\mathcal{O}_{p_1}(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\rangle\right)$$

$$C_{p_0p_1}^{O_{\gamma,\underline{\lambda}}} = \sum_{\underline{t}\vdash\gamma} |\langle \mathcal{O}_{p_0}\mathcal{O}_{p_1}O_{\underline{t}}\rangle|\cdot(|\langle O_{\underline{t}}O_{\gamma,\underline{\lambda}}\rangle|)^{-1}$$

$$\langle \mathcal{O}_{p_0}(z_0)\mathcal{O}_{p_1}(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\rangle = free + N^{-3+\frac{1}{2}\sum_i p_i} \oint\!\!\!\!\!\!\sum_{01234} \widetilde{\mathcal{M}}_{\vec{p}}^{(5)}(\tilde{\delta}_{ij},\tilde{d}_{ij})$$

$$\oint\!\!\!\!\!\!\sum_{01234} \equiv \sum_{[\tilde{d}]}\prod_{i<j}\frac{\vec{Y}_{ij}^{2\tilde{d}_{ij}}}{\Gamma[\tilde{d}_{ij}+1]}\oint_{-i\infty}^{i\infty}\frac{[d\tilde{\delta}]}{(2\pi i)^5}\prod_{i<j}\frac{\Gamma[\tilde{\delta}_{ij}]}{\vec{X}_{ij}^{2\tilde{\delta}_{ij}}}$$

$$\sum_{j=0}^{4}\tilde{\delta}_{ij} = p_i\,;\ \sum_{j=0}^{4}\tilde{d}_{ij} = p_i$$

$$\lim_{\vec{Y}_0\to\vec{Y}_1}\langle \mathcal{O}_{p_0}(z_0)\mathcal{O}_{p_1}(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\mathcal{O}_{p_4}(z_4)\rangle =$$

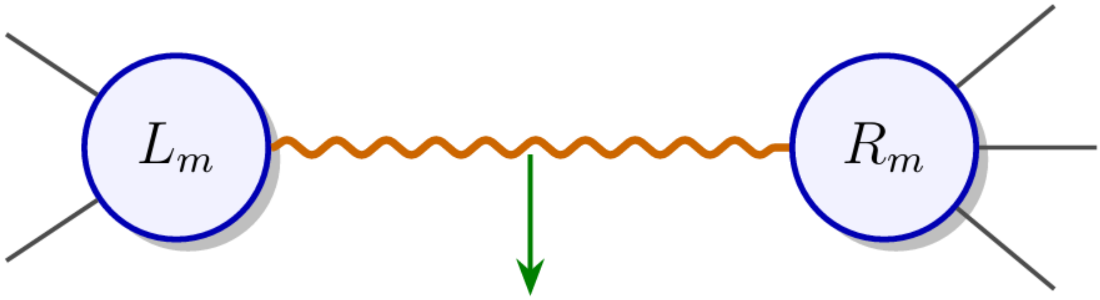
$$\mathcal{P}_{[p_0p_1]p_2p_3p_4}[g_{ij}]\sum_{\gamma=\gamma_{\min}}^{\gamma_{\max}}\left(\sum_{i+j=\frac{1}{2}(\gamma-\gamma_{\min})}(\hat{\sigma})^i(\hat{\tau})^j\mathcal{C}_{i,j}(\vec{X}_0,\vec{X}_1,\vec{X}_2,\vec{X}_3,\vec{X}_4)\right),$$

$$\lim_{\vec{X}_0\to\vec{X}_1}D_{\Delta_0,\Delta_1,\Delta_2,\Delta_3,\Delta_4}(\vec{X}_0,\vec{X}_1,\vec{X}_2,\vec{X}_3,\vec{X}_4) = D_{\Delta_0+\Delta_1,\Delta_2,\Delta_3,\Delta_4}(\vec{X}_1,\vec{X}_2,\vec{X}_3,\vec{X}_4)$$

$$\widetilde{\mathcal{M}}_{\vec{p}}^{(5)}(\tilde{\delta}_{ij},\tilde{d}_{ij}) = \sum_{k_1,k_2=0}^{2}\frac{P_{k_1,k_2}(\tilde{\delta}_{ij},\tilde{d}_{mn},\vec{p})}{(\tilde{\boldsymbol{\rho}}_{12}+k_1)(\tilde{\boldsymbol{\rho}}_{45}+k_2)} + \sum_{k_3=0}^{2}\frac{P_{k_3}(\tilde{\delta}_{ij},\tilde{d}_{mn},\vec{p})}{(\tilde{\boldsymbol{\rho}}_{12}+k_3)} + P(\tilde{\delta}_{ij},\tilde{d}_{mn},\vec{p}) + (\text{ perms })$$

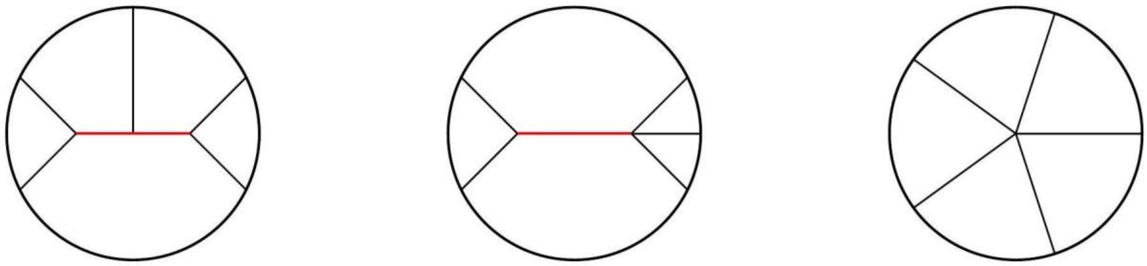
$$\widetilde{\mathcal{M}}(\tilde{\delta}_{ij}) \approx \sum_m \frac{\mathcal{Q}_m(\tilde{\delta}_{ij})}{\tilde{\delta}_{LR}-(\Delta-J)-2m}$$

$$\tilde{\delta}_{LR} = \sum_{a=1}^{k}\sum_{b=k+1}^{n}\tilde{\delta}_{ab}$$



$$\frac{1}{\tilde{\delta}_{LR} - \Delta + J - 2m}$$

$$\mathcal{Q}_m = \frac{-2\Gamma(\Delta)m!}{(\Delta-1)_m} L_m R_m$$



$$\widetilde{\mathcal{M}}_{\vec{p}}^{(5)}(\tilde{\delta}_{ij}, \tilde{d}_{ij})$$

$$\mathcal{O}_M(z) = \sum_p \mathcal{O}_p(z),$$

$$\widetilde{\mathcal{M}}_{\vec{p}}(\tilde{\delta}_{ij}, \tilde{d}_{ij}) \approx \sum_q \sum_{m+r=q} A_{(\Delta,m|[a,b,a],r)} \frac{\widetilde{\mathcal{M}}_{L,m,r} \times \widetilde{\mathcal{M}}_{R,m,r}}{\tilde{\rho}_{LR} - (\Delta - J - b - 2a) - 2q}$$

$$\widetilde{\boldsymbol{\rho}}_{LR} = \sum_{a=1}^{k} \sum_{b=k+1}^{n} \widetilde{\boldsymbol{\rho}}_{ab} \,.$$

$$\widetilde{\mathcal{M}}_{L,m,r} = \frac{(\hat{\delta}_L)^m}{m!} \frac{(\hat{d}_L)^r}{r!} \circ \widetilde{\mathcal{M}}_L(\tilde{\delta}_{ab}, \tilde{d}_{ab})$$

$$\hat{\delta}_L \circ \widetilde{\mathcal{M}} \equiv \sum_{a,b \in L, a<b} \tilde{\delta}_{ab} \widetilde{\mathcal{M}}(\tilde{\delta}_{ab} + 1, \tilde{d}_{ab})\,;\; \hat{d}_L \circ \widetilde{\mathcal{M}} \equiv \sum_{a,b \in L, a<b} \tilde{d}_{ab} \widetilde{\mathcal{M}}(\tilde{\delta}_{ab}, \tilde{d}_{ab} - 1),$$

$$A_{(\Delta,m|[a,b,a],r)} = \frac{1}{|O|^2} \frac{-2\Gamma(\Delta)m!}{(\Delta-1)_m} \frac{(-1)^r (b+2a+1-r)!\, r!}{(b+2a+1)!\,(b+2a)!}$$

$$\sum_{j \neq i} \gamma_{ij} = \sum_{j \neq i} n_{ij}.$$

$$\widetilde{\mathcal{M}}_5 = \sum_{k_1,k_2=0}^{2} \frac{P_{k_1,k_2}(\gamma_{ij}, n_{ij})}{(\tilde{\boldsymbol{\rho}}_{12} + k_1)(\tilde{\boldsymbol{\rho}}_{45} + k_2)} + \sum_{k_3=0}^{2} \frac{P_{k_3}(\gamma_{ij}, n_{ij})}{(\tilde{\boldsymbol{\rho}}_{12} + k_3)} + P(\gamma_{ij}, n_{ij}) + (perms),$$

$$\lim_{\vec{X}_0 \to \vec{X}_1} \left[\lim_{\vec{Y}_0 \to \vec{Y}_1} \mathop{\sum\!\!\!\!\!\!\int}_{01234} \widetilde{\mathcal{M}}^{(5)}_{\vec{p}}(\tilde{\delta}_{ij}, \tilde{d}_{ij}) \right]$$

$$\lim_{\vec{Y}_0 \to \vec{Y}_1} \mathop{\sum\!\!\!\!\!\!\int}_{01234} \widetilde{\mathcal{M}}^{(5)}_{222pp} = \vec{Y}_{12}^{4} \vec{Y}_{14}^{2} \vec{Y}_{13}^{2} \vec{Y}_{34}^{2p-2} \sum_{0 \le m+n \le 2} (\hat{\sigma})^m (\hat{\tau})^n \oint_{-i\infty}^{+i\infty} \frac{[d\tilde{\delta}]}{(2\pi i)^5} \prod_{i<j} \frac{\Gamma[\tilde{\delta}_{ij}]}{X_{ij}^{2\tilde{\delta}_{ij}}} \widetilde{\mathcal{M}}_{\hat{\sigma}^m \hat{\tau}^n}(\tilde{\delta}_{ij})$$

$$\widetilde{\mathcal{M}}_{\hat{\tau}^2} = \frac{2p}{(p-3)!} \left[\frac{\tilde{\delta}_{34}(-1-2p+p(\tilde{\delta}_{02}+\tilde{\delta}_{04}-\tilde{\delta}_{13}+\tilde{\delta}_{23})-\tilde{\delta}_{34})}{(-1+\tilde{\delta}_{04})(-1+\tilde{\delta}_{23})} - \frac{p\tilde{\delta}_{24}\tilde{\delta}_{34}}{(-1+\tilde{\delta}_{03})(-1+\tilde{\delta}_{14})} + (0 \leftrightarrow 1) \right],$$

$$\widetilde{\mathcal{M}}_{\hat{\sigma}^2} = \widetilde{\mathcal{M}}_{\tau^2}\Big|_{3 \leftrightarrow 4},$$

$$\widetilde{\mathcal{M}}_{\hat{\sigma}\hat{\tau}} = \frac{2p}{(p-3)!} \left[\frac{\tilde{\delta}_{34}(-1-2p+p^2+p(\tilde{\delta}_{01}+\tilde{\delta}_{04}+\tilde{\delta}_{13})-\tilde{\delta}_{34})}{(-1+\tilde{\delta}_{04})(-1+\tilde{\delta}_{13})} + \left[\frac{p(p-\tilde{\delta}_{13})\tilde{\delta}_{34}}{(-1+\tilde{\delta}_{03})(-1+\tilde{\delta}_{24})} + (3 \leftrightarrow 4) \right] + (0 \leftrightarrow 1) \right]$$
$$+ \frac{2p}{(p-3)!} \left[\frac{2p\tilde{\delta}_{34}}{(1-\tilde{\delta}_{03})} + \frac{2p\tilde{\delta}_{34}}{(1-\tilde{\delta}_{04})} + (0 \leftrightarrow 1) + (0 \leftrightarrow 2) \right]$$

$$\begin{cases} \widetilde{\mathcal{M}}_{\hat{\tau}} = \dfrac{(\dots) + (p-2)(\dots)}{(p-2)!} \\ \widetilde{\mathcal{M}}_{\hat{\sigma}} = \mathcal{M}_{\hat{\tau}}|_{3 \leftrightarrow 4} \end{cases} ; \ \widetilde{\mathcal{M}}_1 = \frac{(\dots) + (p-2)(\dots)}{(p-2)!}$$

$$\frac{\tilde{\delta}}{-k+\tilde{\delta}} = 1 + \frac{k}{-k+\tilde{\delta}}.$$

$$\frac{1}{-k+\tilde{\delta}} = \sum_{j=1}^{k} \frac{\Gamma[k]}{\Gamma[k-j+1]} \frac{\Gamma[\tilde{\delta}-j]}{\Gamma[\tilde{\delta}]}$$

$$\tilde{\delta}^k = \sum_{j=0}^{k} (-1)^{k-j} S_{k,j} \frac{\Gamma[\tilde{\delta}+j]}{\Gamma[\tilde{\delta}]}$$

$$S_{k,j} = \sum_{i=0}^{j} \frac{(-1)^{j-i} i^k}{i!\,(j-i)!}$$

$$\frac{(p-3)!}{2p}\prod_{i<j}\Gamma[\tilde{\delta}_{ij}]\widetilde{\mathcal{M}}_{\tau^2}=p\Gamma[\tilde{\delta}_{01}]\Gamma[\tilde{\delta}_{12}]\Gamma[\tilde{\delta}_{03}]\Gamma[1+\tilde{\delta}_{02}]\Gamma[-1+\tilde{\delta}_{04}]\Gamma[\tilde{\delta}_{14}]\Gamma[\tilde{\delta}_{13}]\Gamma[-1+\tilde{\delta}_{23}]\Gamma[\tilde{\delta}_{24}]\Gamma[1+\tilde{\delta}_{34}]+\cdots$$

$$\oint_{-i\infty}^{+i\infty}\frac{[d\tilde{\delta}]}{(2\pi i)^5}\prod_{i<j}\frac{\Gamma[\tilde{\delta}_{ij}]}{X_{ij}^{2\tilde{\delta}_{ij}}}\widetilde{\mathcal{M}}_{\hat{\tau}^2}(\tilde{\delta}_{ij})=\frac{1}{X_{12}^4X_{14}^2X_{13}^2X_{34}^{2p-2}}U\mathcal{H}_{[2^2]2pp}$$

$$\mathcal{H}_{[2^2]2pp}=2p\frac{(p-1)}{(p-3)!}\oiiint_{-i\infty}^{+i\infty}\frac{dsdt}{(2\pi i)^2}U^{s+2}V^t\Gamma[-s]\Gamma[-s+p-3]\Gamma[-t]^2\Gamma[-u]^2\frac{3+s}{(s+1)(t+1)(u+1)}$$

$$\lim_{z_0\to z_1}\langle\mathcal{O}_2(z_0)\mathcal{O}_2(z_1)\mathcal{O}_2(z_2)\mathcal{O}_p(z_3)\mathcal{O}_p(z_4)\rangle\Big|_{\frac{N^{\frac{1}{2}\sum p_i}}{N^3}}=\text{ free }+\mathcal{P}_{[2^2]2pp}\mathcal{I}(U,V,\sigma,\tau)\mathcal{H}_{[2^2]2pp}(U,V)$$

$$\lim_{\vec{Y}_3\to\vec{Y}_4}\langle\mathcal{O}_2(z_0)\mathcal{O}_2(z_1)\mathcal{O}_2(z_2)\mathcal{O}_p(z_3)\mathcal{O}_p(z_4)\rangle=0\ \text{ for }\ p\geq 4$$

$$\frac{1}{N^3}\mathcal{C}^{\left(\frac{3}{2}\right)}_{[r_1r_2]spq}=\frac{1}{N}\mathcal{C}^{\left(\frac{1}{2}\right)}_{[r_1r_2]s;\mathcal{D}_{\tau,l}}\times\begin{cases}\frac{1}{N^2}\eta_{\mathcal{D}}\mathcal{C}^{(0)}_{pq;\mathcal{D}}\log(U) & \text{if } p+q\leq\tau<r_1+r_2+s\\ \frac{1}{N^2}\mathcal{C}^{(1)}_{pq;\mathcal{D}} & \text{if } \tau_{\text{unitary}}\leq\tau<p+q\end{cases}$$

$$\mathcal{H}^{\left(\frac{3}{2}\right)}_{[r_1r_2]spq}\Big|_{[a,b,a],\tau,l}\equiv -A^{\left(\frac{3}{2}\right)}_{[r_1r_2]spq,[aba],\tau}+\sum_{ij}\frac{\left(\mathcal{G}^{\left(\frac{1}{2}\right)}_{[r_1r_2]sij}\Big|_{[aba],\tau}\right)\left(\mathcal{C}^{(1)}_{ijpq}\Big|_{[aba],\tau}\right)}{\left(\mathcal{G}^{(0)}_{ijij}\Big|_{[aba],\tau}\right)}$$

$$\tau=2a+b+2,\mathcal{C}^{(1)}_{ijpq}=\mathcal{G}^{(1)}_{ijpq}$$
$$2a+b+4\leq\tau<\max(i+j,p+q),\mathcal{C}^{(1)}_{ijpq}=\mathcal{G}^{(1)}_{ijpq}+\mathcal{I}(U,V,\hat{\sigma},\tau)\mathcal{H}^{(1)}_{ijpq}\Big|_{\log^0(U)}$$

$$\begin{pmatrix}\langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}_a\mathcal{O}_b\rangle & V^T_{pq}\\ V_{[r_1r_2]s} & \langle[\mathcal{O}_{r_1}\mathcal{O}_{r_2}]\mathcal{O}_s\mathcal{O}_p\mathcal{O}_q\rangle\end{pmatrix}$$

$$\sum_{ij}\frac{\left(\mathcal{G}^{\left(\frac{1}{2}\right)}_{[r_1r_2]sij}\Big|_{[aba],\tau}\right)\left(\mathcal{C}^{(1)}_{ijpq}\Big|_{[aba],\tau}\right)}{\left(\mathcal{G}^{(0)}_{ijij}\Big|_{[aba],\tau}\right)}$$

$[a,b,a]$	$su(4)$ harmonic
$[0,p-2,0]$	1
$[0,p,0]$	$-\frac{2(p-2)}{p(1+p)}+\frac{2}{p}\hat{\sigma}+\frac{(p-2)}{p}\hat{\tau}$
$[1,p-2,1]$	$-\frac{p-4}{p+2}+\hat{\sigma}-\hat{\tau}$

$$\mathcal{P}_{[2^2],p,3,p+1}=\sqrt{24p(p+1)}N^{p+4}g_{12}^3g_{14}g_{24}^{p-3}g_{34}^3$$

$$\mathcal{G}^{\left(\frac{3}{2}\right)}_{[2^2],p,3,p+1}=\sqrt{24p(p+1)}\left[(p-1)U\hat{\sigma}+\frac{(p-2)(p-1)U\hat{\tau}}{2V}+2U^2\hat{\sigma}^2+\frac{2(p-1)U^2\hat{\sigma}\hat{\tau}}{V}+\frac{U^3\hat{\sigma}^2\hat{\tau}}{V}\right]$$

$$\mathcal{G}_{[2^2],4,3,5}=\sqrt{\frac{480(a-15)}{(a-3)a^2(a+6)}}\left[\frac{U^3\hat{\sigma}^2\hat{\tau}}{V}+\left[3U\hat{\sigma}+\frac{3U\hat{\tau}}{V}\right]+\left[2U^2\hat{\sigma}^2+\frac{6U^2\hat{\sigma}\hat{\tau}}{V}\right]\right]$$

$$|\mathcal{O}_4|^2=\frac{4a(a-3)(a-8)}{(a+2)},|\mathcal{O}_5|^2=\frac{5a(a-3)(a-8)(a-15)}{(a+6)\sqrt{a+1}}.$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],p,3,p+1}=\sqrt{24p(p+1)}\left[(p-1)M^{4,p,3,p+1}_{k=0,\gamma=p,[2+l]}+\frac{(p-2)(p-1)}{2}M^{4,p,3,p+1}_{k=1,\gamma=p,[2+l]}\right].$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],p,3,p+1,[0,p-2,0],\tau=p+2}=\sqrt{24p(p+1)}\left[2M^{4,p,3,p+1}_{0,p+2,[2+l,2]}+2(p-1)M^{4,p,3,p+1}_{1,p+2,[2+l,2]}\right].$$

$$\mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2]p2p}\Big|_{[0,p-2,0],p+2}=2pM^{4,p,2,p}_{0,p+2,[2+l,2]}$$

$$\mathcal{G}_{2,p,3,p+1}=\frac{\sqrt{6p(p+1)}}{a}\left[U\hat{\sigma}+(p-1)\frac{U\hat{\tau}}{V}+2\frac{U^2\hat{\sigma}\hat{\tau}}{V}\right],\mathcal{H}^{(1)}_{2,p,3,p+1}=-\frac{\sqrt{3(p+1)2p}}{(p-2)!}U^3\bar{D}_{3,p+3,2,p}$$

$$\frac{\mathcal{C}^{(1)}_{2,p,3,p+1}}{\sqrt{6p(p+1)}}\Bigg|_{[0,p-2,0],p+2}=\frac{1}{(p-2)!}\left(2+\frac{(-1)^l(l+3)!\,p!}{(1+l+p)!}\right)\frac{(p+4)}{(p+2)}\frac{(l+p+1)!^2}{(2l+p+4)!}+2M^{2,p,3,p+1}_{1,p+2,[2+l,2]}$$

$$\mathcal{G}^{(0)}_{2p2p}\Big|_{[0,p-2,0],p+2}=M^{2p2p}_{0,p+2,[2+l,2]}$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],p,3,p+1,[0,p,0],\tau=p+2}=\sqrt{24p(p+1)}\left[2M^{4,p,3,p+1}_{0,p+2,[2+l]}+2(p-1)M^{4,p,3,p+1}_{1,p+2,[2+l]}\right]$$

$$\mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2]p2p}\Big|_{[0,p,0],p+2} = 2pM^{4,p,2,p}_{0,p+2,[2+l]},\ \mathcal{G}^{(0)}_{2,p,2,p}\Big|_{[0,p,0],p+2} = M^{2,p,2,p}_{0,p+2,[2+l]},$$

$$\mathcal{G}^{(1)}_{2,p,3,p+1}\Big|_{[0,p,0],p+2} = \sqrt{6p(p+1)}(2)M^{2,p,3,p+1}_{1,p+2,[2+l]}.$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],p,3,p+1,[0,p,0],\tau=p+2}$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],p,3,p+1,[0,p,0],\tau=p+2} = \sqrt{24p(p+1)}\Big[-(p-1)M^{4,p,3,p+1}_{0,p,[3+l]} - \frac{(p-2)(p-1)}{2}M^{4,p,3,p+1}_{1,p,[3+l]}$$
$$+2M^{4,p,3,p+1}_{0,p+2,[2+l,1]} + 2(p-1)M^{4,p,3,p+1}_{1,p+2,[2+l,1]}\Big]$$

$$\mathcal{G}^{(1)}_{2,p,3,p+1}\Big|_{[1,p,1],p+2} = \sqrt{6p(p+1)}\left[-M^{2,p,3,p+1}_{0,p,[3+l]} - (p-1)M^{2,p,3,p+1}_{1,p,[3+l]} + 2M^{2,p,3,p+1}_{1,p+2,[2+l,1]}\right]$$

$$\mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2]p2p}\Big|_{[0,p,0],p+2} = 2pM^{4,p,2,p}_{0,p+2,[2+l,1]},\ \mathcal{G}^{(0)}_{2,p,2,p}\Big|_{[0,p,0],p+2} = M^{2,p,2,p}_{0,p+2,[2+l,1]}$$

$[a,b,a]$	$su(4)$ harmonic
$[0,1,0]$	1
$[0,3,0]$	$\frac{1}{6}(-1+4\sigma+2\tau)$
$[1,1,1]$	$\frac{1}{5}(1+5\sigma-5\tau)$

$$\mathcal{P}_{[2^2],3,p,p+1} = \sqrt{24p(p+1)}N^{4+p}g_{13}^3 g_{14} g_{34}^p$$

$$\mathcal{G}^{\left(\frac{3}{2}\right)}_{[2^2],3,p,p+1} = \sqrt{24p(p+1)}\left[2U\hat{\sigma} + \frac{U\hat{\tau}}{V} + (p-1)U^2\hat{\sigma}^2 + \frac{2(p-1)U^2\hat{\sigma}\hat{\tau}}{V} + \frac{(p-2)(p-1)U^3\sigma^2\tau}{2V}\right]$$

$$\mathcal{P}_{23p(p+1)} = \left(\sqrt{6p(p+1)}N^{3+p} + \cdots\right)g_{12}^2 g_{24} g_{34}^p$$

$$\mathcal{G}_{23p(p+1)} = \frac{1}{a}\left[U\hat{\sigma} + \frac{2U\hat{\tau}}{V} + \frac{(p-1)U^2\hat{\sigma}\hat{\tau}}{V}\right]\ \mathcal{H}^{(1)}_{23p(p+1)} = \frac{\sqrt{6p(p+1)}}{(p-2)!}U^p\bar{D}_{p+2,p+1,4,1}$$

$$\mathcal{H}(U,V,\hat{\sigma},\hat{\tau}) = \frac{1}{V^2}\mathcal{H}\left(\frac{U}{V},\frac{1}{V},\hat{\tau},\hat{\sigma}\right)$$

$$\mathcal{H}(U,V,\hat{\sigma},\hat{\tau}) = \left(\frac{U}{V}\right)^4\hat{\tau}^2\mathcal{H}\left(V,U,\frac{\hat{\sigma}}{\hat{\tau}},\frac{1}{\hat{\tau}}\right)\ \mathcal{H}(U,V,\hat{\sigma},\hat{\tau}) = (U\hat{\sigma})^2\mathcal{H}\left(\frac{1}{U},\frac{V}{U},\frac{1}{\hat{\sigma}},\frac{\hat{\tau}}{\hat{\sigma}}\right)$$

$$\mathcal{H}_{[2^2]444} = \mathcal{F}(U,V) + \hat{\sigma}^2U^2\mathcal{F}\left(\frac{1}{U},\frac{V}{U}\right) + \frac{\hat{\tau}^2U^4}{V^4}\mathcal{F}(V,U) + \hat{\sigma}\hat{\tau}\tilde{\mathcal{F}}(U,V) + \hat{\tau}U^2\tilde{\mathcal{F}}\left(\frac{1}{U},\frac{V}{U}\right) + \frac{\hat{\sigma}U^4}{V^4}\tilde{\mathcal{F}}(V,U)$$

$$\mathcal{F}(U,V)=\frac{1}{V^2}\mathcal{F}\left(\frac{U}{V},\frac{1}{V}\right),\tilde{\mathcal{F}}(U,V)=\frac{1}{V^2}\tilde{\mathcal{F}}\left(\frac{U}{V},\frac{1}{V}\right)$$

$[a,b,a]$	su(4) harmonic
[0, 0, 0]	1
[0, 2, 0]	$-\frac{1}{6}+\frac{1}{2}(\hat{\sigma}+\hat{\tau})$
[1, 0, 1]	$\hat{\sigma}-\hat{\tau}$
[0, 4, 0]	$\frac{1}{60}-\frac{2}{15}(\hat{\sigma}+\hat{\tau})+\frac{1}{6}(\hat{\sigma}+\tau)^2+\frac{1}{3}\hat{\sigma}\hat{\tau}$
[1, 2, 1]	$\frac{1}{4}(\hat{\sigma}-\hat{\tau})(-1+2\hat{\sigma}+2\hat{\tau})$
[2, 0, 2]	$\frac{1}{10}-\frac{1}{2}(\sigma+\tau)+(\sigma+\tau)^2-4\sigma\tau$

$$\mathcal{G}_{[2^2],4,4,4}=\frac{16\sqrt{2}\left(1-\frac{5}{a}-\frac{15}{a+2}\right)\left[U^2\hat{\sigma}^2+\frac{U^2\hat{\tau}^2}{V^2}+4\left[\frac{U^2\hat{\sigma}\hat{\tau}}{V}+\frac{U^3\hat{\sigma}^2\hat{\tau}}{V}+\frac{U^3\hat{\tau}^2\sigma}{V^2}\right]+\frac{U^4\hat{\sigma}^2\hat{\tau}^2}{V^2}\right]}{\sqrt{(-8+a)(-3+a)(1+a)}}$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],4,4,4,[0,0,0],\tau=4}=16\sqrt{2}\left[M^{4,4,4,4}_{0,4,[2+l,2]}+M^{4,4,4,4}_{2,4,[2+l,2]}+4M^{4,4,4,4}_{1,4,[2+l,2]}\right]$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],4,4,4,[0,0,0],\tau=6}=16\sqrt{2}\left[M^{4,4,4,4}_{0,4,[3+l,3]}+M^{4,4,4,4}_{2,4,[3+l,3]}+4M^{4,4,4,4}_{1,4,[3+l,3]}+4M^{4,4,4,4}_{1,6,[2+l,2,2]}+4M^{4,4,4,4}_{2,6,[2+l,2,2]}\right]$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],4,4,4,[1,0,1],\tau=4}=16\sqrt{2}\left[M^{4,4,4,4}_{0,4,[2+l,1]}+M^{4,4,4,4}_{2,4,[2+l,1]}+4M^{4,4,4,4}_{1,4,[2+l,1]}\right]$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],4,4,4,[1,0,1],\tau=6}=16\sqrt{2}\left[4M^{4,4,4,4}_{1,6,[2+l,2,1]}+4M^{4,4,4,4}_{1,6,[2+l,2,1]}\right]$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],4,4,4,[0,2,0],\tau=4}=16\sqrt{2}\left[M^{4,4,4,4}_{0,4,[2+l]}+M^{4,4,4,4}_{2,4,[2+l]}+4M^{4,4,4,4}_{1,4,[2+l]}\right]$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],4,4,4,[0,2,0],\tau=6}=16\sqrt{2}\left[4M^{4,4,4,4}_{1,6,[2+l,2]}+4M^{4,4,4,4}_{2,6,[2+l,2]}\right]$$

$$\mathcal{G}_{2,4,4,4}=\frac{16\sqrt{2}}{a}\left[U\hat{\sigma}+\frac{U\hat{\tau}}{V}+\frac{U^2\hat{\sigma}\hat{\tau}}{V}\right],\mathcal{H}^{(1)}_{2,4,4,4}=-8\sqrt{2}U^3V\bar{D}_{5,5,5,1}$$

$$\mathcal{C}^{(1)}_{2,4,4,4}\Big|_{[0,2,0],6} = 16\sqrt{2}\left[8(l+5)\frac{(l+4)!^2}{(2l+8)!}\frac{1+(-1)^l}{2} + M^{2,4,4,4}_{1,6,[2+l,2]}\right]$$

$$\mathcal{G}^{(0)}_{2424}\Big|_{[0,2,0],6} = M^{2424}_{0,6,[2+l,2]},\ \mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2],4,2,4}\Big|_{[0,2,0],6} = 8M^{4424}_{0,2,[2+l,2]}$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],4,4,4,[2,0,2],\tau=6} = 16\sqrt{2}\left[-M^{4,4,4,4}_{0,4,[3+l,1]} - M^{4,4,4,4}_{2,4,[3+l,1]} - 4M^{4,4,4,4}_{1,4,[3+l,1]} + 4M^{4,4,4,4}_{1,6,[2+l,1,1]} + 4M^{4,4,4,4}_{2,6,[2+l,1,1]}\right]$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],4,4,4,[1,2,1],\tau=6} = 16\sqrt{2}\left[-M^{4,4,4,4}_{0,4,[3+l]} - M^{4,4,4,4}_{2,4,[3+l]} - 4M^{4,4,4,4}_{1,4,[3+l]} + 4M^{4,4,4,4}_{1,6,[2+l,1]} + 4M^{4,4,4,4}_{2,6,[2+l,1]}\right]$$

$$\mathcal{G}^{(1)}_{2444}\Big|_{[1,2,1],6} = 16\sqrt{2}\left[-M^{2,4,4,4}_{0,4,[3+l]} - M^{2,4,4,4}_{1,4,[3+l]} + M^{2,4,4,4}_{1,6,[2+l,1]}\right]$$

$$\mathcal{G}^{(0)}_{2424}\Big|_{[1,2,1],6} = M^{2424}_{0,6,[2+l,1]},\ \mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2],4,2,4}\Big|_{[1,2,1],6} = 8M^{4424}_{0,2,[2+l,1]}$$

$$A^{\left(\frac{3}{2}\right)}_{[2^2],4,4,4,[0,4,0],\tau=6} = 16\sqrt{2}\left[4M^{4,4,4,4}_{1,6,[2+l]} + 4M^{4,4,4,4}_{2,6,[2+l]}\right]$$

$$\mathcal{G}^{(1)}_{2444}\Big|_{[0,4,0],6} = 16\sqrt{2}M^{2,4,4,4}_{1,6,[2+l]},\ \mathcal{G}^{(0)}_{2424}\Big|_{[0,4,0],6} = M^{2424}_{0,6,[2+l]},\ \mathcal{G}^{\left(\frac{1}{2}\right)}_{[2^2],4,2,4}\Big|_{[0,4,0],6} = 8M^{4424}_{0,2,[2+l]}$$

$$\mathcal{F} = 8\sqrt{2}\oiiint_{-i\infty}^{+\infty}\frac{dsdt}{(2\pi i)^2}U^{s+4}V^t\Gamma[-s]^2\Gamma[-t]^2\Gamma[-u]^2\frac{1}{(1+s)(2+s)(1+t)(u+1)}$$

$$\tilde{\mathcal{F}} = 8\sqrt{2}\oiiint_{-i\infty}^{+\infty}\frac{dsdt}{(2\pi i)^2}U^{s+4}V^t\Gamma[-s]^2\Gamma[-t]^2\Gamma[-u]^2\frac{(8+3s)}{(1+s)(1+t)(2+t)(u+1)(u+2)}$$

$$\lim_{\vec{X}_0\to\vec{X}_1}\lim_{\vec{Y}_0\to\vec{Y}_1}\langle\mathcal{O}_{r_1}(z_0)\mathcal{O}_{r_2}(z_1)\mathcal{O}_p(z_2)\mathcal{O}_q(z_3)\rangle = \langle[\mathcal{O}_{r_1}\mathcal{O}_{r_2}](z_1)\mathcal{O}_p(z_2)\mathcal{O}_q(z_3)\rangle$$

$$\langle[\mathcal{O}_{r_1}\mathcal{O}_{r_2}](\vec{X}_1,\vec{Y}_1)\mathcal{O}_p(\vec{X}_2,\vec{Y}_2)\mathcal{O}_q(\vec{X}_3,\vec{Y}_3)\rangle = C_{[r_1r_2]pq}\prod_{i\neq j} g_{ij}^{b_{ij}}$$

$$b_{12} = \frac{r_1+r_2}{2} + \frac{p-q}{2}\ ;\ b_{13} = \frac{r_1+r_2}{2} + \frac{q-p}{2}\ ;\ b_{23} = \frac{p+q}{2} - \frac{r_1+r_2}{2}.$$

$$\mathcal{C}_{\vec{p}}(U,V,\hat{\sigma},\tau) = \mathcal{G}^{free}_{\vec{p}}(U,V,\hat{\sigma},\hat{\tau}) + \mathcal{I}(U,V,\hat{\sigma},\hat{\tau})\mathcal{H}_{\vec{p}}(U,V,\hat{\sigma},\hat{\tau})$$

$$c_s = \frac{r_1+r_2}{2} - \frac{p+q}{2}\ ;\ c_t = \frac{r_1-r_2+q-p}{2} \geq 0\ ;\ c_u = \frac{-r_1+r_2+q-p}{2} \geq 0$$

$$\mathcal{P}_{r_1,r_2,p,q} = g_{01}^{c_s}g_{03}^{c_t}g_{13}^{c_u}(g_{01}g_{23})^p$$

$$\mathcal{H}_{r_1,r_2,p,q} = \sum_{\tilde{s},\tilde{t}}\tilde{U}^{\tilde{s}-p+2}\tilde{V}^{\tilde{t}}\oiiint\frac{dsdt}{(2\pi i)^2}U^{s+p}V^t\frac{\Gamma_{\mathrm{AdS}\times\mathrm{S}}}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)}$$

$$\Gamma_{\mathrm{AdS}\times\mathrm{S}} = \frac{\Gamma[-s]\Gamma[-s+c_s]}{\Gamma[\tilde{s}+1]\Gamma[\tilde{s}+1+c_s]} \times (t\text{-channel }) \times (u\text{-channel })$$

$$U = \frac{\vec{X}_{01}^2 \vec{X}_{23}^2}{\vec{X}_{02}^2 \vec{X}_{13}^2} \ ; \ V = \frac{\vec{X}_{03}^2 \vec{X}_{12}^2}{\vec{X}_{02}^2 \vec{X}_{13}^2} \ ; \ \tilde{U} = \frac{\vec{Y}_{01}^2 \vec{Y}_{23}^2}{\vec{Y}_{02}^2 \vec{Y}_{13}^2} \ ; \ \tilde{V} = \frac{\vec{Y}_{03}^2 \vec{Y}_{12}^2}{\vec{Y}_{02}^2 \vec{Y}_{13}^2}$$

$$\lim_{y_0 \to y_1} \mathcal{P}_{r_1,r_2,p,q} \mathcal{I}(U,V,\hat{\sigma},\tau) \tilde{U}^{\tilde{s}-p+2} = \left(\vec{Y}_{12}^2\right)^{\frac{r_1+r_2}{2}+\frac{p-q}{2}} \left(\vec{Y}_{13}^2\right)^{\frac{r_1+r_2}{2}+\frac{q-p}{2}} \left(\vec{Y}_{23}^2\right)^{\frac{p+q}{2}-\frac{r_1+r_2}{2}}$$

$$\mathcal{P}_{r_1,r_2,p,q} \tilde{U}^{\tilde{s}-p} \propto \left(\vec{Y}^2\right)_{01}^{c_s} \left(\vec{Y}^2\right)_{03}^{c_t} \left(\vec{Y}^2\right)_{13}^{c_u} \left(\vec{Y}_{01}^2 \vec{Y}_{23}^2\right)^p \left(\frac{\vec{Y}_{01}^2 \vec{Y}_{23}^2}{\vec{Y}_{02}^2 \vec{Y}_{13}^2}\right)^{\tilde{s}-p}$$

$$\gamma = (c_t + c_u) + 2(-\tilde{s} + p) = q + p - 2\tilde{s}$$

$$\tilde{s} = \frac{-r_1 - r_2 + p + q}{2} = -c_s$$

$$\mathrm{Res}_{s=-1+c_s} \left[\lim_{\vec{Y}_0 \to \vec{Y}_1} \mathcal{P}_{r_1,r_2,p,q} \mathcal{I}(U,V,\hat{\sigma},\hat{\tau}) \mathcal{H}_{r_1,r_2,p,q} \right] \propto U (g_{12})^{\frac{r_1+r_2}{2}+\frac{p-q}{2}} (g_{13})^{\frac{r_1+r_2}{2}+\frac{q-p}{2}} (g_{23})^{\frac{p+q}{2}-\frac{r_1+r_2}{2}}$$

$$\langle \mathcal{O}_2(X_0,Y_0) \mathcal{O}_2(X_1,Y_1) \mathcal{O}_p(X_2,Y_2) \mathcal{O}_p(X_3,Y_3) \rangle = \sqrt{2N^2.2N^2.pN^p.pN^p}\, g_{01}^2 g_{23}^p \mathcal{C}_{22pp} \qquad (B.13)$$

$$\mathcal{C}_{22pp} = a(U,V) + \left[\hat{\sigma} U b_1(U,V) + \frac{\hat{\tau} U}{V} b_2(U,V) \right] + \left[\hat{\sigma}^2 U^2 c_1(U,V) + \frac{\hat{\tau}^2 U^2}{V^2} c_2(U,V) + \frac{\hat{\sigma}\hat{\tau} U^2}{V} d(U,V) \right]$$

$$g_{01}^2 g_{23}^p \left[\hat{\sigma}^2 U^2 c_1(U,V) + \frac{\hat{\tau}^2 U^2}{V^2} c_2(U,V) + \frac{\hat{\sigma}\hat{\tau} U^2}{V} d(U,V) \right]$$

$$\lim_{z_0 \to z_1} \frac{\langle \mathcal{O}_2(z_0) \mathcal{O}_2(z_1) \mathcal{O}_p(z_2) \mathcal{O}_p(z_3) \rangle}{\sqrt{2N^2 \cdot 2N^2 \cdot pN^p \cdot pN^p}} = g_{12}^2 g_{13}^p g_{23}^{p-2} \times \lim_{X_0 \to X_1} [c_1(U,V) + c_2(U,V) + d(U,V)]$$

$$g_{12}^2 g_{13}^p g_{23}^{p-2} \lim_{X_0 \to X_1} [c_1(U,V) + c_2(U,V) + d(U,V)]_{\frac{1}{N^2}} = \left[\frac{\langle \mathcal{O}_2^2 \mathcal{O}_p \mathcal{O}_p \rangle}{\sqrt{2N^2 \cdot 2N^2 \cdot pN^p \cdot pN^p}} \right]_{\frac{1}{N^2}}$$

$$g_{12}^2 g_{13}^p g_{23}^{p-2} \times \lim_{X_0 \to X_1} [c_1(U,V) + c_2(U,V) + d(U,V)] = g_{12}^2 g_{13}^p g_{23}^{p-2} \times \frac{2p(p-1)}{N^2}$$

$$C_{\mathcal{O}_3,\mathcal{O}_3,\mathcal{O}_2^2} = C_{T_3,T_3,T_2^2} = \frac{72(a-3)(a)}{\sqrt{a+1}}$$

$$C_{\mathcal{O}_4,\mathcal{O}_4,\mathcal{O}_2^2} = C_{\mathcal{O}_4,T_4,T_2^2} = C_{T_4,T_4,T_2^2} + \alpha_4 C_{T_2^2,T_4,T_2^2} = \frac{192(a-8)(a-3)(a)}{a+2}$$

$$C_{\mathcal{O}_5,\mathcal{O}_5,\mathcal{O}_2^2} = C_{\mathcal{O}_5,T_5,T_2^2} = C_{T_5,T_5,T_2^2} + \alpha_5 C_{T_2 T_3,T_5,T_2^2} = \frac{400(a-15)(a-8)(a-3)a}{(a+6)\sqrt{a+1}}$$

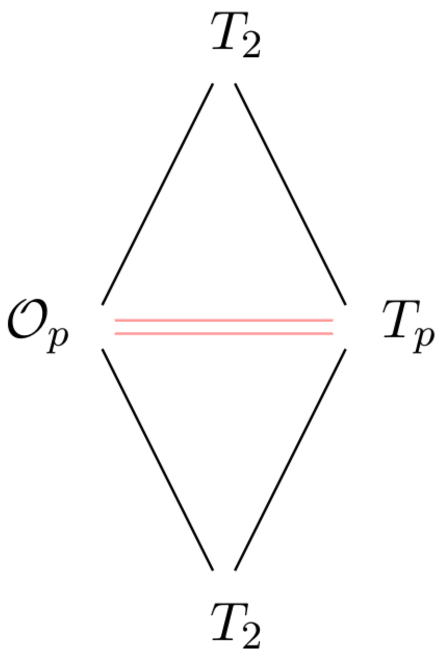
$$\mathcal{O}_p = T_p + \sum \alpha_i (\text{ multi-traces })_i C_{\mathcal{O}_p,T_p,T_2^2}$$

$$C_{\mathcal{O}_p,T_p,T_2} \neq C_{T_p,T_p,T_2^2}$$

$$\mathcal{I}(U,V,\hat{\sigma},\hat{\tau}) \to U^2/\tilde{U}^2 - 2U/\tilde{U} + 1$$

$$C_{p,p,[2^2]} = (2p)(2(p-1))\langle \mathcal{O}_p \mathcal{O}_p \rangle.$$

$$T_2(\vec{X}_3) = \mathcal{O}_p(\vec{X}_1) = T_p(\vec{X}_2) = T_2(\vec{X}_3)$$



$$c_1|_{\frac{1}{N^2}} = \frac{V}{U}\mathcal{H},\ c_2|_{\frac{1}{N^2}} = \frac{V^2}{U}\mathcal{H},\ d|_{\frac{1}{N^2}} = 2p(p-1) + \frac{V}{U}(U-V-1)\mathcal{H}$$

$$\mathcal{H} = -\frac{2p}{(p-2)!}U^2\bar{D}_{2,p+2,2,p}$$

$$D_{\Delta_1,\ldots,\Delta_n} = \int \frac{dZ_0 d^d Z}{Z_0^{d+1}} \prod_{i=1}^{n} \left(\frac{Z_0}{Z_0^2 + (\vec{Z} - \vec{X}_i)^2} \right)^{\Delta_i}$$

$$D_{\Delta_1,\ldots,\Delta_i+1,\ldots,\Delta_j+1,\ldots,\Delta_n} = \frac{d/2 - \Sigma}{\Delta_i \Delta_j} \frac{\partial}{\partial X_{ij}^2} D_{\Delta_1,\ldots,\Delta_n}$$

$$\frac{\prod_{i=1}^4 \Gamma(\Delta_i)}{\Gamma\left(\Sigma - \frac{1}{2}d\right)} \frac{2}{\pi^{d/2}} D_{\Delta_1\Delta_2\Delta_3\Delta_4}(\vec{X}_1,\vec{X}_2,\vec{X}_3,\vec{X}_4) = \frac{(X_{14}^2)^{\Sigma-\Delta_1-\Delta_4}(X_{34}^2)^{\Sigma-\Delta_3-\Delta_4}}{(X_{13}^2)^{\Sigma-\Delta_4}(X_{24}^2)^{\Delta_2}} \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(U,V)$$

$$\Sigma = \frac{1}{2}\sum_{i=1}^{n} \Delta_i$$

$$\bar{D}_{1111} = \frac{1}{x_1 - x_2}\left[2\mathrm{Li}_2(x_1) - 2\mathrm{Li}_2(x_2) + \log(x_1 x_2)\log\left(\frac{1-x_1}{1-x_2}\right)\right]$$

$$\partial_{x_1}\bar{D}_{1111} = \frac{\bar{D}_{1111}}{x_2 - x_1} + \frac{\log[(1-x_1)(1-x_2)]}{x_1(x_2 - x_1)} + \frac{\log(x_1 x_2)}{(x_1 - 1)(x_1 - x_2)}$$

$$\partial_{x_2}\bar{D}_{1111} = \frac{\bar{D}_{1111}}{x_1 - x_2} + \frac{\log[(1-x_1)(1-x_2)]}{x_2(x_1 - x_2)} + \frac{\log(x_1 x_2)}{(x_2 - 1)(x_2 - x_1)}$$

$$\lim_{\vec{X}_0 \to \vec{X}_1} D_{\Delta_0,\Delta_1,\dots,\Delta_4} = D_{\Delta_0+\Delta_1,\dots,\Delta_4}$$

$$\prod_{i<j} \left(x_{ij}^2\right)^{-\alpha_{ij}} D_{\tilde{\Delta}_1\dots\tilde{\Delta}_n} \leftrightarrow \widetilde{\mathcal{M}}(\tilde{\delta}) = \pi^{\frac{d}{2}} \frac{\Gamma\left(\frac{\sum_i \tilde{\Delta}_i - d}{2}\right)}{\prod_i \Gamma(\tilde{\Delta}_i)} \prod_{i<j} \frac{\Gamma(\tilde{\delta}_{ij} - \alpha_{ij})}{\Gamma(\tilde{\delta}_{ij})}$$

$$\tilde{\Delta}_i + \sum_j \alpha_{ij} = \Delta_i$$

$$\widetilde{\mathcal{M}}_{\hat{t}^2} = \frac{2p}{(p-3)!}\left[\frac{\tilde{\delta}_{34}\left(-1-2p+p(\tilde{\delta}_{02}+\tilde{\delta}_{04}-\tilde{\delta}_{13}+\tilde{\delta}_{23})-\tilde{\delta}_{34}\right)}{(-1+\tilde{\delta}_{04})(-1+\tilde{\delta}_{23})} - \frac{p\tilde{\delta}_{24}\tilde{\delta}_{34}}{(-1+\tilde{\delta}_{03})(-1+\tilde{\delta}_{14})} + (0\leftrightarrow 1)\right]$$

$$\widetilde{\mathcal{M}}_{\hat{\sigma}\hat{t}} = \frac{2p}{(p-3)!}\left[\frac{\tilde{\delta}_{34}\left(-1-2p+p^2+p(\tilde{\delta}_{01}+\tilde{\delta}_{04}+\tilde{\delta}_{13})-\tilde{\delta}_{34}\right)}{(-1+\tilde{\delta}_{04})(-1+\tilde{\delta}_{13})} + \left[\frac{p(p-\tilde{\delta}_{13})\tilde{\delta}_{34}}{(-1+\tilde{\delta}_{03})(-1+\tilde{\delta}_{24})} + (3\leftrightarrow 4)\right] + (0\leftrightarrow 1)\right]$$
$$+ \frac{2p}{(p-3)!}\left[\frac{2p\tilde{\delta}_{34}}{(1-\tilde{\delta}_{03})} + \frac{2p\tilde{\delta}_{34}}{(1-\tilde{\delta}_{04})} + (0\leftrightarrow 1) + (0\leftrightarrow 2)\right]$$

$$\widetilde{\mathcal{M}}_{\hat{\sigma}^2} = \widetilde{\mathcal{M}}_{\hat{t}^2}\Big|_{3\leftrightarrow 4}$$

$$\widetilde{\mathcal{M}}_{\hat{t}} = \frac{2p}{(p-2)!}\left[\widetilde{\mathcal{M}}_{\hat{t}}^{\text{curvature}} + \widetilde{\mathcal{M}}_{\hat{t}}^{\text{supercurvature}} + (0\leftrightarrow 1) + 2p(p-2)\right];\ \widetilde{\mathcal{M}}_{\hat{\sigma}} = \widetilde{\mathcal{M}}_{\hat{t}}\Big|_{3\leftrightarrow 4}$$

$$\widetilde{\mathcal{M}}_{\hat{t}}^{\text{curvature}} = \frac{p(p-1)\tilde{\delta}_{2,4}}{(1-\tilde{\delta}_{03})} + \frac{2(p+1)\tilde{\delta}_{3,4}}{(1-\tilde{\delta}_{02})} - \frac{p\tilde{\delta}_{01} + p(p-2)(1+\tilde{\delta}_{34})}{1-\tilde{\delta}_{23}}$$
$$- \frac{2(\tilde{\delta}_{04}+\tilde{\delta}_{34}) + p\left(2(p-2) - (p-1)(\tilde{\delta}_{13}+\tilde{\delta}_{34}) - \tilde{\delta}_{04} + \tilde{\delta}_{12}\right)}{(1-\tilde{\delta}_{04})}$$

$$\widetilde{\mathcal{M}}_{\hat{t}}^{\text{supercurvature}} = \frac{2p\tilde{\delta}_{24}\tilde{\delta}_{34}}{(-1+\tilde{\delta}_{02})(-1+\tilde{\delta}_{13})} + \frac{p(\tilde{\delta}_{02}+\tilde{\delta}_{03})(\tilde{\delta}_{24}+\tilde{\delta}_{34}) + p(p-2)\tilde{\delta}_{34}(\tilde{\delta}_{14}-p) + p^2(\tilde{\delta}_{34}+\tilde{\delta}_{24})^2}{(-1+\tilde{\delta}_{04})(-1+\tilde{\delta}_{23})} +$$
$$\frac{2p\tilde{\delta}_{13}\tilde{\delta}_{34} - 2(\tilde{\delta}_{01}+\tilde{\delta}_{02}+p-2)(\tilde{\delta}_{14}+\tilde{\delta}_{24})}{(-1+\tilde{\delta}_{04})(-1+\tilde{\delta}_{12})} + \frac{p(\tilde{\delta}_{01}+\tilde{\delta}_{13})\left(\tilde{\delta}_{34}+p\tilde{\delta}_{13}-(p-1)\tilde{\delta}_{04}\right) + p(p-2)\tilde{\delta}_{34}\tilde{\delta}_{24}}{(-1+\tilde{\delta}_{03})(-1+\tilde{\delta}_{14})}$$

$$\widetilde{\mathcal{M}}_1 = \frac{2p}{(p-2)!}\Bigg[\left[\frac{2p(\tilde{\delta}_{01}+\tilde{\delta}_{04})(\tilde{\delta}_{04}+\tilde{\delta}_{24})}{(\tilde{\delta}_{02}-1)(\tilde{\delta}_{14}-1)} + (0\leftrightarrow 1{,}3\leftrightarrow 4)\right] + \frac{p\tilde{\delta}_{24}}{\tilde{\delta}_{03}-1} + \frac{p\tilde{\delta}_{23}}{\tilde{\delta}_{14}-1} - 7p + 6$$

$$+(p-2)\left[\frac{(3-p)\tilde{\delta}_{01}}{\tilde{\delta}_{34}-p+3} + \frac{2(\tilde{\delta}_{23}+\tilde{\delta}_{24}-1)\left(\frac{\tilde{\delta}_{04}+\tilde{\delta}_{24}}{\tilde{\delta}_{02}-1} + \frac{\tilde{\delta}_{13}+\tilde{\delta}_{23}}{\tilde{\delta}_{12}-1} - 1\right)}{\tilde{\delta}_{34}-p+2}\right]$$

$$+\frac{2\left(\tilde{\delta}_{01} + 2(\tilde{\delta}_{23}+\tilde{\delta}_{24})\left(\frac{\tilde{\delta}_{04}+\tilde{\delta}_{24}}{\tilde{\delta}_{02}-1} + \frac{\tilde{\delta}_{13}+\tilde{\delta}_{23}}{\tilde{\delta}_{12}-1}\right)\right)}{\tilde{\delta}_{34}-p+1} + \frac{p^2(\tilde{\delta}_{01}+\tilde{\delta}_{04})(\tilde{\delta}_{01}+\tilde{\delta}_{13})}{(\tilde{\delta}_{03}-1)(\tilde{\delta}_{14}-1)}$$

$$+2(p-1)\left[\frac{\tilde{\delta}_{13}}{\tilde{\delta}_{02}-1} + \frac{\tilde{\delta}_{04}}{\tilde{\delta}_{12}-1}\right] + (3\leftrightarrow 4)\Bigg] + \frac{2p}{(p-2)!\,(1-\tilde{\delta}_{02})}\Bigg[\frac{2(\tilde{\delta}_{03}+\tilde{\delta}_{04})(\tilde{\delta}_{23}+\tilde{\delta}_{24})}{\tilde{\delta}_{34}-p+1}$$

$$+(p-1)p\left[\frac{(\tilde{\delta}_{01}+\tilde{\delta}_{03})(\tilde{\delta}_{12}+\tilde{\delta}_{23})}{\tilde{\delta}_{13}-1} + (3\leftrightarrow 4)\right] + (p-2)\Big[2(p(1-\tilde{\delta}_{02})+1)$$

$$+\frac{4(\tilde{\delta}_{03}+\tilde{\delta}_{04}-1)(\tilde{\delta}_{23}+\tilde{\delta}_{24}-1)}{\tilde{\delta}_{34}-p+2} + \frac{(p-3)(\tilde{\delta}_{03}+\tilde{\delta}_{04}-2)(\tilde{\delta}_{23}+\tilde{\delta}_{24}-2)}{\tilde{\delta}_{34}-p+3}\Big] + (0\leftrightarrow 1)\Bigg]$$

$$\frac{(p-3)!}{2p}\prod_{i<j}\Gamma[\tilde{\delta}_{ij}]\mathcal{M}_{\hat{t}^2} =$$

$$\begin{aligned}
&+p\Gamma[\tilde{\delta}_{01}]\Gamma[\tilde{\delta}_{12}]\Gamma[\tilde{\delta}_{03}](\Gamma[1+\tilde{\delta}_{02}]\Gamma[-1+\tilde{\delta}_{04}] + \Gamma[\tilde{\delta}_{02}]\Gamma[\tilde{\delta}_{04}])\Gamma[\tilde{\delta}_{14}]\Gamma[\tilde{\delta}_{13}]\Gamma[-1+\tilde{\delta}_{23}]\Gamma[\tilde{\delta}_{24}]\Gamma[1+\tilde{\delta}_{34}] +\\
&+p\Gamma[\tilde{\delta}_{01}]\Gamma[\tilde{\delta}_{12}]\Gamma[\tilde{\delta}_{03}]\Gamma[\tilde{\delta}_{02}]\Gamma[-1+\tilde{\delta}_{04}]\Gamma[\tilde{\delta}_{14}](\Gamma[\tilde{\delta}_{13}]\Gamma[\tilde{\delta}_{23}] - \Gamma[1+\tilde{\delta}_{13}]\Gamma[-1+\tilde{\delta}_{23}])\Gamma[\tilde{\delta}_{24}]\Gamma[1+\tilde{\delta}_{34}] +\\
&-p\Gamma[\tilde{\delta}_{01}]\Gamma[\tilde{\delta}_{12}]\Gamma[-1+\tilde{\delta}_{03}]\Gamma[\tilde{\delta}_{02}]\Gamma[\tilde{\delta}_{04}]\Gamma[-1+\tilde{\delta}_{14}]\Gamma[\tilde{\delta}_{13}]\Gamma[\tilde{\delta}_{23}]\Gamma[1+\delta_{24}]\Gamma[1+\tilde{\delta}_{34}] +\\
&-\Gamma[\tilde{\delta}_{01}]\Gamma[\tilde{\delta}_{12}]\Gamma[\tilde{\delta}_{03}]\Gamma[\tilde{\delta}_{02}]\Gamma[-1+\tilde{\delta}_{04}]\Gamma[\tilde{\delta}_{14}]\Gamma[\tilde{\delta}_{13}]\Gamma[-1+\tilde{\delta}_{23}]\Gamma[\tilde{\delta}_{24}]\Gamma[2+\tilde{\delta}_{34}] + (0\leftrightarrow 1),
\end{aligned}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)ds = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

$$\left(\prod_{0\le i<j\le 4} X_{ij}^{-2\delta_{ij}}\right)\times X_{12}^4 X_{13}^2 X_{14}^2 X_{34}^{2p-2} \to U^{p-1-\tilde{\delta}_{34}}V^{-\tilde{\delta}_{23}}$$

$$\mathcal{H}_{[2^2]233}(x_1,x_2) = \mathcal{H}_{[2^2]323}\left(\frac{1}{x_1},\frac{1}{x_2}\right)$$

$$\mathcal{H}_{2p2p}^{(1)} = -2p/(p-2)!\,U^2\bar{D}_{2,p+2,2,p}$$

$$\mathcal{H}_{[2^2]2pp}(x_1,x_2) = \mathcal{H}_{[2^2]p2p}\left(\frac{1}{x_1},\frac{1}{x_2}\right)$$

$$(U\hat{\sigma})\mathcal{P}_{[2^2]p3p+1}\Big|_{p=2} = \mathcal{P}_{[2^2]233}$$

$$\mathcal{M}^1 \equiv \mathcal{M}(d_{12}=1, d_{13}=0), \mathcal{M}^{\hat{\sigma}} \equiv \mathcal{M}(d_{12}=0, d_{13}=1), \mathcal{M}^{\hat{t}} \equiv \mathcal{M}(d_{12}=0, d_{13}=0)$$

$$\prod_{i<j}\Gamma[d_{ij}+1]$$

$$(p_2-3)!\,(p_3-2)!,(p_2-2)!\,(p_3-3)!\,,2(p_2-3)!\,(p_3-3)!$$

$$n=k=2,\tilde{\delta}_{LR}=\tilde{\delta}_{13}+\tilde{\delta}_{14}+\tilde{\delta}_{23}+\tilde{\delta}_{24}=p_1+p_2-2\tilde{\delta}_{12}$$

$$\widetilde{M}_{\vec{p}}^{(5),\text{ there}}=\sqrt{2}/\sqrt{p_1p_2\dots p_5}\widetilde{M}_{\vec{p}}^{(5),\text{ here}}$$

$$\langle W(C)\rangle\sim e^{-TV(L)}=e^{-S_{\text{NG}}[\Sigma]},$$

$$V(L)=\lim_{T\to\infty}\frac{S_{\text{NG}}[\Sigma]}{T}$$

$$ds^2=G_{MN}dx^Mdx^N$$

$$S_{\text{NG}}=\frac{1}{2\pi\alpha'}\int\ \mathrm{d}\tau\ \mathrm{d}\sigma\sqrt{-g}$$

$$g_{\alpha\beta}=G_{MN}\partial_\alpha X^M\partial_\beta X^N$$

$$\begin{gathered}x_0=\tau,x_p=\sigma,x_i=x_i(\sigma),\\ x_i(\sigma=-L/2)=x_i(\sigma=+L/2)=x_{i_{\max}}\end{gathered}$$

$$L=\int_{-L/2}^{L/2}\ \mathrm{d}\sigma$$

$$\begin{aligned}g_{\tau\tau}&=G_{00}\\ g_{\sigma\sigma}&=G_{pp}+2\sum_i\ G_{pi}x_i'+\sum_i\ \sum_j\ G_{ij}x_i'x_j'\\ g_{\tau\sigma}&=g_{\sigma\tau}=G_{p0}+\sum_i\ G_{0i}x_i'\end{aligned}$$

$$\mathcal{L}=\sqrt{-g}=\sqrt{g_{\tau\sigma}^2-g_{\tau\tau}g_{\sigma\sigma}}=\sqrt{-g_p-\sum_i\ x_i'\left[2h_{pi}+G_{00}\sum_j\ G_{ij}x_j'\right]+\left(\sum_i\ G_{0i}x_i'\right)^2}.$$

$$\begin{aligned}h_{ab}&=G_{00}G_{ab}-G_{0a}G_{0b}\\ g_a&=h_{aa}=G_{00}G_{aa}-G_{0a}^2\end{aligned}$$

$$\mathcal{H}=\sum_i\ \mathcal{P}_ix_i'-\mathcal{L}=\frac{g_p+\sum_i\ h_{pi}x_i'}{\mathcal{L}}=\mathcal{C}$$

$$\mathcal{P}_i=\frac{\partial\mathcal{L}}{\partial x_i'}=-\frac{h_{pi}+G_{00}\sum_j\ G_{ij}x_j'-G_{0i}\sum_j\ G_{0j}x_j'}{\mathcal{L}}$$

$$\mathcal{C}=-\sqrt{-g_p}\Big|_{x_i=x_{i_{\min}}}=-\sqrt{G_{0p}^2-G_{00}G_{pp}}\Big|_{x_i=x_{i_{\min}}}$$

$$\mathcal{P}_i|_{x_i=x_{i_{\min}}}=-\frac{h_{pi}}{\sqrt{-g_p}}\Bigg|_{x_i=x_{i_{\min}}}=\frac{1}{\mathcal{C}}h_{pi}\Big|_{x_i=x_{i_{\min}}}$$

$$\frac{\left(g_p^{(T)}\right)^2+2g_p^{(T)}h_{pr}x_r'+h_{pr}^2x_r'^2}{\mathcal{L}_T^2-g_rx_r'^2-2x_r'\left[h_{pr}-G_{0r}\sum_{i\neq r}G_{0i}x_i'\right]}=\mathcal{C}^2$$

$$g_r\ =h_{rr}=G_{00}G_{rr}-G_{0r}^2$$
$$g_p^{(T)}\ =g_p+\sum_{i\neq r}h_{pi}x_i'$$

$$\mathcal{L}_T^2\ =\mathcal{L}^2|_{x_r'=0}=-g_p-\sum_{i\neq r}x_i'\left[2h_{pi}+G_{00}\sum_{j\neq r}G_{ij}x_j'\right]+\left(\sum_{i\neq r}G_{0i}x_i'\right)^2.$$

$$ax_r'^2+bx_r'+c=0$$

$$a=h_{pr}^2+\mathcal{C}^2g_r$$
$$b=2\left[g_p^{(T)}h_{pr}+\mathcal{C}^2\left(h_{pr}-G_{0r}\sum_{i\neq r}G_{0i}x_i'\right)\right]$$
$$c=\left(g_p^{(T)}\right)^2-\mathcal{C}^2\mathcal{L}_T^2$$

$$x_r'^2=-\frac{g_p\left(g_p+\mathcal{C}^2\right)}{\mathcal{C}^2g_r},$$

$$x_r'^2=-\frac{G_{pp}\left(G_{00}G_{pp}+\mathcal{C}^2\right)}{\mathcal{C}^2G_{rr}}$$

$$x_r'=\frac{\sqrt{g_p+\mathcal{C}^2}}{h_{pr}^2+\mathcal{C}^2g_r}\left(-h_{pr}\sqrt{g_p+\mathcal{C}^2}\pm|\mathcal{C}|\sqrt{-h}\right)$$

$$h=-\left(h_{pr}^2-g_rg_p\right)=-\left(h_{pr}^2-h_{rr}h_{pp}\right)$$

$$\mathrm{d}s^2=G_{00}(\mathrm{d}t+A_a\,\mathrm{d}x^a)^2+\frac{h_{ab}}{G_{00}}\,\mathrm{d}x^a\,\mathrm{d}x^b$$

$$h=-\left(h_{pr}^2-h_{rr}h_{pp}\right)=\det\begin{pmatrix}h_{pp}&h_{pr}\\h_{pr}&h_{rr}\end{pmatrix}=G_{00}\det\begin{pmatrix}G_{00}&G_{0p}&G_{0r}\\G_{0p}&G_{pp}&G_{pr}\\G_{0r}&G_{pr}&G_{rr}\end{pmatrix}$$

$$S_{\mathrm{NG}}=\frac{\mathcal{T}}{2\pi\alpha'\mathcal{C}}\int_{-L/2}^{L/2}\mathrm{d}\sigma\left(g_p^{(T)}+h_{pr}x_r'\right)$$

$$L\ =\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\left(\frac{1}{x_{r_+}'}+\frac{1}{x_{r_-}'}\right)$$
$$S_{\mathrm{NG}}\ =\frac{\mathcal{T}}{2\pi\alpha'\mathcal{C}}\left[\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\left(\frac{g_p^{(T)}}{x_{r_+}'}+\frac{g_p^{(T)}}{x_{r_-}'}+2h_{pr}\right)\right],$$

$$S_0=\frac{\mathcal{T}}{\pi\alpha'}\int_{x_{r_0}}^{x_{r_{\max}}}\mathrm{d}x_r\sqrt{-g_r}=\frac{\mathcal{T}}{\pi\alpha'}\left(\int_{x_{r_0}}^{x_{r_{\min}}}+\int_{x_{r_{\min}}}^{x_{r_{\max}}}\right)\mathrm{d}x_r\sqrt{-g_r},$$

$$S_{\mathrm{NG}}^{\mathrm{ren}}=S_{\mathrm{NG}}-S_0=\frac{\mathcal{T}}{\pi\alpha'}\left[\frac{1}{2\mathcal{C}}\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\left(\frac{g_p^{(T)}}{x'_{r_+}}+\frac{g_p^{(T)}}{x'_{r_-}}+2h_{pr}-2\mathcal{C}\sqrt{-g_r}\right)-\int_{x_{r_0}}^{x_{r_{\min}}}\mathrm{d}x_r\sqrt{-g_r}\right]$$

$$\pi\alpha' V=\frac{1}{2\mathcal{C}}\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\left(\frac{g_p^{(T)}}{x'_{r_+}}+\frac{g_p^{(T)}}{x'_{r_-}}+2h_{pr}-2\mathcal{C}\sqrt{-g_r}\right)-\int_{x_{r_0}}^{x_{r_{\min}}}\mathrm{d}x_r\sqrt{-g_r}$$

$$L\ =2|\mathcal{C}|\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\sqrt{-G_{00}G_{rr}}\frac{1}{\sqrt{G_{00}G_{pp}\left(G_{00}G_{pp}+\mathcal{C}^2\right)}}$$

$$V\ =\frac{1}{\pi\alpha'}\left[\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\sqrt{-G_{00}G_{rr}}\left(\frac{\sqrt{G_{00}G_{pp}}}{\sqrt{G_{00}G_{pp}+\mathcal{C}^2}}-1\right)-\int_{x_{r_0}}^{x_{r_{\min}}}\mathrm{d}x_r\sqrt{-G_{00}G_{rr}}\right]$$

$$=\frac{1}{\pi\alpha'}\left[-\frac{\mathcal{C}L}{2}+\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\sqrt{-G_{00}G_{rr}}\left(\sqrt{1+\frac{\mathcal{C}^2}{G_{00}G_{pp}}}-1\right)-\int_{x_{r_0}}^{x_{r_{\min}}}\mathrm{d}x_r\sqrt{-G_{00}G_{rr}}\right]$$

$$\frac{x}{\sqrt{x^2+1}}=\frac{\sqrt{x^2+1}}{x}-\frac{1}{x\sqrt{x^2+1}}$$

$$L\ =2|\mathcal{C}|\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\sqrt{-g_r}\frac{1}{\sqrt{g_p\left(g_p+\mathcal{C}^2\right)}}$$

$$V\ =\frac{1}{\pi\alpha'}\left[\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\sqrt{-g_r}\left(\frac{\sqrt{g_p}}{\sqrt{g_p+\mathcal{C}^2}}-1\right)-\int_{x_{r_0}}^{x_{r_{\min}}}\mathrm{d}x_r\sqrt{-g_r}\right]$$

$$=\frac{1}{\pi\alpha'}\left[-\frac{\mathcal{C}L}{2}+\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\sqrt{-g_r}\left(\sqrt{1+\frac{\mathcal{C}^2}{g_p}}-1\right)-\int_{x_{r_0}}^{x_{r_{\min}}}\mathrm{d}x_r\sqrt{-g_r}\right].$$

$$S_{\mathrm{NG}}=\frac{\mathcal{T}}{\pi\alpha'}\left(-\frac{\mathcal{C}L}{2}+\int_{x_{r_{\min}}}^{x_{r_{\max}}}\mathrm{d}x_r\sqrt{-g_r}\sqrt{1+\frac{\mathcal{C}^2}{g_p}}\right)$$

$$\mathrm{d}s^2\ =-a_c^2\xi^2\ \mathrm{d}t^2+\frac{\mathrm{d}\xi^2}{1+a_c^2\xi^2}+\left(1+a_c^2\xi^2\right)\left[\mathrm{d}\chi^2+\frac{1}{a_c^2}\sinh^2\left(a_c\chi\right)\mathrm{d}\Omega_{d-2}^2\right]$$

$$\mathrm{d}s^2\ =-a_c^2\xi^2\ \mathrm{d}t^2+\mathrm{d}\xi^2+\mathrm{d}\chi^2+\chi^2\ \mathrm{d}\Omega_{d-2}^2$$

$(X^1)^2-(X^0)^2>0$ and $(X^1)^2-\left(X^{d+1}\right)^2>0$ in a $(d+2)$-dimensional embedding Einstein - Hilbert space

$$-(X^0)^2+(X^1)^2+\cdots+\left(X^d\right)^2-\left(X^{d+1}\right)^2=-\mathcal{R}^2$$

$$\mathrm{d}s_{\text{boundary}}^2\ =-\mathrm{d}t^2+\mathrm{d}\chi^2+\frac{1}{a_c^2}\sinh^2\left(a_c\chi\right)\mathrm{d}\Omega_{d-2}^2$$

$$a_{\text{prop}}^2 = a_{\text{loc}}^2 + a_c^2 = \frac{1}{\xi^2} + \frac{1}{\mathcal{R}^2}$$

$$T_{\text{loc}} = \frac{1}{2\pi}\sqrt{\frac{2\Lambda}{d(d-1)} + a_{\text{prop}}^2} = \frac{a_{\text{embed}}}{2\pi},$$

$$\Lambda = -\frac{1}{\mathcal{R}^2}\frac{d(d-1)}{2}$$

$$T(\xi) = T_{\text{loc}} = \frac{1}{2\pi}\sqrt{a_{\text{prop}}^2 - a_{\text{crit}}^2}$$

$$T(\xi) = T_{\text{loc}} = \frac{1}{2\pi\xi}$$

$$T(\xi) = \frac{T_{\text{H}}}{\sqrt{-G_{tt}(\xi)}} = \frac{1}{2\pi\xi}$$

$$\tau = t, \sigma = \chi, \xi = \xi(\sigma), \xi(\sigma = \pm L/2) = \xi_{\max} = \infty$$

$$g_p = G_{tt}G_{\chi\chi} = -a^2\xi^2(1 + a^2\xi^2)$$
$$g_r = G_{tt}G_{\xi\xi} = -\frac{a^2\xi^2}{1 + a^2\xi^2}$$

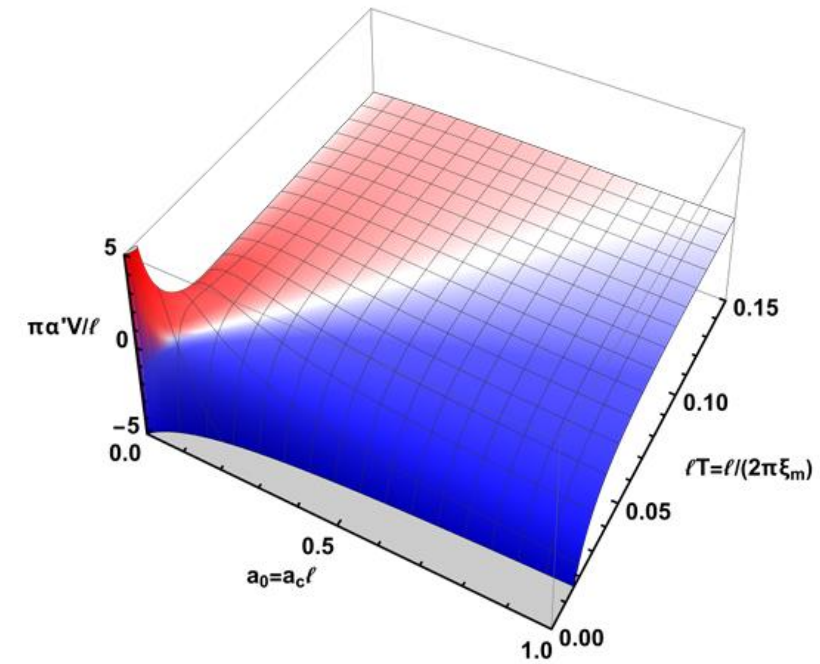
$$\mathcal{C} = -a\xi_m\sqrt{1 + a^2\xi_m^2}$$

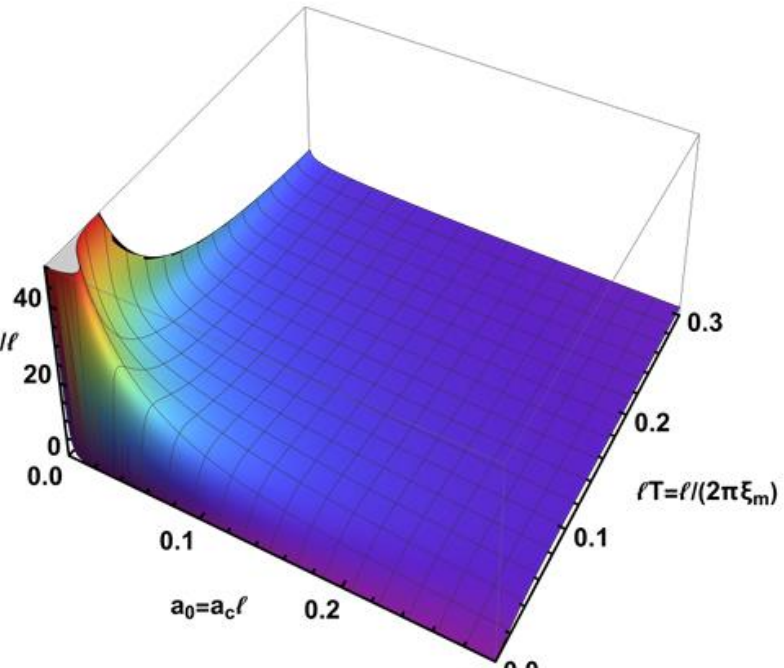
$$\xi_m = \frac{\sqrt{\sqrt{4\mathcal{C}^2 + 1} - 1}}{\sqrt{2}a}$$

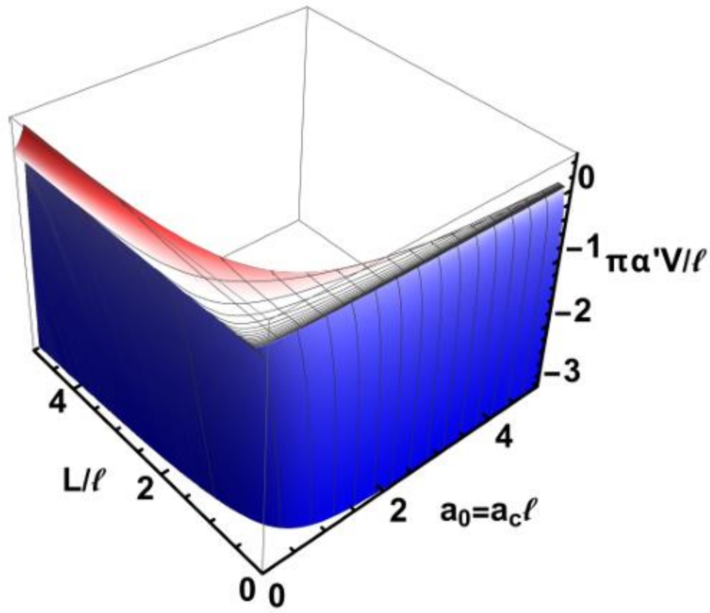
$$L = 2\xi_m \text{Re}\left[\tilde{\Pi}(n, m) - \frac{K(1-m)}{1-n}\right]$$
$$\pi\alpha' V = \frac{1}{a} + \sqrt{\frac{1}{a^2} + \xi_m^2}\,\text{Re}[iK(m) - \tilde{E}(m)]$$
$$= \frac{1}{a} - \frac{\mathcal{C}L}{2} + \frac{\xi_m\mathcal{C}}{n}\text{Re}[\tilde{K}(m) - n\tilde{\Pi}(n, m) - \tilde{E}(m)]$$

$$\tilde{\Pi}(n, m) = m\Pi(1-n, 1-m) + i\Pi(n, m)$$
$$\tilde{K}(m) = mK(1-m) + iK(m)$$
$$\tilde{E}(m) = E(1-m) + iE(m)$$

$$n = -a^2\xi_m^2, m = k^2 = \frac{n}{1-n}$$


5
πα'V/ℓ
0
−5
0.0
0.5
a0=acℓ
1.0
0.00
0.05
0.10
0.15
ℓT=ℓ/(2πξm)


40
L/ℓ
20
0
0.0
0.1
a0=acℓ
0.2
0.3
0.0
0.1
0.2
0.3
ℓT=ℓ/(2πξm)



$$L = 2|\mathcal{C}| \int_{\xi_m}^{\infty} \frac{\mathrm{d}\xi}{1+a^2\xi^2} \frac{1}{\sqrt{a^2\xi^2(1+a^2\xi^2)-\mathcal{C}^2}}$$
$$= /\!/ x = a\xi; x_m = a\xi_m /\!/ = \frac{2|\mathcal{C}|}{a} \int_{a\xi_m}^{\infty} \frac{\mathrm{d}x}{1+x^2} \frac{1}{\sqrt{(x^2-x_m^2)(x^2+x_m^2+1)}}$$
$$= /\!/ y = \frac{x}{x_m}; n = -x_m^2; m = k^2 = \frac{n}{1-n} /\!/ = 2i\xi_m \int_1^{\infty} \frac{\mathrm{d}y}{1-ny^2} \frac{1}{\sqrt{(1-y^2)(1-my^2)}}$$
$$= 2i\xi_m \Pi\left(\arcsin y; n, m\right)\Big|_{y=1}^{y\longrightarrow\infty},$$

$$\Pi(z;n,m) = \int_0^z \mathrm{d}x \frac{1}{(1-nx^2)\sqrt{(1-x^2)(1-mx^2)}}$$
$$\Pi(n,m) = \Pi(z=\pi/2;n,m)$$

$$\lim_{\alpha\longrightarrow\infty} \Pi(-i\alpha;n,m) = i\frac{F(-\pi/2;1-m) - n\Pi(-\pi/2;1-n,1-m)}{1-n}$$
$$= -i\frac{n\Pi(1-n,1-m) - K(1-m)}{1-n}$$

$$L = 2\xi_m \mathrm{Re}\left[\frac{n\Pi(1-n,1-m) - K(1-m)}{1-n} + i\Pi(n,m)\right]$$
$$= 2\xi_m \mathrm{Re}\left[\tilde{\Pi}(n,m) - \frac{K(1-m)}{1-n}\right]$$

$$\tilde{\Pi}(n,m) = m\Pi(1-n,1-m) + i\Pi(n,m).$$

$$\pi\alpha' V = \int_{\xi_m}^{\infty} \mathrm{d}\xi \frac{a^2\xi^2}{\sqrt{\mathcal{C}^2 - a^2\xi^2(1+a^2\xi^2)}} - \int_0^{\infty} \mathrm{d}\xi \frac{a\xi}{\sqrt{1+a^2\xi^2}}$$

$$\int_0^\infty \mathrm{d}\xi \frac{a\xi}{\sqrt{1+a^2\xi^2}} = \sqrt{\frac{1}{a^2}+\xi^2}\Bigg|_{\xi=0}^{\xi\to\infty}$$

$$\begin{aligned}\int_{\xi_m}^\infty \mathrm{d}\xi \frac{a^2\xi^2}{\sqrt{\mathcal{C}^2 - a^2\xi^2(1+a^2\xi^2)}} &= /\!/ y = \xi/\xi_m, \quad n = -a^2\xi_m^2, \quad m = k^2 = \frac{n}{1-n} /\!/ \\ &= i\sqrt{\frac{1}{a^2}+\xi_m^2}\left[\int_1^\infty \mathrm{d}y \frac{\sqrt{1-my^2}}{\sqrt{1-y^2}} - \int_1^\infty \mathrm{d}y \frac{1}{\sqrt{(1-y^2)(1-my^2)}}\right] \\ &= i\sqrt{\frac{1}{a^2}+\xi_m^2}\,[E(\arcsin y; m) - F(\arcsin y; m)]\Big|_{y=1}^{y\longrightarrow\infty},\end{aligned}$$

$$F(z;m) = \int_0^z \mathrm{d}x \frac{1}{\sqrt{(1-x^2)(1-mx^2)}}$$

$$E(z;m) = \int_0^z \mathrm{d}x \frac{\sqrt{1-mx^2}}{\sqrt{1-x^2}}$$

$$\begin{aligned}K(m) &= F(z=\pi/2;m) \\ E(m) &= E(z=\pi/2;m)\end{aligned}$$

$$\lim_{y\longrightarrow\infty} F(\arcsin\ y; m) = -iK(1-m),$$

$$E(i\phi;m) = i\left(F(\psi;1-m) - E(\psi;1-m) + \tan\ \psi\sqrt{1-(1-m)\sin^2\ \psi}\right).$$

$$\begin{aligned}\pi\alpha' V &= \frac{1}{a} + \sqrt{\frac{1}{a^2}+\xi_m^2}\,\mathrm{Re}[iK(m) - E(1-m) - iE(m)] \\ &= \frac{1}{a} + \sqrt{\frac{1}{a^2}+\xi_m^2}\,\mathrm{Re}[iK(m) - \tilde{E}(m)]\end{aligned}$$

$$\tilde{E}(m) = E(1-m) + iE(m).$$

$$\pi\alpha' V = \frac{1}{a} - \frac{\mathcal{C}L}{2} + \frac{\xi_m\mathcal{C}}{n}\mathrm{Re}[\tilde{K}(m) - n\tilde{\Pi}(n,m) - \tilde{E}(m)]$$

$$\tilde{K}(m) = mK(1-m) + iK(m).$$

$$A = \alpha d\psi = \begin{pmatrix} \alpha_1\mathbb{I}_{N_1} & 0 & s & 0 \\ 0 & \alpha_2\mathbb{I}_{N_2} & s & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & s & \alpha_M\mathbb{I}_{N_M} \end{pmatrix} d\psi$$

$$\Phi = \frac{1}{\sqrt{2}z}\begin{pmatrix} (\beta_1+i\gamma_1)\mathbb{I}_{N_1} & 0 & s & 0 \\ 0 & (\beta_2+i\gamma_2)\mathbb{I}_{N_2} & s & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & s & (\beta_M+i\gamma_M)\mathbb{I}_{N_M} \end{pmatrix}$$

$$\eta = \begin{pmatrix} \eta_1 \mathbb{I}_{N_1} & 0 & s & 0 \\ 0 & \eta_2 \mathbb{I}_{N_2} & s & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & s & \eta_M \mathbb{I}_{N_M} \end{pmatrix}$$

$$\delta_\sigma \log \langle \mathcal{O}_\Sigma \rangle = \frac{1}{24\pi} \int_\Sigma d^2 x \sqrt{h} \delta\sigma \left(bR_\Sigma + c_1 g_{mn} h^{\mu\sigma} h^{\nu\rho} \widehat{K}^m_{\mu\nu} \widehat{K}^n_{\rho\sigma} - c_2 W_{\mu\nu\rho\sigma} h^{\mu\rho} h^{\nu\sigma} \right)$$

$$\langle \mathcal{O}_{S^2} \rangle = \left(\frac{r}{r_0} \right)^{\frac{b}{3}}$$

$$b = 3(\dim G - \dim \mathbb{L})$$

$$\log \langle \mathcal{O}_{S^2} \rangle = \left(N^2 - \sum_{l=1}^{M} N_l^2 \right) \log (r\Lambda)$$

$$S_{\text{supergravity}}^{\text{on-shell}} = \left(N^2 - \sum_{l=1}^{M} N_l^2 \right) \log (r\Lambda),$$

$$ds^2 = f_1^2 ds^2_{AdS_3} + f_2^2 ds^2_{S^3} + f_3^2 (d\psi + V)^2 + f_3^2 ds^2_X$$
$$F_{(5)} = F \wedge \mathrm{vol}_{AdS_3} + \tilde{F} \wedge \mathrm{vol}_{S^3}$$

$$ds^2 = y \sqrt{\frac{2z+1}{2z-1}} ds^2_{AdS_3} + y \sqrt{\frac{2z-1}{2z+1}} ds^2_{S^3} + $$
$$+ \frac{2y}{\sqrt{4z^2-1}} (d\chi + V)^2 + \frac{\sqrt{4z^2-1}}{2y} ds^2_X$$

$$ds^2_X = dy^2 + dx_i dx_i$$

$$dV = \frac{1}{y} \star_X dz$$

$$\partial_i \partial_i z(y, \vec{x}) + y \partial_y \left(\frac{1}{y} \partial_y z(y, \vec{x}) \right) = \sum_{l=1}^{M} Q_l \delta(y - y_l) \delta^2 (\vec{x} - \vec{x}_l)$$

$$z(y, \vec{x}) = \frac{1}{2} + \sum_{l=1}^{M} z_l(y, \vec{x}), z_l(y, \vec{x}) = \frac{Q_l}{4\pi y_l} \left(\frac{(\vec{x} - \vec{x}_l)^2 + y^2 + y_l^2}{\sqrt{((\vec{x} - \vec{x}_l)^2 + y^2 + y_l^2)^2 - 4y^2 y_l^2}} - 1 \right)$$

$$V_I dx^I = -\sum_{l=1}^{M} \sum_{I,J=1}^{2} \frac{Q_l}{4\pi y_l} \epsilon_{IJ} \frac{(x_J - (x_l)_J)}{(\vec{x} - \vec{x}_l)^2} \frac{(\vec{x} - \vec{x}_l)^2 + y^2 - y_l^2}{\sqrt{((\vec{x} - \vec{x}_l)^2 + y^2 + y_l^2)^2 - 4y^2 y_l^2}} dx^I$$

$$F_{(5)}= -\frac{1}{4}\left[d\left(y^2\frac{2z+1}{2z-1}(d\chi+V)\right)-y^3\star_X d\left(\frac{1}{y^2}\left(z+\frac{1}{2}\right)\right)\right]\wedge \mathrm{vol}_{AdS_3}+$$
$$-\frac{1}{4}\left[d\left(y^2\frac{2z-1}{2z+1}(d\chi+V)\right)-y^3\star_X d\left(\frac{1}{y^2}\left(z-\frac{1}{2}\right)\right)\right]\wedge \mathrm{vol}_{S^3}$$

$$N_l=\frac{y_l^2}{4\pi l_P^4}=\frac{y_l^2}{L^4}N$$

$$\sum_{l=1}^{M} y_l^2=L^4$$

$$(\beta_l,\gamma_l)=\frac{\vec{x}_l}{2\pi l_s^2}$$

$$\alpha_l=\int_{D_l} B_{\mathrm{NS}},\eta_l=\int_{D_l} B_{\mathrm{R}}$$

$$\chi=\frac{1}{2}(\psi-\phi),\alpha=\psi+\phi$$
$$y=y_0\sqrt{\rho^2+1}\cos\ \theta$$
$$x_1=(x_1)_0+y_0\rho\sin\ \theta\cos\ \alpha$$
$$x_2=(x_2)_0+y_0\rho\sin\ \theta\sin\ \alpha$$

$$z^{(0)}=\frac{\cos^2\ \theta+\rho^2+1}{2(\sin^2\ \theta+\rho^2)}$$
$$V^{(0)}=\frac{(\rho^2-\sin^2\ \theta)}{2(\sin^2\ \theta+\rho^2)}(d\psi+d\phi)$$

$$ds^2\ =L^2\left(\frac{d\rho^2}{\rho^2+1}+(\rho^2+1)ds_{AdS_3}^2+\rho^2d\psi^2+d\theta^2+\sin^2\ \theta d\phi^2+\cos^2\ \theta ds_{S^3}^2\right)$$
$$F_{(5)}\ =-L^4\left(\rho(\rho^2+1)d\rho\wedge d\psi\wedge\mathrm{vol}_{AdS_3}+\sin\ \theta\cos^3\ \theta d\theta\wedge d\phi\wedge\mathrm{vol}_{S^3}\right)$$

$$ds_{AdS_3}^2=\frac{1}{z^2}(dz^2+dl^2+dm^2)$$

$$ds_{AdS_3}^2=dr^2+\sinh^2\ r d\Omega_2^2$$

$$\vec{x}^{(0)}=\sum_{l=1}^{M} y_l^2\vec{x}_l,\left(y^{(0)}\right)^2=\sum_{l=1}^{M} y_l^2=L^4$$

$$\chi=\frac{1}{2}(\psi-\phi),\alpha=\psi+\phi$$
$$y=L^2\sqrt{\rho^2+1}\cos\ \theta$$
$$x_1=\left(\vec{x}^{(0)}\right)_1+L^2\rho\sin\ \theta\cos\ \alpha$$
$$x_2=\left(\vec{x}^{(0)}\right)_2+L^2\rho\sin\ \theta\sin\ \alpha$$

$$m_{abc}=\sum_{l=1}^{M} y_l^a x_{l1}^b x_{l2}^c$$

$$ds_X^2=dy^2+dx_i dx_i=\frac{\rho^2+\sin^2\theta}{\rho^2+1}d\rho^2+(\rho^2+\sin^2\theta)d\theta^2+\rho^2\sin^2\theta d\alpha^2$$

$$\begin{aligned}\omega= &-\frac{1}{2}\alpha(M-1)+\frac{1}{2L^2\rho}\csc\theta(m_{001}\cos\alpha-m_{010}\sin\alpha)\\ &+\frac{1}{4L^4\rho^2}\csc^2\theta\big((m_{002}-m_{020})\sin 2\alpha+2m_{011}\cos 2\alpha\big)\\ &+\frac{\sin\theta}{6L^6\rho^3}\big(\csc^4\theta\big((3m_{012}-m_{030})\sin 3\alpha-(m_{003}-3m_{021})\cos 3\alpha\big)\\ &+2m_{210}\sin\alpha-2m_{201}\cos\alpha)+\frac{\sin^2\theta}{8L^8\rho^4}(\csc^6\theta(4(m_{031}-m_{013})\cos 4\alpha\\ &-(m_{004}-6m_{022}+m_{040})\sin 4\alpha)+4(m_{220}-m_{202})\sin 2\alpha-8m_{211}\cos 2\alpha)+\mathcal{O}(\rho^{-5})\end{aligned}$$

$$\begin{aligned}ds^2= &\frac{1}{(\rho^2+1)}\big(1+F_\rho\big)d\rho^2+(\rho^2+1)(1+F_1)ds_{AdS_3}^2+\rho^2(1+F_2)d\psi^2\\ &+\cos^2\theta(1+F_3)ds_{S^3}^2+(1+F_4)d\theta^2+\sin^2\theta(1+F_5)d\phi^2\\ &+F_6 d\theta d\psi+F_7 d\psi d\phi+F_8 d\theta d\phi\end{aligned}$$

$$\begin{aligned}ds^2 = &\frac{1}{u^2}\big(du^2+\alpha_1 ds_{AdS_3}^2+\alpha_2 d\tilde{\psi}^2\big)+\alpha_3 ds_{S^3}^2+\alpha_4 d\tilde{\theta}^2+\alpha_5 d\tilde{\phi}^2\\ &+\alpha_6 d\tilde{\theta}d\tilde{\psi}+\alpha_7 d\tilde{\psi}d\tilde{\phi}+\alpha_8 d\tilde{\theta}d\tilde{\phi}\end{aligned}$$

$$\begin{aligned}&\alpha_1=1+\sum_{n=1}^{\infty}\alpha_1^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})u^n, \alpha_2=1+\sum_{n=1}^{\infty}\alpha_2^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})u^n\\ &\alpha_3=\cos^2\tilde{\theta}+\sum_{n=1}^{\infty}\alpha_3^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})u^n, \alpha_4=1+\sum_{n=1}^{\infty}\alpha_4^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})u^n,\\ &\alpha_5=\sin^2\tilde{\theta}+\sum_{n=1}^{\infty}\alpha_5^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})u^n, \alpha_6=0+\sum_{n=1}^{\infty}\alpha_6^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})\\ &\alpha_7=0+\sum_{n=1}^{\infty}\alpha_7^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi}), \alpha_8=0+\sum_{n=1}^{\infty}\alpha_8^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})\end{aligned}$$

$$\begin{aligned}&\rho=\frac{1}{u}\left(1+\sum_{n=1}^{\infty}\rho^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})u^n\right), \psi=\tilde{\psi}+\sum_{n=1}^{\infty}\psi^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})u^n\\ &\theta=\tilde{\theta}+\sum_{n=1}^{\infty}\theta^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})u^n, \phi=\tilde{\phi}+\sum_{n=1}^{\infty}\phi^{(n)}(\tilde{\theta},\tilde{\phi},\tilde{\psi})u^n,\end{aligned}$$

$$\rho(u,\tilde{\theta},\tilde{\phi},\tilde{\psi}) = \frac{1}{u} + \frac{\left(F_\rho^{(2)} - 1\right)}{4}u^2 + \frac{F_\rho^{(3)}}{6}u^3$$
$$+\frac{1}{128}\left(-16\left(F_\rho^{(2)}\right)^2 + 16F_\rho^{(2)} + 3\left(\partial_{\tilde{\theta}}F_\rho^{(2)}\right)^2 + 16F_\rho^{(4)} + 3\csc^2\,\tilde{\theta}\left(\partial_{\tilde{\phi}}F_\rho^{(2)}\right)^2\right)u^4 + \mathcal{O}(u^5)$$
$$\theta(u,\tilde{\theta},\tilde{\phi},\tilde{\psi}) = \tilde{\theta} + \frac{\partial_{\tilde{\theta}}F_\rho^{(2)}}{8}u^2 + \frac{\partial_{\tilde{\theta}}F_\rho^{(3)}}{18}u^3 + \frac{1}{256}\left(\partial_{\tilde{\theta}}F_\rho^{(2)}\left(\partial_{\tilde{\theta}}F_\rho^{(2)} - 16F_4^{(2)} - 12F_\rho^2 + 4\right)\right.$$
$$\left.+8\partial_{\tilde{\theta}}F_\rho^{(4)} + \csc^2\,\tilde{\theta}\partial_{\tilde{\phi}}F_\rho^{(4)}\left(\partial_{\tilde{\theta}}\partial_{\tilde{\phi}}F_\rho^{(4)} + \cot\,\tilde{\theta}\partial_{\tilde{\phi}}F_\rho^{(4)}\right)\right)u^4 + \mathcal{O}(u^5)$$
$$\phi(u,\tilde{\theta},\tilde{\phi},\tilde{\psi}) = \tilde{\phi} + \frac{\partial_{\tilde{\phi}}F_\rho^{(2)}}{8\sin^2\,\tilde{\theta}}u^2 + \frac{\partial_{\tilde{\phi}}F_\rho^{(3)}}{18\sin^2\,\tilde{\theta}}u^3 + \frac{1}{256\sin^2\,\tilde{\theta}}\left(8\partial_{\tilde{\phi}}F_\rho^{(4)} + \partial_{\tilde{\theta}}F_\rho^{(2)}\partial_{\tilde{\theta}\tilde{\phi}}F_\rho^{(2)}\right.$$
$$\left.+\partial_{\tilde{\phi}}F_\rho^{(2)}\left(-4\cot\,\tilde{\theta}\partial_{\tilde{\theta}}F_\rho^{(2)} + \csc^2\,\tilde{\theta}\partial_{\tilde{\phi}}F_\rho^{(2)} - 12F_\rho^{(2)} - 16F_5^{(2)} + 4\right)\right) + \mathcal{O}(u^5)$$
$$\psi(u,\tilde{\theta},\tilde{\phi},\tilde{\psi}) = \tilde{\psi} + \frac{\partial_\psi F_\rho^{(2)}}{16}u^4 + \frac{\partial_\psi F_\rho^{(3)}}{30}u^5 + \mathcal{O}(u^6)$$

$$u = \frac{1}{\rho}\left(1 + \frac{F_\rho^{(2)} - 1}{4\rho^2} + \frac{F_\rho^{(3)}}{6\rho^3} + \frac{16\left(F_\rho^{(4)} - F_\rho^{(2)} + 1\right) - \left(\partial_\theta F_\rho^{(2)}\right)^2 - \left(\partial_\phi F_\rho^{(2)}\right)^2\csc^2\,\theta}{128\rho^4} + \mathcal{O}(\rho^{-5})\right)$$
$$\tilde{\psi} = \psi - \frac{\partial_\psi F_\rho^{(2)}}{16\rho^4} - \frac{\partial_\psi F_\rho^{(3)}}{30\rho^5} + \mathcal{O}(\rho^{-6})$$
$$\tilde{\theta} = \theta - \frac{\partial_\theta F_\rho^{(2)}}{8\rho^2} - \frac{\partial_\theta F_\rho^{(3)}}{18\rho^3} + \frac{1}{256\rho^4}\left[-8\partial_\theta F_\rho^{(4)} + 3\partial_\phi F_\rho^{(2)}\partial_\theta\partial_\phi F_\rho^{(2)}\csc^2\,\theta\right.$$
$$\left.-\left(\partial_\phi F_\rho^{(2)}\right)^2\cot\,\theta\csc^2\,\theta + \partial_\theta F_\rho^{(2)}\left(12 - 4F_\rho^{(2)} + 16F_4^{(2)} + 3\partial_\theta^2 F_\rho^{(2)}\right)\right] + \mathcal{O}(\rho^{-5})$$
$$\tilde{\phi} = \phi - \frac{\partial_\phi F_\rho^{(2)}}{8\sin^2\,\theta\rho^2} - \frac{\partial_\phi F_\rho^{(3)}}{18\sin^2\,\theta\rho^3} + \frac{1}{256\sin^2\,\theta\rho^4}\left[-8\partial_\phi F_\rho^{(4)} + 3\partial_\theta F_\rho^{(2)}\partial_\theta\partial_\phi F_\rho^{(2)}\right.$$
$$\left.+\partial_\phi F_\rho^{(2)}\left(12 - 4F_\rho^{(2)} + 16F_5^{(2)} + 3\partial_\phi^2 F_\rho^{(2)}\csc^2\,\theta - 4\partial_\theta F_\rho^{(2)}\cot\,\theta\right)\right] + \mathcal{O}(\rho^{-5})$$

$$\rho_{cut-off}(\epsilon,\psi,\theta,\phi)=\frac{1}{\epsilon}+\frac{F_\rho^{(2)}-1}{4}\epsilon+\frac{F_\rho^{(3)}}{6}\epsilon^2$$
$$+\frac{16\left(F_\rho^{(4)}-F_\rho^{(2)}\left(F_\rho^{(2)}-1\right)\right)\left(\partial_\theta F_\rho^{(2)}\right)^2-\left(\partial_\phi F_\rho^{(2)}\right)^2\csc^2\theta}{128}\epsilon^3+\mathcal{O}(\epsilon^4)=$$
$$\frac{1}{\epsilon}+\frac{\epsilon}{16}\big(12\sin^2\theta\big(2\sin 2\alpha m_{211}+\cos 2\alpha(m_{220}-m_{202})\big)$$
$$-3\cos 2\theta(2m_{202}+2m_{220}-m_{400}+1)+2m_{202}+2m_{220}-m_{400}-3)$$
$$+\frac{\epsilon^2}{2}(3\sin\alpha(\sin\theta-\sin 3\theta)(m_{203}+m_{221}-m_{401})$$
$$+4\sin^3\theta\big(\cos 3\alpha(m_{230}-3m_{212})-\sin 3\alpha(m_{203}-3m_{221})\big)$$
$$+3\cos\alpha(\sin\theta-\sin 3\theta)(m_{212}+m_{230}-m_{410})\big)$$
$$+\epsilon^3(40\sin 4\alpha(-8m_{213}+9m_{211}(m_{202}-m_{220})+8m_{231})\sin^4\theta$$
$$-10\cos 4\alpha\left(9(m_{202}-m_{220})^2-4\left(9m_{211}^2+2m_{204}+2(m_{240}-6m_{222})\right)\right)\sin^4\theta$$
$$+4\sin 2\alpha\sin^2(\theta)(-16(m_{213}+m_{231})-3m_{211}(18m_{202}+18m_{220}-9m_{400}+1)$$
$$+24m_{411}+\cos 2\theta(-80(m_{213}+m_{231})+m_{211}(90m_{202}+90m_{220}-45m_{400}-3)+120m_{411}))$$
$$-2\cos 2\alpha\sin^2\theta(-54m_{202}^2+3(9m_{400}-1)m_{202}-16m_{204}$$
$$+3m_{220}(18m_{220}-9m_{400}+1)+8(2m_{240}+3m_{402}-3m_{420})$$
$$+\cos 2\theta(90m_{202}^2-3(15m_{400}+1)m_{202}-80m_{204}+3m_{220}(-30m_{220}+15m_{400}+1)$$
$$+40(2m_{240}+3m_{402}-3m_{420})))$$
$$-\frac{3}{8}(238m_{202}^2-4(19m_{220}+25m_{400}-9)m_{202}+552m_{211}^2+238m_{220}^2$$
$$-48m_{204}-96m_{222}-48m_{240}+4m_{220}(9-25m_{400})+m_{400}(25m_{400}-18)$$
$$+96m_{402}+96m_{420}+9)$$
$$-\frac{1}{8}\cos 4\theta(270m_{202}^2+12(15m_{220}-15m_{400}-1)m_{202}+360m_{211}^2+270m_{220}^2$$
$$+45m_{400}^2-240m_{204}-480m_{222}-240m_{240}+6m_{400}-12m_{220}(15m_{400}+1)$$
$$+480m_{402}+480m_{420}-80m_{600}+29)$$
$$+\cos 2\theta(99m_{202}^2-6(9m_{220}+6m_{400}+2)m_{202}+252m_{211}^2+99m_{220}^2+9m_{400}^2$$
$$-24m_{204}-48m_{222}-24m_{240}+6m_{400}-12m_{220}(3m_{400}+1)$$
$$+48m_{402}+48m_{420}-8m_{600}-7)+6m_{600})+\mathcal{O}(\epsilon^4)$$

$$\mathcal{L}_I\psi^I=\mathcal{L}_{IJ}\psi^I\psi^J+\mathcal{L}_{IJK}\psi^I\psi^J\psi^K+\cdots$$

$$\widehat{\Psi}^I=s^I+\mathcal{K}_{JL}^I s^J s^K+\cdots$$

$$S=\frac{1}{2\kappa^2}\left(\int d^{10}x\sqrt{g}R-4\int F_{(5)}\wedge\star F_{(5)}+2\int d^9x\sqrt{h}K\right)$$

$$-\frac{1}{2\kappa^2}\left(\int_{10d}24F_{(5)}^{(el)}\wedge\star F_{(5)}^{(el)}\right)$$

$$F_{(5)}^{(el)}=-\frac{1}{4}\left[d\left(y^2\frac{2z+1}{2z-1}(d\chi+V)\right)-y^3\star_X d\left(\frac{1}{y^2}\left(z+\frac{1}{2}\right)\right)\right]\wedge\mathrm{vol}_{AdS_3}$$
$$\star F_{(5)}^{(el)}=F_{(5)}^{(mag)}=-\frac{1}{4}\left[d\left(y^2\frac{2z-1}{2z+1}(d\chi+V)\right)-y^3\star_X d\left(\frac{1}{y^2}\left(z-\frac{1}{2}\right)\right)\right]\wedge\mathrm{vol}_{S^3}$$

$$F_5^{(el)}\wedge\star F_5^{(el)}=-\frac{yz}{2(1-4z^2)^2}\left((1-4z^2)^2+\frac{2y}{z}(1-4z^2)\partial_y z+4y^2\partial_I z\partial^I z\right)$$
$$\times \text{vol}_{AdS_3}\wedge\text{vol}_{S^3}\wedge d\chi\wedge\text{vol}_X=$$
$$=\left(-\frac{1}{2}yz+\partial_I u_I+\frac{y^3}{4(1-4z^2)}\left(yd\left(\frac{1}{y}\star_X dz\right)\right)\right)\times\text{vol}_{AdS_3}\wedge\text{vol}_{S^3}\wedge d\chi\wedge\text{vol}_X$$

$$u_I=-\frac{y^3}{4(1-4z^2)}\partial_I z, I=x_1,x_2,y$$

$$I_1=-\frac{4}{\kappa^2}\text{vol}(AdS_3)\text{vol}(S^3)\text{vol}(S^1)\int_X -\frac{1}{2}yzdx_1dx_2dy$$
$$I_2=-\frac{4}{\kappa^2}\text{vol}(AdS_3)\text{vol}(S^3)\text{vol}(S^1)\int_X \partial_I u^I dx_1dx_2dy$$
$$I_3=\frac{1}{\kappa^2}\int d^9x\sqrt{h}K$$

$$S_{on-shell}=\frac{\pi}{2\kappa^2}\text{Vol}(AdS_3)\text{Vol}(S^3)\text{Vol}(S^1)\left(\frac{5}{\epsilon^4}+\frac{2}{\epsilon^2}+\frac{3}{8}-m_{400}-\mathcal{F}\right)$$

$$\mathcal{F}=\frac{3}{32}(1+4m_{220}+4m_{202}-2m_{400}+10(m_{220}^2+m_{202}^2)+$$
$$+24m_{211}^2-4(m_{220}+m_{202})m_{400}+m_{400}^2-4m_{220}m_{202})$$

$$\frac{a}{2\kappa^2L}\int d^9x\sqrt{h}$$

$$\frac{a}{2\kappa^2L}\int d^9x\sqrt{h}=\frac{a}{2\kappa^2L}\text{vol}(AdS_3)\text{vol}(S^3)\text{vol}(S^1)\pi\left(\frac{\epsilon^4}{2}+\frac{\epsilon^2}{4}+\mathcal{C}\right)$$

$$\mathcal{C}=\frac{1}{32}(-10m_{2,0,2}^2+4(m_{2,2,0}+m_{4,0,0}-1)m_{2,0,2}-24m_{2,1,1}^2-10m_{2,2,0}^2-m_{4,0,0}^2$$
$$-4m_{2,2,0}+4m_{2,2,0}m_{4,0,0}+2m_{4,0,0}-1)$$

$$S_{\text{on-shell}}+S_{\text{counterterm}}=\frac{\pi}{2\kappa^2}\text{Vol}(AdS_3)\text{Vol}(S^3)\text{Vol}(S^1)\left(\frac{7}{2\epsilon^4}+\frac{5}{4\epsilon^2}+\frac{3}{8}-m_{400}\right)$$

$$S_{on-shell}-S_{on-shell}^{\text{vacuum}}=\frac{\pi}{2\kappa^2}L^8\text{Vol}(AdS_3)\text{Vol}(S^3)\text{Vol}(S^1)(1-L^{-8}m_{400})=$$
$$=\log\Lambda\left(N^2-\sum_l N_l^2\right)$$

$$2\kappa^2=(2\pi)^7g_s^2(\alpha')^4, L^4=4\pi g_s(\alpha')^2N, \text{vol}(AdS_3)=2\pi\log\Lambda$$

$$F_a(\rho,\theta,\alpha)=\sum_{n=1}^{\infty}\frac{F_a^{(n)}(\theta,\alpha)}{\rho^n}$$

$$4F_\rho^{(2)} = (1-3\cos\,2\theta)[1+2(m_{220}+m_{202})-m_{400}]$$
$$+12[\cos\,2\alpha(m_{220}-m_{202})+2m_{211}\sin\,2\alpha]\sin^2\,\theta$$
$$F_\rho^{(3)} = 3(\sin\,\theta-\sin\,3\theta)[(m_{212}+m_{230}-m_{410})\cos\,\alpha+(m_{221}+m_{203}-m_{401})\sin\,\alpha]$$
$$+4\sin^3\,\theta[(-3m_{212}+m_{230})\cos\,3\alpha-(-3m_{221}+m_{203})\sin\,3\alpha]$$
$$32F_\rho^{(4)} = -4\cos^4\,\theta+(5-12\cos\,2\theta+15\cos\,4\theta)(2m_{202}+2m_{220}-m_{400})$$
$$-16(1+5\cos\,2\theta)\sin^2\theta\sin\,2\alpha[3m_{211}+8(m_{213}+m_{231})-12m_{411}]$$
$$-8(1+5\cos\,2\theta)\sin^2\,\theta\cos\,2\alpha[3(m_{220}-m_{202})+8(m_{240}-m_{204})+12(m_{402}-m_{420})]$$
$$-640\sin\,4\alpha\sin^4\,\theta(m_{213}-m_{231})+24(3-4\cos\,2\theta+5\cos\,4\theta-40\cos\,4\alpha\sin^4\,\theta)m_{222}$$
$$+4(9-12\cos\,2\theta+15\cos\,4\theta+40\cos\,4\alpha\sin^4\,\theta)(m_{204}+m_{240})$$
$$-4(3-4\cos\,2\theta+5\cos\,4\theta)[6(m_{402}+m_{420})-m_{600}]$$
$$-(12\sin^2\,\theta[\cos\,2\alpha(m_{202}-m_{220})-2\sin\,2\alpha m_{211}]+$$
$$-(1-3\cos\,2\theta)(2m_{202}+2m_{220}-m_{400}))^2$$

$$F_1^{(2)} = 3\sin^2\,\theta(\cos\,2\alpha(m_{202}-m_{220})-2\sin\,2\alpha m_{211})$$
$$+\frac{1}{4}(3\cos\,2\theta-1)(2m_{202}+2m_{220}-m_{400}+1)$$
$$F_1^{(3)} = 3\sin\,\alpha(\sin\,3\theta-\sin\,\theta)(m_{203}+m_{221}-m_{401})$$
$$+4\sin^3\,\theta\big(\sin\,3\alpha(m_{203}-3m_{221})+\cos\,3\alpha(3m_{212}-m_{230})\big)$$
$$+3\cos\,\alpha(\sin\,3\theta-\sin\,\theta)(m_{212}+m_{230}-m_{410}),$$

$$F_4^{(2)} = 6\sin\,2\alpha\sin^2\,\theta m_{211}+3\cos\,2\alpha\sin^2\,\theta(m_{220}-m_{202})$$
$$-\frac{1}{4}(3\cos\,(2\theta)-1)(2m_{202}+2m_{220}-m_{400}+1)$$
$$F_4^{(3)} = \sin\,(\alpha)(\sin\,\theta-\sin\,3\theta)(m_{203}+m_{221}-m_{401})$$
$$+4\sin^3\,\theta\big(\cos\,(3\alpha)(m_{230}-3m_{212})-\sin\,3\alpha(m_{203}-3m_{221})\big)$$
$$+3\cos\,\alpha(\sin\,(\theta)-\sin\,3\theta)(m_{212}+m_{230}-m_{410})$$

$$F_6^{(2)} = \cot\,\theta\csc^4\,\theta\big(4\cos\,4\alpha(m_{031}-m_{013})-\sin\,4\alpha(m_{004}-6m_{022}+m_{040})\big)$$
$$-2\sin\,2\theta(\sin\,2\alpha(m_{202}-m_{220})+2\cos\,2\alpha m_{211})$$
$$F_6^{(3)} = 0$$
$$F_7^{(2)} = \sin^2\,\theta\big(\csc^6\,\theta\big(4\sin\,4\alpha(m_{031}-m_{013})+\cos\,4\alpha(m_{004}-6m_{022}+m_{040})\big)$$
$$+8\sin\,2\alpha m_{211}+4\cos\,2\alpha(m_{220}-m_{202})+4m_{202}+4m_{220}-2m_{400}+2),$$
$$F_7^{(3)} = 8\sin^3\,\theta(-3\cos\,\alpha\cos\,2\theta(m_{212}+m_{230}-m_{410})+$$
$$+2\sin^2\,\theta\big(\cos\,3\alpha(m_{230}-3m_{212})-\sin\,3\alpha(m_{203}-3m_{221})\big)$$
$$-3\sin\,\alpha\cos\,(2\theta)(m_{203}+m_{221}-m_{401})\big)$$
$$F_8^{(4)} = 4\sin^3\,\theta\cos\,\theta(\sin\,2\alpha(m_{202}-m_{220})+2\cos\,2\alpha m_{211})+$$
$$+\cot\,\theta\csc^2\,\theta\big(\sin\,4\alpha(m_{004}-6m_{022}+m_{040})+4\cos\,4\alpha(m_{013}-m_{031})\big)$$

$$I_1 = -\frac{4}{\kappa^2}\mathrm{vol}(AdS_3)\mathrm{vol}(S^3)\mathrm{vol}(S^1)\left(\frac{1}{2}\int_X y\,dx_1dx_2dy+\sum_{l=1}^{M}\int_X yz_l\,dx_1dx_2dy\right)$$

$$\frac{1}{2}\int_X y dx_1 dx_2 dy = \frac{1}{2}\int_0^{2\pi} d\alpha \int_0^{\pi/2} d\theta \int_0^{\rho_c} d\rho\rho \sin\,\theta \cos\,\theta (\sin^2\,\theta + \rho^2)$$
$$= \frac{1}{2}\int_0^{2\pi} d\alpha \int_0^{\pi/2} d\theta \left[\frac{1}{8}\rho^2 \sin\,\theta \cos\,\theta (\rho^2 + 2\sin^2\,\theta)\right]_0^{\rho_c} =$$
$$= \frac{\pi}{8\epsilon^4} + \frac{\pi}{32\epsilon^2}(1 + 2m_{202} + 2m_{220} - m_{400}) +$$
$$+\frac{\pi}{1536}(-288m_{202}^2 + 144m_{220}m_{202} + 108m_{400}m_{202} + 12m_{202} - 720m_{211}^2 - 288m_{220}^2$$
$$-27m_{400}^2 + 48m_{204} + 12m_{220} + 96m_{222} + 48m_{240} + 108m_{220}m_{400} - 6m_{400} - 96m_{402}$$
$$-96m_{420} + 16m_{600} - 7) + O(\epsilon^2)$$

$$dx_1 dx_2 dy = \frac{\rho \sin\,\theta (\sin^2\,\theta + \rho^2)}{\sqrt{\rho^2 + 1}} d\rho d\theta d\alpha$$

$$\chi = \frac{1}{2}(\bar{\psi} - \bar{\phi}), \bar{\alpha} = \bar{\psi} + \bar{\phi}$$
$$y = y_l \sqrt{\bar{\rho}^2 + 1} \cos\,\bar{\theta}$$
$$x_1 = (\vec{x}_l)_1 + y_l \bar{\rho} \sin\,\bar{\theta} \cos\,\bar{\alpha}$$
$$x_2 = (\vec{x}_l)_2 + y_l \bar{\rho} \sin\,\bar{\theta} \sin\,\bar{\alpha}$$

$$\int_X y z_l dx_1 dx_2 dy = y_l^4 \int d\bar{\rho} d\bar{\alpha} d\bar{\theta}\, \bar{\rho} \cos^3\,\bar{\theta} \sin\,\bar{\theta} =$$
$$= \frac{y_l^4}{2}\int d\bar{\alpha} d\bar{\theta}\, [\bar{\rho}^2 \cos^3\,\bar{\theta} \sin\,\bar{\theta}]_0^{\bar{\rho}_c}$$

$$\rho = \bar{\rho} y_l + r_l \sin\,\bar{\theta} \cos\,(\bar{\alpha} + \beta_l)$$
$$+\frac{r_l^2\left(\cos\,(2\bar{\theta}) - 2\sin^2\,\bar{\theta} \cos\,\left(2(\bar{\alpha} + \beta_l)\right) + 3\right) + 4\left(y_l^2 - 1\right)\cos^2\,\bar{\theta}}{8\bar{\rho} y_l} + \mathcal{O}\left(\frac{1}{\bar{\rho}^2}\right)$$
$$\theta = \bar{\theta} + \frac{r_l \cos\,\bar{\theta} \cos\,(\bar{\alpha} + \beta_l)}{\bar{\rho} y_l} - \frac{\cot\,\bar{\theta}}{2\bar{\rho}^2 y_l^2}\left(-\sin^2\,\bar{\theta} + y_l^2 \sin^2\,\bar{\theta} +\right.$$
$$\left.+ r_l^2\left(\cos\,\left(2(\bar{\alpha} + \beta_l)\right) - \cos\,(2\bar{\theta}) \cos^2\,(\bar{\alpha} + \beta_l)\right)\right) + \mathcal{O}\left(\frac{1}{\bar{\rho}^3}\right)$$
$$\alpha = \bar{\alpha} - \frac{r_l \csc\,\bar{\theta} \sin\,(\bar{\alpha} + \beta_l)}{\bar{\rho} y_l} + \frac{r_l^2 \csc^2\,\bar{\theta} \sin\,\left(2(\bar{\alpha} + \beta_l)\right)}{2\bar{\rho}^2 y_l^2} + \mathcal{O}\left(\frac{1}{\bar{\rho}^3}\right)$$

$$(\vec{x}_l)_1 = r_l \cos\,\beta_l, (\vec{x}_l)_2 = -r_l \sin\,\beta_l$$

$$\bar{\rho}_{cut-off} = \frac{1}{\epsilon y_l} - \frac{r_l \sin\,\bar{\theta} \cos\,(\bar{\alpha} + \beta_l)}{y_l} + \frac{\epsilon}{16 y_l}\left(-2r_l^2\left(\cos\,(2\bar{\theta}) - 2\sin^2\,\bar{\theta} \cos\,\left(2(\bar{\alpha} + \beta_l)\right) + 3\right)\right.$$
$$-8y_l^2 \cos^2\,\bar{\theta} + 12\sin^2\bar{\theta}(2m_{211} \sin\,(2\bar{\alpha}) + (m_{220} - m_{202})\cos\,(2\bar{\alpha}))$$
$$\left.+(-6m_{202} - 6m_{220} + 3m_{400} + 1)\cos\,(2\bar{\theta}) + 2m_{202} + 2m_{220} - m_{400} + 1\right) + \mathcal{O}(\epsilon^2)$$

$$\frac{1}{2}\int_X y z_l dx_1 dx_2 dy = \frac{\pi y_l^2}{4\epsilon^2} - \frac{1}{24}\pi y_l^2\left(4((\vec{x}_l)_1^2 + (\vec{x}_l)_2^2) + 4y_l^2 - 1\right) + \mathcal{O}(\epsilon^1)$$

$$\sum_{l=1}^{M} \int_X y z_l dx_1 dx_2 dy = \frac{\pi}{4\varepsilon^2} + \frac{\pi}{24}\left(1 - 4(m_{220} + m_{202} + m_{400})\right) + \mathcal{O}(\epsilon)$$

$$I_1 = -\frac{4}{\kappa^2}\mathrm{vol}(AdS_3)\mathrm{vol}(S^3)\mathrm{vol}(S^1)\Big(\frac{\pi}{8\epsilon^4} + \frac{\pi}{32\epsilon^2}(1 + 2m_{202} + 2m_{220} - m_{400})$$
$$+ \frac{\pi}{1536}(-288m_{202}^2 + 144m_{220}m_{202} + 108m_{400}m_{202} + 12m_{202} - 720m_{211}^2 - 288m_{220}^2$$
$$-27m_{400}^2 + 48m_{204} + 12m_{220} + 96m_{222} + 48m_{240} + 108m_{220}m_{400} - 6m_{400} - 96m_{402}$$
$$-96m_{420} + 16m_{600} - 7)\Big) + O(\epsilon^2)$$

$$I_2 = -\frac{4}{\kappa^2}\mathrm{vol}(AdS_3)\mathrm{vol}(S^3)\mathrm{vol}(S^1)\int_X \partial_I u^I dx_1 dx_2 dy =$$
$$= -\frac{4}{\kappa^2}\mathrm{vol}(AdS_3)\mathrm{vol}(S^3)\mathrm{vol}(S^1)\int_{\partial X} \tilde{n}_I u^I \sqrt{\gamma} d^2x$$

$$\tilde{n}_\rho = \frac{1}{\mathcal{D}}, \tilde{n}_\theta = -\frac{1}{\mathcal{D}}\partial_\theta \rho_c(\epsilon,\theta,\alpha), \tilde{n}_\alpha = -\frac{1}{\mathcal{D}}\partial_\alpha \rho_c(\epsilon,\theta,\alpha),$$

$$\mathcal{D}^2 = \frac{\rho^2 + 1 + (\partial_\theta \rho_c)^2}{\rho^2 + \sin^2\theta} + \frac{(\partial_\alpha \rho_c)^2}{\rho^2 \sin^2\theta}$$

$$\gamma_{ab} = g_{IJ}\frac{\partial x^I}{\partial y_a}\frac{\partial x^J}{\partial y_b}$$

$$\sqrt{\gamma} = \sqrt{g}\mathcal{D}^2$$

$$I_2 = -\frac{4}{\kappa^2}\mathrm{vol}(AdS_3)\mathrm{vol}(S^3)\mathrm{vol}(S^1)\int_{\partial X} \tilde{n}_I u^I \sqrt{\gamma} d^2x$$
$$= -\frac{4}{\kappa^2}\mathrm{vol}(AdS_3)\mathrm{vol}(S^3)\mathrm{vol}(S^1)\Big(-\frac{\pi}{16\epsilon^4} + \frac{\pi}{64\epsilon^2}(1 + 2m_{220} + 2m_{202} - m_{400})$$
$$+ \frac{\pi}{3072}(-51 - 100m_{220} - 100m_{202} + 50m_{400}$$
$$+72(m_{220}^2 + m_{202}^2) + 144m_{211}^2 - 36(m_{220} + m_{202})m_{400} + 9m_{400}^2$$
$$+48(m_{240} + m_{204}) + 96(m_{222} - m_{402} - m_{420}) + 16m_{600})\Big) + O(\epsilon^1)$$

$$I_3 = \frac{1}{\kappa^2}\int d^9x\sqrt{h}K$$

$$K = \nabla_\mu n^\mu$$

$$\sqrt{h}K=\frac{4\sin\ \theta\cos^3\ \theta}{\epsilon^4}+\frac{1}{\epsilon^2}\Big(\frac{9}{4}\sin^3\ 2\theta\big(2\sin\ 2\alpha m_{211}+\cos\ 2\alpha(m_{220}-m_{202})\big)$$
$$+\frac{1}{2}\sin\ \theta\cos^3\ \theta(-9\cos\ 2\theta(2m_{202}+2m_{220}-m_{400}+1)+6m_{202}+6m_{220}-3m_{400}+5)\Big)$$
$$+\frac{28\sin^2\ \theta\cos^3\ \theta}{3\epsilon}\big(-3\cos\ 2\theta\big(\cos\ \alpha(m_{212}+m_{230}-m_{410})-\sin\ \alpha(m_{203}+m_{221}-m_{401})\big)$$
$$+2\sin^2\ \theta\big(\cos\ 3\alpha(m_{230}-3m_{212})-\sin\ (3\alpha)(m_{203}-3m_{221})\big)\big)$$
$$+\frac{1}{64}\big[256\cos^3\ \theta\sin\ 4\alpha\big(9m_{211}(m_{202}-m_{220})+20(m_{231}-m_{213})\big)\sin^5\ \theta$$
$$-64\cos\ 4\alpha\cos^3\ \theta\left(9(m_{202}-m_{220})^2-4\left(9m_{211}^2+5m_{204}+5(m_{240}-6m_{222})\right)\right)\sin^5\ \theta$$
$$+4\sin\ 2\alpha\sin^3\ 2\theta\big(-32(m_{213}+m_{231})+9m_{211}\big(m_{400}-2(m_{202}+m_{220}-1)\big)+48m_{411}$$
$$+4\cos\ 2\theta(-40(m_{213}+m_{231})+3m_{211}(6m_{202}+6m_{220}-3m_{400}-5)+60m_{411})\big)$$
$$-2\cos\ 2\alpha\sin^3\ 2\theta(-18m_{202}^2+9(m_{400}+2)m_{202}-32m_{204}+9m_{220}(2m_{220}-m_{400}-2)$$
$$+4\cos\ 2\theta(18m_{202}^2-3(3m_{400}+5)m_{202}-40m_{204}+3m_{220}(-6m_{220}+3m_{400}+5)$$
$$+20(2m_{240}+3m_{402}-3m_{420})\big)+16(2m_{240}+3m_{402}-3m_{420})\big)$$
$$+\frac{1}{16}(\sin 2\theta(-144m_{211}^2-192m_{204}-384m_{222}-192m_{240}+36m_{202}(4m_{220}-m_{400}-1)$$
$$-36m_{220}(m_{400}+1)+9m_{400}(m_{400}+2)+384m_{402}+384m_{420}-64m_{600}+37)$$
$$-3\sin\ (6\theta)(-144m_{211}^2-192m_{204}-384m_{222}-192m_{240}+36m_{202}(4m_{220}-m_{400}-1)$$
$$-36m_{220}(m_{400}+1)+9m_{400}(m_{400}+2)+384m_{402}+384m_{420}-64m_{600}+37)$$
$$-4\sin\ (8\theta)(54m_{202}^2+12(3m_{220}-3m_{400}-5)m_{202}+72m_{211}^2+54m_{220}^2+9m_{400}^2$$
$$-120m_{204}-240m_{222}-120m_{240}+30m_{400}-12m_{220}(3m_{400}+5)$$
$$+240m_{402}+240m_{420}-40m_{600}+1)$$
$$+4\sin\ 4\theta(108m_{202}^2-12(6m_{220}+3m_{400}+7)m_{202}+288m_{211}^2+108m_{220}^2$$
$$+9m_{400}^2-48m_{204}-96m_{222}-48m_{240}+42m_{400}-12m_{220}(3m_{400}+7)$$
$$+96m_{402}+96m_{420}-16m_{600}-35))+O(\epsilon^1)$$

$$I_3=\frac{1}{\kappa^2}\int\ d^9x\sqrt{h}K=\frac{\pi}{\kappa^2}\mathrm{vol}(AdS_3)\mathrm{vol}(S^3)\mathrm{vol}(S^1)\left(\frac{4}{\epsilon^4}+\frac{1}{\epsilon^2}\right)+\mathcal{O}(\epsilon^1)$$

$$\rho=\rho_{cut-off}(\epsilon,\theta,\phi,\psi)$$

$$h_{\mu\nu}=g_{\mu\nu}-n_\mu n_\nu, h_{ab}=g_{\mu\nu}e_a^\mu e_b^\nu$$

$$e_a^\mu=\frac{\partial x^\mu}{\partial y^a}$$

$$\rho_{cut-off}(\epsilon,\psi,\theta,\alpha)=\frac{1}{\epsilon}+\frac{F_\rho^{(2)}-1}{4}\epsilon+\frac{F_\rho^{(3)}}{6}\epsilon^2$$
$$+\frac{16\left(F_\rho^{(4)}-F_\rho^{(2)}\left(F_\rho^{(2)}-1\right)\right)-\left(\partial_\theta F_\rho^{(2)}\right)^2-\left(\partial_\alpha F_\rho^{(2)}\right)^2\csc^2\ \theta}{128}\epsilon^3+O(\epsilon^4)$$

$$n_\rho=\frac{1}{\mathcal{N}}, n_\theta=-\frac{1}{\mathcal{N}}\partial_\theta\rho_c(\epsilon,\theta,\alpha), n_\alpha=-\frac{1}{\mathcal{N}}\partial_\alpha\rho_c(\epsilon,\theta,\alpha)$$

$$\frac{2y}{\sqrt{4z^2-1}}\left(\frac{4z^2-1}{4y^2}\frac{\rho^2+\sin^2\ \theta}{\rho^2+1}d\rho^2+d\psi^2+\left(\frac{4z^2-1}{4y^2}(\rho^2+\sin^2\ \theta)+V_\theta^2\right)d\theta^2\right.$$
$$+\left(\frac{4z^2-1}{4y^2}\rho^2\sin^2\ \theta+\left(V_\alpha-\frac{1}{2}\right)^2\right)d\alpha^2$$
$$\left.+2V_\theta d\psi d\theta+(2V_\alpha-1)d\psi d\alpha+V_\theta(2V_\alpha-1)d\theta d\alpha\right)$$

$$\mathcal{N}^2=\frac{2\sqrt{\rho^2+1}\cos\ \theta}{\sqrt{4z^2-1}}\left(\frac{\rho^2+1}{\rho^2+\sin^2\ \theta}+\frac{(\partial_\theta\rho_c)^2}{\rho^2+\sin^2\ \theta}+\frac{(\partial_\alpha\rho_c)^2}{\rho^2\sin^2\ \theta}\right)$$

$$\det h=\frac{1}{4}y^5(4z^2-1)^{1/2}(\rho^2+\sin^2\ \theta)\rho^2\sin^2\theta$$
$$\times\left(1+\frac{(\partial_\theta\rho_c)^2}{\rho^2+1}+\frac{\rho^2+\sin^2\ \theta}{\rho^2\sin^2\ \theta(\rho^2+1)}(\partial_\alpha\rho_c)^2\right)\frac{1}{Z^6}\sin^4\ \phi_2\sin^2\ \phi_3\Big|_{\rho=\rho_c}$$

$$\sqrt{h}=\sin\ \theta\cos^3\ \theta\left(\frac{1}{\epsilon^4}+\frac{1}{2\epsilon^2}\left(3F_1^{(2)}+F_2^{(2)}+3F_3^{(2)}+F_4^{(2)}+F_5^{(2)}+2F_\rho^{(2)}+1\right)\right.$$
$$+\frac{1}{6\epsilon}\left(9F_1^{(3)}+3F_2^{(3)}+9F_3^{(3)}+3F_4^{(3)}+3F_5^{(3)}+4F_\rho^{(3)}\right)$$
$$+\frac{1}{8}\left(3\left(F_1^{(2)}\right)^2+12F_1^{(4)}+6F_1^{(2)}\left(F_2^{(2)}+3F_3^{(2)}+F_4^{(2)}+F_5^{(2)}+F_\rho^{(2)}+2\right)\right.$$
$$-\left(F_2^{(2)}\right)^2+4F_2^{(2)}+4F_2^{(4)}+6F_2^{(2)}F_3^{(2)}+2F_2^{(2)}F_4^{(2)}+2F_2^{(2)}F_5^{(2)}+2F_2^{(2)}F_\rho^{(2)}$$
$$+3\left(F_3^{(2)}\right)^2+12F_3^{(2)}+12F_3^{(4)}+6F_3^{(2)}F_4^{(2)}+6F_3^{(2)}F_5^{(2)}+6F_3^{(2)}F_\rho^{(2)}-\left(F_4^{(2)}\right)^2$$
$$+4F_4^{(2)}+4F_4^{(4)}+2F_4^{(2)}F_5^{(2)}+2F_4^{(2)}F_\rho^{(2)}-\left(F_5^{(2)}\right)^2+4F_5^{(2)}+4F_5^{(4)}+2F_5^{(2)}F_\rho^{(2)}$$
$$\left.\left.-\left(F_\rho^{(2)}\right)^2+4F_\rho^{(2)}+4F_\rho^{(4)}\right)\right)+\mathcal{O}(\epsilon^1)$$

$$
\sqrt{h}=\frac{1}{\epsilon^4}\sin\,\theta\cos^3\,\theta+\frac{1}{8\epsilon^2}\Big(\frac{9}{2}\sin^3\,2\theta\big(2\sin\,2\alpha m_{211}+\cos\,2\alpha(m_{220}-m_{202})\big)
$$

$$
+\sin\,\theta\cos^3\,\theta(-9\cos\,2\theta(2m_{202}+2m_{220}-m_{400}+1)+6m_{202}+6m_{220}-3m_{400}+7)\Big)
$$

$$
+\frac{7\sin^2\,\theta\cos^3\,\theta}{3\epsilon}(-3\cos\,\alpha\cos\,2\theta(m_{212}+m_{230}-m_{410})
$$

$$
+2\sin^2\,\theta\big(\cos\,3\alpha(m_{230}-3m_{212})-\sin\,3\alpha(m_{203}-3m_{221})\big)
$$

$$
-3\sin\,\alpha\cos\,2\theta(m_{203}+m_{221}-m_{401}))
$$

$$
+\frac{1}{64}\big[64\cos^3\,\theta\sin\,4\alpha\sin^5\,\theta\big(9m_{211}(m_{202}-m_{220})+20(m_{231}-m_{213})\big)
$$

$$
-16\cos\,4\alpha\cos^3\,\theta\sin^5\,\theta\left(9(m_{202}-m_{220})^2-4\left(9m_{211}^2+5m_{204}+5(m_{240}-6m_{222})\right)\right)
$$

$$
-\cos^3\,\theta\sin\,\theta(98m_{202}^2-20m_{220}m_{202}-52m_{202}+216m_{211}^2+98m_{220}^2+11m_{400}^2+\cos\,4\theta
$$

$$
-72m_{204}-52m_{220}-144m_{222}-72m_{240}-44(m_{202}+m_{220})m_{400}+26m_{400}+144m_{402}
$$

$$
+144m_{420}+\cos^2_{4\theta(54m_{202}}+12(3m_{220}-3m_{400}-5)m_{202}+72m_{211}^2+54m_{220}^2+9m_{400}^2
$$

$$
-120m_{204}-12m_{220}(3m_{400}+5)+30\big(-8m_{222}-4m_{240}+m_{400}+8(m_{402}+m_{420})\big)-40m_{600})
$$

$$
-4\cos\,2\theta(30m_{202}^2-12(m_{220}+m_{400}+3)m_{202}+72m_{211}^2+30m_{220}^2-24m_{204}-48m_{222}
$$

$$
-24m_{240}-12m_{220}(m_{400}+3)+3m_{400}(m_{400}+6)+48m_{402}+48m_{420}-8m_{600}-13)
$$

$$
-24m_{600}-13)+4\sin_{2\alpha}\,\sin^3\,2\theta(-8(m_{213}+m_{231})
$$

$$
+3m_{211}(-2m_{202}-2m_{220}+m_{400}+3)+12m_{411}
$$

$$
+\cos\,2\theta(-40(m_{213}+m_{231})+3m_{211}(6m_{202}+6m_{220}-3m_{400}-5)+60m_{411}))
$$

$$
-2\cos\,2\alpha\sin^3\,2\theta(-6m_{202}^2+3(m_{400}+3)m_{202}-8m_{204}+3m_{220}(2m_{220}-m_{400}-3)
$$

$$
+\cos\,2\theta(18m_{202}^2-3(3m_{400}+5)m_{202}-40m_{204}+3m_{220}(-6m_{220}+3m_{400}+5)
$$

$$
+20(2m_{240}+3m_{402}-3m_{420}))+4(2m_{240}+3m_{402}-3m_{420}))\big]+O(\epsilon^1)
$$

$$
a\frac{1}{2\kappa^2L}\int\,d^9x\sqrt{h}=a\frac{1}{2\kappa^2L}\mathrm{vol}(AdS_3)\mathrm{vol}(S^3)\mathrm{vol}(S^1)\pi\left(\frac{\epsilon^4}{2}+\frac{\epsilon^2}{4}+\mathcal{C}\right)
$$

$$
\mathcal{C}\;=\frac{1}{32}\big(-10m_{2,0,2}^2+4\big(m_{2,2,0}+m_{4,0,0}-1\big)m_{2,0,2}-24m_{2,1,1}^2-10m_{2,2,0}^2-m_{4,0,0}^2
$$

$$
-4m_{2,2,0}+4m_{2,2,0}m_{4,0,0}+2m_{4,0,0}-1\big)
$$

$$
\oint_{\mathrm{JK}}\prod_{I=1}^{k}\frac{d\phi_I}{2\pi i}\mathcal{I}(\phi)=\sum_{\phi_*}\mathrm{JK}_{\phi=\phi_*}-\mathrm{Res}(\eta)\mathcal{I}(\phi)
$$

$$
(\phi_{1*},\dots,\phi_{k*})=\big(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}_{\mathcal{A},1}),\dots,\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}_{\mathcal{A},n_{\mathcal{A}}}),\mathcal{X}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B},1}),\dots,\mathcal{X}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B},n_{\mathcal{B}}}),\dots\big)
$$

$$
\mathcal{X}_{\mathcal{A}}(\boldsymbol{x})\equiv v_{\mathcal{A}}+(\boldsymbol{x}-\mathbf{1})\cdot\boldsymbol{\epsilon}_A=v_{\mathcal{A}}+\sum_{i=1}^{d}\,(x_i-1)\epsilon_{a_i}
$$

$$
\underset{\phi=\phi_*}{\mathrm{JK}-\mathrm{Res}}(\eta)\mathcal{I}(\phi)\supset\frac{1}{\mathrm{sh}(\phi_{i*}-\mathcal{X}_{\mathcal{A}}(\mathbf{1}))}\prod_{\boldsymbol{y}\in\mathbf{Y}_{\mathcal{A}}}\frac{\prod_{\boldsymbol{b}\in\mathbf{B}_{\mathrm{even}}}\,\mathrm{sh}(\phi_{i*}-\mathcal{X}_{\mathcal{A}}(\boldsymbol{y})-\boldsymbol{b}\cdot\boldsymbol{\epsilon}_A)}{\prod_{\boldsymbol{b}\in\mathbf{B}_{\mathrm{odd}}}\,\mathrm{sh}(\phi_{i*}-\mathcal{X}_{\mathcal{A}}(\boldsymbol{y})-\boldsymbol{b}\cdot\boldsymbol{\epsilon}_A)}
$$

$$
\mathcal{S}(\mathbf{Y})\equiv(\mathbf{Y}+\mathbf{B}_d)\setminus\mathbf{Y}
$$

$$|\boldsymbol{b}| \equiv \sum_{i=1}^{d} b_i$$

$$\mathrm{Q}_{\mathbf{Y}}(\boldsymbol{x}) = \sum_{\substack{\boldsymbol{b}\in\mathbf{B}_d \\ \boldsymbol{x}-\boldsymbol{b}\in\mathbf{Y}}} (-1)^{|\boldsymbol{b}|}$$

$$\mathcal{J}(x \mid \mathbf{Y}_{\mathcal{A}}) \equiv \prod_{\boldsymbol{y}\in\mathcal{S}(\mathbf{Y}_{\mathcal{A}})} \mathrm{sh}(x - \mathcal{X}_{\mathcal{A}}(\boldsymbol{y}))^{\mathrm{Q}_{\mathbf{Y}_{\mathcal{A}}}(\boldsymbol{y})}$$

$$\mathcal{J}(x \mid \emptyset_{\mathcal{A}}) \equiv \frac{1}{\mathrm{sh}(x - \mathcal{X}_{\mathcal{A}}(\mathbf{1}))}$$

$$\mathcal{J}(x \mid \mathbf{Y}_{\mathcal{A}}) = \frac{1}{\mathrm{sh}(x - \mathcal{X}_{\mathcal{A}}(\mathbf{1}))} \prod_{\boldsymbol{y}\in\mathbf{Y}_{\mathcal{A}}} \frac{\prod_{\boldsymbol{b}\in\mathbf{B}_{\mathrm{even}}} \mathrm{sh}(x - \mathcal{X}_{\mathcal{A}}(\boldsymbol{y}) - \boldsymbol{b}\cdot\boldsymbol{\epsilon}_A)}{\prod_{\boldsymbol{b}\in\mathbf{B}_{\mathrm{odd}}} \mathrm{sh}(x - \mathcal{X}_{\mathcal{A}}(\boldsymbol{y}) - \boldsymbol{b}\cdot\boldsymbol{\epsilon}_A)}$$

$$\mathcal{J}(x \mid \lambda_{12,\alpha}) = \frac{1}{\mathrm{sh}\big(x - \mathcal{X}_{12,\alpha}(\mathbf{1})\big)} \prod_{\boldsymbol{y}\in\lambda_{12,\alpha}} \frac{\mathrm{sh}\big(x - \mathcal{X}_{12,\alpha}(\boldsymbol{y}) - (0,0)\cdot\boldsymbol{\epsilon}_{12}\big)\mathrm{sh}\big(x - \mathcal{X}_{12,\alpha}(\boldsymbol{y}) - (1,1)\cdot\boldsymbol{\epsilon}_{12}\big)}{\mathrm{sh}\big(x - \mathcal{X}_{12,\alpha}(\boldsymbol{y}) - (0,1)\cdot\boldsymbol{\epsilon}_{12}\big)\mathrm{sh}\big(x - \mathcal{X}_{12,\alpha}(\boldsymbol{y}) - (1,0)\cdot\boldsymbol{\epsilon}_{12}\big)}$$

$$= \frac{1}{\mathrm{sh}\big(x - v_{12,\alpha}\big)} \prod_{\boldsymbol{y}\in\lambda_{12,\alpha}} \frac{\mathrm{sh}\big(x - \mathcal{X}_{12,\alpha}(\boldsymbol{y})\big)\mathrm{sh}\big(x - \mathcal{X}_{12,\alpha}(\boldsymbol{y}) - \epsilon_{12}\big)}{\mathrm{sh}\big(x - \mathcal{X}_{12,\alpha}(\boldsymbol{y}) - \epsilon_{1,2}\big)}$$

$$\mathcal{J}\big(\mathcal{X}_{A,\alpha}(\boldsymbol{x}) \mid \mathbf{Y}_{A,\beta}\big) = \mathcal{J}\big(\mathcal{X}_{A,\alpha}(\boldsymbol{x}+\boldsymbol{y}) \mid \mathbf{Y}_{A,\beta} + \boldsymbol{y}\big)$$

$$\prod_{\boldsymbol{x}\in\mathbf{Y}_{A,\alpha}} \mathrm{sh}\big(\mathcal{X}_{A,\alpha}(\boldsymbol{x}) - \mathcal{X}_{A,\beta}(\mathbf{1})\big)\mathcal{J}\big(\mathcal{X}_{A,\alpha}(\boldsymbol{x}) \mid \mathbf{Y}_{A,\beta}\big)$$

$$= \prod_{\boldsymbol{y}\in\mathbf{Y}_{A,\beta}} \Big(\mathrm{sh}\big(\mathcal{X}_{A,\beta}(\boldsymbol{y}+\mathbf{1}) - \mathcal{X}_{A,\alpha}(\mathbf{1})\big)\mathcal{J}\big(\mathcal{X}_{A,\beta}(\boldsymbol{y}+\mathbf{1}) \mid \mathbf{Y}_{A,\alpha}\big)\Big)^{(-1)^d}$$

$$= \prod_{\boldsymbol{y}\in\mathbf{Y}_{A,\beta}+\mathbf{1}} \Big(\mathrm{sh}\big(\mathcal{X}_{A,\beta}(\boldsymbol{y}) - \mathcal{X}_{A,\alpha}(\mathbf{1})\big)\mathcal{J}\big(\mathcal{X}_{A,\beta}(\boldsymbol{y}) \mid \mathbf{Y}_{A,\alpha}\big)\Big)^{(-1)^d}$$

$$\prod_{\boldsymbol{x}\in\lambda_{12,1}} \mathrm{sh}\big(\mathcal{X}_{12,1}(\boldsymbol{x}) - \mathcal{X}_{12,2}(\mathbf{1})\big)\mathcal{J}\big(\mathcal{X}_{12,1}(\boldsymbol{x}) \mid \lambda_{12,2}\big)$$

$$= \prod_{\boldsymbol{y}\in\lambda_{12,2}+\mathbf{1}} \mathrm{sh}\big(\mathcal{X}_{12,2}(\boldsymbol{y}) - \mathcal{X}_{12,1}(\mathbf{1})\big)\mathcal{J}\big(\mathcal{X}_{12,2}(\boldsymbol{y}) \mid \lambda_{12,1}\big)$$

$$\prod_{\boldsymbol{x}\in\pi_{123,1}} \mathrm{sh}\big(\mathcal{X}_{123,1}(\boldsymbol{x}) - \mathcal{X}_{123,2}(\mathbf{1})\big)\mathcal{J}\big(\mathcal{X}_{123,1}(\boldsymbol{x}) \mid \pi_{123,2}\big)$$

$$= \prod_{\boldsymbol{y}\in\pi_{123,2}+\mathbf{1}} \frac{1}{\mathrm{sh}\big(\mathcal{X}_{123,2}(\boldsymbol{y}) - \mathcal{X}_{123,1}(\mathbf{1})\big)\mathcal{J}\big(\mathcal{X}_{123,2}(\boldsymbol{y}) \mid \pi_{123,1}\big)}$$

$$\frac{\prod_{\boldsymbol{x}\in\mathbf{Y}_{\mathcal{A}}\cup\boldsymbol{w}} \mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) \mid \mathbf{Y}_{\mathcal{A}}\cup\boldsymbol{w})}{\prod_{\boldsymbol{x}\in\mathbf{Y}_{\mathcal{A}}} \mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) \mid \mathbf{Y}_{\mathcal{A}})}$$
$$=\mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{w}) \mid \mathbf{Y}_{\mathcal{A}}\cup\boldsymbol{w})\frac{\prod_{\boldsymbol{x}\in\mathbf{Y}_{\mathcal{A}}} \mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) \mid \mathbf{Y}_{\mathcal{A}}\cup\boldsymbol{w})}{\prod_{\boldsymbol{x}\in\mathbf{Y}_{\mathcal{A}}} \mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) \mid \mathbf{Y}_{\mathcal{A}})}$$
$$=\mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{w}) \mid \mathbf{Y}_{\mathcal{A}}\cup\boldsymbol{w})\frac{\prod_{\boldsymbol{x}\in\mathbf{Y}_{\mathcal{A}}\cup\boldsymbol{w}} \left(\operatorname{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}+\mathbf{1})-\mathcal{X}_{\mathcal{A}}(\mathbf{1}))\mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}+\mathbf{1}) \mid \mathbf{Y}_{\mathcal{A}})\right)^{(-1)^d}}{\prod_{\boldsymbol{x}\in\mathbf{Y}_{\mathcal{A}}} \mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) \mid \mathbf{Y}_{\mathcal{A}})\operatorname{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x})-\mathcal{X}_{\mathcal{A}}(\mathbf{1}))}$$
$$=\mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{w}) \mid \mathbf{Y}_{\mathcal{A}}\cup\boldsymbol{w})\left(\operatorname{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{w})-\mathcal{X}_{\mathcal{A}}(\mathbf{0}))\mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{w}+\mathbf{1}) \mid \mathbf{Y}_{\mathcal{A}})\right)^{(-1)^d}$$
$$\times\frac{\prod_{\boldsymbol{x}\in\mathbf{Y}_{\mathcal{A}}} \left(\operatorname{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x})-\mathcal{X}_{\mathcal{A}}(\mathbf{0}))\mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}+\mathbf{1}) \mid \mathbf{Y}_{\mathcal{A}})\right)^{(-1)^d}}{\prod_{\boldsymbol{x}\in\mathbf{Y}_{\mathcal{A}}} \mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) \mid \mathbf{Y}_{\mathcal{A}})\operatorname{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x})-\mathcal{X}_{\mathcal{A}}(\mathbf{1}))}$$
$$=\mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{w}) \mid \mathbf{Y}_{\mathcal{A}}\cup\boldsymbol{w})\left(\operatorname{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{w})-\mathcal{X}_{\mathcal{A}}(\mathbf{0}))\mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{w}+\mathbf{1}) \mid \mathbf{Y}_{\mathcal{A}})\right)^{(-1)^d}$$

$$\frac{\prod_{\boldsymbol{x}\in\pi_{123,1}\cup\boldsymbol{w}} \mathcal{J}\left(\mathcal{X}_{123,1}(\boldsymbol{x}) \mid \pi_{123,1}\cup\boldsymbol{w}\right)}{\prod_{\boldsymbol{x}\in\pi_{123,1}} \mathcal{J}\left(\mathcal{X}_{123,1}(\boldsymbol{x}) \mid \pi_{123,1}\right)}$$
$$=\frac{\mathcal{J}\left(\mathcal{X}_{123,1}(\boldsymbol{w}) \mid \pi_{123,1}\cup\boldsymbol{w}\right)}{\mathcal{J}\left(\mathcal{X}_{123,1}(\boldsymbol{w}+\mathbf{1}) \mid \pi_{123,1}\right)\operatorname{sh}\left(\mathcal{X}_{123,1}(\boldsymbol{w})-\mathcal{X}_{123,1}(\mathbf{0})\right)}$$

$$\mathbf{Y}_{\mathcal{A}} = \mathbf{Y}'_{\mathcal{A}}\cup\mathbf{Y}''_{\mathcal{A}}$$

$$\mathcal{J}(x \mid \mathbf{Y}_{\mathcal{A}}) = \mathcal{J}(x \mid \mathbf{Y}'_{\mathcal{A}})\mathcal{J}(x \mid \mathbf{Y}''_{\mathcal{A}})\operatorname{sh}(x-\mathcal{X}_{\mathcal{A}}(\boldsymbol{y}))$$

$$\{(1,1),(1,2)\} = \{(1,1)\}\cup\{(1,2)\}$$

$$\mathcal{J}\left(x \mid \{(1,1),(1,2)\}_{12,1}\right) = \mathcal{J}\left(x \mid \{(1,1)\}_{12,1}\right)\mathcal{J}\left(x \mid \{(1,2)\}_{12,1}\right)\operatorname{sh}\left(x-\mathcal{X}_{12,1}(1,2)\right)$$

$$\mathcal{M}^{\text{inst}}_{\mathrm{U}(N),k} = \{(B_1,B_2,I,J) \mid \mu_{\mathbb{R}} = \mu_{\mathbb{C}} = 0\}/\mathrm{U}(k)$$

$$\mu_{\mathbb{R}} = \sum_{i=1}^{2} [B_i,B_i^\dagger] + II^\dagger - JJ^\dagger$$
$$\mu_{\mathbb{C}} = [B_1,B_2] + IJ$$

$$(B_1,B_2,I,J,\Lambda_1) \overset{\mathrm{U}(1)_{\epsilon_1}\times\mathrm{U}(1)_{\epsilon_2}}{\rightarrow} (q_1^{-1}B_1,q_2^{-1}B_2,I,q_{12}^{-1}J,q_{12}^{-1}\Lambda_1)$$

$$\mathcal{Z}^{\mathrm{U}(N)}_{\text{inst}}(v_1,\dots,v_N) = \sum_{k=0}^{\infty} \mathfrak{q}^k\mathcal{Z}^{\mathrm{U}}_{N,k}(v_1,\dots,v_N)$$

$$\mathcal{Z}^{\mathrm{U}}_{N,k}(v_1,\dots,v_N) = \oint_{\mathrm{JK}}\prod_{i=1}^{k}\frac{d\phi_i}{2\pi i}\mathcal{I}^{\mathrm{U}}_{N,k}$$
$$\mathcal{I}^{\mathrm{U}}_{N,k} = \frac{1}{k!}\prod_{i\neq j}^{k}\operatorname{sh}\left(\phi_i-\phi_j\right)\prod_{i,j=1}^{k}\frac{\operatorname{sh}\left(\phi_i-\phi_j-\epsilon_{12}\right)}{\operatorname{sh}\left(\phi_i-\phi_j-\epsilon_{1,2}\right)}\prod_{i=1}^{k}\prod_{\alpha=1}^{N}\frac{1}{\operatorname{sh}(\phi_i-v_\alpha)\operatorname{sh}(v_\alpha-\epsilon_{12}-\phi_i)}$$

$$\mathcal{X}_\alpha(\boldsymbol{x}) = v_\alpha + (\boldsymbol{x}-\mathbf{1})\cdot\boldsymbol{\epsilon}_{12} = v_\alpha + i\epsilon_1 + j\epsilon_2 - \epsilon_{12}$$

$$\mathrm{N}^{\vec{\lambda}}_{\alpha,\beta}(\boldsymbol{x}) \equiv v_\alpha - v_\beta + L_{\lambda_\alpha}(\boldsymbol{x})\epsilon_1 - A_{\lambda_\beta}(\boldsymbol{x})\epsilon_2 - \epsilon_2$$

$$\mathcal{Z}^{\mathrm{U}}_{N,k}(v_1, \dots, v_N) = \sum_{\|\vec{\lambda}\|=k} \mathcal{Z}^{\mathrm{U}}(\vec{\lambda}) = \sum_{\|\vec{\lambda}\|=k} \prod_{\alpha,\beta=1}^{N} \prod_{x\in\lambda_\alpha} \frac{1}{\mathrm{sh}\left(-\mathrm{N}^{\vec{\lambda}}_{\alpha,\beta}(\boldsymbol{x})\right)\mathrm{sh}\left(\mathrm{N}^{\vec{\lambda}}_{\alpha,\beta}(\boldsymbol{x}) + \epsilon_{12}\right)}$$

$$\mathcal{Z}^{\mathrm{U}}_{N,k}(v_1, \dots, v_N) = \sum_{\|\vec{\lambda}\|=k} \prod_{\alpha,\beta=1}^{N} \prod_{x\in\lambda_\alpha} \frac{\mathcal{J}\left(\mathcal{X}_\alpha(\boldsymbol{x}) \mid \lambda_\beta\right)}{\mathrm{sh}\left(-\mathcal{X}_\alpha(\boldsymbol{x}) + \mathcal{X}_\beta(\boldsymbol{0})\right)}$$

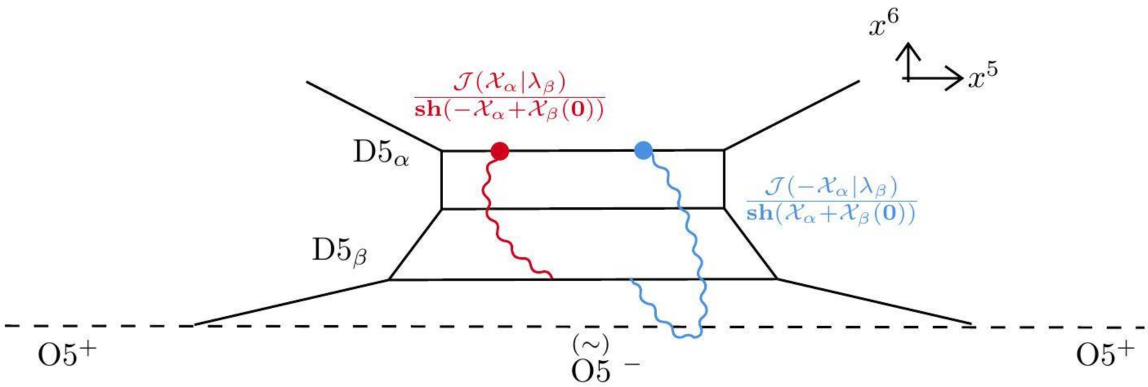
$$\frac{\mathcal{J}\left(\mathcal{X}_\alpha(\boldsymbol{x}) \mid \lambda_\beta\right)}{\mathrm{sh}\left(-\mathcal{X}_\alpha(\boldsymbol{x}) + \mathcal{X}_\beta(\boldsymbol{0})\right)}$$

$$\mathcal{J}\left(\mathcal{X}_\alpha(\boldsymbol{x}) \mid \lambda_\beta\right) = \frac{\mathrm{sh}\left(-\mathcal{X}_\alpha(\boldsymbol{x}) + \mathcal{X}_\beta(\boldsymbol{0})\right)}{\mathrm{sh}\left(-\mathrm{N}^{\vec{\lambda}}_{\alpha,\beta}(\boldsymbol{x})\right)\mathrm{sh}\left(\mathrm{N}^{\vec{\lambda}}_{\alpha,\beta}(\boldsymbol{x}) + \epsilon_{12}\right)}$$

$$\begin{aligned}\frac{\mathcal{Z}^{\mathrm{U}}(\lambda_\alpha \cup \square)}{\mathcal{Z}^{\mathrm{U}}(\lambda_\alpha)} &= \left(\prod_{x\in\lambda_\alpha\cup\square} \frac{\mathcal{J}\left(\mathcal{X}_\alpha(\boldsymbol{x}) \mid \lambda_\alpha \cup \square\right)}{\mathrm{sh}\left(-\mathcal{X}_\alpha(\boldsymbol{x}) + \mathcal{X}_\alpha(\boldsymbol{0})\right)}\right) \Big/ \left(\prod_{y\in\lambda_\alpha} \frac{\mathcal{J}\left(\mathcal{X}_\alpha(\boldsymbol{y}) \mid \lambda_\alpha\right)}{\mathrm{sh}\left(-\mathcal{X}_\alpha(\boldsymbol{y}) + \mathcal{X}_\alpha(\boldsymbol{0})\right)}\right) \\ &= -\mathcal{J}\left(\mathcal{X}_\alpha(\square) \mid \lambda_\alpha \cup \square\right) \times \mathcal{J}\left(\mathcal{X}_\alpha(\square + \boldsymbol{1}) \mid \lambda_\alpha\right)\end{aligned}$$

$$\begin{aligned}\mathcal{J}^{\mathrm{SO}}_{N,k} &= \frac{1}{k!\,2^k} \prod_{i\neq j}^{k} \mathrm{sh}\left(\phi_i - \phi_j\right) \prod_{i,j=1}^{k} \frac{\mathrm{sh}\left(\phi_i - \phi_j - \epsilon_{12}\right)}{\mathrm{sh}\left(\phi_i - \phi_j - \epsilon_{1,2}\right)} \frac{\prod_{i\leq j}^{k} \mathrm{sh}\left(\pm(\phi_i + \phi_j)\right)\mathrm{sh}\left(\pm(\phi_i + \phi_j) - \epsilon_{12}\right)}{\prod_{i<j}^{k} \mathrm{sh}\left(\pm(\phi_i + \phi_j) - \epsilon_{1,2}\right)} \\ &\times \prod_{i=1}^{k} \frac{1}{\prod_{\alpha=1}^{n} \mathrm{sh}\left(\pm\phi_i \pm v_\alpha - \frac{1}{2}\epsilon_{12}\right)} \left(\frac{1}{\mathrm{sh}\left(\pm\phi_i - \frac{1}{2}\epsilon_{12}\right)}\right)^{\chi}\end{aligned}$$

$$\begin{aligned}\mathcal{Z}^{\mathrm{SO}}_{N=2n+\chi,k}(v_1, \dots, v_n) &= \lim_{\epsilon_2\to-\epsilon_1} \sum_{\|\vec{\lambda}\|=k} \prod_{\alpha=1}^{n} \prod_{x\in\lambda_\alpha} \frac{\mathrm{sh}\left(2\mathcal{X}_\alpha(\boldsymbol{x}) + \epsilon_{1,2}\right)\mathrm{sh}^2\left(2\mathcal{X}_\alpha(\boldsymbol{x})\right)}{\mathrm{sh}^{\chi}\left(\pm\mathcal{X}_\alpha(\boldsymbol{x})\right)} \\ &\times \prod_{\beta=1}^{n} \frac{\mathcal{J}\left(\pm\mathcal{X}_\alpha(\boldsymbol{x}) \mid \lambda_\beta\right)}{\mathrm{sh}\left(\pm\mathcal{X}_\alpha(\boldsymbol{x}) + \mathcal{X}_\beta(\boldsymbol{0})\right)}\end{aligned}$$



$$\mathcal{Z}^{\mathrm{Sp},\theta=0}_{2N,k} = \mathcal{Z}^{\mathrm{Sp},+}_{2N,k} + \mathcal{Z}^{\mathrm{Sp},-}_{2N,k}$$
$$\mathcal{Z}^{\mathrm{Sp},\theta=\pi}_{2N,k} = \mathcal{Z}^{\mathrm{Sp},+}_{2N,k} - \mathcal{Z}^{\mathrm{Sp},-}_{2N,k}$$

$$\mathcal{J}^{\mathrm{Sp},+}_{2N,k} = \frac{1}{l!\,2^{l+\chi}} \prod_{i\neq j}^{l} \mathrm{sh}(\phi_i - \phi_j) \prod_{i,j=1}^{l} \frac{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{12})}{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{1,2})} \frac{\prod_{i<j}^{l} \mathrm{sh}\left(\pm(\phi_i + \phi_j)\right) \mathrm{sh}(\pm(\phi_i + \phi_j) - \epsilon_{12})}{\prod_{i\leq j}^{l} \mathrm{sh}(\pm(\phi_i + \phi_j) - \epsilon_{1,2})}$$
$$\times \prod_{i=1}^{l} \frac{1}{\prod_{\alpha=1}^{N} \mathrm{sh}\left(\pm\phi_i \pm v_\alpha - \frac{1}{2}\epsilon_{12}\right)} \left(\frac{1}{\mathrm{sh}(\epsilon_{1,2})} \frac{1}{\prod_{\alpha=1}^{N} \mathrm{sh}\left(\pm v_\alpha - \frac{1}{2}\epsilon_{12}\right)} \prod_{i=1}^{l} \frac{\mathrm{sh}(\pm\phi_i)\mathrm{sh}(\pm\phi_i - \epsilon_{12})}{\mathrm{sh}(\pm\phi_i - \epsilon_{1,2})} \right)^{\chi}$$
$$\mathcal{J}^{\mathrm{Sp},-}_{2N,k} = \frac{1}{(l-1+\chi)!\,2^{l+\chi}} \prod_{i\neq j}^{l-1+\chi} \mathrm{sh}(\phi_i - \phi_j) \prod_{i,j=1}^{l-1+\chi} \frac{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{12})}{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{1,2})} \frac{\prod_{i<j}^{l-1+\chi} \mathrm{sh}\left(\pm(\phi_i + \phi_j)\right) \mathrm{sh}(\pm(\phi_i + \phi_j) - \epsilon_{12})}{\prod_{i\leq j}^{l-1+\chi} \mathrm{sh}(\pm(\phi_i + \phi_j) - \epsilon_{1,2})}$$
$$\times \prod_{i=1}^{l-1+\chi} \frac{1}{\prod_{\alpha=1}^{N} \mathrm{sh}\left(\pm\phi_i \pm v_\alpha - \frac{1}{2}\epsilon_{12}\right)} \left(\frac{1}{\mathrm{sh}(\epsilon_{1,2})} \frac{1}{\prod_{\alpha=1}^{N} \mathrm{ch}\left(\pm v_\alpha - \frac{1}{2}\epsilon_{12}\right)} \prod_{i=1}^{l-1+\chi} \frac{\mathrm{ch}(\pm\phi_i)\mathrm{ch}(\pm\phi_i - \epsilon_{12})}{\mathrm{ch}(\pm\phi_i - \epsilon_{1,2})} \right)$$
$$\times \left(\frac{\mathrm{ch}(\epsilon_{12})}{\mathrm{sh}(2\epsilon_{1,2})} \frac{1}{\prod_{\alpha=1}^{N} \mathrm{sh}\left(\pm v_\alpha - \frac{1}{2}\epsilon_{12}\right)} \prod_{i=1}^{1-\chi} \frac{\mathrm{sh}(\pm\phi_i)\mathrm{sh}(\pm\phi_i - \epsilon_{12})}{\mathrm{sh}(\pm\phi_i - \epsilon_{1,2})} \right)^{1-\chi}$$

$$\mathcal{Z}^{\mathrm{Sp},+}_{2N,k=2l+\chi}(v_1, \dots, v_N) = \lim_{\epsilon_2 \to -\epsilon_1} \sum_{\|\vec{\lambda}\|=l} \left(\prod_{\alpha=1}^{N+4} \prod_{x\in\lambda_\alpha} \frac{\prod_{\beta=1}^{N+4} \mathcal{J}(\pm\mathcal{X}_\alpha(\boldsymbol{x}) \mid \lambda_\beta)}{\prod_{\beta=1}^{N} \mathrm{sh}(\pm\mathcal{X}_\alpha(\boldsymbol{x}) + \mathcal{X}_\beta(\boldsymbol{0}))} \right) \left(\frac{\prod_{\alpha=1}^{N+4} \mathcal{J}(0 \mid \lambda_\alpha)}{\prod_{\alpha=1}^{N} \mathrm{sh}(0 + \mathcal{X}_\alpha(\boldsymbol{0}))} \right)^{\chi}$$

$v_{N+1} = \frac{1}{2}\epsilon_1, v_{N+2} = \frac{1}{2}\epsilon_1 + \pi i, v_{N+3} = \chi\epsilon_1 + \frac{1}{2}(1-\chi)\epsilon_{12}$ and $v_{N+4} = \frac{1}{2}\epsilon_{12} + \pi i$

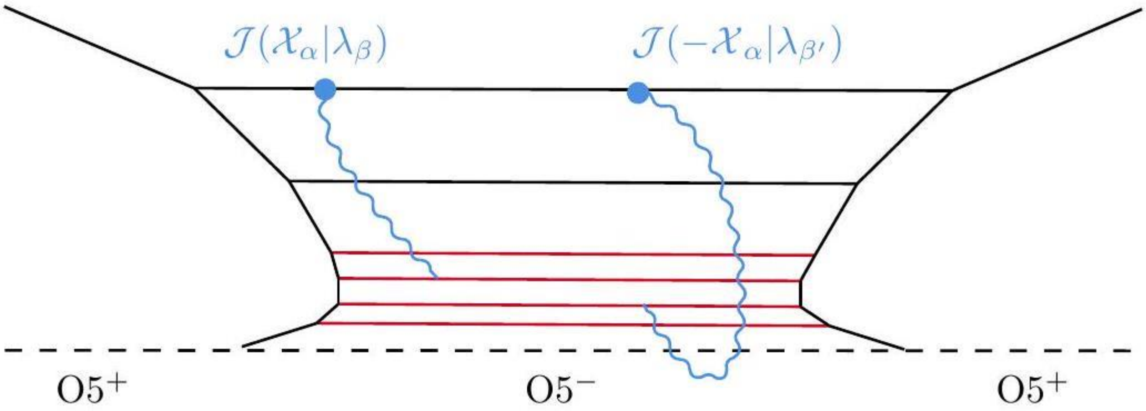
$$\mathcal{Z}^{\mathrm{Sp},\pm}_{2N,k} = \sum_{\|\vec{\lambda}\|=l} C^{\mathrm{Sp}}_{\vec{\lambda},\boldsymbol{v}} \mathcal{Z}^{\mathrm{Sp},\pm}(\vec{\lambda}), C^{\mathrm{Sp}}_{\vec{\lambda},\boldsymbol{v}} = \prod_{\alpha=N+1}^{N+4} C^{\mathrm{Sp}}_{\lambda_\alpha, v_\alpha}$$

$$C^{\mathrm{Sp}}_{\lambda_\alpha, v_\alpha = 0, \pi i, \frac{\epsilon_1}{2}, \frac{\epsilon_1}{2}+\pi i} = \frac{2^{2j-1}}{\binom{2j-1}{j-1}}$$
$$C^{\mathrm{Sp}}_{\lambda_\alpha, v_\alpha = \epsilon_1, \epsilon_1+\pi i} = \frac{2^{2j}}{\binom{2j+1}{j}}$$

$$\mathcal{Z}^{\mathrm{Sp},-}_{2N,k=2l+\chi}(v_1, \dots, v_N) = \lim_{\epsilon_2 \to -\epsilon_1} \sum_{\|\vec{\lambda}\|=l-1+\chi} \left(\prod_{\alpha=1}^{N+4} \prod_{x\in\lambda_\alpha} \frac{\prod_{\beta=1}^{N+4} \mathcal{J}(\pm\mathcal{X}_\alpha(\boldsymbol{x}) \mid \lambda_\beta)}{\prod_{\beta=1}^{N} \mathrm{sh}(\pm\mathcal{X}_\alpha(\boldsymbol{x}) + \mathcal{X}_\beta(\boldsymbol{0}))} \right)$$
$$\times \left(\frac{\prod_{\alpha=1}^{N+4} \mathcal{J}(\pi i \mid \lambda_\alpha)}{\prod_{\alpha=1}^{N} \mathrm{sh}(-\pi i + \mathcal{X}_\alpha(\boldsymbol{0}))} \right) \left(\frac{\prod_{\alpha=1}^{N+4} \mathcal{J}(0 \mid \lambda_\alpha)}{\prod_{\alpha=1}^{N} \mathrm{sh}(0 + \mathcal{X}_\alpha(\boldsymbol{0}))} \right)^{1-\chi}$$

$v_{N+1} = \frac{1}{2}\epsilon_1, v_{N+2} = \frac{1}{2}\epsilon_1 + \pi i, v_{N+3} = (1-\chi)\epsilon_1 + \frac{1}{2}\chi\epsilon_{12}$ and $v_{N+4} = \epsilon_1 + \pi i$

$$\mathcal{Z}^{\mathrm{SU}}_{2,k}(v_1) = \mathcal{Z}^{\mathrm{Sp},\theta=0}_{2,k}(v_1)$$
$$\mathcal{Z}^{\mathrm{SO}}_{5,k}(v_1 + v_2, v_1 - v_2) = \mathcal{Z}^{\mathrm{Sp},\theta=0}_{4,k}(v_1, v_2)$$



$$\mathcal{M}_{N,k}^{\mathrm{D0-D8}} = \{(\boldsymbol{B}, I) \mid \mu_{\mathbb{R}} - \zeta \cdot \mathbf{1}_k = \mu_{ab\in\underline{\mathbf{6}}} = 0\}/\mathrm{U}(k)$$

$$\mu_{\mathbb{R}} = \sum_{a\in\underline{\mathbf{4}}} [B_a, B_a^{\dagger}] + I \cdot I^{\dagger}$$

$$\begin{pmatrix} B_1, B_2, B_3, B_4 \\ I, \Lambda_1, \Lambda_2, \Lambda_3 \end{pmatrix} \overset{\mathrm{U}(1)_{\epsilon_1}\times\mathrm{U}(1)_{\epsilon_2}\times\mathrm{U}(1)_{\epsilon_3}}{\rightarrow} \begin{pmatrix} q_1^{-1}B_1, q_2^{-1}B_2, q_3^{-1}B_3, q_{123}B_4 \\ I, q_{23}\Lambda_1, q_{13}\Lambda_2, q_{12}\Lambda_3 \end{pmatrix}$$

$$\mathcal{Z}_{N,k}^{\mathrm{D0-D8}}(v_{\underline{\mathbf{4}},1}, \dots, v_{\underline{\mathbf{4}},N}) = \oint_{\mathrm{JK}} \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i} \mathcal{I}_{N,k}^{\mathrm{D0-D8}}$$

$$\mathcal{I}_{N,k}^{\mathrm{D0-D8}} = \mathcal{I}_{k}^{\mathrm{D0-D0}} \times \prod_{i=1}^{k} \prod_{\alpha=1}^{N} \frac{1}{\mathrm{sh}(\phi_i - v_{\underline{\mathbf{4}},\alpha})}$$

$$\mathcal{I}_{k}^{\mathrm{D0-D0}} = \frac{1}{k!} \prod_{i\neq j}^{k} \mathrm{sh}(\phi_i - \phi_j) \prod_{i,j}^{k} \frac{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{1,2,3} - \epsilon_4)}{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{1,2,3,4})}$$

$$\prod_{x\in\rho_{\underline{\mathbf{4}}}} \mathcal{J}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x})\rho_{\underline{\mathbf{4}}}) = \prod_{x\in\rho_{\underline{\mathbf{4}}}} \frac{1}{\mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\mathbf{1}))}$$
$$\times \prod_{y\in\rho_{\underline{\mathbf{4}}}} \frac{\mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{y}))\mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{y}) - \epsilon_{\underline{\mathbf{4}}}) \prod_{ab\in\underline{\mathbf{6}}} \mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{y}) - \epsilon_{ab})}{\prod_{a\in\underline{\mathbf{4}}} \mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{y}) - \epsilon_a) \prod_{A\in\underline{\mathbf{4}}} \mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{y}) - \epsilon_A)}$$

$$= \prod_{x\in\rho_{\underline{\mathbf{4}}}} \frac{1}{\mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\mathbf{1}))} \prod_{y\in\rho_{\underline{\mathbf{4}}}} \left(\frac{\mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{y}))\mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{y}) - \epsilon_{1,2,3} - \epsilon_4)}{\mathrm{sh}(\mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{x}) - \mathcal{X}_{\underline{\mathbf{4}}}(\boldsymbol{y}) - \epsilon_{1,2,3,4})}\right)^2$$

$$\mathcal{J}_{\geq}(\mathcal{X}_{\mathcal{B}}(\boldsymbol{x}) \mid \rho_{\mathcal{A}}) \equiv \prod_{\substack{\boldsymbol{y}\in\mathcal{S}(\rho_{\mathcal{A}}) \\ x_d \geq y_d}} \mathrm{sh}(\mathcal{X}_{\mathcal{B}}(\boldsymbol{x}) - \mathcal{X}_{\mathcal{A}}(\boldsymbol{y}))^{\mathrm{Q}_{\rho_{\mathcal{A}}}(\boldsymbol{y})}$$

$$\mathcal{Z}_{N,k}^{\mathrm{D0-D8}}(v_{\underline{\mathbf{4}},1}, \dots, v_{\underline{\mathbf{4}},N}) = \sum_{\|\vec{\rho}\|=k} \mathcal{Z}^{\mathrm{D0-D8}}(\vec{\rho}) = \sum_{\|\vec{\rho}\|=k} \prod_{\alpha,\beta=1}^{N} \prod_{x\in\rho_{\underline{\mathbf{4}},\alpha}} \mathcal{J}_{\geq}(\mathcal{X}_{\underline{\mathbf{4}},\alpha}(\boldsymbol{x}) \mid \rho_{\underline{\mathbf{4}},\beta})$$

$$\mathcal{J}_{\geq}(\mathcal{X}_{\underline{\mathbf{4}},\alpha}(\boldsymbol{x}) \mid \rho_{\underline{\mathbf{4}},\beta})$$

$$\mathcal{I}_{N,k}^{\mathrm{D0-D8-\overline{D8}}} = \mathcal{I}_{N,k}^{\mathrm{D0-D8}} \times \prod_{i=1}^{k} \prod_{\alpha=1}^{N} \mathrm{sh}\left(\phi_i - w_{\underline{\mathbf{4}},\alpha}\right)$$

$$\mathcal{Z}_{N,k}^{\mathrm{D0-D8-\overline{D8}}}\left(\{v_{\underline{\mathbf{4}},\alpha}\},\{w_{\underline{\mathbf{4}},\alpha}\}\right) = \sum_{\|\vec{\rho}\|=k} \prod_{\alpha,\beta=1}^{N} \prod_{x\in\rho_{\underline{\mathbf{4}},\alpha}} \mathrm{sh}\left(-\mathcal{X}_{\underline{\mathbf{4}},\alpha}(\boldsymbol{x}) + w_{\underline{\mathbf{4}},\beta}\right)\mathcal{J}_{\geq}\left(\mathcal{X}_{\underline{\mathbf{4}},\alpha}(\boldsymbol{x}) \mid \rho_{\underline{\mathbf{4}},\beta}\right)$$

$$\mathcal{Z}_{N}^{\mathrm{D0-D8-\overline{D8}}}\left(\{v_{\underline{\mathbf{4}},\alpha}\},\{w_{\underline{\mathbf{4}},\alpha}\}\right) = \sum_{k=0}^{\infty} \mathfrak{q}^k \mathcal{Z}_{N,k}^{\mathrm{D0-D8-\overline{D8}}} = \mathrm{PE}\left(\frac{\mathrm{sh}\left(\epsilon_{12,13,23}\right)}{\mathrm{sh}\left(\epsilon_{1,2,3,4}\right)} \frac{\mathrm{sh}(s)}{\mathrm{sh}\left(p \pm \frac{1}{2}s\right)}\right)$$

$$s \equiv \sum_{i=1}^{N} \left(v_{\underline{\mathbf{4}},i} - w_{\underline{\mathbf{4}},i}\right)$$

$$\mathrm{PE} f(x_1,\dots,x_r) \equiv \exp \sum_{m=1}^{\infty} \frac{1}{m} f(mx_1,\dots,mx_r)$$

$$\frac{\mathcal{Z}^{\mathrm{D0-D8}}\left(\rho_{\underline{\mathbf{4}},\alpha} \cup |\vec{\psi}|\right)}{\mathcal{Z}^{\mathrm{D0-D8}}\left(\rho_{\underline{\mathbf{4}},\alpha}\right)} = \mathcal{J}_{\geq}\left(\mathcal{X}_{\underline{\mathbf{4}},\alpha}(\|\Delta) \mid \rho_{\underline{\mathbf{4}},\alpha} \cup \|\right)\mathcal{J}_{<}\left(\mathcal{X}_{\underline{\mathbf{4}},\alpha}(\|\psi\|) \mid \rho_{\underline{\mathbf{4}},\alpha}\right)$$

$$\begin{pmatrix} I_{\bar{4}}, I_{\bar{3}}, I_{\bar{2}}, I_{\bar{1}} \\ \Lambda_{\bar{1}}, \Lambda_{\bar{2}}, \Lambda_{\bar{3}}, \Lambda_{\bar{4}} \end{pmatrix} \xrightarrow{\mathrm{U}(1)_{\epsilon_1}\times \mathrm{U}(1)_{\epsilon_2}\times \mathrm{U}(1)_{\epsilon_3}} \begin{pmatrix} I_{\bar{4}}, I_{\bar{3}}, I_{\bar{2}}, I_{\bar{1}} \\ q_1^{-1}\Lambda_{\bar{1}}, q_2^{-1}\Lambda_{\bar{2}}, q_3^{-1}\Lambda_{\bar{3}}, q_{123}\Lambda_{\bar{4}} \end{pmatrix}$$

$$\mathcal{M}_{\boldsymbol{N},k}^{\mathrm{D0-D6}} = \left\{(\boldsymbol{B},\boldsymbol{I}) \mid \mu_{\mathbb{R}} - \zeta \cdot \mathbf{1}_k = \mu_{ab\in\underline{\mathbf{6}}} = \sigma_{\bar{a}} = 0\right\}/\mathrm{U}(k)$$

$$\begin{aligned} \mu_{\mathbb{R}} &= \sum_{a\in\underline{\mathbf{4}}} \left[B_a, B_a^\dagger\right] + I_{\bar{a}} I_{\bar{a}}^\dagger \\ \mu_{ab} &= [B_a, B_b] \\ \sigma_{\bar{a}} &= B_a I_{\bar{a}} \end{aligned}$$

$$\mathcal{Z}_{N,k}^{\mathrm{D0-D6}} = \oint_{\mathrm{JK}} \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i} \mathcal{I}_{N,k}^{\mathrm{D0-D6}}, \mathcal{I}_{N,k}^{\mathrm{D0-D6}} = \mathcal{I}_{k}^{\mathrm{D0-D0}} \times \prod_{i=1}^{k} \prod_{\mathcal{A}} \frac{\mathrm{sh}(\phi_i - v_{\mathcal{A}} + \epsilon_{\mathcal{A}})}{\mathrm{sh}(\phi_i - v_{\mathcal{A}})}$$

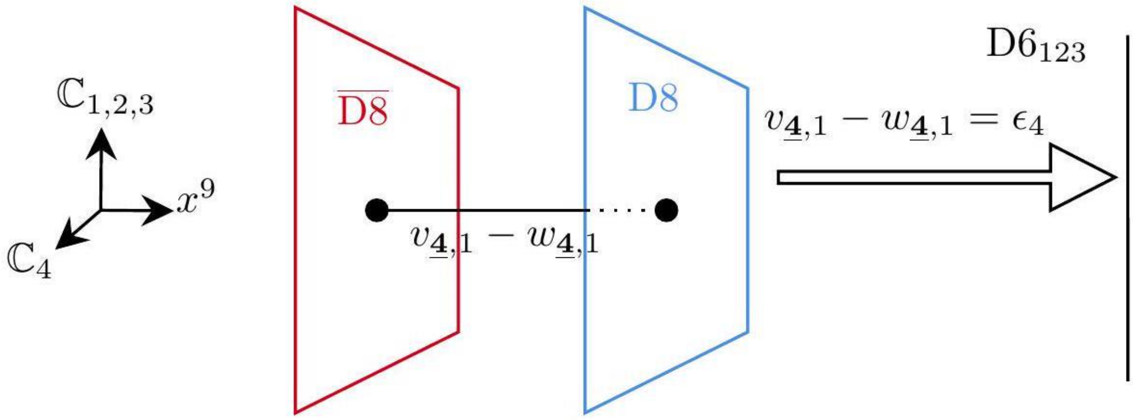
$$\mathcal{Z}_{\boldsymbol{N},k}^{\mathrm{D0-D6}}$$

$$= \sum_{\|\vec{\pi}\|=k} \left(\prod_{\mathcal{A},\mathcal{B}} \prod_{\boldsymbol{x}\in\pi_{\mathcal{A}}} \mathrm{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) - \mathcal{X}_{\mathcal{B}}(\mathbf{0}))\mathcal{J}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) \mid \pi_{\mathcal{B}})\right)$$

$$\times \left(\prod_{\mathcal{A},\mathcal{B}} \prod_{\substack{\boldsymbol{x}\in\pi_{\mathcal{A}} \\ \boldsymbol{y}\in\pi_{\mathcal{B}}}} \frac{\mathrm{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) - \mathcal{X}_{\mathcal{B}}(\boldsymbol{y}) + \epsilon_B)}{\mathrm{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) - \mathcal{X}_{\mathcal{B}}(\boldsymbol{y}) + \epsilon_A)}\right)\left(\prod_{\substack{\mathcal{A},\mathcal{B} \\ A<B}} \prod_{\substack{\boldsymbol{x}\in\pi_{\mathcal{A}} \\ \boldsymbol{y}\in\pi_{\mathcal{B}}}} \prod_{\substack{ab\in\underline{\mathbf{6}}^n \\ \in A}} \frac{\mathrm{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) - \mathcal{X}_{\mathcal{B}}(\boldsymbol{y}) + \epsilon_{ab})}{\mathrm{tc}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) - \mathcal{X}_{\mathcal{B}}(\boldsymbol{y}) - \epsilon_{ab})}\right)$$

$$\frac{\mathcal{Z}^{\text{D0-D6}}(\pi_{\mathcal{A}}\cup\square)}{\mathcal{Z}^{\text{D0-D6}}(\pi_{\mathcal{A}})}=\frac{\mathcal{J}\left(\mathcal{X}_{\mathcal{A}}(\square)\middle|\pi_{\mathcal{A}}\cup\square\right)}{\mathcal{J}\left(\mathcal{X}_{\mathcal{A}}(\square+\mathbf{1})\middle|\pi_{\mathcal{A}}\right)}$$

$$\mathcal{J}_{1,k}^{\text{D0-D8-}\overline{\text{D8}}}=\mathcal{J}_{k}^{\text{D0-D0}}\times\prod_{i=1}^{k}\frac{\text{sh}(\phi_i-w_{\underline{\mathbf{4}},1})}{\text{sh}(\phi_i-v_{\underline{\mathbf{4}},1})}$$

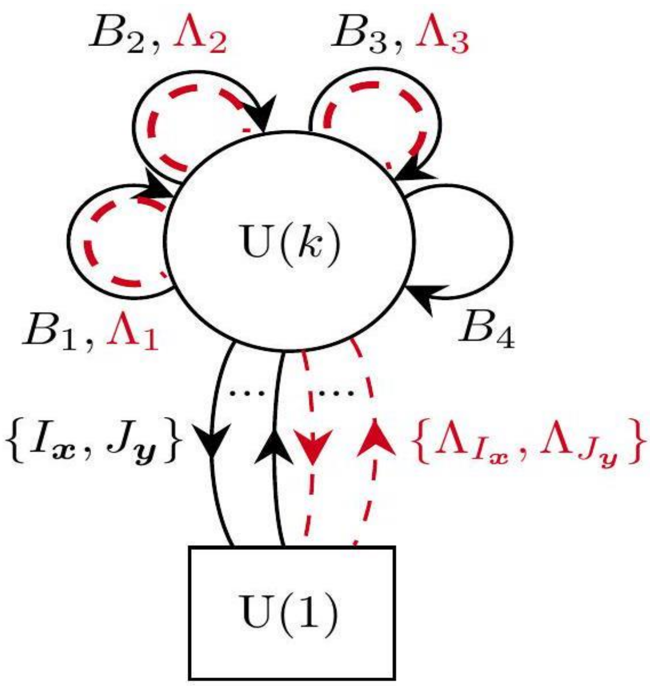


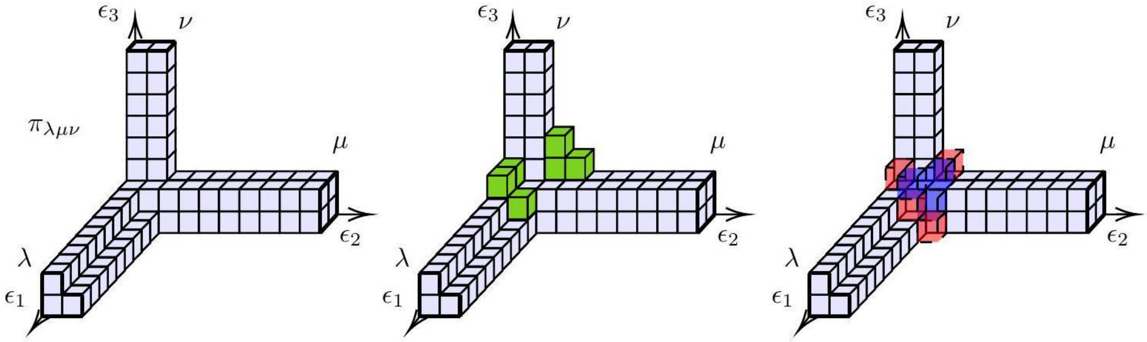
$$\mathcal{Z}_{N}^{\text{D0-D6}}(\{v_{\mathcal{A}}\})=\sum_{k=0}^{\infty}\mathfrak{q}^k\mathcal{Z}_{N,k}^{\text{D0-D6}}=\text{PE}\left(\frac{\text{sh}(\epsilon_{12,13,23})}{\text{sh}(\epsilon_{1,2,3,4})}\frac{\text{sh}(s)}{\text{sh}\left(p\pm\frac{1}{2}s\right)}\right)$$

$$s\equiv\sum_{i=1}^{N}\left(v_{\underline{\mathbf{4}},i}-w_{\underline{\mathbf{4}},i}\right)$$

$$\begin{aligned}s&=N_{\bar{4}}\epsilon_{123}+N_{\bar{3}}\epsilon_{124}+N_{\bar{2}}\epsilon_{134}+N_{\bar{1}}\epsilon_{234}\\&=(N_{123}-N_{234})\epsilon_1+(N_{123}-N_{134})\epsilon_2+(N_{123}-N_{124})\epsilon_3\end{aligned}$$

$$\mathcal{J}_{N,M,k}^{\text{D0-D6-}\overline{\text{D6}}}=\mathcal{J}_{N,k}^{\text{D0-D6}}\times\prod_{i=1}^{k}\prod_{\mathcal{B}}\frac{\text{sh}(-\phi_i+w_{\mathcal{B}})}{\text{sh}(-\phi_i+w_{\mathcal{B}}-\epsilon_{\mathcal{B}})}$$





$$\mathcal{J}^{\mathrm{D0-D2-D6}}_{\lambda,\mu,\nu;k} = \mathcal{J}^{\mathrm{D0-D0}}_{k} \times \prod_{i=1}^{k} \frac{\mathcal{J}(\phi_i \mid \pi_{\lambda\mu\nu})}{\mathcal{J}_{-}(\phi_i + \epsilon_{123} \mid \pi_{\lambda\mu\nu})}$$

$$\mathcal{J}_{-}(x \mid \mathbf{Y}_{\mathcal{A}}) \equiv \prod_{\boldsymbol{y} \in \mathcal{S}(\mathbf{Y}_{\mathcal{A}})} - \mathrm{sh}(\mathcal{X}_{\mathcal{A}}(\boldsymbol{y}) - x)^{\mathrm{Q}_{\mathbf{Y}_{\mathcal{A}}}(\boldsymbol{y})}$$

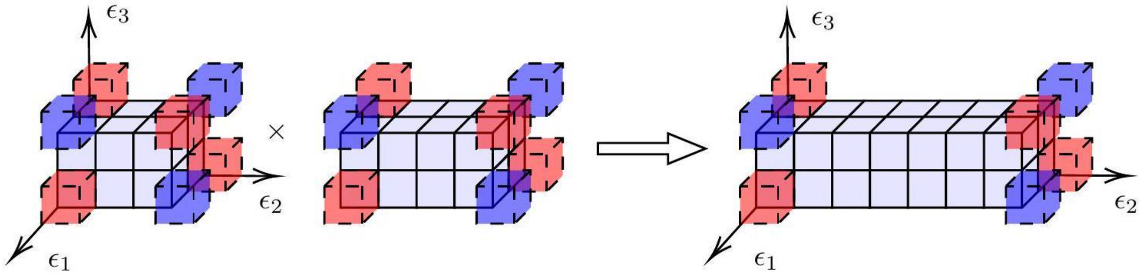
$$\mathcal{J}_{-}(x \mid \emptyset_{\mathcal{A}}) \equiv \frac{-1}{\mathrm{sh}(\mathcal{X}_{\mathcal{A}}(\mathbf{1}) - x)}$$

$$\mathcal{J}(x \mid \pi_{\lambda\mu\nu}) = \frac{\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset})\mathcal{J}(x \mid \pi_{\emptyset\mu\emptyset})\mathcal{J}(x \mid \pi_{\emptyset\emptyset\nu})\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\mu\emptyset} \cap \pi_{\emptyset\emptyset\nu})}{\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\mu\emptyset})\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\emptyset\nu})\mathcal{J}(x \mid \pi_{\emptyset\mu\emptyset} \cap \pi_{\emptyset\emptyset\nu\nu})}$$

$$\pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\mu\emptyset}, \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\emptyset\nu}, \pi_{\emptyset\mu\emptyset} \cap \pi_{\emptyset\emptyset\nu}$$

$$\frac{\mathrm{sh}(\infty\epsilon_2 + \cdots)}{\mathrm{sh}(\infty\epsilon_2 + \cdots)} \frac{\mathrm{sh}(\infty\epsilon_2 + \cdots)}{\mathrm{sh}(\infty\epsilon_2 + \cdots)} \frac{\mathrm{sh}(\infty\epsilon_2 + \cdots)}{\mathrm{sh}(\infty\epsilon_2 + \cdots)} \times \ldots = 1$$

$$\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset}) = \mathcal{J}(x + v_{23,1} - v \mid \lambda_{23,1})$$
$$\mathcal{J}(x \mid \pi_{\emptyset\mu\emptyset}) = \mathcal{J}(x + v_{13,1} - v \mid \mu_{13,1})$$
$$\mathcal{J}(x \mid \pi_{\emptyset\emptyset\nu}) = \mathcal{J}(x + v_{12,1} - v \mid \nu_{12,1})$$



$$\mathcal{Z}^{\mathrm{D0-D2-D6}}_{\lambda,\mu,\nu;k} = \oint_{\mathrm{JK}} \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i} \mathcal{J}^{\mathrm{D0-D2-D6}}_{\lambda,\mu,\nu;k}$$
$$= \sum_{|\tilde{\pi}_{\lambda\mu\nu}|=k} \prod_{\boldsymbol{x}\in\tilde{\pi}_{\lambda\mu\nu}} \frac{\mathcal{J}(\mathcal{X}(\boldsymbol{x}) \mid \pi_{\lambda\mu\nu})}{\mathcal{J}_-(\mathcal{X}(\boldsymbol{x}) + \epsilon_{123} \mid \pi_{\lambda\mu\nu})} \prod_{\boldsymbol{y}\in\tilde{\pi}_{\lambda\mu\nu}} \mathrm{sh}(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{y}))\mathcal{J}(\mathcal{X}(\boldsymbol{x}) \mid \{\boldsymbol{y}\})$$

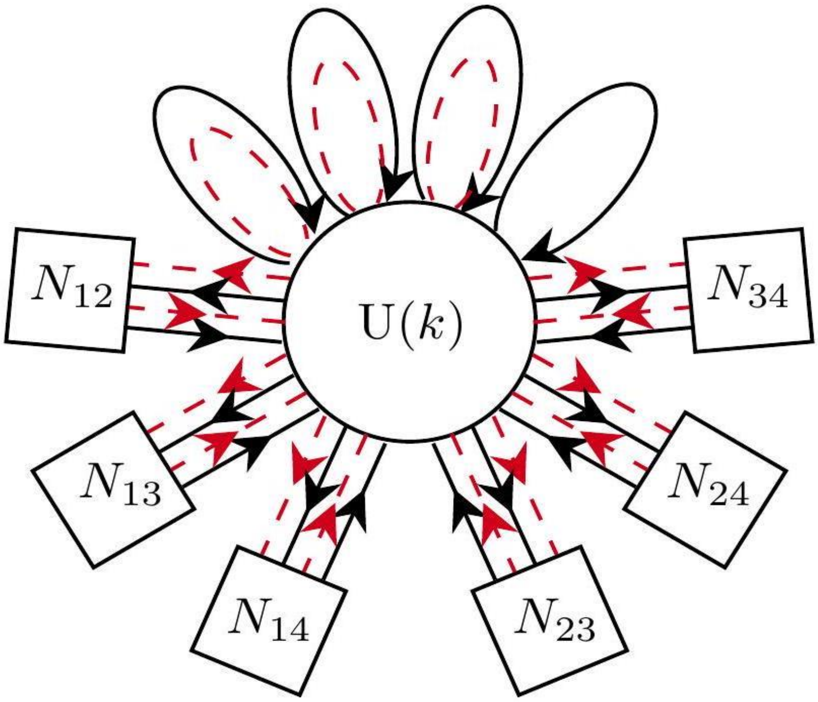
$$\mathcal{Z}^{\mathrm{D0-D2-D6}}_{\emptyset,\emptyset,\emptyset;k} = \sum_{|\pi|=k} \prod_{\boldsymbol{x}\in\pi} \frac{\mathrm{sh}(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{0}))}{\mathrm{sh}(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{1}))} \prod_{\boldsymbol{y}\in\pi} \mathrm{sh}(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{y}))\mathcal{J}(\mathcal{X}(\boldsymbol{x}) \mid \{\boldsymbol{y}\})$$
$$= \sum_{|\pi|=k} \prod_{\boldsymbol{x}\in\pi} \frac{\mathrm{sh}(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{0}))}{\mathrm{sh}(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{1}))} \prod_{\boldsymbol{y}\in\pi} \frac{\mathrm{sh}(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{y}))\mathrm{sh}\big(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{y}) - \epsilon_{12,13,23}\big)}{\mathrm{sh}\big((\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{y}) - \epsilon_{1,2,3}\big)\mathrm{sh}\big((\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{y}) - \epsilon_{123}\big)}$$
$$= \sum_{|\pi|=k} \prod_{\boldsymbol{x}\in\pi} \mathrm{sh}(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{0}))\mathcal{J}(\mathcal{X}(\boldsymbol{x}) \mid \pi)$$

$$\mathcal{Z}^{\mathrm{DO-D2-D6}}_{\emptyset,\emptyset,\emptyset;k} = \mathcal{Z}^{\mathrm{D0-D6}}_{(1,0,0,0),k}$$

$$\mathcal{M}^{\mathrm{D0-D4}}_{N,k} = \big\{(\boldsymbol{B},\boldsymbol{I},\boldsymbol{J}) \mid \mu_{\mathbb{R}} - \zeta \cdot \boldsymbol{1}_k = \mu_{ab} = \sigma_{a;bc} = \tilde{\sigma}_{a;bc} = 0\big\}/\mathrm{U}(k)$$

$$\mu_{\mathbb{R}} = \sum_{a\in\underline{\boldsymbol{4}}} \big[B_a, B_a^\dagger\big] + \sum_{ab\in\underline{\boldsymbol{6}}} \big(I_{ab}I_{ab}^\dagger - J_{ab}^\dagger J_{ab}\big)$$

$$\mu_{ab} = [B_a, B_b] + I_{ab}J_{ab}$$
$$\sigma_{a;bc} = B_a I_{bc}$$
$$\tilde{\sigma}_{a;bc} = J_{bc}B_a$$



$$\mathcal{Z}_{N,k}^{\mathrm{D0-D4}} = \oint_{\mathrm{JK}} \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i} \mathcal{I}_{N,k}^{\mathrm{D0-D4}}$$

$$\mathcal{I}_{N,k}^{\mathrm{D0-D4}} = \mathcal{I}_{k}^{\mathrm{D0-D0}} \times \prod_{i=1}^{k} \prod_{\mathbf{ab}} \frac{\prod_{c\in\overline{ab}} \mathrm{sh}(\phi_i - v_{\mathbf{ab}} - \epsilon_c)}{\mathrm{sh}(\phi_i - v_{\mathbf{ab}})\mathrm{sh}(-\phi_i + v_{\mathbf{ab}} - \epsilon_{ab})}$$

$$\mathcal{Z}_{N,k}^{\mathrm{D4},\mathbb{C}^4} = \sum_{\|\vec{\lambda}\|=k} (-1)^k \prod_{\mathbf{ab},\mathbf{ab}'} \left(\prod_{\boldsymbol{x}\in\lambda_{\mathbf{ab}}} \frac{\mathcal{J}(\mathcal{X}_{\mathbf{ab}}(\boldsymbol{x}) \mid \lambda_{\mathbf{ab}'})}{\mathrm{sh}(-\mathcal{X}_{\mathbf{ab}}(\boldsymbol{x}) + \mathcal{X}_{\mathbf{ab}'}(\mathbf{0}))} \prod_{c\in\overline{ab}} \mathrm{sh}(\mathcal{X}_{\mathbf{ab}}(\boldsymbol{x}) - \mathcal{X}_{\mathbf{ab}'}(\mathbf{0}) + \epsilon_c) \right)$$

$$\times \left(\prod_{\substack{\boldsymbol{x}\in\lambda_{\mathbf{ab}} \\ \boldsymbol{y}\in\lambda_{\mathbf{ab}'}}} \frac{\mathrm{sh}\big(\mathcal{X}_{\mathbf{ab}}(\boldsymbol{x}) - \mathcal{X}_{\mathbf{ab}'}(\boldsymbol{y}) - \epsilon_{1,2,3} - \epsilon_4\big)}{\mathrm{sh}(\mathcal{X}_{\mathbf{ab}}(\boldsymbol{x}+\mathbf{1}) - \mathcal{X}_{\mathbf{ab}'}(\boldsymbol{y})) \prod_{c\in\overline{ab}} \mathrm{sh}(\mathcal{X}_{\mathbf{ab}}(\boldsymbol{x}) - \mathcal{X}_{\mathbf{ab}'}(\boldsymbol{y}) + \epsilon_c)} \right)$$

$$\mathcal{Z}_{k}^{\mathrm{D0-D4,U,SO,Sp}} \supset \prod_{i=1}^{k} \frac{\prod_{j\neq i}^{k} \mathrm{sh}(\phi_i - \phi_j)}{\mathrm{sh}(\phi_i - \mathcal{X}_\alpha(\mathbf{1}))} \prod_{i,j}^{k} \frac{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{12})}{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{1,2})}$$

$$\mathcal{Z}_{k}^{\mathrm{D0-D6,D0-D2-D6}} \supset \prod_{i=1}^{k} \frac{\prod_{j\neq i}^{k} \mathrm{sh}(\phi_i - \phi_j)}{\mathrm{sh}(\phi_i - \mathcal{X}_\alpha(\mathbf{1}))} \prod_{i,j}^{k} \frac{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{12,13,23})}{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{1,2,3})\mathrm{sh}(\phi_i - \phi_j - \epsilon_{123})}$$

$$\mathcal{Z}_{k}^{\mathrm{D0-D8}} \supset \left(\prod_{i=1}^{k} \frac{1}{\mathrm{sh}(\phi_i - \mathcal{X}_\alpha(\mathbf{1}))} \right) \left(\prod_{i\neq j}^{k} \mathrm{sh}(\phi_i - \phi_j) \prod_{i,j}^{k} \frac{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{1234}) \prod_{ab\in\underline{\mathbf{G}}'} \mathrm{sh}(\phi_i - \phi_j - \epsilon_{ab})}{\mathrm{sh}(\phi_i - \phi_j - \epsilon_{1,2,3,4}) \prod_{A\in\underline{\underline{\mathbf{4}}}} \mathrm{sh}(\phi_i - \phi_j - \epsilon_A)} \right)^{1/2}$$

$$\mathcal{Z}_{k}^{\mathrm{D0-D4,U,SO,Sp}} \supset \prod_{\boldsymbol{x}\in\lambda} \frac{\mathcal{J}(\mathcal{X}(\boldsymbol{x}) \mid \lambda)}{\mathrm{sh}(-\mathcal{X}(\boldsymbol{x}) + \mathcal{X}(\mathbf{0}))}$$

$$\mathcal{Z}_k^{\mathrm{D0-D6}} \supset \prod_{x\in\pi} \mathrm{sh}(\mathcal{X}(\boldsymbol{x}) - \mathcal{X}(\boldsymbol{0}))\mathcal{J}(\mathcal{X}(\boldsymbol{x}) \mid \pi)$$

$$\mathcal{Z}_k^{\mathrm{D0-D2-D6}} \supset \prod_{x\in\tilde{\pi}_{\lambda\mu\nu}} \frac{\mathcal{J}(\mathcal{X}(\boldsymbol{x}) \mid \pi_{\lambda\mu\nu})}{\mathcal{J}(\mathcal{X}(\boldsymbol{x}) + \epsilon_{123} \mid \pi_{\lambda\mu\nu})}$$

$$\mathcal{Z}_k^{\mathrm{D0-D8}} \supset \prod_{x\in\rho} \mathcal{J}_{\geq}(\mathcal{X}(\boldsymbol{x}) \mid \rho)$$

$$q_i \equiv e^{-\epsilon_i}, q_{i_1\ldots i_s} = q_{i_1} \ldots q_{i_s} = e^{-(\epsilon_{i_1}+\cdots+\epsilon_{i_s})}$$

$$\mathcal{X}_{\mathcal{A}}(\boldsymbol{x}) \equiv v_{\mathcal{A}} + (\boldsymbol{x} - \boldsymbol{1})\cdot\boldsymbol{\epsilon}_A = v_{\mathcal{A}} + \sum_{i=1}^{d}(x_i - 1)\epsilon_{a_i}$$

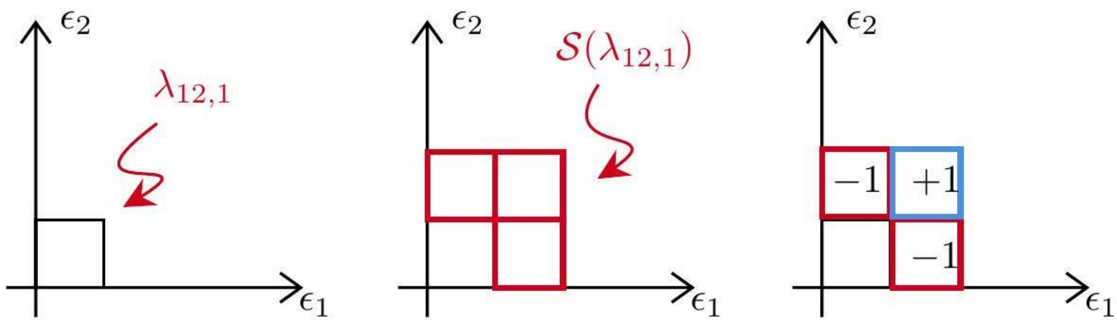
$$\mathrm{sh}(x) \equiv e^{x/2} - e^{-x/2}, \mathrm{ch}(x) \equiv e^{x/2} + e^{-x/2}$$

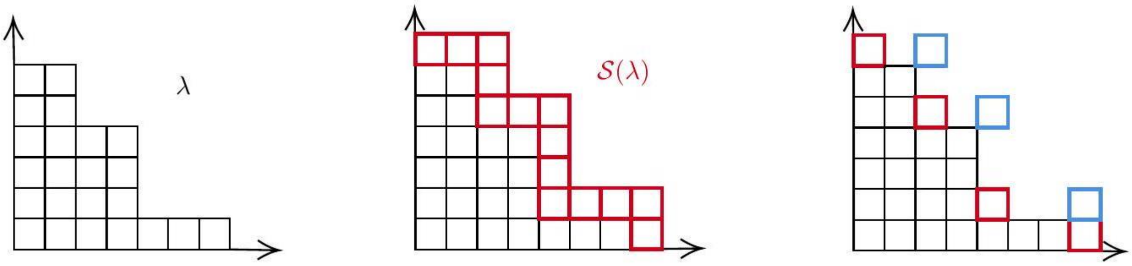
$$\mathrm{sh}(\pm x \pm y) = \mathrm{sh}(x+y)\mathrm{sh}(x-y)\mathrm{sh}(-x+y)\mathrm{sh}(-x-y)$$
$$\mathrm{sh}(x + \epsilon_{12,13,\ldots}) = \mathrm{sh}(x+\epsilon_{12})\mathrm{sh}(x+\epsilon_{13}) \times \ldots$$

$$\begin{aligned}\mathcal{S}(\lambda_{12,1}) &= (\lambda_{12,1} + \mathbf{B}_2) \setminus \lambda_{12,1} \\ &= \{(1,1),(1,2),(2,1),(2,2)\} \setminus \{(1,1)\} \\ &= \{(1,2),(2,1),(2,2)\}\end{aligned}$$

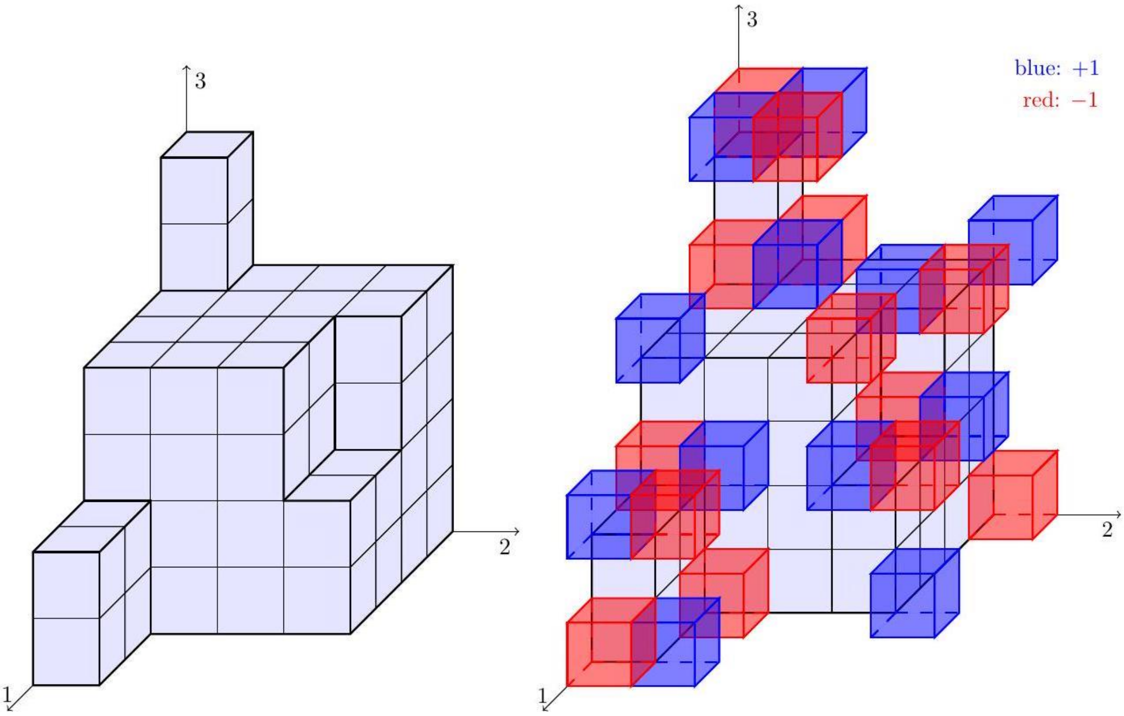
$$\begin{aligned}\mathrm{Q}_{\lambda_{12,1}}(1,2) &= (-1)^{|(0,1)|} = -1 \\ \mathrm{Q}_{\lambda_{12,1}}(2,1) &= (-1)^{|(1,0)|} = -1 \\ \mathrm{Q}_{\lambda_{12,1}}(2,2) &= (-1)^{|(1,1)|} = 1\end{aligned}$$

$$\begin{aligned}\mathcal{J}(x \mid \lambda_{12,1}) &= \frac{\mathrm{sh}(x - \mathcal{X}_{12,1}(2,2))}{\mathrm{sh}(x - \mathcal{X}_{12,1}(1,2))\mathrm{sh}(x - \mathcal{X}_{12,1}(2,1))} \\ &= \frac{\mathrm{sh}(x - v_{12,1} - \epsilon_{12})}{\mathrm{sh}(x - v_{12,1} - \epsilon_1)\mathrm{sh}(x - v_{12,1} - \epsilon_2)}\end{aligned}$$

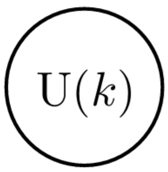




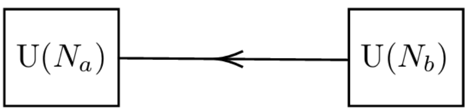
$$\mathcal{J}(x \mid \lambda_{\mathbf{ab}}) = \frac{\prod_{\boldsymbol{y}\in\mathfrak{R}} \operatorname{sh}(x - \mathcal{X}_{\mathrm{ab}}(\boldsymbol{y}+\mathbf{1}))}{\prod_{\boldsymbol{y}\in\mathfrak{A}} \operatorname{sh}(x - \mathcal{X}_{\mathrm{ab}}(\boldsymbol{y}))}$$



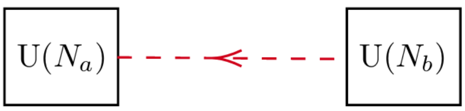
$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H - \sum_i T_i u_i}$$



$$\Longrightarrow \quad \left(\prod_{i=1}^{k} \frac{d\phi_i}{2\pi i}\right) \prod_{i\neq j}^{k} \operatorname{sh}(\phi_i - \phi_j)$$



$$\Longrightarrow \quad \prod_{\alpha=1}^{N_a} \prod_{\beta=1}^{N_b} \frac{1}{\operatorname{sh}(a_\alpha - b_\beta + q(\Phi_\alpha^\beta))}$$



$$\Longrightarrow \quad \prod_{\alpha=1}^{N_a} \prod_{\beta=1}^{N_b} \operatorname{sh}(a_\alpha - b_\beta + q(\Psi_\alpha^\beta))$$

$$\mathcal{I}(B_1) = \prod_{i,j}^{k} \frac{1}{\operatorname{sh}(\phi_i - \phi_j - \epsilon_1)}$$

$$\mathcal{Z} = \oint_{\mathrm{JK}} \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \mathcal{I}(\phi) = \oint_{\text{specific cycles}} \mathcal{I}(\phi) d\phi_1 \wedge \cdots \wedge d\phi_k$$

$$\mathcal{I}(\phi) \propto \prod_{a} \frac{1}{\operatorname{sh}\left(\sum_{I=1}^{k} Q_a^I \phi_I + m_a\right)^{N_a}},$$

$$\sum_{I=1}^{k} Q_a^I \phi_I + m_a = 2\pi i n_a, n_a \in \mathbb{Z}, a = 1, \dots, k$$

$$\begin{pmatrix} n_1 \\ \vdots \\ n_k \end{pmatrix} = \frac{Q}{|\det Q|} \cdot \begin{pmatrix} l_1 \\ \vdots \\ l_k \end{pmatrix}, l_I \in 0,1, \dots, |\det Q| - 1$$

$$\{0\} \subset F_1 \subset \cdots \subset F_k = \mathbb{R}^k, F_\ell = \operatorname{span}\{Q_{a_1}, \dots, Q_{a_\ell}\}$$

$$\kappa(F, Q_*) \equiv (\kappa_1, \dots, \kappa_k), \text{ where } \kappa_\ell = \sum_{\substack{Q \in Q_* \\ Q \in F_\ell}} Q$$

$$\kappa(F, Q_*)^T \cdot \boldsymbol{\lambda} = \eta, \text{ where } \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k$$

$$\delta(F, \eta) = \begin{cases} 1, & \kappa(F, Q_*) \\ 0, & \text{else} \end{cases}$$

$$\mathrm{JK} - \operatorname{Res}(\eta)\mathcal{I} = \sum_{F} \delta(F, \eta) \frac{\operatorname{sgndet}\kappa(F, Q_*)}{\det \mathcal{B}(F, Q_*)}_{\phi = \phi_*} \operatorname*{Res}_{\varepsilon_k = 0} \dots \operatorname*{Res}_{\varepsilon_1 = 0} \mathcal{I} \Bigg|_{\substack{Q_{a_1}\phi_* + m_{a_1} + \varepsilon_1 = n_{a_1} 2\pi i \, \vdots \\ Q_{a_k}\phi_* + m_{a_k} + \varepsilon_k = n_{a_k} 2\pi i}}$$

$$\oiiint_{\mathrm{JK}} \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \mathcal{I}(\phi) = \sum_{\phi_*} \mathrm{JK}_{\phi = \phi_*} - \operatorname{Res}(\eta)\mathcal{I}(\phi)$$

$$\mathcal{Z}^{\mathrm{Sp},+}_{2,k=1} = \lim_{\epsilon_2\to-\epsilon_1} \frac{\prod_{\alpha=1}^{5} \mathcal{J}(0 \mid \emptyset_\alpha)}{\mathrm{sh}(0+v_1-\epsilon_{12})}$$
$$= \lim_{\epsilon_2\to-\epsilon_1} \frac{1}{\mathrm{sh}(v_1-\epsilon_{12})\prod_{\alpha=1}^{5} \mathrm{sh}(-v_\alpha)}$$
$$= \lim_{\epsilon_2\to-\epsilon_1} \frac{-1}{\mathrm{sh}\left(\frac{\epsilon_1}{2}\right)\mathrm{sh}\left(\frac{\epsilon_1}{2}+\pi i\right)\mathrm{sh}(\epsilon_1)\mathrm{sh}\left(\frac{\epsilon_{12}}{2}+\pi i\right)\mathrm{sh}(v_1)\mathrm{sh}(v_1-\epsilon_{12})}$$
$$= \frac{1}{2\mathrm{sh}(v_1)^2\mathrm{sh}(\epsilon_1)^2}$$
$$\mathcal{Z}^{\mathrm{Sp},-}_{2,k=1} = \lim_{\epsilon_2\to-\epsilon_1} \frac{\prod_{\alpha=1}^{5} \mathcal{J}(\pi i \mid \emptyset_\alpha)}{\mathrm{sh}(-\pi i+v_1-\epsilon_{12})}$$
$$= \lim_{\epsilon_2\to-\epsilon_1} \frac{-1}{\mathrm{sh}\left(\frac{\epsilon_1}{2}\right)\mathrm{sh}\left(\frac{\epsilon_1}{2}+\pi i\right)\mathrm{sh}(\epsilon_1)\mathrm{sh}\left(\frac{\epsilon_{12}}{2}+\pi i\right)\mathrm{sh}(v_1+\pi i)\mathrm{sh}(v_1-\epsilon_{12}+\pi i)}$$
$$= -\frac{\mathrm{sh}(v_1)^2}{2\mathrm{sh}(2v_1)^2\mathrm{sh}(\epsilon_1)^2}$$

$$\mathcal{Z}^{\mathrm{Sp}}_{2,k=1} = \mathcal{Z}^{\mathrm{Sp},+}_{2,k=1} + \mathcal{Z}^{\mathrm{Sp},-}_{2,k=1} = \frac{2}{\mathrm{sh}(2v_1)^2\mathrm{sh}(\epsilon_1)^2}$$

$$\mathcal{Z}^{\mathrm{Sp},-}_{2,k=2} = \lim_{\epsilon_2\to-\epsilon_1} \frac{\prod_{\alpha=1}^{5} \mathcal{J}(0 \mid \emptyset_\alpha)}{\mathrm{sh}(0+v_1-\epsilon_{12})}\frac{\prod_{\alpha=1}^{5} \mathcal{J}(\pi i \mid \emptyset_\alpha)}{\mathrm{sh}(-\pi i+v_1-\epsilon_{12})}$$
$$= \frac{-1}{\mathrm{sh}(2v_1)^2\mathrm{sh}(\epsilon_1)^2\mathrm{sh}(2\epsilon_1)^2}$$

$$(\{(1,1)\}_1,\emptyset_2,\emptyset_3,\emptyset_4,\emptyset_5),\quad (\emptyset_1,\{(1,1)\}_2,\emptyset_3,\emptyset_4,\emptyset_5),\quad (\emptyset_1,\emptyset_2,\{(1,1)\}_3,\emptyset_4,\emptyset_5)$$
$$(\emptyset_1,\emptyset_2,\emptyset_3,\{(1,1)\}_4,\emptyset_5),\quad (\emptyset_1,\emptyset_2,\emptyset_3,\emptyset_4,\{(1,1)\}_5)$$

$$\mathcal{Z}^{\mathrm{Sp},+}_{2,k=2} = \lim_{\epsilon_2\to-\epsilon_1} \sum_{i=1}^{5} \frac{\mathcal{J}(\pm v_i \mid \{(1,1)\}_i)}{\mathrm{sh}(v_i \pm v_i - \epsilon_{12})\prod_{j\neq i}^{5} \mathrm{sh}(\pm v_i - v_j)}$$
$$= \lim_{\epsilon_2\to-\epsilon_1} \Bigg(-\frac{1}{\mathrm{sh}(2v_1\pm\epsilon_1)\mathrm{sh}(2v_1+\epsilon_{1,2})\mathrm{sh}(\epsilon_{1,2})\mathrm{sh}(2v_1-\epsilon_{12})^2}$$
$$+\frac{1}{2\mathrm{sh}\left(v_1\pm\frac{\epsilon_1}{2}\right)\mathrm{sh}\left(v_1-\epsilon_{12}\pm\frac{\epsilon_1}{2}\right)\mathrm{sh}(2\epsilon_1)^2\mathrm{sh}(\epsilon_2)^2}$$
$$+\frac{\mathrm{sh}\left(v_1\pm\frac{\epsilon_1}{2}\right)\mathrm{sh}\left(v_1-\epsilon_{12}\pm\frac{\epsilon_1}{2}\right)}{2\mathrm{sh}(2v_1\pm\epsilon_1)\mathrm{sh}(2v_1-2\epsilon_{12}\pm\epsilon_1)\mathrm{sh}(2\epsilon_1)^2\mathrm{sh}(\epsilon_2)^2}$$
$$-\frac{\mathrm{sh}(\epsilon_{12}^2)}{2\mathrm{sh}\left(v_1\pm\frac{\epsilon_{12}}{2}\right)\mathrm{sh}\left(v_1-\epsilon_{12}\pm\frac{\epsilon_{12}}{2}\right)\mathrm{sh}(\epsilon_1)\mathrm{sh}(\epsilon_2)^2\mathrm{sh}(2\epsilon_1+\epsilon_2)^2\mathrm{sh}(\epsilon_1+2\epsilon_2)}$$
$$-\frac{\mathrm{sh}(\epsilon_{12})^2\mathrm{sh}\left(v_1\pm\frac{\epsilon_1 2}{2}\right)\mathrm{sh}\left(v_1-\epsilon_{12}\pm\frac{\epsilon_{12}}{2}\right)}{2\mathrm{sh}(2v_1\pm\epsilon_{12})\mathrm{sh}(2v_1-2\epsilon_{12}\pm\epsilon_{12})\mathrm{sh}(\epsilon_1)\mathrm{sh}(\epsilon_2)^2\mathrm{sh}(\epsilon_1+2\epsilon_2)\mathrm{sh}(2\epsilon_1+\epsilon_2)^2}\Bigg)$$
$$= \frac{1}{\mathrm{sh}(2v_1)^2\mathrm{sh}(2v_1\pm\epsilon_1)^2\mathrm{sh}(\epsilon_1)^2} + \frac{1}{2\mathrm{sh}\left(v_1\pm\frac{\epsilon_1}{2}\right)^2\mathrm{sh}(\epsilon_1)^2\mathrm{sh}(2\epsilon_1)^2} + \frac{\mathrm{sh}\left(v_1\pm\frac{\epsilon_1}{2}\right)^2}{2\mathrm{sh}(2v_1\pm\epsilon_1)^2\mathrm{sh}(\epsilon_1)^2\mathrm{sh}(2\epsilon_1)^2}$$

$$\mathcal{Z}^{\mathrm{Sp}}_{2,k=2} = \frac{1}{\mathrm{sh}(2v_1)^2\mathrm{sh}(2v_1 \pm \epsilon_1)^2\mathrm{sh}(\epsilon_1)^2} + \frac{1}{2\mathrm{sh}\left(v_1 \pm \frac{\epsilon_1}{2}\right)^2 \mathrm{sh}(\epsilon_1)^2\mathrm{sh}(2\epsilon_1)^2}$$
$$+ \frac{\mathrm{sh}\left(v_1 \pm \frac{\epsilon_1}{2}\right)^2}{2\mathrm{sh}(2v_1 \pm \epsilon_1)^2\mathrm{sh}(\epsilon_1)^2\mathrm{sh}(2\epsilon_1)^2} - \frac{1}{\mathrm{sh}(2v_1)^2\mathrm{sh}(\epsilon_1)^2\mathrm{sh}(2\epsilon_1)^2}$$

$$\lambda_2 = \{(1,1),(1,2),(1,3),(2,1),(3,1),(2,2)\}$$

$$\mathcal{J}(x \mid \lambda_2) = \frac{\mathrm{sh}(x - v_2 - \epsilon_1 - 3\epsilon_2)\mathrm{sh}(x - v_2 - 2\epsilon_{12})\mathrm{sh}(x - v_2 - 3\epsilon_1 - \epsilon_2)}{\mathrm{sh}(x - v_2 - 3\epsilon_{1,2})\mathrm{sh}(x - v_2 - \epsilon_1 - 2\epsilon_2)\mathrm{sh}(x - v_2 - 2\epsilon_1 - \epsilon_2)}$$

$$\mathcal{Z}^{\mathrm{Sp},+}(\vec{\lambda}) \propto \lim_{\epsilon_2 \to -\epsilon_1} \frac{\mathrm{sh}(\epsilon_{12})^3}{2\mathrm{sh}(2\epsilon_{12})^2\mathrm{sh}(3\epsilon_{12})} = \frac{1}{24}$$

$$\mathcal{Z}^{\mathrm{Sp},+}(\vec{\lambda}) = \frac{1}{24\mathrm{sh}(\epsilon_1)^2\mathrm{sh}(2\epsilon_1)^4\mathrm{sh}(3\epsilon_1)^6\mathrm{sh}(4\epsilon_1)^6\mathrm{sh}(5\epsilon_1)^5\mathrm{sh}(6\epsilon_1)^2}$$
$$\times \frac{1}{\mathrm{sh}\left(v_1 \pm \frac{5}{2}\epsilon_1\right)^2 \mathrm{sh}\left(v_1 \pm \frac{3}{2}\epsilon_1\right)^4 \mathrm{sh}\left(v_1 \pm \frac{1}{2}\epsilon_1\right)^6}$$

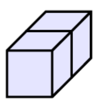
$$\mathcal{Z}^{\mathrm{Sp},+}(\vec{\lambda}) = C^{\mathrm{Sp}}_{\vec{\lambda},v} \times \frac{1}{64\mathrm{sh}(\epsilon_1)^2\mathrm{sh}(2\epsilon_1)^4\mathrm{sh}(3\epsilon_1)^6\mathrm{sh}(4\epsilon_1)^6\mathrm{sh}(5\epsilon_1)^5\mathrm{sh}(6\epsilon_1)^2}$$
$$\times \frac{1}{\mathrm{sh}\left(v_1 \pm \frac{5}{2}\epsilon_1\right)^2 \mathrm{sh}\left(v_1 \pm \frac{3}{2}\epsilon_1\right)^4 \mathrm{sh}\left(v_1 \pm \frac{1}{2}\epsilon_1\right)^6}$$

$$C^{\mathrm{Sp}}_{\vec{\lambda},v} = C^{\mathrm{Sp}}_{\emptyset,\frac{\epsilon_1}{2}} C^{\mathrm{Sp}}_{\lambda_2,\frac{\epsilon_1}{2}+\pi i} C^{\mathrm{Sp}}_{\emptyset,0} C^{\mathrm{Sp}}_{\emptyset,\pi i} = \frac{2^{2j-1}}{\binom{2j-1}{j-1}} = \frac{8}{3}$$

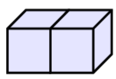
$$\mathcal{J}(x \mid \{(1,1,1)\}_{\bar{4},1}) = \frac{\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_{12})\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_{13})\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_{23})}{\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_{1,2,3})\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_{123})}$$

$$\mathcal{Z}^{\mathrm{D0-D6}}_{(1,0,0,0),k=1} = \mathrm{sh}(\mathcal{X}_{\bar{4},1}(1,1,1) - \mathcal{X}_{\bar{4},1}(0,0,0))\mathcal{J}(\mathcal{X}_{\bar{4},1}(1,1,1) \mid \{(1,1,1)\}_{\bar{4},1})$$
$$= -\frac{\mathrm{sh}(\epsilon_{12})\mathrm{sh}(\epsilon_{13})\mathrm{sh}(\epsilon_{23})}{\mathrm{sh}(\epsilon_1)\mathrm{sh}(\epsilon_2)\mathrm{sh}(\epsilon_3)}$$

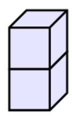
$$\prod_{k=1}^{\infty} \frac{1}{(1 - \mathrm{q}^k)^k} = 1 + \mathrm{q} + 3\mathrm{q}^2 + 6\mathrm{q}^3 + 13\mathrm{q}^4 + \cdots$$



$\{(1,1,1),(2,1,1)\}$



$\{(1,1,1),(1,2,1)\}$



$\{(1,1,1),(1,1,2)\}$

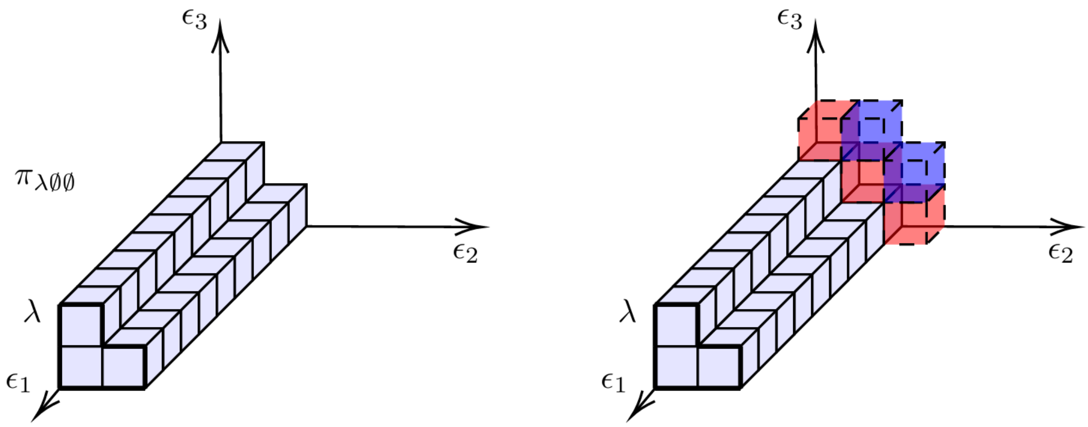
$$\mathcal{J}(x \mid \{(1,1,1),(1,1,2)\}_{\bar{4},1}) = \frac{\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_{12})\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_1 - 2\epsilon_3)\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_2 - 2\epsilon_3)}{\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_{1,2})\mathrm{sh}(x - v_{\bar{4},1} - 2\epsilon_3)\mathrm{sh}(x - v_{\bar{4},1} - \epsilon_{12} - 2\epsilon_3)}$$

$$\mathcal{Z}^{\mathrm{D0-D6}}_{(1,0,0,0)}\left(\{(1,1,1),(1,1,2)\}_{\underline{4},1}\right)=\frac{\mathrm{sh}(\epsilon_{12})\mathrm{sh}(\epsilon_{13})\mathrm{sh}(\epsilon_{23})\mathrm{sh}(\epsilon_{12}-\epsilon_3)\mathrm{sh}(\epsilon_1+2\epsilon_3)\mathrm{sh}(\epsilon_2+2\epsilon_3)}{\mathrm{sh}(\epsilon_1)\mathrm{sh}(\epsilon_2)\mathrm{sh}(\epsilon_3)\mathrm{sh}(2\epsilon_3)\mathrm{sh}(\epsilon_1-\epsilon_3)\mathrm{sh}(\epsilon_2-\epsilon_3)}$$

$$\frac{\mathcal{Z}^{\mathrm{D0-D6}}\left(\{(1,1,1),(1,1,2)\}_{\bar{4},1}\right)}{\mathcal{Z}^{\mathrm{D0-D6}}\left(\{(1,1,1)\}_{\bar{4},1}\right)}=-\frac{\mathrm{sh}(\epsilon_{12}-\epsilon_3)\mathrm{sh}(\epsilon_1+2\epsilon_3)\mathrm{sh}(\epsilon_2+2\epsilon_3)}{\mathrm{sh}(\epsilon_1-\epsilon_3)\mathrm{sh}(\epsilon_2-\epsilon_3)\mathrm{sh}(2\epsilon_3)}$$

$$\mathcal{J}\left(\mathcal{X}_{\bar{4},1}(1,1,2)\mid\{(1,1,1),(1,1,2)\}_{\bar{4},1}\right)=-\frac{\mathrm{sh}(\epsilon_{13})\mathrm{sh}(\epsilon_{23})\mathrm{sh}(\epsilon_{12}-\epsilon_3)}{\mathrm{sh}(\epsilon_3)\mathrm{sh}(\epsilon_{123})\mathrm{sh}(\epsilon_1-\epsilon_3)\mathrm{sh}(\epsilon_2-\epsilon_3)}$$
$$\mathcal{J}\left(\mathcal{X}_{\bar{4},1}(2,2,3)\mid\{(1,1,1)\}_{\bar{4},1}\right)=\frac{\mathrm{sh}(2\epsilon_3)\mathrm{sh}(\epsilon_{13})\mathrm{sh}(\epsilon_{23})}{\mathrm{sh}(\epsilon_3)\mathrm{sh}(\epsilon_{123})\mathrm{sh}(\epsilon_1+2\epsilon_3)\mathrm{sh}(\epsilon_2+2\epsilon_3)}$$

$$\frac{\mathcal{Z}^{\mathrm{D0-D6}}\left(\{(1,1,1),(1,1,2)\}_{\bar{4},1}\right)}{\mathcal{Z}^{\mathrm{D0-D6}}\left(\{(1,1,1)\}_{\bar{4},1}\right)}=\frac{\mathcal{J}\left(\mathcal{X}_{\bar{4},1}(1,1,2)\mid\{(1,1,1),(1,1,2)\}_{\bar{4},1}\right)}{\mathcal{J}\left(\mathcal{X}_{\bar{4},1}(2,2,3)\mid\{(1,1,1)\}_{\bar{4},1}\right)}$$



$$\frac{\mathrm{sh}(x-\infty\epsilon_1-2\epsilon_3)\mathrm{sh}(x-\infty\epsilon_1-\epsilon_{23})\mathrm{sh}(x-\infty\epsilon_1-2\epsilon_2)}{\mathrm{sh}(x-\infty\epsilon_1)\mathrm{sh}(x-\infty\epsilon_1-\epsilon_2-2\epsilon_3)\mathrm{sh}(x-\infty\epsilon_1-2\epsilon_2-\epsilon_3)}=1$$

$$\begin{aligned}\mathcal{J}(x\mid\pi_{\lambda\emptyset\emptyset})&=\frac{\mathrm{sh}(x-v-\epsilon_2-2\epsilon_3)\mathrm{sh}(x-v-2\epsilon_2-\epsilon_3)}{\mathrm{sh}(x-v-2\epsilon_2)\mathrm{sh}(x-v-2\epsilon_3)\mathrm{sh}(x-v-\epsilon_{23})}\\&=\mathcal{J}\left(x+v_{23,1}-v\mid\lambda_{23,1}\right)\end{aligned}$$

$$\mathcal{J}^{\mathrm{D0-D2-D6}}_{\lambda,\emptyset,\emptyset;k}=(-1)^k\mathcal{J}^{\mathrm{D0-D0}}_{k}\times\prod_{i=1}^{k}\frac{\mathrm{sh}(\phi_i-v-\epsilon_{234})}{\mathrm{sh}(\phi_i-v-\epsilon_{23})}\frac{\mathrm{sh}\left(\phi_i-v-2\epsilon_{2,3}-\epsilon_4\right)}{\mathrm{sh}\left(\phi_i-v-2\epsilon_{2,3}\right)}\frac{\mathrm{sh}\left(-\phi_i+v+\epsilon_{2,3}-\epsilon_{14}\right)}{\mathrm{sh}\left(-\phi_i+v+\epsilon_{2,3}-\epsilon_1\right)}$$

$$\tilde{\pi}_{\lambda\emptyset\emptyset}=\pi\setminus\pi_{\lambda\emptyset\emptyset}=\{(1,2,2)\},\{(1,3,1)\},\text{ and }\{(1,1,3)\}$$

$$\mathcal{Z}_{\lambda,\emptyset,\emptyset}^{\mathrm{D0-D2-D6}}(1,2,2) = \frac{\mathrm{sh}(\epsilon_{23})\mathrm{sh}(\epsilon_1 + 2\epsilon_{2,3})}{\mathrm{sh}(\epsilon_1)\mathrm{sh}(\epsilon_2 - \epsilon_3)^2}$$
$$\mathcal{Z}_{\lambda,\emptyset,\emptyset}^{\mathrm{D0-D2-D6}}(1,3,1) = \frac{\mathrm{sh}(\epsilon_{13})\mathrm{sh}(\epsilon_{23})\mathrm{sh}(\epsilon_1 + 2\epsilon_2)\mathrm{sh}(\epsilon_2 - 2\epsilon_3)\mathrm{sh}(\epsilon_1 + 3\epsilon_2 - \epsilon_3)}{\mathrm{sh}(\epsilon_{1,2})\mathrm{sh}(\epsilon_2 - \epsilon_3)\mathrm{sh}(2\epsilon_2 - 2\epsilon_3)\mathrm{sh}(\epsilon_1 + 2\epsilon_2 - \epsilon_3)}$$
$$\mathcal{Z}_{\lambda,\emptyset,\emptyset}^{\mathrm{D0-D2-D6}}(1,1,3) = -\frac{\mathrm{sh}(\epsilon_{12})\mathrm{sh}(\epsilon_{23})\mathrm{sh}(2\epsilon_2 - \epsilon_3)\mathrm{sh}(\epsilon_1 + 2\epsilon_3)\mathrm{sh}(\epsilon_1 - \epsilon_2 + 3\epsilon_3)}{\mathrm{sh}(\epsilon_{1,3})\mathrm{sh}(\epsilon_2 - \epsilon_3)\mathrm{sh}(2\epsilon_2 - 2\epsilon_3)\mathrm{sh}(\epsilon_1 - \epsilon_2 + 2\epsilon_3)}$$
$$\mathcal{Z}_{\lambda,\emptyset,\emptyset;k=1}^{\mathrm{D0-D2-D6}} = \mathcal{Z}_{\lambda,\emptyset,\emptyset}^{\mathrm{D0-D2-D6}}(1,2,2) + \mathcal{Z}_{\lambda,\emptyset,\emptyset}^{\mathrm{D0-D2-D6}}(1,3,1) + \mathcal{Z}_{\lambda,\emptyset,\emptyset}^{\mathrm{D0-D2-D6}}(1,1,3)$$

$$\begin{aligned}\mathcal{Z}_{\lambda,\emptyset,\emptyset}^{\mathrm{D0-D2-D6}}(1,2,2) &= \frac{\mathcal{J}(\mathcal{X}(1,2,2) \mid \pi_{\lambda\emptyset\emptyset} \cup \{(1,2,2)\})}{\mathcal{J}(\mathcal{X}(1,2,2) + \epsilon_{123} \mid \pi_{\lambda\emptyset\emptyset})} \\ &= \frac{\mathcal{J}(\mathcal{X}(1,2,2) \mid \pi_{\lambda\emptyset\emptyset})}{\mathcal{J}(\mathcal{X}(1,2,2) + \epsilon_{123} \mid \pi_{\lambda\emptyset\emptyset})}\mathrm{sh}(0)\mathcal{J}(\mathcal{X}(1,2,2) \mid \{(1,2,2)\}) \\ &= \frac{\mathrm{sh}(\epsilon_{23})\mathrm{sh}(\epsilon_1 + 2\epsilon_{2,3})}{\mathrm{sh}(\epsilon_1)\mathrm{sh}(\epsilon_2 - \epsilon_3)^2}\end{aligned}$$

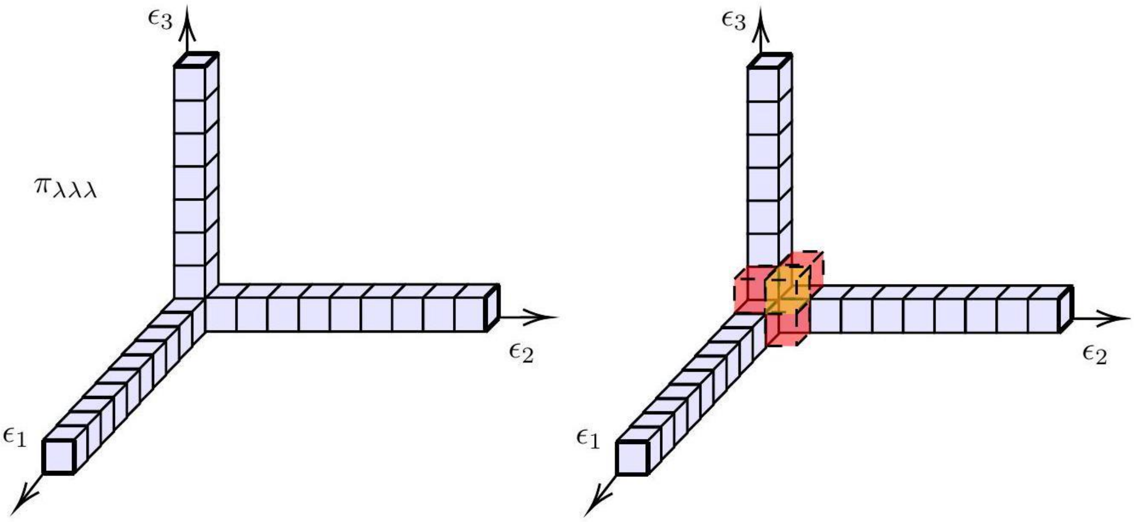
$$\mathcal{Z}_{\lambda,\emptyset,\emptyset}^{\mathrm{D0-D2-D6}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_k) = \frac{\mathcal{J}(\mathcal{X}(\boldsymbol{x}_1) \mid \pi_{\lambda\emptyset\emptyset} \cup \{\boldsymbol{x}_1\})}{\mathcal{J}(\mathcal{X}(\boldsymbol{x}_1) + \epsilon_{123} \mid \pi_{\lambda\emptyset\emptyset})}\frac{\mathcal{J}(\mathcal{X}(\boldsymbol{x}_2) \mid \pi_{\lambda\emptyset\emptyset} \cup \{\boldsymbol{x}_1\} \cup \{\boldsymbol{x}_2\})}{\mathcal{J}(\mathcal{X}(\boldsymbol{x}_2) + \epsilon_{123} \mid \pi_{\lambda\emptyset\emptyset} \cup \{\boldsymbol{x}_1\})} \times \dots$$

$$= \left(\prod_{i=1}^{k} \frac{\mathcal{J}(\mathcal{X}(\boldsymbol{x}_i) \mid \pi_{\lambda\emptyset\emptyset})}{\mathcal{J}(\mathcal{X}(\boldsymbol{x}_i) + \epsilon_{123} \mid \pi_{\lambda\emptyset\emptyset})}\right)\left(\prod_{i,j}^{k} \mathrm{sh}\left(\mathcal{X}(\boldsymbol{x}_i) - \mathcal{X}(\boldsymbol{x}_j)\right)\mathcal{J}(\mathcal{X}(\boldsymbol{x}_i) \mid \boldsymbol{x}_j)\right)$$

$$= \left(\prod_{i=1}^{k} \frac{\mathcal{J}(\mathcal{X}(\boldsymbol{x}_i) \mid \pi_{\lambda\emptyset\emptyset})}{\mathcal{J}(\mathcal{X}(\boldsymbol{x}_i) + \epsilon_{123} \mid \pi_{\lambda\emptyset\emptyset})}\right)$$

$$\times \left(\prod_{i,j}^{k} \frac{\mathrm{sh}\left(\mathcal{X}(\boldsymbol{x}_i) - \mathcal{X}(\boldsymbol{x}_j)\right)\mathrm{sh}\left(\mathcal{X}(\boldsymbol{x}_i) - \mathcal{X}(\boldsymbol{x}_j) - \epsilon_{12,13,23}\right)}{\mathrm{sh}(\boldsymbol{x}_i) - \mathcal{X}(\boldsymbol{x}_j) - \epsilon_{1,2,3})\mathrm{sh}\left(\mathcal{X}(\boldsymbol{x}_i) - \mathcal{X}(\boldsymbol{x}_j) - \epsilon_{123}\right)}\right)$$

$$\begin{aligned}\mathcal{J}(x \mid \pi_{\lambda\lambda\lambda}) &= \frac{\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset})\mathcal{J}(x \mid \pi_{\emptyset\lambda\emptyset})\mathcal{J}(x \mid \pi_{\emptyset\emptyset\lambda})\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\lambda\emptyset} \cap \pi_{\emptyset\emptyset\lambda})}{\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\lambda\emptyset})\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\emptyset\lambda})\mathcal{J}(x \mid \pi_{\emptyset\lambda\emptyset} \cap \pi_{\emptyset\emptyset\lambda})} \\ &= \frac{\mathrm{sh}(x - v - \epsilon_{123})^2}{\mathrm{sh}(x - v - \epsilon_{12})\mathrm{sh}(x - v - \epsilon_{13})\mathrm{sh}(x - v - \epsilon_{23})}\end{aligned}$$



$$\pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\lambda\emptyset} \cap \pi_{\emptyset\emptyset\lambda} = \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\lambda\emptyset} = \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\emptyset\lambda} = \pi_{\emptyset\lambda\emptyset} \cap \pi_{\emptyset\emptyset\lambda} = \{(1,1,1)\}$$

$$\mathcal{J}^{\text{D0-D2-D6}}_{\lambda,\lambda,\lambda;k} = (-1)^k \mathcal{J}^{\text{D0-D0}}_k \times \prod_{i=1}^{k} \frac{\operatorname{sh}(v-\epsilon_{1,2,3}-\phi_i)\operatorname{sh}(\phi_i - v - \epsilon_{123})^2}{\operatorname{sh}(\phi_i - v - \epsilon_{12,13,23})\operatorname{sh}(v-\phi_i)^2}$$

$$\mathcal{Z}^{\text{D0-D2-D6}}_{\lambda,\lambda,\lambda}(1,2,2) = -\frac{\operatorname{sh}(\epsilon_1)\operatorname{sh}(\epsilon_{12})\operatorname{sh}(\epsilon_{13})\operatorname{sh}(2\epsilon_2+\epsilon_3)\operatorname{sh}(\epsilon_2+2\epsilon_3)}{\operatorname{sh}(\epsilon_{2,3})\operatorname{sh}(\epsilon_{23})\operatorname{sh}(\epsilon_1-\epsilon_2)\operatorname{sh}(\epsilon_1-\epsilon_3)}$$

$$\mathcal{Z}^{\text{D0-D2-D6}}_{\lambda,\lambda,\lambda}(2,1,2) = \frac{\operatorname{sh}(\epsilon_2)\operatorname{sh}(\epsilon_{12})\operatorname{sh}(\epsilon_{23})\operatorname{sh}(2\epsilon_1+\epsilon_3)\operatorname{sh}(\epsilon_1+2\epsilon_3)}{\operatorname{sh}(\epsilon_{1,3})\operatorname{sh}(\epsilon_{13})\operatorname{sh}(\epsilon_1-\epsilon_2)\operatorname{sh}(\epsilon_2-\epsilon_3)}$$

$$\mathcal{Z}^{\text{D0-D2-D6}}_{\lambda,\lambda,\lambda}(2,2,1) = -\frac{\operatorname{sh}(\epsilon_3)\operatorname{sh}(\epsilon_{13})\operatorname{sh}(\epsilon_{23})\operatorname{sh}(2\epsilon_1+\epsilon_2)\operatorname{sh}(\epsilon_1+2\epsilon_2)}{\operatorname{sh}(\epsilon_{1,2})\operatorname{sh}(\epsilon_{12})\operatorname{sh}(\epsilon_1-\epsilon_3)\operatorname{sh}(\epsilon_2-\epsilon_3)}$$

$$\begin{aligned}\mathcal{J}(x \mid \pi_{\lambda\mu\emptyset}) &= \frac{\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset})\mathcal{J}(x \mid \pi_{\emptyset\mu\emptyset})\mathcal{J}(x \mid \pi_{\emptyset\emptyset\emptyset})\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\mu\emptyset\emptyset} \cap \pi_{\emptyset\emptyset\emptyset})}{\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\mu\emptyset})\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\emptyset\emptyset})\mathcal{J}(x \mid \pi_{\emptyset\mu\emptyset} \cap \pi_{\emptyset\emptyset\emptyset})} \\ &= \frac{\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset})\mathcal{J}(x \mid \pi_{\emptyset,\mu\emptyset})}{\mathcal{J}(x \mid \pi_{\lambda\emptyset\emptyset} \cap \pi_{\emptyset\mu\emptyset})}\end{aligned}$$

$$\begin{aligned}\mathcal{Z}^{\text{D0-D8}}_{N=1,k=1} = \mathcal{Z}^{\text{D0-D8}}(\{(1,1,1,1)\}) &= \mathcal{J}_{\geq}(\mathcal{X}_{\underline{\mathbf{4}},1}(1,1,1,1) \mid \{(1,1,1,1)\}_{\underline{\mathbf{4}},1}) \\ &= -\frac{\operatorname{sh}(\epsilon_{12})\operatorname{sh}(\epsilon_{13})\operatorname{sh}(\epsilon_{23})}{\operatorname{sh}(\epsilon_{1,2,3})\operatorname{sh}(\epsilon_{123})}\end{aligned}$$

$$\begin{aligned}\mathcal{Z}^{\text{D0-D8}}\left(\{(1,1,1,1),(2,1,1,1)\}_{\underline{\mathbf{4}},1}\right) &= -\frac{\operatorname{sh}(2\epsilon_1+\epsilon_2)\operatorname{sh}(2\epsilon_1+\epsilon_3)\operatorname{sh}(\epsilon_{23})}{\operatorname{sh}(2\epsilon_1)\operatorname{sh}(\epsilon_2)\operatorname{sh}(\epsilon_3)\operatorname{sh}(2\epsilon_1+\epsilon_{23})} \\ &\times \frac{\operatorname{sh}(\epsilon_{12})\operatorname{sh}(\epsilon_1-\epsilon_{23})\operatorname{sh}(\epsilon_{13})}{\operatorname{sh}(\epsilon_1)\operatorname{sh}(\epsilon_1-\epsilon_2)\operatorname{sh}(\epsilon_1-\epsilon_3)\operatorname{sh}(\epsilon_{123})}\end{aligned}$$

$$\mathcal{Z}^{\text{D0-D8}}(\{(1,1,1,1),(1,1,1,2)\}_{\underline{\mathbf{4}},1}) = \frac{\operatorname{sh}(\epsilon_{12})\operatorname{sh}(\epsilon_{13})\operatorname{sh}(\epsilon_{23})}{\operatorname{sh}(\epsilon_{1,2,3})\operatorname{sh}(\epsilon_{123})} \frac{\operatorname{sh}(\epsilon_{12}-\epsilon_4)\operatorname{sh}(\epsilon_{13}-\epsilon_4)\operatorname{sh}(\epsilon_{23}-\epsilon_4)}{\operatorname{sh}(\epsilon_{1,2,3}-\epsilon_4)\operatorname{sh}(\epsilon_{123}-\epsilon_4)}$$

$$\frac{\mathcal{Z}^{\text{D0-D8}}(\{(1,1,1,1),(2,1,1,1)\}_{\underline{\mathbf{4}},1})}{\mathcal{Z}^{\text{D0-D8}}(\{(1,1,1,1)\}_{\underline{\mathbf{4}},1})} = \frac{\operatorname{sh}(2\epsilon_1+\epsilon_2)\operatorname{sh}(\epsilon_1-\epsilon_2-\epsilon_3)\operatorname{sh}(2\epsilon_1+\epsilon_3)}{\operatorname{sh}(2\epsilon_1)\operatorname{sh}(\epsilon_1-\epsilon_2)\operatorname{sh}(\epsilon_1-\epsilon_3)\operatorname{sh}(2\epsilon_1+\epsilon_2+\epsilon_3)}$$

$$\mathcal{J}_{\geq}\left(\mathcal{X}_{\underline{\mathbf{4}},1}(2,1,1,1) \mid \{(1,1,1,1),(2,1,1,1)\}_{\underline{\mathbf{4}},1}\right)=\frac{\mathrm{sh}(\epsilon_{12})\mathrm{sh}(\epsilon_1-\epsilon_{23})\mathrm{sh}(\epsilon_{13})}{\mathrm{sh}(\epsilon_1)\mathrm{sh}(\epsilon_1-\epsilon_{2,3})\mathrm{sh}(\epsilon_{123})}$$

$$\mathcal{J}_{<}\left(\mathcal{X}_{\underline{\mathbf{4}},1}(2,1,1,1) \mid \{(1,1,1,1)\}_{\underline{\mathbf{4}},1}\right)=\frac{\mathrm{sh}(\epsilon_1-\epsilon_{24})\mathrm{sh}(\epsilon_1-\epsilon_{34})\mathrm{sh}(\epsilon_4)\mathrm{sh}(\epsilon_{234})}{\mathrm{sh}(\epsilon_1-\epsilon_4)\mathrm{sh}(\epsilon_1-\epsilon_{234})\mathrm{sh}(\epsilon_{24})\mathrm{sh}(\epsilon_{34})}$$

$$\frac{\mathcal{Z}^{\mathrm{D0-D8}}\left(\{(1,1,1,1),(2,1,1,1)\}_{\underline{\mathbf{4}},1}\right)}{\mathcal{Z}^{\mathrm{D0-D8}}\left(\{(1,1,1,1)\}_{\underline{\mathbf{4}},1}\right)}=\mathcal{J}_{\geq}\left(\mathcal{X}_{\underline{\mathbf{4}},1}(2,1,1,1) \mid \{(1,1,1,1),(2,1,1,1)\}_{\underline{\mathbf{4}},1}\right) \times \mathcal{J}_{<}\left(\mathcal{X}_{\underline{\mathbf{4}},1}(2,1,1,1) \mid \{(1,1,1,1)\}_{\underline{\mathbf{4}},1}\right)$$

$$\tilde{\mathcal{I}}_k^{\mathrm{D0-D0}}=\frac{1}{k!}\prod_{i\neq j}^{k}\mathrm{sh}(\phi_i-\phi_j)\prod_{i,j}^{k}\frac{\mathrm{sh}(\phi_i-\phi_j-\epsilon_{1,2,3}-\epsilon_4)}{\mathrm{sh}(\phi_i-\phi_j-\epsilon_{1,2,3})\mathrm{sh}(\phi_i-\phi_j+\epsilon_4)}$$

$$\tilde{\mathcal{I}}_{N,k}^{\mathrm{D0-D8-\overline{D8}}}=\tilde{\mathcal{I}}_k^{\mathrm{D0-D0}}\times\prod_{i=1}^{k}\prod_{\alpha=1}^{N}\frac{\mathrm{sh}(\phi_i-w_{\underline{\mathbf{4}},\alpha})}{\mathrm{sh}(\phi_i-v_{\underline{\mathbf{4}},\alpha})},$$

$$\rho_1=\{(1,1,1,1),(2,1,1,1)\},\quad \rho_2=\{(1,1,1,1),(1,2,1,1)\},$$
$$\rho_3=\{(1,1,1,1),(1,1,2,1)\},\quad \rho_4=\{(1,1,1,1),(1,1,1,2)\}$$

$$\mathrm{Res}_{\mathcal{X}(\boldsymbol{x}\in\rho)}\left(\prod_{i=1}^{k}\frac{d\phi_i}{2\pi i}\right)\tilde{\mathcal{I}}_{N,k}^{\mathrm{D0-D8-\overline{D8}}}=(-1)^{h(\rho)}\mathrm{Res}_{\mathcal{X}(\boldsymbol{x}\in\rho)}\left(\prod_{i=1}^{k}\frac{d\phi_i}{2\pi i}\right)_{\mathcal{I}_{N,k}^{\mathrm{D0-D8-\overline{D8}}}}$$

$$h(\rho)=1+|\rho|+\#\{(a,d)\big|(a,a,a,d)\in\rho \text{ and } a\leq d\}$$

$$\tilde{\mathcal{I}}_{1,2}^{\mathrm{D0-D8-\overline{D8}}}\supset\tilde{\mathcal{I}}=\frac{1}{\mathrm{sh}(\phi_{1,2}-v)\mathrm{sh}(\phi_1-\phi_2+\epsilon_4)\mathrm{sh}(\phi_2-\phi_1+\epsilon_4)}$$

$$\mathcal{I}_{1,2}^{\mathrm{D0-D8-\overline{D8}}}\supset\mathcal{I}=\frac{1}{\mathrm{sh}(\phi_{1,2}-v)\mathrm{sh}(\phi_1-\phi_2-\epsilon_4)\mathrm{sh}(\phi_2-\phi_1-\epsilon_4)}$$

$$\mathrm{Res}_{\mathcal{X}(\boldsymbol{x}\in\rho_4)}\tilde{\mathcal{I}}=\frac{1}{\mathrm{sh}(\epsilon_4)\mathrm{sh}(2\epsilon_4)}$$

$$\mathrm{Res}_{\mathcal{X}(\boldsymbol{x}\in\rho_4)}\mathcal{I}=\frac{1}{\mathrm{sh}(\epsilon_4)\mathrm{sh}(-2\epsilon_4)}=-\mathrm{Res}_{\mathcal{X}(\boldsymbol{x}\in\rho_4)}\tilde{\mathcal{I}}$$

$$\int d^8zu(\Phi)\left(-\frac{D^2}{4\,\square}\right)v(\Phi)=\int d^6zu(\Phi)v(\Phi)$$

$$\mathcal{S}_0=\mathcal{S}_c+\mathcal{S}_{GF}+\mathcal{S}_{FP}$$

$$\mathcal{S}_c=\int d^4x\mathcal{L}=\mathrm{tr}\int d^8ze^{-gV}\bar{\Phi}_ie^{gV}\Phi_i+\frac{1}{g^2}\mathrm{tr}\int d^6z\mathcal{W}^2+ +\frac{i}{3!}g\epsilon_{ijk}\mathrm{tr}\int d^6z\left[\Phi_i,\left[\Phi_j,\Phi_k\right]\right]+\text{ h.c.}$$

$$\mathcal{S}_{GF} = -\frac{1}{16\xi}\mathrm{tr}\int d^8 z D^2 V \bar{D}^2 V$$

$$\mathcal{S}_{FP} = \mathrm{tr}\int d^8 z \left[\bar{c}' c - c'\bar{c} + \frac{1}{2}(c' + \bar{c}')[V, c + \bar{c}] + \cdots\right]$$

$$W_{\text{tree}} = \frac{i}{3!} g \epsilon_{ijk} \mathrm{tr}\int d^6 z \left[\Phi_i, [\Phi_j, \Phi_k]\right]$$

$$\Phi_i \to \Phi_i + \sqrt{\hbar}\phi_i,$$

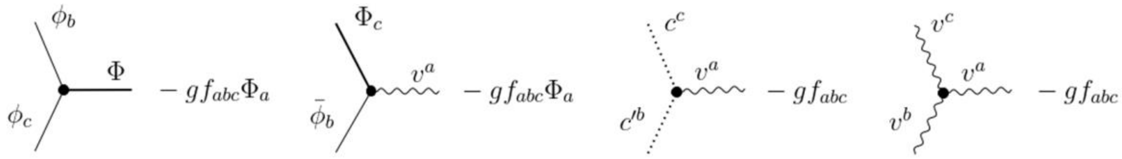
$$\begin{aligned}\mathcal{S}^{(2)} = {} & \mathrm{tr}\int d^8 z(\bar{\phi}\phi - 2g^2\bar{\Phi}[v,\phi] + 2g^2\bar{\phi}[v,\Phi]) \\ & g\mathrm{tr}\int d^6 z(\phi_1\Phi_2\phi_3 + \Phi_1\phi_2\phi_3 + \phi_1\phi_2\Phi_3 + \text{ h.c. }) \\ & +\mathrm{tr}\int d^8 z\left(-\frac{1}{2} v \,\hat{\Box}\, v + cc' + c'c\right)\end{aligned}$$

$$\Gamma[\Phi_i] = \sum_{L=1}^{\infty} \hbar^L \Gamma^{(L)}[\Phi_i]$$

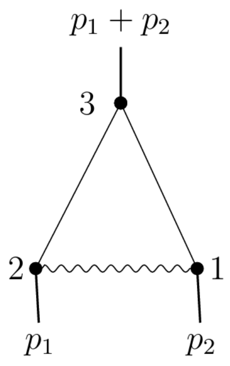
$$= \langle \phi_a \bar{\phi}_b \rangle = -\frac{\bar{D}_1^2 D_2^2}{16\Box}\delta_{ab}\delta^8(z_1 - z_2)$$

$$= \langle v_a v_b \rangle = \left(-\frac{D^\alpha \bar{D}^2 D_\alpha}{8\Box^2} + \xi\frac{\{D^2, \bar{D}^2\}}{16\Box^2}\right)\delta_{ab}\delta^8(z_1 - z_2) \overset{\xi=1}{=} \frac{1}{16\Box}\delta_{ab}\delta^8(z_1 - z_2)$$

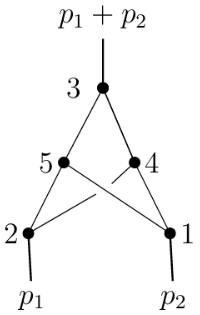
$$= \langle c'_a c_b \rangle = \langle c'_b c_a \rangle = \frac{1}{16\Box}\delta_{ab}\delta^8(z_1 - z_2)$$



$$\mathbf{W}[\Phi_i] = \sum_{L=1}^{\infty} \hbar^L \mathbf{W}^{(L)}[\Phi_i]$$



$$\begin{aligned} &= \lim_{p_1,p_2\to 0}\frac{i}{12}g^3 C_A\int\prod_{l=1}^{3} d^8 z_l \epsilon_{ijk}[\Phi_i(z_1)[\Phi_j(z_2), \Phi_k(z_3)]]\times \\ &\times\left\{\frac{1}{\Box_2}\delta_{1,2}\frac{D_1^2\bar{D}_3}{16\Box_1}\delta_{1,3}\frac{D_2^2}{4}\delta_{2,3}\frac{1}{\Box_2}\right\} = \frac{\hbar}{(4\pi)^2}\frac{1}{2}g^2 C_A \Upsilon^{(1)} W_{tree}.\end{aligned}$$



$$= \lim_{p_1,p_2\to 0} -\frac{i}{12} g^5 (C_A - C_A) C_A \int \prod_{l=1}^{5} d^8 z_l \epsilon_{ijk} [\Phi_i(z_1)[\Phi_j(z_2), \Phi_k(z_3)]] \times$$

$$\times \left\{ \frac{1}{\Box_1}\delta_{1,3} \frac{D_2^2 \bar{D}_3^2}{16\Box_2}\delta_{3,2} \frac{1}{16\Box_2}\delta_{2,4} \frac{D_1^2 \bar{D}_4^2}{16\Box_1}\delta_{1,4} \frac{D_1^2 \bar{D}_5^2}{16\Box_1}\delta_{1,5} \frac{D_2^2}{4\Box_2}\delta_{2,5} \right\} = 0.$$

$$\mathbf{W}_{\text{fin}}^{(2)} = C_A^2 \left[g^4 \Upsilon^{(2)} \right] \times W_{\text{tree}}$$

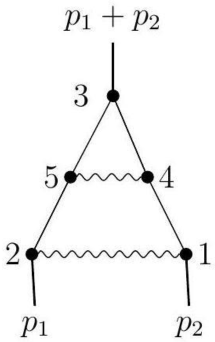
$$\mathbf{W}'^{(m)} = \lim_{p_1,p_2\to 0} (-1)^{m+1} \frac{g^{2m}}{24} C_A^m \int \prod_{l=1}^{2m+1} d^8 z_l g \epsilon_{ijk} \left[\Phi_i(z_1) \left[\Phi_j(z_2), \Phi_k(z_3) \right] \right]$$

$$\times \left\{ \frac{1}{\Box_5}\delta_{5,1} \frac{D_1^2 \bar{D}_3^2}{16\,\Box_1}\delta_{1,3} \frac{D_4^2}{4\,\Box_3}\delta_{3,4} \frac{\bar{D}_4^2 D_6^2}{16\,\Box_4}\delta_{4,6} \cdots \right.$$

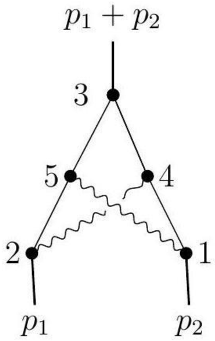
$$\left. \cdots \frac{\bar{D}_{2m-2}^2 D_{2m}^2}{\Box_{2m-2}}\delta_{2m-2,2m} \frac{\bar{D}_{2m}^2 D_2^2}{\Box_{2m}}\delta_{2m,2} \frac{1}{\Box_{2m+1}}\delta_{2m+1,2} \cdots \frac{1}{\Box_{2m}}\delta_{2m+1,2m} \cdots \frac{1}{\Box_4}\delta_{5,4} \right\}$$

$$= (-1)^{m-1} \frac{g^{2m}}{4} C_A^m \Upsilon^{(m)} \times W_{\text{tree}}\,.$$

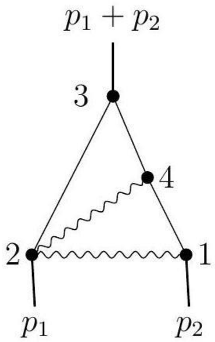
$$\mathbf{W}'^{lead} = \frac{g}{2} \int_0^1 d\tau \frac{\log(\tau)(1-\tau)}{(1+y\log^2(\tau))(1+\tau^3)} \times W_{\text{tree}} = \Upsilon^{tot} \times W_{\text{tree}}$$



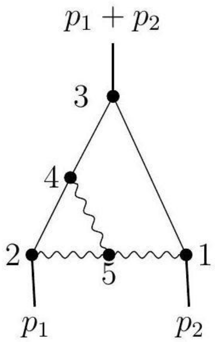
$$W^{(A)} = \lim_{p_1,p_2\to 0} \frac{i}{24} g^5 C_A^2 \int \prod_{l=1}^{5} d^8 z_l \epsilon_{ijk} [\Phi_i(z_1)[\Phi_j(z_2), \Phi_k(z_3)]] \times$$
$$\times \left\{ \frac{1}{\Box_1} \delta_{2,1} \frac{D_1^2 \bar{D}_4^2}{16\Box_4} \delta_{1,4} \frac{D_4^2 \bar{D}_3^2}{16\Box_4} \delta_{4,3} \frac{D_5^2}{4\Box_5} \delta_{3,5} \frac{\bar{D}_5^2 D_2^2}{16\Box_5} \delta_{5,2} \frac{1}{\Box_4} \delta_{4,5} \right\} =$$
$$= \frac{1}{4} 6 g^4 (C_A)^2 \zeta(3) \times W_{tree}.$$



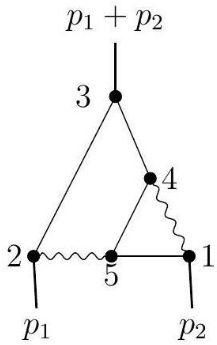
$$W^{(B)} = 0.$$



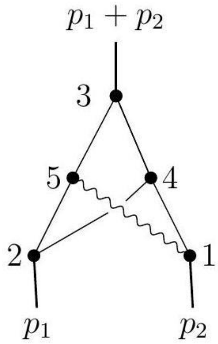
$$W^{(C)} = \lim_{p_1,p_2\to 0} -\frac{i}{24} g^5 C_A^2 \int \prod_{l=1}^{5} d^8 z_l \epsilon_{ijk} [\Phi_i(z_1)[\Phi_j(z_2), \Phi_k(z_3)]] \times$$
$$\times \left\{ \frac{1}{\Box_1} \delta_{2,1} \frac{D_1^2 \bar{D}_4^2}{16\Box_4} \delta_{1,4} \frac{\bar{D}_4^2 D_3^2}{16\Box_4} \delta_{4,3} \frac{D_2^2}{4\Box_3} \delta_{3,2} \frac{1}{\Box_4} \delta_{2,4} \right\} =$$
$$= \frac{6}{4} g^4 C_A^2 \zeta(3) \times W_{tree}.$$



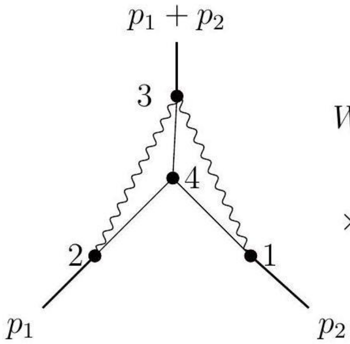
$$W^{(D)} = \lim_{p_1,p_2\to 0} \frac{i}{12} g^5 C_A^2 \int \prod_{l=1}^{5} d^8 z_l \epsilon_{ijk} [\Phi_i(z_1)[\Phi_j(z_2), \Phi_k(z_3)]] \times$$
$$\times \left\{ \frac{1}{\Box_5} \delta_{5,1} \frac{D_1^2 \bar{D}_3^2}{16\Box_1} \delta_{1,3} \frac{D_4^2}{4\Box_3} \delta_{3,4} \frac{\bar{D}_4^2 D_2^2}{16\Box_4} \delta_{4,2} \frac{1}{\Box_4} \delta_{2,5} \frac{1}{\Box_4} \delta_{5,4} \right\} =$$
$$= \frac{1}{2} g^4 C_A^2 \Upsilon^{(2)} \times W_{tree}, \Upsilon^{(2)} = \int_0^1 d\tau \frac{2\log^3(\tau)}{\tau^2 - \tau + 1}.$$



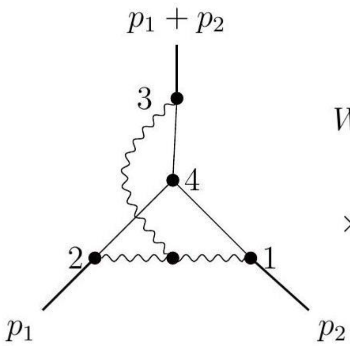
$$W^{(E)} = \lim_{p_1, p_2 \to 0} \frac{i}{12} g^5 C_A^2 \int \prod_{l=1}^{5} d^8 z_l \epsilon_{ijk} [\Phi_i(z_1)[\Phi_j(z_2), \Phi_k(z_3)]] \times$$
$$\times \left\{ \frac{1}{\Box_4} \delta_{1,4} \frac{D_1^2 \bar{D}_5^2}{16\Box_5} \delta_{1,5} \frac{D_5^2 \bar{D}_4^2}{16\Box_5} \delta_{5,4} \frac{D_4^2 \bar{D}_3^2}{16\Box_4} \delta_{4,3} \frac{D_2^2}{4\Box_3} \delta_{3,2} \frac{1}{\Box_5} \delta_{2,5} \right\} =$$
$$= -\frac{1}{2} 6 g^4 (C_A)^2 \zeta(3) \times W_{tree}.$$



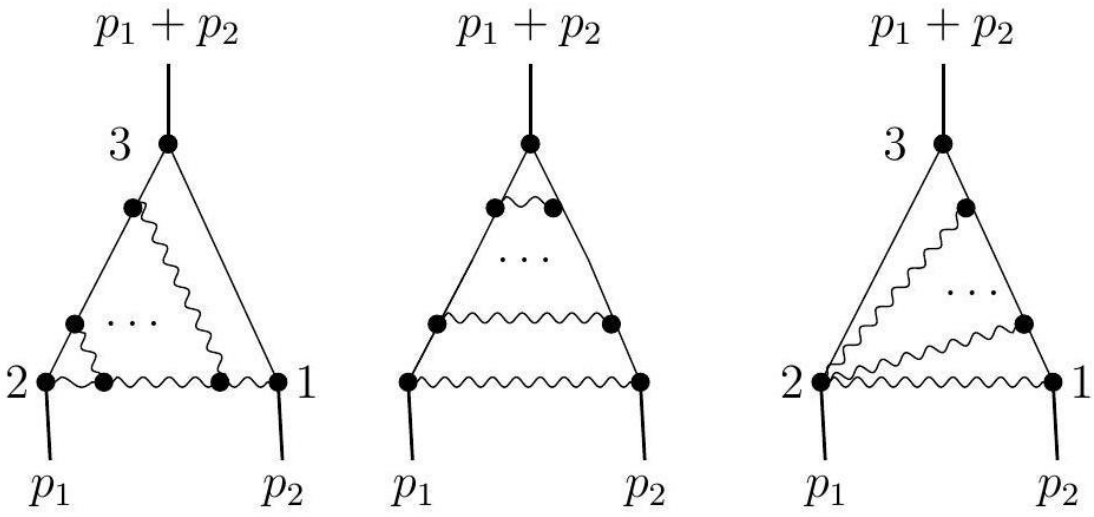
$$W^{(F)} \sim \lim_{p_1, p_2 \to 0} 4 g^4 \int \prod_{l=1}^{5} d^8 z_l \epsilon_{ijk} [\Phi_i(z_1)[\Phi_j(z_2), \Phi_k(z_3)]] \times$$
$$\times \left\{ \frac{1}{\Box_4} \delta_{2,4} \frac{D_1^2 \bar{D}_4^2}{16\Box_4} \delta_{1,4} \frac{\bar{D}_4^2 D_3^2}{16\Box_4} \delta_{4,3} \frac{D_5^2}{4\Box_5} \delta_{3,5} \frac{\bar{D}_5^2 D_2^2}{16\Box_5} \delta_{5,2} \frac{1}{\Box_5} \delta_{1,5} \right\} = 0.$$



$$W^{(G)} \sim \lim_{p_1, p_2 \to 0} 4 g^4 \int \prod_{l=1}^{5} d^8 z_l \epsilon_{ijk} [\Phi_i(z_1)[\Phi_j(z_2), \Phi_k(z_3)]] \times$$
$$\times \left\{ \frac{D_1^2 \bar{D}_4^2}{16\Box_4} \delta_{1,4} \frac{\bar{D}_4^2 D_2^2}{16\Box_4} \delta_{4,2} \frac{1}{\Box_3} \delta_{2,3} \frac{1}{\Box_1} \delta_{3,1} \frac{D_3^2}{4\Box_4} \delta_{3,4} \right\} = 0.$$



$$W^{(H)} \sim \lim_{p_1, p_2 \to 0} 2 g^4 \int \prod_{l=1}^{5} d^8 z_l \epsilon_{ijk} [\Phi_i(z_1)[\Phi_j(z_2), \Phi_k(z_3)]] \times$$
$$\times \left\{ \frac{1}{\Box_5} \delta_{3,5} \frac{D_1^2 \bar{D}_4^2}{16\Box_4} \delta_{1,4} \frac{\bar{D}_4^2 D_2^2}{16\Box_4} \delta_{4,2} \frac{1}{\Box_3} \delta_{2,5} \frac{1}{\Box_1} \delta_{5,3} \frac{D_3^2}{4\Box_4} \delta_{3,4} \right\} = 0.$$



(a) (b) (c)

$$\Upsilon^{tot} = \frac{g}{4}\sum_{m=1}^{\infty} ((\pi - 2\mathrm{Si}(x))\sin(x) - 2\mathrm{Ci}(x)\cos(x))U_m(1/2),$$

$$\tau := \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\mathrm{YM}}^2},$$

$$\hat{\mathcal{O}}_p = [\mathcal{O}_2]^{\frac{p}{2}}, \text{ with } p \in 2\mathbb{N}$$

$$\mathcal{O}_H = \exp\left(\frac{\alpha}{\sqrt{2}}\mathcal{O}_2\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha}{\sqrt{2}}\right)^n \mathcal{O}_2^n, \text{ with } |\alpha|^2 \in \left[0,\frac{1}{2}\right)$$

$$\langle \hat{\mathcal{O}}_p \hat{\mathcal{O}}_p \mathcal{O}_2 \mathcal{O}_2 \rangle, \langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_2 \mathcal{O}_2 \rangle$$

$$\exp\left(-2L_+^* \sqrt{\lambda}\right) \ll \exp\left(-2\sqrt{\lambda}\right) \ll \exp\left(-2L_-^* \sqrt{\lambda}\right)$$

$$L_+^* = \sqrt{\frac{1+\sqrt{2}\alpha}{1-\sqrt{2}\alpha}}, \qquad L_-^* = \sqrt{\frac{1-\sqrt{2}\alpha}{1+\sqrt{2}\alpha}}$$

$$T_p(x,Y) := \frac{1}{p} Y_{I_1} \cdots Y_{I_p} \mathrm{Tr}[\Phi^{I_1}(x) \cdots \Phi^{I_p}(x)] = \frac{1}{p}\mathrm{Tr}[(Y \cdot \Phi(x))^p]$$

$$T_{p_1,\dots,p_n}(x,Y) := \frac{p_1 \cdots p_n}{p} T_{p_1}(x,Y) \cdots T_{p_n}(x,Y)$$

$$\hat{\mathcal{O}}_p(x,Y):=\begin{cases} T_{(2,\dots,2)}(x,Y)=\dfrac{[2T_2(x,Y)]^{\frac{p}{2}}}{p}, & p=2r\equiv 0\text{mod}2 \\ T_{(3,2,\dots,2)}(x,Y)=\dfrac{3T_3(x,Y)[2T_2(x,Y)]^{\frac{p-3}{2}}}{p}, & p=2r+3\equiv 1\text{mod}2 \end{cases}$$

$$\hat{\mathcal{O}}_2(x,Y)=\mathcal{O}_2(x,Y)=T_2(x,Y)$$
$$\hat{\mathcal{O}}_3(x,Y)=\mathcal{O}_3(x,Y)=T_3(x,Y)$$

$$\langle Y_1\cdot\Phi_n^m(x_1)Y_2\cdot\Phi_s^r(x_2)\rangle=\left(\delta_s^m\delta_n^r-\frac{1}{N}\delta_n^m\delta_s^r\right)d_{12},\ \text{with}\ d_{ij}:=\frac{Y_{ij}}{x_{ij}^2}$$

$$\left\langle\hat{\mathcal{O}}_p(x_1,Y_1)\hat{\mathcal{O}}_q(x_2,Y_2)\right\rangle=\delta_{p,q}R_p(N)d_{12}^p$$

$$R_{2r}(N):=\frac{4^r r!}{(2r)^2}\left(\frac{N^2-1}{2}\right)_r,R_{2r+3}(N):=\frac{3(N^2-1)(N^2-4)4^r r!}{N(3+2r)^2}\left(\frac{N^2+5}{2}\right)_r,$$

$$(a)_n:=\prod_{k=0}^{n-1}(a+k)$$

$$\left\langle\mathcal{O}_2(x_1,Y_1)\mathcal{O}_2(x_2,Y_2)\hat{\mathcal{O}}_p(x_3,Y_3)\hat{\mathcal{O}}_p(x_4,Y_4)\right\rangle$$

$$\left\langle\mathcal{O}_2(x_1,Y_1)\mathcal{O}_2(x_2,Y_2)\hat{\mathcal{O}}_p(x_3,Y_3)\hat{\mathcal{O}}_p(x_4,Y_4)\right\rangle=\mathcal{G}_{p,\text{ free}}+d_{12}^2d_{34}^p\mathcal{I}(z,\bar{z},y,\bar{y})\mathcal{H}_p(u,v;N,\tau)$$

$$u=z\bar{z}:=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\quad v=(1-z)(1-\bar{z}):=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$
$$\tilde{u}=y\bar{y}:=\frac{Y_{12}^2Y_{34}^2}{Y_{13}^2Y_{24}^2},\quad \tilde{v}=(1-y)(1-\bar{y}):=\frac{Y_{14}^2Y_{23}^2}{Y_{13}^2Y_{24}^2}$$

$$\mathcal{I}(z,\bar{z},y,\bar{y}):=\frac{(z-y)(z-\bar{y})(\bar{z}-y)(\bar{z}-\bar{y})}{(y\bar{y})^2}$$

$$\tau:=\tau_1+i\tau_2=\frac{\theta}{2\pi}+i\frac{4\pi}{g_{\text{YM}}^2}$$

$$\mathcal{C}_{p,N}(\tau):=-\frac{2}{\pi}\int_0^\infty \mathrm{dr}\int_0^\pi \mathrm{d}\theta\frac{r^3\sin^2(\theta)}{u^2}\frac{\mathcal{H}_p(u,v;N,\tau)}{R_p(N)}\bigg|_{u=1-2r\cos\theta+r^2,v=r^2}$$

$$|Y;\alpha\rangle:=\sum_{n=0}^\infty\frac{1}{n!}\left(\frac{\alpha}{\sqrt{2}}\right)^n|Y;n\rangle,|Y;n\rangle:=\lim_{x\to 0}[\mathcal{O}_2^n(x,Y)|0\rangle]$$

$$\mathcal{O}_H(x=0,Y;\alpha):=\exp\left(\frac{\alpha}{\sqrt{2}}\mathcal{O}_2(x=0,Y)\right)$$

$$\langle Y;\alpha|=\sum_{n=0}^\infty\frac{1}{n!}\left(\frac{\bar{\alpha}}{\sqrt{2}}\right)^n\lim_{x\to\infty}[|x|^{4n}\langle 0|\mathcal{O}_2^n(x,\bar{Y})]$$

$$\lim_{\substack{x_1\to 0\\ x_2\to\infty}} \langle \mathcal{O}_H(x_1,Y;\alpha)\mathcal{O}_H(x_2,Y;\alpha)\rangle = \langle Y;\alpha \mid Y;\alpha\rangle = \left(1-\frac{|Y|^4|\alpha|^2}{2}\right)^{-\frac{N^2-1}{2}}$$

$$\langle \mathcal{O}_2(x_1,Y_1)\mathcal{O}_2(x_2,Y_2)\mathcal{O}_H(0,Y_3;\alpha)\mathcal{O}_H(\infty,Y_4;\alpha)\rangle = \langle Y_4;\alpha|\mathcal{O}_2(x_1,Y_1)\mathcal{O}_2(x_2,Y_2)|Y_3;\alpha\rangle$$

$$O_L(x) \coloneqq \mathcal{O}_2(x,Y_\bullet) = \frac{1}{2}\mathrm{Tr}(\Phi_1(x)+i\Phi_2(x))^2\,, \qquad O_H(x=0;\alpha) = \exp\left(\frac{\alpha}{\sqrt{2}}O_L(0)\right)$$

$$\langle O_L(x_1)\bar{O}_L(x_2)O_H(0;\alpha)\bar{O}_H(\infty;\alpha)\rangle = \langle \alpha|O_L(x_1)\bar{O}_L(x_2)|\alpha\rangle,$$

$$\langle \alpha|O_L(x_1)\bar{O}_L(x_2)|\alpha\rangle = \sum_{r,s=0}^{\infty}\frac{1}{r!\,s!}\left(\frac{\alpha}{\sqrt{2}}\right)^{r+s}\lim_{\substack{x_3\to 0\\ x_4\to\infty}}[|x_4|^{4r}\langle O_L(x_1)\bar{O}_L(x_2)O_L^s(x_3)\bar{O}_L^r(x_4)\rangle]$$

$$d = \sum_{j=2}^{4}\min\left(\frac{p_1+p_j}{2}, S-\frac{p_1+p_j}{2}\right)-S,\ \text{with}\ S := \frac{p_1+p_2+p_3+p_4}{2}$$

$$\langle \alpha|O_L(x_1)\bar{O}_L(x_2)|\alpha\rangle = \mathcal{G}_{\alpha,\,\text{free}} + d_{12}^2\sum_{r=1}^{\infty}\frac{1}{(r!)^2}\lim_{\substack{x_3\to 0\\ x_4\to\infty}}\left[\left(\frac{\alpha|x_4|^2 d_{34}}{\sqrt{2}}\right)^{2r}\binom{2r}{2r}^2\mathcal{H}_{2r}(u,v;N,\tau)\right].$$

$$\mathcal{H}_\alpha(u,v;N,\tau) := \sum_{r=1}^{\infty}\frac{1}{(r!)^2}\lim_{\substack{x_3\to 0\\ x_4\to\infty}}\left[\left(\frac{\alpha|x_4|^2 d_{34}}{\sqrt{2}}\right)^{2r}\binom{2r}{2r}^2\mathcal{H}_{2r}(u,v;N,\tau)\right]$$

$$\mathcal{H}_\alpha(u,v;N,\tau) = 4\sum_{r=1}^{\infty}\frac{r^2}{(r!)^2}\left(\frac{\alpha}{\sqrt{2}}\right)^{2r}\mathcal{H}_{2r}(u,v;N,\tau)$$

$$\mathcal{C}_{\mathrm{HHLL}}(N,\alpha;\tau) := -\frac{2}{\pi}\int_0^\infty \mathrm{d}r\int_0^\pi \mathrm{d}\theta\,\frac{r^3\sin^2(\theta)}{u^2}\,\frac{\mathcal{H}_\alpha(u,v;N,\tau)}{\langle Y_\bullet;\alpha|Y_\bullet;\alpha\rangle}\Bigg|_{u=1-2r\cos\theta+r^2,\,v=r^2}.$$

$$\mathcal{C}_{\mathrm{HHLL}}(N,\alpha;\tau) = 4(1-2\alpha^2)^{\frac{N^2-1}{2}}\sum_{r=1}^{\infty}\frac{r^2}{(r!)^2}\left(\frac{\alpha}{\sqrt{2}}\right)^{2r}R_{2r}(N)\mathcal{C}_{2r,N}(\tau)$$

$$\mathcal{C}_{p,N}(\tau) := -\frac{2}{\pi}\int_0^\infty \mathrm{d}r\int_0^\pi \mathrm{d}\theta\,\frac{r^3\sin^2(\theta)}{u^2}\,\frac{\mathcal{H}_p(u,v;N,\tau)}{R_p(N)}\Bigg|_{u=1-2r\cos\theta+r^2,\,v=r^2}$$

$$\mathcal{C}_{p,N}(\gamma\cdot\tau) = \mathcal{C}_{p,N}(\tau),\ \forall\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z})$$
$$\text{with}\ \gamma\cdot\tau := \frac{a\tau+b}{c\tau+d}$$

$$\mathcal{C}_{p,N}(\tau) = \langle \mathcal{C}_{p,N}\rangle + \int_{\mathrm{Re}(s)=\frac{1}{2}} c_{p,N}(s)E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$E^*(s;\tau) := \frac{\Gamma(s)}{2}\sum_{(m,n)\neq(0,0)} Y_{mn}(\tau)^{-s} = \frac{1}{2}\sum_{(m,n)\neq(0,0)}\int_0^\infty e^{-tY_{mn}(\tau)}t^{s-1}\,\mathrm{d}t$$

$$= \xi(2s)\tau_2^s + \xi(2s-1)\tau_2^{1-s} + \sum_{k\neq 0} e^{2\pi i k\tau_1}2\sqrt{\tau_2}|k|^{s-\frac{1}{2}}\sigma_{1-2s}(k)K_{s-\frac{1}{2}}(2\pi|k|\tau_2)$$

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s)$$

$$Y_{mn}(\tau) := \pi\frac{|n\tau+m|^2}{\tau_2}$$

$$\langle\mathcal{C}_{p,N}\rangle = \lim_{s\to 1}[c_{p,N}(s)]$$

$$c_{p,N}(s) = \frac{\pi}{\sin(\pi s)}P_{p,N}(s)$$

$$\mathcal{C}_{p,N}(\tau) = \frac{1}{2}c_{p,N}(1) + \sum_{k=2}^{\infty}(-1)^{k-1}P_{p,N}(k)E^*(k;\tau)$$

$$\mathcal{C}_{p,N}(\tau) = \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}\int_0^\infty e^{-tY_{mn}(\tau)}B_{p,N}(t)\mathrm{d}t$$

$$B_{p,N}(t) := \sum_{k=1}^{\infty}(-1)^k P_{p,N}(k+1)t^k$$

$$\Phi_P(\Lambda;\tau) = \sum_{k=0}^{\infty} b_k\Lambda^{-k-\frac{3}{2}}E^*\left(k+\frac{3}{2};\tau\right)$$

$$\Lambda^{-s}E^*(s;\tau) = \int_0^\infty \mathcal{E}(\sqrt{\Lambda}t;\tau)\frac{2\Gamma(s)}{\Gamma(2s)}(4t)^{2s-1}\,\mathrm{d}t, \mathcal{E}(t;\tau) := \sum_{(m,n)\neq(0,0)} e^{-4t\sqrt{Y_{mn}(\tau)}}$$

$$\mathcal{B}[\Phi_P](t) := \sum_{k=0}^{\infty} b_k\frac{2\Gamma\left(k+\frac{3}{2}\right)}{\Gamma(2k+3)}(4t)^{2k+2}$$

$$\Phi_P(\Lambda;\tau) \stackrel{?}{=} \int_0^\infty \mathcal{E}(\sqrt{\Lambda}t;\tau)\mathcal{B}[\Phi_P](t)\mathrm{d}t$$

$$\mathcal{S}_\theta[\Phi_P](\Lambda;\tau) := \int_0^{e^{i\theta}\infty}\mathcal{E}(\sqrt{\Lambda}t;\tau)\mathcal{B}[\Phi_P](t)\mathrm{d}t$$

$$\Phi(\Lambda;\tau) = \Phi_P(\Lambda;\tau) + \sigma\Phi_{NP}(\Lambda;\tau)$$

$$\Phi_{NP}(\Lambda;\tau) = \sum_{k=-M}^{\infty} d_k \Lambda^{-\frac{k+1}{2}} D_\Lambda\left(\frac{k+1}{2};\tau\right)$$

$$D_\Lambda(s;\tau) = \sum_{(m,n)\neq(0,0)} \exp\left(-4\sqrt{\Lambda Y_{mn}(\tau)}\right)(16Y_{mn}(\tau))^{-s}$$

$$(\mathcal{S}_+ - \mathcal{S}_-)[\Phi_P] := \lim_{\theta\to 0^+}[\mathcal{S}_{+\theta} - \mathcal{S}_{-\theta}][\Phi_P](\Lambda;\tau) = \int_\gamma \mathcal{E}(\sqrt{\Lambda}t;\tau)\mathcal{B}[\Phi_P](t)\mathrm{d}t$$

$$(\mathcal{S}_+ - \mathcal{S}_-)[\Phi_P](\Lambda;\tau) = -2i\mathcal{S}_0[\Phi_{NP}](\Lambda;\tau)$$

$$\mathcal{S}_\theta[\Phi_{NP}](\Lambda;\tau) := \sum_{k=-M}^{-1} d_k \Lambda^{-\frac{k+1}{2}} D_\Lambda\left(\frac{k+1}{2};\tau\right) + \int_0^{e^{i\theta}\infty} \mathcal{E}(\sqrt{\Lambda}(t+1);\tau)\mathcal{B}[\Phi_{NP}](t)\mathrm{d}t$$

$$\mathcal{B}[\Phi_{NP}](t) := \sum_{k=0}^{\infty} \frac{d_k}{\Gamma(k+1)} t^k$$

$$\mathcal{S}_+[\Phi_P + \sigma_+\Phi_{NP}](\Lambda;\tau) = \mathcal{S}_-[\Phi_P + \sigma_-\Phi_{NP}](\Lambda;\tau)$$

$$M_N(s) = \frac{N(N-1)}{4} \cdot \frac{\pi s(1-s)(2s-1)^2}{\sin(\pi s)}\ {}_3F_2(2-N, s, 1-s; 3,2 \mid 1)$$

$$F_{r,N}(s) = \frac{1}{s(1-s)}\left[1 - {}_3F_2\left(-r, s, 1-s; 1, \frac{N^2-1}{2}\middle| 1\right)\right].$$

$$M(z;s) := \sum_{N=2}^{\infty} M_N(s) z^{N-1} = -\frac{\pi s(1-s)(2s-1)^2}{2\sin(\pi s)} \cdot \frac{z}{(z-1)^3}\ {}_2F_1\left(s, 1-s; 2\middle|\frac{z}{z-1}\right)$$

$$M_N(s) = \oint_{|z|<1} \frac{M(z;s)}{z^N}\frac{\mathrm{d}z}{2\pi i}$$

$$M_N(s) = \frac{\pi s(1-s)(2s-1)^2}{2\sin(\pi s)} \oint_{|z|>1} \frac{z^N}{(z-1)^3}\ {}_2F_1\left(s, 1-s; 2\middle|\frac{1}{1-z}\right)\frac{\mathrm{d}z}{2\pi i}$$

$${}_2F_1\left(s, 1-s; 2\middle|\frac{1}{1-z}\right) = -\frac{4^{-s}\Gamma(s-1)\tan(\pi s)}{\sqrt{\pi}\Gamma\left(s+\frac{1}{2}\right)}(z-1)^s\ {}_2F_1(s-1, s; 2s \mid 1-z) + (s \leftrightarrow 1-s)$$

$$M_N(s) = \frac{4^{-s}(2s-1)\sqrt{\pi}\Gamma(s+1)}{\Gamma\left(s-\frac{1}{2}\right)\cos(\pi s)} N^{2-s} \int_\gamma z^{s-3}\left(1+\frac{z}{N}\right)^N {}_2F_1\left(s-1, s; 2s\middle|-\frac{z}{N}\right)\frac{\mathrm{d}z}{2\pi i} + (s \leftrightarrow 1-s)$$

$$z^{s-3}\left(1+\frac{z}{N}\right)_2^N F_1\left(s-1, s; 2s\middle|-\frac{z}{N}\right) = e^z z^{s-3}\left[1 - \frac{z(z+s-1)}{2N} + O(N^{-2})\right].$$

$$\frac{1}{\Gamma(a)} = \int_\gamma e^z z^{-a}\frac{\mathrm{d}z}{2\pi i}$$

$$M_N(s)=\sum_{g=0}^{\infty} N^{2-2g}\left[N^{-s}\mathcal{M}^{(g)}(s)+N^{s-1}\mathcal{M}^{(g)}(1-s)\right]$$

$$\mathcal{M}^{(0)}(s) = \frac{4^{-s}(2s-1)\Gamma(s-2)\Gamma(s+1)}{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}\tan(\pi s)$$

$$\mathcal{M}^{(1)}(s) = -\frac{4^{-s-1}(s+5)(2s-1)\Gamma(s+1)^2}{6\sqrt{\pi}(2s+1)\Gamma\left(s-\frac{1}{2}\right)}\tan(\pi s)$$

$$F_N(y;s):=\sum_{r=1}^{\infty} F_{r,N}(s)y^r=\frac{1}{s(1-s)(y-1)}\left[{}_2F_1\left(s,1-s;\frac{N^2-1}{2}\middle|\frac{y}{y-1}\right)-1\right].$$

$$F_{r,N}(s)=\oint_{|y|<1}\frac{F_N(y;s)}{y^{r+1}}\frac{\mathrm{d}y}{2\pi i}$$

$${}_2F_1\left(s,1-s;\frac{N^2-1}{2}\middle|\frac{1}{1-y}\right)=\frac{4^{-s}\sqrt{\pi}\Gamma\left(\frac{N^2-1}{2}\right)(y-1)^s}{\Gamma\left(\frac{N^2-1-2s}{2}\right)\Gamma\left(s+\frac{1}{2}\right)\cos(\pi s)}\,{}_2F_1\left(s,\frac{3-N^2}{2}+s;2s\mid 1-y\right)$$
$$+(s\leftrightarrow 1-s)$$

$$F_{r,N}(s)=\frac{1}{s(1-s)}+\left[\frac{\Gamma\left(\frac{N^2-1}{2}\right)\Gamma(s-1)}{\Gamma\left(\frac{N^2-1-2s}{2}\right)\Gamma(2s+1)\cos(\pi s)}\times\right.$$
$$\left.r^{-s}\int_{\gamma} y^{s-1}\left(1+\frac{y}{r}\right)^r {}_2F_1\left(s,s-\frac{N^2-3}{2};2s\middle|-\frac{y}{r}\right)\frac{\mathrm{d}y}{2\pi i}+(s\leftrightarrow 1-s)\right]$$

$$y^{s-1}\left(1+\frac{y}{r}\right)^r{}_2F_1\left(s,s-\frac{N^2-3}{2};2s\middle|-\frac{y}{r}\right)=e^y y^{s-1}\left[1-\frac{y(2s+2y+3-N^2)}{4r}+O(r^{-2})\right],$$

$$F_{r,N}(s)=\frac{1}{s(1-s)}+\sum_{g=0}^{\infty} r^{-g}\left[r^{-s}\mathcal{F}_N^{(g)}(s)+r^{s-1}\mathcal{F}_N^{(g)}(1-s)\right]$$

$$\mathcal{F}_N^{(0)}(s)=\frac{2^{-2s}\Gamma\left(\frac{N^2-1}{2}\right)\Gamma(s-1)}{\sqrt{\pi}s\Gamma\left(\frac{N^2-1}{2}-s\right)\Gamma\left(s+\frac{1}{2}\right)}\tan(\pi s)$$

$$\mathcal{F}_N^{(1)}(s)=\frac{2^{-1-2s}\Gamma\left(\frac{N^2+1}{2}\right)\Gamma(s-1)}{\sqrt{\pi}\Gamma\left(\frac{N^2-1}{2}-s\right)\Gamma\left(s+\frac{1}{2}\right)}\tan(\pi s)$$

$$\mathcal{C}_{2r,N}(\tau)\overset{r\to\infty}{=}\langle\mathcal{C}_{2r,N}\rangle+\int_{\mathrm{Re}(s)=\frac{1}{2}}\frac{M_N(s)}{s(1-s)}E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$
$$+2\sum_{g=0}^{\infty} r^{-g}\int_{\mathrm{Re}(s)=\frac{1}{2}} r^{-s}\mathcal{F}_N^{(g)}(s)M_N(s)E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}_{2r,N}(\tau) = \langle \mathcal{C}_{2r,N} \rangle + H_N(\tau) + 2\sum_{g=0}^{\infty} r^{-g} \int_{\mathrm{Re}(s)=\frac{1}{2}} r^{-s}\mathcal{F}_N^{(g)}(s) M_N(s) E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$H_N(\tau) := \int_{\mathrm{Re}(s)=\frac{1}{2}} \frac{M_N(s)}{s(1-s)} E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\langle \mathcal{C}_{2r,N} \rangle = \lim_{s\to 1} c_{2r,N}(s) = \lim_{s\to 1}[M_N(s) F_{r,N}(s)]$$

$$\lim_{s\to 1} M_N(s) = \frac{N(N-1)}{4}$$

$$\lim_{s\to 1} F_{r,N}(s) = \psi\left(\frac{N^2-1}{2}+r\right) - \psi\left(\frac{N^2-1}{2}\right)$$

$$\mathcal{C}_{2r,N}(\tau) = \frac{N(N-1)}{4}\left[F_{r,N}(1) - \sum_{g=0}^{\infty} r^{-g-1}\mathcal{F}_N^{(g)}(1)\right] + H_N(\tau) + \sum_{g=0}^{\infty} r^{-g}\mathcal{C}_N^{(g)}(r;\tau)$$

$$\mathcal{C}_N^{(g)}(r;\tau) := 2\int_{\mathrm{Re}(s)=1+\epsilon} r^{-s}\mathcal{F}_N^{(g)}(s) M_N(s) E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}_N^{(0)}(r;\tau) \sim \mathcal{C}_{N,P}^{(0)}(r;\tau) := (N-1)N\Gamma\left(\frac{N^2-1}{2}\right)\sin\left(\frac{\pi N^2}{2}\right)\times$$

$$\sum_{k=0}^{\infty} \frac{(k+1)\,{}_3F_2\left(-k-\frac{1}{2}, k+\frac{3}{2}, 2-N; 2,3 \mid 1\right)}{4^{k+1}\pi^{\frac{3}{2}}} \cdot \frac{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+3-\frac{N^2}{2}\right)}{\Gamma(k+1)} r^{-k-\frac{3}{2}} E^*\left(k+\frac{3}{2};\tau\right)$$

$$\mathcal{C}_3^{(0)}(r;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \frac{3\times 2^{1-2s}\sqrt{\pi}(2s-1)[s(1-s)-6]\Gamma(s)}{\Gamma(4-s)\Gamma\left(s-\frac{1}{2}\right)\cos(\pi s)} r^{-s} E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}_3^{(0)}(r;\tau) \sim \mathcal{C}_{3,P}^{(0)}(r;\tau) = \sum_{k=0}^{\infty} \frac{3(k+1)[27+4k(k+2)]}{2^{2k+3}\pi^{\frac{3}{2}}} \cdot \frac{\Gamma\left(k-\frac{3}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}{\Gamma(1+k)} r^{-k-\frac{3}{2}} E^*\left(k+\frac{3}{2};\tau\right)$$

$$\mathcal{B}\left[\mathcal{C}_{3,P}^{(0)}\right](t) = 18t^2\left[3\,{}_2F_1\left(-\frac{3}{2},\frac{3}{2};1 \mid t^2\right) - t^2\,{}_2F_1\left(-\frac{1}{2},\frac{5}{2};1 \mid t^2\right) - 2t^2\,{}_2F_1\left(-\frac{1}{2},\frac{5}{2};2 \mid t^2\right)\right]$$

$$\mathcal{B}\left[\mathcal{C}_{3,P}^{(0)}\right](t) \sim -\frac{(-6)}{\pi(1-t)} + \mathcal{B}\left[\mathcal{C}_{3,NP}^{(0)}\right](t-1)\frac{\log(1-t)}{\pi} + \mathrm{reg}(t-1)$$

$$\mathcal{B}\left[\mathcal{C}_{3,NP}^{(0)}\right](t) = -3(t+1)^2\left[8\,{}_2F_1\left(-\frac{3}{2},\frac{3}{2};1 \mid -t(t+2)\right) + 9(1+4t(t+2))\,{}_2F_1\left(-\frac{1}{2},\frac{5}{2};2 \mid -t(t+2)\right)\right]$$

$$\mathcal{C}_{3,NP}^{(0)}(r;\tau) = \sum_{k=-1}^{\infty} d_{3,k}^{(0)} r^{-\frac{k+1}{2}} D_r\left(\frac{k+1}{2};\tau\right)$$

$$\mathcal{B}\left[\mathcal{C}_{3,NP}^{(0)}\right](t) = \sum_{k=0}^{\infty} \frac{d_{3,k}^{(0)}}{k!} t^k = -51 - \frac{1839}{4}t - \frac{20211}{8}\frac{t^2}{2} + \cdots$$

$$\mathcal{C}_2^{(0)}(r;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} (4r)^{-s}(2s-1)\Gamma(s)E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}_{2+\epsilon}^{(0)}(r;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \frac{\pi s(1-s)(2s-1)^2}{\sin(\pi s)} r^{-s}\mathcal{F}_{2+\epsilon}^{(0)}(s)E^{\star}(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}_{2+\epsilon}^{(0)}(r;\tau) \sim \mathcal{C}_{2+\epsilon,P}^{(0)}(r;\tau) = \frac{2}{\pi}\sin(2\pi\epsilon)\sum_{k=0}^{\infty} 2^{-3-2k}(k+1)\Gamma\left(k+\frac{3}{2}\right)r^{-k-\frac{3}{2}}E^*\left(k+\frac{3}{2};\tau\right)$$

$$\mathcal{B}\left[\mathcal{C}_{2+\epsilon,P}^{(0)}\right](t) = \sin(2\pi\epsilon)t^2(1-t^2)^{-\frac{3}{2}}$$

$$\begin{aligned}(\mathcal{S}_+ - \mathcal{S}_-)\left[\mathcal{C}_{2+\epsilon,P}^{(0)}\right](r;\tau) &= i\sin(2\pi\epsilon)\int_1^{\infty} t^2(t^2-1)^{-\frac{3}{2}}\mathcal{E}(\sqrt{r}t;\tau)\mathrm{d}t \\ &= i\sin(2\pi\epsilon)\mathcal{S}_0\left[\mathcal{C}_{2+\epsilon,NP}^{(0)}\right](r;\tau)\end{aligned}$$

$$\mathcal{C}_{2+\epsilon}^{(0)}(r;\tau) = \mathcal{C}_{2+\epsilon,P}^{(0)}(r;\tau) + \sigma(\epsilon)\mathcal{C}_{2+\epsilon,NP}^{(0)}(r;\tau)$$

$$\sigma(\epsilon) = e^{\pm 2i\pi\epsilon}$$

$$\begin{aligned}\mathcal{C}_2^{(0)}(r;\tau) &= \int_1^{\infty} t^2(t^2-1)^{-\frac{3}{2}}\mathcal{E}(\sqrt{r}t;\tau)\mathrm{d}t = \int_0^{\infty} (t+1)^2[t(t+2)]^{-\frac{3}{2}}\mathcal{E}(\sqrt{r}(t+1);\tau)\mathrm{d}t \\ &= \sqrt{\frac{\pi}{2}}r^{\frac{1}{4}}D_r\left(-\frac{1}{4};\tau\right) - \frac{5}{8}\sqrt{\frac{\pi}{2}}r^{-\frac{1}{4}}D_r\left(\frac{1}{4};\tau\right) + \cdots\end{aligned}$$

$$F_{r,N}(s) = -\sum_{k=1}^{\infty}\frac{(s)_k(1-s)_k}{\left(\frac{N^2-1}{2}\right)_k k!\,s(1-s)}\oint_{|y-1|<1}\frac{y^{k-r-1}}{(y-1)^{k+1}}\frac{\mathrm{d}y}{2\pi i}$$

$$\begin{aligned}F_{r,N}(s) &= \sum_{k=1}^{\infty}\frac{(-r)_k(2-s)_{k-1}(s+1)_{k-1}}{(k!)^2\left(\frac{a}{2}\right)_k} \\ &= \frac{2r}{a} + \frac{(r-1)r(s-2)(s+1)}{a(a+2)} + \frac{2(r-2)(r-1)r(s-3)(s-2)(s+1)(s+2)}{9a(a+2)(a+4)} + \cdots\end{aligned}$$

$$F_{\frac{\alpha N}{2},N}(s) = \frac{\alpha}{N} + \frac{\alpha^2(s+1)(s-2)}{4N^2} + \frac{[\alpha^3(s-3)(s-2)(s+1)(s+2) - 18\alpha(s^2-s-4)]}{36N^3} + O(N^{-3})$$

$$c_{\alpha N,N}(s) = \sum_{g=0}^{\infty} N^{1-g}\left[N^{-s}\mathcal{K}^{(g)}(s;\alpha) + N^{s-1}\mathcal{K}^{(g)}(1-s;\alpha)\right]$$

$$\begin{aligned}\mathcal{K}^{(0)}(s;\alpha) &= \alpha\mathcal{M}^{(0)}(s) \\ \mathcal{K}^{(1)}(s;\alpha) &= \frac{(s-2)(s+1)\alpha^2}{4}\mathcal{M}^{(0)}(s) \\ \mathcal{K}^{(2)}(s;\alpha) &= \alpha\mathcal{M}^{(1)}(s) + \frac{[\alpha^3(s-3)(s-2)(s+1)(s+2) - 18\alpha(s^2-s-4)]}{36}\mathcal{M}^{(0)}(s)\end{aligned}$$

$$F_{r,N}(s) = \frac{1}{s(1-s)} \oint_{|y|>r} \frac{\left(\frac{y}{r}+1\right)^r}{y} \left[1 - {}_2F_1\left(s, 1-s; \frac{N^2-1}{2} \middle| -\frac{r}{y}\right)\right] \frac{\mathrm{d}y}{2\pi i}$$

$${}_2F_1\left(s, 1-s; \frac{N^2-1}{2} \middle| -\frac{\alpha N^2}{2y}\right) = \frac{\Gamma\left(\frac{N^2-1}{2}\right)}{\Gamma(s)\Gamma(1-s)} \int_{-i\infty}^{i\infty} \frac{\Gamma(s+t)\Gamma(1-s+t)\Gamma(-t)}{\Gamma\left(\frac{N^2-1}{2}+t\right)} \left(\frac{\alpha N^2}{2y}\right)^t \frac{\mathrm{d}t}{2\pi i}$$

$$F_{\frac{\alpha N^2}{2},N}(s) =$$

$$\frac{1}{s(1-s)} \left[1 - \int_{-i\infty}^{i\infty} \frac{\Gamma(s+t)\Gamma(1-s+t)\Gamma(-t)}{\Gamma(s)\Gamma(1-s)} \cdot \frac{\Gamma\left(\frac{N^2-1}{2}\right)}{\Gamma\left(\frac{N^2-1}{2}+t\right)} \left(\frac{\alpha N^2}{2}\right)^t \left(\int_\gamma \frac{\left(1+\frac{2y}{\alpha N^2}\right)^{\frac{\alpha N^2}{2}}}{y^{t+1}} \frac{\mathrm{d}y}{2\pi i}\right) \frac{\mathrm{d}t}{2\pi i}\right]$$

$$\frac{\left(1+\frac{2y}{\alpha N^2}\right)^{\frac{\alpha N^2}{2}}}{y^{t+1}} = e^y y^{-t-1} \left[1 - \frac{y^2}{\alpha N^2} + \frac{y^3(3y+8)}{6\alpha^2 N^4} + O(N^{-6})\right]$$

$$\frac{\Gamma\left(\frac{N^2-1}{2}\right)}{\Gamma\left(\frac{N^2-1}{2}+t\right)} \frac{N^{2t}}{2^t} = 1 + \frac{(2-t)t}{N^2} + O(N^{-4})$$

$$F_{\frac{\alpha N^2}{2},N}(s) = \sum_{g=0}^{\infty} N^{-2g} \mathcal{G}^{(g)}(s;\alpha)$$

$$\mathcal{G}^{(0)}(s;\alpha) = \frac{1 - {}_2F_1(s, 1-s; 1 \mid -\alpha)}{s(1-s)}$$
$$\mathcal{G}^{(1)}(s;\alpha) = (2\alpha+1)\, {}_2F_1(2-s, s+1; 2; -\alpha) - (\alpha+1)\, {}_2F_1(2-s, s+1; 1; -\alpha)$$

$$\mathcal{G}^{(g)}(s;\alpha) = \mathcal{G}^{(g)}(1-s;\alpha)$$

$$c_{\alpha N^2,N}(s) = \sum_{g=0}^{\infty} N^{2-2g} \left[N^{-s} \mathcal{A}^{(g)}(s;\alpha) + N^{s-1} \mathcal{A}^{(g)}(1-s;\alpha)\right]$$

$$\mathcal{A}^{(g)}(s;\alpha) = \sum_{g_1=0}^{g} \mathcal{M}^{(g_1)}(s) \mathcal{G}^{(g-g_1)}(s;\alpha)$$

$$\mathcal{A}^{(0)}(s;\alpha) = \mathcal{M}^{(0)}(s) \mathcal{G}^{(0)}(s;\alpha)$$
$$\mathcal{A}^{(1)}(s;\alpha) = \mathcal{M}^{(0)}(s) \mathcal{G}^{(1)}(s;\alpha) + \mathcal{M}^{(1)}(s) \mathcal{G}^{(0)}(s;\alpha)$$

$$\mathcal{A}^{(0)}(s;\alpha) = \frac{(2s-1)\Gamma(s-2)\Gamma(s-1)}{2^{2s}\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)} \tan(\pi s) \left[{}_2F_1(s, 1-s; 1; -\alpha) - 1\right]$$

$$\mathcal{C}_{\alpha N^2,N}(\tau) = \langle \mathcal{C}_{\alpha N^2,N} \rangle + \sum_{g=0}^{\infty} N^{2-2g} 2 \int_{\mathrm{Re}(s)=\frac{1}{2}} \mathcal{A}^{(g)}(s;\alpha) N^{-s} E^{\star}(s;\tau) \frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}_{\alpha N^2,N}(\tau) = \langle \mathcal{C}_{\alpha N^2,N} \rangle - \sum_{g=0}^{\infty} \frac{\mathcal{A}^{(g)}(1;\alpha)}{N^{2g-1}} + \sum_{g=0}^{\infty} N^{2-2g} \mathcal{C}_{\alpha}^{(g)}(N;\tau)$$

$$\mathcal{C}_{\alpha}^{(g)}(N;\tau) := 2 \int_{\mathrm{Re}(s)=1+\epsilon} \mathcal{A}^{(g)}(s;\alpha) N^{-s} E^{*}(s;\tau) \frac{\mathrm{d}s}{2\pi i}$$

$$\begin{aligned} \langle \mathcal{C}_{\alpha N^2,N} \rangle &= \lim_{s\to 1} \left[\mathcal{M}_N(s) F_{\frac{\alpha N^2}{2},N}(s) \right] = \frac{N(N-1)}{4} \left[\psi\left(\frac{N^2(\alpha+1)-1}{2} \right) - \psi\left(\frac{N^2-1}{2} \right) \right] \\ &\sim \frac{N(N-1)}{4} \log(1+\alpha) + \frac{(N-1)}{2N} \frac{\alpha}{(1+\alpha)} + O(N^{-2}) \end{aligned}$$

$$\begin{aligned} \sum_{g=0}^{\infty} \frac{\mathcal{A}^{(g)}(1;\alpha)}{N^{2g-1}} &= \mathcal{M}^{(0)}(1) \sum_{g=0}^{\infty} \frac{\mathcal{G}^{(g)}(1;\alpha)}{N^{2g-1}} = N \mathcal{M}^{(0)}(1) F_{\frac{\alpha N^2}{2},N}(1) \\ &= -\frac{N}{4} \left[\psi\left(\frac{N^2(\alpha+1)-1}{2} \right) - \psi\left(\frac{N^2-1}{2} \right) \right] \end{aligned}$$

$$\begin{aligned} \langle \mathcal{C}_{\alpha N^2,N} \rangle - \sum_{g=0}^{\infty} \frac{\mathcal{A}^{(g)}(1;\alpha)}{N^{2g-1}} &= \frac{N^2}{4} \left[\psi\left(\frac{N^2(\alpha+1)-1}{2} \right) - \psi\left(\frac{N^2-1}{2} \right) \right] \\ &\sim \frac{N^2}{4} \log(\alpha+1) + \frac{\alpha}{2(\alpha+1)} + O(N^{-2}) \end{aligned}$$

$$\mathcal{C}_{\alpha}^{(0)}(N;\tau) \sim \mathcal{C}_{\alpha}^{(0)}(N;\tau)_P =$$
$$\sum_{k=0}^{\infty} \frac{(k+1)\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{2^{2k+1}\pi^{\frac{3}{2}}\Gamma(k+1)} \left[{}_2F_1\left(k+\frac{3}{2}, -k-\frac{1}{2}; 1 \mid -\alpha\right) - 1 \right] N^{-k-\frac{3}{2}} E^{*}\left(k+\frac{3}{2};\tau\right)$$

$$\mathcal{B}\left[\mathcal{C}_{\alpha,P}^{(0)}\right](t) = \sum_{k=0}^{\infty} \frac{4\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\pi\Gamma(k+1)^2} \left[{}_2F_1\left(k+\frac{3}{2}, -k-\frac{1}{2}; 1 \mid -\alpha\right) - 1 \right] t^{2k+2}$$

$$\sum_{k=0}^{\infty} \frac{4\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\pi\Gamma(k+1)^2} t^{2k+2} = -\frac{16t^2 E(t^2)}{\pi}$$

$$\begin{aligned} L_- &= \sqrt{\alpha+1} - \sqrt{\alpha} \\ L_+ &= \frac{1}{L_-} = \sqrt{\alpha+1} + \sqrt{\alpha} \end{aligned}$$

$$\mathcal{S}_{\theta}\left[\mathcal{C}_{\alpha,P}^{(0)}\right](N;\tau) = \int_0^{e^{i\theta}\infty} \mathcal{E}(\sqrt{N}t;\tau) \mathcal{B}\left[\mathcal{C}_{\alpha,P}^{(0)}\right](t)\mathrm{d}t$$

$$\mathcal{S}_{0^+}\left[\mathcal{C}_{\alpha,P}^{(0)}\right](N;\tau) - \mathcal{S}_{0^+}\left[\mathcal{C}_{\alpha,P}^{(0)}\right](N;\tau) \neq 0$$

$$\mathcal{C}^{(0)}_{\alpha,NP}(N;\tau) = \mathcal{C}^{(0),\mathrm{I}}_{\alpha,NP}(N;\tau) + \mathcal{C}^{(0),-}_{\alpha,NP}(N;\tau) + \mathcal{C}^{(0),+}_{\alpha,NP}(N;\tau)$$

$$\mathcal{C}^{(0),i}_{\alpha,NP}(N;\tau) = \sum_{k=0}^{\infty} d^i_k \left(L_i^2 N\right)^{-\frac{k+1}{2}} D_{L_i^2 N}\left(\frac{k+1}{2};\tau\right), \text{ with } i \in \{\mathrm{I},-,+\}$$

$$\mathcal{C}^{(0),\mathrm{I}}_{\alpha,NP}(N;\tau) \sim \exp\left(-4\sqrt{\pi N}\frac{|n\tau+m|}{\sqrt{\tau_2}}\right)$$
$$\mathcal{C}^{(0),\pm}_{\alpha,NP}(N;\tau) \sim \exp\left(-4\sqrt{\pi N}(\sqrt{\alpha+1} \pm \sqrt{\alpha})\frac{|n\tau+m|}{\sqrt{\tau_2}}\right)$$

$$e^{-4(\sqrt{\alpha+1}+\sqrt{\alpha})|n\tau+m|\sqrt{\frac{\pi N}{\tau_2}}} \ll e^{-4|n\tau+m|\sqrt{\frac{\pi N}{\tau_2}}} \ll e^{-4(\sqrt{\alpha+1}-\sqrt{\alpha})|n\tau+m|\sqrt{\frac{\pi N}{\tau_2}}}$$

$$e^{-2|m|(\sqrt{\alpha+1}+\sqrt{\alpha})\sqrt{\lambda}} \ll e^{-2|m|\sqrt{\lambda}} \ll e^{-2|m|(\sqrt{\alpha+1}-\sqrt{\alpha})\sqrt{\lambda}}$$

$$c_{2r,N}(s) = \frac{1}{s(1-s)}\sum_{g=0}^{\infty} N^{2-2g-s}\mathcal{M}^{(g)}(s)$$
$$+\sum_{g=0}^{\infty}\sum_{\ell=0}^{\infty} N^{2-2g}r^{-\ell}\left[(Nr)^{-s}\mathcal{M}^{(g)}(s)\mathcal{F}^{(\ell)}_N(s) + N^{s-1}r^{-s}\mathcal{M}^{(g)}(1-s)\mathcal{F}^{(\ell)}_N(s)\right] + (s \leftrightarrow 1-s)$$

$$\mathcal{F}^{(\ell)}_N(s) = \sum_{k=0}^{\infty} N^{2\ell+2s-2k}\mathcal{F}^{(\ell,k)}(s)$$

$$\mathcal{F}^{(0,0)}(s) = \frac{8^{-s}\Gamma(s-1)}{\sqrt{\pi}s\Gamma\left(s+\frac{1}{2}\right)}\tan(\pi s)$$

$$\mathcal{F}^{(0,1)}(s) = -s(s+2)\mathcal{F}^{(0,0)}(s), \mathcal{F}^{(0,1)}(s) = \frac{s}{4}\mathcal{F}^{(0,0)}(s), \mathcal{F}^{(1,1)}(s) = -\frac{s(s+1)^2}{4}\mathcal{F}^{(0,0)}(s)$$

$$c_{2r,N}(s) = \frac{1}{s(1-s)}\sum_{g=0}^{\infty} N^{2-2g-s}\mathcal{M}^{(g)}(s)$$
$$+\sum_{g,\ell,k=0}^{\infty} N^{2-2g-2k-\ell(\gamma-2)}\left(\frac{\alpha}{2}\right)^{-\ell}\left[\left(\frac{\alpha N^{\gamma-1}}{2}\right)^{-s}\mathcal{M}^{(g)}(s)\mathcal{F}^{(\ell,k)}(s) + \frac{1}{N}\left(\frac{\alpha N^{\gamma-3}}{2}\right)^{-s}\mathcal{M}^{(g)}(1-s)\mathcal{F}^{(\ell,k)}(s)\right]$$
$$+(s \leftrightarrow 1-s)$$

$$\mathcal{C}^{(g,\ell,k)}_1(N,\alpha,\gamma;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} 2^{3s}\mathcal{M}^{(g)}(s)\mathcal{F}^{(\ell,k)}(s)\Lambda_1^{-s}E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\Lambda_1 := 4\alpha N^{\gamma-1} \gg 1$$

$$\mathcal{C}^{(0,0,0)}_1(\Lambda_1;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \frac{2^{1-2s}\Gamma(s-2)\Gamma(s-1)\Gamma(s)\tan(\pi s)^2}{\pi\Gamma\left(s-\frac{1}{2}\right)^2}\Lambda_1^{-s}E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\tilde{\mathcal{C}}^{(0,0,0)}_{1,NP}(\Lambda_1;\tau) \sim \exp\left(-4\sqrt{\Lambda_1 Y_{mn}(\tau)}\right) = \exp\left(-8N^{\frac{\gamma-1}{2}}\sqrt{\alpha Y_{mn}(\tau)}\right),$$

$$\mathcal{C}_2^{(g,\ell,k)}(N,\alpha,\gamma;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \mathcal{M}^{(g)}(1-s)\mathcal{F}^{(\ell,k)}(s)\left(\frac{\alpha N^{\gamma-3}}{2}\right)^{-s} E^{\star}(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}_2^{(0,0,0)}(N,\alpha,\gamma;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \frac{(1-2s)\sec(\pi s)}{4s\Gamma(s+2)}\left(\frac{\alpha N^{\gamma-3}}{2}\right)^{-s} E^{*}(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\tilde{\mathcal{C}}_2^{(g,\ell,k)}(N,\alpha,\gamma;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} 2^{-3s}\mathcal{M}^{(g)}(s)\mathcal{F}^{(\ell,k)}(1-s)\Lambda_2^{-s}E^{*}(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\Lambda_2 := \frac{N^{3-\gamma}}{4\alpha} \gg 1$$

$$\tilde{\mathcal{C}}_2^{(0,0,0)}(\Lambda_2;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \frac{2^{-2s-3}(2s-1)\Gamma(s-2)\tan(\pi s)}{\pi(s-1)}\Lambda_2^{-s}E^{*}(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\tilde{\mathcal{C}}_{2,NP}^{(0,0,0)}(\Lambda_2;\tau) \sim \exp\left(-4\sqrt{\Lambda_2 Y_{mn}(\tau)}\right) = \exp\left(-2N^{\frac{3-\gamma}{2}}\sqrt{\frac{Y_{mn}(\tau)}{\alpha}}\right),$$

$$|\alpha\rangle = O_H(0;\alpha)|0\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha}{\sqrt{2}}\right)^n \lim_{x\to 0}\left[O_L^n(0)|0\rangle\right]$$

$$\langle O_L(x_1)\bar{O}_L(x_2)O_H(0;\alpha)\bar{O}_H(\infty;\alpha)\rangle = \langle\alpha|O_L(x_1)\bar{O}_L(x_2)|\alpha\rangle$$

$$\mathcal{C}_{\mathrm{HHLL}}(N,\alpha;\tau) = 4(1-2\alpha^2)^{\frac{N^2-1}{2}}\sum_{r=1}^{\infty}\frac{r^2}{(r!)^2}\left(\frac{\alpha}{\sqrt{2}}\right)^{2r} R_{2r}(N)\mathcal{C}_{2r,N}(\tau)$$

$$\mathcal{C}_{\mathrm{HHLL}}(N,\alpha;\tau) = \langle\mathcal{C}_{\mathrm{HHLL}}\rangle + \int_{\mathrm{Re}(s)=\frac{1}{2}} c_{\mathrm{HHLL}}(N,\alpha;s)E^{*}(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$c_{\mathrm{HHLL}}(N,\alpha;s) = M_N(s)(1-2\alpha^2)^{\frac{N^2-1}{2}}\sum_{r=1}^{\infty}\frac{2^r\alpha^{2r}}{r!}\left(\frac{N^2-1}{2}\right)_r F_{r,N}(s)$$

$$F_{r,N}(s) = \frac{r!}{s(s-1)}\sum_{k=1}^{r}\frac{(s)_k(1-s)_k(-1)^k}{\left(\frac{N^2-1}{2}\right)_k (k!)^2(r-k)!}$$

$$c_{\mathrm{HHLL}}(N,\alpha;s) = \frac{M_N(s)}{s(1-s)}\left[1 - {}_2F_1\left(s,1-s;1\,\middle|\,\frac{2\alpha^2}{2\alpha^2-1}\right)\right].$$

$$\mathcal{C}_{\mathrm{HHLL}}(N,\alpha;\tau) = \langle\mathcal{C}_{\mathrm{HHLL}}\rangle + \log(1-2\alpha^2)\sum_{g=0}^{\infty} N^{1-2g}\mathcal{M}^{(g)}(1) + \sum_{g=0}^{\infty} N^{2-2g}\mathcal{C}_{\mathrm{HHLL}}^{(g)}(N,\alpha;\tau)$$

$$\mathcal{C}_{\mathrm{HHLL}}^{(g)}(N,\alpha;\tau) := 2\int_{\mathrm{Re}(s)=1+\epsilon}\frac{\mathcal{M}^{(g)}(s)}{s(1-s)}\left[1 - {}_2F_1\left(s,1-s;1\,\middle|\,\frac{2\alpha^2}{2\alpha^2-1}\right)\right]N^{-s}E^{*}(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\langle \mathcal{C}_{\mathrm{HHLL}} \rangle = \lim_{s\to 1}[c_{\mathrm{HHLL}}(N,\alpha;s)] = -\frac{N(N-1)}{4}\log(1-2\alpha^2)$$

$$\mathcal{M}^{(0)}(1) = -\frac{1}{4}, \mathcal{M}^{(g\geq 1)}(1) = 0$$

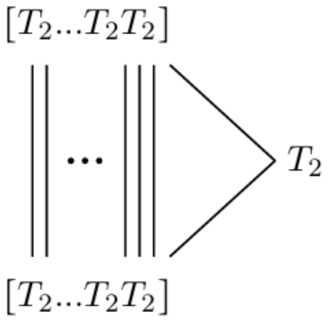
$$\mathcal{C}_{\mathrm{HHLL}}(N,\alpha;\tau) = -\frac{N^2}{4}\log(1-2\alpha^2) + \sum_{g=0}^{\infty} N^{2-2g}\mathcal{C}^{(g)}_{\mathrm{HHLL}}(N,\alpha;\tau)$$

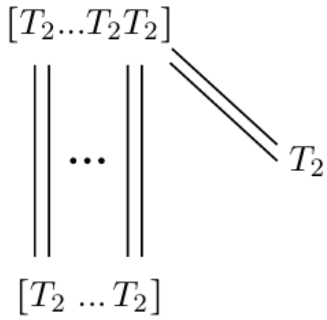
$$\mathcal{C}^{(0)}_{\mathrm{HHLL}}(N,\alpha;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \frac{(2s-1)\Gamma(s-2)\Gamma(s-1)}{2^{2s-1}\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}\tan(\pi s)\left[{}_2F_1\left(s,1-s;1\,\middle|\,\frac{2\alpha^2}{2\alpha^2-1}\right)-1\right]N^{-s}E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}^{(0)}_{\tilde{\alpha}}(N;\tau) := 2\int_{\mathrm{Re}(s)=1+\epsilon} \mathcal{A}^{(0)}(s;\tilde{\alpha})N^{-s}E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}^{(0)}_{\mathrm{HHLL}}(N,\alpha;\tau) = \mathcal{C}^{(0)}_{\tilde{\alpha}}(N;\tau) \text{ with } \tilde{\alpha} = \frac{2\alpha^2}{1-2\alpha^2}$$

$$-\frac{N^2}{4}\log(1-2\alpha^2) = \frac{N^2}{4}\log(\tilde{\alpha}+1) \text{ with } \tilde{\alpha} = \frac{2\alpha^2}{1-2\alpha^2}$$





$$\frac{\langle \hat{\mathcal{O}}_p(Y_1,x_1)\hat{\mathcal{O}}_p(Y_2,x_2)T_2(Y_3,x_3)\rangle}{\langle \hat{\mathcal{O}}_p(Y_1,x_1)\hat{\mathcal{O}}_p(Y_2,x_2)\rangle} = (N^2-1)\tilde{\alpha}\frac{d_{13}d_{23}}{d_{12}},$$

$$\langle \hat{\mathcal{O}}_p(Y_1,x_1)\hat{\mathcal{O}}_p(Y_2,x_2)\rangle = \frac{2^p\left(\frac{p}{2}\right)!}{p^2}\left(\frac{N^2-1}{2}\right)_{\frac{p}{2}} d_{12}^p$$

$$\frac{\langle \mathcal{O}_H(Y_1,x_1)\mathcal{O}_H(Y_2,x_2)T_2(Y_3,x_3)\rangle}{\langle \mathcal{O}_H(Y_1,x_1)\mathcal{O}_H(Y_2,x_2)\rangle} = (N^2-1)\left[\frac{2\alpha^2}{(1-2\alpha^2)}\cdot\frac{d_{13}d_{23}}{d_{12}} + \frac{1}{\sqrt{2}}\frac{\alpha}{(1-2\alpha^2)}\cdot\frac{d_{13}^2+d_{23}^2}{d_{12}}\right].$$

$$\frac{\langle \hat{\mathcal{O}}_p\hat{\mathcal{O}}_p T_2\rangle}{\langle \hat{\mathcal{O}}_p\hat{\mathcal{O}}_p\rangle} = \frac{p}{2}\frac{\langle T_2T_2T_2\rangle}{\langle T_2T_2\rangle}(1+O(N^{-2})),$$

$$\frac{\left\langle \hat{\mathcal{O}}_p \hat{\mathcal{O}}_p \left[T_{q_1} \partial^\ell \, \square^{\frac{1}{2}(t-q_1-q_2)} \, T_{q_2} \right] \right\rangle}{\left\langle \hat{\mathcal{O}}_p \hat{\mathcal{O}}_p \right\rangle} = \frac{p}{2} \frac{\left\langle T_2 T_2 \left[T_{q_1} \partial^\ell \, \square^{\frac{1}{2}(t-q_1-q_2)} \, T_{q_2} \right] \right\rangle}{\langle T_2 T_2 \rangle} (1 + O(N^{-2}))$$

$$\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_H \mathcal{O}_H \rangle \sim \langle \mathcal{O}_2 \mathcal{O}_2 T_2 \rangle \langle T_2 \mathcal{O}_H \mathcal{O}_H \rangle$$

$$\mathcal{C}_{\mathrm{HHLL}}^{(g>0)}(N, \alpha; \tau) \neq \mathcal{C}_{\tilde{\alpha}}^{(g>0)}(N; \tau) \ \text{ with } \ \tilde{\alpha} = \frac{2\alpha^2}{1 - 2\alpha^2}$$

$$E^*(s; \tau) \to \xi(2s)(4\pi N)^s \lambda^{-s} + \xi(2s-1)(4\pi N)^{1-s} \lambda^{s-1}$$

$$\mathcal{C}_{\mathrm{HHLL}}(N, \alpha; \lambda) \sim -\frac{N^2}{4} \log(1 - 2\alpha^2) + \sum_{g=0}^{\infty} N^{2-2g} \left[\sum_{k=0}^{\infty} \tilde{\mathcal{B}}^{(g)}\left(k + \frac{3}{2}; \alpha\right) \xi(2k+3) \left(\frac{\lambda}{4\pi}\right)^{-k-\frac{3}{2}} \right.$$
$$\left. + \sum_{k=0}^{g-1} \tilde{\mathcal{B}}^{(g-k-1)}\left(k + \frac{3}{2}; \alpha\right) \xi(2k+2) \left(\frac{\lambda}{4\pi}\right)^{k+\frac{1}{2}} \right]$$

$$\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle - \frac{N^2}{4} \log(1 - 2\alpha^2)$$

$$\sum_{k=0}^{\infty} \tilde{\mathcal{B}}^{(g)}\left(k + \frac{3}{2}; \alpha\right) \xi(2k+3) \left(\frac{\lambda}{4\pi}\right)^{-k-\frac{3}{2}}$$

$$\sum_{k=0}^{g-1} \tilde{\mathcal{B}}^{(g-k-1)}\left(k + \frac{3}{2}; \alpha\right) \xi(2k+2) \left(\frac{\lambda}{4\pi}\right)^{k+\frac{1}{2}}$$

$$\tilde{\mathcal{B}}^{(0)}\left(k + \frac{3}{2}; \alpha\right) = \frac{(k+1)\Gamma\left(k - \frac{1}{2}\right)\Gamma\left(k + \frac{1}{2}\right)}{2^{2k+1}\pi^{\frac{3}{2}}\Gamma(k+1)} \left[1 - {}_2F_1\left(k + \frac{3}{2}, -k - \frac{1}{2}; 1 \,\middle|\, \frac{2\alpha^2}{2\alpha^2 - 1}\right)\right]$$

$$\left.\frac{\langle \alpha | O_L(1) \bar{O}_L(x, \bar{x}) | \alpha \rangle}{\langle \alpha \mid \alpha \rangle}\right|_{\text{tree}} = N^2 \sum_{\ell=0}^{\infty} \oiiint (\ell+1)(x\bar{x})^{\frac{-\omega-\ell-4}{2}} \left(\frac{x^{\ell+1} - \bar{x}^{\ell+1}}{x - \bar{x}}\right) E_{\text{gravity}}(\omega, \ell; \alpha) \frac{d\omega}{2\pi i},$$

$$E_{\text{gravity}}(\omega, \ell; \alpha) = \sum_{\kappa \in D} \left(\frac{1}{\kappa + \frac{\ell}{2} + 2 + a(\omega, \ell; \alpha)} + \frac{1}{\kappa + \frac{\ell}{2} + 2 - a(\omega, \ell; \alpha)} \right)$$

$$u = -\frac{1}{4} - a^2 + a_t^2 + a_0^2 + t\partial_t F(a, a_0, a_t, a_1, a_\infty, t)$$

$$a_t = a_\infty = 1, a_0^2 = \frac{(\ell+1)^2}{4}$$

$$a_1^2 = \frac{1 + 2\omega^2 - \ell(\ell+2)}{4}, u = \frac{4(2\alpha^2+1) - (1-2\alpha^2)(\omega^2 - \ell(\ell+2))}{4}, t = -\frac{2\alpha^2}{1 - 2\alpha^2}$$

$$\left.\frac{\langle \alpha | O_L(1) \bar{O}_L(x, \bar{x}) | \alpha \rangle}{\langle \alpha \mid \alpha \rangle}\right|_{\text{tree}} = \langle 0 | O_L(1) \bar{O}_L(x, \bar{x}) | 0 \rangle + N^2 \mathcal{K}_\alpha(u, v)$$

$$\mathcal{H}_\alpha(u,v;N,\tau)|_{\text{tree}} = N^2\langle\alpha \mid \alpha\rangle \frac{u^2}{4}\mathcal{K}_\alpha(u,v),$$

$$\mathcal{C}_{\text{HHLL}}|_{\text{tree}} = -\frac{N^2}{2\pi}\int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\theta r^3 \sin^2(\theta)\mathcal{K}_\alpha(u,v)\Big|_{u=1+r^2-2r\cos\theta, v=r^2}$$

$$a(\omega,\ell;\alpha) = \frac{\omega}{2} + \sum_{n=1}^{\infty} \alpha^{2n}\gamma_n(\omega,\ell)$$

$$\langle 0|O_L(1)\bar{O}_L(x,\bar{x})|0\rangle = N^2 \sum_{\kappa=0}^{\infty}\sum_{\ell=0}^{\infty} 2(\ell+1)(x\bar{x})^\kappa \left(\frac{x^{\ell+1}-\bar{x}^{\ell+1}}{x-\bar{x}}\right) = N^2\left(\frac{2}{u^2}\right).$$

$$\mathcal{K}_\alpha(u,v) = \sum_{n=1}^{\infty}\frac{\alpha^{2n}}{n!}K_\alpha^{(n)}(u,v)$$

$$\gamma_1(\omega,l) = \frac{(\omega-\ell-2)(\omega-\ell)(\omega+\ell)(\omega+\ell+2)}{4(\omega-1)\omega(\omega+1)}$$

$$K_\alpha^{(1)}(u,v) = -8u^2\bar{D}_{2422}(u,v),$$

$$\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) O_H(\alpha) = e^{\frac{\alpha}{\sqrt{2}}O_L} \to \alpha/\sqrt{2}O_L$$

$$I_n := -\frac{1}{2\pi}\int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\theta r^3 \sin^2(\theta) K_\alpha^{(n)}(u,v)\Big|_{u=1+r^2-2r\cos\theta, v=r^2}$$

$$I_1 = \frac{1}{2}, I_2 = 1, I_3 = 4$$

$$-\frac{N^2}{4}\log(1-2\alpha^2) = N^2\sum_{n=1}^{\infty}\frac{\alpha^{2n}}{n!}I_n, \text{ with } I_n = 2^{n-2}(n-1)!,$$

$$\mathcal{C}_{\tilde{\alpha}N^2}(N;\lambda) \sim \frac{N^2}{4}\log(1+\tilde{\alpha}) + \sum_{g=0}^{\infty} N^{2-2g}\Bigg[c_g(\alpha) + \sum_{k=0}^{\infty}\tilde{\mathcal{A}}^{(g)}\left(k+\frac{3}{2};\tilde{\alpha}\right)\xi(2k+3)\left(\frac{\lambda}{4\pi}\right)^{-k-\frac{3}{2}} + \sum_{k=0}^{g-1}\tilde{\mathcal{A}}^{(g-k-1)}\left(k+\frac{3}{2};\tilde{\alpha}\right)\xi(2k+2)\left(\frac{\lambda}{4\pi}\right)^{k+\frac{1}{2}}\Bigg].$$

$$\tilde{\alpha} = \frac{2\alpha^2}{1-2\alpha^2}$$

$$\tilde{\mathcal{A}}^{(0)}\left(k+\frac{3}{2};\tilde{\alpha}\right) = \tilde{\mathcal{B}}^{(0)}\left(k+\frac{3}{2};\sqrt{\frac{\tilde{\alpha}}{2(1+\tilde{\alpha})}}\right),$$

$$\mathcal{C}^{(0)}_{\mathrm{HHLL}}(N,\alpha;\tau) \sim \mathcal{C}^{(0)}_{\mathrm{HHLL},P}(N,\alpha;\tau) :=$$
$$\sum_{k=0}^{\infty} \frac{(k+1)\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{2^{2k+1}\pi^{\frac{3}{2}}\Gamma(k+1)}\left[{}_2F_1\left(k+\frac{3}{2},-k-\frac{1}{2};1\,\middle|\,\frac{2\alpha^2}{2\alpha^2-1}\right)-1\right]N^{-k-\frac{3}{2}}E^*\left(k+\frac{3}{2};\tau\right)$$

$$\mathcal{B}\left[\mathcal{C}^{(0)}_{\mathrm{HHLL},P}\right](\alpha,t) = \sum_{k=0}^{\infty} \frac{4\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\pi\Gamma(k+1)^2}\left[{}_2F_1\left(k+\frac{3}{2},-k-\frac{1}{2};1\,\middle|\,\frac{2\alpha^2}{2\alpha^2-1}\right)-1\right]t^{2k+2}$$

$$\mathcal{B}\left[\mathcal{C}^{(0)}_{\mathrm{HHLL},P}\right](\alpha,t) = \mathcal{B}\left[\mathcal{C}^{(0)}_{\tilde{\alpha},P}\right](t)\Big|_{\tilde{\alpha}=\frac{2\alpha^2}{1-2\alpha^2}}$$

$$L^*_{\mathrm{I}} = 1, L^*_{-} = \sqrt{\frac{1-\sqrt{2}\alpha}{1+\sqrt{2}\alpha}}, L^*_{+} = \frac{1}{L^*_{-}} = \sqrt{\frac{1+\sqrt{2}\alpha}{1-\sqrt{2}\alpha}}.$$

$$\mathcal{C}^{(0),\mathrm{I}}_{\mathrm{HHLL},NP}(N;\tau) \sim \exp\left(-4\sqrt{\pi N}\frac{|n\tau+m|}{\sqrt{\tau_2}}\right)$$
$$\mathcal{C}^{(0),-}_{\mathrm{HHLL},NP}(N;\tau) \sim \exp\left(-4\sqrt{\pi N}\sqrt{\frac{1-\sqrt{2}\alpha}{1+\sqrt{2}\alpha}}\frac{|n\tau+m|}{\sqrt{\tau_2}}\right)$$
$$\mathcal{C}^{(0),+}_{\mathrm{HHLL},NP}(N;\tau) \sim \exp\left(-4\sqrt{\pi N}\sqrt{\frac{1+\sqrt{2}\alpha}{1-\sqrt{2}\alpha}}\frac{|n\tau+m|}{\sqrt{\tau_2}}\right)$$

$$\exp\left(-2|m|\sqrt{\lambda}\sqrt{\frac{1+\sqrt{2}\alpha}{1-\sqrt{2}\alpha}}\right) \ll \exp\left(-2|m|\sqrt{\lambda}\right) \ll \exp\left(-2|m|\sqrt{\lambda}\sqrt{\frac{1-\sqrt{2}\alpha}{1+\sqrt{2}\alpha}}\right),$$

$$C_{O,\mathcal{O}_2\mathcal{O}_2} \sim 2^{-\Delta}\Delta^3/\sin\left(\frac{\pi}{2}\Delta\right)\cdot$$

$$ds^2_{10} = G_{\mu\nu}dX^{\mu}dX^{\nu} = \sqrt{\Delta}ds^2_5 + \frac{1}{\sqrt{\Delta}}\left(T_{ij}\right)^{-1}D\mu^i D\mu^j$$

$$\mu^0 + i\mu^3 = \cos\,\theta\cos\,\chi e^{i\psi_1}, \mu^1 + i\mu^2 = \cos\,\theta\sin\,\chi e^{i\psi_2}, \mu^4 + i\mu^5 = \sin\,\theta e^{i\phi}$$

$$ds^2_5 = -H(r)^{-\frac{2}{3}}\left(1+\frac{r^2}{L^2}H(r)\right)dt^2 + H(r)^{\frac{1}{3}}\left(\frac{dr^2}{1+\frac{r^2}{L^2}H(r)} + r^2 d\Omega^2_3\right)$$

$$D\mu^i = d\mu^i + A^{ij}\mu^j$$

$$S = \frac{\sqrt{\lambda}}{2\pi}\int \mathrm{d}t\,\mathrm{d}\sigma\sqrt{-\det(g_{ab})},\ \text{with}\ g_{ab} = G_{\mu\nu}\partial_a X^{\mu}\partial_b X^{\nu}$$

$$S=\frac{\sqrt{\lambda}}{2\pi}\int \mathrm{d}t\,\mathrm{d}\sigma\sqrt{L_-^{*2}\left(\frac{d\mu^4}{d\sigma}\right)^2+L_+^{*2}\left(\frac{d\mu^5}{d\sigma}\right)^2}\,,\text{ with } L_\pm^*=\sqrt{\frac{1\pm\sqrt{2}\alpha}{1\mp\sqrt{2}\alpha}}$$

$$\Delta-J=\frac{\sqrt{\lambda}}{\pi}\cos\,\theta_0$$

$$\begin{cases}\mu_+^4=\sin\,\theta_0\\ \mu_+^5=\sin\,\theta_0\tan\,\sigma\end{cases}\text{ with } \Delta-J=\frac{\sqrt{\lambda}}{\pi}L_+^*\cos\,\theta_0$$

$$\begin{cases}\mu_-^4=\sin\,\theta_0\tan\,\sigma\\ \mu_-^5=\sin\,\theta_0\end{cases}\text{ with } \Delta-J=\frac{\sqrt{\lambda}}{\pi}L_-^*\cos\,\theta_0$$

$$\langle\mathcal{O}_2(x_1,Y_1)\mathcal{O}_2(x_2,Y_2)\tilde{\mathcal{O}}_H(0,Y_3)\tilde{\mathcal{O}}_H(\infty,Y_4)\rangle,$$

$$\tilde{\mathcal{O}}_H(0,Y;\alpha)=\exp\left(\alpha\mathcal{O}_3(0,Y)\right)$$

$${}_2F_1\left(k+\frac{3}{2},-k-\frac{1}{2};1\mid-\alpha\right)=(\sqrt{\alpha+1}-\sqrt{\alpha})^{2k+3}\,{}_2F_1\left(k+\frac{3}{2},\frac{1}{2};1\,\middle|\,\frac{4\sqrt{\alpha(\alpha+1)}}{(\sqrt{\alpha}+\sqrt{\alpha+1})^2}\right)$$

$${}_2F_1\left(k+\frac{3}{2},\frac{1}{2};1\,\middle|\,\frac{4\sqrt{\alpha(\alpha+1)}}{(\sqrt{\alpha}+\sqrt{\alpha+1})^2}\right)=\int_0^1\frac{\left(1-\frac{4x\sqrt{\alpha(\alpha+1)}}{(\sqrt{\alpha}+\sqrt{\alpha+1})^2}\right)^{-k-\frac{3}{2}}}{\pi\sqrt{x(1-x)}}\mathrm{d}x$$

$$\mathcal{B}\left[\mathcal{C}_{\alpha,P}^{(0)}\right](t)=\frac{16t^2}{\pi}E(t^2)-\frac{16t^2}{\pi}\int_0^1\frac{v_\alpha(x)^{-\frac{3}{2}}}{\pi\sqrt{x(1-x)}}E(t^2v_\alpha(x))\mathrm{d}x$$

$$E(z)=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1\mid z\right)$$

$$v_\alpha(x):=(\sqrt{\alpha}+\sqrt{\alpha+1})^2-4x\sqrt{\alpha(\alpha+1)}$$

$$t^2=(\sqrt{\alpha}+\sqrt{\alpha+1})^2-4x\sqrt{\alpha(\alpha+1)}$$

$$(\mathcal{S}_+-\mathcal{S}_-)\left[\mathcal{C}_{\alpha,P}^{(0)}\right](N;\tau)=-2i\left[\mathcal{S}_0\left[\mathcal{C}_{\alpha,NP}^{\mathrm{I},(0)}\right](N;\tau)+\mathcal{I}_\alpha(N;\tau)\right]$$

$$\begin{aligned}\mathcal{S}_0\left[\mathcal{C}_{\alpha,NP}^{\mathrm{I},(0)}\right](N;\tau)&=\int_0^\infty 4t(t+2)(t+1)^2\,{}_2F_1\left(\frac{1}{2},\frac{3}{2};2\mid-t(2+t)\right)\mathcal{E}(\sqrt{N}(t+1);\tau)\mathrm{d}t\\&=\mathcal{C}_{\alpha,NP}^{\mathrm{I},(0)}=\frac{8}{N}D_N(1;\tau)+\frac{28}{N^{\frac{3}{2}}}D_N\left(\frac{3}{2};\tau\right)+\frac{33}{N^2}D_N(2;\tau)+\cdots\end{aligned}$$

$$\lim_{\epsilon\to0^+}[E(z+i\epsilon)-E(z-i\epsilon)]=\frac{i\pi}{2}(1-z){}_2F_1\left(\frac{1}{2},\frac{3}{2};2\mid1-z\right)=-2i[E(1-z)-K(1-z)]$$

$$\mathcal{I}_\alpha(N;\tau)=\int_0^1\frac{4v_\alpha(x)^{-\frac{3}{2}}}{\pi\sqrt{x(1-x)}}\int_{v_\alpha(x)^{\frac{1}{2}}}^\infty\frac{t^2(v_\alpha(x)-t^2)}{v_\alpha(x)}\,{}_2F_1\left(\frac{1}{2},\frac{3}{2};2\,\middle|\,\frac{v_\alpha(x)-t^2}{v_\alpha(x)}\right)\mathcal{E}(\sqrt{N}t;\tau)\mathrm{d}t\,\mathrm{d}x$$

$$\mathcal{I}_\alpha(N;\tau)=\frac{4}{\pi}\int_1^\infty t^2(1-t^2)\,{}_2F_1\left(\frac{1}{2},\frac{3}{2};2\mid 1-t^2\right)\int_0^1\frac{\mathcal{E}\left(\sqrt{Nv_\alpha(x)}t;\tau\right)}{\sqrt{x(1-x)}}\mathrm{d}x\,\mathrm{d}t$$

$$\mathcal{I}_\alpha(N;\tau)=\sum_{\ell=2}^{\infty}c_\ell\sum_{(m,n)\neq(0,0)}(NY_{mn}(\tau))^{-\frac{\ell}{2}}\int_0^1\frac{e^{-4\sqrt{NY_{mn}(\tau)v_\alpha(x)}}}{\sqrt{x(1-x)v_\alpha(x)^\ell}}\mathrm{d}x$$

$$c_2=-\frac{1}{2\pi},c_3=-\frac{7}{16\pi},c_4=-\frac{33}{256\pi}$$

$$t^2:=v_\alpha(x)=(\sqrt{\alpha}+\sqrt{\alpha+1})^2-4x\sqrt{\alpha(\alpha+1)}$$

$$\mathcal{I}_\alpha(N;\tau)=\sum_{\ell=2}^{\infty}4c_\ell\sum_{(m,n)\neq(0,0)}(NY_{mn}(\tau))^{-\frac{\ell}{2}}\int_{L_-}^{L_+}e^{-4t\sqrt{NY_{mn}(\tau)}}t^{-\ell}[(t-L_-)(L_+-t)]^{-\frac{1}{2}}\,\mathrm{d}t$$

$$I_{\alpha,\ell}(N;\tau):=\int_{L_-}^{L_+}e^{-4t\sqrt{NY_{mn}(\tau)}}t^{-\ell}[(t-L_-)(L_+-t)]^{-\frac{1}{2}}\mathrm{d}t=$$
$$\int_{L_-}^{\infty\pm i\epsilon}e^{-4t\sqrt{NY_{mn}(\tau)}}t^{-\ell}[(t-L_-)(L_+-t)]^{-\frac{1}{2}}\,\mathrm{d}t\mp i\int_{L_+}^{\infty}e^{-4t\sqrt{NY_{mn}(\tau)}}t^{-\ell}[(t-L_-)(t-L_+)]^{-\frac{1}{2}}\,\mathrm{d}t$$

$$I_{\alpha,\ell}(N;\tau)=e^{-4L_-\sqrt{NY_{mn}(\tau)}}\int_0^{\infty\pm i\epsilon}e^{-4t\sqrt{NY_{mn}(\tau)}}(t+L_-)^{-\ell}[t(L_+-L_--t)]^{-\frac{1}{2}}\,\mathrm{d}t$$
$$\mp ie^{-4L_+\sqrt{NY_{mn}(\tau)}}\int_0^{\infty}e^{-4t\sqrt{NY_{mn}(\tau)}}(t+L_+)^{-\ell}[t(t+L_+-L_-)]^{-\frac{1}{2}}\,\mathrm{d}t$$

$$I_{\alpha,\ell}(N;\tau)=$$
$$\sum_{k=0}^{\infty}a_k(\alpha,\ell)e^{-4L_-\sqrt{NY_{mn}(\tau)}}[NY_{mn}(\tau)]^{-k-\frac{1}{2}}\mp i\sum_{k=0}^{\infty}\tilde{a}_k(\alpha,\ell)e^{-4L_+\sqrt{NY_{mn}(\tau)}}[NY_{mn}(\tau)]^{-k-\frac{1}{2}}$$

$$e^{-4t_1\sqrt{NY_{mn}(\tau)}}=e^{-4\sqrt{NY_{mn}(\tau)}}\mathcal{C}_{\alpha,NP}^{\mathrm{I},(0)}$$

$$\mathcal{C}_{\alpha,NP}^{\pm,(0)}e^{-4L_+}\sqrt{NY_{mn}(\tau)}e^{-4L_-\sqrt{NY_{mn}(\tau)}}$$

$${}_2F_1(1-s,s;1;-\alpha)\sim\frac{L_-^{1-2s}}{2\sqrt{\pi}[\alpha(\alpha+1)]^{\frac{1}{4}}\sqrt{s}}\left(1+O(s^{-1})\right)+i\frac{L_+^{1-2s}}{2\sqrt{\pi}[\alpha(\alpha+1)]^{\frac{1}{4}}\sqrt{s}}\left(1+O(s^{-1})\right)$$

$$s=s_-^\star=2L_-\sqrt{NY_{mn}(\tau)}$$

$$e^{-4L_-\sqrt{NY_{mn}(\tau)}}$$

$$\tan(\pi s_-^\star)\to\pm i,\arg(N)\gtrless 0$$

$$s=s_+^\star=2L_+\sqrt{NY_{mn}(\tau)},$$

$$e^{-4L_+\sqrt{NY_{mn}(\tau)}}$$

$$\mathcal{C}(\Lambda;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \frac{2^{1-2s}\Gamma(s-2)\Gamma(s-1)\Gamma(s)\tan(\pi s)^2}{\pi\Gamma\left(s-\frac{1}{2}\right)^2} \Lambda^{-s} E^*(s;\tau) \frac{\mathrm{d}s}{2\pi i}$$

$$\Lambda = 4\alpha N^{\gamma-1}$$

$$\mathcal{C}(\Lambda;\tau) \sim \mathcal{C}_P(\Lambda;\tau) = \sum_{k=0}^{\infty} \tilde{b}_k \Lambda^{-k-\frac{3}{2}} \left[\tilde{E}^*\left(k+\frac{3}{2};\tau\right) - \log(\Lambda) E^*\left(k+\frac{3}{2};\tau\right)\right] + \sum_{k=0}^{\infty} b_k \Lambda^{-k-\frac{3}{2}} E^*\left(k+\frac{3}{2};\tau\right),$$

$$\tilde{b}_k = -\frac{\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}{\pi^3 2^{2k+1}\Gamma(k+1)^2}$$

$$b_k = -\frac{\partial}{\partial s}\left(\frac{\Gamma\left(s-\frac{1}{2}\right)\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{\pi^3 2^{2s+1}\Gamma(s+1)^2}\right)\Bigg|_{s=k}$$

$$\tilde{E}^*(s;\tau) := \partial_s E^*(s;\tau)$$

$$\mathcal{C}(\Lambda,\epsilon;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \frac{2^{1-2s}\Gamma(s-2)\Gamma(s-1)\Gamma(s)\tan(\pi s)\tan(\pi(s+\epsilon))}{\pi\Gamma\left(s-\frac{1}{2}\right)^2} \Lambda^{-s} E^*(s;\tau) \frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{C}(\Lambda,\epsilon;\tau) \sim \mathcal{C}_P(\Lambda,\epsilon;\tau) = \mathcal{C}_P^{(1)}(\Lambda,\epsilon;\tau) + \mathcal{C}_P^{(2)}(\Lambda,\epsilon;\tau)$$

$$\mathcal{C}_P^{(1)}(\Lambda,\epsilon;\tau) = -\frac{\cot(\pi\epsilon)}{\pi^2}\sum_{k=0}^{\infty} \frac{\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}{4^{k+1}\Gamma(k+1)^2} \Lambda^{-k-\frac{3}{2}} E^*\left(k+\frac{3}{2};\tau\right)$$

$$\mathcal{C}_P^{(2)}(\Lambda,\epsilon;\tau) = \frac{\cot(\pi\epsilon)}{\pi^2}\sum_{k=0}^{\infty} \frac{\Gamma\left(k-\frac{1}{2}-\epsilon\right)\Gamma\left(k+\frac{1}{2}-\epsilon\right)\Gamma\left(k+\frac{3}{2}-\epsilon\right)}{4^{k+1-\epsilon}\Gamma(k+1-\epsilon)^2} \Lambda^{-k-\frac{3}{2}+\epsilon} E^*\left(k+\frac{3}{2}-\epsilon;\tau\right)$$

$$\mathcal{B}[\mathcal{C}_P^{(1)}](t;\epsilon) = 2t^2\cot(\pi\epsilon)\,{}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};1,2 \mid t^2\right)$$

$$\mathcal{B}[\mathcal{C}_P^{(2)}](t;\epsilon) = \frac{2t^{2-2\epsilon}\cot(\pi\epsilon)}{\pi^{\frac{3}{2}}} \frac{\Gamma\left(-\frac{1}{2}-\epsilon\right)\Gamma\left(\frac{1}{2}-\epsilon\right)\Gamma\left(\frac{3}{2}-\epsilon\right)}{\Gamma(1-\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}\, {}_4F_3\left(1,-\frac{1}{2}-\epsilon,\frac{1}{2}-\epsilon,\frac{3}{2}-\epsilon;1-\epsilon,1-\epsilon,2-\epsilon \mid t^2\right)$$

$$\mathcal{B}[\mathcal{C}_P](t) = \lim_{\epsilon\to 0}\left(\mathcal{B}\left[\mathcal{C}_P^{(1)}\right](t;\epsilon) + \mathcal{B}\left[\mathcal{C}_P^{(2)}\right](t;\epsilon)\right) = f_1(t) + \log(t) f_2(t)$$

$$f_1(t) = \sum_{k=0}^{\infty}\left[b_k \frac{\Gamma\left(k+\frac{3}{2}\right)}{\Gamma(2k+3)} 4^{2k+2} + \tilde{b}_k \chi\left(k+\frac{3}{2}\right)\right] t^{2k+2}, f_2(t) = \sum_{k=0}^{\infty} \tilde{b}_k \frac{2\Gamma\left(k+\frac{3}{2}\right)}{\Gamma(2k+3)} (4t)^{2k+2}$$

$$f_1(t) = \frac{-2t^2}{\pi^{\frac{5}{2}}}\left[\left(\frac{\partial}{\partial\epsilon_1} + \frac{\partial}{\partial\epsilon_2}\right) 2^{2\epsilon_1-2\epsilon_2}\Gamma\left(\epsilon_2-\frac{1}{2}\right)\Gamma\left(\epsilon_2+\frac{1}{2}\right)\Gamma\left(\epsilon_2+\frac{3}{2}\right)\right.$$
$$\left.\times\ {}_4\tilde{F}_3\left(1,\epsilon_2-\frac{1}{2},\epsilon_2+\frac{1}{2},\epsilon_2+\frac{3}{2};\epsilon_2+1,\epsilon_2+1,\epsilon_1+2;t^2\right)\right]\Bigg|_{\epsilon_1=\epsilon_2=0}$$

$$f_2(t) = \frac{4t^2}{\pi}\ {}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};1,2 \mid t^2\right)$$

$$\mathcal{S}_\theta[\mathcal{C}_P](\Lambda;\tau)=\int_0^{e^{i\theta}\infty}\mathcal{B}[\mathcal{C}_P](t)\mathcal{E}(\sqrt{\Lambda}t;\tau)\mathrm{d}t$$

$$\mathrm{Disc}_0 f_a(t):=\lim_{\epsilon\to 0^+}[f_a(t+i\epsilon)-f_a(t-i\epsilon)]=-2\pi i g_a(1-t)=-2\pi i\sum_{k=0}^{\infty}\frac{d_k^{(a)}}{k!}(t-1)^k$$

$$\begin{aligned}(\mathcal{S}_+-\mathcal{S}_-)[\mathcal{C}_P](\Lambda;\tau)\ &=(-2\pi i)\int_1^\infty(g_1(1-t)+\log(t)g_2(1-t))\mathcal{E}(\sqrt{\Lambda}t;\tau)\mathrm{d}t\\ &=:(-2\pi i)\mathcal{S}_0[\mathcal{C}_{NP}](\Lambda;\tau)\end{aligned}$$

$$\begin{aligned}\mathcal{S}_0[\mathcal{C}_{NP}](\Lambda;\tau)&=\int_0^\infty[g_1(-t)+\log(t+1)g_2(-t)]\mathcal{E}(\sqrt{\Lambda}(t+1);\tau)\mathrm{d}t\\ &=\sum_{k=0}^{\infty}d_k D_\Lambda\left(\frac{k+1}{2};\tau\right)+\sum_{k=1}^{\infty}\sum_{\ell=1}^{k}\frac{(-1)^{\ell+1}k!}{\ell(k-\ell)!}\tilde{d}_{k-\ell}D_\Lambda\left(\frac{k+1}{2};\tau\right)\end{aligned}$$

$$D_\Lambda\left(\frac{k+1}{2};\tau\right)$$

$$\mathcal{C}_{NP}(N,\alpha;\tau)\sim\exp\left(-8N^{\frac{\gamma-1}{2}}\sqrt{\alpha Y_{mn}(\tau)}\right)$$

$$Y_{mn}(\tau)=\pi|n\tau+m|^2/\tau_2 \text{ and } (m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}$$

$$\mathcal{C}^{(1)}_{\mathrm{HHLL}}(N,\alpha;\tau)\sim\mathcal{C}^{(1)}_{\mathrm{HHLL},P}(N,\alpha;\tau)=$$
$$\sum_{k=0}^{\infty}\frac{(2k+13)(k+1)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{5}{2}\right)}{24\pi^{\frac{3}{2}}\cdot 2^{2k+3}\Gamma(k+3)}\left[1-{}_2F_1\left(k+\frac{3}{2},-k-\frac{1}{2};1\,\middle|\,\frac{2\alpha^2}{2\alpha^2-1}\right)\right]N^{-k-\frac{3}{2}}E^*\left(k+\frac{3}{2};\tau\right).$$

$$\mathcal{B}\left[\mathcal{C}^{(1)}_{\mathrm{HHLL},P}\right](\alpha,t)=\sum_{k=0}^{\infty}\frac{(2k+13)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{5}{2}\right)}{24\pi(k+2)\Gamma(k+1)^2}\left[1-{}_2F_1\left(k+\frac{3}{2},-k-\frac{1}{2};1\,\middle|\,\frac{2\alpha^2}{2\alpha^2-1}\right)\right]t^{2k+2}$$

$$\mathcal{C}^{(1)}_{\tilde{\alpha}}(N;\tau)=2\int_{\mathrm{Re}(s)=1+\epsilon}\mathcal{A}^{(1)}(s;\tilde{\alpha})N^{-s}E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}$$

$$\mathcal{A}^{(1)}(s;\tilde{\alpha})=\frac{2^{-3-2s}(2s-1)\Gamma(s-2)\Gamma(s+1)}{3\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}\{24(2\tilde{\alpha}+1)\,{}_2F_1(2-s,s+1;2\mid-\tilde{\alpha})$$
$$-24(\tilde{\alpha}+1)\,{}_2F_1(2-s,s+1;1\mid-\tilde{\alpha})+\frac{(2-s)(5+s)}{1+2s}[\,{}_2F_1(s,1-s;1\mid-\tilde{\alpha})-1]\Big\}\tan(\pi s)$$

$$\mathcal{B}[\mathcal{C}^{(1)}_{\tilde{\alpha},P}](t)=\mathcal{B}[\mathcal{C}^{(1)}_{\mathrm{HHLL}}](\alpha,t)\Big|_{\alpha=\sqrt{\frac{\tilde{\alpha}}{2(\tilde{\alpha}+1)}}}+$$
$$\sum_{k=0}^{\infty}\frac{4\Gamma(k-\frac{1}{2})\Gamma(k+\frac{5}{2})}{\pi\Gamma(k+1)^2}\left[(2\tilde{\alpha}+1){}_2F_1\left(k+\tfrac{5}{2},\tfrac{1}{2}-k;2|-\tilde{\alpha}\right)-(\tilde{\alpha}+1){}_2F_1\left(k+\tfrac{5}{2},\tfrac{1}{2}-k;1|-\tilde{\alpha}\right)\right]t^{2k+2}.$$

$$A=\begin{pmatrix}A_3 & & \\ & A_2 & \\ & & A_1\end{pmatrix}\otimes i\sigma_2.$$

$$T = \begin{pmatrix} X_3\mathcal{M}_3 & & \\ & X_2\mathcal{M}_2 & \\ & & X_1\mathcal{M}_1 \end{pmatrix}, \mathcal{M}_i = \begin{pmatrix} \cosh\,\eta_i + \sinh\,\eta_i \cos\,\theta_i & \sinh\,\eta_i \sin\,\theta_i \\ \sinh\,\eta_i \sin\,\theta_i & \cosh\,\eta_i - \sinh\,\eta_i \cos\,\theta_i \end{pmatrix}$$

$$X_1 = e^{-\frac{2}{\sqrt{6}}\varphi_1}, X_2 = e^{\frac{1}{\sqrt{6}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}, X_3 = e^{\frac{1}{\sqrt{6}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}$$

$$\mathcal{L}_5 = \sqrt{g}\left[R - \frac{1}{2}\sum_{a=1}^{2} (\partial\phi_a)^2 - \frac{1}{4}\sum_{i=1}^{3} X_i^{-2} F^{i,\mu\nu} F^i_{\mu\nu} - \frac{1}{2}\sum_{i=1}^{3} [(\partial\eta_i)^2 + \sinh^2\,\eta_i(\partial\theta_i + qA_i)^2] - V\right],$$

$$V := 2\left[\sum_{i=1}^{3} \left(\frac{\partial W}{\partial\eta_i}\right)^2 + \left(\frac{\partial W}{\partial\varphi_1}\right)^2 + \left(\frac{\partial W}{\partial\varphi_2}\right)^2\right] - \frac{4}{3}W^2, \text{ with } W := \frac{1}{2}\sum_{i=1}^{3} qX_i \cosh\,(\eta_i)$$

$$V = \frac{2}{L^2}\left[\sum_{i=1}^{3} X_i^2 \sinh^2\,\eta_i - \sum_{i\neq j} X_i X_j \cosh\,\eta_i \cosh\,\eta_j\right]$$

$$A_1 = \frac{dt}{H(r)}, \varphi_1 = \sqrt{\frac{2}{3}}\log\,H(r), \eta_1 = \operatorname{arccosh}\left[\frac{1}{2r}\frac{d}{dr}[r^2 H(r)]\right], \theta_1 = 0$$

$$ds_5^2 = g_{\mu\nu}dx^\mu dx^\nu = -H(r)^{-\frac{2}{3}}(1 + r^2 H(r))dt^2 + H(r)^{\frac{1}{3}}\left(\frac{dr^2}{1 + r^2 H(r)} + r^2 d\Omega_3^2\right)$$

$$H(r) = -\frac{1}{r^2} + \sqrt{\left(1 + \frac{1}{r^2}\right)^2 + \frac{2Q}{r^2}},$$

$$H(r) = 1 + \frac{Q}{r^2} + O(r^{-4}), r \to \infty$$

$$H(r) = 1 + Q - \frac{1}{2}[Q(2 + Q)]r^2 + O(r^4), r \to 0$$

$$H_C(r) = -\frac{1}{r^2} + \sqrt{\left(1 + \frac{1}{r^2}\right)^2 + \frac{2Q_\star}{r^2} + \frac{C^2 - 1}{r^4}}$$

$$\eta_1 = 0 \;\rightarrow\; H_\star(r) = 1 + \frac{Q_\star}{r^2}$$

$$t=\frac{i}{2}\log\left(z^2+R^2\right),\qquad r=\frac{R}{z}$$

$$ds_5^2=\frac{1}{z^2}\left[\frac{dz^2}{h}+h\left(dR+\frac{R}{z}v_z dz\right)^2+H^{\frac{1}{3}}R^2 d\Omega_3^2\right]$$

$$h=h(z,R):=\left(\frac{1}{g_{tt}}+\frac{r^2H^{\frac{1}{3}}g_{tt}}{(1+r^2)^2}\right)\Bigg|_{r=\frac{R}{z}},v_z=v_z(z,R)=\frac{1}{h}\left(-\frac{1}{g_{tt}}+\frac{H^{\frac{1}{3}}g_{tt}}{(1+r^2)^2}\right)\Bigg|_{r=\frac{R}{z}}.$$

$$A_1(z,R)=-\frac{i}{H\left(\frac{R}{z}\right)}\frac{RdR+zdz}{(R^2+z^2)}=-\frac{idR}{H\left(\frac{R}{z}\right)}+d\left(\int_1^{\frac{R}{z}}\frac{i}{H(r)(r^2+1)}\frac{\mathrm{d}r}{r}\right)$$

$$ds_5^2=\frac{d\mathbb{Z}^2}{\mathbb{Z}^2}+\mathfrak{h}_{ij}(\mathbb{Z},\mathbb{r})d\mathbb{R}^i d\mathbb{R}^j$$

$$z(\mathbb{Z},\mathfrak{r})\ =\mathbb{Z}\left[1+\frac{Q}{6}\frac{\mathbb{Z}^2}{\mathbb{r}^2}-\frac{Q(12+Q)}{36}\frac{\mathbb{Z}^4}{\mathbb{r}^4}+\frac{35Q^2(6+Q)+24Q(45-2Q)}{1944}\frac{\mathbb{Z}^6}{\mathbb{r}^6}+O\left(\frac{\mathbb{Z}^8}{\mathbb{r}^8}\right)\right]$$

$$R(\mathbb{Z},\mathfrak{r})\ =\mathfrak{r}\left[1+0\cdot\frac{\mathbb{Z}^2}{\mathbb{r}^2}-\frac{Q}{6}\frac{\mathbb{Z}^4}{\mathbb{r}^4}+\frac{Q(12+Q)}{27}\frac{\mathbb{z}^6}{\mathbb{r}^6}+O\left(\frac{\mathbb{Z}^8}{\mathbb{r}^8}\right)\right].$$

$$\mathbb{K}_{ij}:=\frac{1}{2}\mathbb{z}\partial_{\mathbb{Z}}\mathfrak{h}_{ij},\ \text{and}\ \mathbb{K}:=\mathbb{h}^{ij}\mathbb{K}_{ij}$$

$$\left\langle\left\langle T_{ij}(\mathfrak{r})\right\rangle\right\rangle_{\mathrm{AdS}}=\frac{1}{8\pi G_N}\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\left[\mathbb{K}_{ij}-\mathbb{K}\mathbb{h}_{ij}-\mathbb{h}_{ij}W\right]_{\mathbb{Z}=\epsilon}$$

$$\langle\langle T(\mathfrak{r})\rangle\rangle_{\mathrm{Ads}}=\frac{Q}{8\pi G_N}\frac{1}{\mathfrak{r}^4}\left(-d\mathbb{r}^2+\frac{\mathbb{m}^2}{3}d\Omega_3^2\right)$$

$$\langle\langle J(\mathfrak{r})\rangle\rangle_{\mathrm{Ads}}=\frac{1}{8\pi G_N}\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}[A_1]_{\mathbb{Z}=\epsilon}$$

$$A_1(z,R)=-\frac{idR}{H\left(\frac{R}{z}\right)}$$

$$A_1(z,R) \to A_1(z,R) = -\frac{idR}{H\left(\frac{R}{z}\right)} + d(i\log(R)) = \frac{i\left(-1+H\left(\frac{R}{z}\right)\right)}{H\left(\frac{R}{z}\right)}\frac{dR}{R}$$

$$\langle\langle J(\mathfrak{r})\rangle\rangle_{\text{Ads}} = \frac{Q}{8\pi G_N}\frac{idr}{\mathfrak{r}^3}$$

$$\langle\langle \varphi_1(\mathfrak{r})\rangle\rangle_{\text{AdS}} = \frac{1}{8\pi G_N}\sqrt{\frac{2}{3}\frac{Q}{\mathfrak{r}^2}}, \langle\langle \eta_1(\mathfrak{r})\rangle\rangle_{\text{AdS}} = \frac{1}{8\pi G_N}\frac{\sqrt{Q(2+Q)}}{\mathfrak{r}^2}$$

$$\frac{\langle \mathbb{O}_\Delta(x_1)\overline{\mathbb{O}}_\Delta(x_2)J_i(x_3)\rangle}{\langle \mathbb{O}_\Delta(x_1)\overline{\mathbb{O}}_\Delta(x_2)\rangle} = ig_{HHJ}\frac{(x_{12}^2)^{\frac{d-1}{2}}}{(x_{13}^2x_{23}^2)^{\frac{d-1}{2}}}(X_{12})_i$$

$$\frac{\langle \mathbb{O}_\Delta(x_1)\overline{\mathbb{O}}_\Delta(x_2)T_{ij}(x_3)\rangle}{\langle \mathbb{O}_\Delta(x_1)\overline{\mathbb{O}}_\Delta(x_2)\rangle} = g_{HHT}\frac{(x_{12}^2)^{\frac{d}{2}}}{(x_{13}^2x_{23}^2)^{\frac{d}{2}}}\left[-(X_{12})_i(X_{12})_j + \frac{1}{d}\delta_{ij}(X_{12})^2\right]$$

$$(X_{12})_i := \frac{(x_{13}^2x_{23}^2)^{\frac{1}{2}}}{(x_{12}^2)^{\frac{1}{2}}}\left[-\frac{(x_{13})_i}{x_{13}^2} + \frac{(x_{23})_i}{x_{23}^2}\right]$$

$$\frac{\langle \mathbb{O}_\Delta(x_1)\overline{\mathbb{O}}_\Delta(x_2)O_S(x_3)\rangle}{\langle \mathbb{O}_\Delta(x_1)\overline{\mathbb{O}}_\Delta(x_2)\rangle} = g_{HHO}\frac{(x_{12}^2)^{\frac{\Delta_S}{2}}}{(x_{13}^2x_{23}^2)^{\frac{\Delta_S}{2}}}$$

$$\langle\langle \mathcal{O}(\mathfrak{x})\rangle\rangle_{\text{CFT}} := \lim_{x\to\infty}\frac{\langle \mathbb{O}_H(0)\overline{\mathbb{O}}_H(x)\mathcal{O}(\mathfrak{x})\rangle}{\langle \mathbb{O}_H(0)\overline{\mathbb{O}}_H(x)\rangle}.$$

$$\langle\langle J_i(\mathrm{x})\rangle\rangle_{\text{CFT}} = \frac{ig_{HHJ}}{|\mathrm{x}|^{d-1}}\cdot\frac{\mathrm{x}_i}{|\mathrm{x}|}, \qquad \left\langle\langle T_{ij}(\mathrm{x})\rangle\right\rangle_{\text{CFT}} = \frac{g_{HHT}}{|\mathrm{x}|^d}\left(-\frac{\mathrm{x}_i\mathrm{x}_j}{|\mathrm{x}|^2} + \frac{1}{d}\delta_{ij}\right).$$

$$g_{HHJ} = \frac{p}{S_d}, \qquad g_{HHT} = \frac{d}{d-1}\frac{\Delta_H}{S_d}, \text{ with } S_d := \frac{2\pi^{\frac{d}{2}}}{\Gamma\left[\frac{d}{2}\right]}.$$

$$^\triangle\langle\langle \varphi_1\rangle\rangle_{\text{CFT}} \propto g_{HH\varphi_1} \text{ and } \langle\langle \eta_1\rangle\rangle_{\text{CFT}} \propto g_{HH\eta_1}$$

$$\frac{p}{S_4} = \frac{\Delta_H}{S_4} = \frac{Q}{8\pi G_N} \Rightarrow \Delta_H = p = N^2\frac{Q}{2}$$

$$Q = \frac{4\alpha^2}{(1-2\alpha^2)}$$

$$\langle\langle O(\mathrm{x})\rangle\rangle_{\text{CFT}} = \frac{g_{HHO}}{|\mathrm{x}|^{\Delta_S}}$$

$$g_{HH\varphi_1} = \sqrt{\frac{2}{3}\frac{Q}{8\pi G_N}}, \qquad g_{HH\eta_1} = \frac{\sqrt{Q(2+Q)}}{8\pi G_N}$$

$$g_{HH\eta_1} = \frac{2}{8\pi G_N}\sqrt{\frac{Q}{2}\left(1+\frac{Q}{2}\right)} = \frac{N^2}{2\pi^2}\sqrt{\tilde{\alpha}(1+\tilde{\alpha})}$$

$$g_{HH\eta_1} = \frac{\sqrt{2}N^2}{\pi^2}\left(\frac{\alpha}{1-2\alpha^2}\right)$$

$$ds_{10}^2 = G_{\mu\nu}dX^\mu dX^\nu = \sqrt{\Delta}ds_5^2 + \frac{1}{\sqrt{\Delta}}\left(T_{ij}\right)^{-1}D\mu^i D\mu^j$$

$$D\mu^i := d\mu^i + A^{ij}\mu^j, \Delta := T_{ij}\mu^i\mu^j$$

$$T = H^{\frac{1}{3}}\begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & \frac{1}{H}\begin{bmatrix} e^{\eta_1} & 0 \\ 0 & e^{-\eta_1}\end{bmatrix}\end{pmatrix}.$$

$$\mu^0 + i\mu^3 = \cos\theta\cos\chi e^{i\psi_1}, \mu^1 + i\mu^2 = \cos\theta\sin\chi e^{i\psi_2}, \mu^4 + i\mu^5 = \sin\theta e^{i\phi}$$

$$d\tilde{\Omega}_3^2 = d\chi^2 + \cos^2\chi d\psi_1^2 + \sin^2\chi d\psi_2^2$$

$$\frac{1}{\sqrt{\Delta}}\left(T_{ij}\right)^{-1}D\mu^i D\mu^j = \frac{\cos^2\theta d\tilde{\Omega}_3^2 + \sin^2\theta d\theta^2 + H(d\theta, D\phi)\cdot\mathbb{T}\cdot(d\theta, D\phi)}{\sqrt{H\cos^2\theta + \mathbb{E}_{\eta_1}(\phi)\sin^2\theta}}$$

$$\mathbb{T} = \begin{pmatrix} \cos^2\theta\,\mathbb{E}_{\eta_1}\left(\phi+\frac{\pi}{2}\right) & \frac{1}{2}\sin(2\theta)\sin(2\phi)\sinh(\eta_1) \\ \frac{1}{2}\sin(2\theta)\sin(2\phi)\sinh(\eta_1) & \sin^2\theta\,\mathbb{E}_{\eta_1}(\phi)\end{pmatrix}$$

$$\mathbb{E}_{\eta_1}(\phi) = e^{\eta_1}\cos^2\phi + e^{-\eta_1}\sin^2\phi$$

$$S = \frac{\sqrt{\lambda}}{2\pi}\int \mathrm{d}t\,\mathrm{d}\sigma\sqrt{-\det(g_{ab})}, \text{ with } g_{ab} = G_{\mu\nu}\partial_a X^\mu\partial_b X^\nu$$

$$ds^2\big|_{\text{giant-particle}} = -\sqrt{H\cos^2\theta + \mathbb{E}_{\eta_1}(\phi)\sin^2\theta}\,\frac{f dt^2}{H} + \frac{\sin^2\theta d\theta^2 + H(d\theta, D\phi)\cdot\mathbb{T}\cdot(d\theta, D\phi)}{\sqrt{H\cos^2\theta + \mathbb{E}_{\eta_1}(\phi)\sin^2\theta}}$$

$$S = \frac{\sqrt{\lambda}}{2\pi}\int \mathrm{d}t\,\mathrm{d}\sigma\sqrt{f}\sqrt{\mathbb{E}_{\eta_1}(\sigma)s^2(\sigma) - \mathbb{E}'_{\eta_1}(\sigma)s(\sigma)s'(\sigma) + \mathbb{E}_{\eta_1}\left(\sigma+\frac{\pi}{2}\right)s'(\sigma)^2 + \frac{(f-1)}{fH}s'(\sigma)^2}$$

$$\mathbb{E}_{\eta_1}(\sigma)\big|_{r=0} = \frac{1+2\alpha^2+2\sqrt{2}\alpha\cos(2\sigma)}{1-2\alpha^2}$$

$$S = \frac{\sqrt{\lambda}}{2\pi}\int \mathrm{d}t\,\mathrm{d}\sigma\sqrt{L_-^{*2}\left(\frac{d\mu^4}{d\sigma}\right)^2 + L_+^{*2}\left(\frac{d\mu^5}{d\sigma}\right)^2}, \text{ with } L_\pm^* = \sqrt{\frac{1\pm\sqrt{2}\alpha}{1\mp\sqrt{2}\alpha}}$$

$$L_\pm^* \rightarrow \tilde{L}_\pm^* = \sqrt{(1+Q)\pm\sqrt{(1+Q)^2 - C^2}}$$

$$\mathrm{d}\mu(\{x_i\}) = \prod_{i=1}^{4} \mathrm{d}^4 x_i/\mathrm{vol}(SO(2,4))$$

$$\langle T_2^n T_2^n T_2\rangle = (n!)^2/(n-1)!\,\langle T_2T_2T_2\rangle\langle T_2T_2\rangle^{n-1}\big(1+O(N^{-2})\big)$$

$$\langle T_2^n T_2^n\rangle = n!\,\langle T_2T_2\rangle^{n}\big(1+O(N^{-2})\big)$$

$$\langle T_2T_2T_2\rangle = 8N^2 + O(N^0)$$

$$n = \frac{p}{2} = \frac{\tilde{\alpha}}{2}N^2 + O(N^0)$$

$$\mathcal{A}_n^{(0),\mathrm{closed}} = \alpha'^{n-2}\int_{\mathcal{M}_{0,n}} \frac{d^{2n}\sigma}{\mathrm{volSL}(2,\mathbb{C})}\left|\sum_{\rho\in S_{n-2}} \frac{N(1,\rho(2),\cdots,\rho(n-1),n)}{\sigma_{1\rho(2)}\sigma_{\rho(2)\rho(3)}\cdots\sigma_{\rho(n-1)n}\sigma_{n1}}\right|^2 \prod_{i<j}\left|\sigma_{ij}\right|^{\alpha' p_i\cdot p_j},$$

$$\mathcal{A}_n^{(1),\,\mathrm{closed}} = \alpha'^n \int_{\mathbb{R}^D} d^D\ell \int_{\mathcal{F}} d^2\tau \int_{T^{n-1}} d^2z_2\cdots d^2z_n\, \mathcal{J}_n(\ell)\tilde{\mathcal{J}}_n(\ell)|\mathrm{KN}_n(\ell)|^2$$

$$\mathcal{A}_n^{(1),\,\mathrm{open}} = \alpha'^n \sum_{\mathrm{top}} C_{\mathrm{top}} \int_{\mathbb{R}^D} d^D\ell \int_{\mathcal{D}_{\mathrm{top}}} d\tau \int_{O_{\mathrm{top}}^{n-1}} d^2z_2\cdots d^2z_n \mathcal{J}_n(\ell)\left|\mathrm{KN}_n^{(\alpha'\mapsto 4\alpha')}(\ell)\right|$$

$$\mathrm{KN}_n(\ell) = \exp\frac{\alpha'}{2}\left(\sum_{1\le i<j\le n} p_i\cdot p_j \ln\,\theta_1(z_{ij},\tau) + 2i\pi\ell\cdot\sum_{j=1}^{n} z_jp_j + i\pi\tau\ell^2\right),$$

$$\theta_1(z,\tau) := 2q^{1/8}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-q^ne^{2\pi iz})(1-q^ne^{-2\pi iz}),\ \text{with}\ q := e^{2\pi i\tau}$$

$$g_{ij}^{(w)} = g^{(w)}(z_i - z_j,\tau),$$

$$F(z,\eta,\tau) := \frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(\eta,\tau)\theta_1(z,\tau)} = \sum_{w=0}^{\infty}\eta^{w-1}g^{(w)}(z,\tau).$$

$$g^{(1)}(z,\tau) = \partial_z \ln\,\theta_1(z,\tau), g^{(2)}(z,\tau) = \frac{1}{2}((\partial_z\ln\,\theta_1(z,\tau))^2 + \partial_z^2\ln\,\theta_1(z,\tau)) - \frac{\partial_z^3\theta_1(0,\tau)}{3!\,\partial_z\theta_1(0,\tau)}$$

$$g_{ij}^{(w)} = (-1)^w g_{ji}^{(w)}$$

$$F(z,\eta_1,\tau)F(z',\eta_2,\tau) = F(z,\eta_1+\eta_2,\tau)F(z'-z,\eta_2,\tau) + F(z',\eta_1+\eta_2,\tau)F(z-z',\eta_1,\tau)$$

$$g^{(w)}(z+\tau,\tau) = \sum_{m=0}^{w}\frac{(-2\pi i)^m}{m!}g^{(w-m)}(z,\tau)$$

$$\mathrm{weight}\ (\mathcal{J}_n) = n-4$$

i-particle supermassive: $\ell_\mu \mapsto \ell_\mu - p_{i,\mu}, z_i \mapsto z_i + \tau.$

$$g_{i_1 i_2}^{(1)} g_{i_2 i_3}^{(1)} \cdots g_{i_m i_1}^{(1)}$$

$$g_{i_1 i_2}^{(w_1)} g_{i_2 i_3}^{(w_2)} \cdots g_{i_m i_1}^{(w_m)}$$

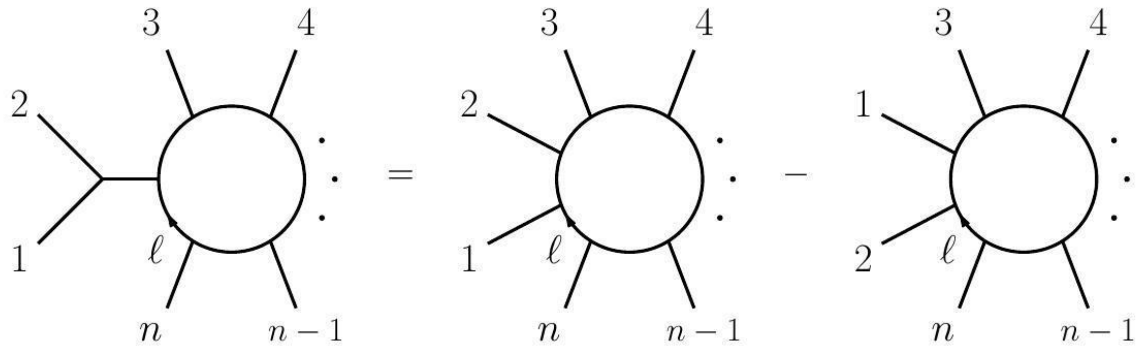
$$G_{2K}(\tau) := \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m+\tau n)^{2K}} = -g^{(2K)}(0,\tau), K \geq 2$$

$$G_2(\tau) := -g^{(2)}(0,\tau) = -\frac{\partial_z^3 \theta_1(0,\tau)}{3\partial_z \theta_1(0,\tau)}$$

$$e^{2i\pi z} = \frac{(\sigma-\sigma_+)(\sigma_* - \sigma_-)}{(\sigma-\sigma_-)(\sigma_* - \sigma_+)}, dz = \left(\frac{1}{\sigma-\sigma_+} - \frac{1}{\sigma-\sigma_-}\right)\frac{d\sigma}{2\pi i}$$

$$(2\pi i)^{n-1} \prod_{i=2}^{n} dz_i = \prod_{i=2}^{n} \left(d\sigma_i \frac{\sigma_{+-}}{\sigma_{i+}\sigma_{i-}}\right) = (-1)^n \frac{d^n\sigma}{\text{volSL}(2)} \sum_{\rho\in S_n} \frac{1}{\sigma_{+\rho(1)}\sigma_{\rho(1)\rho(2)}\cdots\sigma_{\rho(n)-}\sigma_{-+}},$$

$$\frac{d^n\sigma}{\text{volSL}(2)} := (\sigma_{+-}\sigma_{-1}\sigma_{1+}) \prod_{i=2}^{n} d\sigma_i$$



$$g_{ij}^{(1)} \to \pi i \frac{\sigma_{i+}\sigma_{j-} + \sigma_{i-}\sigma_{j+}}{\sigma_{ij}\sigma_{+-}}, g_{ij}^{(2K)} \to -2\zeta(2K), g_{ij}^{(2K+1)} \to 0$$

$$(-1)^n (2\pi i)^3 \mathcal{J}_n(\ell) \prod_{i=2}^{n} dz_i \to \frac{d^n\sigma}{\text{volSL}(2)} \sum_{\rho\in S_n} \frac{N(\rho(1),\rho(2),\cdots,\rho(n);\ell)}{\sigma_{+\rho(1)}\sigma_{\rho(1)\rho(2)}\cdots\sigma_{\rho(n)-}\sigma_{-+}}$$

$$N([1,2],3,\cdots,n;\ell) = N(1,2,3,\cdots,n;\ell) - N(2,1,3,\cdots,n;\ell)$$
$$N([[1,2],3],\cdots,n;\ell) = N([1,2],3,\cdots,n;\ell) - N(3,[1,2],\cdots,n;\ell)$$
$$N([1,2],[3,4],\cdots,n;\ell) = N([1,2],3,4,\cdots,n;\ell) - N([1,2],4,3,\cdots,n;\ell),$$

$$\mathcal{A}_n^{(1),\text{SYM}} = \int_{\mathbb{R}^D} d^D\ell \sum_{\gamma\in\Gamma_{n,3}^{(1)}} \frac{N_\gamma(\ell)c_\gamma}{D_\gamma}, \mathcal{A}_n^{(1),\text{SUGRA}} = \int_{\mathbb{R}^D} d^D\ell \sum_{\gamma\in\Gamma_{n,3}^{(1)}} \frac{N_\gamma(\ell)\tilde{N}_\gamma(\ell)}{D_\gamma}.$$

$$\ell_\mu \mapsto \ell_\mu - p_{1,\mu}, z_1 \mapsto z_1 + \tau, z_{i>1} \mapsto z_i$$

$$\frac{N(1,2,\cdots,n;\ell)}{\sigma_{+1}\sigma_{12}\sigma_{23}\cdots\sigma_{n-}\sigma_{-+}} \mapsto \frac{N(1,2,\cdots,n;\ell-p_1)}{\sigma_{+2}\sigma_{23}\cdots\sigma_{n1}\sigma_{1-}\sigma_{-+}}.$$

$$\frac{N(2,3,\cdots,n,1;\ell)}{\sigma_{+2}\sigma_{23}\cdots\sigma_{n1}\sigma_{1-}\sigma_{-+}}$$

$$\frac{1}{\ell^2}\left[\frac{N(A,B;\ell)}{2\ell\cdot p_A+p_A^2}+\frac{N(B,A;\ell)}{-2\ell\cdot p_A+p_A^2}\right]\overset{\text{shift}}{\longmapsto}\frac{N(A,B;\ell)}{\ell^2(2\ell\cdot p_A+p_A^2)}+\frac{N(B,A;\ell+p_A)}{(\ell+p_A)^2(-2\ell\cdot p_A-p_A^2)}$$
$$=\frac{N(A,B;\ell)}{\ell^2(2\ell\cdot p_A+p_A^2)}+\frac{N(A,B;\ell)}{(\ell+p_A)^2(-2\ell\cdot p_A-p_A^2)}=\frac{N(A,B;\ell)}{\ell^2(\ell+p_A)^2}$$

$$L_\mu := 2\pi i\ell_\mu+\sum_{i=2}^{n}p_{i,\mu}g_{1,i}^{(1)}$$

$$V_{1|i,j} := g_{1,i}^{(1)}+g_{i,j}^{(1)}-g_{1,j}^{(1)}, 2\leqslant i<j\leqslant n$$

$$2\pi i\ell_\mu+\sum_{i=1}^{n-1}p_{i,\mu}g_{n,i}^{(1)}=L_\mu-\sum_{i=2}^{n-1}p_{i,\mu}V_{1|j,n}$$

$$V_{1|i,j,k} := g_{1,i}^{(2)}+g_{i,j}^{(2)}+g_{j,k}^{(2)}+g_{k,1}^{(2)}+g_{1,i}^{(1)}g_{i,j}^{(1)}+g_{1,i}^{(1)}g_{j,k}^{(1)}+g_{1,i}^{(1)}g_{k,1}^{(1)}+g_{i,j}^{(1)}g_{j,k}^{(1)}+g_{i,j}^{(1)}g_{k,1}^{(1)}$$
$$+g_{j,k}^{(1)}g_{k,1}^{(1)}$$

$$V_{1|i_1,i_2,\cdots,i_s} := F(z_1-z_{i_1},\eta,\tau)F(z_{i_1}-z_{i_2},\eta,\tau)F(z_{i_2}-z_{i_3},\eta,\tau)\cdots F(z_{i_s}-z_1,\eta,\tau)\Big|_{\eta^{-2}}$$

$$F(z,\eta,\tau)=\frac{1}{\eta}e^{\eta g^{(1)}(z,\tau)}\Big|_{\left(g^{(1)}(z,\tau)\right)^m\mapsto m!g^{(m)}(z,\tau)}.$$

$$V_{1|i_1,i_2,\cdots,i_s}=\frac{1}{(s-1)!}\left(g_{1,i_1}^{(1)}+g_{i_1,i_2}^{(1)}+\cdots+g_{i_{s-1},i_s}^{(1)}+g_{i_s,1}^{(1)}\right)^{s-1}\Big|_{\left(g_{a,b}^{(1)}\right)^m\mapsto m!g_{a,b}^{(m)}}$$

$$V_{1|i_1,i_2,\cdots,i_{s_1}}V_{1|j_1,j_2,\cdots,j_{s_2}}V_{1|k_1,k_2,\cdots,k_{s_3}},$$

$$1+\sum_{r=1}^{3}s_r=n$$

$$i_1<i_{\mathrm{a}}, j_1<j_{\mathrm{a}}, k_1<k_{\mathrm{a}},\ \text{for a}>1$$

$$E_{1|2|3,4,5,6} := \frac{1}{2\alpha'}\partial_1 g_{12}^{(1)}+p_1\cdot p_2\left(g_{12}^{(1)}\right)^2-2p_1\cdot p_2 g_{12}^{(2)}$$

$$E_{1|2|3,4,5,6}\simeq\frac{1}{p_1\cdot p_2}\left(p_1^\mu p_2^\nu\mathbf{Reg}[L_\mu L_\nu]+\sum_{i=3}^{n}p_1^\mu(p_2\cdot p_i)\mathbf{Reg}[L_\mu V_{1|2,i}]\right),$$

$$\mathcal{J}_5=2\pi i C_5^\mu\ell_\mu+\sum_{i<j}C_{5,ij}g_{ij}^{(1)}$$
$$=C_5^\mu L_\mu+\sum_{2\leqslant i<j\leqslant 5}C_{5,ij}V_{1|i,j}$$

$$C_5^{\mu} p_{i\mu} + \sum_{j \neq i} C_{5,ij} = 0$$

$$C_5^{\mu} = N(\cdots;\ell)|_{\ell_\mu} =: N_5^{\mu}, C_{5,ij} = -N(\cdots[i,j]\cdots;\ell) =: -N_5([i,j])$$

$$\begin{aligned} N(12345;\ell) &= N(\cdots;\ell)|_{\ell_\mu}\ell_\mu + N(12345;\ell)|_{\ell^0} \\ &= N_5^{\mu}\ell_\mu + \frac{1}{2}\sum_{i<j} N_5([i,j]) \end{aligned}$$

$$0 = N(i,\rho_4;\ell) - N(\rho_4,i;\ell+p_i) = -N_5^{\mu} p_{i\mu} + \sum_{j \neq i} N_5([i,j])$$

$$\mathcal{J}_5 = N_5^{\mu} L_\mu - \sum_{2 \leqslant i<j \leqslant 5} N_5([i,j]) V_{1|i,j}$$

$$\begin{aligned} \mathcal{J}_6 = {} & N_6^{\mu\nu}\mathbf{Reg}[L_\mu L_\nu] - \sum_{2 \leqslant i<j \leqslant 6} N_6^{\mu}([i,j])\mathbf{Reg}[L_\mu V_{1|i,j}] \\ & + \sum_{\substack{2 \leq i<j,k \leq 6 \\ j \neq k}} N_6([[i,j],k]) V_{1|i,j,k} + \sum_{\substack{2 \leqslant i<j \leqslant 6 \\ 2 \leqslant i<k<l \leqslant 6 \\ j \neq k,l}} N_6([i,j],[k,l]) V_{1|i,j} V_{1|k,l}. \end{aligned}$$

$$N_6^{\mu\nu} := N(\cdots;\ell)|_{\ell_\mu \ell_\nu}, N_6^{\mu}([i,j]) := N(\cdots[i,j]\cdots;\ell)|_{\ell_\mu},$$

$$\begin{aligned} &\mathbf{Reg}\left[g_{i,j}^{(1)} g_{k,l}^{(1)}\right] := g_{i,j}^{(1)} g_{k,l}^{(1)}, \{i,j\} \neq \{k,l\}, \\ &\mathbf{Reg}\left[\left(g_{i,j}^{(1)}\right)^2\right] := \left(g_{i,j}^{(1)}\right)^2 + \partial_i g_{i,j}^{(1)} + G_2(\tau) = 2g_{i,j}^{(2)}. \end{aligned}$$

$$\begin{aligned} N(123456;\ell) &= N(123456;\ell)|_{\ell_\mu \ell_\nu}\ell_\mu \ell_\nu + N(123456;\ell)|_{\ell_\mu}\ell_\mu + N(123456;\ell)|_{\ell^0} \\ &= N_6^{\mu\nu}\ell_\mu \ell_\nu + \frac{1}{2}\sum_{i<j} N_6^{\mu}([i,j])\ell_\mu + N(123456;\ell)|_{\ell^0} \end{aligned}$$

$$\begin{aligned} N(123456;\ell)\big|_{\ell^0} = {} & \frac{1}{6}\sum_{i<j<k}\left(N([[i,j],k] + N([i,[j,k]]\right) + \frac{1}{4}\sum_{\substack{i<j\,;\,k<l \\ i<k\,;\,j \neq k,l}} N([i,j],[k,l]) \\ & + \frac{1}{6!}\sum_{\rho \in S_6} N(\rho\,;\ell)\big|_{\ell^0}\,. \end{aligned}$$

$$\frac{1}{6!}\sum_{\rho \in S_6} N(\rho;\ell)\Bigg|_{\ell=0} = \frac{1}{12} N_{6,\mu\nu}\sum_{i=1}^{6} p_i^{\mu} p_i^{\nu}$$

$$\mathcal{I}_7 = N_7^{\mu\nu\rho}\,\mathbf{Reg}\left[L_\mu L_\nu L_\rho\right] - \sum_{2\leqslant i<j\leqslant 7} N_7^{\mu\nu}([i,j])\,\mathbf{Reg}\left[L_\mu L_\nu V_{1|i,j}\right]$$

$$+ \sum_{2\leqslant i<j,k\leqslant 7} N_7^{\mu}([[i,j],k])\,\mathbf{Reg}\left[L_\mu V_{1|i,j,k}\right] + \sum_{\substack{2\leqslant i<j\leqslant 7\\ 2\leqslant k<l\leqslant 7\\ i<k,\, j\neq l}} N_7^{\mu}([i,j],[k,l])\,\mathbf{Reg}\left[L_\mu V_{1|i,j} V_{1|k,l}\right]$$

$$- \sum_{\substack{2\leqslant i<j,k,l\\ j,k,l\,\mathrm{dist}}} N_7([[[i,j],k],l])\,V_{1|i,j,k,l} - \sum_{\substack{2\leqslant i<j,k\leqslant 7\\ 2\leqslant l<m\leqslant 7\\ i,j,k,l,m\,\mathrm{dist}}} N_7([[i,j],k],[l,m])\,V_{1|i,j,k} V_{1|l,m}$$

$$- \sum_{\substack{2\leqslant i<j\leqslant 7\\ 2\leqslant i<k<l\leqslant 7\\ 2\leqslant i<k<m<o\leqslant 7\\ j,k,l,m,o\,\mathrm{dist}}} N_7([i,j],[k,l],[m,o])\,V_{1|i,j} V_{1|k,l} V_{1|m,o}\,,$$

$$N_7^{\mu\nu\rho}:=N(\cdots;\ell)|_{\ell_\mu\ell_\nu\ell_\rho}, N_7^{\mu\nu}([i,j]):=N(\cdots[i,j]\cdots;\ell)|_{\ell_\mu\ell_\nu},$$
$$N_7^{\mu}([[i,j],k]):=N(\cdots[[i,j],k]\cdots;\ell)|_{\ell_\mu}, N_7^{\mu}([i,j],[k,l]):=N(\cdots[i,j]\cdots[k,l]\cdots;\ell)|_{\ell_\mu}.$$

$$\begin{aligned}
\mathrm{Reg}\left[\left(g_{i,j}^{(1)}\right)^3\right] &:= 6g_{i,j}^{(3)},\\
\mathrm{Reg}\left[g_{i,j}^{(2)} g_{i,j}^{(1)}\right] &:= 3g_{i,j}^{(3)},\\
\mathrm{Reg}\left[\left(g_{i,j}^{(1)}\right)^2 g_{k,l}^{(1)}\right] &:= 2g_{i,j}^{(2)} g_{k,l}^{(1)},\\
\mathrm{Reg}\left[g_{i,j}^{(1)} g_{j,k}^{(1)} g_{i,k}^{(1)}\right] &:= g_{i,j}^{(1)} g_{i,k}^{(2)} + g_{j,k}^{(1)} g_{i,k}^{(2)} - g_{i,j}^{(2)} g_{i,k}^{(1)} - g_{j,k}^{(2)} g_{i,k}^{(1)} - 3g_{i,k}^{(3)},\\
\mathrm{Reg}\left[g_{i,j}^{(1)} g_{k,l}^{(1)} g_{m,n}^{(1)}\right] &:= g_{i,j}^{(1)} g_{k,l}^{(1)} g_{m,n}^{(1)}.
\end{aligned}$$

$$\frac{1}{7!}\sum_{\rho\in S_7} N(\rho;\ell)\Bigg|_{\ell^1} = \frac{1}{4} N_7^{\mu\nu\rho}\ell_\mu \sum_{i=1}^{7} p_{i,\nu} p_{i,\rho}, \forall \ell$$

$$\frac{1}{6!}\sum_{\rho\in S_{6;[i,j]}} N(\rho;\ell)\Bigg|_{\ell^0} = \frac{1}{12} N_7^{\mu\nu}([i,j])\left(2p_{i,\mu}p_{j,\nu} + \sum_{k=1}^{7} p_{k,\mu}p_{k,\nu}\right) \quad (7.4)$$

$$G_2(\tau)\left(C_{7,G_2L}^{\mu} L_\mu + \sum_{2\leqslant i<j\leqslant 7} C_{7,G_2V_{1|ij}} V_{1|ij}\right).$$

$$\begin{aligned}
\mathcal{I}_8 =\ & N_8^{\mu\nu\rho\sigma}\,\mathbf{Reg}\,[L_\mu L_\nu L_\rho L_\sigma\,] - \sum_{2\leqslant i<j\leqslant 7} N_8^{\mu\nu\rho}([i,j])\,\mathbf{Reg}\,\big[L_\mu L_\nu L_\rho V_{1|i,j}\big] \\
& + \sum_{2\leqslant i<j,k\leqslant 8} N_8^{\mu\nu}([[i,j],k])\,\mathbf{Reg}\,\big[L_\mu L_\nu V_{1|i,j,k}\big] + \sum_{\substack{2\leqslant i<j\leqslant 8\\ 2\leqslant k<l\leqslant 8\\ i<k,j\neq l}} N_8^{\mu\nu}([i,j],[k,l])\,\mathbf{Reg}\,\big[L_\mu L_\nu V_{1|i,j}V_{1|k,l}\big] \\
& - \sum_{\substack{2\leqslant i<j,k,l\\ j,k,l\,\mathrm{dist}}} N_8^{\mu}([[[i,j],k],l])\,\mathbf{Reg}\,\big[L_\mu V_{1|i,j,k,l}\big] - \sum_{\substack{2\leqslant i<j,k\leqslant 8\\ 2\leqslant l<m\leqslant 8\\ i,j,k,l,m\,\mathrm{dist}}} N_8^{\mu}([[i,j],k],[l,m])\,\mathbf{Reg}\,\big[L_\mu V_{1|i,j,k}V_{1|l,m}\big] \\
& - \sum_{\substack{2\leqslant i<j\leqslant 8\\ 2\leqslant i<k<l\leqslant 8\\ 2\leqslant i<k<m<n\leqslant 8\\ j,k,l,m,o\,\mathrm{dist}}} N_8^{\mu}([i,j],[k,l],[m,o])\,\mathbf{Reg}\,\big[L_\mu V_{1|i,j}V_{1|k,l}V_{1|m,o}\big] \\
& + \sum_{\substack{2\leqslant i<j,k,l,m\leqslant 8\\ j,k,l,m\,\mathrm{dist}}} N_8([[[[i,j],k],l],m])\,V_{1|i,j,k,l,m} \\
& + \sum_{\substack{2\leqslant i<j,k,l\leqslant 8\\ 2\leqslant m<o\leqslant 8\\ i,j,k,l,m,o\,\mathrm{dist}}} N_8([[[i,j],k],l],[m,o])\,V_{1|i,j,k,l}V_{1|m,o} + \sum_{\substack{2\leqslant i<j,k\leqslant 8\\ 2\leqslant i<l,m,o\leqslant 8\\ j,k,l,m,o\,\mathrm{dist}}} N_8([[i,j],k],[[l,m],o])\,V_{1|i,j,k}V_{l,m,o} \\
& + \sum_{\substack{2\leqslant i<j,k\leqslant 8\\ 2\leqslant l<m\leqslant 8\\ 2\leqslant l<o<q\leqslant 8\\ i,j,k,l,m,o,q\,\mathrm{dist}}} N_8([[i,j],k],[l,m],[o,q])\,V_{1|i,j,k}V_{1|l,m}V_{1|o,q} \\
& + \frac{(2\pi i)^4}{7!}\sum_{\rho\in S_7} N(1\rho(2)\cdots\rho(8);-\frac{p_1}{2})\,\mathcal{E}_4(\tau)\,,
\end{aligned}$$

$$\mathcal{E}_{2k}(\tau) := \frac{G_{2k}(\tau)}{2\zeta(2k)},$$

$$\frac{1}{7!}\sum_{\rho\in S_7} N(1,\rho;\ell) = N_8^{\mu\nu\rho\sigma}\left(\ell+\frac{1}{2}p_1\right)_\mu\left(\ell+\frac{1}{2}p_1\right)_\nu\left(\ell+\frac{1}{2}p_1\right)_\rho\left(\ell+\frac{1}{2}p_1\right)_\sigma + \frac{1}{(2\pi i)^4}C_{8,\mathcal{E}_4}$$

$$C_{8,\mathcal{E}_4} = \frac{(2\pi i)^4}{7!}\sum_{\rho\in S_7} N\left(1,\rho;-\frac{p_1}{2}\right)$$

$$\sum_{\rho\in S_7} N\left(1,\rho;\ell-\frac{1}{2}p_1\right) = \sum_{\rho\in S_7} N\left(\rho,1;-\ell+\frac{1}{2}p_1\right) = \sum_{\rho\in S_7} N\left(\rho,1;\ell+\frac{1}{2}p_1\right)$$

$$\frac{1}{8!}\sum_{\rho\in S_8} N(\rho;\ell)\Bigg|_{\ell^2} = \frac{1}{2}N_8^{\mu\nu\rho\sigma}\ell_\mu\ell_\nu\sum_{i=1}^{8} p_{i,\rho}p_{i,\sigma}, \forall\ell,$$

$$\frac{1}{7!}\sum_{\rho\in S_{7;[i,j]}} N(\rho;\ell)\Bigg|_{\ell^1} = \frac{1}{4}N_8^{\mu\nu\rho}([i,j])\ell_\mu\left(2p_{i,\nu}p_{j,\rho}+\sum_{k=1}^{8} p_{k,\nu}p_{k,\rho}\right), \forall\ell,$$

$$\frac{1}{6!}\sum_{\rho\in S_{6;[[i,j],k]}} N(\rho;\ell)\Bigg|_{\ell^0} = \frac{1}{12}N_8^{\mu\nu}([[i,j],k])\left(2p_{i,\mu}p_{j,\nu}+2p_{i,\mu}p_{k,\nu}+2p_{j,\mu}p_{k,\nu}+\sum_{l=1}^{8} p_{l,\mu}p_{l,\nu}\right),$$

$$\frac{1}{6!}\sum_{\rho\in S_{6;[i,j],[k,l]}} N(\rho;\ell)\Bigg|_{\ell^0} = \frac{1}{12}N_8^{\mu\nu}([i,j],[k,l])\left(2p_{i,\mu}p_{j,\nu}+2p_{k,\mu}p_{l,\nu}+\sum_{m=1}^{8} p_{m,\mu}p_{m,\nu}\right).$$

$$G_2(\tau)\Bigg(C_{8,G_2LL}^{\mu\nu}\mathbf{Reg}\big[L_{(\mu}L_{\nu)}\big]-\sum_{2\leqslant i<j\leqslant 6} C_{8,G_2LV_{1|i,j}}^{\mu}\mathbf{Reg}\big[L_\mu V_{1|i,j}\big]$$

$$+\sum_{\substack{2\leq i<j,k\leq 6\\ j\neq k}} C_{8,G_2V_{1|i,j,k}}V_{1|i,j,k}+\sum_{\substack{2\leqslant i<j\leqslant 6\\ 2\leqslant i<k<l\leqslant 6\\ j\neq k,l}} C_{8,G_2V_{1|i,j}V_{1|k,l}}V_{1|i,j}V_{1|k,l}\Bigg).$$

$$\gamma_{2K}(\tau)\mathbf{Reg}\left[L_{\mu_1}L_{\mu_2}\cdots L_{\mu_{w_L}}\prod_{r=1}^{v} V_{1|i_1^{(r)}i_2^{(r)}\cdots i_{s_r}^{(r)}}\right],$$

$$2K+w_L+\sum_{r=1}^{v}(s_r-1)=n-4.$$

$$(-1)^{n-w_L}N_n^{\mu_1\mu_2\cdots\mu_{w_L}}(I_1,I_2,\cdots,I_v),\ \text{for } K=0$$

$$I_r := \left[\left[\cdots\left[\left[\left[i_1^{(r)},i_2^{(r)}\right],i_3^{(r)}\right],i_4^{(r)}\right],\cdots\right],i_{s_r}^{(r)}\right]$$

$$V_{1|i_1^{(r)}i_2^{(r)}i_3^{(r)}i_4^{(r)}\cdots i_{sr}^{(r)}}N_n^{\mu_1\mu_2\cdots\mu_{w_L}}(I_1,I_2,\cdots,I_v) := \frac{1}{w_L!}\frac{\partial^{w_L}}{\partial\ell_{\mu_1}\partial\ell_{\mu_1}\cdots\partial\ell_{\mu_{w_L}}}N(1,I_1,I_2,\cdots,I_v;\ell)$$

$$(-1)^{n-w_L}\frac{(2\pi i)^{2K}}{v!}\sum_{\rho\in S_{v;I_1,I_2,\cdots,I_v}} N_n^{\mu_1\mu_2\cdots\mu_{w_L}}\left(\rho;-\frac{p_1}{2}\right),\ \text{for } \gamma_{2K}(\tau)=\mathcal{E}_{2K}(\tau)$$

where

$$N_n^{\mu_1\mu_2\cdots\mu_{w_L}}\left(I_1,I_2,\cdots,I_v;-\frac{p_1}{2}\right) := \frac{1}{w_L!}\frac{\partial^{w_L}}{\partial\ell_{\mu_1}\partial\ell_{\mu_1}\cdots\partial\ell_{\mu_{w_L}}}N(1,I_1,I_2,\cdots,I_v;\ell)\Bigg|_{\ell=-\frac{p_1}{2}}.$$

$$\hat{L}_\mu := 2\pi i \ell_\mu + \sum_{i=2}^{n} p_{i,\mu} \left(g_{1,i}^{(1)} - \hat{D}_i \right)$$

$$\mathbf{Reg}^{(\text{naive})} \left[L_{\mu_1} \cdots L_{\mu_{w_L}} \prod_S V_{1|S} \right] := \hat{L}_{\mu_1} \cdots \hat{L}_{\mu_{w_L}} \prod_S V_{1|S},$$

$$\left(g_{1,i}^{(1)} - \partial_i \right) g_{1,i}^{(m)} = (m+1) g_{1,i}^{(m+1)} - \sum_{\substack{k=2 \\ k \text{ even}}}^{m+1} G_k(\tau) g_{1,i}^{(m+1-k)}$$

$$\partial_i g_{i,j}^{(m)} = -g_{i,j}^{(1)} g_{i,j}^{(m)} + (m+1) g_{i,j}^{(m+1)} - \sum_{\substack{k=2 \\ k \text{ even}}}^{m+1} G_k(\tau) g_{i,j}^{(m+1-k)}$$

$$\sum_{m=1}^{\infty} \frac{\partial g_{1,i}^{(m)}}{\partial z_i} \frac{\partial}{\partial g_{1,i}^{(m)}}$$

$$g_{1,i_1}^{(m_0)} g_{i_1,i_2}^{(m_1)} \cdots g_{i_k,i}^{(m_k)}$$

$$g_{1,j}^{(m)} g_{i,j}^{(m')} = -(-1)^{m'} g_{1,i}^{(m+m')} + \sum_{a=0}^{m} (-1)^a \binom{m'+a-1}{a} g_{1,i}^{(m-a)} g_{i,j}^{(m'+a)}$$
$$+ \sum_{a=0}^{m'} (-1)^{m'} \binom{m+a-1}{a} g_{1,j}^{(m+a)} g_{1,i}^{(m'-a)}$$

$$g_{1,i_1}^{(m_0)} g_{i_1,i_2}^{(m_1)} \cdots g_{i_k,i}^{(m_k)}$$
$$= \sum_{\substack{a_p \in \{-m_p\} \cup \mathbb{Z}^{\geqslant 0} \\ \sum a_p \leqslant 0 \\ a_p \text{ not all zero}}} (-1)^{1+\sum_p a_p} \prod_{p=0}^{m} \binom{m_p + a_p - 1}{a_p} g_{1,i}^{(-a_0 - \cdots - a_k)} g_{1,i_1}^{(m_0+a_0)} g_{i_1,i_2}^{(m_1+a_1)} \cdots g_{i_k,i}^{(m_k+a_k)}$$

$$g_{1,i}^{(\cdot)} g_{1,i_1}^{(\cdot)} g_{i_1,i_2}^{(\cdot)} \cdots g_{i_k,i}^{(\cdot)} = g_{i_p,i_{p+1}}^{(m_p+a_p)} g_{i_p,i_{p+1}}^{(0)} = \binom{m_p + a_p - 1}{a_p} = (-1)^{m_p - 1}$$

$$\hat{D}_i(f) := \sum_{m=1}^{\infty} \frac{\partial g_{1,i}^{(m)}}{\partial z_i} \frac{\partial}{\partial g_{1,i}^{(m)}} \left(f|_{\blacksquare_\star^\dagger} \right)$$

$$\mathcal{P}_G \left[\prod_s G_{2K_s}(\tau) \right] := \frac{\prod_s \left(2\zeta(2K_s) \right)}{2\zeta(2 \sum_s K_s)} G_{2\sum_s K_s}(\tau).$$

$$\mathbf{Reg}\left[L_{\mu_1}\cdots L_{\mu_m}\prod_S V_{1|S}\right]:=\mathcal{P}_G\left[\hat{L}_{\mu_1}\cdots\hat{L}_{\mu_m}\prod_S V_{1|S}\right]\Bigg|_{G_2(\tau)\mapsto 0}$$

$$\mathbf{Reg}[L_\mu L_\nu]=\hat{L}_\mu\hat{L}_\nu\big|_{G_2\mapsto 0}=\hat{L}_\mu L_\nu\big|_{G_2\mapsto 0}$$
$$=(2\pi i)^2\ell_\mu\ell_\nu+4\pi i\ell_{(\mu}\sum_{i=2}^{6}p_{i,\nu)}g_{1,i}^{(1)}+\sum_{i\neq j}p_{i,\mu}p_{j,\nu}g_{1,i}^{(1)}g_{1,j}^{(1)}+\sum_{i=2}^{6}p_{i,\mu}p_{i,\nu}\left(\left(g_{1,i}^{(1)}\right)^2-\partial_i g_{1,i}^{(1)}\right)\Bigg|_{G_2\mapsto 0}$$
$$=(2\pi i)^2\ell_\mu\ell_\nu+4\pi i\ell_{(\mu}\sum_{i=2}^{6}p_{i,\nu)}g_{1,i}^{(1)}+\sum_{i\neq j}p_{i,\mu}p_{j,\nu}g_{1,i}^{(1)}g_{1,j}^{(1)}+\sum_{i=2}^{6}2p_{i,\mu}p_{i,\nu}g_{1,i}^{(2)}$$

$$\mathrm{Reg}[L_\mu V_{1|i,j}]=\hat{L}_\mu V_{1|i,j}\big|_{G_2\mapsto 0}$$
$$=2\pi i\ell_\mu V_{1|i,j}+\sum_{k\neq i,j}p_{k,\mu}g_{1,k}^{(1)}V_{1|i,j}+p_{i,\mu}\left(\left(g_{1,i}^{(1)}\right)^2-\partial_i g_{1,i}^{(1)}\right)-p_{j,\mu}\left(\left(g_{1,j}^{(1)}\right)^2-\partial_j g_{1,j}^{(1)}\right)\Bigg|_{G_2\mapsto 0}$$
$$=2\pi i\ell_\mu V_{1|i,j}+\sum_{k\neq i,j}p_{k,\mu}g_{1,k}^{(1)}V_{1|i,j}+2p_{i,\mu}g_{1,i}^{(2)}-2p_{j,\mu}g_{1,j}^{(2)}$$

$$\mathbf{Reg}[L_{\mu_1}L_{\mu_2}L_{\mu_3}L_{\mu_4}]:=\mathcal{P}_G[\hat{L}_{\mu_1}\hat{L}_{\mu_2}\hat{L}_{\mu_3}\hat{L}_{\mu_4}]\big|_{G_2(\tau)\mapsto 0}$$
$$=\mathcal{P}_G\left[\left(2\pi i\ell+\sum_{i=2}^{n}p_i\left(g_{1,i}^{(1)}-\partial_i\right)\right)_{\mu_1\mu_2\mu_3\mu_4}\right]\Bigg|_{G_2(\tau)\mapsto 0}$$
$$=\left(2\pi i\ell+\sum_{i=2}^{n}p_i g_{1,i}^{(1)}\right)_{\mu_1\mu_2\mu_3\mu_4}\Bigg|_{\left(g_{a,b}^{(1)}\right)^m\mapsto m!g_{a,b}^{(m)}}$$
$$+\left(\sum_{i=2}^{n}9p_{i,\mu_1}p_{i,\mu_2}p_{i,\mu_3}p_{i,\mu_4}+\sum_{2\leqslant i<j\leqslant n}60p_{i,(\mu_1}p_{i,\mu_2}p_{j,\mu_3}p_{j,\mu_4)}\right)G_4(\tau),$$

$$(\cdot)_{\mu_1\mu_2\mu_3\mu_4}:=(\cdot)_{\mu_1}(\cdot)_{\mu_2}(\cdot)_{\mu_3}(\cdot)_{\mu_4}$$

$$\Delta(\tau)=\frac{(2\pi)^{12}}{12^3}[(\mathcal{E}_4(\tau))^3-(\mathcal{E}_6(\tau))^2]=(60G_4(\tau))^3-27(140G_6(\tau))^2=(2\pi)^{12}q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

$$C_{16,\mathcal{E}_{12}}\mathcal{E}_{12}(\tau)+C_{16,\Delta}\Delta(\tau)$$

$$C_{16,\mathcal{E}_{12}}=\frac{(2\pi i)^{12}}{11!}\sum_{\rho\in S_11}N\left(1,\rho;-\frac{p_1}{2}\right).$$

$$\dim M_{2K}=\begin{cases}0, & \text{if } K\notin\mathbb{N}_0,\\ \lfloor K/6\rfloor+1, & \text{if } K\geq 0, K\not\equiv 1(\mathrm{mod}6),\\ \lfloor K/6\rfloor, & \text{if } K\geq 0, K\equiv 1(\mathrm{mod}6).\end{cases}$$

$$\mathcal{S}(x_1,\cdots,x_m;\tau):=\sum_{\nu=1,2,3}(-1)^\nu\left(\frac{\theta_{\nu+1}(0,\tau)}{\theta_1'(0,\tau)}\right)^4 S_\nu(x_1,\tau)S_\nu(x_2,\tau)\cdots S_\nu(x_m,\tau),$$

$$S_v(z,\tau):=\frac{\theta_1'(0,\tau)\theta_{v+1}(z,\tau)}{\theta_{v+1}(0,\tau)\theta_1(z,\tau)}$$

$$\mathcal{S}(x_1,\cdots,x_m;\tau)=V_{m-4}(x_1,\cdots,x_m;\tau)+\sum_{K=2}^{\left\lfloor\frac{m}{2}\right\rfloor-2}\mathcal{G}_{2K}(\tau)V_{m-4-2K}(x_1,\cdots,x_m;\tau),$$

$$V_p(x_1,\cdots,x_m;\tau):=\left(\prod_{r=1}^{m}F(x_r,\eta,\tau)\right)\Bigg|_{\eta^{p-m}}$$

$$g_{12}^{(2)}g_{12}^{(2)}-2g_{12}^{(1)}g_{12}^{(3)}+2g_{12}^{(4)}-3G_4(\tau)=0$$

$$G_2(\tau)\mathbf{Reg}\left[L_{\mu_1}L_{\mu_2}\cdots L_{\mu_{w_L}}\prod_{r=1}^{v}V_{1|i_1^{(r)}i_2^{(r)}\cdots i_{s_r}^{(r)}}\right]$$

$$\frac{1}{(v+1)!}\sum_{\rho\in S_{1+v};I_1,\cdots,I_v,I_{v+1}}N(\rho;\ell)\Bigg|_{\ell^{v-5}}$$
$$=\frac{1}{12}\binom{v-3}{2}N(I_1,\cdots,I_v,I_{v+1};\ell)\Big|_{\ell_{\mu_1}\cdots\ell_{\mu_{v-3}}}\ell_{\mu_3}\cdots\ell_{\mu_{v-3}}\left(\sum_{r=1}^{v+1}p_{I_r,\mu_1}p_{I_r,\mu_2}\right)$$

$$V\sim\left(\epsilon\cdot\partial X+\frac{\alpha'}{2}k\cdot\Psi\epsilon\cdot\Psi\right)e^{ik\cdot X}=\alpha'\left(\epsilon\cdot\partial(X/\alpha')+\frac{1}{2}k\cdot\Psi\epsilon\cdot\Psi\right)e^{ik\cdot X}.$$

$$\nabla_i(f)\mathrm{KN}_n=\partial_{z_i}(f\mathrm{KN}_n)$$

$$E_{1|2|3,4,5,6}=\frac{1}{s_{12}}\left(p_1^{\mu}p_2^{\nu}\mathbf{Reg}[L_\mu L_\nu]+\sum_{i=3}^{n}p_1^{\mu}s_{2i}\mathbf{Reg}[L_\mu V_{1|2,i}]\right)-\frac{1}{s_{12}}\nabla_2(p_1\cdot L)$$

$$E_{1|23|4,5,6,7}=-s_{123}V_{1|2,3}^{(3)}+\frac{1}{2\alpha'}\left(g_{12}^{(1)}+g_{31}^{(1)}\right)\partial g_{23}^{(1)}+\frac{1}{2\alpha'}\partial g_{23}^{(2)}$$
$$E_{1|4|23,5,6,7}=\left[\frac{1}{2\alpha'}\partial g_{14}^{(1)}-2s_{14}g_{14}^{(2)}+s_{14}\left(g_{14}^{(1)}\right)^2\right]V_{1|2,3}-s_{24}V_{1|2,4}^{(3)}+s_{34}V_{1|3,4}^{(3)}$$
$$E_{1|2|3,4,5,6,7}^{\mu}=\left[\frac{1}{2\alpha'}\partial g_{12}^{(1)}-2s_{12}g_{12}^{(2)}+s_{12}\left(g_{12}^{(1)}\right)^2\right]\left(2\pi i\ell^{\mu}+\sum_{j\geq3}k_j^{\mu}g_{1j}^{(1)}\right)+\sum_{j\geq3}k_j^{\mu}s_{2j}V_{1|2,j}^{(3)}$$
$$+k_2^{\mu}\left[\frac{1}{2\alpha'}\partial g_{12}^{(2)}+s_{12}\left(g_{12}^{(1)}g_{12}^{(2)}-3g_{12}^{(3)}\right)\right]$$

$$V_{1|i,j}^{(3)}:=\frac{1}{3!}\left(g_{1i}^{(1)}+g_{ij}^{(1)}-g_{1j}^{(1)}\right)^3\Bigg|_{\left(g_{ab}^{(1)}\right)^m\to m!g_{ab}^{(m)}}$$

$$E_{1|4|23,5,6,7} = \frac{1}{s_{14}} \mathbf{Reg}\left[L^{(1)} L^{(4)} V_{1|2,3}\right] - \frac{1}{s_{14}} \nabla_4 \mathbf{Reg}\left[L^{(1)} V_{1|2,3}\right]$$

$$E^{\mu}_{1|2|3,4,5,6,7} = \mathbf{Reg}\left[\frac{1}{s_{12}} L^{\mu} L^{(1)} L^{(2)} - \frac{p_2^{\mu}}{2s_{12}^2} L^{(2)} \left(L^{(1)}\right)^2\right] + \nabla_2 \mathbf{Reg}\left[\frac{p_2^{\mu}}{2s_{12}^2}\left(L^{(1)}\right)^2 - \frac{1}{s_{1,2}} L^{\mu} L^{(1)}\right]$$

$$E_{1|23|4,5,6,7} = \frac{1}{2s_{23}^2(s_{12}+s_{13})} \mathbf{Reg}\left[L^{(2)} L^{(3)}\left(s_{12} L^{(3)} - s_{13} L^{(2)} - 2s_{23}(s_{12}+s_{13}) V_{1|2,3}\right)\right]$$

$$+ \left(\frac{1}{2s_{23}^2(s_{12}+s_{13})} \nabla_2 \mathbf{Reg}\left[s_{12}\left(2s_{23} V_{1|2,3} - L^{(3)}\right) L^{(3)} + 2s_{23} L^{(1)} L^{(3)}\right] - (2 \leftrightarrow 3)\right)$$

$$L^{(a)} := \frac{1}{2\alpha'} \frac{\partial}{\partial z_a} \mathrm{KN} = 2\pi i \ell_{\mu} p_a^{\mu} + \sum_{i \neq a} s_{a,i} g_{a,i}^{(1)}$$

$$L^{(a)} = \begin{cases} p_a^{\mu} L_{\mu} & a = 1 \\ p_a^{\mu}\left[L_{\mu} + \sum\limits_{i \neq 1,a} p_{i,\mu} V_{1|a,i}\right], & a \neq 1 \end{cases}$$

$$\mathbf{Reg}\left[g_{i,j}^{(1)} g_{k,l}^{(1)}\right] := g_{i,j}^{(1)} g_{k,l}^{(1)}, \{i,j\} \neq \{k,l\}$$
$$\mathbf{Reg}[(\boldsymbol{g}_{1,i}^{(1)})^2] := (\boldsymbol{g}_{1,i}^{(1)})^2 + \boldsymbol{\partial}_1 \boldsymbol{g}_{1,i}^{(1)} + \boldsymbol{G}_2(\boldsymbol{\tau}) = 2\boldsymbol{g}_{1,i}^{(2)}$$

$$\mathbf{Reg}'\left[g_{i,j}^{(1)} g_{k,l}^{(1)}\right] := g_{i,j}^{(1)} g_{k,l}^{(1)}, \{i,j\} \neq \{k,l\}$$
$$\mathbf{Reg}'\left[\left(g_{1,i}^{(1)}\right)^2\right] := \left(g_{1,i}^{(1)}\right)^2 + \partial_1 g_{1,i}^{(1)} = 2g_{1,i}^{(2)} - G_2(\tau)$$

$$\mathcal{I}_6 = N_6^{\mu\nu} \mathbf{Reg}'\left[L_{(\mu} L_{\nu)}\right] - \sum_{2 \leqslant i < j \leqslant 6} N_6^{\mu}([i,j]) \mathbf{Reg}'\left[L_{\mu} V_{1|i,j}\right]$$
$$+ \sum_{\substack{2 \leq i < j, k \leq 6 \\ j \neq k}} N_6([[i,j],k]) V_{1|i,j,k} + \sum_{\substack{2 \leqslant i < j \leqslant 6 \\ 2 \leqslant i < k < l \leqslant 6 \\ j \neq k,l}} N_6([i,j],[k,l]) V_{1|i,j} V_{1|k,l}$$
$$+ \frac{(2\pi i)^2}{5!} \sum_{\rho \in S_5} N\left(1\rho(2) \cdots \rho(6); -\frac{p_1}{2}\right) \frac{G_2(\tau)}{2\zeta(2)}$$

$$\mathcal{I}_{n,0}^{(w)} = f^{(w)}(z_1, \cdots, z_{n-1}) + \sum_{i=1}^{n-1} \sum_{m=1}^{w} g_{i,n}^{(m)} f_i^{(w-m)}(z_1, \cdots, z_{n-1})$$

$$0 = \sum_{i=1}^{n-1} \sum_{m=1}^{w} \left(\sum_{k=0}^{m-1} \frac{(-2\pi i)^{m-k}}{(m-k)!} g_{i,n}^{(k)}\right) f_i^{(w-m)}(z_1, \cdots, z_{n-1})$$

$$f_i^{(w-m)}(z_1, \cdots, z_{n-1}) = 0, \forall m > 1$$
$$\sum_{i=1}^{n-1} f_i^{(w-1)}(z_1, \cdots, z_{n-1}) = 0$$

$$\mathcal{J}_{n,0}^{(w)}= f^{(w)}(z_1,\cdots,z_{n-1}) + \sum_{i=2}^{n-1} \left(g_{i,n}^{(1)} - g_{1,n}^{(1)}\right) f_i^{(w-1)}(z_1,\cdots,z_{n-1})$$

$$= \tilde{f}^{(w)}(z_1,\cdots,z_{n-1}) + \sum_{i=2}^{n-1} \left(g_{i,n}^{(1)} - g_{1,n}^{(1)} - g_{1,i}^{(1)} + \hat{D}_i\right) f_i^{(w-1)}(z_1,\cdots,z_{n-1})\Big|_{G_{2K}\mapsto 0}$$

$$\tilde{f}^{(w)}(z_1,\cdots,z_{n-1}) := f^{(w)}(z_1,\cdots,z_{n-1}) + \sum_{i=2}^{n-1} \left(g_{1,i}^{(1)} - \hat{D}_i\right) f_i^{(w-1)}(z_1,\cdots,z_{n-1})\Big|_{G_{2K}\mapsto 0}$$

$$\mathcal{J}_{n,0}^{(w)}\Big|_{z_a\to z_a+\tau} = \left[\tilde{f}^{(w)}(z_1,\cdots,z_{n-1})\Big|_{z_a\to z_a+\tau}\right]$$
$$+ \sum_{i=2}^{n-1} \left(g_{i,n}^{(1)} - g_{1,n}^{(1)} - g_{1,i}^{(1)} + \hat{D}_i\right) \left[f_i^{(w-1)}(z_1,\cdots,z_{n-1})\Big|_{z_a\to z_a+\tau}\right]\Bigg|_{G_{2K}\mapsto 0}$$

$$\prod_{r=1}^{v} V_{1|i_1^{(r)} i_2^{(r)} \cdots i_{s_r}^{(r)}}$$

$$\left(g_{i,n}^{(1)} - g_{1,n}^{(1)} - g_{1,i}^{(1)} + \hat{D}_i\right) \prod_{r=1}^{v} V_{1|i_1^{(r)} i_2^{(r)} \cdots i_{s_r}^{(r)}}\Bigg|_{G_{2K}\mapsto 0}$$

$$\left(g_{i,n}^{(1)} - g_{1,n}^{(1)} - g_{1,i}^{(1)} + \hat{D}_i\right) \prod_{r=1}^{v} V_{1|i_1^{(r)} i_2^{(r)} \cdots i_{s_r}^{(r)}}\Bigg|_{G_{2K}\mapsto 0} = -V_{1|i,n} \prod_{r=1}^{v} V_{1|i_1^{(r)} i_2^{(r)} \ldots i_{s_r}^{(r)}}$$

$$\left(g_{i_k^{(m)},n}^{(1)} - g_{1,n}^{(1)} - g_{1,i_k^{(m)}}^{(1)} + \hat{D}_{i_k^{(m)}}\right) \prod_{r=1}^{v} V_{1|i_1^{(r)} i_2^{(r)} \cdots i_{sr}^{(r)}}\Bigg|_{G_{2K}\mapsto 0}$$
$$= \sum_{p=k}^{s_m} V_{1|i_1^{(m)} \cdots i_p^{(m)} n i_{p+1}^{(m)} \cdots i_{sm}^{(m)}} \prod_{r\neq m} V_{1|i_1^{(r)} i_2^{(r)} \cdots i_{s_r}^{(r)}}.$$

$$F(z+1,\eta,\tau) = F(z,\eta,\tau), F(z+\tau,\eta,\tau) = e^{-2i\pi\eta} F(z,\eta,\tau)$$

$$G_{2K}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2K} G_{2K}(\tau) \text{ for } K \geq 2$$

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \frac{2\pi i c}{c\tau+d}$$

$$N_4 = \mathrm{tr}(f_1 f_2 f_3 f_4) - \frac{1}{4}\mathrm{tr}(f_1 f_2)\mathrm{tr}(f_3 f_4) + \mathrm{cyc}(2,3,4), \text{ where } f_i^{\mu\nu} = p_i^\mu \epsilon_i^\nu - \epsilon_i^\mu p_i^\nu$$

$$N(\rho;\ell)|_{\ell_{\mu_1}\ell_{\mu_2}\cdots\ell_{\mu_p}} = \frac{1}{p!}\frac{\partial^p}{\partial\ell_{\mu_1}\partial\ell_{\mu_2}\cdots\partial\ell_{\mu_p}} N(\rho;\ell)\Big|_{\ell=0}$$

$$(2\pi i)^4 \mathcal{E}_4(\tau)\left(\frac{1}{8!}\sum_{\rho\in S_8} N_9^\mu\left(\rho,-\frac{p_1}{2}\right) L_\mu - \sum_{2\leqslant i<j\leqslant 7} \frac{1}{7!}\sum_{\rho\in S_{7;[i,j]}} N_9\left(\rho,-\frac{p_1}{2}\right) V_{1|ij}\right)$$

$$S = \int_{\Sigma} S_{(Z,\rho_1,A,B)} + S_{(\lambda,\eta,\rho_2,\tau,D)} + S_{\mathrm{aux}}$$

$$S_{(Z,\rho,A,B)} = \int_{\Sigma} Z^{a} \cdot \bar{\partial} Z_{a} + A_{ab} Z^{a} \cdot Z^{b} + \frac{1}{2} \langle \rho_A \bar{\partial} \rho_B \rangle \Omega^{AB}$$
$$+ a \langle \lambda_A \lambda_B \rangle \Omega^{AB} + B_{ab} \lambda_A^{a} \rho_B^{b} \Omega^{AB} + b \langle \lambda_A \rho_B \rangle \Omega^{AB},$$

$$S_{(\lambda,\eta,\rho,\tau,D)} = \int_{\Sigma} \frac{1}{2} \left(\langle \rho_A \bar{\partial} \rho_B \rangle \Omega^{AB} + \langle \tau_{\mathcal{J}} \bar{\partial} \tau_{\mathcal{J}} \rangle \Omega^{\mathcal{J}\mathcal{J}} \right) + \tilde{a} \left(\langle \lambda_A \lambda_B \rangle \Omega^{AB} + \langle \eta_{\mathcal{J}} \eta_{\mathcal{J}} \rangle \Omega^{\mathcal{J}\mathcal{J}} \right)$$
$$+ D_{ab} \left(\lambda_A^{a} \rho_B^{b} \Omega^{AB} + \eta_{\mathcal{J}}^{a} \tau_{\mathcal{J}}^{b} \Omega^{\mathcal{J}\mathcal{J}} \right) + d \left(\langle \lambda_A \rho_B \rangle \Omega^{AB} + \langle \eta_{\mathcal{J}} \tau_{\mathcal{J}} \rangle \Omega^{\mathcal{J}\mathcal{J}} \right)$$
$$S_{\mathrm{aux}} = \sum_{i=1}^{2} \int_{\Sigma} S_{(Z_i,\rho_{i1},A_i,B_i)} + S_{(\lambda_i,\eta_i,\rho_{i2},\tau_i,D_i)} + \tilde{\tau}_i^{\mathcal{J}} \bar{\partial} \tau'_{i\mathcal{J}}$$

$$S = \int_{\Sigma} Z^{\mathrm{a}} \cdot \bar{\partial} Z_{\mathrm{a}} + A_{\mathrm{ab}} Z^{\mathrm{a}} \cdot Z^{\mathrm{b}} + S_{\rho_1} + S_{(\rho_2,\tau)} + S_{\mathrm{aux}}$$
$$S_{\rho} = \int_{\Sigma} \frac{1}{2} \rho_{\mathrm{a}A} \bar{\partial} \rho_B^{\mathrm{a}} \Omega^{AB} + a \lambda_{\mathrm{a}A} \lambda_B^{\mathrm{a}} \Omega^{AB} + B_{\mathrm{ab}} \lambda_A^{\mathrm{a}} \rho_B^{\mathrm{b}} \Omega^{AB} + b \lambda_{\mathrm{a}A} \rho_B^{\mathrm{a}} \Omega^{AB}$$
$$S_{(\rho,\tau)} = \int_{\Sigma} \frac{1}{2} \left(\rho_{\mathrm{a}A} \bar{\partial} \rho_B^{\mathrm{a}} \Omega^{AB} + \tau_{\mathrm{a}\mathcal{J}} \bar{\partial} \tau_{\mathcal{J}}^{\mathrm{a}} \Omega^{\mathcal{J}\mathcal{J}} \right) + \tilde{a} \left(\lambda_{\mathrm{a}A} \lambda_B^{\mathrm{a}} \Omega^{AB} + \eta_{\mathrm{a}\mathcal{J}} \eta_{\mathcal{J}}^{\mathrm{a}} \Omega^{\mathcal{J}\mathcal{J}} \right)$$
$$+ D_{\mathrm{ab}} \left(\lambda_A^{\mathrm{a}} \rho_B^{\mathrm{b}} \Omega^{AB} + \eta_{\mathcal{J}}^{\mathrm{a}} \tau_{\mathcal{J}}^{\mathrm{b}} \Omega^{\mathcal{J}\mathcal{J}} \right) + d \left(\lambda_{\mathrm{a}A} \rho_B^{\mathrm{a}} \Omega^{AB} + \eta_{\mathrm{a}\mathcal{J}} \tau_{\mathcal{J}}^{\mathrm{a}} \Omega^{\mathcal{J}\mathcal{J}} \right),$$
$$S_{\mathrm{aux}} = \sum_{i=1}^{2} \int_{\Sigma} \tilde{\tau}_i^{\mathcal{J}} \bar{\partial} \tau'_{i\mathcal{J}},$$

$$\varepsilon^{\mathrm{ab}} = \varepsilon^{ab} \oplus \varepsilon^{\dot{a}\dot{b}} \oplus \varepsilon^{\ddot{a}\ddot{b}} \text{ and } \varepsilon_{\mathrm{ab}} = \varepsilon_{ab} \oplus \varepsilon_{\dot{a}\dot{b}} \oplus \varepsilon_{\ddot{a}\ddot{b}}$$

$$\xi_a \zeta_b \varepsilon^{ab} = \xi_a \zeta^a \equiv \langle \xi \zeta \rangle$$

$$Z = (\lambda_A, \mu^A, \eta_\iota) \colon \lambda_A = (\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}), \mu^A = (\tilde{\mu}^{\alpha}, \mu^{\dot{\alpha}}), \eta_\iota = (\eta_{\mathcal{J}}, \tilde{\eta}^{\mathcal{J}}), \eta_{\mathcal{J}} = (\eta_I, \tilde{\eta}_{\dot{I}}), \tilde{\eta}^{\mathcal{J}} = (\tilde{\eta}'^{I}, \eta^{\dot{I}}),$$
$$I, \dot{I} = 1 \ldots \frac{\mathcal{N}}{2}, \mathcal{J} = 1, \ldots, \mathcal{N}, \iota = 1, \ldots, 2\mathcal{N}, \mathcal{N} = 8,$$

$$Z_{\mathrm{a}} \cdot Z_{\mathrm{b}} = \frac{1}{2} \left(\tilde{Z}_{\mathrm{a}} \cdot Z_{\mathrm{b}} + \tilde{Z}_{\mathrm{b}} \cdot Z_{\mathrm{a}} + \tilde{\eta}_{\mathrm{a}}^{\mathcal{J}} \eta_{\mathrm{b}\mathcal{J}} + \tilde{\eta}_{\mathrm{b}}^{\mathcal{J}} \eta_{\mathrm{a}\mathcal{J}} \right), \tilde{Z}_{\mathrm{a}} \cdot Z_{\mathrm{b}} = \tilde{\mu}_{\mathrm{a}}^{\alpha} \lambda_{\mathrm{b}\alpha} + \tilde{\lambda}_{\mathrm{a}\dot{\alpha}} \mu_{\mathrm{b}}^{\dot{\alpha}} = \mu_{\mathrm{a}}^{A} \lambda_{\mathrm{b}A},$$

$$\Omega^{AB} = \Omega^{(\alpha,\dot{\alpha})(\beta,\dot{\beta})} \overset{\text{e.g.}}{=} \varepsilon^{\alpha\beta} \oplus \varepsilon^{\dot{\beta}\dot{\alpha}} \text{ and } \Omega_{AB} \overset{\text{e.g.}}{=} \varepsilon_{\beta\alpha} \oplus \varepsilon_{\dot{\alpha}\dot{\beta}} \text{ with } \Omega^{AB} \Omega_{BC} = \delta_C^A$$

$$\Omega^{\mathcal{J}\mathcal{J}} = \begin{pmatrix} 0 & \mathbb{1}^{I\dot{J}} \\ \mathbb{1}^{\dot{I}J} & 0 \end{pmatrix}$$

$$\begin{aligned} \lambda_A^{(\mathrm{a}} \rho_{1B}^{\mathrm{b})} \Omega^{AB} &= 0 = \lambda_{\mathrm{a}A} \rho_{1B}^{\mathrm{a}} \Omega^{AB} \\ \lambda_A^{(\mathrm{a}} \rho_{2B}^{\mathrm{b})} \Omega^{AB} + \eta_{\mathcal{J}}^{(\mathrm{a}} \tau_{\mathcal{J}}^{\mathrm{b})} \Omega^{\mathcal{J}\mathcal{J}} &= 0 = \lambda_{\mathrm{a}A} \rho_{2B}^{\mathrm{a}} \Omega^{AB} + \eta_{\mathrm{a}\mathcal{J}} \tau_{\mathcal{J}}^{\mathrm{a}} \Omega^{\mathcal{J}\mathcal{J}} \end{aligned}$$

$$Z_{\mathrm{a}}^{A}(z) \cdot Z_B^{\mathrm{b}}(0) = \frac{\delta_B^A \delta_{\mathrm{a}}^{\mathrm{b}}}{z} + \cdots \text{ and } \tilde{\tau}_{\mathrm{a}}^{\mathcal{J}}(z) \tau_{\mathcal{J}}^{\mathrm{b}}(0) = \frac{\delta_{\mathcal{J}}^{\mathcal{J}} \delta_{\mathrm{a}}^{\mathrm{b}}}{z} + \cdots$$

$$\left\{ \left(\beta^{\mathrm{ab}}, \gamma_{\mathrm{ab}} \right), (\beta, \gamma), \left(\tilde{\beta}^{\mathrm{ab}}, \tilde{\gamma}_{\mathrm{ab}} \right), (\tilde{\beta}, \tilde{\gamma}) \right\}$$

$$Q_B = \oint \mathrm{d}z J_{BRST}(z)$$

$$
\begin{aligned}
J_{BRST} =& cT + N_{\mathrm{ab}}\left(J^{\mathrm{ab}} + M^{\mathrm{a}}_{\mathrm{c}} N^{\mathrm{bc}}\right) + n\lambda^{\mathrm{a}}_{A}\lambda^{A}_{\mathrm{a}} + \tilde{n}\left(\lambda^{\mathrm{a}}_{A}\lambda^{A}_{\mathrm{a}} + \eta^{\mathrm{a}}_{\mathcal{J}}\eta^{\mathcal{J}}_{\mathrm{a}}\right) + \gamma_{\mathrm{ab}}\lambda^{\mathrm{a}A}\rho^{\mathrm{b}}_{1A} + \gamma\lambda^{A}_{\mathrm{a}}\rho^{\mathrm{a}}_{1A} \\
&+ m\gamma\gamma + \tilde{\gamma}_{\mathrm{ab}}\left(\lambda^{\mathrm{a}A}\rho^{\mathrm{b}}_{2A} + \eta^{\mathrm{a}\mathcal{J}}\tau^{\mathrm{b}}_{\mathcal{J}}\right) + \tilde{\gamma}\left(\lambda^{A}_{\mathrm{a}}\rho^{\mathrm{a}}_{2A} + \eta^{\mathcal{J}}_{\mathrm{a}}\tau^{\mathrm{a}}_{\mathcal{J}}\right) + \tilde{m}\tilde{\gamma}\tilde{\gamma} \\
T =& \mathcal{Z}^{\mathrm{a}} \cdot \partial \mathcal{Z}_{\mathrm{a}} + \frac{1}{2}\sum_{r=1,2} \rho^{\mathrm{a}}_{rA}\partial\rho^{A}_{r\mathrm{a}} + \beta^{\mathrm{ab}}\partial\gamma_{\mathrm{ab}} + \beta\partial\gamma + \frac{1}{2}\tau^{\mathrm{a}}_{\mathcal{J}}\partial\tau^{\mathcal{J}}_{\mathrm{a}} + \tilde{\beta}^{\mathrm{ab}}\partial\tilde{\gamma}_{\mathrm{ab}} + \tilde{\beta}\partial\tilde{\gamma} \\
&+ M^{\mathrm{ab}}\partial N_{\mathrm{ab}} + m\partial n + \tilde{m}\partial\tilde{n} + \frac{1}{2}(b\partial c + \partial(bc)) + \sum_{i=1}^{2} \tilde{\tau}^{\mathcal{J}}_{i}\partial\tau'_{i\mathcal{J}} \\
J^{\mathrm{ab}} =& \mathcal{Z}^{\mathrm{a}} \cdot \mathcal{Z}^{\mathrm{b}} + \frac{1}{2}\sum_{r=1,2} \rho^{(\mathrm{a}}_{rA}\rho^{\mathrm{b})A}_{r} + \beta^{\mathrm{c}(\mathrm{a}}\gamma^{\mathrm{b})}_{\mathrm{c}} + \frac{1}{2}\tau^{(\mathrm{a}}_{\mathcal{J}}\tau^{\mathrm{b})\mathcal{J}} + \tilde{\beta}^{\mathrm{c}(\mathrm{a}}\tilde{\gamma}^{\mathrm{b})}_{\mathrm{c}}
\end{aligned}
$$

$$
\begin{aligned}
c =& -3(8-2\mathcal{N})_{\mathcal{Z}_{\mathrm{a}}} - 26_{bc} - 18_{MN} - 2_{mn} - 2_{\tilde{m}\tilde{n}} + 12_{\rho_1} + 12_{\rho_2} - 3\mathcal{N}_{\tau} + 20_{\beta\gamma} + 20_{\tilde{\beta}\tilde{\gamma}} - \sum_{i=1}^{2} \mathcal{N}_{\tau_i\tilde{\tau}_i} \\
=& -8 + \mathcal{N} = 0
\end{aligned}
$$

$$
\left(\mathrm{tr}_{\mathrm{adj}}\left(t^k t^k\right) = 3\cdot 6 = 18, \mathrm{tr}_{\mathrm{F}}\left(t^k t^k\right) = 3\cdot\frac{3}{2} = \frac{9}{2}\right)
$$

$$
a^G = \frac{1}{2}\mathrm{tr}_{\mathrm{F}}\left(t^k t^k\right)\left((8-2\mathcal{N})_{\mathcal{Z}} - 8_{\rho_{1,2}} + \mathcal{N}_{\tau}\right) + \mathrm{tr}_{\mathrm{adj}}\left(t^k t^k\right)\left(-1_{\mathcal{MN}} + 1_{\beta\gamma} + 1_{\tilde{\beta}\tilde{\gamma}}\right) = \frac{9}{4}(8-\mathcal{N}) = 0
$$

$$
\mathcal{V} = \int \mathrm{d}\mu(u)\mathrm{d}\mu(v)\mathcal{W}(u)\bar{\delta}(v^{\alpha}_{\mathrm{a}}\epsilon^{\mathrm{a}}_{\alpha} - 1)\bar{\delta}(u^{\alpha}_{\mathrm{a}}\lambda^{\mathrm{a}}_{A} - v^{\alpha}_{\mathrm{a}}\kappa^{\mathrm{a}}_{A})\bar{\delta}(u^{\alpha}_{\mathrm{a}}\eta^{\mathrm{a}}_{\mathcal{J}} - v^{\alpha}_{\mathrm{a}}\zeta^{\mathrm{a}}_{\mathcal{J}})e^{u^{\mathrm{a}}_{\alpha}\mu^{A}_{\mathrm{a}}\epsilon^{\alpha}_{A} + u^{\mathrm{a}}_{\alpha}\tilde{\eta}^{\mathcal{J}}_{\mathrm{a}}q^{\alpha}_{\mathcal{J}}}
$$

$$
\mathcal{W}(u) = cn\tilde{n}\delta(\gamma)\delta\left(\gamma_{\mathrm{ab}}u^{\mathrm{a}}_{\alpha}u^{\mathrm{b}\alpha}\right)\delta(\tilde{\gamma})\delta\left(\tilde{\gamma}_{\mathrm{ab}}u^{\mathrm{a}}_{\beta}u^{\mathrm{b}\beta}\right)
$$

$$
\lambda^{\mathrm{a}}_{A}(\sigma) = \sum_{i} \frac{u^{\mathrm{a}}_{i\alpha}\epsilon^{\alpha}_{iA}}{\sigma - \sigma_i}, \eta^{\mathrm{a}}_{\mathcal{J}}(\sigma) = \sum_{i} \frac{u^{\mathrm{a}}_{i\alpha}q^{\alpha}_{i\mathcal{J}}}{\sigma - \sigma_i}
$$

$$
\mathcal{V} = \int \mathrm{d}^2u\, \mathrm{d}^2v\mathcal{W}(u)\bar{\delta}(\langle v\epsilon\rangle - 1)\bar{\delta}^4(\langle u\lambda_A\rangle - \langle v\kappa_A\rangle)\bar{\delta}^{\mathcal{N}}(\langle u\eta_{\mathcal{J}}\rangle - \langle v\zeta_{\mathcal{J}}\rangle)e^{\langle u\mu^A\rangle\epsilon_A + \langle u\tilde{\eta}^{\mathcal{J}}\rangle q_{\mathcal{J}}}
$$

$$
\mathcal{W}(u) = cn\tilde{n}\delta(\gamma)\delta\left(\gamma_{ab}u^a u^b\right)\delta(\tilde{\gamma})\delta\left(\tilde{\gamma}_{ab}u^a u^b\right),
$$

$$
\begin{aligned}
\mathcal{V}'_i &= \mathrm{c}n\tilde{n}N_{\mathrm{ab}}\delta(\gamma)\delta(\gamma_{\mathrm{a'b'}})\delta(\tilde{\gamma})\delta(\tilde{\gamma}_{\mathrm{a''b''}})\int \frac{\mathrm{d}t}{t^3}\bar{\delta}^4(t\tau'_{iI} - \kappa'_I)e^{t\tilde{\tau}'_i\kappa'_I}, \\
\tilde{\mathcal{V}}'_i &= \mathrm{c}n\tilde{n}N_{\mathrm{ab}}\delta(\gamma)\delta(\gamma_{\mathrm{a'b'}})\delta(\tilde{\gamma})\delta(\tilde{\gamma}_{\mathrm{a''b''}})\int \frac{\mathrm{d}t}{t^3}\bar{\delta}^4(t\tau'_{iI} - \kappa'_I)e^{t\tilde{\tau}''_i\kappa'_I},
\end{aligned}
$$

$$
\begin{aligned}
&\Upsilon_{\beta}(z) = \delta(\beta)[Q_B,\beta], \qquad \Upsilon_{\hat{u}\beta^{\mathrm{ab}}}(z) = \delta\left(\beta^{\mathrm{ab}}\hat{u}^{\alpha}_{\mathrm{a}}\hat{u}_{\mathrm{b}\alpha}\right)\left[Q_B,\beta^{\mathrm{ab}}\hat{u}^{\alpha}_{\mathrm{a}}\hat{u}_{\mathrm{b}\alpha}\right] \\
&\Upsilon_{\tilde{\beta}}(z) = \delta(\tilde{\beta})[Q_B,\tilde{\beta}], \qquad \Upsilon_{\hat{u}\tilde{\beta}^{\mathrm{ab}}}(z) = \delta\left(\tilde{\beta}^{\mathrm{ab}}\hat{u}^{\alpha}_{\mathrm{a}}\hat{u}_{\mathrm{b}\alpha}\right)\left[Q_B,\tilde{\beta}^{\mathrm{ab}}\hat{u}^{\alpha}_{\mathrm{a}}\hat{u}_{\mathrm{b}\alpha}\right] \\
&\Upsilon_{m}(z) = m\delta(\{Q_B,m\}), \quad \Upsilon_{\tilde{m}}(z) = \tilde{m}\delta(\{Q_B,\tilde{m}\}), \Upsilon_{M^{\mathrm{ab}}}(z) = M^{\mathrm{ab}}\delta\left(\{Q_B,M^{\mathrm{ab}}\}\right)
\end{aligned}
$$

$$
V = \int \mathrm{d}\sigma\, \mathrm{d}\mu(u)\mathrm{d}\mu(v)w(u)\bar{\delta}(v^{\alpha}_{\mathrm{a}}\epsilon^{\mathrm{a}}_{\alpha} - 1)\bar{\delta}(u^{\alpha}_{\mathrm{a}}\lambda^{\mathrm{a}}_{A} - v^{\alpha}_{\mathrm{a}}\kappa^{\mathrm{a}}_{A})\bar{\delta}(u^{\alpha}_{\mathrm{a}}\eta^{\mathrm{a}}_{\mathcal{J}} - v^{\alpha}_{\mathrm{a}}\zeta^{\mathrm{a}}_{\mathcal{J}})e^{u^{\mathrm{a}}_{\alpha}\mu^{A}_{\mathrm{a}}\epsilon^{\alpha}_{A} + u^{\mathrm{a}}_{\alpha}\tilde{\eta}^{\mathcal{J}}_{\mathrm{a}}q^{\alpha}_{\mathcal{J}}}
$$

$$
\begin{aligned}
w(u) =& \delta\left(\mathrm{Res}_{\sigma}(\lambda_{A\mathrm{a}}\lambda^{\mathrm{a}}_{B}\Omega^{AB})\right)\delta\left(\mathrm{Res}_{\sigma}\left(\eta_{\mathcal{J}\mathrm{a}}\eta^{\mathrm{a}}_{\mathcal{J}}\Omega^{\mathcal{J}\mathcal{J}}\right)\right)\left(\frac{\hat{u}^{\alpha}_{\mathrm{a}}\lambda^{\mathrm{a}}_{A}\epsilon^{A}_{\alpha}}{U} + \epsilon^{\beta A}\frac{u^{\mathrm{a}}_{\beta}\rho_{1\mathrm{a}A}\rho^{\mathrm{b}}_{1B}\hat{u}^{\alpha}_{\mathrm{b}}}{U}\epsilon^{B}_{\alpha}\right) \\
&\left(\frac{\hat{u}^{\alpha}_{\mathrm{a}}\lambda^{\mathrm{a}}_{A}\epsilon^{A}_{\alpha}}{U} + \epsilon^{\beta A}\frac{u^{\mathrm{a}}_{\beta}\rho_{2\mathrm{a}A}\rho^{\mathrm{b}}_{2B}\hat{u}^{\alpha}_{\mathrm{b}}}{U}\epsilon^{B}_{\alpha} + \frac{\hat{u}^{\alpha}_{\mathrm{a}}\eta^{\mathrm{a}}_{\mathcal{J}}q^{\mathcal{J}}_{\alpha}}{U} + q^{\beta\mathcal{J}}\frac{u^{\mathrm{a}}_{\beta}\tau_{\mathrm{a}\mathcal{J}}\tau^{\mathrm{b}}_{\mathcal{J}}\hat{u}^{\alpha}_{\mathrm{b}}}{U}q^{\mathcal{J}}_{\alpha}\right).
\end{aligned}
$$

$$\zeta_I^{\mathrm{a}}\zeta_{\mathrm{a}}^I = -\tilde{\zeta}_I^{\mathrm{a}}\tilde{\zeta}_{\mathrm{a}}^I \prod_{i=2}^{n} \delta\left(\kappa_{i\mathrm{a}}^{A}\kappa_{iA}^{\mathrm{a}}\right)\delta\left(\zeta_{i\mathrm{a}}^{\mathcal{J}}\zeta_{i\mathcal{J}}^{\mathrm{a}}\right)^{\dagger}$$

$$\sum_i \kappa_{i\mathrm{a}}^{A}\kappa_i^{\mathrm{a}B} = \zeta_{1\mathrm{a}}^{\mathcal{J}}\zeta_{1\mathcal{J}}^{\mathrm{a}} = \sum_i \zeta_{i\mathrm{a}}^{\mathcal{J}}\zeta_{i\mathcal{J}}^{\mathrm{a}} = 0$$

$$u_{\mathrm{a}}^{\alpha}\lambda_A^{\mathrm{a}}\epsilon_{\alpha}^{A} = v_{\mathrm{a}}^{\alpha}\kappa_A^{\mathrm{a}}\epsilon_{\alpha}^{\mathrm{b}}\kappa_{\mathrm{b}}^{A} = 0$$

$$\lambda_A^{\mathrm{a}}\epsilon_{\beta}^{A} = -u_{\alpha}^{\mathrm{a}}H_{\beta}^{\alpha}$$

$$H_{\beta}^{\alpha} = \frac{\hat{u}_{\mathrm{a}}^{\alpha}\lambda_A^{\mathrm{a}}\epsilon_{\beta}^{A}}{U}$$

$$\epsilon_{\alpha}^{\mathrm{a}} = \kappa_A^{\mathrm{a}}\tilde{\epsilon}_{\alpha}^{A}e^{AB} = \epsilon^{\alpha[A}\tilde{\epsilon}_{\alpha}^{B]}P_{AB} = \lambda_{A\mathrm{a}}\lambda_B^{\mathrm{a}}$$

$$e\cdot P = e^{AB}P_{AB} = \epsilon_{\alpha}^{[A}\tilde{\epsilon}^{B]\alpha}\lambda_A^{\mathrm{a}}\lambda_{\mathrm{a}B} = -u_{\beta}^{\mathrm{a}}H_{\alpha}^{\beta}\lambda_{\mathrm{a}B}\tilde{\epsilon}^{B\alpha} = -v_{\beta}^{\mathrm{a}}\kappa_{\mathrm{a}B}\tilde{\epsilon}^{B\alpha}H_{\alpha}^{\beta} = -H_{\alpha}^{\alpha} = -\frac{\hat{u}_{\mathrm{a}}^{\alpha}\lambda_A^{\mathrm{a}}\epsilon_{\alpha}^{A}}{U}$$

$$\epsilon_{\alpha}^{\mathrm{a}} = \zeta_{\mathcal{J}}^{\mathrm{a}}\tilde{q}_{\alpha}^{\mathcal{J}} \text{ and } Y_{\mathcal{J}\mathcal{J}} = \eta_{\mathcal{J}\mathrm{a}}\eta_{\mathcal{J}}^{\mathrm{a}}$$

$$q_{\alpha}^{\mathcal{J}}\tilde{q}^{\mathcal{J}\alpha}Y_{\mathcal{J}\mathcal{J}} = -u_{\beta}^{\mathrm{a}}G_{\alpha}^{\beta}\eta_{\mathrm{a}\mathcal{J}}\tilde{q}^{\mathcal{J}\alpha} = -v_{\beta}^{\mathrm{a}}\zeta_{\mathrm{a}\mathcal{J}}\tilde{q}^{\mathcal{J}\alpha}G_{\alpha}^{\beta} = -G_{\alpha}^{\alpha} = -\frac{\hat{u}_{\mathrm{a}}^{\alpha}\eta_{\mathcal{J}}^{\mathrm{a}}q_{\alpha}^{\mathcal{J}}}{U}$$

$$\mathcal{A}_n^{\text{tree}} = \int \mathrm{d}\mu_n^{\text{pol}|\mathcal{N}}\,\mathcal{I}_n\,,$$

$$\mathrm{d}\mu_n^{\text{pol}|\mathcal{N}} := \frac{\prod_l \mathrm{d}\sigma_l \mathrm{d}\mu(u_l)\mathrm{d}\mu(v_l)}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C})_{\sigma}\times \mathrm{G}_{Q_u})} \prod_{r=1}^{n} \bar{\delta}(v_{r\mathrm{a}}^{\alpha}\epsilon_{r\alpha}^{\mathrm{b}}-1) \quad \bar{\delta}(u_{r\mathrm{a}}^{\alpha}\lambda_{rA}^{\mathrm{a}}-v_{r\mathrm{a}}^{\alpha}\kappa_{rA}^{\mathrm{a}})\;\bar{\delta}(u_{r\mathrm{a}}^{\alpha}\eta_{r\mathcal{I}}^{\mathrm{a}}-v_{r\mathrm{a}}^{\alpha}\zeta_{r\mathcal{I}}^{\mathrm{a}})$$

$$\prod_{r=2}^{n}\delta(\kappa_{r\mathrm{a}}^{A}\kappa_{rA}^{\mathrm{a}})\delta(\zeta_{r\mathrm{a}}^{\mathcal{I}}\zeta_{r\mathcal{I}}^{\mathrm{a}}),$$

$$\mathcal{I}_n = \det{}'H\,\det{}'G = \frac{H_{[kl]}^{[kl]}}{(u_{k\alpha}^{\mathrm{a}}u_{l\mathrm{a}}^{\beta})(u_{k\mathrm{b}}^{\alpha}u_{l\beta}^{\mathrm{b}})} \quad \frac{\det G_{[pr]}^{[pr]}}{(u_{p\alpha}^{\mathrm{a}}u_{r\mathrm{a}}^{\beta})(u_{p\mathrm{b}}^{\alpha}u_{r\beta}^{\mathrm{b}})},$$

$$H_{ij} = \frac{\epsilon_{iA}^{\alpha}\epsilon_{j\alpha}^{A}}{\sigma_{ij}}, \qquad H_{ii} = -e_i\cdot P_i$$

$$G_{ij} = \frac{\epsilon_{iA}^{\alpha}\epsilon_{j\alpha}^{A} + q_{i\mathcal{J}}^{\alpha}q_{j\alpha}^{\mathcal{J}}}{\sigma_{ij}}, \quad G_{ii} = -\left(e_i\cdot P_i + q_i^{\alpha\mathcal{J}}\tilde{q}_{i\alpha}^{\mathcal{J}}Y_{i\mathcal{J}\mathcal{J}}\right)$$

$$\sum_i U_iH_{ij} = \hat{u}_{\mathrm{a}}^{\alpha}\lambda_A^{\mathrm{a}}\epsilon_{j\alpha}^{A} + \sum_{i\neq j} U_i\frac{\epsilon_{iA}^{\alpha}\epsilon_{j\alpha}^{A}}{\sigma_{ij}} = \hat{u}_{\mathrm{a}}^{\alpha}\lambda_A^{\mathrm{a}}\epsilon_{j\alpha}^{A} + \left(\sum_{i\neq j}\frac{u_{i\alpha}^{\mathrm{a}}\epsilon_{iA}^{\alpha}}{\sigma_{ij}}\right)\epsilon_{j\beta}^{A}\hat{u}_{\mathrm{a}}^{\beta} = 0$$

$$\oint_{a_i} \omega_j = \delta_{ij}, \oint_{b_i} \omega_j = \tau_{ij}$$

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$$

$$\tau \to \tilde{\tau} = (A\tau + B)/(C\tau + D)$$

$$\theta[\alpha](\mathbf{z}) = \sum_{m \in \mathbb{Z}^g} \exp\left[i\pi(m+\alpha')^T \tau (m+\alpha') + 2\pi i (m+\alpha')^T (\mathbf{z} + \alpha'')\right]$$

$$\theta[\alpha](-\mathbf{z}) = (-1)^{4\alpha' \cdot \alpha''} \theta[\alpha](\mathbf{z}), \alpha = \begin{pmatrix} \alpha' \\ \alpha'' \end{pmatrix}, \alpha', \alpha'' \in (\mathbb{Z}/2\mathbb{Z})^g$$

$$\sum_{j=1}^{n} c_j z_j \to \mathbf{z} \in \mathbb{C}^g, \mathbf{z_i} = \sum_{j=1}^{n} c_j \int_{z_0}^{z_j} \omega_i$$

$$\Delta \in \mathbb{C}^g, \Delta_i = -\frac{1+\tau_{ii}}{2} + \sum_{j \neq i} \oint_{a_j} \omega_j(z) \int_{z_0}^{z} \omega_i$$

$$\theta(z) = 0 \ \Leftrightarrow \ \exists z_1, \dots, z_{g-1} \text{ with } z = \Delta - \sum_{i=1}^{g-1} z_i$$

$$E(z,w) = \frac{\theta[\alpha](z-w)}{h_\alpha(z) h_\alpha(w)}, h_\alpha(z) = \left(\sum_{i=1}^{g} \partial_i \theta[\alpha](0) \omega_i(z)\right)^{\frac{1}{2}}$$

$$S_\alpha(z,w \mid \tau) = \frac{\theta[\alpha](z-w)}{E(z,w)\theta[\alpha](0)}, \qquad \alpha \text{ even}$$

$$S_\alpha(z,w \mid \tau) = \frac{1}{E(z,w)} \frac{\sum_{i=1}^{g} \partial_i \theta[\alpha](z-w)\omega_i(y))}{\sum_{i=1}^{g} \partial_i \theta[\alpha](0)\omega_i(y)}, \qquad \alpha \text{ odd}$$

$$Z_{\lambda,\alpha}(z_1, \dots, z_G) = \int \mathrm{D}b\mathrm{D}c\, b(z_1) \dots b(z_G) \exp\left(-\frac{1}{2\pi}\int \mathrm{d}^2 z \sqrt{g}\, b(z) \bar{\nabla}^{\alpha}_{1-\lambda} c(z)\right), \lambda > 1$$

$$Z_{1,\alpha}(z_1, \dots, z_g, w) = \int \mathrm{D}\beta\mathrm{D}\gamma\, \beta(z_1) \dots \beta(z_g) \gamma(w) \exp\left(-\frac{1}{2\pi}\int \mathrm{d}^2 z \sqrt{g}\, \beta(z) \bar{\nabla}^{\alpha}_{1-\lambda} \gamma(z)\right),$$

$$Z^{\text{even}}_{\frac{1}{2},\alpha} = \int \mathrm{D}\chi\mathrm{D}\psi \exp\left(-\frac{1}{2\pi}\int \mathrm{d}^2 z \sqrt{g}\, \chi(z) \bar{\nabla}^{\alpha}_{\frac{1}{2}} \psi(z)\right),$$

$$Z^{\text{odd}}_{\frac{1}{2},\alpha}(w) = \int \mathrm{D}\chi\mathrm{D}\psi\, \chi(w)\psi(w) \exp\left(-\frac{1}{2\pi}\int \mathrm{d}^2 z \sqrt{g}\, \chi(z) \bar{\nabla}^{\alpha}_{\frac{1}{2}} \psi(z)\right),$$

$$Z_{\lambda>1,\alpha}(z_1,\dots,z_G)=\tilde{Z}_1^{-\frac{1}{2}}\theta[\alpha]\left(\sum_{i=1}^{G} z_i-Q\Delta\right)\prod_{j>i=1}^{G} E(z_i,z_j)\prod_{i=1}^{G}\sigma(z_i)^Q$$

$$Z_{1,\alpha}(z_1,\dots,z_g,w)=\tilde{Z}_1^{-\frac{1}{2}}\theta[\alpha]\left(\sum_{i=1}^{g} z_i-w-\Delta\right)\frac{\prod_{j>i=1}^{g} E(z_i,z_j)\prod_{i=1}^{g}\sigma(z_i)}{\prod_{i=1}^{g} E(z_i,w)\sigma(w)}$$

$$Z_{\frac{1}{2},\alpha}^{\text{even}}=\tilde{Z}_1^{-\frac{1}{2}}\theta[\alpha](0), Z_{\frac{1}{2},\alpha}^{\text{odd}}(z)=\left(\tilde{Z}_1^{\frac{1}{2}}\det{}_\omega\right)^{-1} h_\alpha(z)^2$$

$$\tilde{Z}_1=\frac{Z_{1,0}(z_1,\dots,z_g,w)}{\det_\omega}, \det{}_\omega\equiv\det\|\omega_i(z_j)\|, \int_\Sigma \mathrm{d}^2z\bar{\omega}_i\omega_j(z)=<\omega_i\mid\omega_j>=2i\operatorname{Im}\tau_{ij}$$

$$\frac{\sigma(z)}{\sigma(w)}=\frac{\theta\left(z-\sum_{i=1}^{g} r_i-\Delta\right)}{\theta\left(w-\sum_{i=1}^{g} r_i-\Delta\right)}\prod_{i=1}^{g}\frac{E(w,r_i)}{E(z,r_i)}$$

$$\sigma(z)=\exp\left[\sum_{i=1}^{g}\oiiint_{a_i}\mathrm{d}w\omega_i(w)\ln E(z,w)\right].$$

$$(\nu_1\wedge\dots\wedge\nu_n)\otimes(\phi_1\wedge\dots\phi_{G+n}), G=Q(g-1)$$

$$\det\bar{\nabla}_{1-\lambda}^{\alpha}=\frac{Z_{\lambda,\alpha}(z_1,\dots,z_G)}{\det\|\phi_i(z_j)\|}\cdot\phi_1\wedge\dots\wedge\phi_G,\qquad \lambda>1,$$

$$\det\bar{\nabla}_0^{\alpha}=\frac{Z_{1,\alpha}(z_1,\dots,z_g,w)}{\det_\omega e}\cdot\omega_1\wedge\dots\wedge\omega_g\otimes e,$$

$$\det\bar{\nabla}_{\frac{1}{2},\alpha}^{\text{ev}}=Z_{\frac{1}{2},\alpha}^{\text{even}}, \det\bar{\nabla}_{\frac{1}{2},\alpha}^{\text{odd}}=\frac{Z_{\frac{1}{2},\alpha}^{\text{odd}}(z)}{h_\alpha(z)^2}\cdot h_\alpha\otimes h_\alpha.$$

$$\det{}_{[\text{bos}]}\bar{\nabla}_{1-\lambda}^{\alpha}=\left(\det{}_{[\text{ferm}]}\bar{\nabla}_{1-\lambda}^{\alpha}\right)^{-1}.$$

$$Z_{\lambda,\tilde{\alpha}}(\tilde{z}_1,\dots,\tilde{z}_G)=\epsilon^{\frac{2}{3}}e^{i\pi\phi(\alpha)}Z_{\lambda,\alpha}(z_1,\dots,z_G),$$

$$Z_{\frac{1}{2},\tilde{\alpha}}^{\text{odd}}(\tilde{z})=\epsilon^{\frac{2}{3}}e^{i\pi\phi(\alpha)}\det\|C\tau+D\|Z_{\frac{1}{2},\alpha}^{\text{odd}}(z),$$

$$Z_{\frac{1}{2},\tilde{\alpha}}^{\text{even}}=\epsilon^{\frac{2}{3}}e^{i\pi\phi(\alpha)}Z_{\frac{1}{2},\alpha}^{\text{even}},$$

with $\tilde{z}_i=(\tau C^T+D^T)_{ij}^{-1}z_j, \tilde{\omega}_i=\omega_j(C\tau+D)_{ji}^{-1}, \tilde{\Delta}-\tilde{\tau}\delta_1-\delta_2=(\tau C^T+D^T)^{-1}\Delta$

$$\tilde{\alpha}=\alpha'+\delta, \begin{bmatrix}\alpha_1'\\ \alpha_2'\end{bmatrix}=\begin{bmatrix}D & -C\\ -B & A\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}, \begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\operatorname{diag}CD^T\\ \operatorname{diag}AB^T\end{bmatrix}.$$

$$\epsilon^8=1, \phi(\alpha)=\alpha_1'\cdot\alpha_2'-\alpha_1\cdot\alpha_2+2\alpha_1'\cdot\delta_2\in\frac{\mathbb{Z}_2}{4}$$

$$\tilde{Z}_1=\tilde{Z}_1^{-\frac{1}{2}}\frac{-\theta_1(z-w)}{E(z,w)\det_\omega}=-\tilde{Z}_1^{-\frac{1}{2}}\theta_1'(0)=Z^{\text{odd}}_{\frac{1}{2},\alpha}(z)$$
$$\tilde{Z}_{2,\alpha}\det_\omega=\frac{Z_{2,\alpha}}{\det_\omega}=Z_{1,\alpha}=\tilde{Z}_1^{-\frac{1}{2}}\frac{\theta[\alpha](z-w-\Delta_\alpha)}{E(z,w)}=\tilde{Z}_1^{-\frac{1}{2}}\frac{-\theta_1'(0)\det_\omega}{r_\alpha(z-w)}$$
$$\theta_1(z)=-r_\alpha(z)\theta[\alpha](z-\Delta_\alpha),r_3(z)=e^{\frac{1}{4}\pi i\tau+\pi i\left(z+\frac{1}{2}\right)}=r_4(z),r_2(z)=1=r_1(z)$$
$$\Delta_3=\Delta=-\frac{1}{2}(\tau+1),\Delta_4=-\frac{\tau}{2},\Delta_2=-\frac{1}{2},\Delta_1=0$$

$$\tilde{Z}_{2,\alpha}=\frac{Z_{2,\alpha}}{\det_\omega^2}=\frac{Z_{1,\alpha}}{\det_\omega}=q_\alpha^{-\frac{1}{4}}\eta(\tau)^2,Z^{\text{odd}}_{\frac{1}{2}}=\tilde{Z}_1=\eta(\tau)^2,Z^{\text{even}}_{\frac{1}{2},\alpha}=\frac{\theta[\alpha](0)}{\eta(\tau)}$$

$$q=e^{\pi i\tau},q_\alpha=\begin{cases}q, & \alpha=3,4\\ 1, & \alpha=1,2\end{cases}$$

$$\mathrm{d}(\mathrm{WP})\left(\det\hat{P}_1^\dagger\hat{P}_1\right)^{\frac{1}{2}}=W\wedge\bar{W},W=\mathrm{d}m^{3(g-1)}\int\ \mathrm{d}b\ \mathrm{d}c\prod_{i=1}^{3(g-1)}(\mu_i,b)\exp\left[\iint\ \mathrm{d}^2zb(z)\bar{\nabla}^\alpha_{-1}c(z)\right]$$

$$W=\mathrm{d}m^{3(g-1)}\frac{\det\left\|(\mu_i,\phi_j)\right\|}{\det\left\|\phi_i(z_j)\right\|}Z_{2,\alpha}\left(z_1,\ldots,z_{3(g-1)}\right)=\det\bar{\nabla}^\alpha_{-1}$$

$$\mathrm{d}\mu_g=\frac{\phi_1\wedge\ldots\wedge\phi_{3(g-1)}}{\det\left\|\phi_i(z_j)\right\|}=\mathrm{d}m^{3(g-1)}\frac{\det\left\|(\mu_i,\phi_j)\right\|}{\det\left\|\phi_i(z_j)\right\|}$$

$Z_{g\geq 2}$

$$=\left\langle\prod_{i=1}^{3(g-1)}b(y_i)\prod_{j=1}^{g}\Upsilon_m(x_j)\Upsilon_{\tilde{m}}(\tilde{x}_j)\prod_{\substack{\mathrm{ab}=c_1\oplus c_2\oplus c_3\\ c_i=(0,0),(0,1),(1,1)}}\left(\Upsilon_{M^{\mathrm{ab}}}(w_{\mathrm{ab}j})\Upsilon_{\hat{u}\beta^{\mathrm{ab}}}(z_{\mathrm{ab}j})\Upsilon_{\hat{u}\tilde{\beta}^{\mathrm{ab}}}(\tilde{z}_{\mathrm{ab}j})\right)\Upsilon_\beta(z_j)\Upsilon_{\tilde{\beta}}(\tilde{z}_j)\right\rangle$$

$$=\sum_\alpha\int\ \mathrm{d}\mu_g\frac{Z_{2,\alpha}((y))Z_{1,\alpha}((x))^2Z_{1,\alpha}((w))^9}{Z_{1,\alpha}((z))^{10}Z_{1,\alpha}((\tilde{z}))^{10}}\left\langle\prod^{g}\left(\bar{\delta}_{\mathrm{tot}}j_{\mathrm{tot}}\right)\right\rangle_\alpha,$$
$$\bar{\delta}_{\mathrm{tot}}=\bar{\delta}\left(\lambda^A_{\mathrm{a}}\lambda^{\mathrm{a}}_A+\eta^{\mathcal{J}}_{\mathrm{a}}\eta^{\mathrm{a}}_{\mathcal{J}}\right)\ldots,j_{\mathrm{tot}}=\left(\lambda^A_{\mathrm{a}}\rho^{\mathrm{a}}_{2A}+\eta^{\mathcal{J}}_{\mathrm{a}}\tau^{\mathrm{a}}_{\mathcal{J}}\right)\ldots,$$

$$Z^{\mathrm{ev}}_{g=1}=\sum_\alpha\int\frac{\mathrm{d}\mu_1Z_{2,\alpha}Z_{1,\alpha}^{-9}\left(Z^{\text{even}}_{\frac{1}{2},\alpha}\right)^8}{\mathrm{Vol(CKV)_{chir}}}\left\langle\bar{\delta}_{\mathrm{tot}}j_{\mathrm{tot}}\right\rangle_\alpha=\sum_\alpha\int\frac{\mathrm{d}\mu_1\det_\omega^{-7}}{\mathrm{Vol(CKV)_{chir}}}\left(\frac{\theta_\alpha(0)}{\eta(\tau)}\right)^8\frac{\left\langle\bar{\delta}_{\mathrm{tot}}j_{\mathrm{tot}}\right\rangle_\alpha}{q_\alpha^{-2}\eta(\tau)^{16}},$$
$$Z^{\mathrm{odd}}_{g=1}=\int\frac{\mathrm{d}\mu_1Z_{2,1}Z_{1,1}^{-9}\left(Z^{\text{odd}}_{\frac{1}{2}}\right)^8}{\mathrm{Vol(CKV)_{chir}}}\mathrm{d}^{16}\psi\left\langle\bar{\delta}_{\mathrm{tot}}j_{\mathrm{tot}}\right\rangle=\int\frac{\mathrm{d}\mu_1\det_\omega^{-7}\ \mathrm{d}^{16}\psi}{\mathrm{Vol(CKV)_{chir}}}\left\langle\bar{\delta}_{\mathrm{tot}}j_{\mathrm{tot}}\right\rangle,$$

$$\mathrm{Vol(CKV)}=2\mathrm{Im}\tau=\int_\Sigma\mathrm{d}^2z=\int_\Sigma|\det_\omega|^2$$

$$\mathrm{d}\mu_1=\mathrm{d}\tau\det_\omega^{-2}$$

$$\sum_{\alpha} q_\alpha^2 \theta_\alpha(0)^8 \eta(\tau)^{-24}$$

$$Z_{g=1}^{\mathrm{ev}} = \sum_{\tilde{\alpha}} \int \mathrm{d}\tilde{\tau} \frac{\langle \bar{\delta}_{\mathrm{tot}} j_{\mathrm{tot}} \rangle_{\tilde{\alpha}}}{\det_{\tilde{\omega}}^{10}} \tilde{q}_{\tilde{\alpha}}^2 \left(\frac{\theta_{\tilde{\alpha}}(0)}{\eta(\tilde{\tau})^3}\right)^8 = \sum_{\alpha} \int \frac{\mathrm{d}\tau}{M_\tau^2} \frac{\langle \bar{\delta}_{\mathrm{tot}} j_{\mathrm{tot}} \rangle_{\alpha}}{(\det_\omega M_\tau^{-1})^{10}} q_\alpha^2 \left(\frac{\theta_\alpha(0)}{\eta(\tau)^3 M_\tau}\right)^8$$

$$Z_{g=1}^{\mathrm{odd}} = \int \mathrm{d}\tilde{\tau} \frac{\mathrm{d}^{16}\tilde{\psi}}{\det_{\tilde{\omega}}^{10}} \bar{\delta}_{\mathrm{tot}} j_{\mathrm{tot}} \Big\rangle = \int \frac{\mathrm{d}\tau}{M_\tau^2} \frac{\mathrm{d}^{16}\psi M_\tau^{-8}}{(\det_\omega M_\tau^{-1})^{10}} \bar{\delta}_{\mathrm{tot}} j_{\mathrm{tot}} \Big\rangle$$

$$\tilde{\tau} = \frac{A\tau + B}{C\tau + D}, M_\tau = (C\tau + D), \tilde{\omega} = \frac{\omega}{C\tau + D}, \tilde{z} = \frac{z}{C\tau + D}$$

$$\tilde{q}_{\tilde{\alpha}}^2 = \lim_{\tilde{z}\to 0} \left(\frac{\theta_1(\tilde{z})}{\theta_{\tilde{\alpha}}(\tilde{z} - \tilde{\Delta})}\right)^8 = \lim_{z\to 0} \left(\frac{\theta_1(z)}{\theta_\alpha(z - \Delta)}\right)^8 = q_\alpha^2$$

$$A_{g,n} = \langle V_1 \dots V_n \rangle = \sum_{\alpha} \int \mathrm{d}\mu_g Z_{g,\alpha}^{\mathrm{chir}}((z), (\tilde{z}), (y), (x), (w)) A_g^\alpha((z), (\tilde{z}), (x), (w))$$

$$Z_{g,\alpha}^{\mathrm{chir}} = \frac{Z_{2,\alpha}((y)) Z_{1,\alpha}((x))^2 Z_{1,\alpha}((w))^9}{Z_{1,\alpha}((z))^{10} Z_{1,\alpha}((\tilde{z}))^{10} \tilde{Z}_1^4} = \frac{Y_3(y) Y(x) Y(\tilde{x}) \prod_{\mathrm{ab}} Y(w_{\mathrm{ab}})}{Y(z) Y(\tilde{z}) \prod_{\mathrm{ab}} Y(z_{\mathrm{ab}}) Y(\tilde{z}_{\mathrm{ab}})}$$

$$A_g^\alpha((z), (\tilde{z}), (x), (w)) = \int \mathrm{d}m_0 \tilde{Z}_1^4 \left\langle \prod_{j=1}^{g} \left(\bar{\delta}_{x_j} \bar{\delta}'_{w_j} j_{z_j} \tilde{j}_{\tilde{z}_j}\right) V_1 \dots V_n \right\rangle_\alpha$$

$$Y_Q(y) \equiv \theta[\alpha](\Delta_y^Q) \prod_{j>i=1}^{Q(g-1)} E(y_i, y_j) \prod_{i=1}^{Q(g-1)} \sigma(y_i)^Q, Y(a) \equiv \theta[\alpha](\Delta_a) \frac{\prod_{j>i=1}^{g} E(a_i, a_j) \prod_{i=1}^{g} \sigma(a_i)}{\prod_{i=1}^{g} E(a_i, a_0) \sigma(a_0)}$$

$$\Delta_a^Q \equiv \sum_{i=1}^{Q(g-1)} a_i - Q\Delta, \Delta_a \equiv \sum_{i=1}^{g} a_i - a_0 - \Delta$$

$$\bar{\delta}_{x_j} = \bar{\delta}\left(\lambda_{\mathrm{a}}^A(x_j) \lambda_A^{\mathrm{a}}(x_j)\right) \bar{\delta}\left(\lambda_{\mathrm{a}}^A(\tilde{x}_j) \lambda_A^{\mathrm{a}}(\tilde{x}_j) + \eta_{\mathrm{a}}^{\mathcal{J}}(\tilde{x}_j) \eta_{\mathcal{J}}^{\mathrm{a}}(\tilde{x}_j)\right)$$

$$\bar{\delta}'_{w_j} = \prod_{\mathrm{ab}} \bar{\delta}\left(\{Q_B, M^{\mathrm{ab}}(w_{\mathrm{ab}j})\}\right)$$

$$j_{z_j} = \lambda_{\mathrm{a}}^A(z_j) \rho_{1A}^{\mathrm{a}}(z_j) \prod_{\mathrm{ab}} \left(\lambda^{A(\mathrm{a}}(z_{\mathrm{ab}j}) \rho_{1A}^{\mathrm{b})}(z_{\mathrm{ab}j})\right)$$

$$\tilde{j}_{\tilde{z}_j} = \left(\lambda_{\mathrm{a}}^A(\tilde{z}_j) \rho_{2A}^{\mathrm{a}}(\tilde{z}_j) + \eta_{\mathrm{a}}^{\mathcal{J}}(\tilde{z}_j) \tau_{\mathcal{J}}^{\mathrm{a}}(\tilde{z}_j)\right) \prod_{\mathrm{ab}} \left(\lambda^{A(\mathrm{a}}(\tilde{z}_{\mathrm{ab}j}) \rho_{2A}^{\mathrm{b})}(\tilde{z}_{\mathrm{ab}j}) + \eta^{\mathcal{J}(\mathrm{a}}(\tilde{z}_{\mathrm{ab}j}) \tau_{\mathcal{J}}^{\mathrm{b})}(\tilde{z}_{\mathrm{ab}j})\right)$$

$$\int \mathrm{d}m_0 Y(z) Y(\tilde{z}) \prod_{\mathrm{ab}} Y(z_{\mathrm{ab}}) Y(\tilde{z}_{\mathrm{ab}})$$

$$\lambda_{\mathrm{a}}^A(z_j) \lambda_A^{\mathrm{a}}(z_j) \text{ or } \lambda_{\mathrm{a}}^A(\tilde{z}_j) \lambda_A^{\mathrm{a}}(\tilde{z}_j) + \eta_{\mathrm{a}}^{\mathcal{J}}(\tilde{z}_j) \eta_{\mathcal{J}}^{\mathrm{a}}(\tilde{z}_j)$$

$$\bar{\delta}'_{w_j} j_{z_j} \tilde{j}_{\tilde{z}_j} \to j^u_{z_j} \tilde{j}^u_{\tilde{z}_j}$$

$$j^u_{z_j} = \lambda^A_{\mathrm{a}}(z_j)\rho^{\mathrm{a}}_{1A}(z_j) \int \frac{\prod_{\mathrm{ab}=c_1\oplus c_2\oplus c_3} \mathrm{d}^2 u_{(\mathrm{ab})j} e^{\|u_{(\mathrm{ab})j}\|^2} u^{\mathrm{a}}_{(\mathrm{ab})j}\lambda_{\mathrm{a}A}(z_{\mathrm{ab}j}) u^{\mathrm{b}}_{(\mathrm{ab})j}\rho^A_{1\,\mathrm{b}}(z_{\mathrm{ab}j})}{\pi^{18}\prod_{i=1}^3 \prod_{c_{i1}\in c_i}^{c_{i3}} \left(u^{\mathrm{a}}_{(c_{i1})j} u_{(c_{i2})j\mathrm{a}}\right)\left(u^{\mathrm{a}}_{(c_{i2})j} u_{(c_{i3})j\mathrm{a}}\right)\left(u^{\mathrm{a}}_{(c_{i3})j} u_{(c_{i1})j\mathrm{a}}\right)}$$

$$\tilde{j}^u_{\tilde{z}_j} = \left(\lambda^A_{\mathrm{a}}(\tilde{z}_j)\rho^{\mathrm{a}}_{2A}(\tilde{z}_j) + \eta^{\mathcal{J}}_{\mathrm{a}}(\tilde{z}_j)\tau^{\mathrm{a}}_{\mathcal{J}}(\tilde{z}_j)\right) \int \frac{\pi^{-18}\prod_{\mathrm{ab}=c_1\oplus c_2\oplus c_3} \mathrm{d}^2 \tilde{u}_{(\mathrm{ab})j} e^{\|\tilde{u}_{(\mathrm{ab})j}\|^2}}{\prod_{i=1}^3 \prod_{c_{i1}\in c_i}^{c_{i3}} \left(\tilde{u}^{\mathrm{a}}_{(c_{i1})j} \tilde{u}_{(c_{i2})j\mathrm{a}}\right)\left(\tilde{u}^{\mathrm{a}}_{(c_{i2})j} \tilde{u}_{(c_{i3})j\mathrm{a}}\right)\left(\tilde{u}^{\mathrm{a}}_{(c_{i3})j} \tilde{u}_{(c_{i1})j\mathrm{a}}\right)}$$

$$\prod_{\mathrm{ab}} \left(\tilde{u}^{\mathrm{a}}_{(\mathrm{ab})j}\lambda_{\mathrm{a}A}(z_{\mathrm{ab}j})\tilde{u}^{\mathrm{b}}_{(\mathrm{ab})j}\rho^A_{2\,\mathrm{b}}(z_{\mathrm{ab}j}) + \tilde{u}^{\mathrm{a}}_{(\mathrm{ab})j}\eta^{\mathcal{J}}_{\mathrm{a}}(\tilde{z}_{\mathrm{ab}j})\tilde{u}^{\mathrm{b}}_{(\mathrm{ab})j}\mathcal{J}_{\mathrm{bI}}(\tilde{z}_{\mathrm{ab}j})\right)$$

$$u^{\mathrm{a}}_{(c_{1k})j} = \left(u^{a}_{(c_{1k})j}, 0,0\right), u^{\mathrm{a}}_{(c_{2k})j} = \left(0, u^{\dot{a}}_{(c_{2k})j}, 0\right), u^{\mathrm{a}}_{(c_{3k})j} = \left(0,0, u^{\ddot{a}}_{(c_{3k})j}\right), k = 1,2,3$$

$$v^{\alpha}_{r\mathrm{a}}\kappa^{\mathrm{a}}_{rA} = u^{\alpha}_{r\mathrm{a}}\lambda^{\mathrm{a}}_A(\sigma_r), r = 1,\ldots,n, \lambda^{\mathrm{a}}_A(\sigma) = \lambda^{\mathrm{a}}_{0A}\frac{h_\alpha(\sigma)}{\sqrt{\mathrm{d}\sigma}} + \sum_{l=1}^{n} u^{\mathrm{a}\beta}_l \epsilon_{lA\beta}\frac{S_\alpha(\sigma,\sigma_l \mid \tau)}{\sqrt{\mathrm{d}\sigma}\sqrt{\mathrm{d}\sigma_l}},$$

$$v^{\alpha}_{r\mathrm{a}}\zeta^{\mathrm{a}}_{r\mathcal{J}} = u^{\alpha}_{r\mathrm{a}}\eta^{\mathrm{a}}_{\mathcal{J}}(\sigma_r), r = 1,\ldots,n, \eta^{\mathrm{a}}_{\mathcal{J}}(\sigma) = \eta^{\mathrm{a}}_{0\mathcal{J}}\frac{h_\alpha(\sigma)}{\sqrt{\mathrm{d}\sigma}} + \sum_{l=1}^{n} u^{\mathrm{a}\beta}_l q_{l\mathcal{J}\beta}\frac{S_\alpha(\sigma,\sigma_l \mid \tau)}{\sqrt{\mathrm{d}\sigma}\sqrt{\mathrm{d}\sigma_l}},$$

$$\frac{1}{U}u^{[\mathrm{a}}_\alpha \hat{u}^{\mathrm{b}]}_\beta = \varepsilon^{ab}\delta^0_\alpha\delta^0_\beta + \varepsilon^{\dot{a}\dot{b}}\delta^2_\alpha\delta^2_\beta + \varepsilon^{\ddot{a}\ddot{b}}\delta^3_\alpha\delta^3_\beta \text{ with } U = u_{\mathrm{a}\alpha}\hat{u}^{\mathrm{a}}_\beta D^{\alpha\beta} = u_\alpha\hat{u}^a \equiv \langle u\hat{u}\rangle.$$

$$\epsilon^\alpha_{iA}\epsilon^A_{j\alpha} = \epsilon_{iA}\epsilon^A_j, q^\alpha_{i\mathcal{J}}q^{\mathcal{J}}_{j\alpha} = q_{i\mathcal{J}}q^{\mathcal{J}}_j$$

$$\left(u^{\mathrm{a}}_{k\alpha}u^\beta_{l\mathrm{a}}\right)\left(u^\alpha_{k\,\mathrm{b}}u^{\mathrm{b}}_{l\beta}\right) = \langle u_k u_l\rangle^2$$

$$\begin{aligned} w(u) = \delta\left(\mathrm{Res}_\sigma(\lambda_{A\mathrm{a}}\lambda^{\mathrm{a}}_B\Omega^{AB})\right)\delta\left(\mathrm{Res}_\sigma\left(\eta_{\mathcal{J}\mathrm{a}}\eta^{\mathrm{a}}_{\mathcal{J}}\Omega^{\mathcal{J}\mathcal{J}}\right)\right) \\ \left(\frac{\hat{u}^\alpha_{\mathrm{a}}\lambda^{\mathrm{a}}_A\epsilon^A_\alpha}{U} + \epsilon^A_0\rho_{1aA}\rho^a_{1B}\epsilon^B_0 + \epsilon^A_2\rho_{1\dot{a}A}\rho^{\dot{a}}_{1B}\epsilon^B_2 + \epsilon^A_3\rho_{1\ddot{a}A}\rho^{\ddot{a}}_{1B}\epsilon^B_3\right) \\ \left(\frac{\hat{u}^\alpha_{\mathrm{a}}\lambda^{\mathrm{a}}_A\epsilon^A_\alpha}{U} + \epsilon^A_0\rho_{2aA}\rho^a_{2B}\epsilon^B_0 + \epsilon^A_2\rho_{2\dot{a}A}\rho^{\dot{a}}_{2B}\epsilon^B_2 + \epsilon^A_3\rho_{2\ddot{a}A}\rho^{\ddot{a}}_{2B}\epsilon^B_3\right. \\ \left. + \frac{\hat{u}^\alpha_{\mathrm{a}}\eta^{\mathrm{a}}_{\mathcal{J}}q^{\mathcal{J}}_\alpha}{U} + q^{\mathcal{J}}_0\tau_{a\mathcal{J}}\tau^a_{\mathcal{J}}q^{\mathcal{J}}_2 + q^{\mathcal{J}}_2\tau_{\dot{a}\mathcal{J}}\tau^{\dot{a}}_{\mathcal{J}}q^{\mathcal{J}}_2 + q^{\mathcal{J}}_3\tau_{\ddot{a}\mathcal{J}}\tau^{\ddot{a}}_{\mathcal{J}}q^{\mathcal{J}}_3\right). \end{aligned}$$

$$\hat{u}^{\dot{b}}_2 = u^{\dot{b}}_{(c_{2k})j}$$

$$\mathcal{B} = \begin{pmatrix} A & C \\ -C^T & H \end{pmatrix},$$

$$A_{ij} = \Lambda^{\mathrm{a}}_{iA}(z'_i)\Lambda^{A}_{j\mathrm{a}}(z'_j)\frac{S_\alpha(z'_i, z'_j \mid \tau)}{\sqrt{\mathrm{d}z'_i}\sqrt{\mathrm{d}z'_j}}, A = 10g \times 10g$$

$$C^{\mathrm{a}\beta}_{ij} = \Lambda^{\mathrm{a}}_{iA}(z'_i)\epsilon^{A\beta}_{j}\frac{S_\alpha(z'_i, \sigma_j \mid \tau)}{\sqrt{\mathrm{d}z'_i}\sqrt{\mathrm{d}\sigma_j}},$$

$$H_{ij} = \epsilon^{\beta}_{iA}\epsilon^{A}_{j\beta}\frac{S_\alpha(\sigma_i, \sigma_j \mid \tau)}{\sqrt{\mathrm{d}\sigma_i}\sqrt{\mathrm{d}\sigma_j}}, (i \neq j = 1, \ldots, n), H_{ii} = -e \cdot P(\sigma_i), k = 1, \ldots, g$$

$$\Lambda^{\mathrm{a}}_{iA}(z'_i) = \begin{cases} \lambda^{\mathrm{a}}_{A}(z_k), i = 10k \\ u^{\mathrm{a}}_{(k)l}\left(u^{\mathrm{b}}_{(k)l}\lambda_{A\,\mathrm{b}}(z_{(k)l})\right), i = 10k + l, l = 1, \ldots, 9, \end{cases}$$

$$\sum_i U_i C^{\mathrm{b}}_{ri\alpha} C^{\alpha}_{kd\,\mathrm{b}} = \Lambda^{\mathrm{b}}_{kA}(z'_k)\epsilon^{A}_{d\gamma}\hat{u}^{\gamma}_{\mathrm{a}}\frac{S_\alpha(z'_k, \sigma_d \mid \tau)}{\sqrt{\mathrm{d}z'_k}\sqrt{\mathrm{d}\sigma_d}}\Lambda_{rB\,\mathrm{b}}(z'_r)\sum_i u^{\mathrm{a}}_{i\beta}\epsilon^{B\beta}_{i}\frac{S_\alpha(z'_r, \sigma_i \mid \tau)}{\sqrt{\mathrm{d}z'_r}\sqrt{\mathrm{d}\sigma_i}}$$
$$= \frac{S_\alpha(z'_k, \sigma_d \mid \tau)}{\sqrt{\mathrm{d}z'_k}\sqrt{\mathrm{d}\sigma_d}}\epsilon^{A}_{d\gamma}\hat{u}^{\gamma}_{\mathrm{a}}\Lambda^{\mathrm{b}}_{kA}(z'_k)\Lambda_{rB\,\mathrm{b}}(z'_r)\lambda^{B\mathrm{a}}(z'_r)$$

$$\sum_i U_i C^{\mathrm{b}}_{ki\beta} C^{\beta}_{rd\,\mathrm{b}}$$

$$\sum_i U_i \left(C^{\mathrm{b}}_{ri\beta} C^{\beta}_{kd\,\mathrm{b}} - C^{\mathrm{b}}_{ki\beta} C^{\beta}_{rd\,\mathrm{b}}\right) \sim \varepsilon_{ABCD}\epsilon^{D}_{d\gamma}\lambda^{C}_{\mathrm{c}}(z'_r)\lambda^{B}_{\mathrm{b}}(z'_r)\lambda^{A}_{\mathrm{a}}(z'_r) = 0$$

$$\tilde{\mathcal{B}} = \begin{pmatrix} \tilde{A} & \tilde{C} \\ -\tilde{C}^T & G \end{pmatrix}$$

$$\tilde{A}_{ij} = (\tilde{\Lambda}^{\mathrm{a}}_{iA}(z'_i)\tilde{\Lambda}^{A}_{j\mathrm{a}}(z'_j) + \tilde{H}^{\mathrm{a}}_{i\mathcal{I}}(\tilde{z}'_i)\tilde{H}^{\mathcal{I}}_{j\mathrm{a}}(\tilde{z}'_j))\frac{S_\alpha(\tilde{z}'_i, \tilde{z}'_j|\tau)}{\sqrt{\mathrm{d}\tilde{z}'_i}\sqrt{\mathrm{d}\tilde{z}'_j}}, \tilde{A} = 10g \times 10g$$

$$\tilde{C}^{\mathrm{a}\beta}_{ij} = (\tilde{\Lambda}^{\mathrm{a}}_{iA}(\tilde{z}'_i)\epsilon^{A\beta}_{j} + \tilde{H}^{\mathrm{a}}_{i\mathcal{I}}(\tilde{z}'_i)q^{\mathcal{I}\beta}_{j})\ \frac{S_\alpha(\tilde{z}'_i, \sigma_j|\tau)}{\sqrt{\mathrm{d}\tilde{z}'_i}\sqrt{\mathrm{d}\sigma_j}},$$

$$\tilde{C}^{\mathrm{a}}_{ij\beta}\tilde{C}^{\beta}_{kl\mathrm{a}} \equiv \left(\epsilon^{A}_{j\beta}\tilde{\Lambda}^{\mathrm{a}}_{iA}(\tilde{z}'_i)\tilde{\Lambda}_{kB\mathrm{a}}(\tilde{z}'_k)\epsilon^{B\beta}_{l} + q^{\mathcal{I}}_{j\beta}\tilde{H}^{\mathrm{a}}_{i\mathcal{I}}(\tilde{z}'_i)\tilde{H}_{k\mathcal{J}\mathrm{a}}(\tilde{z}'_k)q^{\mathcal{J}\beta}_{l}\right)\frac{S_\alpha(\tilde{z}'_i, \sigma_j|\tau)}{\sqrt{\mathrm{d}\tilde{z}'_i}\sqrt{\mathrm{d}\sigma_j}}\frac{S_\alpha(\tilde{z}'_k, \sigma_l|\tau)}{\sqrt{\mathrm{d}\tilde{z}'_k}\sqrt{\mathrm{d}\sigma_l}},$$

$$G_{ij} = (\epsilon^{\beta}_{iA}\epsilon^{A}_{j\beta} + q^{\beta}_{i\mathcal{I}}q^{\mathcal{I}}_{j\beta})\frac{S_\alpha(\sigma_i, \sigma_j|\tau)}{\sqrt{\mathrm{d}\sigma_i}\sqrt{\mathrm{d}\sigma_j}},\quad G_{ii} = -(e_i \cdot P_i + q^{\beta\mathcal{I}}_{i}\tilde{q}^{\mathcal{J}}_{i\beta}Y_{i\mathcal{I}\mathcal{J}}),$$

$$\tilde{\Lambda}^{\mathrm{a}}_{iA}(\tilde{z}'_i) = \begin{cases} \lambda^{\mathrm{a}}_{A}(\tilde{z}_k), \\ \tilde{u}^{\mathrm{a}}_{(k)l}\tilde{u}^{\mathrm{b}}_{(k)l}\lambda_{A\mathrm{b}}(\tilde{z}_{(k)l}), \end{cases},\quad \tilde{H}^{\mathrm{a}}_{i\mathcal{I}}(\tilde{z}'_i) = \begin{cases} \eta^{\mathrm{a}}_{\mathcal{I}}(\tilde{z}_k), & i = 10k, \\ \tilde{u}^{\mathrm{a}}_{(k)l}\tilde{u}^{\mathrm{b}}_{(k)l}\eta_{\mathcal{I}\mathrm{b}}(\tilde{z}_{(k)l}), & i = 10k + l, l = 1, \ldots, 9. \end{cases}$$

$$A_g^\alpha = \int \mathrm{d}m_0 \tilde{Z}_1^4 \left(Z_{\frac{1}{2},\alpha}\right)^8 \prod_{j=1}^{g} \bar{\delta}_{x_j} A_{g,V,n}^\alpha$$

$$A_{g,V,n}^\alpha = \left\langle \prod_{j=1}^{g} \left(j_{z_j}^u \tilde{j}_{\tilde{z}_j}^u\right) V_1 \dots V_n \right\rangle_\alpha$$

$$= \prod_{l=1}^{n} \frac{\mathrm{d}\sigma_l\, \mathrm{d}\mu(u_l)\mathrm{d}\mu(v_l)}{\mathrm{vol}\left(\mathrm{G}_Q\right)_u} \prod_{r=1}^{n} \bar{\delta}(v_{r\mathrm{a}}^\alpha \epsilon_{r\alpha}^{\mathrm{a}} - 1)\bar{\delta}(u_{r\mathrm{a}}^\alpha \lambda_{rA}^{\mathrm{a}} - v_{r\mathrm{a}}^\alpha \kappa_{rA}^{\mathrm{a}})\bar{\delta}(u_{r\mathrm{a}}^\alpha \eta_{r\mathcal{J}}^{\mathrm{a}} - v_{r\mathrm{a}}^\alpha \zeta_{r\mathcal{J}}^{\mathrm{a}})\bar{\delta}_r \mathcal{H}\mathcal{G}$$

$$\bar{\delta}_r = \prod_{r=2}^{n} \bar{\delta}\left(\mathrm{Res}_{\sigma_r}(\lambda_A^{\mathrm{a}} \lambda_{B\mathrm{a}} \Omega^{AB})\right) \bar{\delta}\left(\mathrm{Res}_{\sigma_r}(\eta_{\mathcal{J}}^{\mathrm{a}} \eta_{\mathcal{J}\mathrm{a}} \Omega^{\mathcal{J}\mathcal{J}})\right)$$

$$\mathcal{H} = \prod_{j=1}^{g} \int \frac{\prod_{\mathrm{ab}=c_1\oplus c_2 \oplus c_3} \mathrm{d}^2 u_{(\mathrm{ab})j} e^{\|u_{(\mathrm{ab})j}\|^2} \sum_{I,J} \mathrm{Pf}(A_J^I) \sum_D \mathrm{qDet}\left(H_{[kl]}^{[ij]}\right) I}{\pi^{18} \prod_{i=1}^{3} \prod_{c_{i1}\in c_i}^{c_{i3}} \left(u_{(c_{i1})j}^{\mathrm{a}} u_{(c_{i2})j\mathrm{a}}\right)\left(u_{(c_{i2})j}^{\mathrm{a}} u_{(c_{i3})j\mathrm{a}}\right)\left(u_{(c_{i3})j}^{\mathrm{a}} u_{(c_{i1})j\mathrm{a}}\right)}$$

$$\mathcal{G} = \prod_{j=1}^{g} \int \frac{\prod_{\mathrm{ab}=c_1\oplus c_2 \oplus c_3} \mathrm{d}^2 \tilde{u}_{(\mathrm{ab})j} e^{\|\tilde{u}_{(\mathrm{ab})j}\|^2} \sum_{I,J} \mathrm{Pf}(\tilde{A}_J^I) \sum_D \mathrm{qDet}\left(G_{[st]}^{[pr]}\right)_{DJ}^I}{\pi^{18} \prod_{i=1}^{3} \prod_{c_{i1}\in c_i}^{c_{i3}} \left(\tilde{u}_{(c_{i1})j}^{\mathrm{a}} \tilde{u}_{(c_{i2})j\mathrm{a}}\right)\left(\tilde{u}_{(c_{i2})j}^{\mathrm{a}} \tilde{u}_{(c_{i3})j\mathrm{a}}\right)\left(\tilde{u}_{(c_{i3})j}^{\mathrm{a}} \tilde{u}_{(c_{i1})j\mathrm{a}}\right)}$$

$$H_{ij} = \epsilon_{iA}^\beta \epsilon_{j\beta}^A \frac{S_\alpha\left(\sigma_i, \sigma_j \mid \tau\right)}{\sqrt{\mathrm{d}\sigma_i}\sqrt{\mathrm{d}\sigma_j}}, H_{ii} = -e_i \cdot P_i$$

$$G_{ij} = \left(\epsilon_{iA}^\beta \epsilon_\beta^A + q_{i\mathcal{J}}^\beta q_{j\beta}^{\mathcal{J}}\right) \frac{S_\alpha\left(\sigma_i, \sigma_j \mid \tau\right)}{\sqrt{\mathrm{d}\sigma_i}\sqrt{\mathrm{d}\sigma_j}}, G_{ii} = -\left(e_i \cdot P_i + q_i^{\beta\mathcal{J}} \tilde{q}_{i\beta}^{\mathcal{J}} Y_{i\mathcal{J}\mathcal{J}}\right)$$

$$Z_{g,\alpha}^{\mathrm{chir}} \tilde{Z}_1^4 \left(Z_{\frac{1}{2},\alpha}^{\mathrm{even}}\right)^8 \bar{\delta}'_{w_j} j_{z_j} \tilde{j}_{\tilde{z}_j} \to j_{z_j}^u \tilde{j}_{\tilde{z}_j}^u$$

$$\bar{\delta}\left(\lambda_{\mathrm{a}}^A(x_j)\lambda_A^{\mathrm{a}}(x_j)\right) \bar{\delta}\left(\lambda_{\mathrm{a}}^A(\tilde{x}_j)\lambda_A^{\mathrm{a}}(\tilde{x}_j) + \eta_{\mathrm{a}}^{\mathcal{J}}(\tilde{x}_j)\eta_{\mathcal{J}}^{\mathrm{a}}(\tilde{x}_j)\right)$$

$$0 = \sum_i \epsilon_{i\gamma}^{[A}\left(v_i^{\mathrm{a}\gamma} \kappa_{i\mathrm{a}}^{B]} - u_i^{\mathrm{a}\gamma} \lambda_{\mathrm{a}}^{B]}(\sigma_i)\right) = \sum_i \kappa_{i\mathrm{a}}^A \kappa_i^{\mathrm{a}B} - \sum_{ij} u_i^{\mathrm{a}\gamma} u_{j\mathrm{a}}^\beta \epsilon_{i\gamma}^{[A} \epsilon_{j\beta}^{B]} S_\alpha\left(\sigma_i, \sigma_j \mid \tau\right) = \sum_i k_i^{AB}$$

$$\sum_i \kappa_i^{A\mathrm{a}} \zeta_{i\mathrm{a}}^{\mathcal{J}} = \sum_i \left((\kappa_i^{A\mathrm{a}} v_{i\mathrm{a}}^\gamma)(\epsilon_{i\gamma}^{\mathrm{b}} \zeta_{i\,\mathrm{b}}^{\mathcal{J}}) - (\kappa_i^{A\mathrm{a}} \epsilon_{i\mathrm{a}}^\gamma)(v_{i\gamma}^{\mathrm{b}} \zeta_{i\,\mathrm{b}}^{\mathcal{J}})\right) = \sum_i \left(u_{i\mathrm{a}}^\gamma \lambda_{\mathrm{a}}^A(\sigma_i)\right) q_{i\gamma}^{\mathcal{J}} - \epsilon_i^{A\gamma}\left(u_{i\gamma}^{\mathrm{b}} \eta_{\mathrm{b}}^{\mathcal{J}}(\sigma_i)\right)$$

$$= \sum_{ij} \left(\epsilon_j^{A\beta} q_i^{\mathcal{J}\gamma} - \epsilon_i^{A\gamma} q_j^{\mathcal{J}\beta}\right) u_{i\gamma}^{\mathrm{a}} u_{j\mathrm{a}\beta} S_\alpha\left(\sigma_i, \sigma_j \mid \tau\right) = 0$$

$$A_{g,n}^{\mathrm{even}} = \sum_\alpha \int \mathrm{d}\mu_g Z_g^\alpha((z),(\tilde{z}),(y),(x),(w)) \left(\prod_{j=1}^{g} \bar{\delta}_{x_j}\right) A_{g,V,n}^\alpha$$

$$Z_g^\alpha = \frac{Z_{2,\alpha}((y))Z_{1,\alpha}((x))^2 Z_{1,\alpha}((w))^9 \left(Z_{\frac{1}{2},\alpha}^{\mathrm{even}}\right)^8}{Z_{1,\alpha}((z))^{10} Z_{1,\alpha}((\tilde{z}))^{10}} = \frac{Y_3(y)Y(x)Y(\tilde{x})\prod_{\mathrm{ab}} Y(w_{\mathrm{ab}})\theta[\alpha](0)^8}{Y(z)Y(\tilde{z})\prod_{\mathrm{ab}} Y(z_{\mathrm{ab}})Y(\tilde{z}_{\mathrm{ab}})}$$

$$\bar{\delta}_r = \prod_{r=2}^{n} \bar{\delta}(\kappa_{ra}^A \kappa_{rA}^{\mathrm{a}}) \bar{\delta}(\zeta_{ra}^{\mathcal{J}} \zeta_{r\mathcal{J}}^{\mathrm{a}})$$

$$\lambda_A^{\mathrm{a}}(\sigma)\lambda_{\mathrm{a}}^A(\sigma)=\sum_{r=1}^{n} k_{rA}^A S_\alpha(\sigma,\sigma_r\mid\tau)=0$$

$$\mathrm{d}z=\det_\omega=\mathrm{Vol}(\mathrm{CKV})_{\mathrm{chir}}$$

$$A_{1,n}^{\mathrm{even}}=\sum_\alpha\int\frac{\mathrm{d}\tau}{\det_\omega^{10}}q_\alpha^2\left(\frac{\theta[\alpha](0)}{\eta(\tau)^3}\right)^8\bar{\delta}_{x_1}\frac{A_{1,V,n}^\alpha}{\mathrm{vol}(\mathrm{GL}(1,\mathbb{C}))_\sigma},q_\alpha^2=\lim_{z\to 0}\left(\frac{\theta_1(z)}{\theta_\alpha(z-\Delta)}\right)^8$$

$$\mu_0^{\mathrm{a}\dot{\beta}}\frac{h_\alpha}{\sqrt{\mathrm{d}\sigma}}=x_0^{\beta\dot{\beta}}\lambda_\beta^{\mathrm{a}}-\Theta_0^{\mathcal{J}\dot{\beta}}\eta_{\mathcal{J}}^{\mathrm{a}},\tilde{\mu}_0^{\mathrm{a}\beta}\frac{h_\alpha}{\sqrt{\mathrm{d}\sigma}}=-x_0^{\beta\dot{\beta}}\tilde{\lambda}_{\dot{\beta}}^{\mathrm{a}}-\Theta_0^{\mathcal{J}\beta}\eta_{\mathcal{J}}^{\mathrm{a}},\tilde{\eta}_0^{\mathrm{a}\mathcal{J}}\frac{h_\alpha}{\sqrt{\mathrm{d}\sigma}}=\Theta_0^{\mathcal{J}A}\lambda_A^{\mathrm{a}},$$

$$\int\frac{\mathrm{d}\mu(\mu_0^{\mathrm{a}A})\mathrm{d}\mu(\tilde{\eta}_0^{\mathrm{a}\mathcal{J}})}{\mathrm{vol}(\mathrm{G}_Q\times\mathbb{C})_{\mu_0}}\exp\left[-\sum_{i=1}^{n}U_i\right]$$

$$\begin{aligned}
U_i&=\left((u_{i\xi}^{\mathrm{a}}\mu_{0\mathrm{a}}^A)\epsilon_{iA}^\xi+(u_{i\xi}^{\mathrm{a}}\tilde{\eta}_{0\mathrm{a}}^{\mathcal{J}})q_{i\mathcal{J}}^\xi\right)\frac{h_\alpha}{\sqrt{\mathrm{d}\sigma}}\\
&=x_0^{\beta\dot{\beta}}\left(\epsilon_{i\dot{\beta}}^\xi(u_{i\xi}^{\mathrm{a}}\lambda_{i\beta}^{\mathrm{a}})-\epsilon_{i\beta}^\xi\left(u_{i\xi}^{\mathrm{a}}\tilde{\lambda}_{i\dot{\beta}}^{\mathrm{a}}\right)\right)-\Theta_0^{\mathcal{J}A}(u_{i\xi}^{\mathrm{a}}\eta_{\mathcal{J}\mathrm{a}})\epsilon_{iA}^\xi+\Theta_0^{\mathcal{J}A}(u_{i\xi}^{\mathrm{a}}\lambda_{iA\mathrm{a}})q_{i\mathcal{J}}^\xi\\
&=x_0^{\beta\dot{\beta}}\left(\left(v_{i\mathrm{a}\xi}\kappa_{i\beta}^{\mathrm{a}}\right)\left(\epsilon_{i\,\mathrm{b}}^\xi\tilde{\kappa}_{i\dot{\beta}}^{\mathrm{b}}\right)-\left(\epsilon_{i\mathrm{a}}^\xi\kappa_{i\beta}^{\mathrm{a}}\right)\left(v_{i\,\mathrm{b}\xi}\tilde{\kappa}_{i\dot{\beta}}^{\mathrm{b}}\right)\right)+\Theta_0^{\mathcal{J}A}\left(-\left(\epsilon_{i\mathrm{a}}^\xi\kappa_{i\beta}^{\mathrm{a}}\right)\left(v_{i\,\mathrm{b}\xi}\zeta_{\mathcal{J}}^{\mathrm{b}}\right)+\left(v_{i\mathrm{a}\xi}\kappa_{i\beta}^{\mathrm{a}}\right)\left(\epsilon_{i\,\mathrm{b}}^\xi\zeta_{\mathcal{J}}^{\mathrm{b}}\right)\right)\\
&=x_0^{\beta\dot{\beta}}\kappa_{i\beta}^{\mathrm{a}}\tilde{\kappa}_{i\dot{\beta}\mathrm{a}}+\Theta_0^{\mathcal{J}A}\kappa_{iA\mathrm{a}}\zeta_{i\mathcal{J}}^{\mathrm{a}}=x_0\cdot k_i+\Theta_0^{\mathcal{J}A}\kappa_{iA\mathrm{a}}\zeta_{i\mathcal{J}}^{\mathrm{a}}
\end{aligned}$$

$$\tilde{\eta}_0^{\mathrm{a}\mathcal{J}},k_i^{\beta\dot{\beta}}=\kappa_{i\beta}^{\mathrm{a}}\tilde{\kappa}_{i\dot{\beta}\mathrm{a}}=k_i^\nu\sigma_\nu^{\beta\dot{\beta}}$$

$$A_{g,n}^{\mathrm{odd}}=\delta\left(\sum_{i=1}^{n}k_i\right)\delta\left(\sum_{i=1}^{n}\kappa_{iA\mathrm{a}}\zeta_{i\mathcal{J}}^{\mathrm{a}}\right)\sum_\alpha\int\mathrm{d}\mu_g Z_g^\alpha((z),(\tilde{z}),(y),(x),(w))\int\mathrm{d}m_0\left(\prod_{j=1}^{g}\bar{\delta}_{x_j}\right)A_{g,V,n}^\alpha$$

$$Z_g^\alpha=\frac{Z_{2,\alpha}((y))Z_{1,\alpha}((x))^2Z_{1,\alpha}((w))^9\left(Z_{\frac{1}{2},\alpha}^{\mathrm{odd}}\right)^8}{Z_{1,\alpha}((z))^{10}Z_{1,\alpha}((\tilde{z}))^{10}}=\frac{Y_3(y)Y(x)Y(\tilde{x})\prod_{\mathrm{ab}}Y(w_{\mathrm{ab}})}{Y(z)Y(\tilde{z})\prod_{\mathrm{ab}}Y(z_{\mathrm{ab}})Y(\tilde{z}_{\mathrm{ab}})}\left(\frac{(h_\alpha)^2}{\det_\omega}\right)^8$$

$$\mathrm{d}m_0=\frac{\mathrm{d}\mu(\lambda_{0A}^{\mathrm{a}})\mathrm{d}\mu(\eta_{0\mathcal{J}}^{\mathrm{a}})}{\mathrm{vol}(\mathrm{G}_Q\times\mathbb{C})_{\lambda_0}}$$

$$A_{1,n}^{\mathrm{odd}}=\delta\left(\sum_{i=1}^{n}k_i\right)\delta\left(\sum_{i=1}^{n}\kappa_{iA}^{\mathrm{a}}\zeta_{i\mathrm{a}\mathcal{J}}\right)\int\frac{\mathrm{d}\tau}{\det_\omega^{10}}\frac{\mathrm{d}\mu(\lambda_{0A}^{\mathrm{a}})\mathrm{d}\mu(\eta_{0\mathcal{J}}^{\mathrm{a}})}{\mathrm{vol}(\mathrm{G}_Q\times\mathbb{C})_{\lambda_0}}\bar{\delta}_{x_1}\frac{A_{1,V,n}}{\mathrm{vol}(\mathrm{GL}(1,\mathbb{C}))_\sigma}.$$

$$A_{1,n}^{\mathrm{odd}}=\delta\left(\sum_{i=1}^{n}k_i\right)\delta\left(\sum_{i=1}^{n}\kappa_{iA}^{\mathrm{a}}\zeta_{i\mathrm{a}\mathcal{J}}\right)\int\frac{\mathrm{d}\tau}{\mathrm{d}^2z}\frac{\mathrm{d}\mu\left(\lambda_{0A}^{\mathrm{a}}\sqrt{\mathrm{d}z}\right)\mathrm{d}\mu\left(\eta_{0\mathcal{J}}^{\mathrm{a}}\sqrt{\mathrm{d}z}\right)}{\left(\mathrm{vol}(\mathrm{G}_Q\times\mathbb{C})_{\lambda_0}\mathrm{d}^4z\right)}\bar{\delta}_{x_1}\frac{A_{1,V,n}^\alpha}{\mathrm{vol}(\mathrm{GL}(1,\mathbb{C}))_\sigma}.$$

$$\oint\frac{\mathrm{d}q_{gg}}{q_{gg}}$$

$$\tau=\begin{pmatrix}\tau_{ij}^{(g-1)} & \int_{\sigma_-}^{\sigma_+}\omega_i^{(g-1)}\\ \int_{\sigma_-}^{\sigma_+}\omega_j^{(g-1)} & \frac{1}{\pi i}\ln q_{gg}+\mathrm{const}\end{pmatrix}+O\left(q_{gg}^2\right)$$

$$\omega_i = \omega_i^{(g-1)} + O\left(q_{gg}^2\right) \text{ for } i < g$$
$$\omega_g = \omega_{+-}^{(g-1)} + O\left(q_{gg}^2\right)$$
$$S_\alpha(\sigma, \sigma_* \mid \tau) = S_{\alpha^{(g-1)}}\left(\sigma, \sigma_* \mid \tau^{(g-1)}\right) + O\left(q_{gg}\right) \equiv S_\alpha^{(g-1)}(\sigma, \sigma_* \mid \tau) + O\left(q_{gg}\right)$$

$$P^\mu = \sum_{I=1}^{g} c_i^\mu \omega_i + \sum_{j=1}^{n} k_j^\mu \omega_{j*}, \omega_{j*} = S_\alpha\left(\sigma, \sigma_j \mid \tau\right) - S_\alpha(\sigma, \sigma_* \mid \tau)$$

$$P^\mu \to \sum_{I=1}^{g-1} c_i^\mu \omega_i^{(g-1)} + k_+^\mu S_\alpha^{(g-1)}(\sigma, \sigma_+ \mid \tau) + k_-^\mu S_\alpha^{(g-1)}(\sigma, \sigma_- \mid \tau) + \sum_{j=1}^{n} k_j^\mu S_\alpha^{(g-1)}\left(\sigma, \sigma_j \mid \tau\right)$$
$$= \sum_{I=1}^{g-1} c_i^\mu \omega_i^{(g-1)} + \sum_{j=1}^{n+2} k_j^\mu S_\alpha^{(g-1)}\left(\sigma, \sigma_j \mid \tau\right)$$

$$\lambda_A^{\mathrm{a}}(\sigma) \to \sum_{i=1}^{n+2} u_{i\beta}^{\mathrm{a}} \epsilon_{iA}^\beta \frac{S_\alpha^{(g-1)}(\sigma, \sigma_i \mid \tau)}{\sqrt{\mathrm{d}\sigma}\sqrt{\mathrm{d}\sigma_i}}, \kappa_{n+1A}^{\mathrm{a}} = \kappa_{+A}^{\mathrm{a}}, \kappa_{n+2A}^{\mathrm{a}} = \kappa_{-A}^{\mathrm{a}}, \kappa_{+A}^{\mathrm{a}}\kappa_{+B\mathrm{a}} = -\kappa_{-A}^{\mathrm{a}}\kappa_{-B\mathrm{a}}$$
$$\eta_{\mathcal{J}}^{\mathrm{a}}(\sigma) \to \sum_{i=1}^{n+2} u_{i\beta}^{\mathrm{a}} q_{i\mathcal{J}}^\beta \frac{S_\alpha^{(g-1)}(\sigma, \sigma_i \mid \tau)}{\sqrt{\mathrm{d}\sigma}\sqrt{\mathrm{d}\sigma_i}}, \zeta_{n+1\mathcal{J}}^{\mathrm{a}} = \zeta_{+\mathcal{J}}^{\mathrm{a}}, \zeta_{n+2\mathcal{J}}^{\mathrm{a}} = \zeta_{-\mathcal{J}}^{\mathrm{a}}, \zeta_{+\mathcal{J}}^{\mathrm{a}}\zeta_{+\mathcal{J}\mathrm{a}} = -\zeta_{-\mathcal{J}}^{\mathrm{a}}\zeta_{-\mathcal{J}\mathrm{a}}$$
$$\kappa_{+A}^{\mathrm{a}}\zeta_{+\mathcal{J}\mathrm{a}} = -\kappa_{-A}^{\mathrm{a}}\zeta_{-\mathcal{J}\mathrm{a}}$$

$$\epsilon_{n+1,2\beta}^A = \epsilon_{\pm\beta}^A = \epsilon_{\pm\beta}^{\mathrm{a}}\kappa_{\pm\mathrm{a}}^A$$

$$P^{AB}(\sigma) = \lambda_{\mathrm{a}}^A(\sigma)\lambda^{B\mathrm{a}}(\sigma)$$

$$P^{AB}(\sigma) = \epsilon_+^{\beta[A}\left(u_{+\beta}^{\mathrm{a}}\lambda_{\mathrm{a}}^{B]}(\sigma_+)\right)\frac{S_\alpha(\sigma, \sigma_+ \mid \tau)}{\sqrt{\mathrm{d}\sigma}\sqrt{\mathrm{d}\sigma_+}} + \epsilon_-^{\beta[A}\left(u_{-\beta}^{\mathrm{a}}\lambda_{\mathrm{a}}^{B]}(\sigma_-)\right)\frac{S_\alpha(\sigma, \sigma_- \mid \tau)}{\sqrt{\mathrm{d}\sigma}\sqrt{\mathrm{d}\sigma_-}} + \sum_{i=1}^{n} k_i^{AB}\frac{S_\alpha(\sigma, \sigma_i \mid \tau)}{\sqrt{\mathrm{d}\sigma}\sqrt{\mathrm{d}\sigma_i}}$$
$$P_A^A(\sigma) = 0, P^2(\sigma) = \varepsilon_{ABCD}\lambda_{\mathrm{a}}^A(\sigma)\lambda^{B\mathrm{a}}(\sigma)\lambda_{\mathrm{b}}^C(\sigma)\lambda^{D\,\mathrm{b}}(\sigma) = 0$$

$$\epsilon_{\pm\alpha}^A\left(u_{\pm\mathrm{a}}^\alpha\lambda_A^{\mathrm{a}}(\sigma_\pm)\right) = 0$$

$$\epsilon_\pm^\beta\left[A\left(u_{\pm\beta}^{\mathrm{a}}\lambda_{\mathrm{a}}^{B]}(\sigma_\pm)\right)P_{AB}(\sigma_\pm) = 0\right.$$

$$v_{\pm\alpha}^{\mathrm{a}}\kappa_{\pm\mathrm{a}}^A = u_{\pm\alpha}^{\mathrm{a}}\lambda_{\mathrm{a}}^A(\sigma_\pm), k_\pm \cdot P(\sigma_\pm) = 0$$

$$\delta^4\left(\sum_{i=1}^{n} k_i^{\mu}\right)\delta^{4\mathcal{N}}\left(\sum_{i=1}^{n}\kappa_{iA}^{\mathrm{a}}\zeta_{ia\mathcal{J}}\right)\frac{\mathrm{d}\mu(\lambda_{0A}^{\mathrm{a}})\mathrm{d}\mu(\eta_{0\mathcal{J}}^{\mathrm{a}})}{\mathrm{vol}\left(\mathrm{G}_Q\times\mathbb{C}\right)_{\lambda_0}}\frac{1}{(2\pi i)^3}\frac{\mathrm{d}q_{gg}}{q_{gg}}\frac{\mathrm{d}q_1}{q_1}\frac{\mathrm{d}q_2}{q_2}\bar{\delta}\left(\lambda_{\mathrm{a}}^{A}(x_j)\lambda_{A}^{\mathrm{a}}(x_j)\right)$$

$$\bar{\delta}\left(\lambda_{\mathrm{a}}^{A}(\tilde{x}_j)\lambda_{A}^{\mathrm{a}}(\tilde{x}_j)\right)\bar{\delta}\left(\eta_{\mathrm{a}}^{\mathcal{J}}(\tilde{x}_j)\eta_{\mathcal{J}}^{\mathrm{a}}(\tilde{x}_j)\right)$$

$$=\int\frac{\mathrm{d}\mu(\kappa_{n+1\mathrm{a}}^{A})\mathrm{d}\mu(\zeta_{n+1\mathrm{a}}^{\mathcal{J}})}{\mathrm{vol}\left(\mathrm{G}_Q\times\mathbb{C}\right)_{\kappa}}\frac{\prod_{l=n+1}^{n+2}\ \mathrm{d}\sigma_l\ \mathrm{d}\mu(u_l)\mathrm{d}\mu(v_l)}{\mathrm{vol}\left(\mathrm{SL}(2,\mathbb{C})_{\sigma}\times\mathrm{G}_{Q_u}\right)}\frac{1}{(u_{n+1\alpha}^{\mathrm{a}}u_{n+2\mathrm{a}}^{\alpha})^4}$$

$$\prod_{r=n+1}^{n+2}\ \bar{\delta}(v_{r\mathrm{a}}^{\alpha}\epsilon_{r\alpha}^{\mathrm{a}}-1)\bar{\delta}(u_{r\mathrm{a}}^{\alpha}\lambda_{rA}^{\mathrm{a}}-v_{r\mathrm{a}}^{\alpha}\kappa_{rA}^{\mathrm{a}})\bar{\delta}(u_{r\mathrm{a}}^{\alpha}\eta_{r\mathcal{J}}^{\mathrm{a}}-v_{r\mathrm{a}}^{\alpha}\zeta_{r\mathcal{J}}^{\mathrm{a}})$$

$$\bar{\delta}\left(\lambda_{\mathrm{a}}^{A}(x_j)\lambda_{A}^{\mathrm{a}}(x_j)\right)\bar{\delta}\left(\lambda_{\mathrm{a}}^{A}(\tilde{x}_j)\lambda_{A}^{\mathrm{a}}(\tilde{x}_j)+\eta_{\mathrm{a}}^{\mathcal{J}}(\tilde{x}_j)\eta_{\mathcal{J}}^{\mathrm{a}}(\tilde{x}_j)\right)$$

$$(u_{n+1\alpha}^{\mathrm{a}}u_{n+2\mathrm{a}}^{\alpha})^{4-\mathcal{N}}=(u_{n+1\alpha}^{\mathrm{a}}u_{n+2\mathrm{a}}^{\alpha})^{-4}$$

$$\mathrm{d}\mu(\lambda_{0A}^{\mathrm{a}})\mathrm{d}\mu(\eta_{0\mathcal{J}}^{\mathrm{a}})\text{ to }\prod_l\ \mathrm{d}\mu(\kappa_{l\mathrm{a}}^{A})\mathrm{d}\mu(\zeta_{l\mathrm{a}}^{\mathcal{J}})$$

$$1=\int\frac{\mathrm{d}\mu(\kappa_{n+1\mathrm{a}}^{A})\mathrm{d}\mu(\zeta_{n+1\mathrm{a}}^{\mathcal{J}})}{\mathrm{vol}\left(\mathrm{G}_Q\times\mathbb{C}\right)_{\kappa}}\frac{\mathcal{W}_{\mathrm{vac}}}{\mathrm{d}\sigma_*\ \mathrm{d}^3u}\prod_{l=n+1}^{n+2}\ \mathrm{d}\mu(u_l)\mathrm{d}\mu(v_l)\bar{\delta}(v_{l\mathrm{a}}^{\alpha}\epsilon_{l\alpha}^{\mathrm{a}}-1)$$

$$\bar{\delta}(u_{l\mathrm{a}}^{\alpha}\lambda_{lA}^{\mathrm{a}}-v_{l\mathrm{a}}^{\alpha}\kappa_{lA}^{\mathrm{a}})\bar{\delta}(u_{l\mathrm{a}}^{\alpha}\eta_{l\mathcal{J}}^{\mathrm{a}}-v_{l\mathrm{a}}^{\alpha}\zeta_{l\mathcal{J}}^{\mathrm{a}})e_{l\alpha}^{u_{l\alpha}^{\mathrm{a}}}\mu_{\mathrm{a}}^{A}(\sigma_l)\epsilon_{A}^{l\alpha}+u_{l\alpha}^{\mathrm{a}}\tilde{\eta}_{\mathrm{a}}^{\mathcal{J}}(\sigma_l)q_{\mathcal{J}}^{l\alpha}$$

$$\kappa_{n+1}^{A\mathrm{a}}\kappa_{n+1\mathrm{a}}^{B}=-\kappa_{n+2}^{A\mathrm{a}}\kappa_{n+2\mathrm{a}}^{B},\zeta_{n+1}^{\mathcal{J}\mathrm{a}}\zeta_{n+1\mathrm{a}}^{\mathcal{J}}=-\zeta_{n+2}^{\mathcal{J}\mathrm{a}}\zeta_{n+2\mathrm{a}}^{\mathcal{J}},\kappa_{n+1A}^{\mathrm{a}}\zeta_{n+1\mathcal{J}\mathrm{a}}=-\kappa_{n+2A}^{\mathrm{a}}\zeta_{n+2\mathcal{J}\mathrm{a}},$$

$$\mathcal{W}_{\mathrm{vac}}(\sigma_{n+1},\sigma_{n+2},\sigma_*)=c(\sigma_{n+1})c(\sigma_{n+2})c(\sigma_*)n\tilde{n}\delta(\gamma)\delta(\tilde{\gamma})\prod_{\mathrm{ab}}\ (\delta(\gamma_{\mathrm{ab}})\delta(\tilde{\gamma}_{\mathrm{ab}})N_{\mathrm{ab}})\Bigg),$$

$$\langle 0|\mathcal{W}_{\mathrm{vac}}(0,0,0)|0\rangle=1,\epsilon_{n+1\mathrm{a}}^{\alpha}\epsilon_{n+2\alpha}^{\mathrm{a}}=1,$$

$$(\delta(\gamma_{\mathrm{ab}}u_{1\alpha}^{\mathrm{a}}u_1^{\mathrm{b}\alpha})\delta(\beta_{\mathrm{ab}}\hat{u}_{\alpha}^{\mathrm{a}}\hat{u}^{\mathrm{b}\alpha})\lambda_A^{\mathrm{a}}\rho_{1\mathrm{a}}^{A}\hat{u}_{\mathrm{a}}^{\alpha}\hat{u}_{\alpha}^{\mathrm{a}})(\delta(\gamma)\delta(\beta)\lambda_A^{\mathrm{a}}(\sigma_1)\rho_{1\mathrm{a}}^{A})e_{1\alpha}^{u_{1\alpha}^{\mathrm{a}}\mu_{\mathrm{a}}^{A}(\sigma_1)\epsilon_{1A}^{\alpha}}\Rightarrow$$

$$\left(\frac{\hat{u}_{\mathrm{a}}^{\alpha}\lambda_A^{\mathrm{a}}(\sigma_1)\epsilon_{1\alpha}^{A}}{u_{1\alpha}^{\mathrm{a}}\hat{u}_{\mathrm{a}}^{\alpha}}+\frac{\rho_{1\mathrm{a}}^{A}(\sigma_1)\rho_1^{B\mathrm{a}}(\sigma_1)}{2}\epsilon_{1A}^{\alpha}\epsilon_{1B\alpha}\right)e_{1\alpha}^{u_{\mathrm{a}}^{\mathrm{a}}\mu_{\mathrm{a}}^{A}(\sigma_1)\epsilon_{1A}^{\alpha}}$$

$$(\delta(\gamma_{\mathrm{ab}}u_{2\alpha}^{\mathrm{a}}u_2^{\mathrm{b}\alpha})\delta(\beta_{\mathrm{ab}}\hat{u}_{\alpha}^{\mathrm{a}}\hat{u}^{\mathrm{b}\alpha})\lambda_A^{\mathrm{a}}\rho_{1\mathrm{a}}^{A}\hat{u}_{\mathrm{a}}^{\alpha}\hat{u}_{\alpha}^{\mathrm{a}})$$

$$\left((u_{2\mathrm{a}}^{\alpha}\hat{u}_{\alpha}^{\mathrm{a}})\delta(\gamma_{\mathrm{ab}}u_2^{\mathrm{a}}\hat{u}^{\mathrm{b}})\delta(\beta_{\mathrm{ab}}\hat{u}^{\mathrm{a}}u_2^{\mathrm{b}})\lambda_A^{\mathrm{a}}\hat{u}_{\mathrm{a}}^{\alpha}\rho_{1\mathrm{a}}^{A}u_{2\alpha}^{\mathrm{a}}\right)e^{u_{2\alpha}^{\mathrm{a}}\mu_{\mathrm{a}}^{A}(\sigma_2)\epsilon_{2A}^{\alpha}}\Rightarrow$$

$$\left(\frac{\hat{u}_{\mathrm{a}}^{\alpha}\lambda_A^{\mathrm{a}}(\sigma_2)\epsilon_{2\alpha}^{A}}{u_{2\alpha}^{\mathrm{a}}\hat{u}_{\mathrm{a}}^{\alpha}}+\frac{\rho_{1\mathrm{a}}^{A}(\sigma_2)\rho_1^{B\mathrm{a}}(\sigma_2)}{2}\epsilon_{2A}^{\alpha}\epsilon_{2B\alpha}\right)e^{u_{2\alpha}^{\mathrm{a}}\mu_{\mathrm{a}}^{A}(\sigma_2)\epsilon_{2A}^{\alpha}}$$

$$\bar{\delta}\left(\mathrm{Res}_{\sigma_{n+1}}(\lambda_A^{\mathrm{a}}\lambda_{\mathrm{a}}^{A})\right)\zeta_{n+1\mathrm{a}}^{\mathcal{J}}=\pm\zeta_{n+2\mathrm{a}}^{\mathcal{J}}\zeta_{n+1}^{\mathcal{J}\mathrm{a}}\zeta_{n+1\mathrm{a}}^{\mathcal{J}}=-\zeta_{n+2}^{\mathcal{J}\mathrm{a}}\zeta_{n+2\mathrm{a}}^{\mathcal{J}}$$

$$S_{\alpha}^{(g)}\to S_{\alpha}^{(g-1)}+O(q_{gg})$$

$$\delta\left(\sum_{i=1}^{n}k_i\right)\delta\left(\sum_{i=1}^{n}\kappa_{iA\mathrm{a}}\zeta_{i\mathcal{J}}^{\mathrm{a}}\right)\int\ \mathrm{d}m_0\bar{\delta}(x_g)A_{g,V,n}^{\alpha}\to\int\ \frac{\mathrm{d}\mu(\kappa_{n+1\mathrm{a}}^{A})\mathrm{d}\mu(\zeta_{n+1\mathrm{a}}^{\mathcal{J}})}{\mathrm{vol}\left(\mathrm{G}_Q\right)_{\kappa}\delta(k_{n+2A}^{A})}A_{g-1,V,n+2}^{\alpha}$$

$$\left(A_{g,n}^{\text{odd}}\right)_{\text{cyc}}^{\text{odd}} = \frac{1}{4}\int \mathrm{d}m_{\text{fw}} A_{g-1,n+2}^{\text{even}}$$

$$= \frac{1}{4}\sum_{\alpha} \int \mathrm{d}\mu_{g-1} Z_{g-1}^{\alpha}((z),(\tilde{z}),(y),(x),(w)) \int \mathrm{d}m_{\text{fw}} \prod_{j=1}^{g-1} \bar{\delta}_{x_j} A_{g-1,V,n+2}^{\alpha}$$

$$\mathrm{d}m_{\text{fw}}\delta\left(k_{n+2A}^{A}\right) = \frac{\mathrm{d}\mu\left(\kappa_{n+1\mathrm{a}}^{A}\right)\mathrm{d}\mu\left(\zeta_{n+1\mathrm{a}}^{\mathcal{J}}\right)}{\mathrm{vol}\left(\mathrm{G}_{Q}\right)_{\kappa}}$$

$$\left(A_{g,n}^{\text{odd}}\right)_{\text{cyc}}^{\text{even}} = \frac{3}{4}\int \mathrm{d}m_{\text{fw}} A_{g-1,n+2}^{\text{odd}} = \delta\left(\sum_{i=1}^{n+2} k_i\right)\delta\left(\sum_{i=1}^{n+2} \kappa_{iA}^{\mathrm{a}}\zeta_{i\mathcal{J}\mathrm{a}}\right)$$

$$\frac{3}{4}\sum_{\alpha} \int \mathrm{d}\mu_{g-1} Z_{g-1}^{\alpha}((z),(\tilde{z}),(y),(x),(w)) \int \mathrm{d}m_0\, \mathrm{d}m_{\text{fw}} \prod_{j=1}^{g-1} \bar{\delta}_{x_j} A_{g-1,V,n+2}^{\alpha}$$

$$\mathrm{d}m_0 = \frac{\mathrm{d}\mu\left(\lambda_{0A}^{\mathrm{a}}\right)\mathrm{d}\mu\left(\eta_{0\mathcal{J}}^{\mathrm{a}}\right)}{\mathrm{vol}\left(\mathrm{G}_{Q}\times\mathbb{C}\right)_{\lambda_0}},\ \mathrm{d}m_{\text{fw}}\delta\left(k_{n+2A}^{A}\right) = \frac{\mathrm{d}\mu\left(\kappa_{n+1\mathrm{a}}^{A}\right)\mathrm{d}\mu\left(\zeta_{n+1\mathrm{a}}^{\mathcal{J}}\right)}{\mathrm{vol}\left(\mathrm{G}_{Q}\right)_{\kappa}}$$

$$\left(A_{g,n}^{\text{even}}\right)_{\text{cyc}}^{\text{even}} = \frac{3}{4}\int \mathrm{d}m_{\text{fw}} A_{g-1,n+2}^{\text{even}}$$

$$= \frac{3}{4}\sum_{\alpha} \int \mathrm{d}\mu_{g-1} Z_{g-1}^{\alpha}((z),(\tilde{z}),(y),(x),(w)) \int \mathrm{d}m_{\text{fw}} \prod_{j=1}^{g-1} \bar{\delta}_{x_j} A_{g-1,V,n+2}^{\alpha}$$

$$A_{1,n}^{\text{even}} = \frac{3}{4}\int \mathrm{d}m_{\text{fw}} A_{0,V,n+2}^{\alpha}$$

$$\left(A_{g,n}^{\text{even}}\right)_{\text{cyc}}^{\text{odd}} = \frac{1}{4}\int \mathrm{d}m_{\text{fw}} A_{g-1,n+2}^{\text{odd}} = \delta\left(\sum_{i=1}^{n+2} k_i\right)\delta\left(\sum_{i=1}^{n+2} \kappa_{iA}^{\mathrm{a}}\zeta_{i\mathcal{J}\mathrm{a}}\right)$$

$$\frac{1}{4}\sum_{\alpha} \int \mathrm{d}\mu_{g-1} Z_{g-1}^{\alpha}((z),(\tilde{z}),(y),(x),(w)) \int \mathrm{d}m_0\, \mathrm{d}m_{\text{fw}} \prod_{j=1}^{g-1} \bar{\delta}_{x_j} A_{g-1,V,n+2}^{\alpha}$$

$$\widehat{\mathcal{M}}_{g_1,n_1+1} \times \widehat{\mathcal{M}}_{g_2,n_2+1} \text{ with } g = g_1 + g_2, n = n_1 + n_2$$

$$\int \mathrm{d}m_{\text{fw}} A_{g,n}^{\text{even}}$$

$$p^{AB} = p^{\mu}\gamma_{\mu}^{AB} = \kappa_{n+1\mathrm{a}}^{A}\kappa_{n+1}^{B\mathrm{a}} = k_{n+1}^{AB} = -k_{n+2}^{AB}$$

$$\mathrm{d}m_{\text{fw}}\delta\left(k_{n+1A}^{A}\right)\delta\left(k_{n+2A}^{A}\right) \to \frac{\mathrm{d}^4 p^{\mu}}{p^2}\, \mathrm{d}^{\mathcal{N}} q_{n+1}^{\mathcal{J}} \mathrm{d}^{\mathcal{N}} q_{n+2}^{\mathcal{J}}$$

$$\det{}' H_{DJ}^{I} \det{}' G_{DJ}^{I} = \frac{\det\left(H_{[+-]}^{[+-]}\right)_{DJ}^{I} \det\left(G_{[+-]}^{[+-]}\right)_{DJ}^{I}}{\langle u_{+}u_{-}\rangle^{4}}$$

$$(cu_{+a}+u_{-a})\sum_i u_i^a\left(\left(H_{[+-]}^{[+-]}\right)_{DJ}^I\right)_{ij} = (cu_{+a}+u_{-a})\left(-\frac{u_+^a\epsilon_+^B\epsilon_{jB}}{\sigma_{+j}}-\frac{u_-^a\epsilon_-^B\epsilon_{jB}}{\sigma_{-j}}\right)$$

$$= \langle u_+u_-\rangle\left(\frac{\epsilon_+^B\epsilon_{jB}}{\sigma_{+j}}-c\frac{\epsilon_-^B\epsilon_{jB}}{\sigma_{-j}}\right), \frac{1}{\sigma_{\pm j}} = \frac{S_\alpha(\sigma_\pm,\sigma_j \mid \tau)}{\sqrt{\sigma_\pm}\sqrt{\sigma_j}}$$

$$\epsilon_+^a = (1/\sqrt{2})(1,i), \epsilon_-^a = (1/\sqrt{2})(i,1), \quad \langle\epsilon_+\epsilon_-\rangle = 1$$

$$\kappa_{-0}^A = -\sin(\phi)\kappa_{+0}^A - \cos(\phi)\kappa_{+1}^A, \qquad \kappa_{-1}^A = -\cos(\phi)\kappa_{+0}^A + \sin(\phi)\kappa_{+1}^A$$

$$(cu_{+a}+u_{-a})\sum_i u_i^a\left(\left(H_{[+-]}^{[+-]}\right)_{DJ}^I\right)_{ij} = \langle u_+u_-\rangle\epsilon_+^B\epsilon_{jB}\left(\frac{1}{\sigma_{+j}}-\frac{1}{\sigma_{-j}}\right) = \langle u_+u_-\rangle\epsilon_+^B\epsilon_{jB}O(\Lambda^{-1})$$

$$\sum_i u_i^a\left(C_{ri0}^bC_{kjb}^0 - C_{ki0}^bC_{rjb}^0\right) = \epsilon_j^A\Xi_{AB}(z)\left(-\frac{u_+^a\epsilon_+^B}{\sigma_{z+}}-\frac{u_-^a\epsilon_-^B}{\sigma_{z-}}\right), \Xi_{AB}(z) = \frac{1}{\sigma_{zj}}\Lambda_{A[k}^b(z)\Lambda_{r]Bb}(z)$$

$$(cu_{+a}+u_{-a})\sum_i u_i^a\left(C_{ri0}^bC_{kjb}^0 - C_{ki0}^bC_{rjb}^0\right) = (cu_{+a}+u_{-a})\epsilon_j^A\Xi_{AB}(z)\left(-\frac{u_+^a\epsilon_+^B}{\sigma_{z+}}-\frac{u_-^a\epsilon_-^B}{\sigma_{z-}}\right)$$

$$= \left(\epsilon_j^A\Xi_{AB}(z)\epsilon_+^B\right)\langle u_+u_-\rangle O(\Lambda^{-1})$$

$$u_{i\alpha}^{\mathrm{a}} = \left(u_i^a, 0, u_i^{\dot a}, u_i^{\ddot a}\right)_\alpha, \qquad u_{j\alpha}^{\mathrm{a}} = \left(u_j^a, 0, u_j^{\dot a}, u_j^{\ddot a}\right)_\alpha,$$

$$v_{i\alpha}^{\mathrm{a}} = \left(v_i^a, 0, \frac{1}{\sqrt{2}}(\dot{1}, -i), \frac{1}{\sqrt{2}}(\ddot{1}, -\ddot{\imath})\right)_\alpha, \qquad v_{j\alpha}^{\mathrm{a}} = \left(v_j^a, 0, \frac{1}{\sqrt{2}}(i, \dot{1}), \frac{1}{\sqrt{2}}(\ddot{\imath}, \ddot{1})\right)_\alpha,$$

$$\epsilon_{i\alpha}^{\mathrm{a}} = \left(\epsilon_i^a, 0, \frac{1}{\sqrt{2}}(-i, \dot{1}), \frac{1}{\sqrt{2}}(-\ddot{\imath}, \ddot{1})\right)_\alpha, \qquad \epsilon_{j\alpha}^{\mathrm{a}} = \left(\epsilon_j^a, 0, -\frac{1}{\sqrt{2}}(\dot{1}, i), -\frac{1}{\sqrt{2}}(\ddot{1}, \ddot{\imath})\right)_\alpha,$$

$$\hat{u}_{x\alpha}^{\mathrm{a}} = U_x\left(\frac{\hat{u}^a}{U_x}, 0, (-\dot{1}, \dot{0}), (-\ddot{1}, \ddot{0})\right)_\alpha, \qquad U_x = u_{x\alpha}^{\mathrm{a}}u_{x\mathrm{a}}^\alpha = \langle u_x\hat{u}\rangle, x = i, j,$$

$\kappa_A^{\ddot a} = \kappa_A^{\dot a}, \kappa_A^{\dot a}$ arbitrary with $\qquad k_A^A = \kappa_A^{\dot a}\kappa_{\dot a}^A \neq 0 \neq \kappa_A^{\ddot a}\kappa_{\ddot a}^A,$

$$\kappa_{iA}^{\dot a} = \kappa_A^{\dot a}, \kappa_{iA}^{\ddot a} = \kappa_A^{\ddot a}, \qquad \kappa_{jA}^{\dot 0} = \kappa_A^{\dot 0}, \kappa_{jA}^{\dot 1} = -\kappa_A^{\dot 0}, \kappa_{jA}^{\ddot 0} = \kappa_A^{\ddot 0}, \kappa_{jA}^{\ddot 1} = -\kappa_A^{\ddot 1},$$

$$u_i^{\dot 1} = \left(\dot{\sigma_{ij}}, \dot{1}\right), u_i^{\ddot a} = (\ddot{0}, \ddot{1}), u_j^{\dot a} = (\dot{0}, \dot{1}), u_j^{\ddot a} = \left(\tilde{\sigma}_{ji}, \ddot{1}\right), \quad \sigma_{ij} \equiv \frac{\sqrt{\mathrm{d}\sigma_i}\sqrt{\mathrm{d}\sigma_j}}{S_\alpha\left(\sigma_i, \sigma_j \mid \tau\right)}, u_i^{\dot a}u_{j\dot a} = u_i^{\ddot a}u_{j\ddot a} = \sigma_{ij},$$

$$k^{AB} = \kappa_i^{\ddot a[A}\kappa_{i\ddot a}^{B]} = \kappa_i^{\dot a[A}\kappa_{i\dot a}^{B]} = -\kappa_j^{\dot a[A}\kappa_{j\dot a}^{B]} = \kappa_j^{\ddot a[A}\kappa_{j\ddot a}^{B]},$$

$$\epsilon_{i\alpha}^{[A}\epsilon_j^{B]\alpha} = \epsilon_i^{[A}\epsilon_j^{B]}, \dot{\epsilon}_{i2}^{[A}\dot{\epsilon}_{j2}^{B]} = -k^{AB} = \ddot{\epsilon}_{i3}^{[A}\ddot{\epsilon}_{j3}^{B]} \qquad \epsilon_{i\alpha}^{[A}\tilde{\epsilon}_i^{B]\alpha} = \epsilon_i^{[A}\tilde{\epsilon}_i^{B]}, \epsilon_{j\alpha}^{[A}\tilde{\epsilon}_j^{B]\alpha} = \epsilon_j^{[A}\tilde{\epsilon}_j^{B]},$$

$$u_{i\alpha}^{\mathrm{a}}u_{j\mathrm{a}}^\alpha = u_i^a u_{ja} \equiv \langle u_iu_j\rangle, \qquad \epsilon_j^{A\alpha}u_{j\alpha}^{\mathrm{a}}\hat{u}_{\mathrm{a}\beta} = \epsilon_{j\beta}^AU_j,$$

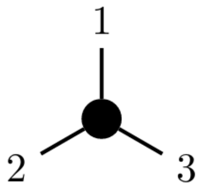
$$\epsilon_i^{A\alpha}u_{i\alpha}^{\mathrm{a}}\hat{u}_{\mathrm{a}\beta} = \epsilon_{i\beta}^AU_i, \qquad \frac{1}{U_j}\hat{u}_{\mathrm{a}\alpha}\lambda_A^{\mathrm{a}}(\sigma)\epsilon_j^{A\alpha} = \frac{1}{\langle u_j\hat{u}\rangle}\langle\hat{u}\lambda_A(\sigma)\rangle\epsilon_j^A,$$

$$\frac{1}{U_i}\hat{u}_{\mathrm{a}\alpha}\lambda_A^{\mathrm{a}}(\sigma)\epsilon_i^{A\alpha} = \frac{1}{\langle u_i\hat{u}\rangle}\langle\hat{u}\lambda_A(\sigma)\rangle\epsilon_i^A,$$

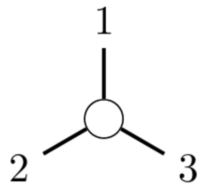
$$\mathrm{d}\mu(u_i) = \mathrm{d}^2u_i^a\ \mathrm{d}^2u_i^{\dot a}\ \mathrm{d}^2u_i^{\ddot a}\delta\left(u_i^{\dot a} - (\sigma_{ij},1)\right)\delta\left(u_i^{\ddot a} - (0,1)\right),$$
$$\mathrm{d}\mu(u_j) = \mathrm{d}^2u_j^a\ \mathrm{d}^2u_j^{\dot a}\ \mathrm{d}^2u_j^{\ddot a}\delta\left(u_j^{\dot a} - (0,1)\right)\delta\left(u_j^{\ddot a} - (\sigma_{ji},1)\right),$$
$$\mathrm{d}\mu(v_i) = \mathrm{d}^2v_i^a\ \mathrm{d}^2v_i^{\dot a}\ \mathrm{d}^2v_i^{\ddot a}\delta\left(v_i^{\dot a} - \frac{1}{\sqrt{2}}(1,-i)\right)\delta\left(v_i^{\ddot a} - \frac{1}{\sqrt{2}}(1,-i)\right),$$
$$\mathrm{d}\mu(v_j) = \mathrm{d}^2v_j^a\ \mathrm{d}^2v_j^{\dot a}\ \mathrm{d}^2v_j^{\ddot a}\delta\left(v_j^{\dot a} - \frac{1}{\sqrt{2}}(i,1)\right)\delta\left(v_j^{\ddot a} - \frac{1}{\sqrt{2}}(i,1)\right).$$

$$\mathrm{d}m_{\mathrm{fw}}\delta\left(k_{n+2A}^{A}\right) = \frac{\mathrm{d}\mu(\kappa_{n+1\mathrm{a}}^{A})\mathrm{d}\mu\left(\zeta_{n+1\mathrm{a}}^{\mathcal{J}}\right)}{\mathrm{vol}\left(\mathrm{G}_Q\right)\kappa} \to \frac{\prod_{l=n+1}^{n+2}\ \mathrm{d}^4\epsilon_l^A\ \mathrm{d}^{\mathcal{N}}q_l^{\mathcal{J}}}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C}))_\epsilon}$$

$$p_1^2 = p_2^2 = p_3^2 = 0, p_1 + p_2 + p_3 = 0$$

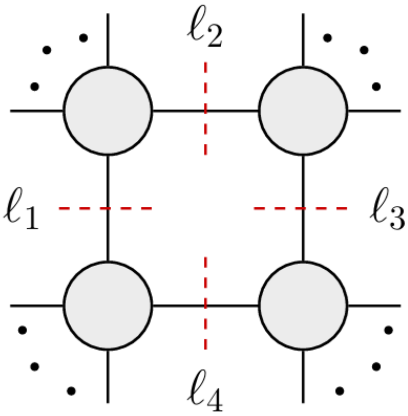


$$= \frac{\delta(P)\delta(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle},$$

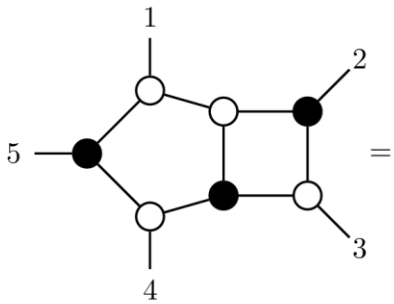


$$= \frac{\delta(P)\delta(\tilde{\mathcal{Q}})}{[12][23][31]}$$

$$\delta(\mathcal{Q}) \equiv \delta^{2\times 4}(\lambda_1\tilde\eta_1 + \lambda_2\tilde\eta_2 + \lambda_3\tilde\eta_3), \delta(\tilde{\mathcal{Q}}) \equiv \delta^{1\times 4}([12]\tilde\eta_3 + [23]\tilde\eta_1 + [31]\tilde\eta_2).$$

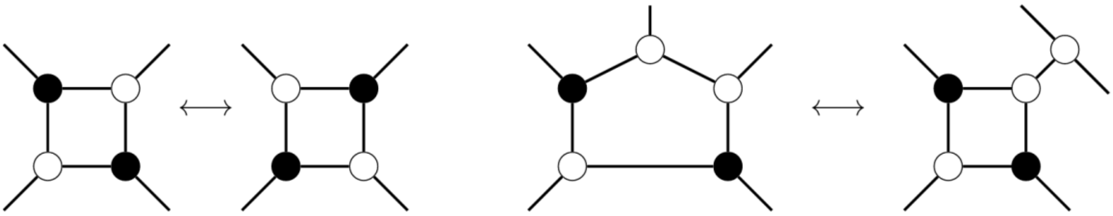


$$= \int_{\mathrm{Cut}} \mathcal{I}\ d^4\ell_1 \ldots d^4\ell_4$$



$$= \prod_k \int d^{\mathcal{N}}\tilde\eta_k \int \frac{d^2\lambda_k\, d^2\tilde\lambda_k}{\mathrm{GL}(1)} \left(\prod_j A_3^{(j)}\right)$$

$$\mathcal{F}_G = F(\lambda,\tilde\lambda,\tilde\eta)\delta^4(Q)\delta^4(P)\ \text{ where }\ \delta(P) \equiv \delta^{2\times 2}(\lambda\cdot\tilde\lambda), \delta(Q) \equiv \delta^{2\times 4}(\lambda\cdot\tilde\eta)$$



$$\omega_G = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\dots\frac{d\alpha_m}{\alpha_m}$$

$$\omega_G = \oint \frac{dC}{(12\dots k)(23\dots k+1)\dots(n1\dots k-1)}$$

$$dC \equiv \frac{d^{k\times n}C}{\mathrm{GL}(k)}$$

$$\int dx f(x)\delta(x-a) = f(a)$$

$$\mathcal{F}_G = \int \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\dots\frac{d\alpha_m}{\alpha_m}\delta^{k\times 2}(C\cdot\tilde{\lambda})\delta^{(n-k)\times 2}(C^{\perp}\cdot\lambda)\delta^{k\times 4}(C\cdot\tilde{\eta})$$

$$\sum_{i=1}^{n} p_i^{\mu} = 0 \leftrightarrow \lambda\cdot\tilde{\lambda} = 0$$

$$\delta^{k\times 2}(C\cdot\tilde{\lambda})\delta^{(n-k)\times 2}(C^{\perp}\cdot\lambda)$$

$$\delta^{k\times 2}(C\cdot\tilde{\lambda})\delta^{(n-k)\times 2}(C^{\perp}\cdot\lambda) = \delta^{2\times 2}(\lambda\cdot\tilde{\lambda})\prod_{i=1}^{2n-4}\delta(\alpha_i-\alpha_i^{*})$$

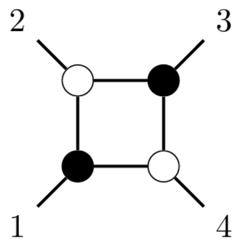
$$\delta^{k\times 4}(C\cdot\tilde{\eta}) = \delta^{2\times 4}(Q)\delta^{(k-2)\times 4}(\Xi)$$

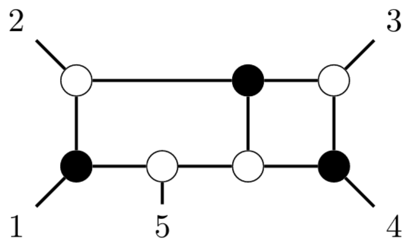
$$(ij) = \langle ij\rangle\,\langle ij\rangle = \epsilon_{ab}\lambda_a^{(i)}\lambda_b^{(j)}$$

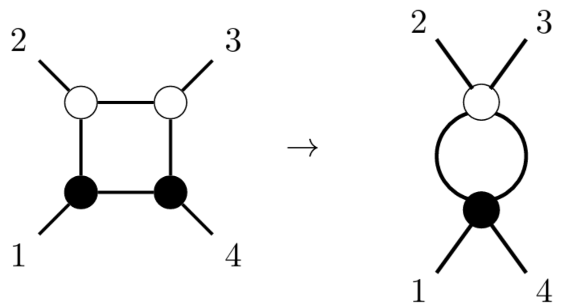
$$\omega_G = \frac{dC}{(12)(23)(34)\dots(n-1n)(n1)}$$

$$\mathcal{F}_G = \frac{\delta(P)\delta(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\dots\langle n1\rangle}$$

$$f_G = \frac{1}{(12)(23)(34)\dots(n1)}$$

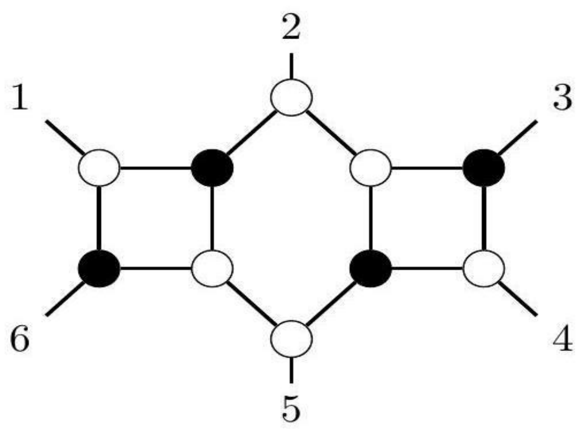


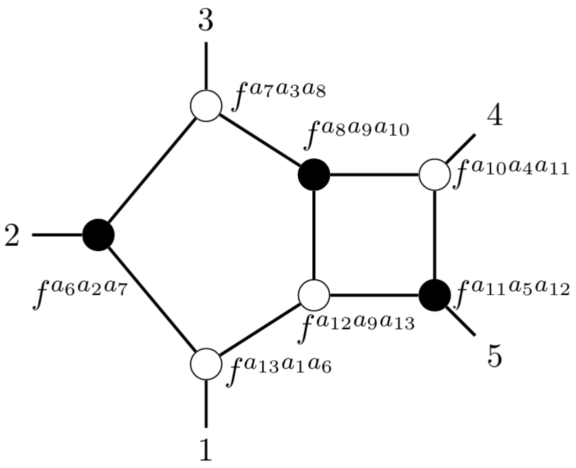




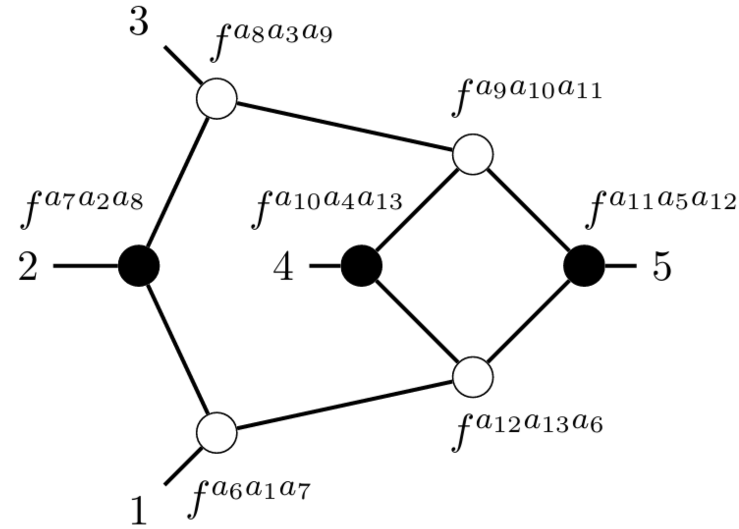
$$\mathcal{F}_G = \int \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\delta^{2\times 2}(C\cdot\tilde{\lambda})\delta^{2\times 2}(C^{\perp}\cdot\lambda)\delta^{2\times 4}(C\cdot\tilde{\eta})$$
$$= \int \frac{d\alpha_4}{\alpha_4}\frac{\delta(P)\delta(Q)}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\delta(\langle 12\rangle)$$

$$C' = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \end{pmatrix}$$

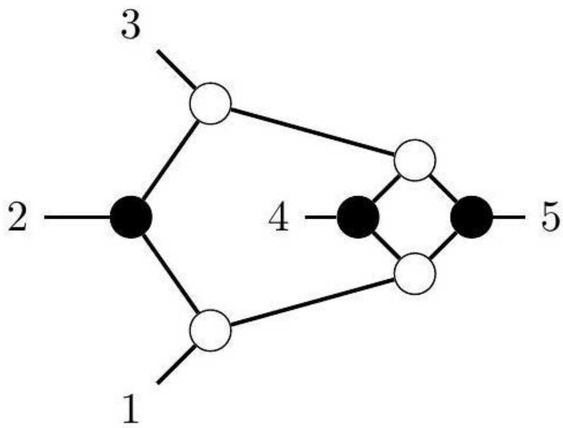




$$= [\mathrm{Tr}(T^{a_1}T^{a_2}\dots T^{a_n}) + \dots] \times \mathcal{F}_G$$



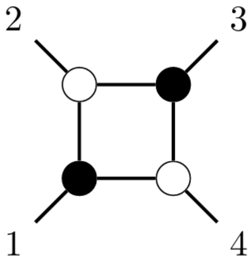
$$= \mathcal{O}\left(\frac{1}{N}\right) \times \mathcal{F}_G$$



$$= \frac{\langle 13\rangle\, \delta(P)\delta(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 14\rangle\langle 15\rangle\langle 34\rangle\langle 35\rangle}$$

$$\mathcal{F}_G = \frac{\mathcal{N}}{\prod_{ij}\ \langle ij\rangle}\delta(P)\delta(\mathcal{Q})$$

$$f_G = \frac{(13)}{(12)(23)(14)(15)(34)(35)}$$



$$C^{\perp} = \begin{pmatrix} \langle 24\rangle & \langle 14\rangle & 0 & \langle 12\rangle \\ 0 & \langle 34\rangle & \langle 24\rangle & \langle 23\rangle \end{pmatrix}$$

$$T = \{(124), (234)\}$$

$$M_{13} = \begin{pmatrix} \langle 14\rangle & \langle 12\rangle \\ \langle 34\rangle & \langle 23\rangle \end{pmatrix}$$

$$\frac{\det(M_{ab})}{\langle ab\rangle}.$$

$$\mathcal{F}_G = \frac{(\det(M_{ab})/\langle ab\rangle)^2}{\prod_{\tau\in T} \langle \tau_1\tau_2\rangle\langle \tau_2\tau_3\rangle\langle \tau_3\tau_1\rangle}\delta(P)\delta(Q)$$

$$\mathcal{F}_G = \frac{\langle 13\rangle^2\langle 24\rangle^2}{\langle 13\rangle^2}\frac{\delta(P)\delta(Q)}{\langle 12\rangle\langle 14\rangle\langle 24\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle} = \frac{\delta(P)\delta(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 14\rangle}$$

$$M_{13} = \begin{pmatrix} \langle 13\rangle & 0 & 0 \\ 0 & \langle 13\rangle & 0 \\ 0 & 0 & \langle 13\rangle \end{pmatrix}$$

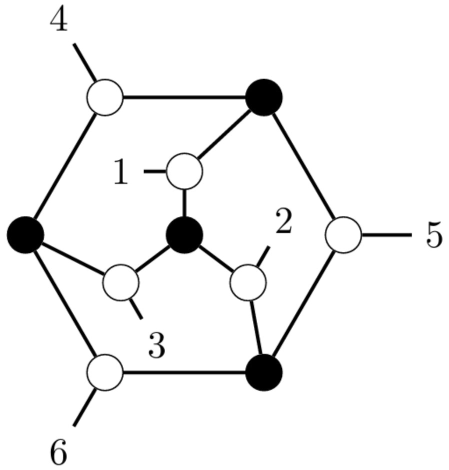
$$f_G = \frac{(13)^4}{(13)^3(12)(14)(15)(23)(34)(35)} = \frac{(13)}{(12)(23)(14)(15)(34)(35)}$$

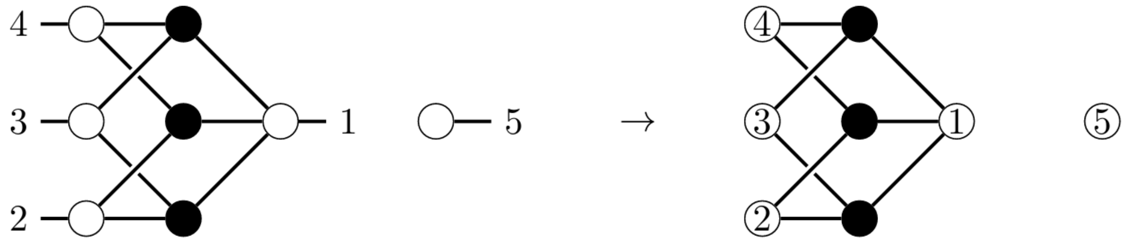
$$\mathrm{Res}_{1=\alpha 2}\,\omega_G = \frac{(23)\alpha^2 d\alpha \wedge d^4C'}{\alpha^3(23)(24)(25)(34)(35)} = \frac{d\alpha \wedge d^4C'}{\alpha(24)(25)(34)(35)}$$

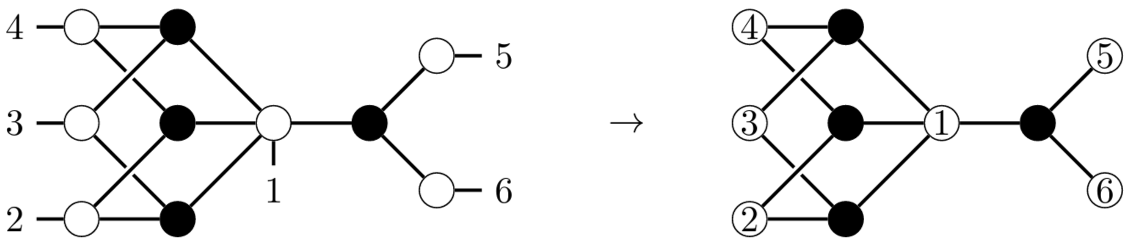
$$f_T = \sum_{\sigma} \mathrm{PT}(\sigma) \text{ where } \mathrm{PT}(\sigma) = \frac{1}{(\sigma_1\sigma_2)(\sigma_2\sigma_3)(\sigma_3\sigma_4)\ldots(\sigma_n\sigma_1)}$$

$$f_T = \sum_{\sigma\in S_n^{(T)}} \mathrm{PT}(\sigma)$$

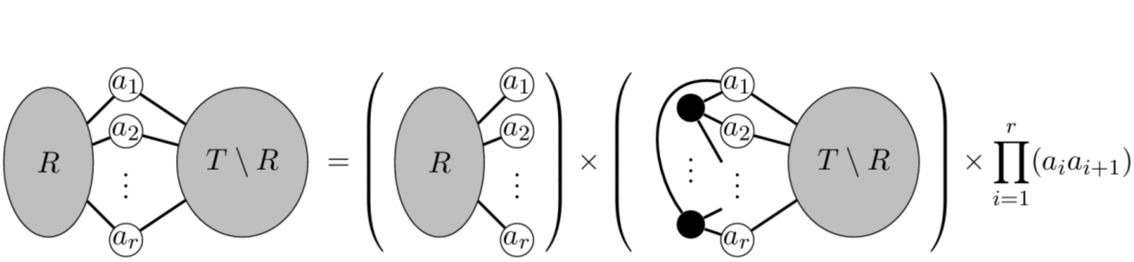
$$S_n^{(T)} = \{\sigma \in S_n : \sigma_1 = 1, \forall (ijk) \in T i, j, k\}.$$







$$\left|\bigcup_{i\in S}\tau_i\right| \geq |S| + 2$$



$$f_T = f_R \times f_{(T\setminus R)\cup P} \times \prod_{i=1}^{r} (a_i a_{i+1})$$

$$f_T = f_R \times f_{T\setminus R} \times (ij)^2$$
$$= f_{T\setminus R} \times \frac{(ij)}{(ik)(kj)}$$

$$f_T = \sum_{\sigma \in S_n^{(T)}} \mathrm{PT}(\sigma) \rightsquigarrow R_T = \bigcup_{\sigma \in S_n^{(T)}} R(\varepsilon_\sigma \sigma).$$

$$f_T = f_R \times f_{(T\setminus R)\cup P} \times \prod_{i=1}^{r} (a_i a_{i+1})$$

$$M_T = \begin{pmatrix} M_{R,1} & M_{R,2} & 0 \\ 0 & M_{S,1} & M_{S,2} \end{pmatrix}$$

$$f_T = \frac{\left[\det\left(M_{T,ab}\right)/(ab)\right]^2}{\prod_{\tau \in T} (\tau_1\tau_2)(\tau_2\tau_3)(\tau_3\tau_1)}$$
$$= \frac{\left[\det\left(M_{R,ab}\right)/(ab)\right]^2}{\prod_{\tau \in R} (\tau_1\tau_2)(\tau_2\tau_3)(\tau_3\tau_1)} \times \frac{\left[\det\left(M_{S,2}\right)\right]^2}{\prod_{\tau \in S} (\tau_1\tau_2)(\tau_2\tau_3)(\tau_3\tau_1)}$$
$$= f_R \times \frac{\left[\det\left(M_{S,2}\right)\right]^2}{\prod_{\tau \in S} (\tau_1\tau_2)(\tau_2\tau_3)(\tau_3\tau_1)}$$

$$\det\left(M_{P\cup S,a_1a_2}\right)/(a_1a_2) = \det\begin{pmatrix} M_{P,a_1}a_2 & 0 \\ M_{S,1,a_1a_2} & M_{S,2} \end{pmatrix}/(a_1a_2)$$
$$= \det\left(M_{S,2}\right) \times \left[\det\left(M_{P,a_1a_2}\right)/(a_1a_2)\right]$$
$$= \det\left(M_{S,2}\right) \times \prod_{i=2}^{r-1} (a_1a_i)$$

$$f_T = f_R \times \frac{\left[\det\left(M_{S,2}\right)\right]^2}{\prod_{\tau \in S} (\tau_1\tau_2)(\tau_2\tau_3)(\tau_3\tau_1)}$$
$$= f_R \times \frac{\left[\det\left(M_{P\cup S,a_1a_2}\right)/(a_1a_2)\right]^2}{\prod_{\tau \in S} (\tau_1\tau_2)(\tau_2\tau_3)(\tau_3\tau_1) \times \prod_{i=2}^{r+1} (a_1a_i)^2}$$
$$= f_R \times \frac{\left[\det\left(M_{P\cup S,a_1a_2}\right)/(a_1a_2)\right]^2}{\prod_{\tau \in P\cup S} (\tau_1\tau_2)(\tau_2\tau_3)(\tau_3\tau_1)} \times \prod_{i=1}^{r} (a_ia_{i+1})$$
$$= f_R \times f_{P\cup S} \times \prod_{i=1}^{r} (a_ia_{i+1})$$

$$[(ij) \times f_T]|_{j\mapsto i} = -f_{T'}$$

$$(ij)\times f_T=\frac{\left[\det\left(M_{ij}\right)/(ij)\right]^2}{(jk)(ki)\prod_{(a,b,c)\in T\setminus\{(ijk)\}}(ab)(bc)(ca)}$$
$$=\frac{\left[\det\begin{pmatrix}(ij) & 0\\ \vdots & M_{ijk}\end{pmatrix}/ij\right]^2}{(jk)(ki)\prod_{(a,b,c)\in T\setminus\{(ijk)\}}(ab)(bc)(ca)}$$
$$=\frac{\det\left(M_{ijk}\right)^2}{(jk)(ki)\prod_{(a,b,c)\in T\setminus\{(ijk)\}}(ab)(bc)(ca)}$$

$$[(ij)\times f_T]|_{j\mapsto i}=-\left.\frac{\left[\det\left(M_{ijk}\right)/(ik)\right]^2}{\prod_{(a,b,c)\in T\setminus\{(ijk)\}}(ab)(bc)(ca)}\right|_{j\mapsto i}$$
$$=-\frac{[\det(M'_{ik})/(ik)^2]}{\prod_{(a,b,c)\in T'}(ab)(bc)(ca)}$$
$$=-f_{T'}$$

$$\left|\bigcup_{(abc)\in R'}\{a,b,c\}\right|<|R'|+2$$

$$\left|\bigcup_{(abc)\in R}\{a,b,c\}\right|=\left|\bigcup_{(abc)\in R'}\{a,b,c\}\right|+1\leq |R'|+2=|R|+2$$

$$f_T=f_R\times\frac{(ij)}{(ik)(kj)},\text{ or } R\cup\{(ijk)\}\subset T$$

$$f_T=f_R\times f_{(T\setminus R)\cup P}\times\prod_{i=1}^{r}(a_ia_{i+1})$$

$$ij\in D([R\cup R']\cup[T\setminus R]\cup[T'\setminus R'])=D(T\cup T'),$$

$$S_n^{(P)}=\left\{\sigma\in S_n\colon \sigma_i<\sigma_j\forall i\prec j\in P\right\}.$$

$$R_P(\varepsilon)=\bigcup_{\sigma\in S_n^{(P)}}R(\varepsilon\sigma)$$

$$\mathrm{d}s_{10}^2=r^2(-\mathrm{d}t^2+\mathrm{d}x_1^2+\mathrm{d}x_2^2+f(r)\mathrm{d}\phi^2)+\frac{\mathrm{d}r^2}{r^2f(r)}+\sum_{i=1}^{3}\mathrm{d}\mu_i^2+\mu_i^2(\mathrm{d}\phi_i+\mathcal{A})^2$$
$$f(r)=1-\frac{Q^6}{r^6},\mathcal{A}=Q^3\left(\frac{1}{r^2}-\frac{1}{Q^2}\right)\mathrm{d}\phi$$
$$F_5=(1+\star_{10})G_5,G_5=-4r^3\,\mathrm{d}t\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}r\wedge\mathrm{d}\phi-2Q^3J_2\wedge\mathrm{d}t\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2$$
$$J_2=\sum_{i=1}^{3}\mu_i\,\mathrm{d}\mu_i\wedge(\mathrm{d}\phi_i+\mathcal{A})$$

$$\mu_1=\sin\theta\sin\varphi,\mu_2=\sin\theta\cos\varphi,\mu_3=\cos\theta,\sum_{i=1}^{3}\mu_i^2=1$$

$$\phi \sim \phi + 2\pi R, R = \frac{2}{r^2 f'(r)}\Big|_Q = \frac{1}{3Q},$$

$$Q_{D3} = \frac{1}{(2\pi)^4 g_s \alpha^2}\int \star_{10} G_5 = \frac{l^4}{4\pi g_s \alpha'^2} = N$$

$$ds^2_{S^5} = \sum_{i=1}^{3} d\mu_i^2 + \mu_1^2(D\psi - d\varphi_1)^2 + \mu_2^2(D\psi + d\varphi_1 + d\varphi_2)^2 + \mu_3^2(D\psi - d\varphi_2)^2$$

$$\phi_1 = \psi - \varphi_1, \phi_2 = \psi + \varphi_1 + \varphi_2, \phi_3 = \psi - \varphi_2$$

$$D\psi = d\psi + \mathcal{A}$$

$$ds^2_{S^5} = \sum_{i=1}^{3} d\mu_i^2 + 9\frac{\mu_1^2\mu_2^2\mu_3^2}{g_0}D\psi^2 + (\mu_1^2 + \mu_2^2)\left(D\tilde{\varphi}_1 + \frac{\mu_2^2}{\mu_1^2 + \mu_2^2}D\tilde{\varphi}_2\right)^2 + \frac{g_0}{\mu_1^2 + \mu_2^2}D\tilde{\varphi}_2^2$$

$$g_0 = \mu_1^2\mu_2^2 + \mu_1^2\mu_3^2 + \mu_2^2\mu_3^2$$
$$D\tilde{\varphi}_1 = d\varphi_1 + \left(3\frac{\mu_2^2\mu_3^2}{g_0} - 1\right)D\psi = D\varphi_1 + \left(3\frac{\mu_2^2\mu_3^2}{g_0} - 1\right)\mathcal{A},$$
$$D\tilde{\varphi}_2 = d\varphi_2 + \left(3\frac{\mu_2^2\mu_1^2}{g_0} - 1\right)D\psi = D\varphi_2 + \left(3\frac{\mu_2^2\mu_1^2}{g_0} - 1\right)\mathcal{A}.$$

$$ds^2_{10} = r^2(-dt^2 + dx_1^2 + dx_2^2 + f(r)d\phi^2) + \frac{dr^2}{r^2 f(r)} + ds^2_{S^5_\beta}$$
$$ds^2_{S^5_\beta} = \sum_{i=1}^{3} d\mu_i^2 + 9\frac{\mu_1^2\mu_2^2\mu_3^2}{g_0}D\psi^2 + G(\mu_1^2 + \mu_2^2)\left(D\tilde{\varphi}_1 + \frac{\mu_2^2}{\mu_1^2 + \mu_2^2}D\tilde{\varphi}_2\right)^2 + \frac{g_0 G}{\mu_1^2 + \mu_2^2}D\tilde{\varphi}_2^2$$
$$B = \gamma g_0 G D\tilde{\varphi}_1 \wedge D\tilde{\varphi}_2, e^{2\Phi} = G$$
$$F_3 = \frac{\gamma}{G} i_{\varphi_2} i_{\varphi_1} \star_{10\beta} G_5, F_5 = (1 + \star_{10\beta})G_5, F_7 = G_5 \wedge B$$

$$Q'_{D3} = \frac{1}{(2\pi)^4}\int_{\Sigma_5} \star_{10\beta} G_5 + B \wedge F_3 = \frac{1}{(2\pi)^4}\int_{\Sigma_5} \star_{10\beta} G_5(1 + \gamma^2 g_0) = \frac{1}{(2\pi)^4}\int_{\Sigma_5} \star_{10} G_5 = N,$$

$$Q'_{D5} = \frac{1}{(2\pi)^2}\int_{\Sigma_3} F_3 = \frac{1}{(2\pi)^2}\frac{\gamma}{G}\int_{\Sigma_3} i_{\varphi_2} i_{\varphi_1} \star_{10\beta} G_5 = \frac{\gamma}{(2\pi)^4}\int_{\Sigma_5} \star_{10} G_5 = \gamma N,$$

$$Q'_{D1} = \frac{1}{(2\pi)^6}\int_{\Sigma_7} \star_{10\beta} F_3 + B \wedge G_5 = \frac{1}{(2\pi)^6}\int_{\Sigma_7} B \wedge G_5 - F_7 = 0,$$

$$ds^2_{10} = ds^2_8 + r^2 f(r) d\phi^2 + (\mathcal{D}\varphi + \mathcal{A})^2$$
$$ds^2_8 = r^2(-dt^2 + dx_1^2 + dx_2^2) + \frac{dr^2}{r^2 f(r)} + ds^2_{\mathbb{C}P^2}, \mathcal{D}\varphi = d\varphi + \eta$$

$$ds^2_{\mathbb{C}P^2} = d\alpha^2 + \frac{1}{4}\sin^2\alpha[\cos^2\alpha(d\psi - \cos\theta_1\, d\theta_2) + d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2]$$

$$\eta = \frac{1}{2}\sin^2\alpha(d\psi - \cos\theta_1\, d\theta_2)$$

$$F_5 = (1+\star_{10})G_5$$
$$G_5 = -4r^3\, \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}r \wedge \mathrm{d}\phi - 2Q^3 J_2 \wedge \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2, J_2 = \frac{1}{2}\,\mathrm{d}\eta$$

$$\mathrm{d}s_{10}^2 = \mathrm{d}s_8^2 + Gr^2 f(r)\mathrm{d}\phi^2 + G(\mathcal{D}\varphi + \mathcal{A})^2,$$
$$B = \gamma r^2 f(r) G\mathcal{D}\varphi \wedge \mathrm{d}\phi, e^{2\Phi} = G,$$
$$F_5 = (1+\star_{10\beta})G_5, F_3 = \frac{\gamma}{G} i_\phi i_\varphi \star_{10\beta} G_5, F_7 = G_5 \wedge B$$

$$C_2 = -\gamma \frac{Q^3}{r^2} J_2, F_3 = \mathrm{d}C_2 = 2\gamma \frac{Q^3}{r^3}\,\mathrm{d}r \wedge J_2$$

$$Q'_{D3} = \frac{1}{(2\pi)^4}\int_{\Sigma_5} \star_{10\beta} G_5 + B \wedge F_3 = \frac{1}{(2\pi)^4}\int \star_{10} G_5 = N,$$

$$Q'_{D1} = \frac{1}{(2\pi)^6}\int_{\Sigma_7} \star_{10\beta} F_3 + B \wedge G_5 = \frac{1}{(2\pi)^6}\int_{\Sigma_7} B \wedge G_5 - F_7 = 0$$

$$Q'_{\mathrm{NS5}} = \frac{1}{(2\pi)^2}\int_{M_3} H_3$$

$$R = \frac{2\gamma^2\left(Q^{12}(26\gamma^2 r^2 + 28) + Q^6(2r^6 - 34\gamma^2 r^8) + r^{12}(8\gamma^2 r^2 + 15)\right)}{(-\gamma^2 Q^6 r + \gamma^2 r^7 + r^5)^2},$$

$$R|_{r=Q} = 90Q^2\gamma^2,\ R|_{r\to\infty} \sim 16 - \frac{2}{\gamma^2 r^2} + O(r^{-3})$$

$$R_{\mu\nu}R^{\mu\nu}\big|_{r=Q} = 36(81\gamma^4 Q^4 - 2\gamma^2 Q^2 + 6),\ R_{\mu\nu}R^{\mu\nu}\big|_{r\to\infty} \sim 176 - \frac{96}{\gamma^2 r^2} + O(r^{-3}),$$
$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big|_{r=Q} = 36(99\gamma^4 Q^4 + 40\gamma^2 Q^2 + 14),\ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big|_{r\to\infty} \sim 232 - \frac{248}{\gamma^2 r^2} + O(r^{-3}),$$

$$R|_{r\to\infty} \sim \frac{5}{2\sqrt{\gamma r}} + O\left(r^{-\frac{5}{2}}\right),$$
$$R_{\mu\nu}R^{\mu\nu}\big|_{r\to\infty} \sim \frac{821}{4\gamma r} + O(r^{-3}),$$
$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big|_{r\to\infty} \sim \frac{4057}{16\gamma r} + O(r^{-3}).$$

$$G|_{r\to\infty} \sim \frac{1}{\gamma^2 r^2}$$

$$\mathcal{L} = \mathrm{Tr}\Big\{-\frac{1}{2}F_{MN}^2 - 2\overline{D_M Z^i} D^M Z^i + g^2[\bar{Z}^i, Z^i]^2 + 2g^2[\bar{Z}^j, \bar{Z}^k][Z^k, Z^j] +$$
$$+ i\bar{\lambda}\not{D}\lambda + i\bar{\psi}^i\not{D}\psi^i + 2\sqrt{2}g\left(\bar{\lambda}[Z^i, \psi^i] - [\bar{\psi}^i, \bar{Z}^i]\lambda\right) - \sqrt{2}\epsilon_{ijk}\left([\bar{\psi}^i, Z^j]\psi^k - [\bar{\psi}^i, \bar{Z}^j]\psi^k\right)\Big\}$$

$$D_M = \partial_M - ig[A_M, \cdot]$$

$$A_\mu(x,\phi)=\sum_{n\in\mathbb{Z}} e^{i\frac{n\phi}{R}}A_\mu^{(n)}(x)$$

$$A_\phi(x,\phi)\equiv \Theta(x,\phi)=\sum_{n\in\mathbb{Z}} e^{i\frac{n\phi}{R}}\Theta^{(n)}(x)$$

$$Z^j(x,\phi)=\sum_{n\in\mathbb{Z}} e^{i\frac{n\phi}{R}}Z^{j(n)}(x), j{:}\,1,2,3$$

$$\chi(x,\phi)=\sum_{n\in\mathbb{Z}} e^{i\frac{(n+1/2)}{R}\phi}\chi^{(n)}(x), \chi=\lambda,\psi^j, j{:}\,1,2,3$$

$$\begin{aligned} i\bar{\lambda}\not{D}\lambda &= i\bar{\lambda}(x,\phi)\left[\gamma^\mu D_\mu\lambda(x,\phi)+\gamma^\phi\partial_\phi\lambda(x,\phi)-i\gamma^\phi g\left[\Theta(x,\phi),\lambda(x,\phi)\right]\right] \\ &= \sum_{n,m,p\in\mathbb{Z}} e^{i\frac{(m+1/2)}{R}\phi}e^{-i\frac{(n+1/2)}{R}\phi}e^{i\frac{p}{R}\phi}i\bar{\lambda}^{(n)}(x)\Big\{\delta_{p,0}\gamma^\mu\partial_\mu\lambda^{(m)}(x)+ \\ &\quad -i\gamma^\mu g\left[A_\mu^{(p)}(x),\lambda^{(m)}(x)\right]+i\frac{(n+1/2)}{R}\delta_{p,0}\gamma^\phi\lambda^{(m)}(x)-i\gamma^\phi g\left[\Theta^{(p)}(x),\lambda^{(m)}(x)\right]\Big\}, \end{aligned}$$

$$\begin{aligned} \frac{1}{2\pi R}\int_{S^1_\phi}\mathrm{d}\phi\ \mathrm{Tr}\left[i\bar{\lambda}\not{D}\lambda\right] &= \mathrm{Tr}\Big[\sum_{n,m\in\mathbb{Z}}\bar{\lambda}^{(n)}(x)\Big\{\delta_{m,0}i\gamma^\mu\partial_\mu\lambda^{(n)}(x)+\gamma^\mu g\left[A_\mu^{(n-m)}(x),\lambda^{(m)}(x)\right] \\ &\quad -\frac{(n+1/2)}{R}\delta_{m,0}\gamma^\phi\lambda^{(n)}(x)+\gamma^\phi g\left[\Theta^{(n-m)}(x),\lambda^{(m)}(x)\right]\Big\}\Big]. \end{aligned}$$

$$\bar{\lambda}^{(n)}(x)\left[i\gamma^\mu\partial_\mu-\frac{(n+1/2)}{R}\gamma^\phi\right]\lambda^{(n)}(x)$$

$$m_{n,\lambda}=\frac{|n+1/2|}{R}\neq 0, \forall n\in\mathbb{Z}$$

$$\begin{aligned} \frac{1}{2\pi R}\int_{S^1_\phi}d\phi\frac{1}{2}\mathrm{Tr}[F_{MN}^2] &= \frac{1}{2\pi R}\int_{S^1_\phi}\mathrm{d}\phi\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}^2]+\sum_{n\in\mathbb{Z}}\mathrm{Tr}\left[\partial_\mu\Theta^{(-n)}\partial^\mu\Theta^{(n)}-\frac{n^2}{R^2}A_\mu^{(-n)}A^{(n)\mu}\right] \\ &-ig\sum_{n,m\in\mathbb{Z}}\mathrm{Tr}\left[\left(\partial_\mu\Theta^{(-n-m)}+i\frac{n+m}{R}A_\mu^{(-n-m)}\right)\left[A^{(n)\mu},\Theta^{(m)}\right]\right] \\ &-\frac{g^2}{2}\sum_{n,m,p,s\in\mathbb{Z}}\delta_{n+m+p+s,0}\mathrm{Tr}\left[\left[A^{(n)\mu},\Theta^{(m)}\right]\left[A_\mu^{(p)},\Theta^{(s)}\right]\right] \end{aligned}$$

$$\begin{aligned} \int d\phi\mathrm{Tr}\left(D_M Z^j\right)^2 &= \frac{1}{2\pi R}\int_{S^1_\phi}\mathrm{d}\phi\mathrm{Tr}\left[\overline{D_\mu Z^i}D^\mu Z^i\right]+\sum_{n\in\mathbb{Z}}\frac{n^2}{R^2}\mathrm{Tr}\left[\bar{Z}^{i(n)}Z^{i(n)}\right] \\ &-g\sum_{n,m\in\mathbb{Z}}\frac{n}{R}\mathrm{Tr}\left[\bar{Z}^{i(n+m)}\left[\Theta^{(m)},Z^{i(n)}\right]+\left[\bar{Z}^{i(n+m)},\Theta^{(m)}\right]Z^{i(n)}\right] \\ &+g^2\sum_{n,m,p,s\in\mathbb{Z}}\delta_{n,m+p+s}\mathrm{Tr}\left[\left[\bar{Z}^{i(n)},\Theta^{(m)}\right]\left[\Theta^{(p)},Z^{i(s)}\right]\right] \end{aligned}$$

$$D_\mu\Phi\to D_\mu\Phi, D_\phi\to D_\phi\Phi-iR_\Phi\mathcal{A}_\phi\Phi, \mathcal{A}_\phi=Q$$

$$R_\lambda=\frac{3}{2}, R_{\psi^i}=-\frac{1}{2}$$

$$m_{n,Z^i} \to \left|\frac{n}{R} - Q\right|, m_{n,A} \to \left|\frac{n}{R}\right|$$
$$m_{n,\psi^i} \to \left|\frac{n+1/2}{R} + \frac{Q}{2}\right|, m_{n,\lambda} \to \left|\frac{n+1/2}{R} - \frac{3Q}{2}\right|$$

$$m_{n,Z^i} \to \frac{|n-1/3|}{R}, \quad m_{n,A} \to \frac{|n|}{R}$$
$$m_{n,\psi^i} \to \frac{|n+2/3|}{R}, \quad m_{n,\lambda} \to \frac{|n|}{R}.$$

$$\hat{Q}^i \Phi = Q_\Phi^i \Phi, i = 1,2$$

$$\Phi_A \Phi_B \longrightarrow \Phi_A \star \Phi_B \equiv e^{i\pi\gamma\left(Q_{\Phi_A}^1 Q_{\Phi_B}^2 - Q_{\Phi_A}^2 Q_{\Phi_B}^1\right)} \Phi_A \Phi_B, \gamma \in \mathbb{R}$$

$$\hat{Q}^{1,2} A_M = 0, \hat{Q}^{1,2} Z^i = -\hat{Q}^{1,2} \bar{Z}^i = Q_{Z^i}^{1,2} Z^i$$
$$\hat{Q}^{1,2} \lambda = -\hat{Q}^{1,2} \bar{\lambda} = Q_\lambda^{1,2} \lambda, \hat{Q}^{1,2} \psi^i = -\hat{Q}^{1,2} \bar{\psi}^i = Q_{\psi^i}^{1,2} \psi^i$$

$$\Phi A_M \longrightarrow \Phi \star A_M = e^{i\pi\gamma(Q_\Phi^1 Q_A^2 - Q_\Phi^2 Q_A^1)} \Phi A_M = \Phi A_M$$

$$(\Phi_1, \Phi_2, \Phi_3) \longrightarrow \left(\Phi_1, e^{i\varphi_A} \Phi_2, e^{-i\varphi_A} \Phi_3\right)$$
$$(\Phi_1, \Phi_2, \Phi_3) \longrightarrow \left(e^{-i\varphi_B} \Phi_1, e^{i\varphi_B} \Phi_2, \Phi_3\right)$$

$$\bar{\lambda}[\bar{Z}^i, \psi] \longrightarrow \bar{\lambda} \star [\bar{Z}^i, \psi^i]_\star = \bar{\lambda} \star [\bar{Z}^i, \psi^i] = \bar{\lambda}[\bar{Z}^i, \psi^i]$$

$$\Phi_A \Phi_B \to \Phi_A \star \Phi_B \equiv e^{i\pi\gamma\left(R_{\Phi_A} L_{\Phi_B} - R_{\Phi_B} L_{\Phi_A}\right)} \Phi_A \Phi_B$$

$$\mathcal{L}_B = \mathrm{Tr}\left[-\frac{1}{2} F_{MN} \star F^{MN} - 2\overline{D_M Z^i} \star D^M Z^i + g^2 [\bar{Z}^i, Z^i]_\star^2 + 2g^2 [\bar{Z}^j, \bar{Z}^k]_\star \star [Z^k, Z^j]_\star\right]$$

$$D_\mu \Phi = \partial_\mu \Phi - ig\left(A_\mu \star \Phi - \Phi \star A_\mu\right)$$
$$D_\phi \Phi = \partial_\phi \Phi - ig(\Theta \star \Phi - \Phi \star \Theta) - iR_\Phi \mathcal{A}_\phi \Phi$$

$$\frac{1}{2\pi R} \int_{S_\phi^1} \mathrm{d}\phi \mathrm{Tr} \overline{D_M Z^i} \star D^M Z^i = \frac{1}{2\pi R} \int_{S_\phi^1} \mathrm{d}\phi \mathrm{Tr}\left[\overline{D_\mu Z^i} \star D^\mu Z^i + \overline{D_\phi Z^i} \star D^\phi Z^i\right]$$

$$\frac{1}{2\pi R} \int_{S_\phi^1} \mathrm{d}\phi \mathrm{Tr} \overline{D_\mu Z^i} \star D^\mu Z^i = \frac{1}{2\pi R} \int_{S_\phi^1} \mathrm{d}\phi \mathrm{Tr}\left[\partial_\mu \bar{Z}^i \star \partial^\mu Z^i - ig \partial_\mu \bar{Z}^i \star [A^\mu, Z^i]_\star \right.$$
$$\left. - ig[A_\mu, \bar{Z}^i]_\star \star \partial^\mu Z^i - g^2 [A_\mu, \bar{Z}^i]_\star \star [A^\mu, Z^i]_\star\right.$$

$$\bar{Z}^i \star Z^i = e^{i\pi\gamma\left(R_{\bar{Z}^i} L_{Z^i} - R_{Z^i} L_{\bar{Z}^i}\right)} \sum_{n,m \in \mathbb{Z}} e^{-i\frac{n}{R}\phi} e^{i\frac{m}{R}\phi} \bar{Z}^{i(n)} Z^{i(m)}$$
$$= \sum_{n,m \in \mathbb{Z}} e^{i\pi\gamma\frac{n-m}{R}} e^{-i\frac{n}{R}\phi} e^{i\frac{m}{R}\phi} \bar{Z}^{i(n)} Z^{i(m)}$$

$$A_\mu \star Z^i = e^{i\pi\gamma\left(R_A L_{Z^i} - R_{Z^i} L_A\right)} \sum_{n,m \in \mathbb{Z}} e^{i\frac{n}{R}\phi} e^{i\frac{m}{R}\phi} A_\mu^{(n)} Z^{i(m)}$$
$$= \sum_{n,m \in \mathbb{Z}} e^{-i\pi\gamma\frac{n}{R}} e^{i\frac{n}{R}\phi} e^{i\frac{m}{R}\phi} A_\mu^{(n)} Z^{i(m)}$$

$$\left[A_\mu, Z^i\right]_\star = \sum_{n,m\in\mathbb{Z}} e^{i\frac{n+m}{R}\phi}\left[e^{-i\pi\gamma\frac{n}{R}}A_\mu^{(n)}Z^{i(m)} - e^{i\pi\gamma\frac{n}{R}}Z^{i(m)}A_\mu^{(n)}\right]$$
$$\left[A_\mu, \bar{Z}^i\right]_\star = \sum_{n,m\in\mathbb{Z}} e^{i\frac{n-m}{R}\phi}\left[e^{i\pi\gamma\frac{n}{R}}A_\mu^{(n)}\bar{Z}^{i(m)} - e^{-i\pi\gamma\frac{n}{R}}\bar{Z}^{i(m)}A_\mu^{(n)}\right]$$

$$\frac{1}{2\pi R}\int_{S_\phi^1} \mathrm{d}\phi \mathrm{Tr}\partial_\mu\bar{Z}^i\partial^\mu Z^i - ig\sum_{n,m\in\mathbb{Z}}\mathrm{Tr}\left\{\partial^\mu\bar{Z}^{i(n+m)}\left[e^{-i\pi\gamma\frac{n}{R}}A_\mu^{(n)}Z^{i(m)} - e^{i\pi\gamma\frac{n}{R}}Z^{i(m)}A_\mu^{(n)}\right]\right.$$
$$\left.+\left[e^{i\pi\gamma\frac{n}{R}}A_\mu^{(n)}\bar{Z}^{i(n+m)} - e^{-i\pi\gamma\frac{n}{R}}\bar{Z}^{i(n+m)}A_\mu^{(n)}\right]\partial^\mu Z^{i(m)}\right\}$$
$$-g^2\sum_{n,m,s\in\mathbb{Z}}\left[e^{i\pi\gamma\frac{n}{R}}A_\mu^{(n)}\bar{Z}^{i(n+m+s)} - e^{-i\pi\gamma\frac{n}{R}}\bar{Z}^{i(n+m+s)}A_\mu^{(n)}\right]$$
$$\times\left[e^{-i\pi\gamma\frac{m}{R}}A^{(m)\mu}Z^{i(s)} - e^{i\pi\gamma\frac{m}{R}}Z^{i(s)}A^{(m)\mu}\right]\Bigg]$$

$$\frac{1}{2\pi R}\int_{S_\phi^1} \mathrm{d}\phi \mathrm{Tr}\overline{D_\phi Z^i}\star D^\phi Z^i$$

$$\mathrm{Tr}\left[\sum_{n\in\mathbb{Z}}\left(\frac{n}{R}-Q\right)^2\bar{Z}^{i(n)}Z^{i(n)}\right.$$
$$-g\sum_{n,m\in\mathbb{Z}}\left\{\left(\frac{(n+m)}{R}-Q\right)\bar{Z}^{i(n+m)}\left[e^{-i\pi\gamma\frac{n}{R}}\Theta^{(n)}Z^{i(m)} - e^{i\pi\gamma\frac{n}{R}}Z^{i(m)}\Theta^{(n)}\right]\right.$$
$$\left.-\left(\frac{m}{R}-Q\right)\left[e^{i\pi\gamma\frac{n}{R}}\Theta^{(n)}\bar{Z}^{i(n+m)} - e^{-i\pi\gamma\frac{n}{R}}\bar{Z}^{i(n+m)}\Theta^{(n)}\right]Z^{i(m)}\right\}$$
$$\left.-g^2\sum_{n,m,s\in\mathbb{Z}}\left[e^{i\pi\gamma\frac{n}{R}}\Theta^{(n)}\bar{Z}^{i(n+m+s)} - e^{-i\pi\gamma\frac{n}{R}}\bar{Z}^{i(n+m+s)}\Theta^{(n)}\right]\left[e^{-i\pi\gamma\frac{m}{R}\Theta^{(m)}}Z^{i(s)} - e^{i\pi\gamma\frac{m}{R}}Z^{i(s)}\Theta^{(m)}\right]\right]$$

$$\frac{1}{2\pi R}\int_{S_\phi^1} \mathrm{d}\phi \mathrm{Tr}\bar{\lambda}\star\left[\bar{Z}^i,\psi^i\right]_\star$$
$$=\mathrm{Tr}\left[\sum_{n,m,s\in\mathbb{Z}}\delta_{m,n+s}e^{i\pi\gamma\frac{s(R_\psi-R_Z)-(m-n)R_\lambda}{R}}\bar{\lambda}^{(s)}\left[e^{i\pi\gamma\frac{nR_Z-mR_\psi}{R}}\bar{Z}^{i(n)}\psi^{i(m)} - e^{-i\pi\gamma\frac{nR_Z-mR_\psi}{R}}\psi^{i(m)}\bar{Z}^{i(n)}\right]\right],$$

$$\mathrm{Tr}\left(\Phi_i\left[\Phi_j,\Phi_k\right]\right)\longrightarrow\mathrm{Tr}\left(\Phi_i\left[e^{i\pi\gamma}\Phi_j\Phi_k - e^{-i\pi\gamma}\Phi_k\Phi_j\right]\right)$$

$$\left[Z_i,Z_j\right]=0,\sum_{i=1}^{3}\left[Z_i,Z_i^\dagger\right]=0$$

$$\left[Z_i,Z_j\right]_\star=0,\sum_{i=1}^{3}\left[Z_i,Z_i^\dagger\right]_\star=0$$

$$U=\mathrm{diag}\left[1,e^{-\frac{2\pi i}{n}},\ldots,e^{-\frac{2\pi i(n-1)}{n}}\right]\otimes\mathbb{1}_k, V=\hat{V}\otimes\mathbb{1}_k$$

$$U=\mathrm{diag}\left[1,e^{-\frac{2\pi i}{3}},e^{-\frac{4\pi i}{3}}\right], V=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

$$\langle\Theta^{(0)}\rangle = \operatorname{diag}\left[x_1^{(0)},\dots,x_N^{(0)}\right], x_i^{(0)} \in \mathbb{C},$$

$$S_{DBI} = T_{D5}\int_{\mathbb{R}^{1,3}\times T^2} \mathrm{d}^{5+1}\xi e^{-\Phi}\sqrt{-\det(P[g]_{ab}+F_{ab}+P[B]_{ab})}$$

$$S_{WZ} = T_{D5}\int_{\mathbb{R}^{1,3}\times T^2} C_6 + C_4\wedge(F+B)$$

$$\mathrm{d}(C_6 + C_4\wedge B) = \star\,\mathrm{d}C_2 + H\wedge C_4 + \mathrm{d}C_4\wedge B = -F_7 + G_5\wedge B = 0$$

$$S_{WZ} = T_{D5}\int_{\mathbb{R}^{1,3}\times T^2} C_4\wedge F$$

$$T_{D5}\int_{\mathbb{R}^{1,3}\times T^2} \mathrm{d}^{5+1}\xi\left[e^{-\Phi}r^4\sqrt{f(r)}\sqrt{G^2g_0+(F_{12}+B_{12})^2}+F_{12}r^4\left(1-\frac{Q^4}{r^4}\right)\right]$$

$$C_4 = r^4\left(1-\frac{Q^4}{r^4}\right)\mathrm{d}t\wedge\,\mathrm{d}x_1\wedge\,\mathrm{d}x_2\wedge\,\mathrm{d}\phi$$

$$T_{D5}\int_{\mathbb{R}^{1,3}\times T^2} \mathrm{d}^{5+1}\xi\left[e^{-\Phi}r^4\sqrt{f(r)}\sqrt{G^2g_0+\frac{1}{\gamma^2}(1-\gamma^2g_0G)^2}-\frac{1}{\gamma}r^4\left(1-\frac{Q^4}{r^4}\right)\right]$$
$$= T_{D5}\int_{\mathbb{R}^{1,3}\times T^2} \mathrm{d}^{5+1}\xi\frac{1}{\gamma}\left[e^{-\Phi}r^4\sqrt{f(r)}\sqrt{G}-r^4\left(1-\frac{Q^4}{r^4}\right)\right]$$
$$= T_{D5}\int_{\mathbb{R}^{1,3}\times T^2} \mathrm{d}^{5+1}\xi\frac{r^4}{\gamma}\left[\sqrt{\left(1-\frac{Q^6}{r^6}\right)}-\left(1-\frac{Q^4}{r^4}\right)\right]\geq 0$$

$$S_{\mathrm{BI}} + S_{WZ} = T_{D7}\int_{\mathbb{R}^{1,2}\times\widehat{S^5}} \mathrm{d}^8\xi e^{-\Phi}\sqrt{-\det(P[g]_{ab}+F_{ab}+P[B]_{ab})}$$
$$+T_{D7}\int_{\mathbb{R}^{1,2}\times\widehat{S^5}} C_8 + C_6\wedge(F+B)+\frac{1}{2}C_4\wedge(F+B)\wedge(F+B)$$

$$S_{WZ} = T_{D7}\int_{\mathbb{R}^{1,2}\times\widehat{S^5}}\frac{1}{2}C_4\wedge F\wedge F = -T_{D7}\int_{\mathbb{R}^{1,2}\times\widehat{S^5}} F_5\wedge A\wedge F$$
$$= -\frac{N}{(2\pi)^3}\int_{\mathbb{R}^{1,2}} A\wedge F$$

$$S_{WZ} = T_{D3}\int_{\mathbb{R}^{1,2}\times S^1} C_0 F\wedge F = -T_{D3}\int_{S^1} F_1\int_{\mathbb{R}^{1,2}} A\wedge F = -\frac{N}{(2\pi)^3}\int_{\mathbb{R}^{1,2}} A\wedge F$$

$$\mathrm{d}s^2_{\mathrm{ind}} = r^2\left[-\mathrm{d}\tau^2+\left(1+\frac{r'(\sigma)^2}{r^4f(r)}\right)\mathrm{d}\sigma^2\right], r'(\sigma) = \partial_\sigma r(\sigma)$$

$$S_{NG} = \frac{1}{2\pi}\int \mathrm{d}\sigma\,\mathrm{d}\tau\sqrt{F_w^2(r)+G_w(r)^2r'(\sigma)^2} = \frac{T}{2\pi}\int_{-L/2}^{L/2}\mathrm{d}\sigma\sqrt{F_w^2(r)+G_w(r)^2r'(\sigma)^2}$$

$$F_w(r) = r^2, G_w(r) = \frac{1}{\sqrt{f(r)}}$$

$$r'(\sigma) = \pm V_{\text{eff}}(r) = \pm \frac{r^2(\sigma)}{r_0^2}\sqrt{f(r)(r^4(\sigma) - r_0^4)}$$

$$L_{QQ}(r_0) = 2\int_{r_0}^{\infty} \frac{\mathrm{d}r}{V_{\text{eff}}(r)} = 2r_0^2 \int_{r_0}^{\infty} \mathrm{d}r \frac{r}{\sqrt{(r^6 - Q^6)(r^4 - r_0^4)}}$$

$$\tilde{L}_{QQ}(r_0) \equiv \pi \frac{G_w(r)}{\partial_r F_w(r)}\bigg|_{r=r_0} = \frac{\pi}{2} \frac{r_0^2}{\sqrt{r_0^6 - Q^6}}$$

$$S_{NG} = \frac{T}{2\pi} r_0^2 L_{QQ}(r_0) + 2\frac{T}{2\pi}\int_{r_0}^{\infty} \mathrm{d}r \frac{\sqrt{r^4 - r_0^4}}{r^2\sqrt{f(r)}}$$

$$E_{QQ}(r_0) = r_0^2 L_{QQ}(r_0) + 2\int_{r_0}^{\infty} \mathrm{d}r \frac{\sqrt{r^4 - r_0^4}}{r^2\sqrt{f(r)}} - 2\int_{Q}^{\infty} \mathrm{d}r \frac{1}{\sqrt{f(r)}}$$

$$\mathcal{Z}_w(r_0) \equiv \frac{\mathrm{d}\tilde{L}_{QQ}(r_0)}{\mathrm{d}r_0} = -\pi r \frac{(2r^6 + Q^6)}{(r^6 - Q^6)^{\frac{3}{2}}}$$

$$\frac{\mathrm{d}E_{QQ}}{\mathrm{d}L_{QQ}} = F_w(r_0) > 0, \left(\frac{\mathrm{d}^2 E_{QQ}}{\mathrm{d}L_{QQ}^2}\right) \sim \mathcal{Z}_w(r_0)^{-1} \frac{\mathrm{d}F_w(r_0)}{\mathrm{d}r_0} < 0$$

$$S_{D_3} = \int \mathrm{d}^4\xi e^{-\Phi}\sqrt{-\det(P[g]_{ab} + P[B]_{ab})}$$

$$\mathrm{d}s_{D3}^2 = r^2\left[-\mathrm{d}t^2 + \left(1 + \frac{r'(x_1)^2}{r^4 f(r)}\right)\mathrm{d}x_1^2 + f(r)\mathrm{d}\phi^2\right] + G(\mathrm{d}\psi + \mathcal{A})^2, r'(x_1) = \partial_{x_1} r(x_1)$$

$$S_{D3} = \int \mathrm{d}^4\xi e^{-\Phi}\sqrt{-\det g_{D3}}$$

$$S_{\text{eff}} = e^{-\Phi} L_\phi L_\psi \int \mathrm{d}t\, \mathrm{d}x_1 \sqrt{G}\sqrt{r^6 f(r) + r^2 r'^2} = L_\phi L_\psi T \int_{-L/2}^{L/2} \mathrm{d}x_1 \sqrt{F_t(r)^2 + G_t(r)^2 r'^2}$$

$$F_t(r) = r^3\sqrt{f(r)}\; G_t(r) = r$$

$$V_{\text{eff}}(r) = \frac{F_t(r)}{F_t(r_0) G_t(r)}\sqrt{F_t(r)^2 - F_t(r_0)^2}$$

$$L_{MM}(r_0) = 2\int_{r_0}^{\infty} \frac{\mathrm{d}r}{V_{\text{eff}}(r)} = 2F_t(r_0)\int_{r_0}^{\infty} \mathrm{d}r \frac{1}{r^2\sqrt{f(r)}} \frac{1}{\sqrt{r^6 f(r) - F_t(r_0)^2}}$$

$$\tilde{L}_{MM}(r_0) \equiv \pi \frac{G_t(r)}{\partial_r F_t(r)}\bigg|_{r=r_0} = \frac{\pi}{3r_0^4}\sqrt{r_0^6 - Q^6}$$

$$E_{MM}(r_0) = F_t(r_0) L_{MM}(r_0) + 2\int_{r_0}^{\infty} \mathrm{d}r \frac{\sqrt{r^6 f(r) - F_t(r_0)^2}}{r^2\sqrt{f(r)}} - 2\int_{Q}^{\infty} \mathrm{d}r r$$

$$\mathcal{Z}_t(r_0) \equiv \frac{\mathrm{d}\tilde{L}_{MM}(r_0)}{\mathrm{d}r_0} = -\frac{\pi}{3}\frac{r_0^6 - 4Q^6}{r_0^5\sqrt{r_0^6 - Q^6}}$$

$$\mathrm{d}s_{D3}^2 = r^2\left[-\mathrm{d}t^2 + \left(1 + \frac{r'(x_1)^2}{r^4 f(r)}\right)\mathrm{d}x_1^2 + Gf(r)\mathrm{d}\phi^2\right] + G(\,\mathrm{d}\varphi + \mathcal{A})^2$$

$$-\det(P[g]_{ab} + P[B]_{ab}) = G(r)(r^6 f(r) + r^2 r^2)$$

$$S_{\text{eff}} = L_\phi L_\psi \int \; e^{-\Phi}\mathrm{d}t\,\mathrm{d}x_1\sqrt{G(r)(r^6 f(r) + r^2 r'^2)} = L_\phi L_\psi T\int_{-L/2}^{L/2} \mathrm{d}x_1\sqrt{F_t(r)^2 + G_t(r)^2 r'^2}$$

$$S_{\text{EE}} = \frac{1}{4G_N}\int_{\Sigma_8} \mathrm{d}^8 x e^{-2\Phi}\sqrt{\det g_{\Sigma_8}}$$

$$S_{EE} \longrightarrow \frac{1}{4G_N}\int_{\Sigma_8} \mathrm{d}^8 x G^{-1} e^{-2\Phi_0}\sqrt{G^2 \det g_{\Sigma_8}} = S_{EE}$$

$$S_{EE,\beta} = \mathcal{N}_\beta\int_{-L/2}^{L/2} dx_1\sqrt{r^6 f(r) + r^2 r'^2}, S_{EE,\,\text{dip}} = \mathcal{N}_{\text{dip}}\int_{-L/2}^{L/2} dx_1\sqrt{r^6 f(r) + r^2 r'^2}$$

$$S_{EE,NC} = \mathcal{N}_{NC}\int_{-L/2}^{L/2} dx_1\sqrt{r^6 f(r) + r^2 G(r) r'^2}$$

$$ds^2 = -\alpha_0\,\mathrm{d}t^2 + \sum_{n=1}^{d} \alpha_n\,\mathrm{d}x_n^2 + \left(\prod_{n=1}^{d} \alpha_n\right)^{\frac{1}{d}} \beta(r)dr^2 + g_{ij}\left(dy^i - A^i\right)\left(dy^j - A^j\right)$$

$$\hat{g}_{ab}\,\mathrm{d}\xi^a\,\mathrm{d}\xi^b \equiv \sum_{n=1}^{d} \alpha_n\,\mathrm{d}x_n^2 + g_{ij}dy^i dy^j$$

$$H^{\frac{1}{2}}(r) \equiv \int \; \mathrm{d}\xi^a e^{-2\Phi}\sqrt{\det\hat{g}_{ab}}$$

$$c_{\text{flow}} = d^d\frac{\beta(r)^{d/2} H^{(2d+1)/2}}{G_N^{(10)}(H')^d}$$

$$\det\hat{g}_{ab} \longrightarrow_\beta \; G^2\det\hat{g}_{ab}$$

$$\alpha_{1,2} = r^2, \alpha_3 = r^2 f(r), \beta(r) = \frac{1}{r^4} f(r)^{-\frac{4}{3}}$$

$$H^{\frac{1}{2}}(r) = \mathrm{vol}_{S^5} r^3\sqrt{f(r)}$$

$$c_{\text{flow}} = \frac{\mathrm{vol}_{S^5}}{8G_N^{(10)}}\left(\frac{\sqrt{f(r)}}{f(r) + \frac{r}{6}f'(r)}\right)^3 = \frac{\mathrm{vol}_{S^5}}{8G_N^{(10)}}\left(1 - \frac{Q^6}{r^6}\right)^{\frac{3}{2}}.$$

$$c_{\mathrm{UV}}=\frac{\mathrm{vol}_{S^5}}{8G_N^{(10)}}$$

$$\alpha_1=\alpha_2=r^2,\alpha_3=r^2f(r)G(r),\beta(r)=\frac{1}{r^4f(r)^{4/3}G^{1/3}}$$

$$c_{\mathrm{flow}}=\frac{\mathrm{vol}_{S^5}}{8G_N^{(10)}}\left(\frac{\sqrt{f(r)}}{f(r)+\frac{r}{6}f'(r)}\right)^3\times\frac{1}{\sqrt{G(r)}}$$

$$P_\phi=\frac{1}{N}\mathrm{Trexp}\ i\oint_{S_\phi^1}\Theta^{(0)}$$

$$P_\phi\to hP_\phi,h\in\mathbb{Z}_N=\left\{e^{2\pi ki/N},k=0,\ldots,N-1\right\}$$

$$\langle P_\phi\rangle\sim\exp\left(-S_{NG}^{\mathrm{reg}}\right)\sim e^{-a\sqrt{\lambda}}\neq 0$$

$$\mathrm{d}s_{\mathrm{ind}}^2=\frac{1}{r^2f(r)}\mathrm{d}r^2+\left[r^2f(r)+G(1+9\gamma^2\mu_1^2\mu_2^2\mu_3^2)Q^6\left(\frac{1}{r^2}-\frac{1}{Q^2}\right)^2\right]\mathrm{d}\phi^2$$

$$S_{\mathrm{Nambu-Goto}}^{\text{on-shell}}=\frac{1}{2\pi}\int_Q^\infty\mathrm{d}r\int_0^{2\pi R}\mathrm{d}\phi\sqrt{\det g_{\mathrm{ind}}}=R\int_Q^\infty\mathrm{d}r\sqrt{1+G(1+9\gamma^2\mu_1^2\mu_2^2\mu_3^2)\left(\frac{Q^2(r^2-Q^2)^2}{r^6f(r)}\right)}$$
$$=\sqrt{2\lambda}RQ\int_1^\infty\mathrm{d}\rho\sqrt{1+G(1+9\gamma^2\mu_1^2\mu_2^2\mu_3^2)\frac{(\rho^2-1)^2}{(\rho^6-1)}}$$

$$S_{NG}^{\mathrm{reg}}=\lim_{\Lambda\to\infty}\sqrt{2\lambda}RQ\left[\int_1^\Lambda\mathrm{d}\rho\sqrt{1+G(1+9\gamma^2\mu_1^2\mu_2^2\mu_3^2)\frac{(\rho^2-1)^2}{(\rho^6-1)}}-\Lambda\right]$$

$$\mathrm{d}s_{\mathrm{ind}}^2=\frac{1}{r^2f(r)}\mathrm{d}r^2+G(r)\left[r^2f(r)+Q^6\left(\frac{1}{r^2}-\frac{1}{Q^2}\right)^2\right]\mathrm{d}\phi^2$$

$$S_{NG}^{\text{on-shell}}=\frac{1}{2\pi}\int_Q^\infty\mathrm{d}r\int_0^{2\pi R}\mathrm{d}\phi\sqrt{\det g_{\mathrm{ind}}}=R\int_Q^\infty\mathrm{d}r\sqrt{G(r)\left(1+\frac{Q^2(r^2-Q^2)^2}{r^6f(r)}\right)}$$
$$=\sqrt{2\lambda}RQ\int_1^\infty\mathrm{d}\rho\sqrt{\frac{\rho^4}{\rho^4+\gamma^2Q^2\ell^2(\rho^6-1)}\left(1+\frac{(\rho^2-1)^2}{(\rho^6-1)}\right)}$$

$$S_{NG}^{\mathrm{reg}}=\lim_{\Lambda\to\infty}\sqrt{2\lambda}RQ\left[\int_1^\Lambda\mathrm{d}\rho\sqrt{\frac{\rho^4}{\rho^4+\gamma^2Q^2\ell^4(\rho^6-1)}\left(1+\frac{(\rho^2-1)^2}{(\rho^6-1)}\right)}-\frac{1}{\gamma Q\ell^2}\log\Lambda\right]$$

$$S_P=-\frac{1}{4\pi}\int\mathrm{d}^2\sigma\left[\eta^{\alpha\beta}\partial_\alpha X^m\partial_\beta X^n g_{mn}-\epsilon^{\alpha\beta}\partial_\alpha X^m\partial_\beta X^n B_{mn}\right]$$

$$\sqrt{-h}h^{\alpha\beta} \equiv \eta^{\alpha\beta}.$$

$$2\partial_\alpha\left(\eta^{\alpha\beta}\partial_\beta X^m g_{ms} - \epsilon^{\alpha\beta}\partial_\beta X^m B_{sm}\right) = \eta^{\alpha\beta}\partial_\alpha X^m \partial_\beta X^n \partial_s g_{mn} - \epsilon^{\alpha\beta}\partial_\alpha X^m \partial_\beta X^n \partial_s B_{mn}$$

$$g_{sm}\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^m + \frac{1}{2}[\partial_n g_{ms} + \partial_m g_{ns} - \partial_s g_{mn}]\eta^{\alpha\beta}\partial_\alpha X^m \partial_\beta X^n = -\frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha X^m \partial_\beta X^n H_{smn}$$

$$\eta^{\alpha\beta}\left(\partial_\alpha\partial_\beta X^s + \partial_\alpha X^m \partial_\beta X^n \Gamma^s_{mn}\right) = -\frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha X^m \partial_\beta X^n H^s{}_{mn}.$$

$$T_{\alpha\beta} = \partial_\alpha X^m \partial_\beta X^n g_{mn} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\lambda}\partial_\rho X^m \partial_\lambda X^n g_{mn} = 0$$

$$\Sigma = \left\{\theta = \frac{\pi}{2}, \mu_1 = \sin\ \varphi\ \mu_2 = \cos\ \varphi, \mu_3 = 0, x_1, x_2, \phi, \psi = \text{const}\right\}$$

$$\mathrm{d}s^2_{10}\big|_\Sigma = -r^2\ \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 f(r)} + \mathrm{d}\varphi^2 + G(\sin^2\ \varphi\ \mathrm{d}\phi_1^2 + \cos^2\ \varphi\ \mathrm{d}\phi_2^2)$$

$$f(r) = 1 - \frac{Q^6}{r^6}, G^{-1} = 1 + \frac{\gamma^2}{4}\sin^2\ 2\varphi, g_0 = \sin^2\ \varphi\cos^2\ \varphi$$

$$B|_\Sigma = -\gamma g_0 G d\phi_1 \wedge d\phi_2$$

$$\begin{aligned}
&\eta^{\alpha\beta}\left(\partial_\alpha\partial_\beta t + 2\Gamma^t_{tr}\partial_\alpha t \partial_\beta r\right) = 0\\
&\eta^{\alpha\beta}\left(\partial_\alpha\partial_\beta r + \Gamma^r_{rr}\partial_\alpha r \partial_\beta r + \Gamma^r_{tt}\partial_\alpha t \partial_\beta t\right) = 0\\
&\eta^{\alpha\beta}\left(\partial_\alpha\partial_\beta\varphi + \Gamma^\varphi_{\phi_1\phi_1}\partial_\alpha\phi_1\partial_\beta\phi_1 + \Gamma^\varphi_{\phi_2\phi_2}\partial_\alpha\phi_2\partial_\beta\phi_2\right) + \partial_\varphi B_{\phi_1\phi_2}\epsilon^{\alpha\beta}\partial_\alpha\phi_1\partial_\beta\phi_2 = 0\\
&\eta^{\alpha\beta}\left(\partial_\alpha\partial_\beta\phi_1 + 2\Gamma^{\phi_1}_{\phi_1\varphi}\partial_\alpha\phi_1\partial_\beta\varphi\right) - \epsilon^{\alpha\beta}\partial_\alpha\left(B_{\phi_1\phi_2}\partial_\beta\phi_2\right)g^{\phi_1\phi_1} = 0\\
&\eta^{\alpha\beta}\left(\partial_\alpha\partial_\beta\phi_2 + 2\Gamma^{\phi_2}_{\phi_2\varphi}\partial_\alpha\phi_1\partial_\beta\varphi\right) + \epsilon^{\alpha\beta}\partial_\alpha\left(B_{\phi_1\phi_2}\partial_\beta\phi_1\right)g^{\phi_2\phi_2} = 0
\end{aligned}$$

$$\Gamma^r_{tt} = 0, \Gamma^{\phi_1}_{\phi_1\varphi} = -\Gamma^{\phi_2}_{\phi_2\varphi} = 1, \Gamma^\varphi_{\phi_1\phi_1} = -\Gamma^\varphi_{\phi_2\phi_2} = -\frac{2}{4+\gamma^2}$$

$$\begin{aligned}
T_{\tau\tau} &= -\frac{Q^2}{2}\kappa^2 + \frac{G}{4}(\omega_1^2 + \omega_2^2 + m_1^2 + m_2^2) \equiv 0\\
T_{\tau\sigma} &= \frac{G}{2}(\omega_1 m_1 + \omega_2 m_2) \equiv 0
\end{aligned}$$

$$\Pi^\tau_m = \frac{\delta\mathcal{L}}{\delta\partial_\tau X^m} = \frac{1}{2\pi}(\partial_\tau X^n g_{mn} + \partial_\sigma X^n B_{mn})$$

$$\begin{aligned}
\Pi^\tau_t &= -\frac{1}{2\pi}Q^2\partial_\tau t = -\frac{\sqrt{2\lambda}}{2\pi}Q^2\kappa\\
\Pi^\tau_{\phi_1} &= \frac{1}{2\pi}\frac{G}{2}\left(\partial_\tau\phi_1 - \frac{\gamma}{2}\partial_\sigma\phi_2\right) = \frac{\sqrt{2\lambda}}{2\pi}\frac{G}{2}\left(\omega - \frac{\gamma}{2}m\right)\\
\Pi^\tau_{\phi_2} &= \frac{1}{2\pi}\frac{G}{2}\left(\partial_\tau\phi_2 + \frac{\gamma}{2}\partial_\sigma\phi_1\right) = \frac{\sqrt{2\lambda}}{2\pi}\frac{G}{2}\left(\omega - \frac{\gamma}{2}m\right)
\end{aligned}$$

$$P_m = \int_0^{2\pi} \mathrm{d}\sigma\Pi^\tau_m$$

$$E=\sqrt{2\lambda}\mathcal{E}=-P_0=\frac{\sqrt{2\lambda}}{2\pi}\int_0^{2\pi}\ \mathrm{d}\sigma Q^2\kappa=\sqrt{2\lambda}Q^2\kappa$$
$$J_1=J_2=\sqrt{2\lambda}\mathcal{J}_1=\frac{\sqrt{2\lambda}G}{4\pi}\int_0^{2\pi}\ \mathrm{d}\sigma\left(\omega-\frac{\gamma}{2}m\right)=\sqrt{2\lambda}\frac{G}{2}\left(\omega-\frac{\gamma}{2}m\right)$$
$$\mathcal{J}=\mathcal{J}_1+\mathcal{J}_2$$

$$E=\sqrt{2\lambda}Q\sqrt{\mathcal{J}^2+\left(m+\frac{\gamma}{2}\mathcal{J}\right)^2}$$

$$\mathrm{d}s_{10}^2=\mathrm{d}s_9^2+e^{2C}(\ \mathrm{d}z+A_1)^2$$

$$B=B_2+B_1\wedge\ \mathrm{d}z, F=F_\perp+F_\parallel\wedge e^C(\ \mathrm{d}z+A_1)$$

$$\mathrm{d}s_{10}^2(2)=\mathrm{d}s_9^2+e^{-2C}(\ \mathrm{d}z-B_1)^2$$
$$B^{(2)}=B_2-A_1\wedge(\ \mathrm{d}z-B_1),\Phi^{(2)}=\Phi-C,$$
$$F^{(2)}=F_\perp^{(2)}+F_\parallel^{(2)}\wedge e^{-C}(\ \mathrm{d}z-B_1),F_\perp^{(2)}=e^CF_\parallel,F_\parallel^{(2)}=e^CF_\perp$$

$$\mathrm{d}s_{10}^2=\mathrm{d}s_{AdS_5}^2+\sum_{i=1}^{3}\ \mathrm{d}\mu_i^2+9\frac{\mu_1^2\mu_2^2\mu_3^2}{g_0}\ \mathrm{d}\psi^2+(\mu_1^2+\mu_2^2)\left(D\varphi_1+\frac{\mu_2^2}{\mu_1^2+\mu_2^2}D\varphi_2\right)^2+\frac{g_0}{\mu_1^2+\mu_2^2}D\varphi_2^2$$
$$e^{2\Phi}=1,B=0,\omega_{S^5}=3\ \mathrm{d}\omega_1\wedge\ \mathrm{d}\psi\wedge\ \mathrm{d}\varphi_2\wedge\mathrm{d}\varphi_1,\ \ \mathrm{d}\omega_1=\sqrt{\mu_1^2+\mu_2^2}\ \mu_1\mu_2\mu_3\ \mathrm{d}\theta\wedge\ \mathrm{d}\varphi$$
$$F_5=4(\omega_{AdS_5}+\omega_{S^5}),$$

$$g_0=\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2$$
$$D\varphi_1=\mathrm{d}\varphi_1+\left(3\frac{\mu_2^2\mu_3^2}{g_0}-1\right)\mathrm{d}\psi, D\varphi_2=\mathrm{d}\varphi_2+\left(3\frac{\mu_1^2\mu_2^2}{g_0}-1\right)\mathrm{d}\psi$$

$$\mathrm{d}z=\ \mathrm{d}\varphi_1\ A_1=\left(3\frac{\mu_2^2\mu_3^2}{g_0}-1\right)\mathrm{d}\psi+\frac{\mu_2^2}{\mu_1^2+\mu_2^2}D\varphi_2$$
$$\mathrm{d}s_9^2=\mathrm{d}s_{AdS_5}^2+\sum_{i=1}^{3}\ \mathrm{d}\mu_i^2+9\frac{\mu_1^2\mu_2^2\mu_3^2}{g_0}\ \mathrm{d}\psi^2+\frac{g_0}{\mu_1^2+\mu_2^2}D\varphi_2^2$$
$$e^{2C}=(\mu_1^2+\mu_2^2),B_1=B_2=0,F_\perp=4\omega_{AdS_5},F_\perp=4e^{-C}i_{\varphi_1\omega_{S^5}},$$

$$\mathrm{d}s_{10}^2(2)=\mathrm{d}s_9^2+e^{-2C}\ \mathrm{d}\varphi_1^2$$
$$B^{(2)}=-A_1\wedge\ \mathrm{d}\varphi_1,\Phi^{(2)}=\Phi-C$$
$$F^{(2)}=F_\perp^{(2)}+F_\parallel^{(2)}\wedge e^{-C}\ \mathrm{d}\varphi_1,F_\parallel^{(2)}=e^C4\omega_{AdS_5},F_\perp^{(2)}=4i_{\varphi_1}\omega_{S^5}$$

$$\mathrm{d}s_{10}^2{}^{(3)}=\mathrm{d}s_{AdS_5}^2+\sum_{i=1}^{3}\ \mathrm{d}\mu_i^2+9\frac{\mu_1^2\mu_2^2\mu_3^2}{g_0}\ \mathrm{d}\psi^2+\frac{g_0}{\mu_1^2+\mu_2^2}(D\varphi_2+\gamma\mathrm{d}\varphi_1)^2+\frac{1}{\mu_1^2+\mu_2^2}\ \mathrm{d}\varphi_1^2$$

$$\mathrm{d}s_{10}^{2}{}^{(3)} = \mathrm{d}s_{AdS_5}^2 + \sum_{i=1}^{3} \mathrm{d}\mu_i^2 + 9\frac{\mu_1^2\mu_2^2\mu_3^2}{g_0}\mathrm{d}\psi^2 + \frac{g_0 G}{\mu_1^2+\mu_2^2}D\varphi_2^2 + \frac{G^{-1}}{\mu_1^2+\mu_2^2}\left(\mathrm{d}\varphi_1 + A_1^{(3)}\right)^2$$

$$A_1^{(3)} = \gamma g_0 G D\varphi_2, G^{-1} = 1+\gamma^2 g_0, e^{2C^{(3)}} = \frac{G^{-1}}{\mu_1^2+\mu_2^2},$$

$$B^{(3)} = -A_1 \wedge \mathrm{d}\varphi_1, \Phi^{(3)} = \Phi - C,$$

$$F^{(3)} = F_\perp^{(3)} + F_\parallel^{(3)} \wedge e^{C^{(3)}}\left(\mathrm{d}\varphi_1 + A_1^{(3)}\right)$$

$$F_\parallel^{(3)} = e^{-C^{(3)}}4\left(\omega_{AdS_5} + \gamma i_{\varphi_2} i_{\varphi_1}\omega_{S^5}\right), F_\perp^{(3)} = 4\left(G i_{\varphi_1}\omega_{S^5} - \omega_{AdS_5}\wedge A_1^{(3)}\right).$$

$$\mathrm{d}s_{10}^2(4) = \mathrm{d}s_{AdS_5}^2 + \sum_{i=1}^{3} \mathrm{d}\mu_i^2 + 9\frac{\mu_1^2\mu_2^2\mu_3^2}{g_0}\mathrm{d}\psi^2 + \frac{g_0 G}{\mu_1^2+\mu_2^2}D\varphi_2^2 + G(\mu_1^2+\mu_2^2)(\mathrm{d}\varphi_1 + A_1)^2$$

$$B^{(4)} = -A_1^{(3)} \wedge (\mathrm{d}\varphi_1 + A_1), \Phi^{(4)} = \Phi + \frac{1}{2}\log\, G$$

$$F^{(4)} = F_\perp^{(4)} + F_\parallel^{(4)} \wedge e^{-C^{(3)}}(\mathrm{d}\varphi_1 + A_1)$$

$$F_\perp^{(4)} = 4\left(\omega_{AdS_5} + \gamma i_{\varphi_2} i_{\varphi_1}\omega_{S^5}\right), F_\parallel^{(4)} = 4e^{C^{(3)}}\left(G\omega_{S^5} - \omega_{AdS_5}\wedge A_1^{(3)}\right),$$

$$\mathrm{d}s_{10}^2 = \mathrm{d}s_{AdS_5}^2 + \sum_{i=1}^{3} \mathrm{d}\mu_i^2 + 9\frac{\mu_1^2\mu_2^2\mu_3^2}{g_0}\mathrm{d}\psi^2 + \frac{g_0 G}{\mu_1^2+\mu_2^2}D\varphi_2^2 + G(\mu_1^2+\mu_2^2)\left(D\varphi_1 + \frac{\mu_2^2}{\mu_1^2+\mu_2^2}D\varphi_2\right)^2$$

$$B = \gamma g_0 G D\varphi_1 \wedge D\varphi_2, \Phi^{(4)} = \Phi + \frac{1}{2}\log\, G$$

$$F_5 = 4\left(\omega_{AdS_5} + G\omega_{S^5}\right), F_3 = 4\gamma i_{\varphi_2} i_{\varphi_1}\omega_{S^5}, F_7 = 4\omega_{AdS_5}\wedge B$$

$$G_5 = -4\mathrm{vol}_5 - 2Q^3 J_2 \wedge \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2$$

$$= -4\mathrm{vol}_5 - 2Q^3\left[\sum_{i=1}^{3} \mu_i\,\mathrm{d}\mu_i \wedge D\psi + (\mu_2\,\mathrm{d}\mu_2 - \mu_3\,\mathrm{d}\mu_3)\wedge \mathrm{d}\varphi_2 + (\mu_2\,\mathrm{d}\mu_2 - \mu_1\,\mathrm{d}\mu_1)\wedge \mathrm{d}\varphi_1\right]\wedge \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 =$$

$$-4\mathrm{vol}_5 + \beta_4 \wedge D\psi + \gamma_4 \wedge \mathrm{d}\varphi_2 + \alpha_4 \wedge \mathrm{d}\varphi_1$$

$$\mathrm{vol}_5 \equiv \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}r \wedge \mathrm{d}\phi, \alpha_4 \equiv 2Q^3(\mu_2\,\mathrm{d}\mu_2 - \mu_1\,\mathrm{d}\mu_1) \wedge \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2$$

$$\beta_4 \equiv 2Q^3\sum_{i=1}^{3} \mu_i\,\mathrm{d}\mu_i \wedge \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2, \gamma_4 \equiv 2Q^3(\mu_2\,\mathrm{d}\mu_2 - \mu_3\,\mathrm{d}\mu_3)\wedge \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2$$

$$G_5 = G_{5\perp} + \alpha_4 \wedge (\mathrm{d}\varphi_1 + A_1)$$

$$G_{5\perp} = -4\mathrm{vol}_5 + \beta_4 \wedge D\psi + \gamma_4 \wedge \mathrm{d}\varphi_2 - \alpha_4 \wedge A_1$$

$$A_1 = \left(3\frac{\mu_2^2\mu_3^2}{g_0} - 1\right)D\psi + \frac{\mu_2^2}{\mu_1^2+\mu_2^2}D\tilde{\varphi}_2$$

$$G_4^{(3)} = \alpha_4$$

$$G_6^{(3)} = G_{5\perp} \wedge \left(\mathrm{d}\varphi_1 + A_1^{(3)}\right) - G_{5\perp} \wedge A_1^{(3)}$$

$$G_5^{(4)} = G_5$$

$$G_7^{(4)} = -G_{5\perp} \wedge A_1^{(3)} \wedge (\mathrm{d}\varphi_1 + A_1) \quad \text{(A.19)}$$

$$
\begin{aligned}
G_7^{(4)} &= -\gamma g_0 G G_{5\perp} \wedge D\tilde{\varphi}_2 \wedge \left(D\tilde{\varphi}_1 + \frac{\mu_2^2}{\mu_1^2+\mu_2^2} D\tilde{\varphi}_2 \right) \\
&= -\gamma g_0 G G_{5\perp} \wedge D\tilde{\varphi}_2 \omega D\tilde{\varphi}_1 \\
&= \gamma g_0 G(-4\mathrm{vol}_5 + \beta_4 \wedge D\psi + \gamma_4 \wedge \mathrm{d}\varphi_2 - \alpha_4 \wedge A_1) \wedge D\tilde{\varphi}_1 \wedge D\tilde{\varphi}_2 \\
&= \gamma g_0 G(-4\mathrm{vol}_5 + \beta_4 \wedge D\psi + \gamma_4 \wedge \mathrm{d}\varphi_2 + \alpha_4 \wedge \varphi_1) \wedge D\tilde{\varphi}_1 \wedge D\tilde{\varphi}_2 \\
&= \gamma g_0 G G_5 \wedge D\tilde{\varphi}_1 \wedge D\tilde{\varphi}_2
\end{aligned}
$$

$$
\begin{aligned}
F_3 &= \gamma i_{\varphi_2} i_{\varphi_1} \star_{10} G_5 = \frac{\gamma}{G} i_{\varphi_2} i_{\varphi_1} \star_{10\beta} G_5 \\
F_5 &= (1 + G \star_{10}) G_5 = (1 + \star_{10\beta}) G_5 \\
F_7 &= \gamma g_0 G G_5 \wedge D\tilde{\varphi}_1 \wedge D\tilde{\varphi}_2 = G_5 \wedge B
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}s_{10}^2 &= \mathrm{d}s_8^2 + r^2(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) \\
\mathrm{d}s_8^2 &= r^2(-\mathrm{d}t^2 + f(r)\mathrm{d}\phi^2) + \frac{\mathrm{d}r^2}{r^2 f(r)} + \sum_{i=1}^{3} \mathrm{d}\mu_i^2 + \mu_i^2(\mathrm{d}\phi_i + \mathcal{A})^2
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{d}s_{10}^2 = \mathrm{d}s_8^2 + G r^2(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) \\
&B = \gamma r^4 G\, \mathrm{d}x_1 \wedge \mathrm{d}x_2, e^{2\Phi} = G \\
&F_5 = (1 + \star_{10\beta}) G G_5, F_3 = \gamma i_{x_2} i_{x_1} G_5, F_7 = \star_{10\beta} G_5 \wedge B,
\end{aligned}
$$

$$
\begin{aligned}
G_5 &= -4r^3\, \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}r \wedge \mathrm{d}\phi - 2Q^3 J_2 \wedge \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \\
J_2 &= \sum_{i=1}^{3} \mu_i\, \mathrm{d}\mu_i \wedge (\mathrm{d}\phi_i + \mathcal{A})
\end{aligned}
$$

$$
Q'_{D3} = \frac{1}{(2\pi)^4} \int_{\Sigma_5} \star_{10\beta} G G_5 = \frac{1}{(2\pi)^4} \int_{\Sigma_5} \star_{10} G_5 = N,
$$

$$
Q'_{D1} = \frac{1}{(2\pi)^6} \int_{\Sigma_7} \star_{10\beta} F_3 + B \wedge \star_{10\beta} G_5 = \frac{1}{(2\pi)^6} \int_{\Sigma_7} B \wedge \star_{10\beta} G_5 - F_7 = 0.
$$

$$
R = \frac{8\gamma^2(Q^6(4\gamma^2 r^4 - 3) + 2\gamma^2 r^{10} + 9r^6)}{r^2(\gamma^2 r^4 + 1)^2}
$$

$$
R|_{r=Q} = \frac{48\gamma^2 Q^4}{\gamma^2 Q^4 + 1}, \; R|_{r\to\infty} \sim 16 + O(r^{-3})
$$

$$
\begin{aligned}
R^{\mu\nu} R_{\mu\nu}\big|_{r=Q} &= \frac{24(17\gamma^4 Q^8 + 2\gamma^2 Q^4 + 9)}{(\gamma^2 Q^4 + 1)^2}, \; R^{\mu\nu} R_{\mu\nu}\big|_{r\to\infty} \sim 96 + O(r^{-3}) \\
R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}\big|_{r=Q} &= \frac{72(7\gamma^4 Q^8 + 6\gamma^2 Q^4 + 7)}{(\gamma^2 Q^4 + 1)^2}, \; R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}\big|_{r\to\infty} \sim 80 + O(r^{-3})
\end{aligned}
$$

$$
\begin{aligned}
R|_{r\to\infty} &\sim -\frac{2}{\sqrt{\gamma} r} + O(r^{-3}) \\
R_{\mu\nu} R^{\mu\nu}\big|_{r\to\infty} &\sim \frac{116}{\gamma r^2} + O(r^{-3}) \\
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\big|_{r\to\infty} &\sim \frac{97}{\gamma r^2} + O(r^{-3})
\end{aligned}
$$

$$ds_{10}^2 = ds_8^2 + r^2\, dx_1^2 + \left(r^2 f(r) + \hat{\mathcal{A}}^2\right)\left(d\phi + \frac{\hat{\mathcal{A}}}{r^2 f(r) + \hat{\mathcal{A}}^2} D\varphi\right)^2$$

$$ds_8^2 = r^2(-dt^2 + dx_2^2) + \frac{dr^2}{r^2 f(r)} + ds_{\mathbb{C}P^2}^2 + \frac{r^2 f(r)}{r^2 f(r) + \hat{\mathcal{A}}^2} D\varphi^2$$

$$\mathcal{D}\varphi = d\varphi + \eta$$

$$ds_{10}^2 = ds_8^2 + G r^2\, dx_1^2 + G\left(r^2 f(r) + \hat{\mathcal{A}}^2\right)\left(d\phi + \frac{\hat{\mathcal{A}}}{r^2 f(r) + \hat{\mathcal{A}}^2} D\varphi\right)^2$$

$$B = -\gamma G r^2\left(r^2 f(r) + \hat{\mathcal{A}}^2\right) dx_1 \wedge \left(d\phi + \frac{\hat{\mathcal{A}}}{r^2 f(r) + \hat{\mathcal{A}}^2} D\varphi\right), e^{2\Phi} = G$$

$$F_5 = \left(1 + \star_{10\beta}\right)\tilde{G}_5$$
$$\tilde{G}_5 = -4r^3\, dt \wedge dx_1 \wedge dx_2 \wedge dr \wedge d\phi - Q^3\, d\eta \wedge dt \wedge dx_1 \wedge dx_2 + B \wedge F_3$$
$$F_3 = -4\gamma r^3\, dt \wedge dx_2 \wedge dr, F_7 = -4\gamma r^4 f(r) G\, dx_1 \wedge d\phi \wedge \mathrm{vol}_{\mathbb{C}P^2} \wedge D\varphi$$

$$Q'_{D3} = \frac{1}{(2\pi)^4}\int_{\Sigma_5} \star_{10\beta}\, \tilde{G}_5 = \frac{1}{(2\pi)^4}\int_{\Sigma_5} 4\mathrm{vol}_{\mathbb{C}P^2}\, d\varphi = N$$

$$R = \frac{2\gamma^2}{r^2(2\gamma^2 Q^4 - \gamma^2 Q^2 r^2 - \gamma^2 r^4 - 1)^2}[19\gamma^2 Q^{10} + 22\gamma^2 Q^8 r^2 - 8\gamma^2 Q^6 r^4$$
$$-6Q^6 - 64\gamma^2 Q^4 r^6 + 23\gamma^2 Q^2 r^8 + 15Q^2 r^4 + 8\gamma^2 r^{10} + 36r^6]$$

$$R|_{r=Q} = 90\gamma^2 Q^4,\ R|_{r\to\infty} \sim 16 + 14\frac{Q^2}{r^2} + O(r^{-3}).$$

$$R^{\mu\nu}R_{\mu\nu}\big|_{r=Q} = 36(81\gamma^4 Q^8 - 14\gamma^2 Q^4 + 6),\ R^{\mu\nu}R_{\mu\nu}\big|_{r\to\infty} \sim 96 + O(r^{-3})$$

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\big|_{r=Q} = 36(99\gamma^4 Q^8 + 24\gamma^2 Q^4 + 14),\ R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\big|_{r\to\infty} \sim 80 - 40\frac{Q^2}{r^2} + O(r^{-3}).$$

$$R|_{r\to\infty} \sim -\frac{2}{\sqrt{\gamma}r} + O(r^{-3})$$
$$R_{\mu\nu}R^{\mu\nu}\big|_{r\to\infty} \sim \frac{116}{\gamma r^2} + O(r^{-3})$$
$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big|_{r\to\infty} \sim \frac{97}{\gamma r^2} + O(r^{-3})$$

$$G|_{r\to\infty} \sim \frac{1}{\gamma^2 r^4}$$

$$\epsilon^{\alpha\beta}\partial_\alpha X^m \partial_\beta X^n\left[\partial_m B_{sn} - \frac{1}{2}\partial_s B_{mn}\right] = -\frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha X^m \partial_\beta X^n H_{smn}$$

$$\mathbb{III}_{\mathcal{B}}^{4\ \mathrm{dG}}(t, z; q) = \mathrm{Tr}(-1)^F q^{J + \frac{H-C}{4}} t^{H-C} z^f$$

$$(a;q)_0 := 1, (a;q)_n := \prod_{k=0}^{n-1} (1-aq^k), (q)_n := \prod_{k=1}^{n} (1-q^k)$$

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1-aq^k), (q)_\infty := \prod_{k=1}^{\infty} (1-q^k)$$

$$\mathbb{II}_{\mathcal{N}}^{U(N)|U(M)}(t;q) = \frac{1}{N!}\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i^{(1)}}{2\pi i s_i^{(1)}}\prod_{i<j}\frac{\left(s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}$$
$$\times\frac{1}{M!}\frac{(q)_\infty^M}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^M}\oint\prod_{i=1}^{M}\frac{ds_i^{(2)}}{2\pi i s_i^{(2)}}\prod_{i<j}\frac{\left(s_i^{(2)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(2)\pm}s_j^{(2)\mp};q\right)_\infty}$$
$$\times\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{\left(q^{\frac{3}{4}}t^{-1}s_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}}ts_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}.$$

$$\mathbb{II}_{\mathcal{D}}^{U(N)|U(M)}(t;q) = \frac{1}{N!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qs_i^{\pm}s_j^{\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty}$$
$$\times\prod_{i=1}^{N}\frac{\left(q^{\frac{3}{4}+\frac{N-M}{4}}t^{-1+(N-M)}s_i^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{N-M}{4}}t^{1+(N-M)}s_i^{\pm};q\right)_\infty}$$
$$\times\prod_{l=1}^{N-M}\frac{\left(q^{\frac{l+1}{2}}t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}}t^{2l};q\right)_\infty}$$

$$\mathbb{II}_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} = \oint\frac{ds}{2\pi i s}i_{4d}(s,t;q)i_{3d}(s,t;q)$$
$$\overset{W_{\mathcal{R}_k}}{\rightarrow}\left\langle\prod_{k=1} W_{\mathcal{R}_k}\right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} = \oint\frac{ds}{2\pi i s}i_{4d}(s,t;q)i_{3d}(s,t;q)\prod_k\chi_{\mathcal{R}_k}$$

$$\mathbb{II}_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} = F(t;q)\oint\frac{ds}{2\pi i s}i_{4d}(s,t;q)i_{3d}(s,t;q)$$
$$\overset{V_B}{\rightarrow}\left\langle\prod_{k=1} V_B\right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} = q^{\frac{\delta}{2}}t^{-2\delta}F(t;q)\oint\frac{ds}{2\pi i s}i_{4d}(q^{\alpha(B)}s,t;q)i_{3d}\left(q^{\frac{1}{2}\rho(B)}s,t;q\right)$$

$$\mathbb{II}_{\mathcal{B}}^{4\text{ d}}G^{(H)}(z;\mathfrak{q}) = \lim_{\substack{\mathfrak{q}:=q^{1/4}t:\text{ fixed}\\ q\to 0}}\mathbb{II}_{\mathcal{B}}^{4\text{ d}}G(t,z;q),$$
$$\mathbb{II}_{\mathcal{B}}^{4\text{ d}}G^{(C)}(z;\mathfrak{q}) = \lim_{\mathfrak{q}:=q^{1/4}t^{-1}:\text{ fixed}}\mathbb{II}_{\mathcal{B}}^{4\text{ d}}G(t,z;q).$$

$$\left\langle \prod_k \mathcal{W}_{\mathcal{R}_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q) = \frac{\langle \prod_k W_{\mathcal{R}_k} \rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q)}{\mathbb{II}_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q)}$$

$$\left\langle \prod_k \mathcal{V}_{B_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q) = \frac{\langle \prod_k V_{B_k} \rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q)}{\mathbb{II}_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q)}$$

$$\left\langle \prod_k \mathcal{W}_{\mathcal{R}_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)^{(H)}} (\mathfrak{q}) = \lim_{\mathfrak{q}:=q^{1/4}:\text{ fixed}} \left\langle \prod_{k\to 0} \mathcal{W}_{\mathcal{R}_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q),$$

$$\left\langle \prod_k \mathcal{W}_{\mathcal{R}_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)^{(C)}} (\mathfrak{q}) = \lim_{\mathfrak{q}:=q^{1/4}t^{-1}:\text{ fixed}} \left\langle \prod_k \mathcal{W}_{\mathcal{R}_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q),$$

$$\left\langle \prod_k \mathcal{V}_{B_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} {}^{(H)}(\mathfrak{q}) = \lim_{\mathfrak{q}:=q^{1/4}:\text{ fixed}} \left\langle \prod_k \mathcal{V}_{B_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q),$$

$$\left\langle \prod_k \mathcal{V}_{B_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} {}^{(C)}(\mathfrak{q}) = \lim_{\mathfrak{q}:=q^{1/4}t^{-1}:\text{ fixed}} \left\langle \prod_k \mathcal{V}_{B_k} \right\rangle_{\mathcal{N}/\mathcal{D}}^{U(N)|U(M)} (t;q).$$

$$\langle W_{n^{(1)};n^{(2)}} \rangle_{\mathcal{N}}^{U(1)|U(1)} (t;q) = \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^2} \oint \frac{ds^{(1)}}{2\pi i s^{(1)}} s^{(1)^{n^{(1)}}} \oint \frac{ds^{(2)}}{2\pi i s^{(2)}} s^{(2)^{n^{(2)}}} \frac{\left(q^{\frac{3}{4}}t^{-1}s^{(1)\mp}s^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}}ts^{(1)\pm}s^{(2)\mp};q\right)_\infty}$$

$$\langle W_{n^{(1)};n^{(2)}} \rangle_{\mathcal{N}}^{U(1)|U(1)} (t;q) = \begin{cases} \frac{\left(q^{\frac{1}{2}}t^2;q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty} \sum_{l=0}^{\infty} \frac{(q^{1+l};q)_\infty^2}{\left(q^{\frac{1}{2}+l}t^2;q\right)_\infty^2} q^{\frac{l}{2}+|n|l+\frac{|n|}{4}} t^{-2l+|n|} & \text{for } n^{(1)} = -n^{(2)} = n \\ 0 & \text{otherwise} \end{cases}$$

$$\langle V_n \rangle_{\mathcal{D}}^{U(1)|U(1)} (t;q) = q^{\frac{|n|}{4}} t^{-|n|} \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty} \oint \frac{ds}{2\pi i s} \frac{\left(q^{\frac{3}{4}+\frac{|n|}{2}}t^{-1}s^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{|n|}{2}}ts^{\pm};q\right)_\infty}$$

$$\langle V_n \rangle_{\mathcal{D}}^{U(1)|U(1)} (t;q) = \sum_{l=0}^{\infty} \frac{(q^{1+l};q)_\infty (q^{1+|n|+l};q)_\infty}{\left(q^{\frac{1}{2}+l}t^2;q\right)_\infty \left(q^{\frac{1}{2}+|n|+l}t^2;q\right)_\infty} q^{\frac{l}{2}+\frac{|n|}{4}} t^{-2l-|n|}$$

$$\langle W_{n;-n} \rangle_{\mathcal{N}}^{U(1)|U(1)} (t;q) = \langle V_n \rangle_{\mathcal{D}}^{U(1)|U(1)} (t^{-1};q)$$

$$\mathbb{II}_{\mathcal{N}}^{U(1)|U(1)^{(C)}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)^2}$$
$$\mathbb{II}_{\mathcal{N}}^{U(1)|U(1)(H)}(\mathfrak{q}) = \frac{1}{1-\mathfrak{q}^2}$$

$$\langle \mathcal{W}_{n;-n}\rangle_{\mathcal{N}}^{U(1)|U(1)^{(C)}}(\mathfrak{q}) = 0$$
$$\langle \mathcal{W}_{n;-n}\rangle_{\mathcal{N}}^{U(1)|U(1)(H)}(\mathfrak{q}) = \mathfrak{q}^n$$

$$\langle V_{n;0}\rangle_{\mathcal{N}}^{U(1)|U(1)}(t;q) = q^{\frac{|n|}{4}} t^{-|n|} \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^2} \oint \frac{ds^{(1)}}{2\pi i s^{(1)}} \oint \frac{ds^{(2)}}{2\pi i s^{(2)}} \frac{\left(q^{\frac{3}{4}+\frac{|n|}{2}}t^{-1}s^{(1)\mp}s^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{|n|}{2}}ts^{(1)\pm}s^{(2)\mp};q\right)_\infty}$$

$$s^{(2)} = q^{\frac{1}{4}+\frac{|n|}{2}+l}ts^{(1)}, l = 0,1,\cdots$$

$$\langle V_{n;0}\rangle_{\mathcal{N}}^{U(1)|U(1)}(t;q) = \frac{\left(q^{\frac{1}{2}}t^2;q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty} \sum_{l=0}^{\infty} \frac{(q^{1+l};q)_\infty (q^{1+|n|+l};q)_\infty}{\left(q^{\frac{1}{2}+l}t^2;q\right)_\infty \left(q^{\frac{1}{2}+|n|+l}t^2;q\right)_\infty} q^{\frac{l}{2}+\frac{|n|}{4}} t^{-2l-|n|}$$

$$\langle W_n\rangle_{\mathcal{D}}^{U(1)|U(1)}(t;q) = \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty} \oint \frac{ds}{2\pi i s} \frac{\left(q^{\frac{3}{4}}t^{-1}s^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}}ts^{\pm};q\right)_\infty} s^n$$

$$\langle W_n\rangle_{\mathcal{D}}^{U(1)|U(1)}(t;q) = \sum_{l=0}^{\infty} \frac{(q^{1+l};q)_\infty^2}{\left(q^{\frac{1}{2}+l}t^2;q\right)_\infty^2} q^{\frac{l}{2}+|n|l+\frac{|n|}{4}} t^{-2l+|n|}$$

$$\langle V_{n;0}\rangle_{\mathcal{N}}^{U(1)|U(1)}(t;q) = \langle W_n\rangle_{\mathcal{D}}^{U(1)|U(1)}(t^{-1};q)$$

$$\langle \mathcal{V}_{n;0}\rangle_{\mathcal{N}}^{U(1)|U(1)(C)}(\mathfrak{q}) = \mathfrak{q}^n$$
$$\langle \mathcal{V}_{n;0}\rangle_{\mathcal{N}}^{U(1)|U(1)(H)}(\mathfrak{q}) = 0$$

$$\begin{aligned}
&\langle W_{\mathcal{R}^{(1)};n^{(2)}}\rangle_{\mathcal{N}}^{U(N)|U(1)}(t;q) \\
&= \frac{1}{N!} \frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i^{(1)}}{2\pi i s_i^{(1)}} \prod_{i<j} \frac{\left(s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty} \chi_{\mathcal{R}^{(1)}}(s^{(1)}) \\
&\times \frac{(q)_\infty}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty} \oint \frac{ds^{(2)}}{2\pi i s^{(2)}} s^{(2)^{n^{(2)}}} \prod_{i=1}^{N} \frac{\left(q^{\frac{3}{4}}t^{-1}s_i^{(1)\mp}s^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}t}s_i^{(1)\pm}s^{(2)\mp};q\right)_\infty}
\end{aligned}$$

$$\langle W_{\mathcal{R}^{(1)};n^{(2)}}\rangle_{\mathcal{N}}^{U(N)|U(1)}(t;q) = 0 \text{ for } \left(\mathcal{R}^{(1)}, n^{(2)}\right) \neq ((n), -n).$$

$$\mathbb{II}_{\mathcal{D}}^{U(N)|U(1)}(t;q)=\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}\frac{\left(q^{\frac{3}{4}+\frac{(N-1)}{4}}t^{-1+(N-1)}s^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{(N-1)}{4}}t^{1+(N-1)}s^{\pm};q\right)_\infty}$$
$$\times\prod_{l=1}^{N-1}\frac{\left(q^{\frac{l+1}{2}}t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}}t^{2l};q\right)_\infty}$$

$$\langle V_n\rangle_{\mathcal{D}}^{U(N)|U(1)}(t;q)=q^{\frac{|n|}{4}}t^{-|n|}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}\frac{\left(q^{\frac{3}{4}+\frac{(N-1)}{4}+\frac{|n|}{2}}\right.}{\left(q^{\frac{1}{4}+\frac{(N-1)}{4}+\frac{|n|}{2}}t^{1+(N-1)}s^{\pm};q\right)_\infty}$$
$$\times\prod_{l=1}^{N-1}\frac{\left(q^{\frac{l+1}{2}}t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}}t^{2l};q\right)_\infty}$$

$$\langle V_n\rangle_{\mathcal{D}}^{U(N)|U(1)}(t;q)=\sum_{m=0}^{\infty}\frac{(q^{1+m};q)_\infty\left(q^{\frac{N+1}{2}+|n|+m}t^{2N-2};q\right)_\infty}{\left(q^{\frac{1}{2}+m}t^2;q\right)_\infty\left(q^{\frac{N}{2}+|n|+m}t^{2N};q\right)_\infty}q^{\frac{|n|}{4}+\frac{m}{2}}t^{-|n|-2m}$$
$$\times\prod_{l=1}^{N-1}\frac{\left(q^{\frac{l+1}{2}}t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}}t^{2l};q\right)_\infty}$$

$$\langle W_{(n);-n}\rangle_{\mathcal{N}}^{U(N)|U(1)}(t;q)=\langle V_n\rangle_{\mathcal{D}}^{U(N)|U(1)}(t^{-1};q)$$

$$\mathbb{II}_{\mathcal{N}}^{U(N)|U(1)^{(C)}}(\mathfrak{q})=\frac{1}{1-\mathfrak{q}^2}\prod_{n=1}^{N}\frac{1}{1-\mathfrak{q}^{2n}}$$
$$\mathbb{II}_{\mathcal{N}}^{U(N)|U(1)^{(H)}}(\mathfrak{q})=\frac{1}{1-\mathfrak{q}^2}$$

$$\langle \mathcal{W}_{(n);-n}\rangle_{\mathcal{N}}^{U(N)|U(1)(C)}(\mathfrak{q})=0$$
$$\langle \mathcal{W}_{(n);-n}\rangle_{\mathcal{N}}^{U(N)|U(1)(H)}(\mathfrak{q})=\mathfrak{q}^n$$

$$\langle V_{0;n}\rangle_{\mathcal{N}}^{U(N)|U(1)}(t;q)=q^{\frac{N|n|}{4}}t^{-N|n|}\frac{1}{N!}\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{\left(s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}$$
$$\times\frac{(q)_\infty}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty}\oint\frac{ds^{(2)}}{2\pi i s^{(2)}}\prod_{i=1}^{N}\frac{\left(q^{\frac{3}{4}+\frac{|n|}{2}}t^{-1}s_i^{(1)\mp}s^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{|n|}{2}}t s_i^{(1)\pm}s^{(2)\mp};q\right)_\infty}$$

$$\langle W_n\rangle_{\mathcal{D}}^{U(N)|U(1)}(t;q)=\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}\frac{\left(q^{\frac{3}{4}+\frac{(N-1)}{4}}t^{-1+(N-1)}s^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{(N-1)}{4}}t^{1+(N-1)}s^{\pm};q\right)_\infty}s^n$$
$$\times\prod_{l=1}^{N-1}\frac{\left(q^{\frac{l+1}{2}}t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}}t^{2l};q\right)_\infty}$$

$$\langle W_n\rangle_{\mathcal{D}}^{U(N)|U(1)}(t;q)=\sum_{m=0}^{\infty}\frac{(q^{1+m};q)_\infty\left(q^{\frac{N+1}{2}+m}t^{2N-2};q\right)_\infty}{\left(q^{\frac{1}{2}+m}t^2;q\right)_\infty\left(q^{\frac{N}{2}+m}t^{2N};q\right)_\infty}q^{\frac{Nn}{4}+nm+\frac{m}{2}}t^{Nn-2m}$$
$$\times\prod_{l=1}^{N-1}\frac{\left(q^{\frac{l+1}{2}}t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}}t^{2l};q\right)_\infty}$$

$$\left\langle V_{0;n}\right\rangle_{\mathcal{N}}^{U(N)|U(1)}(t;q)=\langle W_n\rangle_{\mathcal{D}}^{U(N)|U(1)}(t^{-1};q).$$

$$\left\langle \mathcal{V}_{n;0}\right\rangle_{\mathcal{N}}^{U(N)|U(1)^{(C)}}(\mathfrak{q})=\mathfrak{q}^{Nn}$$
$$\left\langle \mathcal{V}_{n;0}\right\rangle_{\mathcal{N}}^{U(N)|U(1)^{(H)}}(\mathfrak{q})=0$$

$$\left\langle W_{\mathcal{R}^{(1)};\mathcal{R}^{(2)}}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)$$
$$=\frac{1}{N!}\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i^{(1)}}{2\pi i s_i^{(1)}}\prod_{i<j}\frac{\left(s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}\chi_{\mathcal{R}^{(1)}}\left(s^{(1)}\right)$$
$$\times\frac{1}{N!}\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i^{(2)}}{2\pi i s_i^{(2)}}\prod_{i<j}\frac{\left(s_i^{(2)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(2)\pm}s_j^{(2)\mp};q\right)_\infty}\chi_{\mathcal{R}^{(2)}}\left(s^{(2)}\right)$$
$$\times\prod_{i=1}^{N}\prod_{j=1}^{N}\frac{\left(q^{\frac{3}{4}}t^{-1}s_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}}ts_i^{(1)\pm}s_j^{(2)\mp};q\right)_\infty}$$

$$
\begin{aligned}
&\left\langle\prod_{i=1}^{m} W_{\mathcal{R}_i^{(1)};\mathcal{R}_i^{(2)}}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)\\
&=\frac{1}{N!}\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i^{(1)}}{2\pi i s_i^{(1)}}\prod_{i<j}\frac{\left(s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}\prod_{i=1}^{m}\chi_{\mathcal{R}_i^{(1)}}\left(s^{(1)}\right)\\
&\times\frac{1}{N!}\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i^{(2)}}{2\pi i s_i^{(2)}}\prod_{i<j}\frac{\left(s_i^{(2)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(2)\pm}s_j^{(2)\mp};q\right)_\infty}\prod_{i=1}^{m}\chi_{\mathcal{R}_i^{(2)}}\left(s^{(2)}\right)\\
&\times\prod_{i=1}^{N}\prod_{j=1}^{N}\frac{\left(q^{\frac{3}{4}}t^{-1}s_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}}ts_i^{(1)\pm}s_j^{(2)\mp};q\right)_\infty}
\end{aligned}
$$

$$
\begin{aligned}
&\left\langle V_{(1^k,0^{N-k})}\right\rangle_{\mathcal{D}}^{U(N)|U(N)}(t;q)\\
&=q^{\frac{k}{4}}t^{-k}\frac{1}{k!\,(N-k)!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\\
&\times\prod_{i=1}^{k}\prod_{j=k+1}^{N}\frac{\left(q^{\frac{1}{2}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}s_i^{\pm}s_j^{\mp};q\right)_\infty}{\left(qt^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qt^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty}\prod_{\substack{1\le i<j\le k\\ k+1\le i<j\le N}}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qs_i^{\pm}s_j^{\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty}\\
&\times\prod_{i=1}^{k}\frac{\left(q^{\frac{5}{4}}t^{-1}s_i^{\mp};q\right)_\infty}{\left(q^{\frac{3}{4}}ts_i^{\pm};q\right)_\infty}\prod_{i=k+1}^{N}\frac{\left(q^{\frac{3}{4}}t^{-1}s_i^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}}ts_i^{\pm};q\right)_\infty}
\end{aligned}
$$

$(2,1^{N-1})$

$$
\begin{aligned}
&\langle V_{(2,1^{N-1})}\rangle_{\mathcal{D}}^{U(N)|U(N)}(t;q)\\
&=q^{\frac{N+1}{4}}t^{-(N+1)}\frac{1}{(N-1)!}\frac{(q)_\infty^{2N}}{(q^{\frac{1}{2}}t^{\pm 2};q)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\\
&\times\prod_{j=2}^{N}\frac{(q^{\frac{1}{2}}s_1^{\pm}s_j^{\mp};q)_\infty(q^{\frac{3}{2}}s_1^{\pm}s_j^{\mp};q)_\infty}{(qt^2s_1^{\pm}s_j^{\mp};q)_\infty(qt^{-2}s_1^{\pm}s_j^{\mp};q)_\infty}\prod_{2\le i<j\le N}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q)_\infty(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty}\\
&\times\frac{(q^{\frac{7}{4}}t^{-1}s_1^{\mp};q)_\infty}{(q^{\frac{5}{4}}ts_1^{\pm};q)_\infty}\prod_{i=2}^{N}\frac{(q^{\frac{3}{4}}t^{-1}s_i^{\mp};q)_\infty}{(q^{\frac{1}{4}}ts_i^{\pm};q)_\infty}.
\end{aligned}
$$

$$
\begin{aligned}
\left\langle W_{(1^k);\overline{(1^k)}}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)&=\left\langle V_{(1^k,0^{N-k})}\right\rangle_{\mathcal{D}}^{U(N)|U(N)}(t^{-1};q),\\
\left\langle W_{(2,1^{N-1});\overline{(2,1^{N-1})}}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)&=\left\langle V_{(2,1^{N-1})}\right\rangle_{\mathcal{D}}^{U(N)|U(N)}(t^{-1};q).
\end{aligned}
$$

$$\mathbb{II}_{\mathcal{N}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = \prod_{n=1}^{N} \frac{1}{(1-\mathfrak{q}^{2n})^2},$$

$$\mathbb{I}_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \prod_{n=1}^{N} \frac{1}{1-\mathfrak{q}^{2n}}.$$

$$\langle \prod_{i=1}^{m} W_{\mathcal{R}_i^{(1)};\mathcal{R}_i^{(2)}} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = \langle \prod_{i=1}^{m} W_{\mathcal{R}_i^{(1)}} \rangle_{\frac{1}{2}\mathrm{BPS}}^{U(N)}(\mathfrak{q}) \times \langle \prod_{i=1}^{m} W_{\mathcal{R}_i^{(2)}} \rangle_{\frac{1}{2}\mathrm{BPS}}^{U(N)}(\mathfrak{q}),$$

$$\langle \prod_{i=1}^{m} W_{\mathcal{R}_i^{(1)};\mathcal{R}_i^{(2)}} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \frac{1}{(N!)^2} \oint \prod_{I=1}^{2}\prod_{i=1}^{N} \frac{ds_i^{(I)}}{2\pi i s_i^{(I)}} \frac{\prod_{I=1}^{2}\prod_{i<j}(1-s_i^{(I)\pm}s_j^{(I)\mp})}{\prod_{i,j} 1-\mathfrak{q}s_i^{(1)\pm}s_j^{(2)\mp}}$$

$$\times \prod_{i=1}^{m} \chi_{\mathcal{R}_i^{(1)}}(s^{(1)})\chi_{\mathcal{R}_i^{(2)}}(s^{(2)})$$

$$\langle \mathcal{W}_{\square;\overline{\square}} \rangle_{\mathcal{N}}^{U(N)|U(N)(C)}(\mathfrak{q}) = 0$$

$$\langle \mathcal{W}_{\square;\overline{\square}} \rangle_{\mathcal{N}}^{U(N)|U(N)\ (H)}(\mathfrak{q}) = \mathfrak{q}\left(1+\mathfrak{q}^2+\cdots+\mathfrak{q}^{2(N-1)}\right)$$

$$\langle \mathcal{W}_{(1^k);\overline{(1^k)}} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = 0,$$

$$\langle \mathcal{W}_{(1^k);\overline{(1^k)}} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^k \binom{N}{k}_{\mathfrak{q}^2}$$

$$\binom{m}{r}_{\mathfrak{q}} = \frac{(\mathfrak{q})_m}{(\mathfrak{q})_r(\mathfrak{q})_{m-r}} = \frac{(1-\mathfrak{q}^m)(1-\mathfrak{q}^{m-1})\cdots(1-\mathfrak{q}^{m-r+1})}{(1-\mathfrak{q})(1-\mathfrak{q}^2)\cdots(1-\mathfrak{q}^r)}$$

$$\langle \mathcal{W}_{(2,1^{N-1});\overline{(2,1^{N-1})}} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = 0,$$

$$\langle \mathcal{W}_{(2,1^{N-1});\overline{(2,1^{N-1})}} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^{N+1} \binom{N}{1}_{\mathfrak{q}^2}$$

$$\langle \mathcal{W}_{(k);\overline{(k)}} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = 0,$$

$$\langle \mathcal{W}_{(k);\overline{(k)}} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^k \binom{N+k-1}{k}_{\mathfrak{q}^2}$$

$$\langle \mathcal{W}_{\lambda;\overline{\lambda}} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^{|\lambda|} \langle \mathcal{W}_{\lambda} \mathcal{W}_{\overline{\lambda}} \rangle_{\frac{1}{2}\mathrm{BPS}}^{U(N)}(\mathfrak{q})$$

$$\langle \mathcal{W}_{\lambda;\emptyset} \mathcal{W}_{\overline{\lambda};\emptyset} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = \langle \mathcal{W}_{\lambda} \mathcal{W}_{\overline{\lambda}} \rangle_{\frac{1}{2}\mathrm{BPS}}^{U(N)}(\mathfrak{q}),$$

$$\langle \mathcal{W}_{\lambda;\emptyset} \mathcal{W}_{\overline{\lambda};\emptyset} \rangle_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \langle \mathcal{W}_{\lambda} \mathcal{W}_{\overline{\lambda}} \rangle_{\frac{1}{2}\mathrm{BPS}}^{U(N)}(\mathfrak{q}).$$

$$\langle V_{(1^k,0^{N-k})} \rangle_{\mathcal{D}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = \mathfrak{q}^k \binom{N}{k}_{\mathfrak{q}^2}$$

$$\frac{1}{N!} \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \frac{\prod_{i \neq j} 1 - \frac{s_i}{s_j}}{\prod_{i,j} 1 - \mathrm{t}\frac{s_i}{s_j}} P_{\mu}(s;\mathrm{t}) P_{\lambda}(s^{-1};\mathrm{t}) = \frac{\delta_{\mu\lambda}}{(\mathrm{t};\mathrm{t})_{N-l(\mu)} \prod_{j \geq 1} (\mathrm{t};\mathrm{t})_{m_j(\mu)}},$$

$$\left\langle V_{(2,1^{N-1})} \right\rangle_{\mathcal{D}}^{U(N)|U(N)}(\mathfrak{q}) = \mathfrak{q}^{N+1} \binom{N}{1}_{\mathfrak{q}^2}$$

$$\left\langle V_{(1^k,0^{N-k});0}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)$$

$$= q^{\frac{k^2}{4}} t^{-k^2} \frac{1}{k!\,(N-k)!} \frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i^{(1)}}{2\pi i s_i^{(1)}}$$

$$\times \prod_{i=1}^{k} \prod_{j=k+1}^{N} \frac{\left(q^{\frac{1}{2}} s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}{\left(q t^{-2} s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty} \prod_{\substack{1\le i<j\le k \\ k+1\le i<j\le N}} \frac{\left(s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}} t^{-2} s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}$$

$$\times \frac{1}{N!} \frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i^{(2)}}{2\pi i s_i^{(2)}} \prod_{i<j} \frac{\left(s_i^{(2)\pm} s_j^{(2)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}} t^{-2} s_i^{(2)\pm} s_j^{(2)\mp};q\right)_\infty}$$

$$\times \prod_{i=1}^{k} \prod_{j=1}^{N} \frac{\left(q^{\frac{5}{4}} t^{-1} s_i^{(1)\mp} s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{3}{4}} t s_i^{(1)\pm} s_j^{(2)\mp};q\right)_\infty} \prod_{i=k+1}^{N} \prod_{j=1}^{N} \frac{\left(q^{\frac{3}{4}} t^{-1} s_i^{(1)\mp} s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}} t s_i^{(1)\pm} s_j^{(2)\mp};q\right)_\infty}$$

$$\left\langle V_{(2,1^{N-1});0}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)$$

$$= q^{\frac{N^2+1}{4}} t^{-(N^2+1)} \frac{1}{(N-1)!} \frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i^{(1)}}{2\pi i s_i^{(1)}}$$

$$\times \prod_{j=2}^{N} \frac{\left(q^{\frac{1}{2}} s_1^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}{\left(q t^{-2} s_1^{(1)\pm} s_j^{(1)\mp};q\right)_\infty} \prod_{2\le i<j\le N} \frac{\left(s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}} t^{-2} s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}$$

$$\times \frac{1}{N!} \frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i^{(2)}}{2\pi i s_i^{(2)}} \prod_{i<j} \frac{\left(s_i^{(2)\pm} s_j^{(2)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}} t^{-2} s_i^{(2)\pm} s_j^{(2)\mp};q\right)_\infty}$$

$$\times \prod_{j=1}^{N} \frac{\left(q^{\frac{7}{4}} t^{-1} s_1^{(1)\mp} s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{5}{4}} t s_1^{(1)\pm} s_j^{(2)\mp};q\right)_\infty} \prod_{i=2}^{N} \prod_{j=1}^{N} \frac{\left(q^{\frac{5}{4}} t^{-1} s_i^{(1)\mp} s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{3}{4}} t s_i^{(1)\pm} s_j^{(2)\mp};q\right)_\infty}$$

$$
\left\langle V_{(1^k,0^{N-k});(1^k,0^{N-k})}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)
$$

$$
=\frac{1}{k!\,(N-k)!}\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i^{(1)}}{2\pi i s_i^{(1)}}
$$

$$
\times\prod_{i=1}^{k}\prod_{j=k+1}^{N}\frac{\left(q^{\frac{1}{2}}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}{\left(qt^{-2}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}\prod_{\substack{1\le i<j\le k\\ k+1\le i<j\le N}}\frac{\left(s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}
$$

$$
\times\frac{1}{k!\,(N-k)!}\frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i^{(2)}}{2\pi i s_i^{(2)}}
$$

$$
\times\prod_{i=1}^{k}\prod_{j=k+1}^{N}\frac{\left(q^{\frac{1}{2}}s_i^{(2)\pm}s_j^{(2)\mp};q\right)_\infty}{\left(qt^{-2}s_i^{(2)\pm}s_j^{(1)\mp};q\right)_\infty}\prod_{1\le i<j\le k}\frac{\left(s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty}
$$

$$
\times\prod_{i=1}^{k}\prod_{j=k+1}^{N}\frac{\left(q^{\frac{5}{4}}t^{-1}s_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{3}{4}}ts_i^{(1)\pm}s_j^{(2)\mp};q\right)_\infty}\prod_{i=k+1}^{N}\prod_{j=1}^{k}\frac{\left(q^{\frac{5}{4}}t^{-1}s_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{3}{4}}ts_i^{(1)\pm}s_j^{(2)\mp};q\right)_\infty}
$$

$$
\times\prod_{i=1}^{k}\prod_{j=1}^{k}\frac{\left(q^{\frac{3}{4}}t^{-1}s_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}}ts_i^{(1)\pm}s_j^{(2)\mp};q\right)_\infty}\prod_{i=k+1}^{N}\prod_{j=k+1}^{N}\frac{\left(q^{\frac{3}{4}}t^{-1}s_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}}ts_i^{(1)\pm}s_j^{(2)\mp};q\right)_\infty}
$$

$$
\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle_{\mathcal{D}}^{U(N)|U(N)}(t;q)
$$

$$
=\frac{1}{N!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qs_i^{\pm}s_j^{\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty}
$$

$$
\times\prod_{i=1}^{N}\frac{\left(q^{\frac{3}{4}}t^{-1}s_i^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}}ts_i^{\pm};q\right)_\infty}\prod_{j=1}^{k}\chi_{\mathcal{R}_j}(s)
$$

$$
\left\langle V_{(1^k,0^{N-k});0}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)=\left\langle W_{(1^k)}\right\rangle_{\mathcal{D}}^{U(N)|U(N)}(t^{-1};q)
$$

$$
\left\langle V_{(2,1^{N-1});0}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)=\left\langle W_{(2,1^{N-1})}\right\rangle_{\mathcal{D}}^{U(N)|U(N)}(t^{-1};q)
$$

$$
\left\langle V_{(1^k,0^{N-k});(1^k,0^{N-k})}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)=\left\langle W_{(1^k)}W_{\overline{(1^k)}}\right\rangle_{\mathcal{D}}^{U(N)|U(N)}(t^{-1};q)
$$

$$\langle W_{\mathcal{R}}\rangle_{\mathcal{D}}^{U(N)|U(N)}(t^{-1};q)$$
$$=\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{(q^{1+n_1};q)_\infty^2\left(q^{\frac{3}{2}+n_1+n_2}t^{-2};q\right)_\infty^2\cdots\left(q^{\frac{N+1}{2}+n_1+\cdots+n_N}t^{-2(N-1)};q\right)_\infty^2}{\left(q^{\frac{1}{2}+n_1}t^{-2};q\right)_\infty^2(q^{1+n_1+n_2}t^{-4};q)_\infty^2\cdots\left(q^{\frac{N}{2}+n_1+\cdots+n_N}t^{-2N};q\right)_\infty^2}$$
$$\times\left(q^{\frac{1}{2}}t^2\right)^{Nn_1+(N-1)n_2+\cdots+n_N}\chi_{\mathcal{R}}\left(s_i=q^{\sum_{j=1}^{i}n_j+\frac{2i-1}{4}}t^{-(2i-1)}\right)$$

$$\left\langle V_{(1,0^{N-1});0}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)=\langle W_{\square}\rangle_{\mathcal{D}}^{U(N)|U(N)}(t^{-1};q)$$
$$=\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{(q^{1+n_1};q)_\infty^2\left(q^{\frac{3}{2}+n_1+n_2}t^{-2};q\right)_\infty^2\cdots\left(q^{\frac{N+1}{2}+n_1+\cdots+n_N}t^{-2(N-1)};q\right)_\infty^2}{\left(q^{\frac{1}{2}+n_1}t^{-2};q\right)_\infty^2(q^{1+n_1+n_2}t^{-4};q)_\infty^2\cdots\left(q^{\frac{N}{2}+n_1+\cdots+n_N}t^{-2N};q\right)_\infty^2}$$
$$\times\left(q^{\frac{1}{2}}t^2\right)^{Nn_1+(N-1)n_2+\cdots+n_N}\left[q^{n_1+\frac{1}{4}}t^{-1}+q^{n_1+n_2+\frac{3}{4}}t^{-3}+\cdots+q^{n_1+\cdots+n_N+\frac{2N-1}{4}}t^{-(2N-1)}\right]$$

$$\left\langle V_{(1^N);0}\right\rangle_{\mathcal{N}}^{U(N)|U(N)}(t;q)=\left\langle W_{(1^N)}\right\rangle_{\mathcal{D}}^{U(N)|U(N)}(t^{-1};q)$$
$$=\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{(q^{1+n_1};q)_\infty^2\left(q^{\frac{3}{2}+n_1+n_2}t^{-2};q\right)_\infty^2\cdots\left(q^{\frac{N+1}{2}+n_1+\cdots+n_N}t^{-2(N-1)};q\right)_\infty^2}{\left(q^{\frac{1}{2}+n_1}t^{-2};q\right)_\infty^2(q^{1+n_1+n_2}t^{-4};q)_\infty^2\cdots\left(q^{\frac{N}{2}+n_1+\cdots+n_N}t^{-2N};q\right)_\infty^2}$$
$$\times\left(q^{\frac{1}{2}}t^2\right)^{Nn_1+(N-1)n_2+\cdots+n_N}q^{Nn_1+(N-1)n_2+\cdots+n_N+\frac{N^2}{4}}t^{-N^2}$$

$$\langle\mathcal{V}_{(1^k,0^{N-k});0}\rangle_{\mathcal{N}}^{U(N)|U(N)^{(C)}}(\mathfrak{q})=\mathfrak{q}^{k^2}\binom{N}{k}_{\mathfrak{q}^2}$$
$$\langle\mathcal{V}_{(1^k,0^{N-k});0}\rangle_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q})=0.$$

$$\langle\mathcal{V}_{(2,1^{N-1});0}\rangle_{\mathcal{N}}^{U(N)|U(N)^{(C)}}(\mathfrak{q})=\mathfrak{q}^{N^2+1}\binom{N}{1}_{\mathfrak{q}^2}$$
$$\langle\mathcal{V}_{(2,1^{N-1});0}\rangle_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q})=0.$$

$$\langle V_{(1^k,0^{N-k});(1^k,0^{N-k})}\rangle_{\mathcal{N}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = \binom{N}{k}_{\mathfrak{q}^2}^2,$$

$$\langle V_{(1^k,0^{N-k});(1^k,0^{N-k})}\rangle_{\mathcal{N}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \binom{N}{k}_{\mathfrak{q}^2}.$$

$$\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle_{\mathcal{D}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = \langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle^{U(N)}{}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}),$$

$$\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle_{\mathcal{D}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \frac{1}{N!}\frac{1}{(1-\mathfrak{q}^2)^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{1-s_i^{\pm}s_j^{\mp}}{1-\mathfrak{q}^2 s_i^{\pm}s_j^{\mp}}$$
$$\times\prod_{i=1}^{N}\frac{1}{1-\mathfrak{q}s_i^{\pm}}\prod_{j=1}^{k}\chi_{\mathcal{R}_j}(s).$$

$$\langle \mathcal{W}_{\mathcal{R}}\rangle_{\mathcal{D}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \chi_{\mathcal{R}}(s_1=\mathfrak{q}, s_2=\mathfrak{q}^3,\cdots,s_N=\mathfrak{q}^{2N-1})$$

$$\langle \mathcal{W}_{(1)}\rangle_{\mathcal{D}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \mathfrak{q}(1+\mathfrak{q}^2+\cdots+\mathfrak{q}^{2(N-1)}),$$

$$\langle \mathcal{W}_{(1^k)}\rangle_{\mathcal{D}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^{k^2}\binom{N}{k}_{\mathfrak{q}^2},$$

$$\langle \mathcal{W}_{(k)}\rangle_{\mathcal{D}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^{k}\binom{N+k-1}{k}_{\mathfrak{q}^2}.$$

$$\langle \mathcal{W}_{(1^k)}\mathcal{W}_{\overline{(1^k)}}\rangle_{\mathcal{D}}^{U(N)|U(N)^{(C)}}(\mathfrak{q}) = \binom{N}{k}_{\mathfrak{q}^2}$$

$$\langle \mathcal{W}_{(1^k)}\mathcal{W}_{\overline{(1^k)}}\rangle_{\mathcal{D}}^{U(N)|U(N)^{(H)}}(\mathfrak{q}) = \binom{N}{k}_{\mathfrak{q}^2}^2$$

$$\left\langle \prod_{i=1}^{m} W_{\mathcal{R}_i^{(1)};\mathcal{R}_i^{(2)}} \right\rangle_{\mathcal{N}}^{U(N)|U(M)} (t;q)$$

$$= \frac{1}{N!} \frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i^{(1)}}{2\pi i s_i^{(1)}} \prod_{i<j} \frac{\left(s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(1)\pm}s_j^{(1)\mp};q\right)_\infty} \prod_{i=1}^{m} \chi_{\mathcal{R}_i^{(1)}}\left(s^{(1)}\right)$$

$$\times \frac{1}{M!} \frac{(q)_\infty^M}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^M} \oint \prod_{i=1}^{M} \frac{ds_i^{(2)}}{2\pi i s_i^{(2)}} \prod_{i<j} \frac{\left(s_i^{(2)\mp} s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}s_i^{(2)\pm}s_j^{(2)\mp};q\right)_\infty} \prod_{i=1}^{m} \chi_{\mathcal{R}_i^{(2)}}\left(s^{(2)}\right)$$

$$\times \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{\left(q^{\frac{3}{4}}t^{-1}s_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}}t s_i^{(1)\mp}s_j^{(2)\pm};q\right)_\infty}$$

$$\left\langle W_{(1^k);\left(\overline{(1k)}^k\right)} \right\rangle_{\mathcal{N}}^{U(N)|U(M)} (t;q) = \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{-2};q^{\frac{1}{2}}t^{-2}\right)_N \left(q^{\frac{N}{2}+1}t^{-2N};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2};q^{\frac{1}{2}}t^{-2}\right)_M \left(q^{\frac{M}{2}+1}t^{-2M};q\right)_\infty}$$

$$\times \sum_{\ell(\lambda)\leq M} \left(q^{\frac{1}{2}}t^2\right)^{|\lambda|+\frac{k}{2}} \sum_{\mu\in V_M^k(\lambda)} \varphi'_{\mu/\lambda}\left(q,q^{\frac{1}{2}}t^{-2}\right) \psi'_{\mu/\lambda}\left(q,q^{\frac{1}{2}}t^{-2}\right)$$

$$\times \prod_{i=1}^{\ell(\mu)} \frac{\left(q^{\frac{N-i+1}{2}}t^{-2N+2i-2};q\right)_{\mu_i} \left(q^{\frac{M-i+1}{2}}t^{-2M+2i-2};q\right)_{\mu_i}}{\left(q^{\frac{N-i}{2}+1}t^{-2N+2i};q\right)_{\mu_i} \left(q^{\frac{M-i}{2}+1}t^{-2M+2i};q\right)_{\mu_i}}$$

$$V_M^k(\lambda) = \{\mu \vdash |\lambda| + k \mid \ell(\mu) \leq M \text{ and } \mu/\lambda \text{ is a vertical strip }\}$$

$$\psi'_{\mu/\lambda}(\mathbf{q},\mathbf{t}) = \psi_{\mu'/\lambda'}(\mathbf{t},\mathbf{q})$$

$$\psi_{\mu/\lambda}(\mathbf{q},\mathbf{t}) = \prod_{1\leq i\leq j\leq \ell(\lambda)} \frac{\left(\mathrm{t}^{j-i+1}\mathbf{q}^{\lambda_i-\lambda_j};\mathbf{q}\right)_{\mu_i-\lambda_i} \left(\mathrm{t}^{j-i}\mathbf{q}^{\lambda_i-\mu_{j+1}+1};\mathbf{q}\right)_{\mu_i-\lambda_i}}{\left(\mathrm{t}^{j-i}\mathbf{q}^{\lambda_i-\lambda_j+1};\mathbf{q}\right)_{\mu_i-\lambda_i} \left(\mathrm{t}^{j-i+1}\mathbf{q}^{\lambda_i-\mu_{j+1}};\mathbf{q}\right)_{\mu_i-\lambda_i}}$$

$$\varphi'_{\mu/\lambda}(\mathbf{q},\mathrm{t}) = \varphi_{\mu'/\lambda'}(\mathrm{t},\mathbf{q})$$

$$\varphi_{\mu/\lambda}(\mathbf{q},\mathrm{t}) = \prod_{1\leq i\leq j\leq \ell(\mu)} \frac{\left(\mathrm{t}^{j-i+1}\mathbf{q}^{\mu_i-\mu_j};\mathbf{q}\right)_{\mu_j-\lambda_j} \left(\mathrm{t}^{j-i}\mathbf{q}^{\lambda_i-\mu_{j+1}+1};\mathbf{q}\right)_{\mu_{j+1}-\lambda_{j+1}}}{\left(\mathrm{t}^{j-i}\mathbf{q}^{\mu_i-\mu_j+1};\mathbf{q}\right)_{\mu_j-\lambda_j} \left(\mathrm{t}^{j-i+1}\mathbf{q}^{\lambda_i-\mu_{j+1}};\mathbf{q}\right)_{\mu_{j+1}-\lambda_{j+1}}}$$

$$\left\langle V_{(1^k,0^{M-k})}\right\rangle_{\mathcal{D}}^{U(N)|U(M)}(t;q)$$

$$= q^{\frac{k}{4}}t^{-k}\frac{1}{k!\,(M-k)!}\frac{(q)_\infty^{2M}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^M}\oint\prod_{i=1}^{M}\frac{ds_i}{2\pi i s_i}$$

$$\times\prod_{i=1}^{k}\prod_{j=k+1}^{M}\frac{\left(q^{\frac{1}{2}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}s_i^{\pm}s_j^{\mp};q\right)_\infty}{\left(qt^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qt^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty}\prod_{\substack{1\le i<j\le k\\ k+1\le i<j\le M}}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qs_i^{\pm}s_j^{\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty}$$

$$\times\prod_{i=1}^{k}\frac{\left(q^{\frac{3}{4}+\frac{(N-M)}{4}+\frac{1}{2}}t^{-1+(N-M)}s_i^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{(N-M)}{4}+\frac{1}{2}}t^{1+(N-M)}s_i^{\pm};q\right)_\infty}\prod_{i=k+1}^{M}\frac{\left(q^{\frac{3}{4}+\frac{(N-M)}{4}}t^{-1+(N-M)}s_i^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{(N-M)}{4}}t^{1+(N-M)}s_i^{\pm};q\right)_\infty}$$

$$\times\prod_{l=1}^{N-M}\frac{\left(q^{\frac{l+1}{2}}t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}}t^{2l};q\right)_\infty}$$

$$\left\langle V_{(2,1^{M-1})}\right\rangle_{\mathcal{D}}^{U(N)|U(M)}(t;q)$$

$$= q^{\frac{M+1}{4}}t^{-(M+1)}\frac{1}{(M-1)!}\frac{(q)_\infty^{2M}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^M}\oint\prod_{i=1}^{M}\frac{ds_i}{2\pi i s_i}$$

$$\times\prod_{j=2}^{M}\frac{\left(q^{\frac{1}{2}}s_1^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_j^{\mp};q\right)_\infty}{\left(qt^2s_1^{\pm}s_j^{\mp};q\right)_\infty\left(qt^{-2}s_1^{\pm}s_j^{\mp};q\right)_\infty}\prod_{2\le i<j\le M}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qs_i^{\pm}s_j^{\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty}$$

$$\times\frac{\left(q^{\frac{3}{4}+\frac{(N-M)}{4}+1}t^{-1+(N-M)}s_1^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{(N-M)}{4}+1}t^{1+(N-M)}s_1^{\pm};q\right)_\infty}\prod_{i=2}^{M}\frac{\left(q^{\frac{3}{4}+\frac{(N-M)}{4}+\frac{1}{2}}t^{-1+(N-M)}s_i^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{(N-M)}{4}+\frac{1}{2}}t^{1+(N-M)}s_i^{\pm};q\right)_\infty}$$

$$\times\prod_{l=1}^{N-M}\frac{\left(q^{\frac{l+1}{2}}t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}}t^{2l};q\right)_\infty}$$

$$\left\langle W_{(1^k);\overline{(1^k)}}\right\rangle_{\mathcal{N}}^{U(N)|U(M)}(t;q) = \left\langle V_{(1^k,0^{M-k})}\right\rangle_{\mathcal{D}}^{U(N)|U(M)}(t^{-1};q),$$

$$\left\langle W_{(2,1^{M-1});\overline{(2,1^{M-1})}}\right\rangle_{\mathcal{N}}^{U(N)|U(M)}(t;q) = \left\langle V_{(2,1^{M-1})}\right\rangle_{\mathcal{D}}^{U(N)|U(M)}(t^{-1};q).$$

$$\langle\prod_{i=1}^{m}W_{\mathcal{R}_i^{(1)};\mathcal{R}_i^{(2)}}\rangle_{\mathcal{N}}^{U(N)|U(M)\,(C)}(\mathfrak{q}) = \langle\prod_{i=1}^{m}W_{\mathcal{R}_i^{(1)}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{U(N)}(\mathfrak{q})\times\langle\prod_{i=1}^{m}W_{\mathcal{R}_i^{(2)}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{U(M)}(\mathfrak{q}),$$

$$\langle\prod_{i=1}^{m} W_{\mathcal{R}_i^{(1)};\mathcal{R}_i^{(2)}}\rangle_{\mathcal{N}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \frac{1}{N!}\oint\prod_{i=1}^{N}\frac{ds_i^{(1)}}{2\pi i s_i^{(1)}}\prod_{i<j}(1-s_i^{(1)\pm}s_j^{(1)\mp})$$
$$\times\frac{1}{M!}\oint\prod_{i=1}^{M}\frac{ds_i^{(2)}}{2\pi i s_i^{(2)}}\prod_{i<j}(1-s_i^{(2)\pm}s_j^{(2)\mp})$$
$$\times\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1}{1-\mathfrak{q}s_i^{(1)\pm}s_j^{(2)\mp}}\prod_{i=1}^{m}\chi_{\mathcal{R}_i^{(1)}}(s^{(1)})\prod_{i=1}^{m}\chi_{\mathcal{R}_i^{(2)}}(s^{(2)}).$$

$$\langle W_{\square;\bar{\square}}\rangle_{\mathcal{N}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \mathfrak{q}\left(1+\mathfrak{q}^2+\cdots+\mathfrak{q}^{2(M-1)}\right).$$

$$\left\langle W_{(1^k);\overline{(1^k)}}\right\rangle_{\mathcal{N}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^k\binom{M}{k}_{\mathfrak{q}^2}$$

$$\left\langle \mathcal{W}_{(2,1^{M-1});(2,1^{M-1})}\right\rangle_{\mathcal{N}}^{U(N)|U(M)^{(H)}} = \mathfrak{q}^{M+1}\binom{M}{1}_{\mathfrak{q}^2}$$

$$\langle W_{(k);(k)}\rangle_{\mathcal{N}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^k\binom{M+k-1}{k}_{\mathfrak{q}^2}$$

$$\langle W_{\lambda;\overline{\lambda}}\rangle_{\mathcal{N}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^{|\lambda|}\langle W_\lambda W_{\overline{\lambda}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{U(M)}(\mathfrak{q}).$$

$$\langle W_{\lambda;\lambda}\rangle_{\mathcal{N}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \frac{1}{N!}\oint\prod_{i=1}^{N}\frac{ds_i^{(1)}}{2\pi i s_i^{(1)}}\prod_{i<j}\left(1-s_i^{(1)\pm}s_j^{(1)\mp}\right)$$
$$\times\frac{1}{M!}\oint\prod_{i=1}^{M}\frac{ds_i^{(2)}}{2\pi i s_i^{(2)}}\prod_{i<j}\left(1-s_i^{(2)\pm}s_j^{(2)\mp}\right)$$
$$\times\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1}{1-\mathfrak{q}s_i^{(1)\pm}s_j^{(2)\mp}}\chi_\lambda(s^{(1)})\chi_\lambda(s^{(2)-1})$$

$$\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1}{1-\mathfrak{q}x_i/y_j} = \sum_{\ell(\mu)\leq M}\mathfrak{q}^{|\mu|}\chi_\mu(x)\chi_\mu(y^{-1})$$

$$\chi_\lambda(x)\chi_\mu(x) = \sum_{\ell(\nu)\leq M}N_{\lambda,\mu}^{\nu}\chi_\nu(x)$$

$$\langle W_{\lambda;\bar{\lambda}}\rangle_{\mathcal{N}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \sum_{\ell(\mu)\leq M}\sum_{\ell(\nu)\leq M}\mathfrak{q}^{|\mu|+|\nu|}\left(N_{\lambda,\mu}^{\nu}\right)^2.$$

$$\langle W_\lambda W_{\bar{\lambda}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{U(M)}(\mathfrak{q}) = \frac{1}{M!}\oint\prod_{i=1}^{M}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\left(1-s^{\pm}s^{\mp}\right)\prod_{i,j=1}^{M}\frac{1}{1-\mathfrak{q}^2 s_i s_j^{-1}}\chi_\lambda(s)\chi_\lambda(s^{-1})$$

$$\langle W_\lambda W_{\bar{\lambda}} \rangle^{U(M)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \sum_{\ell(\mu)\le M} \sum_{\ell(\nu)\le M} \mathfrak{q}^{2|\mu|} \left(N^{\nu}_{\lambda,\mu}\right)^2.$$

$$\langle \mathcal{V}_{(1^k,0^{M-k})} \rangle^{U(N)|U(M)^{(C)}}_{\mathcal{D}} = \mathfrak{q}^k \binom{M}{k}_{\mathfrak{q}^2},$$

$$\langle \mathcal{V}_{(1^k,0^{M-k})} \rangle^{U(N)|U(M)^{(H)}}_{\mathcal{D}} = 0.$$

$$\langle \mathcal{V}_{(2,1^{M-1})} \rangle^{U(N)|U(M)^{(C)}}_{\mathcal{D}} = \mathfrak{q}^{M+1} \binom{M}{1}_{\mathfrak{q}^2},$$

$$\langle \mathcal{V}_{(2,1^{M-1})} \rangle^{U(N)|U(M)^{(H)}}_{\mathcal{D}} = 0.$$

$$\left\langle V_{0;(1^k,0^{M-k})} \right\rangle^{U(N)|U(M)}_{\mathcal{N}} (t;q)$$
$$= q^{\frac{k^2+(N-M)k}{4}} t^{-2(k^2+(N-M)k)} \frac{1}{N!} \frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i^{(1)}}{2\pi i s_i^{(1)}} \prod_{i<j} \frac{\left(s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2} s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}$$
$$\times \frac{1}{k!\,(M-k)!} \frac{(q)_\infty^M}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^M} \oint \prod_{i=1}^{M} \frac{ds_i^{(2)}}{2\pi i s_i^{(2)}} \times \prod_{i=1}^{k} \prod_{j=k+1}^{N} \frac{\left(q^{\frac{1}{2}} s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}{\left(q t^{-2} s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}$$
$$\prod_{1\le i<j\le k} \frac{\left(s_i^{(2)\pm} s_j^{(2)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2} s_i^{(2)\pm} s_j^{(2)\mp};q\right)_\infty}$$
$$\times \prod_{i=1}^{N} \prod_{j=1}^{k} \frac{\left(q^{\frac{5}{4}}t^{-1} s_i^{(1)\mp} s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{3}{4}}t s_i^{(1)\mp} s_j^{(2)\pm};q\right)_\infty} \prod_{i=1}^{N} \prod_{j=k+1}^{M} \frac{\left(q^{\frac{3}{4}}t^{-1} s_i^{(1)\mp} s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{1}{4}}t s_i^{(1)\mp} s_j^{(2)\pm};q\right)_\infty}$$

$$
\begin{aligned}
&\left\langle V_{0;(2,1^{M-1})}\right\rangle_{\mathcal{N}}^{U(N)|U(M)}(t;q)\\
&= q^{\frac{M^2+1}{4}} t^{-(M^2+1)} \frac{1}{N!} \frac{(q)_\infty^N}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i^{(1)}}{2\pi i s_i^{(1)}} \prod_{i<j} \frac{\left(s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2} s_i^{(1)\pm} s_j^{(1)\mp};q\right)_\infty}\\
&\times \frac{1}{(M-1)!} \frac{(q)_\infty^M}{\left(q^{\frac{1}{2}}t^{-2};q\right)_\infty^M} \oint \prod_{i=1}^{M} \frac{ds_i^{(2)}}{2\pi i s_i^{(2)}}\\
&\times \prod_{j=2}^{M} \frac{\left(q^{\frac{1}{2}} s_1^{(2)\pm} s_j^{(2)\mp};q\right)_\infty}{\left(q t^{-2} s_1^{(2)\pm} s_j^{(2)\mp};q\right)_\infty} \prod_{2\le i<j\le M} \frac{\left(s_i^{(2)\pm} s_j^{(2)\mp};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2} s_i^{(2)\pm} s_j^{(2)\mp};q\right)_\infty}\\
&\times \prod_{j=1}^{N} \frac{\left(q^{\frac{7}{4}} t^{-1} s_1^{(1)\mp} s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{5}{4}} t s_1^{(1)\pm} s_j^{(2)\mp};q\right)_\infty} \prod_{i=2}^{N} \prod_{j=1}^{N} \frac{\left(q^{\frac{5}{4}} t^{-1} s_i^{(1)\mp} s_j^{(2)\pm};q\right)_\infty}{\left(q^{\frac{3}{4}} t s_i^{(1)\pm} s_j^{(2)\mp};q\right)_\infty}
\end{aligned}
$$

$$
\begin{aligned}
&\left\langle \prod_{i=1}^{m} W_{\mathcal{R}_i} \right\rangle_{\mathcal{D}}^{U(N)|U(M)}(t;q)\\
&= \frac{1}{M!} \frac{(q)_\infty^{2M}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^M} \oint \prod_{i=1}^{M} \frac{ds_i}{2\pi i s_i} \prod_{i<j} \frac{\left(s_i^{\pm} s_j^{\mp};q\right)_\infty \left(q s_i^{\pm} s_j^{\mp};q\right)_\infty}{\left(q^{\frac{1}{2}} t^2 s_i^{\pm} s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm} s_j^{\mp};q\right)_\infty}\\
&\times \prod_{i=1}^{M} \frac{\left(q^{\frac{3}{4}+\frac{N-M}{4}} t^{-1+(N-M)} s_i^{\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{N-M}{4}} t^{1+(N-M)} s_i^{\pm};q\right)_\infty} \prod_{i=1}^{m} \chi_{\mathcal{R}_i}(s)\\
&\times \prod_{l=1}^{N-M} \frac{\left(q^{\frac{l+1}{2}} t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}} t^{2l};q\right)_\infty}
\end{aligned}
$$

$$
\left\langle V_{0;(1^k,0^{M-k})}\right\rangle_{\mathcal{N}}^{U(N)|U(M)}(t;q) = \left\langle W_{(1^k)}\right\rangle_{\mathcal{D}}^{U(N)|U(M)}(t^{-1};q)
$$

$$
\left\langle V_{0;(2,1^{M-1})}\right\rangle_{\mathcal{N}}^{U(N)|U(M)}(t;q) = \left\langle W_{(2,1^{M-1})}\right\rangle_{\mathcal{D}}^{U(N)|U(M)}(t^{-1};q)
$$

$$\langle W_{\mathcal{R}}\rangle_{\mathcal{D}}^{U(N)|U(M)}(t^{-1};q)$$

$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} \frac{(q^{1+n_1};q)_\infty \left(q^{1+\frac{N-M}{2}+n_1} t^{-2(N-M)};q\right)_\infty}{\left(q^{\frac{1}{2}+n_1} t^{-2};q\right)_\infty \left(q^{\frac{1}{2}+\frac{N-M}{2}+n_1} t^{-2(N-M+1)};q\right)_\infty}$$

$$\times \frac{\left(q^{\frac{3}{2}+n_1+n_2} t^{-2};q\right)_\infty \left(q^{\frac{3}{2}+\frac{N-M}{2}+n_1+n_2} t^{-2(N-M+1)};q\right)_\infty}{(q^{1+n_1+n_2} t^{-4};q)_\infty \left(q^{1+\frac{N-M}{2}+n_1+n_2} t^{-2(N-M+2)};q\right)_\infty} \times \cdots$$

$$\times \frac{\left(q^{\frac{M+1}{2}+n_1+\cdots+n_M} t^{-2(M-1)};q\right)_\infty \left(q^{\frac{N+1}{2}+n_1+\cdots+n_M} t^{-2(N-1)};q\right)_\infty}{\left(q^{\frac{M}{2}+n_1+\cdots+n_M} t^{-2M};q\right)_\infty q^{\frac{N}{2}+n_1+\cdots+n_M} t^{-2N};q\Big)_\infty} (q^{1/2}t^2)^{Mn_1+(M-1)n_2+\cdots+n_M}$$

$$\times \chi_{\mathcal{R}}\left(s_i = q^{\sum_{j=1}^{i} n_j + \frac{N-M+2i-1}{4}} t^{-(N-M+2i-1)}\right) \prod_{l=1}^{N-M} \frac{\left(q^{\frac{l+1}{2}} t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}} t^{2l};q\right)_\infty}$$

$$\left\langle V_{0;(1,0^{M-1})}\right\rangle_{\mathcal{N}}^{U(N)|U(M)}(t;q) = \langle W_{\square}\rangle_{\mathcal{D}}^{U(N)|U(M)}(t^{-1};q)$$

$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} \frac{(q^{1+n_1};q)_\infty \left(q^{1+\frac{N-M}{2}+n_1} t^{-2(N-M)};q\right)_\infty}{\left(q^{\frac{1}{2}+n_1} t^{-2};q\right)_\infty \left(q^{\frac{1}{2}+\frac{N-M}{2}+n_1} t^{-2(N-M+1)};q\right)_\infty}$$

$$\times \frac{\left(q^{\frac{3}{2}+n_1+n_2} t^{-2};q\right)_\infty \left(q^{\frac{3}{2}+\frac{N-M}{2}+n_1+n_2} t^{-2(N-M+1)};q\right)_\infty}{(q^{1+n_1+n_2} t^{-4};q)_\infty \left(q^{1+\frac{N-M}{2}+n_1+n_2} t^{-2(N-M+2)};q\right)_\infty} \times \cdots$$

$$\times \frac{\left(q^{\frac{M+1}{2}+n_1+\cdots+n_M} t^{-2(M-1)};q\right)_\infty \left(q^{\frac{N+1}{2}+n_1+\cdots+n_M} t^{-2(N-1)};q\right)_\infty}{\left(q^{\frac{M}{2}+n_1+\cdots+n_M} t^{-2M};q\right)_\infty q^{\frac{N}{2}+n_1+\cdots+n_M} t^{-2N};q\Big)_\infty \quad Mn_1+(M-1)n_2+\cdots+n_M} (q^{1/2}t^2)^{Mn_1+(M-1)n_2+\cdots+n_M}$$

$$\times \left[q^{n_1+\frac{N-M+1}{4}} t^{-(N-M+1)} + q^{n_1+n_2+\frac{N-M+3}{4}} t^{-(N-M+3)} + \cdots + q^{n_1+\cdots+n_M+\frac{N+M-1}{4}} t^{-(N+M-1)}\right]$$

$$\times \prod_{l=1}^{N-M} \frac{\left(q^{\frac{l+1}{2}} t^{2(l-1)};q\right)_\infty}{\left(q^{\frac{l}{2}} t^{2l};q\right)_\infty}$$

$$\langle V_{0;(1^k,0^{M-k})}\rangle_{\mathcal{N}}^{U(N)|U(M)^{(C)}}(\mathfrak{q}) = \mathfrak{q}^{k^2+(N-M)k} \binom{M}{k}_{\mathfrak{q}^2}$$

$$\langle V_{0;(2,1^{M-1})}\rangle_{\mathcal{N}}^{U(N)|U(M)^{(C)}}(\mathfrak{q}) = \mathfrak{q}^{M^2+1} \binom{M}{1}_{\mathfrak{q}^2}.$$

$$\langle \prod_{i=1}^{m} W_{\mathcal{R}_i} \rangle_{\mathcal{D}}^{U(N)|U(M)^{(C)}}(\mathfrak{q}) = \langle \prod_{i=1}^{m} W_{\mathcal{R}_i} \rangle_{\frac{1}{2}\mathrm{BPS}}^{U(M)}(\mathfrak{q}),$$

$$\langle \prod_{i=1}^{m} W_{\mathcal{R}_i} \rangle_{\mathcal{D}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \frac{1}{M!}\frac{1}{(1-\mathfrak{q}^2)^M} \oint \prod_{i=1}^{M} \frac{ds_i}{2\pi i s_i} \prod_{i<j} \frac{1-s_i^{\pm} s_j^{\mp}}{1-\mathfrak{q}^2 s_i^{\pm} s_j^{\mp}} \times \prod_{i=1}^{M} \frac{1}{1-\mathfrak{q}^{N-M+1} s_i^{\pm}} \prod_{i=1}^{m} \chi_{\mathcal{R}_i}(s) \prod_{l=1}^{N-M} \frac{1}{1-\mathfrak{q}^{2l}}.$$

$$\langle \mathcal{W}_{\mathcal{R}} \rangle_{\mathcal{D}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \chi_{\mathcal{R}}(\mathfrak{q}^{N-M+1}, \mathfrak{q}^{N-M+3}, \cdots, \mathfrak{q}^{N+M-1}).$$

$$\langle \mathcal{W}_{\mathcal{R}} \rangle_{\mathcal{D}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^{\sum_{i\geq 1}(2i+N-M-1)\lambda_i} \prod_{(i,j)\in\lambda} \frac{1-\mathfrak{q}^{2M+2j-2i}}{1-\mathfrak{q}^{2h(i,j)}},$$

$$n(\lambda) = \sum_{i} (i-1)\lambda_i.$$

$$\prod_{i,j\in\lambda} \frac{M+j-i}{h(i,j)},$$

$$\langle \mathcal{W}_{\yng(3,1)} \rangle_{\mathcal{D}}^{U(5)|U(3)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^{14} + 2\mathfrak{q}^{16} + 3\mathfrak{q}^{18} + 3\mathfrak{q}^{20} + 3\mathfrak{q}^{22} + 2\mathfrak{q}^{24} + \mathfrak{q}^{26}.$$

$$n=1 \quad : \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 0 \\ \cline{1-1}\end{array},$$

$$n=2 \quad : \begin{array}{|c|c|c|}\hline 1&1&0\\\hline 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&0&0\\\hline 0 \\ \cline{1-1}\end{array},$$

$$n=3 \quad : \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&1&0\\\hline 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&0&0\\\hline 1 \\ \cline{1-1}\end{array},$$

$$n=4 \quad : \begin{array}{|c|c|c|}\hline 2&1&1\\\hline 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&1&0\\\hline 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&2&0\\\hline 0 \\ \cline{1-1}\end{array},$$

$$n=5 \quad : \begin{array}{|c|c|c|}\hline 2&2&1\\\hline 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&1&1\\\hline 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&2&0\\\hline 1 \\ \cline{1-1}\end{array},$$

$$n=6 \quad : \begin{array}{|c|c|c|}\hline 2&2&2\\\hline 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&2&1\\\hline 1 \\ \cline{1-1}\end{array},$$

$$n=7 \quad : \begin{array}{|c|c|c|}\hline 2&2&2\\\hline 1 \\ \cline{1-1}\end{array}.$$

$$s_{(1^k)}\left(q^a, q^{a+1}, \cdots, q^{a+(N-1)}\right) = q^{\frac{k(k-1)}{2}+ka}\binom{N}{k}_q.$$

$$\left\langle \mathcal{W}_{(1^k)}\right\rangle_{\mathcal{D}}^{U(N)|U(M)^{(H)}}(\mathfrak{q}) = \mathfrak{q}^{k^2+(N-M)k}\binom{M}{k}_{\mathfrak{q}^2},$$

$\langle W_{\square;-1}\rangle_{\mathcal{N}}^{U(2)\vert U(1)}(t;q) = \langle V_{(1,0)}\rangle_{\mathcal{D}}^{U(2)\vert U(1)}(t^{-1};q)$	$tq^{\frac{1}{4}} + (2t^{-1}+t^3)q^{\frac{3}{4}} + (3t^{-3}-t+t^5)q^{\frac{5}{4}} + (4t^{-5}-t^{-1}+t^7)q^{\frac{7}{4}} +(5t^{-7}+t+t^9)q^{\frac{9}{4}} + (6t^{-9}+t^{-5}-3t^{-1}-t^3+t^{11})q^{\frac{11}{4}}+\cdots$
$\langle W_{\square\square;-2}\rangle_{\mathcal{N}}^{U(2)\vert U(1)}(t;q) = \langle V_{(2,0)}\rangle_{\mathcal{D}}^{U(2)\vert U(1)}(t^{-1};q)$	$t^2q^{\frac{1}{2}}+(2+t^4)q+(3t^{-2}-t^2+t^6)q^{\frac{3}{2}}+(-1+4t^{-4}+t^8)q^2 +(5t^{-6}-t^{-2}+t^2+t^{10})q^{\frac{5}{2}}+(-2+6t^{-8}-t^{-4}-t^4+t^{12})q^3+\cdots$
$\langle W_{\square\square\square;-3}\rangle_{\mathcal{N}}^{U(2)\vert U(1)}(t;q) = \langle V_{(3,0)}\rangle_{\mathcal{D}}^{U(2)\vert U(1)}(t^{-1};q)$	$t^3q^{\frac{3}{4}}+(2t+t^5)q^{\frac{5}{4}}+(3t^{-1}-t^3+t^7)q^{\frac{7}{4}}+(4t^{-3}-t+t^9)q^{\frac{9}{4}} +(5t^{-5}-t^{-1}+t^3+t^{11})q^{\frac{11}{4}}+(6t^{-7}-t^{-3}-2t-t^5+t^{13})q^{\frac{13}{4}}\cdots$

$\langle V_{0;1}\rangle_{\mathcal{N}}^{U(2)\vert U(1)}(t;q) = \langle W_1\rangle_{\mathcal{D}}^{U(2)\vert U(1)}(t^{-1};q)$	$t^{-2}q^{\frac{1}{2}}+2t^{-4}q+(4t^{-6}-2t^{-2})q^{\frac{3}{2}}+(1+6t^{-8}-3t^{-4})q^2 +(9t^{-10}-5t^{-6})q^{\frac{5}{2}}+(-1+12t^{-12}-6t^{-8}-2t^{-4})q^3+\cdots$
$\langle V_{0;2}\rangle_{\mathcal{N}}^{U(2)\vert U(1)}(t;q) = \langle W_2\rangle_{\mathcal{D}}^{U(2)\vert U(1)}(t^{-1};q)$	$t^{-4}q+2t^{-6}q^{\frac{3}{2}}+(4t^{-8}-2t^{-4})q^2+(6t^{-10}-3t^{-6})q^{\frac{5}{2}} +(9t^{-12}-5t^{-8}-t^{-4})q^3+(12t^{-14}-6t^{-10}-3t^{-6}+t^{-2})q^{\frac{7}{2}}+\cdots$
$\langle V_{0;3}\rangle_{\mathcal{N}}^{U(2)\vert U(1)}(t;q) = \langle W_3\rangle_{\mathcal{D}}^{U(2)\vert U(1)}(t^{-1};q)$	$t^{-6}q^{\frac{3}{2}}+2t^{-8}q^2+(4t^{-10}-2t^{-6})q^{\frac{5}{2}}+(6t^{-12}-3t^{-8})q^3 +(9t^{-14}-5t^{-10}-t^{-6})q^{\frac{7}{2}}+(12t^{-16}-6t^{-12}-3t^{-8})q^4+\cdots$

$\langle W_{\square;-1}\rangle_{\mathcal{N}}^{U(3)\vert U(1)}(t;q)$ $= \langle V_{(1,0)}\rangle_{\mathcal{D}}^{U(2)\vert U(1)}(t^{-1};q)$	$tq^{\frac{1}{4}} + (2t^{-1}+t^{3})q^{\frac{3}{4}} + (4t^{-3}-t+t^{5})q^{\frac{5}{4}} + (6t^{-5}-t^{-1}+t^{7})q^{\frac{7}{4}}$ $(9t^{-7}-3t^{-3}+t+t^{9})q^{\frac{9}{4}} + (12t^{-9}-2t^{-5}-t^{-1}-t^{3}+t^{11})q^{\frac{11}{4}} + \cdots$
$\langle W_{\square\square;-2}\rangle_{\mathcal{N}}^{U(3)\vert U(1)}(t;q)$ $= \langle V_{(2,0)}\rangle_{\mathcal{D}}^{U(3)\vert U(1)}(t^{-1};q)$	$t^{2}q^{\frac{1}{2}} + (2+t^{-4})q + (4t^{-2}-t^{2}+t^{6})q^{\frac{3}{2}} + (-1+6t^{-4}+t^{8})q^{2}$ $(9t^{-6}-3t^{-2}+t^{2}+t^{10})q^{\frac{5}{2}} + (-1+12t^{-8}-3t^{-4}-t^{4}+t^{12})q^{3} + \cdots$
$\langle W_{\square\square\square;-3}\rangle_{\mathcal{N}}^{U(3)\vert U(1)}(t;q)$ $= \langle V_{(3,0)}\rangle_{\mathcal{D}}^{U(3)\vert U(1)}(t^{-1};q)$	$t^{3}q^{\frac{3}{4}} + (2t+t^{5})q^{\frac{5}{4}} + (4t^{-1}-t^{3}+t^{7})q^{\frac{7}{4}} + (6t^{-3}-t+t^{9})q^{\frac{9}{4}}$ $+(9t^{-5}-3t^{-1}+t^{3}+t^{11})q^{\frac{11}{4}} + (12t^{-7}-3t^{-3}-t-t^{5}+t^{13})q^{\frac{13}{4}} + \cdots$

$\langle V_{0;1}\rangle_{\mathcal{N}}^{U(3)\vert U(1)}(t;q)$ $= \langle W_{1}\rangle_{\mathcal{D}}^{U(3)\vert U(1)}(t^{-1};q)$	$t^{-3}q^{\frac{3}{4}} + 2t^{-5}q^{\frac{5}{4}} + (4t^{-7}-2t^{-3})q^{\frac{7}{4}} + (7t^{-9}-3t^{-5}+t^{-1})q^{\frac{9}{4}}$ $+(11t^{-11}-6t^{-7})q^{\frac{11}{4}} + (16t^{-13}-9t^{-9}-t^{-5}-t^{-1})q^{\frac{13}{4}} + \cdots$
$\langle V_{0;2}\rangle_{\mathcal{N}}^{U(3)\vert U(1)}(t;q)$ $= \langle W_{2}\rangle_{\mathcal{D}}^{U(3)\vert U(1)}(t^{-1};q)$	$t^{-6}q^{\frac{3}{2}} + 2t^{-8}q^{2} + (4t^{-10}-2t^{-6})q^{\frac{5}{2}} + (7t^{-12}-3t^{-8})q^{3}$ $+(11t^{-14}-6t^{-10}-t^{-6})q^{\frac{7}{2}} + (16t^{-16}-9t^{-12}-3t^{-8}+t^{-4})q^{4} + \cdots$
$\langle V_{0;3}\rangle_{\mathcal{N}}^{U(3)\vert U(1)}(t;q)$ $= \langle W_{3}\rangle_{\mathcal{D}}^{U(3)\vert U(1)}(t^{-1};q)$	$t^{-9}q^{\frac{9}{4}} + 2t^{-11}q^{\frac{11}{4}} + (4t^{-13}-2t^{-9})q^{\frac{13}{4}} + (7t^{-15}-3t^{-11})q^{\frac{15}{4}}$ $+(11t^{-17}-6t^{-13}-t^{-11})q^{\frac{17}{4}} + (16t^{-19}-9t^{-15}-3t^{-11})q^{\frac{19}{4}} + \cdots$

$\langle W_{\square;\overline{\square}}\rangle_{\mathcal{N}}^{U(2)\vert U(2)}(t;q)$ $= \langle V_{(1,0)}\rangle_{\mathcal{D}}^{U(2)\vert U(2)}(t^{-1};q)$	$tq^{\frac{1}{4}} + (3t^{-1}+2t^{2})q^{\frac{3}{4}} + (6t^{-3}+t+3t^{5})q^{\frac{5}{4}}$ $+(10t^{-5}-2t^{-1}+4t^{7})q^{\frac{7}{4}} + (15t^{-7}-4t^{-3}+3t+5t^{9})q^{\frac{9}{4}} + \cdots$
$\langle W_{\substack{\square\\\square};\overline{\substack{\square\\\square}}}\rangle_{\mathcal{N}}^{U(2)\vert U(2)}(t;q)$ $= \langle V_{(1,1)}\rangle_{\mathcal{D}}^{U(2)\vert U(2)}(t^{-1};q)$	$t^{2}q^{\frac{1}{2}} + (1+t^{4})q + (2t^{-2}+2t^{6})q^{\frac{3}{2}} + (1+2t^{-4}+2t^{8})q^{2}$ $+(3t^{-6}+2t^{-2}+3t^{10})q^{\frac{5}{2}} + (3t^{-8}+3t^{-4}+t^{4}+3t^{12})q^{3} + \cdots$
$\langle W_{\substack{\square\square\\\square};\overline{\substack{\square\square\\\square}}}\rangle_{\mathcal{N}}^{U(2)\vert U(2)}(t;q)$ $= \langle V_{(2,1)}\rangle_{\mathcal{D}}^{U(2)\vert U(2)}(t^{-1};q)$	$t^{3}q^{\frac{3}{4}} + (2t+2t^{5})q^{\frac{5}{4}} + (3t^{-1}+t^{3}+3t^{7})q^{\frac{7}{4}} + (4t^{-3}+t+4t^{9})q^{\frac{9}{4}}$ $+(5t^{-5}+2t^{-1}+t^{3}+5t^{11})q^{\frac{11}{4}} + (6t^{-7}+3t^{-3}+2t^{5}+6t^{13})q^{\frac{13}{4}} + \cdots$

$\langle V_{0;(1,0)}\rangle_{\mathcal{N}}^{U(2)\vert U(2)}(t;q)$ $= \langle W_{\square}\rangle_{\mathcal{D}}^{U(2)\vert U(2)}(t^{-1};q)$	$t^{-1}q^{\frac{1}{4}} + (3t^{-3}+t)q^{\frac{3}{4}} + (7t^{-5}+t^{3})q^{\frac{5}{4}}$ $+(13t^{-7}-3t^{-3}+t^{5})q^{\frac{7}{4}} + (22t^{-9}-8t^{-5}+t^{-1}+t^{3}+t^{7})q^{\frac{9}{4}} + \cdots$
$\langle V_{(1,0);(1,0)}\rangle_{\mathcal{N}}^{U(2)\vert U(2)}(t;q)$ $= \langle W_{\square}W_{\overline{\square}}\rangle_{\mathcal{D}}^{U(2)\vert U(2)}(t^{-1};q)$	$1 + (4t^{-2}+2t^{2})q^{\frac{1}{2}} + (2+10t^{-4}+3t^{4})q$ $+(20t^{-6}-2t^{-2}+4t^{6})q^{\frac{3}{2}} + (1+35t^{-8}-10t^{-4}+5t^{8})q^{2} + \cdots$
$\langle V_{0;(1,1)}\rangle_{\mathcal{N}}^{U(2)\vert U(2)}(t;q)$ $= \langle W_{\substack{\square\\\square}}\rangle_{\mathcal{D}}^{U(2)\vert U(2)}(t^{-1};q)$	$t^{-4}q + 2t^{-6}q^{\frac{3}{2}} + (5t^{-8}+2t^{-4})q^{2} + (8t^{-10}-4t^{-6}+t^{-2})q^{\frac{5}{2}}$ $+(14t^{-12}-8t^{-8}+t^{-4})q^{3} + \cdots$
$\langle V_{(1,1);(1,1)}\rangle_{\mathcal{N}}^{U(2)\vert U(2)}(t;q)$ $= \langle W_{\substack{\square\\\square}}W_{\overline{\substack{\square\\\square}}}\rangle_{\mathcal{D}}^{U(2)\vert U(2)}(t^{-1};q)$	$1 + (2t^{-2}+t^{2})q^{\frac{1}{2}} + (5t^{-4}+2t^{4})q + (8t^{-6}-t^{-2}+2t^{6})q^{\frac{3}{2}}$ $+(14t^{-8}-4t^{-4}+3t^{8})q^{2} + \cdots$
$\langle V_{0;(2,1)}\rangle_{\mathcal{N}}^{U(2)\vert U(2)}(t;q)$ $= \langle W_{\substack{\square\square\\\square}}\rangle_{\mathcal{D}}^{U(2)\vert U(2)}(t^{-1};q)$	$t^{-5}q^{\frac{5}{4}} + 3t^{-7}q^{\frac{7}{4}} + (7t^{-9}-2t^{-5})q^{\frac{9}{4}} + (13t^{-11}-6t^{-7}+t^{-3})q^{\frac{11}{4}}$ $+(22t^{-13}-12t^{-9}+t^{-5})q^{\frac{13}{4}} + \cdots$

$\langle W_{\square;\overline{\square}}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle V_{(1,0,0)}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$tq^{\frac{1}{4}}+(3t^{-1}+2t^{3})q^{\frac{3}{4}}+(8t^{-3}+3t+4t^{5})q^{\frac{5}{4}}$ $+(16t^{-5}+5t^{-1}+4t^{3}+6t^{7})q^{\frac{7}{4}}+(30t^{-7}+2t^{-3}+4t+3t^{5}+9t^{9})q^{\frac{9}{4}}+\cdots$
$\langle W_{\ydiagram{1,1};\overline{\ydiagram{1,1}}}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle V_{(1,1,0)}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$t^{2}q^{\frac{1}{2}}+(3+2t^{4})q+(7t^{-2}+2t^{2}+4t^{6})q^{\frac{3}{2}}+(2+13t^{-4}+4t^{4}+6t^{8})q^{2}$ $+(22t^{-6}-t^{-2}+6t^{2}+2t^{6}+9t^{10})q^{\frac{5}{2}}+\cdots$
$\langle W_{\ydiagram{1,1,1};\overline{\ydiagram{1,1,1}}}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle V_{(1,1,1)}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$t^{3}q^{\frac{3}{4}}+(t+t^{5})q^{\frac{5}{4}}+(2t^{-1}+t^{7})q^{\frac{7}{4}}+(3t^{-3}+2t+t^{5}+3t^{9})q^{\frac{9}{4}}$ $+(4t^{-5}+2t^{-1}+t^{3}+4t^{11})q^{\frac{11}{4}}+\cdots$
$\langle W_{\ydiagram{2,1,1};\overline{\ydiagram{2,1,1}}}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle V_{(2,1,1)}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$t^{4}q+(2t^{2}+2t^{4})q^{\frac{3}{2}}+(4+2t^{4}+4t^{8})q^{2}+(6t^{-2}+4t^{2}+3t^{6}+6t^{10})q^{\frac{5}{2}}$ $+(5+9t^{-4}+4t^{4}+2t^{8}+9t^{12})q^{3}+\cdots$

$\langle V_{0;(1,0,0)}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle W_{\square}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$t^{-1}q^{\frac{1}{4}}+(3t^{-3}+t)q^{\frac{3}{4}}+(8t^{-5}+t^{-1}+2t^{3})q^{\frac{5}{4}}$ $+(17t^{-7}+t^{-3}+3t+2t^{5})q^{\frac{7}{4}}+\cdots$
$\langle V_{(1,0,0);(1,0,0)}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle W_{\square}W_{\overline{\square}}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$1+(4t^{-2}+2t^{2})q^{\frac{1}{2}}+(4+12t^{-4}+4t^{4})q$ $+(28t^{-6}+7t^{-2}+6t^{2}+6t^{6})q^{\frac{3}{2}}+\cdots$
$\langle V_{0;(1,1,0)}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle W_{\ydiagram{1,1}}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$t^{-4}q+(3t^{-6}+t^{-2})q^{\frac{3}{2}}+(1+8t^{-8})q^{2}$ $+(17t^{-10}-t^{-6}+t^{2})q^{\frac{5}{2}}+(1+33t^{-12}-8t^{-8}+t^{-4}+t^{4})q^{3}+\cdots$
$\langle V_{(1,1,0);(1,1,0)}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle W_{\ydiagram{1,1}}W_{\overline{\ydiagram{1,1}}}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$1+(4t^{-2}+2t^{2})q^{\frac{1}{2}}+(4+12t^{-4}+4t^{4})q$ $+(28t^{-6}+7t^{-2}+6t^{2}+6t^{6})q^{\frac{3}{2}}+\cdots$
$\langle V_{0;(1,1,1)}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle W_{\ydiagram{1,1,1}}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$t^{-9}q^{\frac{9}{4}}+2t^{-11}q^{\frac{11}{4}}+(5t^{-13}-2t^{-9})q^{\frac{13}{4}}$ $+(10t^{-15}-4t^{-11}+t^{-7})q^{\frac{15}{4}}+\cdots$
$\langle V_{(1,1,1);(1,1,1)}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle W_{\ydiagram{1,1,1}}W_{\overline{\ydiagram{1,1,1}}}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$1+(2t^{-2}+t^{2})q^{\frac{1}{2}}+(5t^{-4}+2t^{4})q$ $(3+18t^{-8}-2t^{-4}+4t^{8})q^{2}+\cdots$
$\langle V_{0;(2,1,1)}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle W_{\ydiagram{2,1,1}}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$t^{-10}q^{\frac{5}{2}}+3t^{-12}q^{3}+(8t^{-14}-2t^{-10})q^{\frac{7}{2}}+\cdots$
$\langle V_{(2,1,1);(2,1,1)}\rangle_{\mathcal{N}}^{U(3)\|U(3)}(t;q)$ $=\langle W_{\ydiagram{2,1,1}}W_{\overline{\ydiagram{2,1,1}}}\rangle_{\mathcal{D}}^{U(3)\|U(3)}(t^{-1};q)$	$1+(4t^{-2}+2t^{2})q^{\frac{1}{2}}+(4+12t^{-4}+4t^{4})q$ $+(28t^{-6}+7t^{-2}+6t^{2}+6t^{6})q^{\frac{3}{2}}+\cdots$

$\langle W_{\square;\overline{\square}}\rangle_{\mathcal{N}}^{U(3)\|U(2)}(t;q)$ $=\langle V_{(1,0)}\rangle_{\mathcal{D}}^{U(3)\|U(2)}(t^{-1};q)$	$tq^{\frac{1}{4}}+(3t^{-1}+2t^{3})q^{\frac{3}{4}}+(7t^{-3}+2t+3t^{5})q^{\frac{5}{4}}$ $+(13t^{-5}+t^{-1}+t^{3}+4t^{7})q^{\frac{7}{4}}+(22t^{-7}-3t^{-3};2t+t^{5}+5t^{9})q^{\frac{9}{4}}+\cdots$
$\langle W_{\ydiagram{1,1};\overline{\ydiagram{1,1}}}\rangle_{\mathcal{N}}^{U(3)\|U(2)}(t;q)$ $=\langle V_{(1,1)}\rangle_{\mathcal{D}}^{U(3)\|U(2)}(t^{-1};q)$	$t^{2}q^{\frac{1}{2}}+(2+t^{4})q+(4t^{-2}+2t^{6})q^{\frac{3}{2}}+(6t^{-4}+t^{4}+2t^{8})q^{2}$ $+(9t^{-6}+t^{2}+3t^{10})q^{\frac{5}{2}}+(1+12t^{-8}+2t^{-4}+2t^{4}+t^{8}+3t^{12})q^{3}+\cdots$
$\langle W_{\ydiagram{2,1};\overline{\ydiagram{2,1}}}\rangle_{\mathcal{N}}^{U(3)\|U(2)}(t;q)$ $=\langle V_{(2,1)}\rangle_{\mathcal{D}}^{U(3)\|U(2)}(t^{-1};q)$	$t^{3}q^{\frac{3}{4}}+(3t+2t^{5})q^{\frac{5}{4}}+(6t^{-1}+2t^{3}+3t^{7})q^{\frac{7}{4}}$ $+(10t^{-3}+t+t^{5}+4t^{9})q^{\frac{9}{4}}+(15t^{-5}+t^{-1}+2t^{3}+t^{7}+5t^{11})q^{\frac{11}{4}}+\cdots$

$\langle V_{0;(1,0)}\rangle_{\mathcal{N}}^{U(3)\vert U(2)}(t;q) = \langle W_{\Box}\rangle_{\mathcal{D}}^{U(3)\vert U(2)}(t^{-1};q)$	$t^{-2}q^{\frac{1}{2}}+(1+3t^{-4})q+(7t^{-6}+t^{2})q^{\frac{3}{2}} +(14t^{-8}-2t^{-4}+t^{4})q^{2}+(25t^{-10}-7t^{-6}+t^{-2}+t^{2}+t^{6})q^{\frac{5}{2}}+\cdots$
$\langle V_{0;(1,1)}\rangle_{\mathcal{N}}^{U(3)\vert U(2)}(t;q) = \langle W_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}\rangle_{\mathcal{D}}^{U(3)\vert U(2)}(t^{-1};q)$	$t^{-6}q^{\frac{3}{2}}+2t^{-8}q^{2}+(5t^{-10}-2t^{-6})q^{\frac{5}{2}} +(9t^{-12}-4t^{-8}+t^{-4})q^{3}+(16t^{-14}-9t^{-10}+t^{-6})q^{\frac{7}{2}}+\cdots$
$\langle V_{0;(2,1)}\rangle_{\mathcal{N}}^{U(3)\vert U(2)}(t;q) = \langle W_{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}}\rangle_{\mathcal{D}}^{U(3)\vert U(2)}(t^{-1};q)$	$t^{-8}q^{2}+3t^{-10}q^{\frac{5}{2}}+(7t^{-12}-2t^{-8})q^{3} +(14t^{-14}-6t^{-10}+t^{-6})q^{\frac{7}{2}}+(25t^{-16}-13t^{-12}+t^{-8})q^{4}+\cdots$

$\langle W_{\Box;\overline{\Box}}\rangle_{\mathcal{N}}^{U(4)\vert U(2)}(t;q) = \langle V_{(1,0)}\rangle_{\mathcal{D}}^{U(4)\vert U(2)}(t^{-1};q)$	$tq^{\frac{1}{4}}+(3t^{-1}+2t^{3})q^{\frac{3}{4}}+(7t^{-3}+2t+3t^{5})q^{\frac{5}{4}} +(14t^{-5}+2t^{-1}+t^{3}+4t^{7})q^{\frac{7}{4}}+(25t^{-7}+3t+t^{5}+5t^{9})q^{\frac{9}{4}}+\cdots$
$\langle W_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix};\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}}\rangle_{\mathcal{N}}^{U(4)\vert U(2)}(t;q) = \langle V_{(1,1)}\rangle_{\mathcal{D}}^{U(4)\vert U(2)}(t^{-1};q)$	$t^{2}q^{\frac{1}{2}}+(2+t^{4})q+(5t^{-2}+2t^{6})q^{\frac{3}{2}} (8t^{-4}+t^{4}+2t^{8})q^{2}+(2+20t^{-8}-2t^{-4}+2t^{4}+t^{8}+3t^{12})q^{3}+\cdots$
$\langle W_{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix};\overline{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}}}\rangle_{\mathcal{N}}^{U(4)\vert U(2)}(t;q) = \langle V_{(2,1)}\rangle_{\mathcal{D}}^{U(4)\vert U(2)}(t^{-1};q)$	$t^{3}q^{\frac{3}{4}}+(3t+2t^{5})q^{\frac{5}{4}}+(7t^{-1}+2t^{3}+3t^{7})q^{\frac{7}{4}} +(13t^{-3}+2t+t^{5}+4t^{9})q^{\frac{9}{4}}+(22t^{-5}+3t^{3}+t^{7}+5t^{11})q^{\frac{11}{4}}+\cdots$

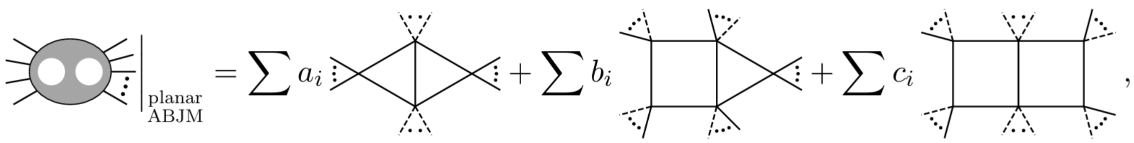
$\langle V_{0;(1,0)}\rangle_{\mathcal{N}}^{U(4)\vert U(2)}(t;q) = \langle W_{\Box}\rangle_{\mathcal{D}}^{U(4)\vert U(2)}(t^{-1};q)$	$t^{-3}q^{\frac{3}{4}}+(3t^{-5}+t^{-1})q^{\frac{5}{4}}+(7t^{-7}+t)q^{\frac{7}{4}} (t^{-3}-2t^{5}+14t^{9})q^{\frac{9}{4}}+(t^{-5}+t^{-1}+t^{3}-6t^{7}+26t^{11})q^{\frac{11}{4}}+\cdots$
$\langle V_{0;(1,1)}\rangle_{\mathcal{N}}^{U(4)\vert U(2)}(t;q) = \langle W_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}\rangle_{\mathcal{D}}^{U(4)\vert U(2)}(t^{-1};q)$	$t^{-8}q^{2}+2t^{-10}q^{\frac{5}{2}}+(5t^{-12}-2t^{-8})q^{3} +(9t^{-14}-4t^{-10}+t^{-6})q^{\frac{7}{2}}+\cdots$
$\langle V_{0;(2,1)}\rangle_{\mathcal{N}}^{U(4)\vert U(2)}(t;q) = \langle W_{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}}\rangle_{\mathcal{D}}^{U(4)\vert U(2)}(t^{-1};q)$	$t^{-11}q^{\frac{11}{4}}+3t^{-13}q^{\frac{13}{4}}+(7t^{-15}-2t^{-11})q^{\frac{15}{4}}+\cdots$

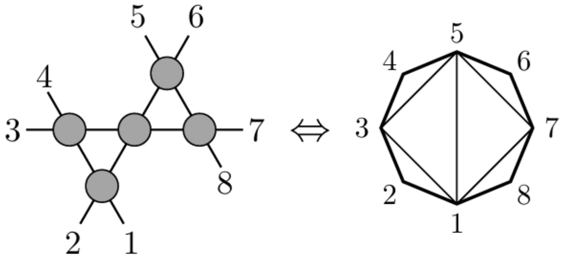
$$\mathrm{BDS}_n^{3\mathrm{D}} := \frac{1}{2}\mathrm{BDS}_n^{4\mathrm{D}} - \frac{(n-3)\pi^2}{6}$$

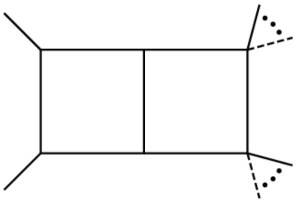
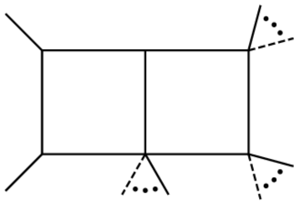
$$\mathrm{BDS}_n^{4\mathrm{D}} = \frac{n}{6}\pi^2 + \frac{1}{2}\sum_{i=1}^{n} \log^2\left(\frac{x_{i,i+3}^2}{x_{i+1,i+3}^2}\right) - \log^2\left(\frac{4\mu_{\mathrm{IR}}^2 x_{i,i+3}^2}{x_{i,i+2}^2 x_{i+1,i+3}^2}\right)$$

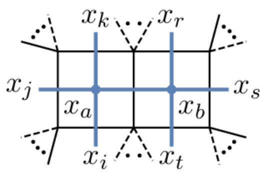
$$-\sum_{i=1}^{n}\sum_{j=0}^{\frac{n}{2}-4} \log\left(\frac{x_{i,i+j+2}^2 x_{i-1,i+j+3}^2}{x_{i,i+j+3}^2 x_{i-1,i+j+2}^2}\right)\log\left(\frac{4\mu_{\mathrm{IR}}^2}{x_{i-1,i+j+2}^2}\right) + \mathrm{Li}_2\left(1-\frac{x_{i,i+j+2}^2 x_{i-1,i+j+3}^2}{x_{i,i+j+3}^2 x_{i-1,i+j+2}^2}\right)$$

$$-\sum_{i=1}^{\frac{n}{2}} \log\left(\frac{x_{i,i+\frac{n}{2}-1}^2 x_{i-1,i+\frac{n}{2}}^2}{x_{i,i+\frac{n}{2}}^2 x_{i-1,i+\frac{n}{2}-1}^2}\right)\log\left(\frac{4\mu_{\mathrm{IR}}^2}{x_{i-1,i+\frac{n}{2}-1}^2}\right) + \mathrm{Li}_2\left(1-\frac{x_{i,i+\frac{n}{2}-1}^2 x_{i-1,i+\frac{n}{2}}^2}{x_{i,i+\frac{n}{2}}^2 x_{i-1,i+\frac{n}{2}-1}^2}\right)$$



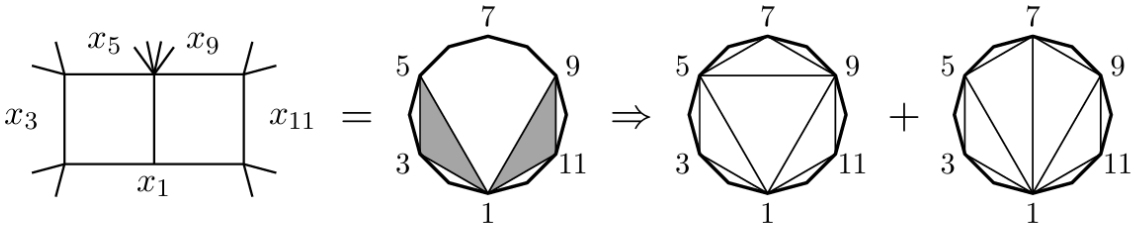


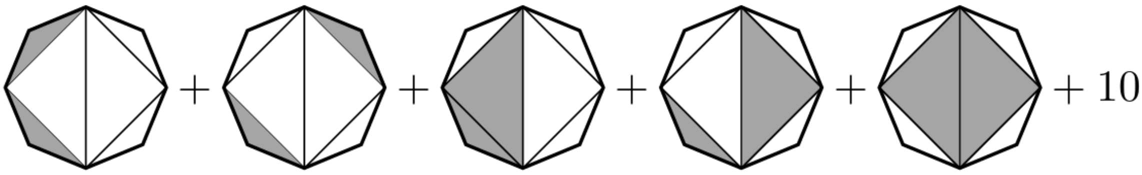
 $\sim \log^2 \mu_{\mathrm{IR}}^2 \,,$  $\sim \log \mu_{\mathrm{IR}}^2 \,.$

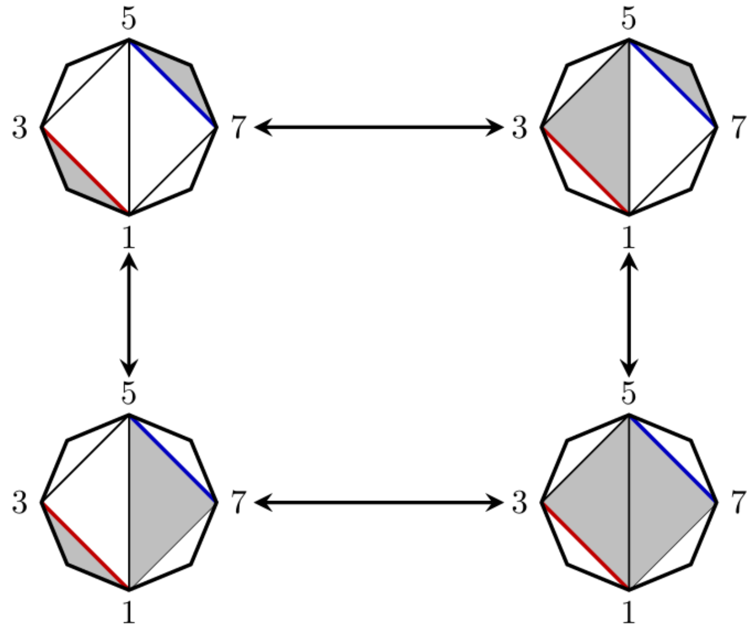
$I^{\mathrm{db}}(i,j,k;r,s,t) :=$ 

$$= \int \frac{\mathcal{G}_{\{a,i,j,k\}}^{\{b,r,s,t\}}\, \mathrm{d}^3 x_a \mathrm{d}^3 x_b}{x_{a,i}^2 x_{a,j}^2 x_{a,k}^2 x_{a,b}^2 x_{b,r}^2 x_{b,s}^2 x_{b,t}^2} \,,$$

$$\mathcal{G}_B^A = \det\left(x_{a,b}^2\right)\big|_{a \in A, b \in B} \text{ and } \mathcal{G}_A := \mathcal{G}_A^B$$

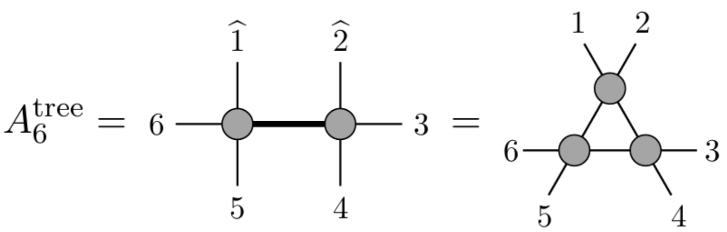


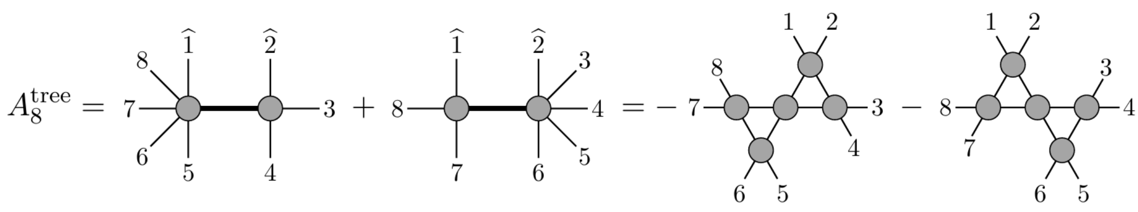


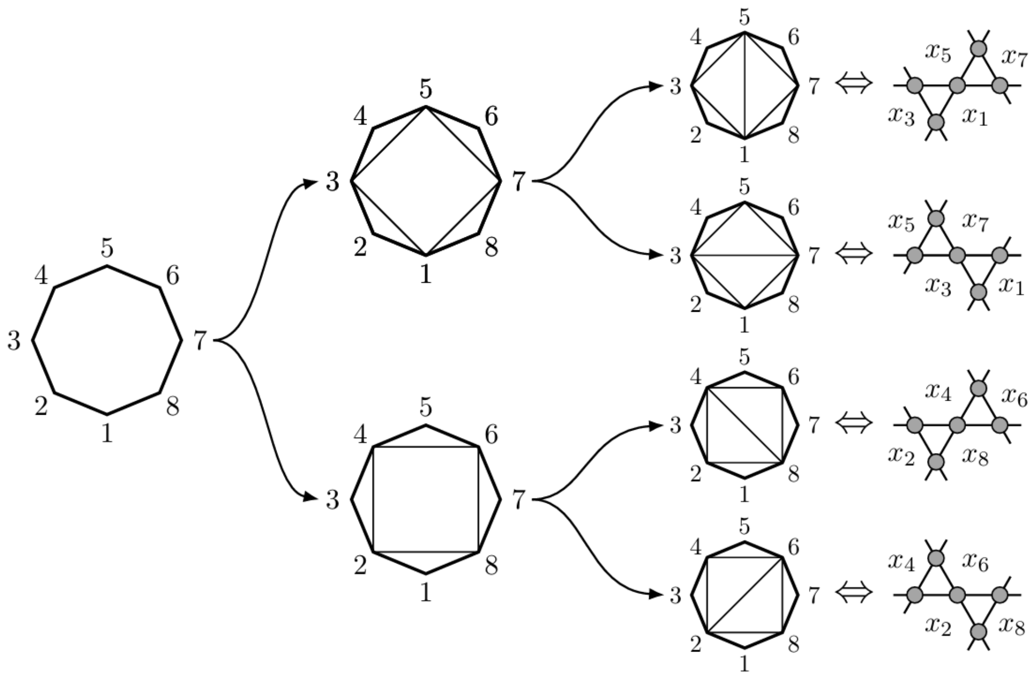


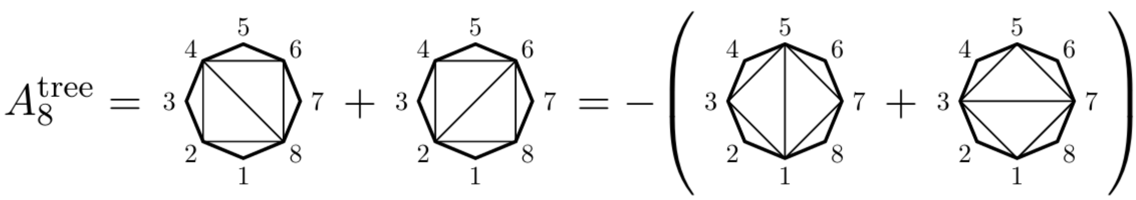
$$I^{\mathrm{db}}(i,j,k;r,s,t) = \mathrm{E}(i,j;r,s) - \mathrm{E}(i,j;r,t) + \mathrm{E}(i,j;s,t) - \mathrm{E}(i,k;r,s) + \mathrm{E}(i,k;r,t) \\ - \mathrm{E}(i,k;s,t) + \mathrm{E}(j,k;r,s) - \mathrm{E}(j,k;r,t) + \mathrm{E}(j,k;s,t)$$

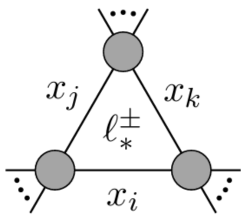
$$\underbrace{\sum_{2\le j-i\le n-2} \mathrm{E}(i,i+1;j,j+1) + \sum_{i=1}^{n} \mathrm{E}(i-1,i;i,i+1)}_{=\frac{1}{2}\mathrm{BDS}_n^{4\mathrm{D}} - n\pi^2/12} - \frac{\pi^2}{6}\left(\frac{n}{2}-3\right),$$









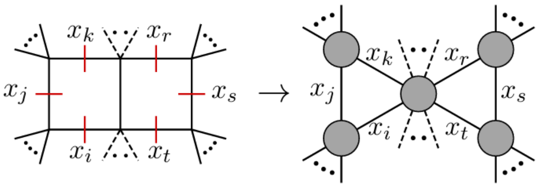


$$\Rightarrow \quad \mathcal{C}^{\pm}_{i,j,k} = \int \prod_{I=1}^{3} \mathrm{d}\eta^{I}_{\ell_i} \mathrm{d}\eta^{I}_{\ell_j} \mathrm{d}\eta^{I}_{\ell_k} \, A^{\text{tree}}_{(ij)} A^{\text{tree}}_{(jk)} A^{\text{tree}}_{(ki)} \Big|_{\ell=\ell^{\pm}_{*}}$$

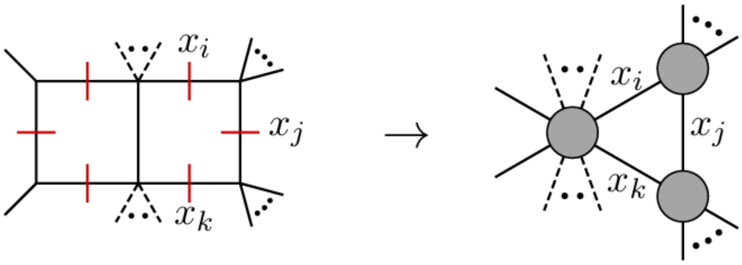
$$I^{\pm}(i,j,k) := \frac{1}{2}\int \left(\frac{\sqrt{\mathcal{G}_{i,j,k}}}{x^2_{a,i} x^2_{a,j} x^2_{a,k}} \mp \frac{\epsilon\left(x_a, x_i, x_j, x_k, x_\rho\right)}{x^2_{a,i} x^2_{a,j} x^2_{a,k} x^2_{a,\rho}} \right) \mathrm{d}^3 x_a,$$

$$\mathcal{D}_{i,j,k} \equiv \frac{\mathcal{C}^{+}_{i,j,k} - \mathcal{C}^{-}_{i,j,k}}{\sqrt{2\mathcal{G}_{\{i,j,k\}}}}.$$

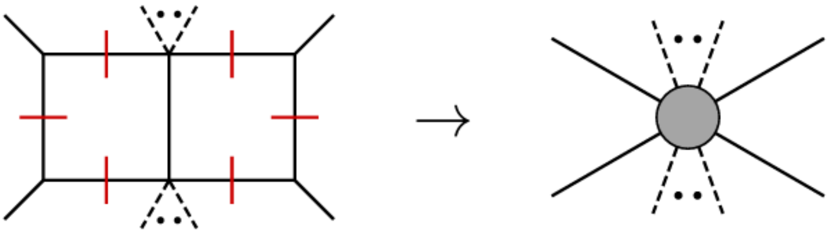
$$A^{\text{tree}}_{n} = (-1)^{i} \sum_{j=1}^{k-2} \mathcal{D}_{i,i+2,i+2+2j}.$$



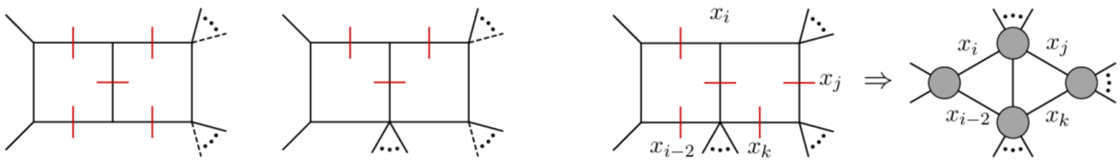
$$= \mathcal{D}^{i,j,k}_{r,s,t} = \sum_{(\sigma,\sigma')\in(\pm,\pm)} \frac{(-1)^{ir}\sigma\sigma'\mathcal{C}^{\sigma,\sigma'}_{i,j,k;r,s,t}}{4\sqrt{\mathcal{G}_{\{i,j,k\}}\mathcal{G}_{\{r,s,t\}}}}$$



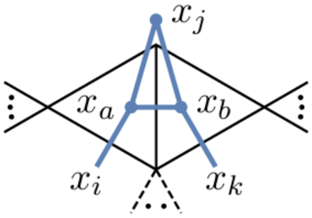
$$= (-1)^i\,\mathcal{D}_{i,j,k}\ ,$$



$$= A_n^{\text{tree}}$$



$$I^{\text{dt}}(i,j;j,k) :=$$



$$= \int \frac{x_{i,j}^2 x_{j,k}^2\, \mathrm{d}^3x_a \mathrm{d}^3x_b}{x_{a,i}^2 x_{a,j}^2 x_{a,b}^2 x_{b,j}^2 x_{b,k}^2}$$

$$\hat{I}^{\text{db}}(i_1,i_2,j;j,k_1,k_2) := I^{\text{db}}(i_1,i_2,j;j,k_1,k_2) - \sum_{\alpha,\beta}(-1)^{\alpha+\beta} I^{\text{dt}}(i_\alpha,j;j,k_\beta)$$

$$\begin{aligned}\hat{I}^{\text{db}}(i,j,k;k,\ell,i) :=\, & I^{\text{db}}(i,j,k;k,\ell,i) + I^{\text{dt}}(i,k;k,i) \\ & - I^{\text{dt}}(i,k;\ell,i) - I^{\text{dt}}(i,j;k,i) + I^{\text{dt}}(i,j;\ell,i) \\ & - I^{\text{dt}}(i,k;k,\ell) + I^{\text{dt}}(j,k;k,\ell) - I^{\text{dt}}(i,k;k,j),\end{aligned}$$

$$\mathrm{E}_{\text{cut}}(i,j;k,\ell): \{x_{a,i}^2 = 0, x_{a,j}^2 = 0, x_{a,b}^2 = 0, x_{b,k}^2 = 0, x_{b,\ell}^2 = 0\}.$$

$$\begin{aligned} I^{\text{db}}(9,10,1;1,3,5)\big|_{\mathrm{E}_{\text{cut}}(9,1;3,5)} &= -I^{\text{db}}(7,9,1;1,3,5)\big|_{\mathrm{E}_{\text{cut}}(9,1;3,5)} \\ &= -I^{\text{db}}(5,9,1;1,3,5)\big|_{\mathrm{E}_{\text{cut}}(9,1;3,5)}\end{aligned}$$

$$\sum_{i\in[n]^{(\mathrm{o})},\{\triangle\}\subset[n]^{(\mathrm{e})}} \mathcal{D}_{\triangle} I^{\mathrm{db}}(i-1,i,i+1;\triangle) + \sum_{\{\triangle_a\}|\{\triangle_b\},\{\triangle_a,\triangle_b\}\subset[n]^{(\mathrm{e})}} \mathcal{D}_{\triangle_a}^{\triangle_b} I^{\mathrm{db}}(\triangle_a;\triangle_b)$$
$$+ \sum_{\{i,j\}\subset[n]^{(\mathrm{o})}} A_n I^{\mathrm{db}}(i-1,i,i+1;j-1,j,j+1) + (\mathrm{e}\leftrightarrow\mathrm{o}),$$

$$\mathcal{D}_{\triangle} = \sum_{\{\triangle\}\in\tau} \mathrm{OSD}_{\tau}, \mathcal{D}_{\triangle_a}^{\triangle_b} = \sum_{\{\{\triangle_a\},\{\triangle_b\}\}\subset\tau} \mathrm{OSD}_{\tau}$$

$$\mathcal{D}_{1,3,5} = \underbrace{\mathrm{OSD}_{\{\{1,3,5\},\{7,9,1\},\{5,7,1\}\}}}_{=\mathcal{D}_{1,3,5}^{7,9,1}} + \underbrace{\mathrm{OSD}_{\{\{1,3,5\},\{5,9,1\},\{5,7,9\}\}}}_{=\mathcal{D}_{1,3,5}^{5,9,1}}$$

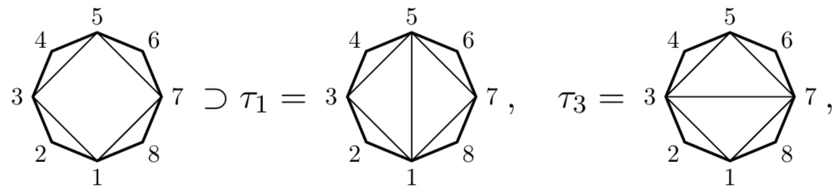
$$\sum_{\tau\in\mathcal{T}_n^{(\mathrm{e})}} \mathrm{ODS}_{\tau} \sum_{\{\{\triangle_a\},\{\triangle_b\}\}\subset\tau_c} I^{\mathrm{db}}(\triangle_a;\triangle_b) + (\mathrm{e}\leftrightarrow\mathrm{o}),$$

$$I^{\mathrm{db}}(i,a,j;d,\ell,k) + I^{\mathrm{db}}(i,a,j;\ell,c,k) + I^{\mathrm{db}}(i,j,b;d,\ell,k) + I^{\mathrm{db}}(i,j,b;\ell,c,k)$$

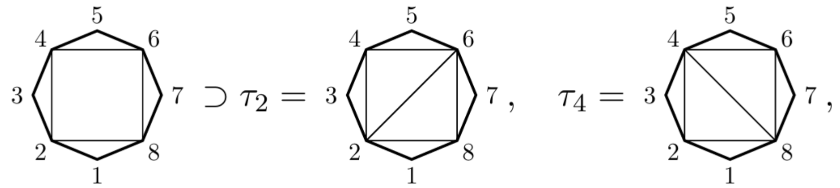
$$\sum_{\tau\in\mathcal{T}_n^{(\mathrm{e})}} \mathrm{ODS}_{\tau} \underbrace{\sum_{\{\{\triangle_a\},\{\triangle_b\}\}\subset\tau_c} \left(I^{\mathrm{db}}(\triangle_a;\triangle_b) + I^{\mathrm{dt}}\right.}_{:=\mathrm{BDS}_{n,\tau}^{3\mathrm{D}}} + (\mathrm{e}\leftrightarrow 0)$$

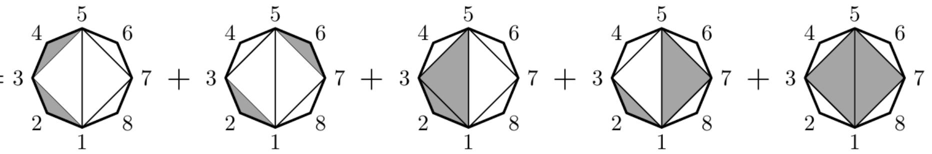
$$\begin{aligned} A_6^{\mathrm{tree}}\,\mathrm{BDS}_6^{3\mathrm{D}} &= \mathrm{OSD}_{\tau_{\mathrm{e}}}\mathrm{BDS}_{6,\tau_{\mathrm{e}}}^{3\mathrm{D}} - \mathrm{OSD}_{\tau_{\mathrm{o}}}\mathrm{BDS}_{6,\tau_{\mathrm{o}}}^{3\mathrm{D}} \\ &= \sum_{\{i,j\}\subset\{2,4,6\}} A_6 \hat{I}^{\mathrm{db}}(i-1,i,i+1;j-1,j,j+1) \\ &\quad + \sum_{i=1,3,5} (-1)^{i-1}\mathcal{D}_{2,4,6}\hat{I}^{\mathrm{db}}\left(i-1,i,i+1;\Delta_{2,4,6}\right) + (\mathrm{e}\leftrightarrow\mathrm{o}) \end{aligned}$$

Odd scaffolding:



Even scaffolding:



$\mathrm{BDS}^{\mathrm{3D}}_{8,\tau_1} =$  $+10$

$$A_8^{\mathrm{tree}}\mathrm{BDS}_8^{\mathrm{3D}} = \sum_{i=1}^{4} \mathrm{OSD}_{\tau_i}\mathrm{BDS}^{\mathrm{3D}}_{8,\tau_i}$$

$$=A_8^{\mathrm{tree}}\sum_{i=1}^{8} \hat{I}^{\mathrm{db}}(i-1,i,i+1;i-3,i-2,i-1) + A_8\sum_{i=1}^{4} I^{\mathrm{db}}(i-1,i,i+1;i+3,i+4,i+5)$$

$$+\sum_{i=1}^{8}(-1)^{i-1}\left[\sum_{j=1}^{2}\mathcal{D}_{i-1,i+1,i+2j+1}\hat{I}^{\mathrm{db}}(i-1,i,i+1;i+1,i+2j+1,i-1)\right.$$

$$\left.+\mathcal{D}_{i-1,i+3,i+5}\hat{I}^{\mathrm{db}}(i-1,i,i+1;i+3,i+5,i-1)+\mathcal{D}_{i+1,i+3,i+5}\hat{I}^{\mathrm{db}}(i-1,i,i+1;i+1,i+3,i+5)\right]$$

$$+\sum_{i=1}^{4}(-1)^{i}\mathcal{D}^{i,i+2,i+4}_{i+4,i+6,i}\hat{I}^{\mathrm{db}}(i,i+2,i+4;i+4,i+6,i)$$

$$I^{\mathrm{db}}(i,j,k;r,s,t) = \mathrm{E}(i,j;r,s) - \mathrm{E}(i,j;r,t) + \mathrm{E}(i,j;s,t) - \mathrm{E}(i,k;r,s) + \mathrm{E}(i,k;r,t)$$
$$-\mathrm{E}(i,k;s,t) + \mathrm{E}(j,k;r,s) - \mathrm{E}(j,k;r,t) + \mathrm{E}(j,k;s,t)$$

$$\mathrm{E}(i,j;k,\ell) = \int \frac{\mathrm{d}c[\,\mathrm{d}\vec{a}_L]\mathrm{d}^2\vec{b}_R}{\pi\sqrt{c}(1+c)}\frac{X_{i\ell}X_{jk}-X_{ik}X_{j\ell}}{4((c+1)X_L^2+2X_L\cdot X_R+X_R^2)^2}$$

$$I^{\mathrm{db}}(i,j,k;r,s,t) = \int \frac{[\mathrm{d}^2\vec{a}_L]\mathrm{d}^3\vec{b}_R}{(-X_L^2)^{1/2}}\frac{3}{16}\left(\frac{5}{2}\frac{\mathcal{G}^{\{r,s,t,Y_{[6]}\}}_{\{i,j,k,Y_{[6]}\}}}{\left(-Y_{[6]}\cdot Y_{[6]}\right)^{7/2}} - \frac{\mathcal{G}^{\{r,s,t\}}_{\{i,j,k\}}}{\left(-Y_{[6]}\cdot Y_{[6]}\right)^{5/2}}\right)$$

$$\frac{1}{A^{(2n-1)/2}B^{1/2}} = \frac{\Gamma(n)}{\Gamma\left(n-\frac{1}{2}\right)\sqrt{\pi}}\int_0^{\infty}\frac{\mathrm{d}c}{\sqrt{c}(A+cB)^n} \quad \text{for } n\in\mathbb{Z}_{\geq 1},$$

$$I^{\mathrm{db}}(i,j,k;r,s,t) = \int \frac{[\mathrm{d}^2\vec{a}_L]\mathrm{d}^3\vec{b}_R\ \mathrm{d}c}{4\pi\sqrt{c}}\left(\frac{6\mathcal{G}^{\{r,s,t,Y_{[6]}\}}_{\{i,j,k,Y_{[6]}\}}}{\left(-Y_{[6]}\cdot Y_{[6]}-cX_L^2\right)^4} - \frac{2\mathcal{G}^{\{r,s,t\}}_{\{i,j,k\}}}{\left(-Y_{[6]}\cdot Y_{[6]}-cX_L^2\right)^3}\right),$$

$$\mathrm{E}(i,j;j,k) = -1+\frac{\pi^2}{24}+\frac{3\log^2 2}{4}+\frac{1}{2}\log\frac{4\mu_{\mathrm{IR}}^2x_{i,k}^2}{x_{i,j}^2x_{j,k}^2}-\frac{1}{2}\log 2\log\frac{4\mu_{\mathrm{IR}}^2x_{i,k}^2}{x_{i,j}^2x_{j,k}^2},$$

$$\mathrm{E}(i,j;j,i) = -\frac{\pi^2}{12}+\frac{3\log^2 2}{2}+\log\frac{4\mu_{\mathrm{IR}}^2}{x_{i,j}^2}-\log 2\log\frac{4\mu_{\mathrm{IR}}^2}{x_{i,j}^2},$$

$$\mathrm{E}(i,j;j,j+1) = -\frac{\pi^2}{24}+\frac{\log^2 2}{4}-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i,j+1}^2}{x_{i,j}^2}\right)+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i,j+1}^2}{2x_{i,j}^2}\right),$$

$$\mathrm{E}(i-1,i;i,j) = -\frac{\pi^2}{24}+\frac{\log^2 2}{4}-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i-1,j}^2}{x_{i,j}^2}\right)+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i-1,j}^2}{2x_{i,j}^2}\right)$$

$$\mathrm{E}(i-1,i;i,i+1) = -\frac{\pi^2}{24}+\frac{\log^2 2}{4}-\frac{1}{2}\log 2\log\frac{4\mu_{\mathrm{IR}}^2}{x_{i-1,i+1}^2}$$

$$\mathrm{E}(i+2,i;i,i+1) = \mathrm{E}(i-1,i;i,i-2) = -\frac{\pi^2}{24}+\frac{\log^2 2}{4}$$

$$\mathrm{E}(i,i+1;k,\ell) = -\frac{1}{8}\log^2\left(\frac{x_{i,\ell}^2 x_{i+1,k}^2}{x_{i,k}^2 x_{i+1,\ell}^2}\right)-\frac{1}{4}\log\left(\frac{x_{i,\ell}^2 x_{i+1,k}^2}{x_{i,k}^2 x_{i+1,\ell}^2}\right)\log\left(\frac{4\mu_{\mathrm{IR}}^2 x_{k,\ell}^2}{x_{i+1,\ell}^2 x_{i+1,k}^2}\right)$$

$$+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i+1,k}^2}{x_{i,k}^2}\right)-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i+1,\ell}^2}{x_{i,\ell}^2}\right)-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i,\ell}^2 x_{i+1,k}^2}{x_{i,k}^2 x_{i+1,\ell}^2}\right),$$

$$\mathrm{E}(i,i+1;j,j+1) = \frac{1}{2}\log\frac{2\mu_{\mathrm{IR}}^2}{x_{i+1,j+1}^2}\log\left(\frac{x_{i,j}^2 x_{i+1,j+1}^2}{x_{i,j+1}^2}x_{i+1,j}^2\right)-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{i,j}^2 x_{i+1,j+1}^2}\right)$$

$$+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i,j+1}^2}{x_{i,j}^2}\right)+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i+1,j}^2}{x_{i,j}^2}\right)$$

$$-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i+1,j+1}^2}{x_{i,j+1}^2}\right)-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i+1,j+1}^2}{x_{i+1,j}^2}\right),$$

$$\mathrm{E}(i-1,i;i+1,j) = \mathrm{E}(j,i-1;i,i+1) = -\frac{\pi^2}{24}-\frac{1}{8}\log^2\left(\frac{4\mu_{\mathrm{IR}}^2 x_{i-1,j}^2 x_{i+1,j}^2}{x_{i-1,i+1}^2 x_{i,j}^4}\right)$$

$$\mathrm{E}(i,i+1;i+2,i+3) = \frac{\pi^2}{24}-\frac{1}{4}\log^2\left(\frac{2\mu_{\mathrm{IR}}^2}{x_{i,i+2}^2 x_{i+1,i+3}^2}\mathrm{Li}_2\left(1-\frac{x_{i,j}^2}{x_{i-1,j}^2}\right)-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i,j}^2}{2x_{i+1,j}^2}\right),\right.$$

$$\left.-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right)-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i+1,i+3}^2}{x_{i,i+3}^2}\right),\right.$$

$$\hat{\mathrm{E}}(i,j;j,k) := \mathrm{E}(i,j;j,k)+\hat{I}^{\mathrm{dt}}(i,j;j,k)$$

$$= \int \frac{\mathrm{d}c[\,\mathrm{d}\vec{a}_L]\mathrm{d}^2\vec{b}_R}{\pi\sqrt{c}}\left(\frac{-1}{1+c}+1\right)\frac{X_{ij}X_{jk}}{4((c+1)A^2+2A\cdot B+B^2)^2}$$

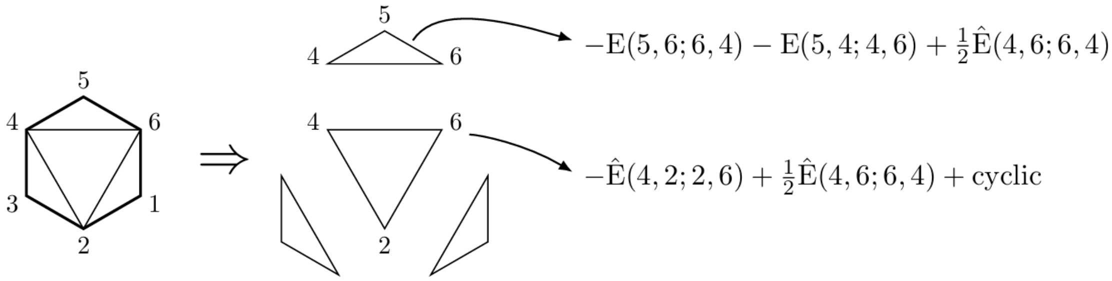
$$\hat{\mathrm{E}}(i,j;j,k) = \frac{\pi^2}{24}+\frac{3\log^2 2}{2}-\frac{1}{2}\log 2\log\frac{4\mu_{\mathrm{IR}}^2 x_{i,k}^2}{x_{i,j}^2 x_{j,k}^2}$$

$$\hat{\mathrm{E}}(i,j;j,i) = -\frac{\pi^2}{12}+\frac{3\log^2 2}{2}-\log 2\log\frac{4\mu_{\mathrm{IR}}^2}{x_{i,j}^2}$$

$$\mathrm{E}(i,j;k,\ell) = \frac{1}{2\pi}\int_0^\infty \frac{(v-1)\mathrm{d}c}{(1+c)\sqrt{c}\Delta(c)}\left[\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar{z})+\frac{1}{2}\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}\right]$$

$$u = \frac{x_{i,j}^2 x_{k,\ell}^2}{x_{i,k}^2 x_{j,\ell}^2}, v = \frac{x_{i,\ell}^2 x_{j,k}^2}{x_{i,k}^2 x_{j,\ell}^2}$$

$$z,\bar{z} = \frac{1-(1+c)u+v\pm\Delta}{2}, \Delta(c) = \sqrt{(1-(1+c)u-v)^2-4(1+c)uv}$$



$$\begin{aligned}\mathrm{BDS}^{3\mathrm{D}}_{n,\tau} = &\sum_{2\le j-i\le n-2} \mathrm{E}(i,i+1;j,j+1) + \sum_{i\in[n]_\mathrm{e}} \mathrm{E}(i-1,i;i,i+1)\\ &+ \sum_{i\in[n]_\mathrm{e}} \left(-\mathrm{E}(i-1,i;i,i-2) - \mathrm{E}(i-1,i-2;i-2,i) + \frac{1}{2}\hat{\mathrm{E}}(i-2,i;i,i-2)\right)\\ &+ \sum_{\{i,j,k\}\in\tau} \left(-\hat{\mathrm{E}}(i,j;j,k) + \frac{1}{2}\hat{\mathrm{E}}(i,k;k,i) + \text{ cyclic in } i,j,k\right)\end{aligned}$$

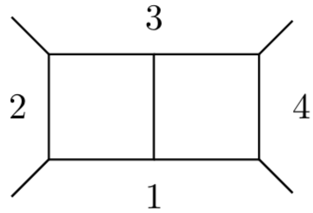
$$-\hat{\mathrm{E}}(i,j;j,k) + \frac{1}{2}\hat{\mathrm{E}}(i,k;k,i) + \text{ cyclic in } i,j,k = -\frac{\pi^2}{4}.$$

$$\begin{aligned}&-\mathrm{E}(i-1,i;i,i-2) - \mathrm{E}(i-1,i-2;i-2,i) + \frac{1}{2}\hat{\mathrm{E}}(i-2,i;i,i-2)\\ =&\frac{\pi^2}{24} + \frac{\log^2 2}{4} - \frac{1}{2}\log 2\log\frac{4\mu_{\mathrm{IR}}^2}{x_{i-2,i}^2} = \mathrm{E}(i-2,i-1;i-1,i) + \frac{\pi^2}{12}.\end{aligned}$$

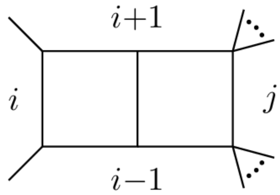
$$\mathrm{BDS}^{3\mathrm{D}}_{n,\tau} = \sum_{2\le j-i\le n-2} \mathrm{E}(i,i+1;j,j+1) + \sum_{i\in[n]} \mathrm{E}(i-1,i;i,i+1) - \frac{\pi^2}{12}(n-6),$$

$$\begin{aligned}&\mathrm{E}(i,i+1;i+2,i+3) + \frac{1}{2}\mathrm{E}(i,i+1;i+1,i+2) + \frac{1}{2}\mathrm{E}(i+1,i+2;i+2,i+3)\\ &= \frac{1}{2}I_{1\mathrm{mb}}(i,i+1,i+2,i+3) - \frac{\pi^2}{12} + \frac{1}{4}\log 2\log\left(\frac{x_{i,i+3}^4}{x_{i,i+2}^2 x_{i+1,i+3}^2}\right)\\ &\mathrm{E}(i,i+1;j,j+1) = \frac{1}{2}I_{2\mathrm{me}}(i,i+1,j,j+1) - \frac{1}{2}\log 2\log\left(\frac{x_{i,j}^2 x_{i+1,j+1}^2}{x_{i,j+1}^2 x_{i+1,j}^2}\right)\end{aligned}$$

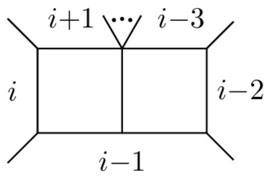
$$\begin{aligned}\mathrm{BDS}^{3\mathrm{D}}_{n} =&\frac{1}{2}\left(\sum_{i\in[n]} I_{1\mathrm{mb}}(i,i+1,i+2,i+3) + \sum_{2<j-i<n-2} I_{2\mathrm{me}}(i,i+1,j,j+1)\right) - \frac{\pi^2}{12}(2n-6)\\ &+ \sum_{i\in[n]} \frac{1}{4}\log 2\log\left(\frac{x_{i,i+3}^4}{x_{i,i+2}^2 x_{i+1,i+3}^2}\right) - \sum_{2<j-i<n-2} \frac{1}{2}\log 2\log\left(\frac{x_{i,j}^2 x_{i+1,j+1}^2}{x_{i,j+1}^2 x_{i+1,j}^2}\right)\\ =&\frac{1}{2}\mathrm{BDS}^{4\mathrm{D}}_n - \frac{\pi^2}{6}(n-3)\end{aligned}$$

$$\hat{I}^{\mathrm{db}}(1,2,3;3,4,1) :=$$ 

$$= \frac{\pi^2}{3} - \log\frac{4\mu_{\mathrm{IR}}^2}{x_{1,3}^2}\log\frac{4\mu_{\mathrm{IR}}^2}{x_{2,4}^2}.$$

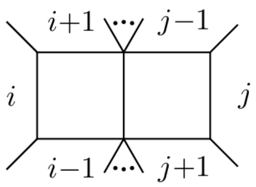
$$\hat{I}^{\mathrm{db}}(i{-}1,i,i{+}1;i{+}1,j,i{-}1) :=$$ 

$$= -\frac{\pi^2}{4} - \frac{1}{4}\log^2\left(\frac{4\mu_{\mathrm{IR}}^2 x_{i-1,j}^2 x_{i+1,j}^2}{x_{i-1,i+1}^2 x_{i,j}^4}\right) - \mathrm{Li}_2\left(1-\frac{x_{i,j}^2}{x_{i-1,j}^2}\right) - \mathrm{Li}_2\left(1-\frac{x_{i,j}^2}{x_{i+1,j}^2}\right).$$

$$\hat{I}^{\mathrm{db}}(i{-}1,i,i{+}1;i{-}3,i{-}2,i{-}1) :=$$ 

$$= \frac{\pi^2}{4} + \frac{1}{2}\log\left(\frac{x_{i,i-2}^2 x_{i+1,i-3}^2}{x_{i,i-3}^2 x_{i+1,i-2}^2}\right)\log\left(\frac{4\mu_{\mathrm{IR}}^2 x_{i+1,i-3}^2}{x_{i-1,i+1}^2 x_{i-1,i-3}^2}\right) - \frac{1}{4}\log^2\left(\frac{4\mu_{\mathrm{IR}}^2 x_{i+1,i-3}^2}{x_{i-1,i+1}^2 x_{i-1,i-3}^2}\right)$$

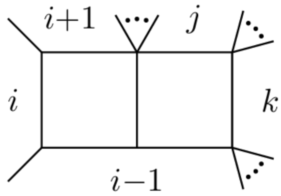
$$- \mathrm{Li}_2\left(1-\frac{x_{i-1,i+1}^2}{x_{i+1,i-2}^2}\right) - \mathrm{Li}_2\left(1-\frac{x_{i-1,i-3}^2}{x_{i,i-3}^2}\right) + \frac{1}{2}\mathrm{Li}_2\left(1-\frac{x_{i,i-3}^2 x_{i+1,i-2}^2}{x_{i,i-2}^2 x_{i+1,i-3}^2}\right)$$

$$I^{\mathrm{db}}(i{-}1,i,i{+}1;j{-}1,j,j{+}1) :=$$ 

$$= \frac{1}{2}\log 2\log\left(\frac{x_{i-1,j+1}^2 x_{i+1,j-1}^2}{x_{i-1,j-1}^2 x_{i+1,j+1}^2}\right) + \frac{1}{2}\log\left(\frac{x_{i-1,j-1}^2 x_{i,j}^2}{x_{i-1,j}^2 x_{i,j-1}^2}\right)\log\left(\frac{x_{i-1,j+1}^2 x_{i+1,j-1}^2}{x_{i-1,j-1}^2 x_{i+1,j+1}^2}\right)$$
$$+ \frac{1}{4}\log^2\left(\frac{x_{i-1,j-1}^2 x_{i,j}^2}{x_{i-1,j}^2 x_{i,j-1}^2}\right) + \frac{1}{4}\log\left(\frac{x_{i-1,j+1}^2 x_{i+1,j-1}^2}{x_{i-1,j-1}^2 x_{i+1,j+1}^2}\right)\log\left(\frac{x_{i-1,i+1}^2 x_{j-1,j+1}^2}{x_{i-1,j-1}^2 x_{i+1,j+1}^2}\right)$$

$$+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i-1,j-1}x^2_{i,j}}{x^2_{i-1,j}x^2_{i,j-1}}\right)-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i-1,j-1}x^2_{i,j+1}}{x^2_{i-1,j+1}x^2_{i,j-1}}\right)+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i-1,j}x^2_{i,j+1}}{x^2_{i-1,j+1}x^2_{i,j}}\right)$$
$$-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i-1,j-1}x^2_{i+1,j}}{x^2_{i-1,j}x^2_{i+1,j-1}}\right)+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i,j-1}x^2_{i+1,j}}{x^2_{i,j}x^2_{i+1,j-1}}\right)+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i,j+1}x^2_{i+1,j-1}}{x^2_{i,j-1}x^2_{i+1,j+1}}\right)$$
$$+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i-1,j+1}x^2_{i+1,j}}{x^2_{i-1,j}x^2_{i+1,j+1}}\right)-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i,j+1}x^2_{i+1,j}}{x^2_{i,j}x^2_{i+1,j+1}}\right)+\mathrm{E}(i-1,i+1;j-1,j+1)\,.$$

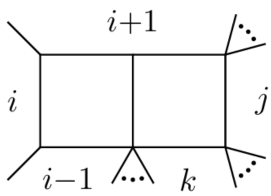
$$\hat{I}^{\mathrm{db}}(i-1,i,i+1;j,k,i-1):=$$



$$=\frac{1}{2}\log 2\log\left(\frac{x^2_{i-1,j}x^2_{i+1,k}}{x^2_{i-1,k}x^2_{i+1,j}}\right)+\frac{1}{2}\log\left(\frac{x^2_{i-1,j}x^2_{i,k}}{x^2_{i-1,k}x^2_{i,j}}\right)\log\left(\frac{4\mu^2_{\mathrm{IR}}x^2_{j,k}}{x^2_{i-1,j}x^2_{i-1,k}}\right)$$

$$+\frac{1}{4}\log\left(\frac{x^2_{i-1,j}x^2_{i+1,k}}{x^2_{i-1,k}x^2_{i+1,j}}\right)\log\left(\frac{x^2_{i-1,i+1}x^2_{j,k}}{x^2_{i-1,j}x^2_{i+1,k}}\right)-\frac{1}{2}\log\left(\frac{x^2_{i-1,j}x^2_{i,k}}{x^2_{i-1,k}x^2_{i,j}}\right)\log\left(\frac{x^2_{i-1,i+1}x^2_{j,k}}{x^2_{i-1,j}x^2_{i+1,k}}\right)$$
$$-\mathrm{Li}_2\left(1-\frac{x^2_{i-1,j}}{x^2_{i,j}}\right)+\mathrm{Li}_2\left(1-\frac{x^2_{i-1,k}}{x^2_{i,k}}\right)+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i-1,j}x^2_{i,k}}{x^2_{i-1,k}x^2_{i,j}}\right)$$
$$-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i,k}x^2_{i+1,j}}{x^2_{i,j}x^2_{i+1,k}}\right)-\mathrm{E}(i-1,i+1;j,k)\,.$$

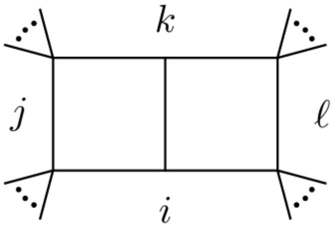
$$\hat{I}^{\mathrm{db}}(i-1,i,i+1;i+1,j,k):=$$



$$=\frac{1}{2}\log 2\log\left(\frac{x^2_{i-1,j}x^2_{i+1,k}}{x^2_{i-1,k}x^2_{i+1,j}}\right)+\frac{1}{2}\log\left(\frac{x^2_{i,j}x^2_{i+1,k}}{x^2_{i,k}x^2_{i+1,j}}\right)\log\left(\frac{4\mu^2_{\mathrm{IR}}x^2_{j,k}}{x^2_{j,i+1}x^2_{k,i+1}}\right)$$

$$+\frac{1}{4}\log\left(\frac{x^2_{i-1,j}x^2_{i+1,k}}{x^2_{i-1,k}x^2_{i+1,j}}\right)\log\left(\frac{x^2_{i-1,i+1}x^2_{j,k}}{x^2_{i-1,j}x^2_{i+1,k}}\right)-\frac{1}{2}\log\left(\frac{x^2_{i,j}x^2_{i+1,k}}{x^2_{i,k}x^2_{i+1,j}}\right)\log\left(\frac{x^2_{i-1,i+1}x^2_{j,k}}{x^2_{i-1,j}x^2_{i+1,k}}\right)$$
$$+\mathrm{Li}_2\left(1-\frac{x^2_{i+1,j}}{x^2_{i,j}}\right)-\mathrm{Li}_2\left(1-\frac{x^2_{i+1,k}}{x^2_{i,k}}\right)-\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i-1,k}x^2_{i,j}}{x^2_{i-1,j}x^2_{i,k}}\right)$$
$$+\frac{1}{2}\mathrm{Li}_2\left(1-\frac{x^2_{i,j}x^2_{i+1,k}}{x^2_{i,k}x^2_{i+1,j}}\right)-\mathrm{E}(i-1,i+1;j,k)\,.$$

$$\hat{I}^{\mathrm{db}}(i,j,k;k,\ell,i) :=$$



$$= -\frac{\pi^2}{6} + \frac{1}{2}\log 2 \log\left(\frac{x_{i,\ell}^2 x_{j,k}^2}{x_{i,k}^2 x_{j,\ell}^2}\right) + \frac{1}{2}\log 2 \log\left(\frac{x_{i,j}^2 x_{k,\ell}^2}{x_{i,k}^2 x_{j,\ell}^2}\right) + \mathrm{E}(i,\ell;k,j) + \mathrm{E}(i,j;k,\ell)$$

$$\mathrm{E}(1,2;3,1) = \int \frac{\mathrm{d}c\ \mathrm{d}a_1\ \mathrm{d}b_3\ \mathrm{d}b_1}{\pi(1+c)\sqrt{c}} \frac{-2\mu_{\mathrm{IR}}^2 x_{1,3}^2}{\left(2(a_1+b_1)b_3 x_{1,3}^2 + 2\mu_{\mathrm{IR}}^2((a_1+1+b_1+b_3)^2 + c(a_1+1)^2)\right)^2}$$

$$\mathrm{E}(1,2;3,1) = \int \frac{\mathrm{d}c\ \mathrm{d}a_1\ \mathrm{d}b_3\ \mathrm{d}b_1}{\pi(1+c)\sqrt{c}} \frac{-x_{1,3}^2}{\left(2(a_1+b_1)b_3 x_{1,3}^2 + ((a_1+1+b_1+2\mu_{\mathrm{IR}}^2 b_3)^2 + c(a_1+1)^2)\right)^2}$$

$$\mathrm{E}(1,2;3,1) = \int \frac{\mathrm{d}c\ \mathrm{d}a_1\ \mathrm{d}b_1}{\pi(1+c)\sqrt{c}} \frac{-1}{2(a_1+b_1)((a_1+1+b_1)^2 + c(a_1+1)^2)},$$

$$\mathrm{E}(1,2;3,1) = -\frac{\pi^2}{24} + \frac{\log^2\ 2}{4}$$

$$I^{\mathrm{dt}}(j-1,j;j,k) = \frac{\pi^2}{12}$$

$$S_4 = \int\ d^4x\sqrt{-g}\left[\frac{R}{2\kappa} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g_4\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + h_4 R^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + f_4\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right]$$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa\Big[& F_\mu^\alpha F_{\nu\alpha} - \frac{1}{4}F^2 g_{\mu\nu} + g_4 F^2\left(F^2 g_{\mu\nu} - 8F_\mu^\alpha F_{\nu\alpha}\right) \\ & + 8f_4\left(\tilde{F}_{\mu\rho}F_\nu^\rho + \tilde{F}_{\nu\rho}F_\mu^\rho\right)F_{\alpha\beta}\tilde{F}^{\alpha\beta} \\ & + \kappa h_4\left(g_{\mu\nu}F_{\alpha\beta}F_{\gamma\delta}R^{\alpha\beta\gamma\delta} - 6F_{\alpha\nu}F^{\gamma\delta}R^\alpha_{\mu\gamma\delta} - 4\nabla_\beta\nabla_\alpha F_\mu^\alpha F_\nu^\beta\right)\Big], \\ \nabla_\mu F^{\mu\nu} = & 8g_4\left(F^2\nabla_\mu F^{\mu\nu} + F^{\mu\nu}\nabla_\mu F^2\right) + 8f_4\left(F_{\alpha\beta}\tilde{F}^{\alpha\beta}\nabla_\rho\tilde{F}^{\nu\rho} + 2\tilde{F}^{\nu\rho}\tilde{F}_{\alpha\beta}\nabla_\rho F^{\alpha\beta}\right) \\ & + 4h_4\nabla_\nu\left(R^{\alpha\beta\mu\nu}F_{\alpha\beta}\right) \end{aligned}$$

$$ds^2 = -G(r)dt^2 + \frac{dr^2}{G(r)} + r^2(d\theta^2 + \sin^2\ \theta d\phi^2)$$

$$G(r) = 1 - \frac{2M}{r} + \frac{\kappa Q^2}{2r^2} + \sum_{n=f_4,g_4,h_4} n\mathcal{S}_n(r)$$

$$F_{\mu\nu} = \left(\frac{Q}{r^2} + \sum_{n=f_4,g_4,h_4} n\delta A_n(r)\right)dt \wedge dr$$

$$\delta A_{g_4}(r) = -\frac{16Q^3}{r^6}\,, \mathcal{S}_{g_4}(r) = -\frac{8Q^4}{5r^6}$$

$$\delta A_{h_4}(r) = \frac{16MQ}{r^5} - \frac{24Q^3}{r^6}\,, \mathcal{S}_{h_4}(r) = -\frac{64Q^4}{5r^6} + \frac{28MQ^2}{r^5} - \frac{16Q^2}{r^4}$$

$$\delta A_{f_4}(r) = 0\,, \mathcal{S}_{f_4}(r) = 0$$

$$R_+^{(1)} = r_+ + \frac{4g_4Q^4}{4r_+^4\sqrt{M^2-Q^2}} + \frac{2h_4Q^2(5Mr_+ - 4Q^2)}{5r_+^4\sqrt{M^2-Q^2}} + O\left(\frac{g_{EFT}^2}{(M^2-Q^2)^{3/2}},\right)$$

$$R_-^{(1)} = r_- - \frac{4g_4Q^4}{5r_-^4\sqrt{M^2-Q^2}} + \frac{2h_4Q^2(4Q^2 - 5Mr_-)}{5r_-^4\sqrt{M^2-Q^2}} + O\left(\frac{g_{EFT}^2}{\left(M^2-Q^2\right)^{3/2}},\right)$$

$$\frac{g_4}{M^2-Q^2}, \frac{h_4}{M^2-Q^2}.$$

$$\frac{g_4}{M^2-Q^2} \ll 1, \frac{h_4}{M^2-Q^2} \ll 1.$$

$$M = Q - \frac{2g_4 + h_4}{5Q}$$

$$2g_4 + h_4 \geq 0$$

$$\Box\,\Phi(t,r,\theta,\phi) = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\right]\Phi(t,r,\theta,\phi) = 0$$

$$\Phi(t,r,\theta,\phi) = \exp\left(-i\omega t + im\phi\right)R(r)S(\theta)$$

$$\left[\frac{1}{\sin(\theta)}\partial_\theta(\sin\,\theta\partial_\theta) - \frac{m^2}{\sin^2\,\theta}\right]S(\theta) = -\ell(\ell+1)S(\theta)$$

$$r^2G(r)R''(r) + r(2G(r) + rG'(r))R'(r) + \left(\frac{r^2\omega^2}{G(r)} - \ell(\ell+1)\right)R(r) = 0$$

$$R(r) = \frac{\psi(r)}{r\sqrt{G(r)}}, \psi''(r) + Q_W(r,\omega)\psi(r) = 0$$

$$Q_W(r,\omega) = \frac{r^2(4\omega^2 + G'(r)^2) - 2G(r)(2\ell(\ell+1) + 2rG'(r) + r^2G''(r))}{4r^2G(r)^2}$$

$$Q_W^{\text{num}} = \frac{8g_4Q^4\left(5r^4\left((\ell^2+\ell+15)r^2 - 24Mr + 10Q^2\right) - 8h_4Q^2(-175Mr + 96Q^2 + 90r^2)\right)}{4r^2\left(-\frac{8g_4Q^4}{5r^6} + h_4\left(\frac{28MQ^2}{r^5} - \frac{64Q^4}{5r^6} - \frac{16Q^2}{r^4}\right) - \frac{2M}{r} + \frac{Q^2}{r^2} + 1\right)^2}$$
$$+ \frac{20h_4Q^2r^4(5r^2(7M-4r)(15M - (\ell^2+\ell+6)r) + 2Q^2r(2(4\ell(\ell+1)+75)r - 297M) + 160Q^4)}{4r^2\left(-\frac{8g_4Q^4}{5r^6} + h_4\left(\frac{28MQ^2}{r^5} - \frac{64Q^4}{5r^6} - \frac{16Q^2}{r^4}\right) - \frac{2M}{r} + \frac{Q^2}{r^2} + 1\right)^2}$$
$$+ \frac{25r^{10}(-(\ell^2+\ell+1)Q^2 + 2\ell(\ell+1)Mr - \ell(\ell+1)r^2 + M^2 + r^4\omega^2)}{4r^2\left(-\frac{8g_4Q^4}{5r^6} + h_4\left(\frac{28MQ^2}{r^5} - \frac{64Q^4}{5r^6} - \frac{16Q^2}{r^4}\right) - \frac{2M}{r} + \frac{Q^2}{r^2} + 1\right)^2}$$

$$(r-r_+)(r-r_-)R''(r)+(2r-r_+-r_-)R'(r)+\left[\frac{r^4\omega^2}{(r-r_+)(r-r_-)}-\ell(\ell+1)\right]R(r)=0$$

$$R(r)=\frac{\psi(r)}{\sqrt{(r-r_+)(r-r_-)}}$$

$$\psi''(r)+Q_{W,RN}(r,\omega)\psi(r)=0$$

$$Q_{W,RN}(r,\omega)=\frac{r^4\omega^2-\ell(\ell+1)(r-r_+)(r-r_-)+\frac{1}{4}(r_+-r_-)^2}{(r-r_+)^2(r-r_-)^2}$$

$$\tau=\frac{\theta_{eff}}{\pi}+\frac{8\pi i}{g_{eff}^2}=\frac{8\pi i}{g_0^2}+\frac{2i}{\pi}\log\left(\frac{a^2}{\Lambda^2}\right)-\frac{i}{\pi}\sum_{i=1}^{\infty}c_i\left(\frac{\Lambda}{a}\right)^{4i}$$

$$qy^2P_L(x)+yP_0(x)+P_R(x)=0,$$

$$P_0(x)=x^2-u+qp_0(x),P_R(x)=(x-m_1)(x-m_2),P_L(x)=(x-m_3)(x-m_4).$$

$$y_\pm=\frac{1}{2qP_L(x)}\left(-P_0\pm\sqrt{P_0^2-4qP_LP_R}\right)$$

$$P_0^2-4qP_LP_R=\prod_{i=1}^{4}(x-e_i)$$

$$a=\oint_\alpha\lambda_0\,,a_D=\oint_\beta\lambda_0$$

$$\lambda_0=\frac{1}{2}(\lambda_+-\lambda_-)\,,\lambda_\pm=\frac{1}{2\pi i}x\partial_x\ln\,y_\pm(x)dx$$

$$[\hat{x},\ln\,\hat{y}]=\hbar$$

$$\left[q\hat{y}^{\frac{1}{2}}P_L(\hat{x})\hat{y}^{\frac{1}{2}}+P_0(\hat{x})+\hat{y}^{-\frac{1}{2}}P_R(\hat{x})\hat{y}^{-\frac{1}{2}}\right]U(x)=0^4$$

$$p_0(x)=x^2-\left(x+\frac{\hbar}{2}\right)\sum_i m_i+u+\sum_{i<j}m_im_j+\frac{\hbar^2}{2}$$

$$\left[qy^2P_L\left(\hat{x}+\frac{\hbar}{2}\right)+yP_0(\hat{x})+P_R\left(\hat{x}-\frac{\hbar}{2}\right)\right]U(y)=[A(y)\hat{x}^2+B(y)\hat{x}+C(y)]U(y)=0$$

$$A=(1+y)(1+qy)\,,B=-m_1-m_2-\hbar+qy\left[y(\hbar-m_3-m_4)-\sum_i m_i\right]$$

$$C=\left(m_1+\frac{\hbar}{2}\right)\left(m_2+\frac{\hbar}{2}\right)-uy+qy\left[u+\sum_{i<j}m_im_j-\frac{\hbar}{2}\sum_i m_i+\frac{\hbar^2}{2}+y\left(m_3-\frac{\hbar}{2}\right)\left(m_4-\frac{\hbar}{2}\right)\right].$$

$$U(y)=\frac{1}{\sqrt{y}}e^{-\frac{1}{2\hbar}\int^{y}\frac{B(y')}{y'A(y')}\frac{B(y')}{y'A(y')}dy'}\Psi(y)$$

$$\Psi''(y)+Q_{SW}(y)\Psi(y)=0$$
$$Q_{SW}(y)=\frac{4CA-B^2+2\hbar y(BA'-AB')+\hbar^2A^2}{4\hbar^2u^2A^2}$$

$$Q_{2,2}(y)=\frac{\hbar^2-(m_1-m_2)^2}{4\hbar^2y^2}+\frac{\hbar^2-(m_1+m_2)^2}{4\hbar^2(1+y)^2}+\frac{q^2(\hbar^2-(m_3+m_4)^2)}{4\hbar^2(1+qy)^2}$$
$$+\frac{1}{4\hbar^2y(1+y)(1+qy)}[q(2m_1^2y+2m_2^2y+4m_3m_4y-2(m_1+m_2+m_3+m_4)\hbar$$
$$+2m_1m_3+2m_2m_3+2m_1m_4+2m_2m_4+4m_3m_4+4u+(1-2y)\hbar^2)+2m_1^2+2m_2^2-4u-\hbar^2]$$

$$q\to 0, m_4\to\infty, q'=-qm_4=\text{const},$$

$$Q_{1,2}(y)=-\frac{q'^2}{4\hbar^2}+\frac{\hbar^2-(m_1-m_2)^2}{4\hbar^2y^2}+\frac{\hbar^2-(m_1+m_2)^2}{4\hbar^2(1+y)^2}$$
$$-\frac{m_3q'}{\hbar^2y}+\frac{2(m_1^2+m_2^2)-\hbar^2-4u+2q'(\hbar-m_1-m_2)}{4\hbar^2y(1+y)}$$

$$\Psi''(y)+Q_{1,2}(y)\Psi(y)=0$$

$$y=\frac{r-r_+}{r_+-r_-}$$

$$q=2i(r_+-r_-)\omega, m_1=\frac{i(r_+^2+r_-^2)\omega}{r_+-r_-}, m_2=-m_3=-i(r_++r_-)\omega$$
$$u=\left(\ell+\frac{1}{2}\right)^2-\omega(i(r_--r_+)+(r_-+r_+)^2\omega)$$

$$\left[qP_L\left(x-\frac{1}{2}\right)\hat{y}+P_0(x)+P_R\left(x+\frac{1}{2}\right)\hat{y}^{-1}\right]\tilde{U}(x)=0$$

$$W(x)=\frac{1}{P_R\left(x+\frac{1}{2}\right)}\frac{\tilde{U}(x)}{\tilde{U}(x+1)}, M(x)=qP_L\left(x-\frac{1}{2}\right)P_R\left(x-\frac{1}{2}\right)$$

$$qM(x)W(x)W(x-1)+P_0(x)W(x)+1=0.$$

$$P_0(a)=\frac{M(a+1)}{P_0(a+1)-\frac{M(a+2)}{P_0(a+2)-\cdots}}+\frac{M(a)}{P_0(a-1)-\frac{M(a-1)}{P_0(a-1)-\cdots}},$$

$$u(a,q)=a^2+q\left[\frac{1}{2}(1-m_1-m_2-m_3)-\frac{2m_1m_2m_3}{4a^2-1}\right]+\frac{q^2}{128(a^2-1)}[4a^2-5+$$
$$+4(m_1^2+m_2^2+m_3^2)-\frac{48(m_1^2m_2^2+m_2^2m_3^2+m_1^2m_3^2)}{4a^2-1}+\frac{64m_1^2m_2^2m_3^2(20a^2+7)}{(4a^2-1)^3}\right]+\cdots$$

$$a(u,q)=\sqrt{u}+\frac{q}{4\sqrt{u}}\left(\frac{4m_1m_2m_3}{4u-1}+m_1+m_2+m_3-1\right)$$
$$-\frac{q^2}{256\sqrt{u}}\left(\frac{1024m_1^2m_2^2m_3^2}{(4u-1)^3}-\frac{256m_1m_2\big(m_1(m_2m_3-2)-2(m_2+m_3-1)\big)m_3}{(1-4u)^2}\right.$$
$$+\frac{8(m_2+m_3+m_1(1-4m_2m_3)-1)^2}{u}+\frac{(4m_1^2-1)(4m_2^2-1)(4m_3^2-1)}{u-1}$$
$$\left.+\frac{64\Big(\big((1-12m_3^2)m_2^2+4m_3m_2+m_3^2\big)m_1^2+4m_2m_3(m_2+m_3-1)m_1+m_2^2m_3^2\Big)}{4u-1}+4\right)+\cdots$$

$$u=-q\frac{\partial\mathcal{F}_{NS}(a,q)}{\partial q}$$

$$\mathcal{F}_{NS}(a,q)=\mathcal{F}_{\text{tree}}(a,q)+\mathcal{F}_{1-\text{ loop}}(a)+\mathcal{F}_{\text{inst}}(a,q)$$

$$\mathcal{F}_{\text{tree}}(a,q)=-a^2\log q$$
$$\mathcal{F}_{\text{inst}}(a,q)=q\frac{1-m_2-m_3+4a^2(m_1+m_2+m_3-1)+m_1(4m_2m_3-1)}{2(4a^2-1)}+\cdots$$
$$\frac{\partial\mathcal{F}_{\text{one-loop}}(a)}{\partial a}=\log\left[\frac{\Gamma^2(1+2a)}{\Gamma^2(1-2a)}\prod_{i=1}^{3}\frac{\Gamma\left(\frac{1}{2}+m_i-a\right)}{\Gamma\left(\frac{1}{2}+m_i+a\right)}\right]$$

$$a_D=-\frac{1}{2\pi i}\frac{\partial\mathcal{F}_{NS}}{\partial a}$$

$$\psi_H(y)=A_1y^{\frac{1}{2}(1+m_1-m_2)}(1+y)^{\frac{1}{2}(1+m_1+m_2)}\ {}_2F_1\left[\frac{1}{2}+m_1-\sqrt{u},\frac{1}{2}+m_1+\sqrt{u},1+m_1-m_2,-y\right]$$
$$+(1\leftrightarrow 2)$$

$$\psi_{H,in}(y)=y^{\frac{1}{2}(1-m_1+m_2)}(1+y)^{\frac{1}{2}(1+m_1+m_2)}\ {}_2F_1\left[\frac{1}{2}+m_2-\sqrt{u},\frac{1}{2}+m_2+\sqrt{u},1-m_1+m_2,-y\right]$$

$$\psi_{H,in}(y)\underset{y\to\infty}{\sim}y^{\frac{1}{2}-\sqrt{u}}+y^{\frac{1}{2}+\sqrt{u}}\frac{\Gamma\left(\frac{1}{2}-m_1-\sqrt{u}\right)\Gamma\left(\frac{1}{2}+m_2-\sqrt{u}\right)\Gamma(2\sqrt{u})}{\Gamma\left(\frac{1}{2}-m_1+\sqrt{u}\right)\Gamma\left(\frac{1}{2}+m_2+\sqrt{u}\right)\Gamma(-2\sqrt{u})}$$

$$Q_{1,2}\underset{y\to\infty}{\sim}-\frac{q^2}{4}-\frac{m_3q}{y}+\frac{1-4u}{4y^2}$$

$$\psi_\infty(y)=\sum_{\alpha=\pm}A_\alpha e^{\alpha\frac{qy}{2}}(qy)^{\frac{1}{2}+\sqrt{u}}U\left[\frac{1}{2}-\alpha m_3+\sqrt{u},1+2\sqrt{u},-\alpha y\right]$$

$$\psi_{\infty,out}(y)=e^{\frac{qy}{2}}(qy)^{\frac{1}{2}+\sqrt{u}}U\left[\frac{1}{2}-m_3+\sqrt{u},1+2\sqrt{u},-y\right]$$

$$\psi_{\infty,out}(y)\underset{y\to 0}{\sim}y^{\frac{1}{2}-\sqrt{u}}+y^{\frac{1}{2}+\sqrt{u}}\frac{(-q)^{2\sqrt{u}}\Gamma\left(\frac{1}{2}-m_3+\sqrt{u}\right)\Gamma(-2\sqrt{u})}{\Gamma\left(\frac{1}{2}-m_3-\sqrt{u}\right)\Gamma(2\sqrt{u})}$$

$$(-q)^{2\sqrt{u}}\frac{\Gamma(-2\sqrt{u})^2\Gamma\left(\frac{1}{2}-m_1+\sqrt{u}\right)\Gamma\left(\frac{1}{2}+m_2+\sqrt{u}\right)\Gamma\left(\frac{1}{2}-m_3+\sqrt{u}\right)}{\Gamma(2\sqrt{u})^2\Gamma\left(\frac{1}{2}-m_1-\sqrt{u}\right)\Gamma\left(\frac{1}{2}+m_2-\sqrt{u}\right)\Gamma\left(\frac{1}{2}-m_3-\sqrt{u}\right)}=1$$

$$\omega_{\text{match}}\sim -i\frac{\ell+1+n}{2r_+^2}(r_+-r_-)$$

$$R(r)\underset{r\to r_+}{\sim}(r-r_+)^{\delta}$$

$$\delta_\pm=\pm i\frac{r_+^2\omega}{r_+-r_-}$$

$$R_L(r)=e^{i\omega r}(r-r_+)^{-i\frac{r_+^2\omega}{r_+-r_-}}(r-r_-)^{\rho}\sum_{n=0}^{\infty}c_n\left(\frac{r-r_+}{r-r_-}\right)^n.$$

$$\rho=-1+i\frac{(2r_+^2-r_-^2)\omega}{r_+-r_-}$$

$$\alpha_n c_{n+1}+\beta_n c_n+\gamma_n c_{n-1}=0.$$

$$\alpha_n=(n+1)\left(1+n-\frac{2ir_+^2\omega}{r_+-r_-}\right)$$

$$\beta_n=-\left(\ell+\frac{1}{2}\right)^2-2\left(n+\frac{1}{2}\right)^2-\frac{1}{4}+2i(1+2n)(2r_++r_-)\omega+8(r_-^2+r_-r_++r_+^2)\omega^2+\frac{2r_-^2\omega(i+2in+4r_-\omega)}{r_+-r_-}$$

$$\gamma_n=\frac{(n(r_--r_+)+2ir_+^2\omega)(n-2i(r_-+r_+)\omega)}{r_--r_+}$$

$$0=\beta_0-\cfrac{\alpha_0\gamma_1}{\beta_1-\cfrac{\alpha_1\gamma_2}{\beta_2-\cfrac{\alpha_2\gamma_3}{\beta_3-\cdots}}}$$

$$-\frac{8g_4Q^4}{5r^6}+h_4\left(\frac{28MQ^2}{r^5}-\frac{64Q^4}{5r^6}-\frac{16Q^2}{r^4}\right)-\frac{2M}{r}+\frac{Q^2}{r^2}+1$$
$$=\frac{(r-R_-)(r-R_+)}{r^6}\times$$
$$\left(-\frac{8g_4(8M^3r^3+Q^4r(2M-r)+4MQ^2r^2(M-r)+Q^6)}{5Q^4}\right.$$
$$\left.+\frac{4h_4(12M^3r^3+Q^4r(3M-4r)+MQ^2r^2(6M-11r)-16Q^6)}{5Q^4}+r^4\right)$$

$$R_{-}^{(1)} = M - \sqrt{M^2 - Q^2} + \frac{4g_4Q^4 - 2h_4Q^2\left(5M\sqrt{M^2 - Q^2} - 5M^2 + 4Q^2\right)}{5\left(\sqrt{M^2 - Q^2} - M\right)^3\left(M\sqrt{M^2 - Q^2} - M^2 + Q^2\right)}$$

$$R_{+}^{(1)} = M + \sqrt{M^2 - Q^2} + \frac{4g_4Q^4 + 2h_4Q^2\left(5M\sqrt{M^2 - Q^2} + 5M^2 - 4Q^2\right)}{5\left(\sqrt{M^2 - Q^2} + M\right)^3\left(M\sqrt{M^2 - Q^2} + M^2 - Q^2\right)}$$

$$Q_W^{\text{num}} = \frac{A}{(r - R_+)^2} + \frac{B}{(r - R_+)} + \frac{C}{(r - R_-)^2} + \frac{D}{(r - R_-)} + E + \frac{F}{r} + \frac{G}{r^2} + O(r^{-3}) + \cdots$$

$$A, C \propto (R_+ - R_-)^{-2}, B, D \propto (R_+ - R_-)^{-1}, F \propto (R_+ - R_-), G \propto (R_+ - R_-)^2.$$

$$Q_W^{\text{Heun}} = \frac{A}{(r - R_+)^2} + \frac{B}{(r - R_+)} + \frac{C}{(r - R_-)^2} + \frac{D}{(r - R_-)} + E$$

$$Q_W^{EFT} = \frac{4r^4\omega^2 - 4(r - r_-)(r - r_+)\ell^2 - 4(r - r_-)(r - r_+)\ell + (r_- - r_+)^2}{4(r - r_-)^2(r - r_+)^2}$$
$$-g_4\frac{4\left(4r_+^4(4r_-^3 + 3r_+r_-^2 + 2r_+^2r_- + r_+^3)\omega^2 - 2r_-^5 + r_+r_-^4 + r_+^5\right)}{5r_-^2(r - r_+)^2(r_- - r_+)^2r_+^3}$$
$$+h_4\frac{4r_+^4(23r_-^3 + r_-^2r_+ - r_-r_+^2 - 3r_+^3)\omega^2 + 6r_-^5 - 13r_-^4r_+ + 5r_-^3r_+^2 + 5r_-r_+^4 - 3r_+^5}{5r_-^2(r - r_+)^2(r_- - r_+)^2r_+^3}$$

$$r_+ = M + \sqrt{M^2 - Q^2}, r_- = M - \sqrt{M^2 - Q^2}$$

$$\frac{g_{EFT}}{M^2 - Q^2} \ll 1^5$$

$$\omega_{ZDM} = -\frac{i(\ell + n + 1)}{Q^2}\sqrt{M^2 - Q^2} - \frac{4ig_4(1 + \ell + n)}{5Q^2\sqrt{M^2 - Q^2}} - \frac{2ih_4(\ell + n + 1)}{5Q^2\sqrt{M^2 - Q^2}}$$

$$+\frac{ig_4\sqrt{M^2 - Q^2}}{5Q^4}\left[-\frac{8((2n + 11)\ell + n(4n + 7) + 8\ell^2)}{2\ell + 1} + \frac{1}{n + \ell + 1} + \frac{1}{(n + \ell + 1)^3} - \frac{24}{2\ell + 1}\right]$$
$$+\frac{ih_4\sqrt{M^2 - Q^2}}{10Q^4}\left[\frac{4(11 - 8n)n}{2\ell + 1} + \ell\left(\frac{184n}{2\ell + 1} + 68\right) + \frac{1}{n + \ell + 1} + \frac{1}{(n + \ell + 1)^3}\right.$$
$$\left. + \frac{4}{2\ell + 1} + 72\right]$$

$$\mathcal{H} = g^{\mu\nu}P_\mu P_\nu = 0\,, P_\mu = \frac{\partial\mathcal{L}}{2\partial\dot{x}^\mu}$$

$$P_t = -E = -G(r)\dot{t}, P_r = \frac{\dot{r}}{G(r)}, P_\theta^2 = r^2\dot{\theta}, P_\phi = J = r^2\sin^2\,\theta\dot{\phi}$$

$$-\frac{E^2}{G(r)} + G(r)P_r^2 + \frac{P_\theta}{r^2} + \frac{J^2}{r^2\sin^2\,\theta} = 0$$

$$P_r^2 = Q_R(r) = \frac{E^2}{G^2(r)} - \frac{K^2}{r^2 G(r)}$$

$$P_\theta^2 = Q_A(\theta) = K^2 - \frac{J^2}{\sin^2\ \theta}$$

$$Q_R(r_c, b_c) = Q_R'(r_c, b_c) = 0$$

$$\omega_{QNM} \sim E_c - \mathrm{i}(2n+1)\lambda\,, E_c = \frac{\ell}{b_c},$$

$$\frac{dr}{dt} \sim -2\lambda(r - r_c).$$

$$\lambda = \left(\sqrt{2}\partial_E Q_R(r_c, E_c)\right)^{-1} \sqrt{\partial_r^2 Q_R(r_c, E_c)}.$$

$$\psi(r) = \frac{1}{\sqrt[4]{Q_W(r,\omega)}} \exp\left(\pm \mathrm{i} \int^{\mathrm{r}} \sqrt{\mathrm{Q_W(r',\omega)}}\,\sqrt{\mathrm{Q_W(r',\omega)}} dr'\right).$$

$$\psi(r) = \frac{1}{\sqrt[4]{-Q_W(r,\omega)}} \exp\left(\pm \int^{\mathrm{r}} \sqrt{-\mathrm{Q_W(r',\omega)}}\,\sqrt{-\mathrm{Q_W(r',\omega)}} dr'\right)$$

$$\int_{r_-}^{r_+} \sqrt{Q_W(r,\omega)} dr = \pi\left(n + \frac{1}{2}\right)$$

$$\int_{r_-}^{r_+} \sqrt{Q_W(r,\omega)} dr \sim \int_{r_-}^{r_+} \sqrt{Q_W(r_c,\omega) + \frac{Q_W''(r_c,\omega)}{2}(r - r_c)^2} dr \sim \frac{i\pi Q_W(r_c,\omega)}{\sqrt{2Q_W''(r_c,\omega)}}$$

$$\ell = \frac{J}{\hbar} - \frac{1}{2}\,, \omega = \frac{E}{\hbar}\,, \psi(r) \sim e^{i\frac{S_0(r)}{\hbar}}$$

$$\begin{aligned}\hbar^2 \left(\frac{S_0(r)}{dr}\right)^2 = Q_{geo}(r,E) = &\frac{E^2 r^4 - J^2 (r - r_-)(r - r_+)}{(r - r_+)^2 (r - r_-)^2} \\ &- g_4 \frac{16 r_+ (4r_-^3 + 3r_-^2 r_+ + 2r_- r_+^2 + r_+^3) E^2}{5 r_-^2 (r - r_+)^2 (r_+ - r_-)^2} \\ &- h_4 \frac{4 r_+ (23 r_-^3 + r_-^2 r_+ - r_- r_+^2 - 3 r_+^3) E^2}{5 r_-^2 (r - r_+)^2 (r_+ - r_-)^2}\end{aligned}$$

$$Q_{geo}(r_c, J_c) = \frac{dQ_{geo}(r_c, J_c)}{dr} = 0$$

$$r_c = \frac{1}{4}\big(\sigma + 3(r_- + r_+)\big)$$
$$+g_4 \frac{32(4r_-^3 + 3r_+r_-^2 + 2r_+^2r_- + r_+^3)(-r_-(\sigma + 7r_+) + 2r_+(\sigma + 3r_+) + 3r_-^2)}{5\sigma r_-^2(r_- - r_+)\big(\sigma + 3(r_+ + r_-)\big)^2}$$
$$+h_4 \frac{8(23r_-^3 + r_-^2r_+ - r_-r_+^2 - 3r_+^3)\big(3r_-^2 + 2r_+(3r_+ + \sigma) - r_-(7r_+ + \sigma)\big)}{5r_-^2\sigma(r_- - r_+)\big(\sigma + 3(r_+ + r_-)\big)^2}$$

$$J_c = \frac{E\big(\sigma + 3(r_- + r_+)\big)^{3/2}}{2\sqrt{2}\sqrt{\sigma + r_- + r_+}} - g_4 \frac{16\sqrt{2}E(4r_-^3 + 3r_+r_-^2 + 2r_+^2r_- + r_+^3)}{5\sigma r_-^2(r_- - r_+)^2(\sigma + r_- + r_+)^{3/2}\big(\sigma + 3(r_- + r_+)\big)^{3/2}}$$
$$\times [\sigma^2(2r_-^2 - 7r_+r_- + 7r_+^2) - 2\sigma(r_- - 3r_+)(3r_-^2 - 5r_+r_- + 4r_+^2)$$
$$+r_+(r_- + r_+)(9r_-^2 - 14r_+r_- + 9r_+^2)]$$
$$-h_4 \frac{4\sqrt{2}E(23r_-^3 + r_+r_-^2 - r_+^2r_{--} - 3r_+^3)}{5\sigma r_-^2(r_- - r_+)^2(\sigma + r_- + r_+)^{3/2}\big(\sigma + 3(r_- + r_+)\big)^{3/2}} \times [\sigma^2(2r_-^2 - 7r_+r_- + 7r_+^2)$$
$$-2\sigma(r_- - 3r_+)(3r_-^2 - 5r_+r_- + 4r_+^2) + r_+(r_- + r_+)(9r_-^2 - 14r_+r_- + 9r_+^2)]$$

$$\sigma = \sqrt{9r_+^2 + 9r_-^2 - 14r_+r_-}$$

$$\lambda = \lambda_{RN} + g_4\lambda_{g_4} + h_4\lambda_{h_4}$$
$$\lambda_{RN} = \frac{4\sqrt{2}\rho(3r_- - r_+ + \sigma)^2(-r_- + 3r_+ + \sigma)^2}{\sqrt{r_- + r_+ + \sigma}(3(r_- + r_+) + \sigma)^4(r_-(\sigma - 2r_+) + r_+(3r_+ + \sigma) + 3r_-^2)^2}$$
$$\lambda_{g_4} = \frac{1}{5\rho r_-^2(r_- - r_+)^2\sigma(3r_- - r_+ + \sigma)^2(r_- + r_+ + \sigma)^{3/2}(-r_- + 3r_+ + \sigma)^2(3(r_- + r_+) + \sigma)^6}$$
$$\times \big[131072\sqrt{2}r_+(4r_-^3 + 3r_+r_-^2 + 2r_+^2r_- + r_+^3)(r_-(\sigma - 2r_+) + r_+(3r_+ + \sigma) + 3r_-^2)^2$$
$$\times (r_-^8(9\sigma - 21r_+) + 12r_+^2r_-^6(9r_+ + \sigma) - 6r_+^3r_-^5(69r_+ + 4\sigma) + 42r_+^4r_-^4(25r_+ + 3\sigma)$$
$$-12r_+^5r_-^3(129r_+ + 20\sigma) + 4r_+^6r_-^2(375r_+ + 79\sigma) - 27r_+^7r_-(31r_+ + 8\sigma) + 81r_+^8(3r_+ + \sigma)$$
$$+27r_-^9 + 20r_+^2r_-^7)]$$
$$\lambda_{h_4} = \lambda_{g_4} \times \frac{23r_-^3 + r_-^2r_+ - r_-r_+^2 - 3r_+^3}{4(4r_-^3 + 3r_-^2r_+ + 2r_-r_+^2 + r_+^3)}$$

$$\rho^2 = 81r_-^6(\sigma - 3r_+) + 3r_+r_-^5(77r_+ - 6\sigma) + r_+^2r_-^4(25r_+ + 47\sigma) + r_+^3r_-^3(25r_+ + 36\sigma)$$
$$+r_+^4r_-^2(231r_+ + 47\sigma) - 9r_+^5r_-(27r_+ + 2\sigma) + 81r_+^6(3r_+ + \sigma) + 243r_-^7$$

$$g_4\lambda_{g_4} + h_4\lambda_{h_4} > 0$$

$$2g_4 + f\left(\frac{r_-}{r_+}\right)h_4 > 0$$

$$f(\gamma) = \frac{23\gamma^3 + \gamma^2 - \gamma - 3}{2(4\gamma^3 + 3\gamma^2 + 2\gamma + 1)}, -\frac{3}{2} < f(\gamma) < 1.$$

$$-2g_4 < h_4 < \frac{4}{3}g_4$$

$$R(r) \underset{r\to r_+}{\sim} (r - r_+)^{\alpha_\pm},$$

$$\alpha_\pm = \pm\frac{\sqrt{P_0 + g_4 P_{g_4} + h_4 P_{h_4}}}{\sqrt{5} r_+^{3/2} r_-(r_+ - r_-)}$$
$$P_0 = -5r_+^7 r_-^2 \omega^2$$
$$P_{h_4} = 4r_+^4(23r_-^3 + r_+ r_-^2 - r_+^2 r_- - 3r_+^3)\omega^2 + 6r_-^5 - 13r_+ r_-^4 + 5r_+^2 r_-^3 + 5r_+^4 r_- - 3r_+^5$$
$$P_{g_4} = 4\left(4r_+^4(4r_-^3 + 3r_+ r_-^2 + 2r_+^2 r_- + r_+^3)\omega^2 - 2r_-^5 + r_+ r_-^4 + r_+^5\right)$$

$$R(r) \underset{r\to\infty}{\sim} e^{\pm i\omega r}$$

$$R(r) = e^{i\omega r}(r - r_+)^{\alpha_-}(r - r_-)^\beta \sum_{n=0}^{\infty} c_n \left(\frac{r - r_+}{r - r_-}\right)^n.$$

$$\beta = -1 + i\omega(r_+ + r_-) - \alpha_-$$

$$z = \frac{r - r_+}{r - r_-}$$

$$\alpha_n = (n+1)(n+1+2\alpha_-)$$
$$\beta_n = -\ell(\ell+1) - 1 - 2n(n+1) - 2\alpha_-^2 + 2ir_+\omega(1+2n) + \frac{2r_+^3(r_+ - 2r_-)\omega^2}{(r_+ - r_-)^2} + 2\alpha_-(2ir_+\omega - 2n - 1)$$
$$\gamma_n = n^2 - 2in(r_+ + r_-)\omega + \alpha_-^2 + \frac{r_+^2(2r_-^2 - r_+^2)\omega^2}{(r_+ - r_-)^2} + 2\alpha_-(n - i(r_+ + r_-)\omega)$$

$$5r_-^2 r_+^3 - 4h_4(23r_-^3 + r_-^2 r_+ - r_- r_+^2 - 3r_+^3) - 16g_4(4r_-^3 + 3r_-^2 r_+ + 2r_- r_+^2 + r_+^3) > 0$$

$$R_\infty(r) \simeq e^{\mathrm{i}\omega \mathrm{r}} r^{2\mathrm{i}M\omega} \sum_{n=0}^{N_\infty} c_n r^{-n}$$

$$R_H(r) \simeq (r - r_H)^\alpha \sum_{n=0}^{N_H} d_n (r - r_H)$$

$$S_{\text{mixed}} = \frac{i}{8\pi^2}\int_{X\times S^1} A^{(I)} \wedge \mathrm{Tr}(F\wedge F) + \cdots$$

$$B\left(w_2(P), c_1\left(L^{(I)}\right) + w_2(X)\right) = 0 \bmod 2.$$

$$V = (A_m, \sigma, \lambda_A, D_{AB})$$

$$S_{\text{SYM}}[V] = \int_{X_5} \mathrm{d}^5 x \sqrt{g}\mathcal{L}_{\text{SYM}}$$

$$\mathcal{L}_{\text{SYM}} = \frac{1}{g_{5d}^2}\mathrm{tr}\left(\frac{1}{2}F_{mn}F^{mn} + D^m\sigma D_m\sigma + \frac{1}{2}D^{AB}D_{AB} + i\lambda^A\Gamma^m D_m\lambda_A + i\lambda^A[\sigma, \lambda_A]\right)$$

$$D_m = \nabla_m - iA_m, F_{mn} = \partial_m A_n - \partial_n A_m - i[A_m, A_n].$$

$$\int [\mathrm{d}V] e^{-S_{\text{SYM}}[V]}$$

$$A_m^\dagger = A_m, \sigma^\dagger = \sigma, (D^{AB})^\dagger = -\epsilon^{AA'}\epsilon^{BB'}D_{A'B'}$$

$$\begin{aligned}
\delta A_m &= i\xi_A\Gamma_m\lambda^A \\
\delta\sigma &= -\xi_A\lambda^A \\
\delta\lambda_A &= \frac{1}{2}\Gamma^{mn}\xi_A F_{mn} + i\Gamma^m\xi_A D_m\sigma - iD_{AB}\xi^B \\
\delta D^{AB} &= \xi^A\Gamma^m D_m\lambda^B + \xi^B\Gamma^m D_m\lambda^A + \xi^A[\sigma,\lambda^B] + \xi^B[\sigma,\lambda^A]
\end{aligned}$$

$$j^{(I)} = \frac{1}{8\pi^2}\mathrm{tr}(F\wedge F).$$

$$V^{(I)} = \left(A_m^{(I)}, \sigma^{(I)}, \lambda_A^{(I)}, D_{AB}^{(I)}\right)$$

$$\boldsymbol{n}_I := \left[\frac{F^{(I)}}{2\pi}\right] = \overline{c_1(L^{(I)})} \in H^2(X_5,\mathbb{Z})/\ \mathrm{Tors}$$

$$\begin{aligned}
S_{\mathrm{mixed}} =& \frac{i}{8\pi^2}\int_{X_5} A^{(I)}\wedge \mathrm{tr}(F\wedge F) \\
&+\frac{1}{8\pi^2}\int_{X_5} \mathrm{d}^5x\sqrt{g}\,\mathrm{tr}\left[\frac{1}{2}\lambda^{(I)A}\Gamma^{mn}\lambda_A F_{mn} - \frac{1}{4}\lambda^A\Gamma^{mn}\lambda_A F_{mn}^{(I)} + \frac{i}{2}\lambda^A\lambda^B D_{AB}^{(I)} + i\lambda^{(I)A}\lambda^B D_{AB}\right. \\
&+\sigma^{(I)}\left(\frac{1}{2}F^{mn}F_{mn} + D^m\sigma D_m\sigma + \frac{1}{2}D^{AB}D_{AB} + i\lambda^A\Gamma^m D_m\lambda_A + i\lambda^A[\sigma,\lambda_A]\right) \\
&\left.+\sigma\left(F_{mn}^{(I)}F^{mn} + 2D^m\sigma^{(I)}D_m\sigma + D^{(I)AB}D_{AB} + i\lambda^{(I)A}\Gamma^m D_m\lambda_A + i\lambda^A\Gamma^m D_m\lambda_A^{(I)}\right)\right]
\end{aligned}$$

$$\sigma^{(I)} = -\frac{8\pi^2}{g_{5d}^2}$$

$$\begin{aligned}
S^{\mathrm{CS}} =& \frac{-i\kappa_{\mathrm{CS}}}{24\pi^2}\int_{X_5} \mathrm{tr}\left[A\wedge F\wedge F + \frac{i}{2}A\wedge A\wedge A\wedge F - \frac{1}{10}A\wedge A\wedge A\wedge A\wedge A\right] \\
&+\frac{\kappa_{\mathrm{CS}}}{2\pi^2}\int_{X_5} \mathrm{d}^5x\sqrt{g}\,\mathrm{tr}\left[\sigma\left(\frac{1}{2}F_{mn}F^{mn} + D^m\sigma D_m\sigma + \frac{1}{2}D^{AB}D_{AB} + i\lambda^A\Gamma^m D_m\lambda_A + i\lambda^A[\sigma,\lambda_A]\right)\right] \\
&+\frac{\kappa_{\mathrm{CS}}}{8\pi^2}\int_{X_5} \mathrm{d}^5x\sqrt{g}\,\mathrm{tr}[\lambda^A\Gamma^{mn}\lambda_A F_{mn} - 2i\lambda^A\lambda^B D_{AB}]
\end{aligned}$$

$$a_5 := \frac{1}{R}\oiiint_{S^1} A_5\,\mathrm{d}x^5$$

$$W_F(p) := \mathrm{tr}_F\mathrm{Pexp}\left[\oiiint_{p\times S^1}(\sigma + iA_5)\mathrm{d}x^5\right]$$

$$\mathcal{R}^4 := \exp\left(-\frac{8\pi^2 R}{g_{5d}^2} + i\theta\right)$$

$$U(\mathfrak{a},\mathcal{R}) = e^{\mathfrak{a}} + e^{-\mathfrak{a}} + \mathcal{R}^4 \frac{e^{\mathfrak{a}} + e^{-\mathfrak{a}}}{(e^{\mathfrak{a}} - e^{-\mathfrak{a}})^2} + \mathcal{R}^8 \frac{5(e^{\mathfrak{a}} + e^{-\mathfrak{a}})}{(e^{\mathfrak{a}} - e^{-\mathfrak{a}})^6} + \mathcal{R}^{12} \frac{(e^{\mathfrak{a}} + e^{-\mathfrak{a}})(7e^{-2\mathfrak{a}} + 58 + 7e^{2\mathfrak{a}})}{(e^{\mathfrak{a}} - e^{-\mathfrak{a}})^{10}} + \mathcal{O}(\mathcal{R}^{16})$$

$$a^{(I)} := \sigma^{(I)} + \frac{i}{R} \oiiint_{S^1} A_m^{(I)} \mathrm{d}x^m$$

$$\mathcal{R}^4 = \exp\left[Ra^{(I)}\right]$$

$$V^{(K)} = \left(A_\mu^{(K)}, \phi^{(K)}, \lambda_A^{(K)}, D_{AB}^{(K)}\right)$$

$$\mathcal{F}(a,R,\Lambda) = \mathcal{F}^{\text{pert}}(a,R,\Lambda) + \mathcal{F}^{\text{inst}}(a,R,\Lambda)$$

$$\mathcal{F}^{\text{pert}}(a,R,\Lambda) = -4a^2 \log(R\Lambda) + \frac{2\left(\zeta(3) - \mathrm{Li}_3(e^{-2Ra})\right)}{R^2} - \frac{2\pi^2}{3R} a + 2\pi i a^2 + \frac{4R}{3} a^3.$$

$$R^2 \mathcal{F}^{\text{pert}}(\mathfrak{a}, R\Lambda) = -4\mathfrak{a}^2 \log(R\Lambda) + 2\left(\zeta(3) - \mathrm{Li}_3(e^{-2\mathfrak{a}})\right) - \frac{2\pi^2}{3} \mathfrak{a} + 2\pi i \mathfrak{a}^2 + \frac{4}{3} \mathfrak{a}^3$$

$$\mathcal{F}^{\text{inst}}(a,R,\Lambda) = \frac{\sinh^2(Ra)}{R^2} \sum_{n=1}^{\infty} \mathcal{F}_n(\sinh^2(Ra)) \left(\frac{\mathcal{R}}{\sinh(Ra)}\right)^{4n}$$

$$R^2 \mathcal{F}^{\text{inst}}(\mathfrak{a},\mathcal{R}) = \sum_{n=1}^{\infty} \frac{\mathcal{F}_n(\sinh^2(\mathfrak{a}))}{\sinh^{4n-2}(\mathfrak{a})} \mathcal{R}^{4n}$$

$$\mathcal{F}_1(x) = -\frac{1}{2}, \mathcal{F}_2(x) = -\frac{5+x}{64}, \cdots$$

$$\mathcal{F}(a,R,\Lambda) \to 4a^2 \left(\log\left(\frac{2a}{\Lambda}\right) - \frac{3}{2}\right) - \frac{\Lambda^4}{2a^2} - \frac{5\Lambda^8}{64a^6} + \cdots$$

$$\mathrm{Li}_3(e^{-2\mathfrak{a}}) = \zeta(3) - \frac{\pi^2 \mathfrak{a}}{3} - 2\mathfrak{a}^2 \log(\mathfrak{a}) + (3 - 2\log 2)\mathfrak{a}^2 + \mathcal{O}(\mathfrak{a}^3),$$

$$\lim_{R\to 0} \frac{U-2}{R^2} = \frac{1}{2} \langle \mathrm{tr}_F \phi^2 \rangle = u$$

$$\phi = \frac{1}{R} \oiiint_{S^1} (\sigma + iA_5) \mathrm{d}x^5$$

$$\frac{U-2}{R^2} \to a^2 \left[1 + \frac{\Lambda^4}{2a^4} + \mathcal{O}\left(\frac{\Lambda^8}{a^8}\right)\right]$$

$$\mathcal{F}^{\text{pert}}\left(\tilde{a}-\frac{i\pi}{R}\middle|\, R,\Lambda\right)=-4\tilde{a}^2\log(R\Lambda)+\frac{2\left(\zeta(3)-\mathrm{Li}_3\left(e^{-2R\tilde{a}}\right)\right)}{R^2}-\frac{2\pi^2\tilde{a}}{3R}-2\pi i\tilde{a}^2+\frac{4R\tilde{a}^3}{3}+\frac{4\pi^2}{R^2}\log(R\Lambda)+\frac{8\pi i\tilde{a}\log(R\Lambda)}{R}$$

$$\frac{U(\tilde{a}-i\pi,\mathcal{R})+2}{R^2}\to-\tilde{a}^2\left[1+\frac{\Lambda^4}{2\tilde{a}^4}+\mathcal{O}\left(\frac{\Lambda^8}{\tilde{a}^8}\right)\right].$$

$$\mathcal{F}^{\text{pert}}\to R\left(\frac{8\pi^2}{g_{5d}^2}\sigma^2+\frac{4}{3}\sigma^3\right)$$

$$w+w^{-1}=-\mathcal{R}^{-2}(\mathcal{X}+U+\mathcal{X}^{-1})$$

$$\tau=-\frac{1}{2\pi i}\frac{\partial^2\mathcal{F}}{\partial a^2}$$

$$\mathcal{Y}^2=P(\mathcal{X})^2-4\mathcal{X}^2\mathcal{R}^4$$

$$R\lambda=\frac{1}{2\pi i}\log(\mathcal{X})\frac{\mathrm{d}w}{w}$$

$$\mathrm{dlog}\ w=(\mathcal{X}-\mathcal{X}^{-1})\frac{\mathrm{d}\mathcal{X}}{\mathcal{Y}}$$

$$R\lambda=\frac{1}{2\pi i}\log(\mathcal{X})(\mathcal{X}-\mathcal{X}^{-1})\frac{\mathrm{d}\mathcal{X}}{\mathcal{Y}}$$

$$(\mathcal{R}^2,U,\mathcal{X},w)\mapsto(\mathcal{R}^2,-U,-\mathcal{X},-w)$$

$$\lambda\to\lambda\pm\frac{1}{2R}\frac{\mathrm{d}w}{w}$$

$$(\mathcal{X}^2+(U-2\mathcal{R}^2)\mathcal{X}+1)(\mathcal{X}^2+(U+2\mathcal{R}^2)\mathcal{X}+1)=0$$

$$\mathcal{X}^{+\pm}=\frac{-(U+2\mathcal{R}^2)\pm\sqrt{(U+2\mathcal{R}^2)^2-4}}{2}$$
$$\mathcal{X}^{-\pm}=\frac{-(U-2\mathcal{R}^2)\pm\sqrt{(U-2\mathcal{R}^2)^2-4}}{2}$$

$$U_1=2-2\mathcal{R}^2,U_2=2+2\mathcal{R}^2,U_3=-2+2\mathcal{R}^2,U_4=-2-2\mathcal{R}^2$$

$$\mathfrak{a}=Ra=\oiiint_{\mathcal{A}}R\lambda,a_D=\oiiint_{\mathcal{B}}\lambda$$

$$\left.\frac{\partial(R\lambda)}{\partial U}\right|_{w,\mathcal{R}}=\frac{1}{2\pi i}\frac{\partial\log\,\mathcal{X}}{\partial U}\frac{\mathrm{d}w}{w}$$

$$\left.\frac{\mathrm{d}(\log\,\mathcal{X})}{\mathrm{d}U}\right|_{w,\mathcal{R}}=-(\mathcal{X}-\mathcal{X}^{-1})^{-1}$$

$$\left.\frac{\partial(R\lambda)}{\partial U}\right|_{w,\mathcal{R}} = -\frac{1}{2\pi i}\frac{\mathrm{d}\mathcal{X}}{\mathcal{Y}}$$

$$x = \alpha\frac{\mathcal{X}+1}{\mathcal{X}-1}, y = \beta\frac{\mathcal{Y}}{(\mathcal{X}-1)^2}$$

$$y^2 = (1-x^2)(1-\mathcal{K}^2x^2)$$

$$\frac{\mathrm{d}x}{y} = -2\frac{\alpha}{\beta}\frac{\mathrm{d}\mathcal{X}}{\mathcal{Y}}$$

$$\mathcal{K} = \frac{\vartheta_2(\tilde{\tau})^2}{\vartheta_3(\tilde{\tau})^2}$$

$$U(\tau)^2 = -8\mathcal{R}^2\mathbf{u}(\tau) + 4\mathcal{R}^4 + 4$$

$$\mathrm{u}(\tau) = \frac{\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{2\vartheta_2(\tau)^2\vartheta_3(\tau)^2}.$$

$$\begin{array}{ccc} \widetilde{\mathbb{H}} & \xrightarrow{\tilde{\pi}_{\mathrm{u}}} & \mathbb{C}_U \\ \downarrow \pi & & \downarrow \pi_U \\ \mathbb{H} & \xrightarrow{\pi_{\mathrm{u}}} & \mathbb{C}_{\mathbf{u}} \end{array}$$

$$\mathrm{u}(\tau) = \frac{1}{2}(\mathcal{R}^2 + \mathcal{R}^{-2}),$$

$$\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} = \pm\mathcal{R}, \pm\mathcal{R}^{-1}$$

$$\tau_{\mathrm{bp}} = \frac{i}{2}\frac{K_{\mathrm{e}}\left(\sqrt{1-\mathcal{R}^4}\right)}{K_{\mathrm{e}}(\mathcal{R}^2)} \bmod 4,$$

$$\tau_{\mathrm{bp}} = \frac{4i}{\pi}\log\left(\frac{2}{\mathcal{R}}\right) - \frac{i}{2\pi}\mathcal{R}^4 + \mathcal{O}(\mathcal{R}^8) \bmod 4.$$

$$\tau_{\mathrm{bp}} \to i\infty \ \text{ for } \mathcal{R} \to 0.$$

$$\Delta_{\mathrm{phys}} = \prod_{j=1}^{4}\left(U - U_j\right) = 64\mathcal{R}^4(\mathrm{u}^2 - 1)$$

$$\frac{\mathrm{da}}{\mathrm{d}U} = \frac{i}{2\mathcal{R}}\vartheta_2(\tau)\vartheta_3(\tau)$$

$$\frac{\mathrm{d}U}{\mathrm{d}\tau}=-\frac{4\mathcal{R}^2}{U}\frac{\mathrm{du}}{\mathrm{d}\tau}.$$

$$\frac{\mathrm{da}}{\mathrm{d}\tau}=-\frac{4\mathcal{R}^2}{U}\frac{\mathrm{du}}{\mathrm{d}\tau}\frac{\mathrm{da}}{\mathrm{d}U}$$

$$\frac{\mathrm{du}}{\mathrm{d}\tau}=-\frac{i\pi}{4}\frac{\vartheta_4^8}{(\vartheta_2\vartheta_3)^2}$$

$$\frac{\mathrm{da}}{\mathrm{d}\tau}=-\frac{\pi\mathcal{R}}{4}\frac{1}{U}\frac{\vartheta_4(\tau)^9}{\eta(\tau)^3}$$

$$v=\frac{R}{4}\frac{\partial a_D}{\partial\log(\mathcal{R})}$$

$$v=-\frac{1}{\pi}\frac{1}{\vartheta_2(\tau)\vartheta_3(\tau)}\int_0^{\mathcal{R}}\frac{\mathrm{d}x}{\sqrt{1-2\mathrm{u}x^2+x^4}}$$

$$x^2=\left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^2,\ \text{and}\ \ x^2=\left(\frac{\vartheta_3(\tau)}{\vartheta_2(\tau)}\right)^2$$

$$\left|\mathcal{R}q^{-\frac{1}{8}}\right|\ll 1$$

$$v\ =-\frac{1}{\pi}\frac{1}{\vartheta_2(\tau)\vartheta_3(\tau)}\sum_{\substack{n\geq 0\\ n\geq k\geq 0}}\begin{pmatrix}-\frac{1}{2}\\ n\end{pmatrix}\binom{n}{k}\frac{(-2\mathrm{u})^k\mathcal{R}^{4n-2k+1}}{4n-2k+1}$$

$$=-\frac{1}{\sqrt{\pi}}\frac{1}{\vartheta_2(\tau)\vartheta_3(\tau)}\sum_{s=0}^{\infty}\sum_{k=0}^{s}\frac{(-2\mathrm{u})^k\mathcal{R}^{2s+1}}{(2s+1)k!\,\Gamma\left(\frac{s-k}{2}+1\right)\Gamma\left(\frac{1-s-k}{2}\right)}$$

$$K=\frac{\vartheta_2^2(\tau)}{\vartheta_3^2(\tau)}$$

$$v=-\frac{i}{\pi\vartheta_3(\tau)^2}\int_0^{\mathcal{R}}\frac{\mathrm{d}x}{\sqrt{(1-Kx^2)(x^2-K)}}$$

$$\frac{1}{\sqrt{1-Kx^2}}=\sum_{\ell=0}^{\infty}\frac{(2\ell-1)!!}{2^\ell\ell!}(Kx^2)^\ell$$

$$\int_0^{\mathcal{R}}\frac{\mathrm{d}x}{\sqrt{x^2-K}}=\frac{1}{2}\log\left(1+\frac{x}{\sqrt{x^2-K}}\right)-\frac{1}{2}\log\left(1-\frac{x}{\sqrt{x^2-K}}\right)\Bigg|_0^{\mathcal{R}}$$

$$\frac{\mathcal{R}}{\sqrt{\mathcal{R}^2-K}}=\sum_{\ell=0}^{\infty}\frac{(2\ell-1)!!}{2^\ell\ell!}(K/\mathcal{R}^2)^\ell$$

$$v=\frac{i}{2\pi}(\log(-K/4)-2\log(\mathcal{R})+\cdots)$$

$$v=-\frac{\tau}{4}-\frac{1}{2}+2s-\frac{i}{\pi}\log(\mathcal{R})+\mathcal{O}\left(q^{\frac{1}{4}}\right)$$

$$v=-\frac{\tau}{4}-\frac{1}{2}-\frac{i}{\pi}\log(\mathcal{R})+\mathcal{O}\left(q^{\frac{1}{4}}\right)\ \mathrm{mod}2$$

$$\lambda=\left(-\frac{1}{2\pi i}x+\frac{\log(\mp 1)}{2\pi i R}\right)\frac{\mathrm{d}w}{w}$$

$$-\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F\wedge F=1$$

$$Z_I:=a^{(I)}=\frac{1}{R}\log(\mathcal{R}^4)$$

$$a_D=-\frac{1}{2\pi i}\frac{\partial\mathcal{F}\left(a,a^{(I)},a^{(K)}\right)}{\partial a}.$$

$$a_D^{(I)}=-\frac{1}{2\pi i}\frac{\partial\mathcal{F}}{\partial a^{(I)}},a_D^{(K)}=-\frac{1}{2\pi i}\frac{\partial\mathcal{F}}{\partial a^{(K)}}$$

$$\Pi=\left(a_D,a,a_D^{(I)},a^{(I)},a_D^{(K)},a^{(K)}\right)^{\mathrm{T}}$$

$$\Omega=\begin{pmatrix}0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\end{pmatrix}$$

$$\begin{aligned}\tau&:=-\frac{1}{2\pi i}\frac{\partial^2\mathcal{F}}{\partial a^2}\\ v_I&:=v=-\frac{1}{2\pi i}\frac{\partial^2\mathcal{F}}{\partial a\partial a^{(I)}}=-\frac{R}{8\pi i}\frac{\partial^2\mathcal{F}}{\partial a\partial\log\mathcal{R}},\\ v_K&:=-\frac{1}{2\pi i}\frac{\partial^2\mathcal{F}}{\partial a\partial a^{(K)}},\\ \xi_{II}&:=-\frac{1}{2\pi i}\frac{\partial^2\mathcal{F}}{\partial(a^{(I)})^2}=-\frac{1}{2\pi i}\frac{R^2}{16}\frac{\partial^2\mathcal{F}}{\partial\log(\mathcal{R})^2},\\ \xi_{KK}&:=-\frac{1}{2\pi i}\frac{\partial^2\mathcal{F}}{\partial(a^{(K)})^2},\\ \xi_{IK}&:=-\frac{1}{2\pi i}\frac{\partial^2\mathcal{F}}{\partial a^{(I)}\partial a^{(K)}}.\end{aligned}$$

$$\begin{aligned}Ra_D&=\frac{1}{2\pi i}\left[-4(\mathrm{Li}_2(e^{-2\mathfrak{a}})-\zeta(2))+2\mathfrak{a}\log(\mathcal{R}^4)-4\pi i\mathfrak{a}-4\mathfrak{a}^2+\cdots\right]\\ Ra_D^{(I)}&=\frac{1}{2\pi i}(\mathfrak{a}^2+\cdots)\\ Ra_D^{(K)}&=\frac{1}{(Ra^{(K)})^2}\left[4(\mathrm{Li}_3(e^{-2\mathfrak{a}})-\zeta(3))+4\mathfrak{a}\mathrm{Li}_2(e^{-2\mathfrak{a}})-\mathfrak{a}^2\log(\mathcal{R}^4)+\frac{2\pi^2\mathfrak{a}}{3}+\frac{4\mathfrak{a}^3}{3}+\cdots\right]\end{aligned}$$

$$\tau = -\frac{1}{2\pi i}\frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{1}{2\pi i}\left(8\log(\mathcal{R}) - 8\log[2\sinh \mathfrak{a}] - 4\pi i + \mathcal{O}(\mathcal{R}^4)\right)$$

$$\frac{2\pi i\tau}{8} = \begin{cases} -\mathfrak{a} + \log \mathcal{R} - \frac{i\pi}{2} + \mathcal{O}(\mathcal{R}, e^{-\mathfrak{a}}), & \mathrm{Re}(\mathfrak{a}) \gg 0 \\ \mathfrak{a} \pm i\pi + \log \mathcal{R} - \frac{i\pi}{2} + \mathcal{O}(\mathcal{R}, e^{\mathfrak{a}}), & \mathrm{Re}(\mathfrak{a}) \ll 0 \end{cases}$$

$$q^{\frac{1}{8}} = \begin{cases} -i\mathcal{R}e^{-\mathfrak{a}}\left(1 + \mathcal{O}(\mathcal{R}, e^{-\mathfrak{a}})\right), & \mathrm{Re}(\mathfrak{a}) \gg 0 \\ i\mathcal{R}e^{\mathfrak{a}}\left(1 + \mathcal{O}(\mathcal{R}, e^{\mathfrak{a}})\right), & \mathrm{Re}(\mathfrak{a}) \ll 0 \end{cases}$$

$$U = \pm i\left(\mathcal{R}e^{-2\pi i\tau/8} - \frac{2}{\mathcal{R}}e^{2\pi i\tau/8} + \mathcal{O}\left(q^{3/8}\right)\right)$$

$$v = R\frac{\partial a_D}{\partial \log(\mathcal{R}^4)} = -\frac{\tau}{4} + \cdots$$

$$\frac{\vartheta_1(\tau, v)}{\vartheta_4(\tau, v)} = \mathcal{R}$$

$$v \notin \mathbb{Z}\tau + \mathbb{Z}, v \notin \frac{\tau}{2} + \mathbb{Z}\tau + \mathbb{Z}$$

$$v = -\frac{\tau}{4} + \frac{1}{2\pi i}\Big[2\log(\mathcal{R}) - \pi i + 2(\mathcal{R}^2 - \mathcal{R}^{-2})q^{\frac{1}{4}} + 3(\mathcal{R}^4 - \mathcal{R}^{-4})q^{\frac{1}{2}} + \left(\frac{20}{3}\mathcal{R}^6 - 4\mathcal{R}^2 + 4\mathcal{R}^{-2} - \frac{20}{3}\mathcal{R}^{-6}\right)q^{\frac{3}{4}} + \cdots\Big]$$

$$\left|\frac{\vartheta_2^2}{\vartheta_3^2}\right| < \min(|\mathcal{R}|^2, |\mathcal{R}|^{-2})$$

$$C_{II} := \exp(-\pi i\xi_{II}), C_{KK} := \exp(-\pi i\xi_{KK}), C_{IK} := \exp(-\pi i\xi_{IK})$$

$$C(\tau, \mathcal{R}) = \frac{1}{\mathcal{R}}\frac{\vartheta_1(\tau, v)}{\vartheta_4(\tau)}$$

$$C(\tau, \mathcal{R}) = \frac{\vartheta_4(\tau, v)}{\vartheta_4(\tau)}$$

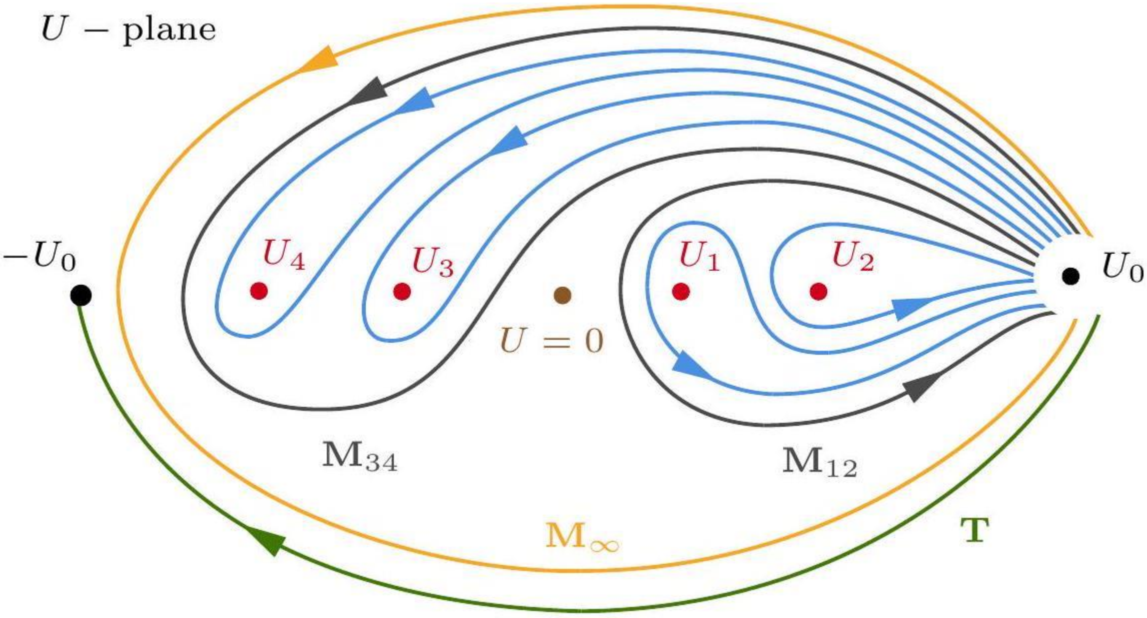
$$w = \mathcal{R}^4 q^{-\frac{1}{2}} + 4\mathcal{R}^2(\mathcal{R}^4 - 1)q^{-\frac{1}{4}} + 2(1 - 8\mathcal{R}^4 + 7\mathcal{R}^8) + 16\mathcal{R}^2(1 - 4\mathcal{R}^4 + 3\mathcal{R}^8)q^{\frac{1}{4}} + (-\mathcal{R}^{-4} + 92\mathcal{R}^4 - 256\mathcal{R}^8 + 165\mathcal{R}^{12})q^{\frac{1}{2}} + \mathcal{O}\left(q^{\frac{3}{4}}\right)$$

$$w = 1 + \left(-2iq^{-\frac{1}{8}} + \mathcal{O}\left(q^{\frac{3}{8}}\right)\right)\mathcal{R} + \left(-2q^{-\frac{1}{4}} + \mathcal{O}\left(q^{\frac{1}{4}}\right)\right)\mathcal{R}^2 + \mathcal{O}(\mathcal{R}^3)$$

$$\mathrm{Ser}_{\mathcal{R}}\mathrm{Ser}_q\left(e^{2\pi ikv_q} + e^{-2\pi ikv_q}\right) = \mathrm{Ser}_q\mathrm{Ser}_{\mathcal{R}}\left(e^{2\pi ikv_{\mathcal{R}}} + e^{-2\pi ikv_{\mathcal{R}}}\right),$$

$$\mathrm{Ser}_{\mathcal{R}}\mathrm{Ser}_{q}C(\tau,\mathcal{R}) = 1+\mathcal{R}^2q^{\frac{1}{4}}+2\mathcal{R}^4q^{\frac{1}{2}}+(-2\mathcal{R}^2+5\mathcal{R}^6)q^{\frac{3}{4}} +(-10\mathcal{R}^4+14\mathcal{R}^8)q+\mathcal{O}\left(q^{\frac{5}{4}},\mathcal{R}^9\right)$$

$$\mathrm{Ser}_{q}\mathrm{Ser}_{\mathcal{R}}C(\tau,\mathcal{R}) = 1+\left(q^{\frac{1}{4}}-2q^{3/4}\right)\mathcal{R}^2+\left(2q^{\frac{1}{2}}-10q\right)\mathcal{R}^4 +5q^{\frac{3}{4}}\mathcal{R}^6+\mathcal{O}\left(q^{\frac{5}{4}},\mathcal{R}^8\right)$$



$$\begin{aligned} a_D &\mapsto a_D+4a-a^{(I)},\\ a &\mapsto a-\frac{1}{2}a_K,\\ a_D^{(I)} &\mapsto a_D^{(I)}-a+\frac{1}{4}a^{(K)},\\ a^{(I)} &\mapsto a^{(I)},\\ a_D^{(K)} &\mapsto \frac{1}{2}a_D+2a-\frac{1}{4}a^{(I)}+a_D^{(K)},\\ a^{(K)} &\mapsto a^{(K)}. \end{aligned}$$

$$\mathbf{T}=\begin{pmatrix} 1 & 4 & 0 & -1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2}\\ 0 & -1 & 1 & 0 & 0 & \frac{1}{4}\\ 0 & 0 & 0 & 1 & 0 & 0\\ \frac{1}{2} & 2 & 0 & -\frac{1}{4} & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
\begin{aligned}
\tau &\mapsto \tau + 4,\\
v_I &\mapsto v_I - 1,\\
v_K &\mapsto v_K + \frac{1}{2}\tau + 2,\\
\xi_{II} &\mapsto \xi_{II},\\
\xi_{IK} &\mapsto \xi_{IK} + \frac{1}{2}v_I - \frac{1}{4},\\
\xi_{KK} &\mapsto \xi_{KK} + v_K + \frac{1}{4}\tau + 1,
\end{aligned}
$$

$$
\begin{aligned}
a_D &\mapsto a_D - 8a + 2a^{(I)} - 6a^{(K)},\\
a &\mapsto a + a^{(K)},\\
a_D^{(I)} &\mapsto a_D^{(I)} + 2a + a^{(K)},\\
a^{(I)} &\mapsto a^{(I)},\\
a_D^{(K)} &\mapsto -a_D + 2a - a^{(I)} + a_D^{(K)} + a^{(K)},\\
a^{(K)} &\mapsto a^{(K)},
\end{aligned}
$$

$$
\mathbf{M}_\infty = \begin{pmatrix} 1 & -8 & 0 & 2 & 0 & -6\\ 0 & 1 & 0 & 0 & 0 & 1\\ 0 & 2 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\\ -1 & 2 & 0 & -1 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\begin{aligned}
\tau &\mapsto \tau - 8,\\
v_I &\mapsto v_I + 2,\\
v_K &\mapsto v_K - \tau + 2,\\
\xi_{II} &\mapsto \xi_{II},\\
\xi_{KK} &\mapsto \xi_{KK} - 2v_K + \tau - 1,\\
\xi_{KI} &\mapsto \xi_{KI} - v_I - 1.
\end{aligned}
$$

$$
\frac{1}{2} + \frac{\log\left(-e^{2\mathfrak{a}}\right)}{2\pi i} = \frac{1}{2} + \frac{\ln\left(e^{\pi i}e^{2\mathfrak{a}}\right)}{2\pi i} = 1 + \frac{\mathfrak{a}}{\pi i}
$$

$$
\begin{aligned}
a_D &\mapsto -a_D + 4a,\\
a &\mapsto e^{\pi i}a = -a,\\
a_D^{(I)} &\mapsto a_D^{(I)},\\
a^{(I)} &\mapsto a^{(I)},\\
a_D^{(K)} &\mapsto a_D^{(K)},\\
a^{(K)} &\mapsto a^{(K)},
\end{aligned}
$$

$$
\mathbf{M}_{12} = \begin{pmatrix} -1 & 4 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

$$\mathbf{M}_{34} = \begin{pmatrix} -1 & 4 & 0 & -2 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} a_D &\mapsto a_D \\ a &\mapsto -a_D + a \\ a_D^{(I)} &\mapsto a_D^{(I)} \\ a^{(I)} &\mapsto a^{(I)} \\ a_D^{(K)} &\mapsto a_D^{(K)} \\ a^{(K)} &\mapsto a^{(K)} \end{aligned}$$

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\frac{\partial\left(a, a^{(I)}, a^{(K)}\right)}{\partial(\tilde{a}, \tilde{a}^{(I)}, \tilde{a}^{(K)})} = \left(\frac{\partial\left(\tilde{a}, \tilde{a}^{(I)}, \tilde{a}^{(K)}\right)}{\partial(a, a^{(I)}, a^{(K)})}\right)^{-1}$$

$$\begin{pmatrix} \partial a/\partial\tilde{a} & \partial a/\partial\tilde{a}^{(I)} & \partial a/\partial\tilde{a}^{(K)} \\ \partial a^{(I)}/\partial\tilde{a} & \partial a^{(I)}/\partial\tilde{a}^{(I)} & \partial a^{(I)}/\partial\tilde{a}^{(K)} \\ \partial a^{(K)}/\partial\tilde{a} & \partial a^{(K)}/\partial\tilde{a}^{(I)} & \partial a^{(K)}/\partial\tilde{a}^{(K)} \end{pmatrix} = \frac{1}{-\tau+1}\begin{pmatrix} 1 & v_I & v_K \\ 0 & -\tau+1 & 0 \\ 0 & 0 & -\tau+1 \end{pmatrix}.$$

$$\tilde{\tau} = \frac{\partial\tilde{a}_D}{\partial a}\frac{\partial a}{\partial\tilde{a}} + \frac{\partial\tilde{a}_D}{\partial a^{(I)}}\frac{\partial a^{(I)}}{\partial\tilde{a}} + \frac{\partial\tilde{a}_D}{\partial a^{(K)}}\frac{\partial a^{(K)}}{\partial\tilde{a}} = \frac{\tau}{-\tau+1},$$

$$\tilde{\xi}_{II} = \frac{\partial\tilde{a}_D^{(I)}}{\partial a}\frac{\partial a}{\partial\tilde{a}^{(I)}} + \frac{\partial\tilde{a}_D^{(I)}}{\partial a^{(I)}}\frac{\partial a^{(I)}}{\partial\tilde{a}^{I}} + \frac{\partial\tilde{a}_D^{(I)}}{\partial a^{(K)}}\frac{\partial a^{(K)}}{\partial\tilde{a}^{(I)}} = \frac{v_I^2}{-\tau+1} + \xi_{II}$$

$$\begin{aligned} \tau &\mapsto \frac{\tau}{-\tau+1}, \\ v_I &\mapsto \frac{v_I}{-\tau+1}, \\ v_K &\mapsto \frac{v_K}{-\tau+1}, \\ \xi_{II} &\mapsto \xi_{II} + \frac{v_I^2}{-\tau+1}, \\ \xi_{KK} &\mapsto \xi_{KK} + \frac{v_K^2}{-\tau+1}, \\ \xi_{KI} &\mapsto \xi_{KI} + \frac{v_K v_I}{-\tau+1}. \end{aligned}$$

$$\mathbf{M}_2 = \begin{pmatrix} -1 & 4 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{M}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned} \tau &\mapsto \frac{\tau}{-\tau+1}, \\ v_I &\mapsto \frac{v_I+\tau}{-\tau+1}, \\ v_K &\mapsto \frac{v_K}{-\tau+1}, \\ \xi_{II} &\mapsto \xi_{II} + \frac{(v_I+1)^2}{-\tau+1}, \\ \xi_{KK} &\mapsto \xi_{KK} + \frac{v_K^2}{-\tau+1}, \\ \xi_{IK} &\mapsto \xi_{IK} + \frac{(v_I+1)v_K}{-\tau+1}. \end{aligned}$$

$$\mathbf{M}_4 = \begin{pmatrix} -1 & 4 & 0 & -2 & 0 & 2 \\ -1 & 3 & 0 & -1 & 0 & 1 \\ 1 & -2 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \tau &\mapsto \frac{\tau-4}{\tau-3}, \\ v_I &\mapsto \frac{v_I+\tau-2}{-\tau+3}, \\ v_K &\mapsto \frac{v_K-\tau+2}{-\tau+3}, \\ \xi_{II} &\mapsto \xi_{II} + \frac{(v_I+1)^2}{3-\tau}, \\ \xi_{IK} &\mapsto \xi_{IK} + \frac{(v_K-1)(v_I+1)}{3-\tau}, \\ \xi_{KK} &\mapsto \xi_{KK} + \frac{(v_K-1)^2}{3-\tau}. \end{aligned}$$

$$\Gamma^0(8) = \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, b = 0 \bmod 8 \right\}.$$

$$\Gamma^0(4) = \Gamma^0(8) \sqcup \Gamma^0(8) \cdot T^4$$

$$\mathbb{H}/\Gamma^0(8) \to \mathbb{H}/\Gamma^0(4)$$

$$U(\tau) = \mp 2 \frac{\vartheta_2(\tau)^2 + \vartheta_3(\tau)^2}{\vartheta_2(\tau)\vartheta_3(\tau)}$$

$$U(\tau) = \mp 2i\frac{\vartheta_2(\tau)^2 - \vartheta_3(\tau)^2}{\vartheta_2(\tau)\vartheta_3(\tau)}$$

$$\tilde{\Gamma}^0(8) = T^{-2}\Gamma^0(8)T^2 := \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A = T^{-2}BT^2, B \in \Gamma^0(8)\}$$

$$\begin{aligned}
[\overline{\mathcal{Q}}, A_X] &= -\psi \\
[\overline{\mathcal{Q}}, A_5] &= \eta \\
[\overline{\mathcal{Q}}, \sigma] &= -i\eta \\
\{\overline{\mathcal{Q}}, \psi\} &= 2\left(D_{A_X}\sigma - i\iota_{\partial_5}F\right) \\
\{\overline{\mathcal{Q}}, \eta\} &= -2D_5\sigma \\
\{\overline{\mathcal{Q}}, \chi\} &= -2iF_+ - iD \\
[\overline{\mathcal{Q}}, D] &= 2\left(D_{A_X}\psi\right)_+ - 2[D_5 - \sigma, \chi]
\end{aligned}$$

$$\begin{aligned}
[\mathcal{K}, A_X] &= \frac{i}{2}\chi \\
[\mathcal{K}, A_5] &= -\frac{i}{4}\psi \\
[\mathcal{K}, \sigma] &= \frac{1}{4}\psi \\
\{\mathcal{K}, \psi\} &= D - 2F_- \\
\{\mathcal{K}, \eta\} &= -\frac{i}{2}\left(D_\mu\sigma + iF_{\mu 5}\right)\mathrm{d}x^\mu \\
\{\mathcal{K}, \chi\} &= \frac{i}{8}\epsilon_{\nu\mu\rho\sigma}(D^\nu\sigma + iF^{\nu 5})\mathrm{d}x^\mu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \\
[\mathcal{K}, D] &= -\frac{i}{8}\epsilon_{\gamma\tau\mu\nu}\left(D^\gamma\eta + 2D^\rho\chi^\gamma_\rho + [D_5 + \sigma, \psi^\gamma]\right)\mathrm{d}x^\tau \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu
\end{aligned}$$

$$F^{(I)} = 0, \partial_\mu \oiiint_{S^1} A^{(I)} = 0, \partial_5 A^{(I)}_\mu = 0$$

$$\sigma^{(I)} = -\frac{8\pi^2}{g^2_{5d}}, (D^{AB})^{(I)} = 0, \lambda^{(I)}_A = 0$$

$$S_{\text{twisted}} = \frac{i}{8\pi^2}\int_{X\times S^1} A^{(I)} \wedge \mathrm{tr}(F \wedge F) - \frac{\sigma^{(I)}}{8\pi^2}\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\mathcal{L}^{\text{twisted}}_{\text{SYM}}$$

$$\begin{aligned}
\mathcal{L}^{\text{twisted}}_{\text{SYM}} = \mathrm{tr}\Big(&\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + F_{\mu 5}F^{\mu 5} - \frac{1}{4}D_{\mu\nu}D^{\mu\nu} + D_\mu\sigma D^\mu\sigma + D_5\sigma D^5\sigma \\
&+2i\chi^{\mu\nu}D_\mu\psi_\nu - \frac{i}{2}\chi^{\mu\nu}[D_5 - \sigma, \chi_{\mu\nu}] - \frac{i}{2}\eta[D_5 - \sigma, \eta] + i\psi^\mu D_\mu\eta + \frac{i}{2}\psi^\mu[D_5 + \sigma, \psi_\mu]\Big)
\end{aligned}$$

$$\mathcal{V} = \frac{1}{g^2_{5d}}\int \mathrm{d}^5x\sqrt{g}\mathrm{tr}\left(\frac{i}{4}\chi^{\mu\nu}(2F_{\mu\nu} - D_{\mu\nu}) - \frac{i}{2}\psi^\mu F_{\mu 5} + \frac{1}{2}\psi^\mu D_\mu\sigma - \frac{1}{2}\eta D_5\sigma\right)$$

$$-\frac{\sigma^{(I)}}{8\pi^2}\int \mathrm{d}^5x\sqrt{g}\mathcal{L}^{\text{twisted}}_{\text{SYM}} - \overline{\mathcal{Q}}\mathcal{V} = -\frac{1}{4g^2_{5d}}\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\mathrm{tr}F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$$

$$S_{\text{twisted}} = \frac{i\theta}{8\pi^2}\int_X \operatorname{tr}(F\wedge F) - \frac{R}{g_{5d}^2}\int_X \operatorname{tr}(F\wedge F) + \overline{\mathcal{Q}}\mathcal{V}$$

$$S_{\text{twisted}} = \overline{\mathcal{Q}}\mathcal{V} + (\log \mathcal{R}^4)\frac{1}{8\pi^2}\int_X F\wedge F$$

$$\begin{aligned}\overline{\mathcal{Q}}\mathcal{V} \to \frac{1}{g_{5d}^2}\int \mathrm{d}^5x\sqrt{g}\operatorname{tr}\Big[&\lambda^{-1}\left(\left(F_+^{\mu\nu}\right)^2 + 2i\chi^{\mu\nu}D_\mu\psi_\nu - \frac{i}{2}\chi_{\mu\nu}[D_5-\sigma,\chi^{\mu\nu}] - \frac{1}{4}D_{\mu\nu}D^{\mu\nu}\right)\\ &+\left(D_\mu\sigma D^\mu\sigma + F_{\mu5}F^{\mu5} + i\psi_\mu D^\mu\eta + \frac{i}{2}\psi_\mu[D_5+\sigma,\psi^\mu]\right)\\ &+\lambda\left(D_5\sigma D^5\sigma - \frac{i}{2}\eta[D_5-\sigma,\eta]\right)\Big]\end{aligned}$$

$$M_{\boldsymbol{\mu}} = \coprod_{k=0}^{\infty} M_{\boldsymbol{\mu},k}$$

$$k = -\frac{1}{8\pi^2}\int_X \operatorname{tr}(F\wedge F)$$

$$\delta_I A = \frac{\partial}{\partial Z_I}A(x,Z) - D_A\alpha_I$$

$$\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\operatorname{tr}\left(F_{\mu5}F^{\mu5} + D_\mu\sigma D^\mu\sigma\right) = \int_{S^1} \mathrm{d}t g_{IJ}\dot{Z}^I\dot{Z}^J + \int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\operatorname{tr}\left(D_\mu\Phi D^\mu\bar{\Phi}\right)$$

$$g_{IJ} = \int_X \operatorname{tr}\left(\delta_I A\wedge * \,\delta_J A\right)$$

$$\begin{aligned}&\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\operatorname{tr}\left(\frac{i}{2}\psi^\mu[D_5+\sigma,\psi_\mu]\right)\\ &= \int_{S^1} \mathrm{d}t g_{IJ}\left(\frac{i}{2}\zeta^I\nabla_t\zeta^J\right) + \int_{X_4\times S^1} \mathrm{d}^5x\sqrt{g}\operatorname{tr}\left(\frac{i}{2}\psi^\mu[\bar{\Phi},\psi_\mu]\right)\end{aligned}$$

$$\Gamma_{IK,J} = \int_{X_4} \mathrm{d}^4x\sqrt{g_4}\operatorname{tr}\left(\delta_I A_\mu\nabla_K\left(\delta_J A^\mu\right)\right)$$

$$\int \mathrm{d}t g_{IJ}\left(\dot{Z}^I\dot{Z}^J + \frac{i}{2}\zeta^I\nabla_t\zeta^J\right) + \int \mathrm{d}t\int_X \mathrm{d}^4x\sqrt{g}\operatorname{tr}\left(D^\mu\Phi D_\mu\bar{\Phi} + \frac{i}{2}\psi^\mu[\bar{\Phi},\psi_\mu]\right)$$

$$D_\mu D^\mu\Phi = \frac{i}{2}[\psi_\mu,\psi^\mu]$$

$$\Phi = \frac{i}{2}[\nabla_I,\nabla_J]\zeta^I\zeta^J$$

$$\int_{S^1} \mathrm{d}t g_{IJ}\left(\dot{Z}^I\dot{Z}^J + \frac{i}{2}\zeta^I\nabla_t\zeta^J\right)$$

$$\begin{aligned}\partial\alpha^{(1,0)} + \left[A^{(1,0)},\alpha^{(1,0)}\right] + N_{jk}^{\bar{\imath}}\alpha_{\bar{\imath}}e^j\wedge e^k = 0\\ \bar{\partial}\alpha^{(0,1)} + \left[A^{(0,1)},\alpha^{(0,1)}\right] + N_{\bar{\jmath},\bar{k}}^{i}\alpha_i e^{\bar{\jmath}}\wedge e^{\bar{k}} = 0\end{aligned}$$

$$S=\frac{i}{8\pi^2}\int F^{(I)}\wedge CS_3(A)+(\mathfrak{S}_{\text{supersymmetric completion}})$$

$$\sigma^{(I)}=-\frac{8\pi^2}{g_{5d}^2},\lambda_A^{(I)}=0,D^{(I)}=F_+^{(I)}$$

$$S|_{\text{superparticle}}=S_{\text{YM}}+S_\Delta,$$

$$S_\Delta=\frac{i}{8\pi^2}\int F^{(I)}\wedge CS_3(A)-\frac{1}{8\pi^2}\int d^5x\sqrt{g}F_{mn}^{(I)}\text{tr}(\sigma F^{mn})$$

$$\int_{X\times S^1}F^{(I)}\wedge CS_3(A-i\sigma\,dx^5)=\int_{X\times S^1}F^{(I)}\wedge CS_3(A)-2i\int_{X\times S^1}F^{(I)}\wedge\text{tr}(F\sigma\,dx^5)$$

$$S_\Delta|_{\text{superparticle}}=\frac{i}{8\pi^2}\int_{X\times S^1}F^{(I)}\wedge CS_3(A-i\sigma\,dx^5)$$

$$A-i\sigma\,dx^5=A_X+(A_5-i\sigma)dx^5=A_X+\alpha_I\dot{Z}^I\,dx^5$$

$$e^{-S_\Delta}\to e^{-i\int_Y\mathcal{A}^{(I)}}$$

$$\mathcal{A}^{(I)}=\frac{1}{8\pi^2}\int_X F^{(I)}\wedge CS_3(\mathbb{A})$$

$$\frac{1}{2\pi}\,d_M\mathcal{A}^{(I)}=\int_X\frac{F^{(I)}}{2\pi}\wedge\frac{\text{tr}(\mathbb{F}\wedge\mathbb{F})}{8\pi^2}$$

$$S=\text{PD}\left(c_1\left(L^{(I)}\right)\right)\in H_2(X,\mathbb{Z})$$

$$\overline{c_1(\mathcal{L}^{(I)})}=-\mu_D(S).$$

$$\frac{i}{8\pi^2}\int_{X\times S^1}F^{(I)}\wedge\text{tr}\left(A\wedge dA+\frac{2}{3}A^3\right)=\frac{i}{8\pi^2}\int_{X_6}F^{(I)}\wedge\text{tr}(F\wedge F)$$

$$\frac{1}{8\pi^2}\int_Y F^{(I)}\wedge\text{tr}(F\wedge F)\in 2\pi\mathbb{Z}$$

$$\frac{1}{8\pi^2}\text{tr}(F\wedge F)=\frac{1}{4}c_1(\tilde{P})^2\ \text{mod}4$$

$$\int_Y \boldsymbol{n}_I\cup c_1(\tilde{P})^2\in 4\mathbb{Z}$$

$$\int_X \boldsymbol{n}_I\cup c_1(P)\in 2\mathbb{Z}$$

$$\exp\left(-i\oiiint_{S^1}\mathcal{A}^{(I)}\right)$$

$$\mathrm{Pfaff}(\not{D}_{\phi^*(TM)})\exp\left(-i\oiiint_{S^1}\mathcal{A}^{(I)}\right)$$

$$Z^J_{\boldsymbol{\mu},\boldsymbol{n}_I}(\mathcal{R})=\sum_l d_{\boldsymbol{\mu},\boldsymbol{n}_I}(l)\mathcal{R}^l$$

$$d_{\boldsymbol{\mu},\boldsymbol{n}_I}(l)=\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-RH}$$

$$l=4k-3\left(\frac{\chi+\sigma}{4}\right)=\frac{1}{2}\mathrm{vdim}_{\mathbb{R}}(M_{\boldsymbol{\mu},k})$$
$$=-4\boldsymbol{\mu}^2-3\chi_{\mathrm{h}}\mathrm{mod}4$$

$$\int_{M_{\boldsymbol{\mu},k}} e^{\mu_D(x)}\hat{A}(M_{\boldsymbol{\mu},k}),x\in H_2(X)$$

$$\int_{M_{\boldsymbol{\mu},k}} e^{\mu_D(x)}$$

$$\chi^H_{c_1}(L_{\mathrm{GNY}},\Lambda_{\mathrm{GNY}})=\sum_{d\geq 0}\chi\big(M^X_H(c_1,d),\mathcal{O}(\mu(L_{\mathrm{GNY}}))\Lambda^d_{\mathrm{GNY}},$$

$$\alpha: S\otimes K_N^{-\frac{1}{2}}\xrightarrow{\sim}\bigoplus_k\Lambda^{0,k}T^*N,$$

$$\mathrm{Ind}(\not{D}_L)=\mathrm{Ind}(\bar{\partial}_{\tilde{L}})$$

$$\tilde{L}\cong L\otimes K_N^{\frac{1}{2}}$$

$$\hat{A}(TN)=e^{-\frac{1}{2}c_1(T^{1,0}N)}\mathrm{Td}(T^{1,0}N).$$

$$Z^J_{\boldsymbol{\mu},\boldsymbol{n}_I}(\mathcal{R})=\chi^H_{c_1}(L_{\mathrm{GNY}},\Lambda_{\mathrm{GNY}})$$

$$\Lambda_{\mathrm{GNY}}=\mathcal{R}$$

$$\mu(L_{\mathrm{GNY}})\cong\mathcal{L}^{(I)}\otimes K^{\frac{1}{2}}_{M_{\boldsymbol{\mu},k}}$$

$$\mu_D(c_1(L_{\mathrm{GNY}}))=c_1(\mu(L_{\mathrm{GNY}}))$$

$$c_1(L_{\mathrm{GNY}})=-\boldsymbol{n}_I+c_1(K_X).$$

$$W_{\mathfrak{R}}(p)=\mathrm{tr}_{\mathfrak{R}}\mathrm{Pexp}\left[\oiiint_{p\times S^1}(\sigma+iA_5)\mathrm{d}x^5\right]$$

$$W_{\mathfrak{R}}^{(j)} = \mathcal{K}^j W_{\mathfrak{R}}, j = 1, \cdots, 4,$$

$$W_{\mathfrak{R}}^{(j)}\left(\Sigma^{(j)}\right) = \int_{\Sigma^{(j)}} W_{\mathfrak{R}}^{(j)}$$

$$\mathrm{tr}_{\mathfrak{R}} \mathrm{Pexp}\left(\int_{\gamma} i\mathcal{A}_p + \Phi \mathrm{d}t\right),$$

$$\mathrm{ch}\left(V_{\mathfrak{R},p}\right) = \left(\mathrm{tr}_{\mathfrak{R}} e^{i\mathbb{F}}\right)/p$$

$$S^+ \otimes \mathcal{L}^{(I)} \text{ by } S^+ \otimes \mathcal{L}^{(I)} \otimes V_{\mathfrak{R},p}$$

$$\langle W_{\mathfrak{R}}(p_1) \cdots W_{\mathfrak{R}}(p_k)\rangle,$$

$$\mathcal{O}(S) = \int_S \frac{1}{\sqrt{8}} \frac{\mathrm{d}U}{\mathrm{d}a}(F_- + D) + \frac{1}{32} \frac{\mathrm{d}^2 U}{\mathrm{d}a^2} \psi^2$$

$$\mathcal{L}_0 = \frac{i}{16\pi}\left(\bar{\tau}_{jk} F_+^j \wedge F_+^k + \tau_{jk} F_-^j \wedge F_-^k\right) - \frac{1}{8\pi} y_{jk} D^j \wedge D^k - \frac{i}{4\pi} \overline{\mathcal{F}}_{jkl} \eta^j \chi^k \wedge (D + F_+)^l$$

$$\tau_{jk} = -\frac{1}{2\pi i} \frac{\partial^2 \mathcal{F}}{\partial a^j \partial a^k}, y_{jk} = \mathrm{Im}\left(\tau_{jk}\right)$$

$$[\overline{\mathcal{Q}}, a] = 0, [\overline{\mathcal{Q}}, \bar{a}] = i\sqrt{2}\eta, [\overline{\mathcal{Q}}, A] = \psi, [\overline{\mathcal{Q}}, D] = (\mathrm{d}\psi)^+$$
$$\{\overline{\mathcal{Q}}, \psi\} = 4\sqrt{2}\, \mathrm{d}a, \{\overline{\mathcal{Q}}, \chi\} = i(F_+ - D), \{\overline{\mathcal{Q}}, \eta\} = 0$$

$$[\mathcal{K}, \bar{a}] = 0, [\mathcal{K}, a] = \frac{1}{4\sqrt{2}}\psi, [\mathcal{K}, A] = -2i\chi, [\mathcal{K}, D] = \frac{3i}{4} * \mathrm{d}\eta + \frac{3i}{2} \mathrm{d}\chi$$

$$\{\mathcal{K}, \psi\} = 2(D + F_-), \{\mathcal{K}, \eta\} = -\frac{i\sqrt{2}}{2} \mathrm{d}\bar{a}, \{\mathcal{K}, \chi\} = \frac{3\sqrt{2}i}{4} * \mathrm{d}\bar{a}$$

$$\boldsymbol{k} = \left[\frac{F}{4\pi}\right] \in \boldsymbol{\mu} + L, \boldsymbol{n}_I = \left[\frac{F^I}{2\pi}\right], \boldsymbol{n}_K = \left[\frac{F^K}{2\pi}\right].$$

$$F_{\mathrm{SU}(4)} = \frac{1}{2}\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}.$$

$$\begin{aligned}
\mathcal{L}_0|_{\text{superparticle}} = & \frac{i}{16\pi}(\bar{\tau} F_+ \wedge F_+ + \tau F_- \wedge F_-) - \frac{y}{8\pi} D \wedge D \\
& + \frac{i}{4\pi}\left(\bar{v}_I F_+^{(I)} \wedge F_+ + v_I F_-^{(I)} \wedge F_-\right) - \frac{\mathrm{Im}(v_I)}{2\pi} D^{(I)} \wedge D \\
& + \frac{i}{4\pi}\left(\bar{\xi}_{II} F_+^{(I)} \wedge F_+^{(I)} + \xi_{II} F_-^{(I)} \wedge F_-^{(I)}\right) - \frac{\mathrm{Im}(\xi_{II})}{2\pi} D^{(I)} \wedge D^{(I)} \\
& + \frac{i}{4\pi}\left(\bar{v}_K F_+^{(K)} \wedge F_+ + v_K F_-^{(K)} \wedge F_-\right) - \frac{\mathrm{Im}(v_K)}{2\pi} D^{(K)} \wedge D \\
& + \frac{i}{4\pi}\left(\bar{\xi}_{KK} F_+^{(K)} \wedge F_+^{(K)} + \xi_{KK} F_-^{(K)} \wedge F_-^{(K)}\right) - \frac{\mathrm{Im}(\xi_{KK})}{2\pi} D^{(K)} \wedge D^{(K)} \\
& + \frac{i}{2\pi}\left(\bar{\xi}_{KI} F_+^{(K)} \wedge F_+^{(I)} + \xi_{KI} F_-^{(K)} \wedge F_-^{(I)}\right) - \frac{\mathrm{Im}(\xi_{KI})}{\pi} D^{(K)} \wedge D^{(I)}
\end{aligned}$$

$$\pm \boldsymbol{n}_I = K_X + c_1(v) + \frac{\mathrm{rk}(v)}{2}(c_1 - K_X),$$

$$\pm \boldsymbol{n}_I = K_X + c_1\big(v(L_{\mathrm{GNY}})\big) = K_X + c_1\big(L_{\mathrm{GNY}}^{-1}\big) = K_X - c_1(L_{\mathrm{GNY}})$$

$$e^{-\int_X \mathcal{L}_0}\Big|_{\text{superparticle}} \sim \exp\big(-\pi i\bar{\tau}\boldsymbol{k}_+^2 - \pi i\tau\boldsymbol{k}_-^2 - 2\pi i\bar{v}_I B(\boldsymbol{k}_+, \boldsymbol{n}_I) - 2\pi i v_I B(\boldsymbol{k}_-, \boldsymbol{n}_I)\big) \times \exp\left(-\pi i\xi_{II}\boldsymbol{n}_I^2 - 2\pi\frac{\mathrm{Im}(v_I)^2}{y}\boldsymbol{n}_{I,+}^2\right)$$

$$K_U\, \mathrm{d}a \wedge \mathrm{d}\bar{a} A^\chi B^\sigma \int \mathrm{d}\eta\, \mathrm{d}\chi\, \mathrm{d}D e^{\pi i B(\boldsymbol{k},K)} e^{-\int_X \mathcal{L}_0}$$

$$A = \alpha R^{-1}\left(\frac{\mathrm{d}U}{\mathrm{d}a}\right)^{\frac{1}{2}}, B = \beta \Lambda^{\frac{1}{2}} \Delta_{\mathrm{phys}}^{\frac{1}{8}},$$

$$\nu_R(\tau) = \Lambda^{-3} 2\sqrt{2}\pi i A^\chi B^\sigma \frac{\mathrm{d}a}{\mathrm{d}\tau}.$$

$$\nu_R(\tau) = \frac{2\nu(\tau)}{U} = -\frac{i}{4U}\frac{\vartheta_4(\tau)^{13-b_2}}{\eta(\tau)^9}.$$

$$\lim_{\epsilon\to 0}\lim_{\tau\to\epsilon+i\infty} U(\tau) = i\mathcal{R}q^{-\frac{1}{8}} + \mathcal{O}\left(q^{\frac{1}{8}}\right).$$

$$\lim_{\tau\to i\infty} \nu_R(\tau) = -\frac{1}{4\mathcal{R}}q^{-\frac{1}{4}} - \frac{1+\mathcal{R}^4}{2\mathcal{R}^3} + \mathcal{O}\left(q^{\frac{1}{4}}\right)$$

$$\lim_{\mathcal{R}\to 0} \nu_R(\tau) = \left(-\frac{i}{8}q^{-\frac{3}{8}} + \mathcal{O}\left(q^{\frac{1}{8}}\right)\right) + \left(-\frac{i}{64}q^{-\frac{5}{8}} + \mathcal{O}\left(q^{-\frac{1}{8}}\right)\right)\mathcal{R}^2 + \mathcal{O}(\mathcal{R}^4)$$

$$\Psi_{\boldsymbol{\mu}}^J(\tau,\bar{\tau}) = \frac{i}{2\sqrt{2y}}\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} B(\boldsymbol{k},J)e^{\pi i B(\boldsymbol{k},K)} q^{-\boldsymbol{k}_-^2/2}\bar{q}_+^2/2$$

$$\Phi_{\boldsymbol{\mu}}^J(\mathcal{R}) = K_U \int_{\mathcal{F}_\mathcal{R}} \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}\nu_R(\tau)\Psi_{\boldsymbol{\mu}}^J(\tau,\bar{\tau})$$

$$(K_U, \alpha, \beta) \sim (\zeta^{-4}K_U, \zeta\alpha, \zeta\beta)$$

$$\Psi_{\boldsymbol{\mu}}^J(\tau,\bar{\tau},\boldsymbol{z},\bar{\boldsymbol{z}}) = \exp(-2\pi y\boldsymbol{b}_+^2)\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \partial_{\bar{\tau}}(\sqrt{2y}B(\boldsymbol{k}+\boldsymbol{b},J))e^{\pi i B(\boldsymbol{k},K)} q^{-\boldsymbol{k}_-^2/2}\bar{q}^{\boldsymbol{k}_+^2/2} \times \exp\big(-2\pi i B(\boldsymbol{k}_-,\boldsymbol{z}) - 2\pi i B(\boldsymbol{k}_+,\bar{\boldsymbol{z}})\big)$$

$$S^{-1}TS{:}\Psi_{\boldsymbol{\mu}}^J\left(\frac{\tau}{-\tau+1},\frac{\bar{\tau}}{-\bar{\tau}+1},\frac{\boldsymbol{z}}{-\tau+1},\frac{\bar{\boldsymbol{z}}}{-\bar{\tau}+1}\right) = (-\tau+1)^{b_2/2}(-\bar{\tau}+1)^2\exp\left(\frac{\pi i\boldsymbol{z}^2}{-\tau+1} - \frac{\pi i}{4}\sigma(X)\right)\Psi_{\boldsymbol{\mu}}^J(\tau,\bar{\tau},\boldsymbol{z},\bar{\boldsymbol{z}})$$

$$T^4{:}\Psi_{\boldsymbol{\mu}}^J(\tau+4,\bar{\tau}+4,\boldsymbol{z},\bar{\boldsymbol{z}}) = e^{2\pi i B(\boldsymbol{\mu},K)}\Psi_{\boldsymbol{\mu}}^J(\tau,\bar{\tau},\boldsymbol{z},\bar{\boldsymbol{z}})$$

$$\Psi_{\boldsymbol{\mu}}^J(\tau,\bar{\tau},-\boldsymbol{z},-\bar{\boldsymbol{z}}) = -e^{2\pi i B(\boldsymbol{\mu},K)}\Psi_{\boldsymbol{\mu}}^J(\tau,\bar{\tau},\boldsymbol{z},\bar{\boldsymbol{z}})$$

$$\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},\boldsymbol{z}+\boldsymbol{\nu},\bar{\boldsymbol{z}}+\boldsymbol{\nu})=e^{-2\pi iB(\boldsymbol{\nu},\boldsymbol{\mu})}\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},\boldsymbol{z},\bar{\boldsymbol{z}})$$

$$\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},\boldsymbol{z}+\boldsymbol{\nu}\tau,\bar{\boldsymbol{z}}+\boldsymbol{\nu}\bar{\tau})=e^{2\pi iB(\boldsymbol{z},\boldsymbol{\nu})}q^{\frac{1}{2}\boldsymbol{\nu}^2}e^{-\pi iB(\boldsymbol{\nu},K)}\Psi_{\boldsymbol{\mu}+\boldsymbol{\nu}}^{J}(\tau,\bar{\tau},\boldsymbol{z},\bar{\boldsymbol{z}})$$

$$\begin{aligned}\left|\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},\boldsymbol{z},\bar{\boldsymbol{z}})\right|\le\ &\exp\left(-2\pi y\boldsymbol{b}_+^2\right)\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}}\partial_y(\sqrt{2y}B(\boldsymbol{k}+\boldsymbol{b},J))\\ &\times\exp\left[\pi y(\boldsymbol{k}+\boldsymbol{b})_-^2-\pi y(\boldsymbol{k}+\boldsymbol{b})_+^2-\pi y(\boldsymbol{b}_-^2-\boldsymbol{b}_+^2)\right]\\ &<\infty\end{aligned}$$

$$\Phi_{\boldsymbol{\mu},\boldsymbol{n}_I}^{J}(\mathcal{R},\overline{\mathcal{R}})=K_U\int_{\mathcal{F}_{\mathcal{R}}}\mathrm{d}\tau\wedge\ \mathrm{d}\bar{\tau}\nu_R(\tau)C^{\boldsymbol{n}_I^2}\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},v_I\boldsymbol{n}_I,\bar{v}_I\boldsymbol{n}_I)$$

$$\Phi_{\boldsymbol{\mu},-\boldsymbol{n}_I}^{J}(\mathcal{R},\overline{\mathcal{R}})=-e^{2\pi iB(\boldsymbol{\mu},K)}\Phi_{\boldsymbol{\mu},\boldsymbol{n}_I}^{J}(\mathcal{R},\overline{\mathcal{R}})$$

$$\begin{aligned}\Phi_{\boldsymbol{\mu},\boldsymbol{n}_I,\boldsymbol{n}_K}^{J}(\mathcal{R},\overline{\mathcal{R}})=K_U\int_{\mathcal{F}_{\mathcal{R}}}&\mathrm{d}\tau\wedge\ \mathrm{d}\bar{\tau}\nu_R(\tau)C_{II}^{\boldsymbol{n}_I^2}C_{KK}^{\boldsymbol{n}_K^2}C_{IK}^{2B(\boldsymbol{n}_I,\boldsymbol{n}_K)}\\ \times\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},&v_I\boldsymbol{n}_I+v_K\boldsymbol{n}_K,\bar{v}_I\boldsymbol{n}_I+\bar{v}_K\boldsymbol{n}_K)e^{-4\pi iB(\boldsymbol{\mu},\boldsymbol{n}_I)}\end{aligned}$$

$$\nu_R(\tau)\to e^{\frac{1}{4}\pi i\sigma}(-\tau+1)^{2-b_2/2}\nu_R(\tau)$$

$$C_{II}\to\exp\left(-\pi i\frac{v_I^2}{-\tau+1}\right)C_{II}$$

$$\Psi_{\boldsymbol{\mu}}^{J}\to(-\tau+1)^{\frac{1}{2}b_2}(-\bar{\tau}+1)^2\exp\left(\pi i\frac{v_I^2\boldsymbol{n}_I^2}{-\tau+1}-\frac{\pi i\sigma}{4}\right)\Psi_{\boldsymbol{\mu}}^{J}$$

$$\nu_R(\tau)\to e^{\frac{1}{4}\pi i\sigma}(-\tau+1)^{2-b_2/2}\nu_R(\tau)$$

$$C_{II}\to\exp\left(-\pi i\frac{(v_I+1)^2}{-\tau+1}\right)C_{II}$$

$$\boldsymbol{n}_I\in L,\Psi_{\boldsymbol{\mu}}^{J}\to(-\tau+1)^{\frac{1}{2}b_2}(-\bar{\tau}+1)^2\exp\left(\pi i\frac{(v_I+1)^2\boldsymbol{n}_I^2}{-\tau+1}-\frac{\pi i\sigma}{4}\right)\Psi_{\boldsymbol{\mu}}^{J}$$

$$\nu_R(\tau)\to e^{4\pi i}\nu_R(\tau)=\nu_R(\tau)$$

$$C_{KK}\to q^{-\frac{1}{2}}e^{2\pi iv_K+\pi i}C_{KK}$$

$$\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},v_K\boldsymbol{n}_K,\bar{v}_K\boldsymbol{n}_K)\to e^{-2\pi i\boldsymbol{n}_K^2v_K}q^{\frac{1}{2}\boldsymbol{n}_K^2}(-1)^{\boldsymbol{n}_K^2}\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},v_K\boldsymbol{n}_K,\bar{v}_K\boldsymbol{n}_K)$$

$$\nu_R(\tau)\to e^{\pi i\sigma/4}(-\tau+1)^{2-b_2/2}\nu_R(\tau)$$

$$C_{KK}\to\exp\left(-\pi i\frac{v_K^2}{-\tau+1}\right)C_{KK}$$

$$\Psi_{\boldsymbol{\mu}}^{J}\to(-\tau+1)^{\frac{1}{2}b_2}(-\bar{\tau}+1)^2\exp\left(\pi i\frac{v_K^2\boldsymbol{n}_K^2}{-\tau+1}-\frac{\pi i\sigma}{4}\right)\Psi_{\boldsymbol{\mu}}^{J}$$

$$\Psi^J_{\boldsymbol{\mu}}(\tau,\bar\tau,v_I\boldsymbol{n}_I+v_K\boldsymbol{n}_K,\bar v_I\boldsymbol{n}_I+\bar v_K\boldsymbol{n}_K)$$
$$\to(-1)^{B(\boldsymbol{n}_K,K)}e^{-4\pi iB(\boldsymbol{\mu},\boldsymbol{n}_I)-\pi iB(v_I\boldsymbol{n}_I+2v_K\boldsymbol{n}_K,\boldsymbol{n}_K)}q^{\frac{1}{2}\boldsymbol{n}_K^2}$$
$$\times\Psi^J_{\boldsymbol{\mu}}(\tau,\bar\tau,v_I\boldsymbol{n}_I+v_I\boldsymbol{n}_I,\bar v_I\boldsymbol{n}_I+\bar v_K\boldsymbol{n}_K)$$

$$\Omega_{\boldsymbol{\mu}}=-\frac{i}{4}\,\mathrm{d}\tau\wedge\,\mathrm{d}\bar\tau\frac{\vartheta_4(\tau)^{13-b_2}}{\eta(\tau)^9}C^{\boldsymbol{n}_I^2}\Psi^J_{\boldsymbol{\mu}}(\tau,\bar\tau,v_I\boldsymbol{n}_I,\bar v_I\boldsymbol{n}_I)$$

$$\Omega_{\boldsymbol{\mu}}\to-\exp\left(i\pi B\big(2\boldsymbol{\mu},\overline{w_2(X)}-\boldsymbol{n}_I\big)\right)\Omega_{\boldsymbol{\mu}}$$

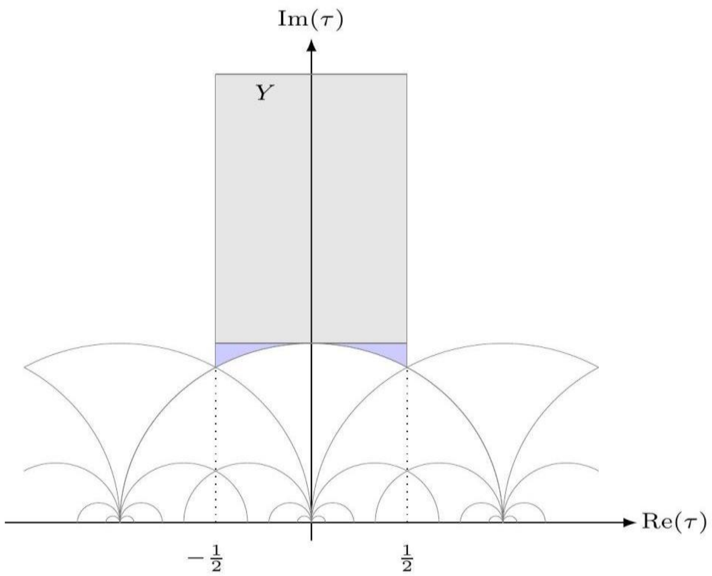
$$\Omega_{\boldsymbol{\mu}}\to-\exp\left(\pi iB(2\boldsymbol{\mu}+\boldsymbol{n}_K,K+\boldsymbol{n}_I)-\frac{1}{2}\pi iB(\boldsymbol{n}_I,\boldsymbol{n}_K)\right)\Omega_{\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{n}_K}$$

$$\mathcal{I}_f=\int_{\mathcal{D}}\mathrm{d}\tau\wedge\,\mathrm{d}\bar\tau y^{-s}f(\tau,\bar\tau)$$

$$L_{m,n,s}(\mathcal{D})=\int_{\mathcal{D}}\mathrm{d}\tau\wedge\,\mathrm{d}\bar\tau y^{-s}q^m\bar q^n$$

$$\begin{aligned}
&(1)\ \tau=\frac{1}{2}+iy, && y\in\left[\frac{1}{2}\sqrt{3},Y\right],\\
&(2)\ \tau=x+iY, && x\in\left[-\frac{1}{2},\frac{1}{2}\right],\\
&(3)\ \tau=-\frac{1}{2}+iy, && y\in\left[\frac{1}{2}\sqrt{3},Y\right],\\
&(4)\ \tau=ie^{i\varphi}, && \varphi\in\left[-\frac{1}{6}\pi,\frac{1}{6}\pi\right].
\end{aligned}$$

$$\mathcal{F}^\infty=\lim_{Y\to\infty}\mathcal{F}^Y$$



$$L_{m,n,s}(\mathcal{F}^\infty)=\lim_{Y\to\infty}L_{m,n,s}(\mathcal{F}^Y)$$

$$L_{m,n,s}(\mathcal{F}^\infty)=L_{m,n,s}(\mathcal{F}^1)-2i\delta_{m,n}E_s(4\pi m)$$

$$E_l(z)=\begin{cases} z^{l-1}\int_z^{\infty} e^{-t}t^{-l}\,\mathrm{d}t, & \text{for } z\in\mathbb{C}^*, l\in\mathbb{Z}/2\\ \frac{1}{l-1}, & \text{for } z=0, l\neq 1\\ 0, & \text{for } z=0, l=1\end{cases}$$

$$\mathrm{Im}(E_l(-z))=\lim_{\delta\to 0}\mathrm{Im}(E_l(-z-i\delta))=\frac{\pi z^{l-1}}{\Gamma(l)}, l\geq 1$$

$$L^{\mathrm{r}}_{m,n,s}(\mathcal{F}^{\infty})=L_{m,n,s}(\mathcal{F}^{1})-2i\delta_{m,n}E_s(4\pi m)$$

$$\partial_{\bar\tau}\hat h(\tau,\bar\tau)=y^{-s}f(\tau,\bar\tau)$$

$$h(\tau)=\sum_n d(n)q^n$$

$$\Phi^J_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R},\overline{\mathcal{R}})=\sum_{l_1,l_2} D_{l_1,l_2}\mathcal{R}^{l_1}\overline{\mathcal{R}}^{l_2}$$

$$D_{l_1,l_2}=\sum_{m,n,s}\tilde D_{l_1,l_2,m,n,s}L_{m,n,s}(\mathbb{H}/\Gamma^0(4))$$

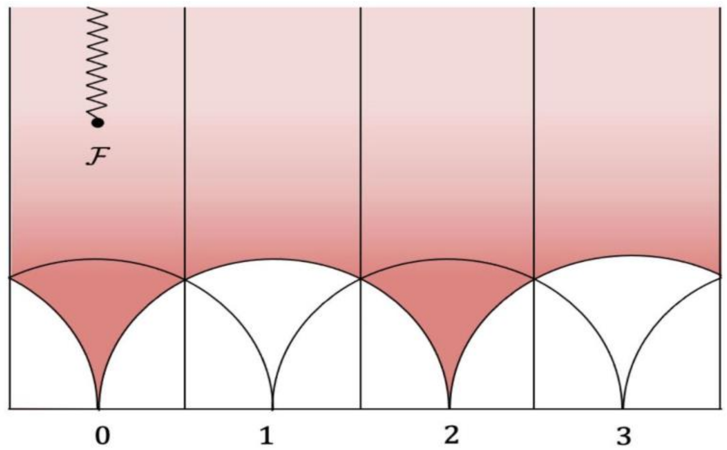
$$\sum_{l_1,l_2,m,n,s}\tilde D_{l_1,l_2,m,n,s}y^{-s}q^m\bar q^n$$

$$D_{l_1,l_2}=\sum_{m,n,s}D_{l_1,l_2,m,n,s}L_{m,n,s}(\mathcal{F}^{\infty})$$

$$\Phi^J_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R},\overline{\mathcal{R}})=\sum_{\substack{l_1,l_2\\ m,n,s}} D_{l_1,l_2,m,n,s}\mathcal{R}^{l_1}\overline{\mathcal{R}}^{l_2}L^{\mathrm{r}}_{m,n,s}(\mathcal{F}^{\infty})$$

$$\Phi^J_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R},\overline{\mathcal{R}})=\sum_{m,n,s} C_{m,n,s}(\mathcal{R},\overline{\mathcal{R}})L_{m,n,s}(\mathcal{F}_{\mathcal{R}})$$

$$\tau_{\mathrm{bp}}\sim\frac{4i}{\pi}\log\,\mathcal{R}^{-1}$$



$$\left.\frac{\mathrm{d}\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},\boldsymbol{z},\overline{\boldsymbol{z}})}{\mathrm{d}\overline{\mathcal{R}}}\right|_{\tau,\bar{\tau}\text{ fixed}} = \partial_{\bar{\tau}}\left(\frac{i}{\sqrt{y}}B\left(\partial_{\overline{\mathcal{R}}}\overline{\boldsymbol{z}},J\right)\Psi_{\boldsymbol{\mu}}^{J}[1](\tau,\bar{\tau},\boldsymbol{z},\overline{\boldsymbol{z}})\right)$$

$$\frac{\partial\Phi_{\boldsymbol{\mu},\boldsymbol{n}}^{J}(\mathcal{R},\overline{\mathcal{R}})}{\partial\overline{\mathcal{R}}} = K_U B(\boldsymbol{n},J)\int_{\mathcal{F}_R} \mathrm{d}\tau\wedge\, \mathrm{d}\bar{\tau}\nu_R(\tau)C^{\boldsymbol{n}^2}\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}\left(\frac{i}{2\sqrt{y}}\frac{\partial\bar{v}}{\partial\overline{\mathcal{R}}}\Psi_{\boldsymbol{\mu}}^{J}[1](\tau,\bar{\tau},\boldsymbol{n}v,\overline{\boldsymbol{n}}\bar{v})\right).$$

$$\frac{\partial\Phi_{\boldsymbol{\mu},\boldsymbol{n}}^{J}(\mathcal{R},\overline{\mathcal{R}})}{\partial\overline{\mathcal{R}}} = 0.$$

$$\frac{\mathrm{d}\widehat{G}_{\boldsymbol{\mu},\boldsymbol{n}}(\tau,\bar{\tau},z,\bar{z})}{\mathrm{d}\bar{\tau}} = \Psi_{\boldsymbol{\mu}}(\tau,\bar{\tau},\boldsymbol{n}z,\boldsymbol{n}\bar{z})$$

$$\widehat{G}_{\mu,n}(\tau,\bar{\tau},z,\bar{z}) = G_{\mu,n}(\tau,z) + \Delta G_{\mu,n}(\tau,\bar{\tau},z,\bar{z})$$

$$\Delta G_{\boldsymbol{\mu},\boldsymbol{n}}(\tau,\bar{\tau},z,\bar{z}) = -\frac{1}{2}\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \sqrt{2y}B(\boldsymbol{k}+\boldsymbol{n}b,J)E_{\frac{1}{2}}(2\pi y(\boldsymbol{k}+\boldsymbol{n}b)_+^2)$$
$$\times\, e^{\pi iB(\boldsymbol{k},K)}q^{-\frac{1}{2}\boldsymbol{k}^2}e^{-2\pi iB(\boldsymbol{k},\boldsymbol{n}z)}$$

$$\Phi_{\boldsymbol{\mu},\boldsymbol{n}}^{J}(\mathcal{R},\overline{\mathcal{R}}) = K_U\lim_{Y\to\infty}\int_{(\mathbb{H}/\Gamma^0(4))_{\mathrm{L}}^{Y}} \mathrm{d}\tau\wedge\, \mathrm{d}\bar{\tau}\mathrm{Ser}_{\mathcal{R}}\left[\nu_R(\tau)C^{\boldsymbol{n}^2}\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}\widehat{G}_{\boldsymbol{\mu},\boldsymbol{n}}(\tau,\bar{\tau},v,\bar{v})\right]$$
$$+K_U\lim_{Y\to\infty}\int_{(\mathbb{H}/\Gamma^0(4))_{\mathrm{R}}^{Y}} \mathrm{d}\tau\wedge\, \mathrm{d}\bar{\tau}\mathrm{Ser}_{\mathcal{R}}\left[\nu_R(\tau)C^{\boldsymbol{n}^2}\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}\widehat{G}_{\boldsymbol{\mu},\boldsymbol{n}}(\tau,\bar{\tau},v,\bar{v})\right]$$

$$\Phi_{\boldsymbol{\mu},\boldsymbol{n}}^{J}(\mathcal{R},\overline{\mathcal{R}}) = K_U\left(1+(-1)^{B(2\boldsymbol{\mu},\overline{w_2(X)}+\boldsymbol{n})}\right)$$
$$\times\lim_{Y\to\infty}\int_{(\mathbb{H}/\Gamma^0(4))^{Y}} \mathrm{d}\tau\wedge\, \mathrm{d}\bar{\tau}\mathrm{Ser}_{\mathcal{R}}\left[\nu_R(\tau)C^{\boldsymbol{n}^2}\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}\widehat{G}_{\boldsymbol{\mu},\boldsymbol{n}}(\tau,\bar{\tau},v,\bar{v})\right]$$

$$\Phi_{\boldsymbol{\mu},\boldsymbol{n}}^{J}(\mathcal{R}) = K_{U,\boldsymbol{\mu},\boldsymbol{n}}\lim_{Y\to\infty}\int_{(\mathbb{H}/\Gamma^0(4))^{Y}} \mathrm{d}\tau\wedge\, \mathrm{d}\bar{\tau}\mathrm{Ser}_{\mathcal{R}}\left[\nu_R(\tau)C^{\boldsymbol{n}^2}\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}\widehat{G}_{\boldsymbol{\mu},\boldsymbol{n}}(\tau,\bar{\tau},v,0)\right]$$

$$K_{U,\boldsymbol{\mu},\boldsymbol{n}} = K_U\left(1+(-1)^{B(2\boldsymbol{\mu},\overline{w_2(X)}+\boldsymbol{n})}\right)\in\{0,1\}$$

$$\Phi_{\boldsymbol{\mu},\boldsymbol{n}}^{J}(\mathcal{R}) = \begin{cases} 0, & B\left(2\boldsymbol{\mu},\overline{w_2(X)}+\boldsymbol{n}\right) = \text{ odd} \\ 4\mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[\nu_R(\tau)C(\tau)^{\boldsymbol{n}^2}G_{\boldsymbol{\mu},\boldsymbol{n}}(\tau,v)\right] & \\ +\mathrm{Coeff}_{q_1^0}\mathrm{Ser}_{\mathcal{R}}\left[\tau_1^{-2}\nu_R(\tau)C(\tau)^{\boldsymbol{n}^2}G_{\boldsymbol{\mu},\boldsymbol{n}}(\tau,v)\right] & B\left(2\boldsymbol{\mu},\overline{w_2(X)}+\boldsymbol{n}\right) = \text{ even} \\ +\mathrm{Coeff}_{q_2^0}\mathrm{Ser}_{\mathcal{R}}\left[\tau_2^{-2}\nu_R(\tau)C(\tau)^{\boldsymbol{n}^2}G_{\boldsymbol{\mu},\boldsymbol{n}}(\tau,v)\right] & \end{cases}$$

$$\Psi_{\boldsymbol{\mu}}^{J}(\tau,\bar{\tau},\boldsymbol{z},\overline{\boldsymbol{z}}) - \Psi_{\boldsymbol{\mu}}^{J'}(\tau,\bar{\tau},\boldsymbol{z},\overline{\boldsymbol{z}}) = \partial_{\bar{\tau}}\widehat{\Theta}_{\boldsymbol{\mu}}^{J,J'}(\tau,\bar{\tau},\boldsymbol{z},\overline{\boldsymbol{z}})$$

$$\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau,\bar{\tau},\boldsymbol{z},\overline{\boldsymbol{z}}) = \sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}}\frac{1}{2}\left[E(\sqrt{2y}B(\boldsymbol{k}+\boldsymbol{b},J)) - E\left(\sqrt{2y}B(\boldsymbol{k}+\boldsymbol{b},J')\right)\right]$$
$$\times\, e^{\pi iB(\boldsymbol{k},K)}q^{-\frac{1}{2}\boldsymbol{k}^2}e^{-2\pi iB(\boldsymbol{k},\boldsymbol{z})}$$

$$\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau+1,\bar{\tau}+1,\boldsymbol{z},\bar{\boldsymbol{z}}) = \exp\left[\pi i(\boldsymbol{\mu}^2 - B(K,\boldsymbol{\mu}))\right]\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}\left(\tau,\bar{\tau},\boldsymbol{z}+\boldsymbol{\mu}-\frac{K}{2},\bar{\boldsymbol{z}}+\boldsymbol{\mu}-\frac{K}{2}\right)$$

$$\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}\left(-\frac{1}{\tau},-\frac{1}{\bar{\tau}},\frac{\boldsymbol{z}}{\tau},\bar{\boldsymbol{z}}\right) = i(-i\tau)^{\frac{b_2}{2}}\exp\left(\pi i B(\boldsymbol{\mu},K) - \frac{\pi i \boldsymbol{z}^2}{\tau}\right) \times \widehat{\Theta}_{\frac{K}{2}}^{JJ'}\left(\tau,\bar{\tau},\boldsymbol{z}-\boldsymbol{\mu}+\frac{K}{2},\bar{\boldsymbol{z}}-\boldsymbol{\mu}+\frac{K}{2}\right)$$

$$\Theta_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{z}) = \sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \frac{1}{2}\left[\operatorname{sgn}(B(\boldsymbol{k}+\boldsymbol{b},J)) - \operatorname{sgn}\big(B(\boldsymbol{k}+\boldsymbol{b},J')\big)\right] \times e^{\pi i B(\boldsymbol{k},K)} q^{-\frac{1}{2}\boldsymbol{k}^2} e^{-2\pi i B(\boldsymbol{k},\boldsymbol{z})}$$

$$\begin{aligned}
\Delta\Phi_{\boldsymbol{\mu},\boldsymbol{n},\infty}^{JJ'}(\mathcal{R}) &= \Phi_{\boldsymbol{\mu},\boldsymbol{n},\infty}^{J}(\mathcal{R}) - \Phi_{\boldsymbol{\mu},\boldsymbol{n},\infty}^{J'}(\mathcal{R}) \\
&= K_U \lim_{Y\to\infty}\int_{iY-\frac{1}{2}}^{iY+\frac{7}{2}} d\tau \operatorname{Ser}_{\mathcal{R}}\left[\nu_R(\tau) C^{\boldsymbol{n}^2}\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau,\bar{\tau},\boldsymbol{n}v,\boldsymbol{n}\bar{v})\right] \\
&+ K_U \lim_{Y\to\infty}\int_{iY+\frac{7}{2}}^{iY+\frac{15}{2}} d\tau \operatorname{Ser}_{\mathcal{R}}\left[\nu_R(\tau) C^{\boldsymbol{n}^2}\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau,\bar{\tau},\boldsymbol{n}v,\boldsymbol{n}\bar{v})\right]
\end{aligned}$$

$$\Delta\Phi_{\boldsymbol{\mu},\boldsymbol{n},\infty}^{JJ'}(\mathcal{R}) = K_{U,\boldsymbol{\mu},\boldsymbol{n}} \lim_{Y\to\infty}\int_{iY-\frac{1}{2}}^{iY+\frac{7}{2}} d\tau \operatorname{Ser}_{\mathcal{R}}\left[\nu_R(\tau) C^{\boldsymbol{n}^2}\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}(\tau,\bar{\tau},\boldsymbol{n}v,\boldsymbol{n}\bar{v})\right]$$

$$\Delta\Phi_{\boldsymbol{\mu},\boldsymbol{n},\infty}^{JJ'}(\mathcal{R}) = 4K_{U,\boldsymbol{\mu},\boldsymbol{n}} \operatorname{Coeff}_{q^0}\operatorname{Ser}_{\mathcal{R}}\left[\nu_R(\tau) C^{\boldsymbol{n}^2}\Theta_{\boldsymbol{\mu}}^{JJ'}(\tau,\boldsymbol{n}v)\right]$$

$$B\left(\boldsymbol{k}+\frac{\operatorname{Im}(v)}{y}\boldsymbol{n},J\right) > 0 \text{ and } B\left(\boldsymbol{k}+\frac{\operatorname{Im}(v)}{y}\boldsymbol{n},J'\right) < 0$$

$$\begin{aligned}
\Delta\Phi_{\boldsymbol{\mu},\boldsymbol{n},1}^{JJ'}(\mathcal{R}) &:= \left[\Phi_{\boldsymbol{\mu},\boldsymbol{n}}^{J}(\mathcal{R}) - \Phi_{\boldsymbol{\mu},\boldsymbol{n}}^{J'}(\mathcal{R})\right]_1 \\
&= -\frac{1}{4}\lim_{Y\to\infty}\int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY} d\tau_1 \frac{\vartheta_2(\tau_1)^{11+\sigma}}{\eta(\tau_1)^9}\frac{1}{U} C_1(\tau_1,v_1)^{\boldsymbol{n}^2} \\
&\quad \times (-1)^{B(\boldsymbol{\mu},K)} \sum_{\boldsymbol{k}\in L+\frac{1}{2}K} \frac{1}{2}\left[\operatorname{sgn}(B(\boldsymbol{k},J)) - \operatorname{sgn}\big(B(\boldsymbol{k},J')\big)\right] \\
&\quad \times q_1^{-\frac{1}{2}\boldsymbol{k}^2} e^{-2\pi i B(\boldsymbol{k},\boldsymbol{n})v_1} e^{-2\pi i B(\boldsymbol{k},\boldsymbol{\mu})}
\end{aligned}$$

$$\begin{aligned}
\Delta\Phi_{\boldsymbol{\mu},\boldsymbol{n},1}^{JJ'}(\mathcal{R}) &= \frac{2}{\Lambda}\operatorname{Res}_{a_1=0}\left[\frac{\vartheta_2(\tau_1)^{2+\sigma}}{\eta(\tau_1)^6} C_1(\tau_1,v_1)^{\boldsymbol{n}^2}\right. \\
&\quad \times (-1)^{B(\boldsymbol{\mu},K)} \sum_{\boldsymbol{k}\in L+\frac{1}{2}K} \frac{1}{2}\left[\operatorname{sgn}(B(\boldsymbol{k},J)) - \operatorname{sgn}\big(B(\boldsymbol{k},J')\big)\right] \\
&\quad \left.\times q_1^{-\frac{1}{2}\boldsymbol{k}^2} e^{-2\pi i B(\boldsymbol{k},\boldsymbol{n})v_1} e^{-2\pi i B(\boldsymbol{k},\boldsymbol{\mu})}\right].
\end{aligned}$$

$$
\begin{aligned}
\Delta\Phi^{JJ'}_{\boldsymbol{\mu},\boldsymbol{n},2}(\mathcal{R}) :&= \left[\Phi^{J}_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R}) - \Phi^{J'}_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R})\right]_2 \\
&= -\frac{1}{4} e^{-\frac{3\pi i}{2} - 2\pi i \boldsymbol{\mu}^2} \lim_{Y\to\infty} \int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY} d\tau_2 \frac{\vartheta_2(\tau_2)^{11+\sigma}}{\eta(\tau_2)^9} \frac{1}{U} \\
&\quad \times (-1)^{B(\boldsymbol{\mu},K)} C_2(\tau_2, v_2)^{\boldsymbol{n}^2} \sum_{\boldsymbol{k}\in L+\frac{1}{2}K} \frac{1}{2}\left[\mathrm{sgn}\big(B(\boldsymbol{k},J) - \mathrm{sgn}\big(B(\boldsymbol{k},J')\big)\right] \\
&\quad \times q_2^{-\frac{1}{2}\boldsymbol{k}^2} e^{-2\pi i B(\boldsymbol{k},\boldsymbol{n}) v_2} e^{-2\pi i B(\boldsymbol{k},\boldsymbol{\mu})}
\end{aligned}
$$

$$
\begin{aligned}
\Delta\Phi^{JJ'}_{\boldsymbol{\mu},\boldsymbol{n},2}(\mathcal{R}) = -\frac{2}{\Lambda} e^{-2\pi i \boldsymbol{\mu}^2} \mathrm{Res}_{a_2=0} \Bigg[& \frac{\vartheta_2(\tau_2)^{2+\sigma}}{\eta(\tau_2)^6} C_2(\tau_2, v_2)^{\boldsymbol{n}^2} \\
& \times (-1)^{B(\boldsymbol{\mu},K)} \sum_{\boldsymbol{k}\in L+\frac{1}{2}K} \frac{1}{2}\left[\mathrm{sgn}\big(B(\boldsymbol{k},J) - \mathrm{sgn}\big(B(\boldsymbol{k},J')\big)\right] \\
& \times q_2^{-\frac{1}{2}\boldsymbol{k}^2} e^{-2\pi i B(\boldsymbol{k},\boldsymbol{n}) v_2} e^{-2\pi i B(\boldsymbol{k},\boldsymbol{\mu})} \Bigg].
\end{aligned}
$$

$$
\Delta\Phi^{JJ'}_{\boldsymbol{\mu},\boldsymbol{n},j}(\mathcal{R}) = (-1)^{B(2\boldsymbol{\mu},K-\boldsymbol{n})} \Delta\Phi^{JJ'}_{\boldsymbol{\mu},\boldsymbol{n},j-2}(\mathcal{R})
$$

$$
\Delta\Phi^{JJ'}_{\boldsymbol{\mu},(n_1,n_2)}(\mathcal{R}) = -\Delta\Phi^{JJ'}_{\boldsymbol{\mu},(-n_1,-n_2)}(\mathcal{R}), \Delta\Phi^{JJ'}_{\boldsymbol{\mu},(n_1,n_2)}(\mathcal{R}) = \Delta\Phi^{JJ'}_{\boldsymbol{\mu},(n_2,n_1)}(\mathcal{R})
$$

$$
\boldsymbol{\mu} = \left(\frac{1}{2}, \frac{1}{2}\right), J = (1+\delta, 1), J' = (\epsilon, 1),
$$

$$
\Delta\Phi^{JJ'}_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R}) = \begin{cases}
\mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (0,0), \quad 0 < \epsilon < 1, \\
-4\mathcal{R}^3 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (0,2), \quad 0 < \epsilon < 1, \\
-35\mathcal{R}^3 + 28\mathcal{R}^7 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (0,4), \quad 0 < \epsilon < \frac{1}{3}, \\
84\mathcal{R}^3 + 64\mathcal{R}^7 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (1,-5), \ 0 < \epsilon < \frac{1}{5}, \\
20\mathcal{R}^3 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (1,-3), \ 0 < \epsilon < \frac{1}{5}, \\
\mathcal{R}^3 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (1,-1), \ 0 < \epsilon < \frac{1}{3}, \\
\mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (1,1), \quad 0 < \epsilon < 1, \\
56\mathcal{R}^3 + 120\mathcal{R}^7 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (2,-4), \ 0 < \epsilon < \frac{1}{5}, \\
10\mathcal{R}^3 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (2,-2), \ 0 < \epsilon < \frac{1}{5}, \\
-\mathcal{R}^3 - \mathcal{R}^7 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (2,2), \quad 0 < \epsilon < \frac{1}{5}, \\
-20\mathcal{R}^3 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (2,4), \quad 0 < \epsilon < 1, \\
120\mathcal{R}^3 + 1100\mathcal{R}^7 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (3,-5), \ 0 < \epsilon < \frac{1}{5}, \\
35\mathcal{R}^3 + 56\mathcal{R}^7 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (3,-3), \ 0 < \epsilon < \frac{1}{5}, \\
-4\mathcal{R}^3 - 10\mathcal{R}^7 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (3,3), \quad 0 < \epsilon < \frac{1}{5}, \\
84\mathcal{R}^3 + 603\mathcal{R}^7 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (4,-4), \ 0 < \epsilon < \frac{1}{5}, \\
-10\mathcal{R}^3 - 54\mathcal{R}^7 + \mathcal{O}(\mathcal{R}^9), & \boldsymbol{n} = (4,4), \quad 0 < \epsilon < \frac{1}{5}.
\end{cases}
$$

$$
\Psi_\mu(\tau, \bar{\tau}, vn, \bar{v}n) = -i(-1)^{\mu(K-1)} f_{\mu,n}(\tau, \bar{\tau}, v, \bar{v}),
$$

$$f_{\mu,n}(\tau,\bar{\tau},v,\bar{v}) = \frac{\mathrm{d}\hat{G}_{\mu,n}(\tau,\bar{\tau},v,\bar{v})}{\mathrm{d}\bar{\tau}}$$

$$f_{\mu,0}(\tau,\bar{\tau}) = \begin{cases} 0, & \mu = 0 \bmod \mathbb{Z} \\ -\frac{i}{2\sqrt{2y}}\overline{\eta(\tau)^3}, & \mu = \frac{1}{2} \bmod \mathbb{Z} \end{cases}$$

$$\hat{G}_{\frac{1}{2},0}(\tau,\bar{\tau}) = G_{\frac{1}{2},0}(\tau) - \frac{i}{2}\int_{-\bar{\tau}}^{i\infty} \frac{\eta(w)^3}{\sqrt{-i(w+\tau)}}dw$$

$$\begin{aligned} G_{\frac{1}{2},0}(\tau) &= -\frac{1}{\vartheta_4(\tau)}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n^2}{2}-\frac{1}{8}}}{1-q^{n-\frac{1}{2}}} \\ &= 2q^{\frac{3}{8}}\left(1 + 3q^{\frac{1}{2}} + 7q + 14q^{\frac{3}{2}} + \cdots\right) \end{aligned}$$

$$f_{\mu,n}(\tau,\bar{\tau},z,\bar{z}) = \frac{1}{2}e^{\pi i v}q^{-\frac{v^2}{2}}w^{-\frac{1}{2}nv}\frac{\mathrm{d}R(\tau,\bar{\tau},nz+v\tau,n\bar{z}+v\bar{\tau})}{\mathrm{d}\bar{\tau}}, v = \mu - \frac{1}{2}$$

$$\hat{G}_{\frac{1}{2},n}(\tau,\bar{\tau},v,\bar{v}) = -i\hat{M}\left(\tau,\bar{\tau},nv+\frac{\tau}{2},n\bar{v}+\frac{\bar{\tau}}{2},\frac{\tau}{2},\frac{\bar{\tau}}{2}\right)$$

$$G_{\frac{1}{2},n}(\tau,v) = -iM\left(\tau,nv+\frac{\tau}{2},\frac{\tau}{2}\right)$$

$$G_{\frac{1}{2},n}(\tau,v) = -\frac{e^{\pi i n v}}{\vartheta_4(\tau)}\sum_{l\in\mathbb{Z}} \frac{(-1)^l q^{\frac{l^2}{2}-\frac{1}{8}}}{1-e^{2\pi i n v}q^{l-\frac{1}{2}}}$$

$$\Phi_{\frac{1}{2},n}(\mathcal{R}) = \begin{cases} \frac{1}{2}\mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[\frac{\vartheta_4(\tau)^{12}}{\eta(\tau)^9}(1-2\mathcal{R}^2\mathbf{u}(\tau)+\mathcal{R}^4)^{-\frac{1}{2}}C^{n^2}G_{\frac{1}{2},n}(\tau,v)\right], & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$\Phi_{\frac{1}{2},n}(\mathcal{R}) = \begin{cases} 1 + \mathcal{O}(\mathcal{R}^{13}), & n = \pm 1, \\ 1 + \mathcal{R}^4 + \mathcal{R}^8 + \mathcal{R}^{12} + \mathcal{O}(\mathcal{R}^{13}), & n = \pm 3, \\ 1 + 6\mathcal{R}^4 + 21\mathcal{R}^8 + 56\mathcal{R}^{12} + \mathcal{O}(\mathcal{R}^{13}), & n = \pm 5, \\ 1 + 21\mathcal{R}^4 + 210\mathcal{R}^8 + 1401\mathcal{R}^{12} + \mathcal{O}(\mathcal{R}^{13}), & n = \pm 7, \\ 1 + 55\mathcal{R}^4 + 1310\mathcal{R}^8 + 19432\mathcal{R}^{12} + \mathcal{O}(\mathcal{R}^{13}), & n = \pm 9. \end{cases}$$

$$\Phi_{0,n}(\mathcal{R}) = -\frac{1}{2}\mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[\frac{\vartheta_4(\tau)^{12}}{\eta(\tau)^9}(1-2\mathcal{R}^2\mathbf{u}(\tau)+\mathcal{R}^4)^{-\frac{1}{2}}C^{n^2}G_{0,n}(\tau,v)\right]$$

$$\begin{aligned} G_{0,n}(\tau,v) &= \frac{1}{\eta(\tau)^3}\sum_{\substack{k_1\in\mathbb{Z} \\ k_2\in\mathbb{Z}+\frac{1}{2}}} \frac{1}{2}[\mathrm{sgn}(k_1+k_2)-\mathrm{sgn}(k_1)]k_2 e^{\pi i(k_1+k_2)}e^{2\pi i n v k_1}q^{\frac{1}{2}(k_2^2-k_1^2)} \\ &= \frac{i}{2} - \frac{i}{\vartheta_4(\tau)}\sum_{l\in\mathbb{Z}} \frac{(-1)^l q^{\frac{l^2}{2}}}{1-e^{2\pi i v n}q^l} - \frac{i}{\vartheta_4(\tau,nv)}\partial_\rho \ln\left(\frac{\vartheta_1(\tau,\rho)}{\vartheta_4(\tau,\rho)}\right)\Bigg|_{\rho=nv} \end{aligned}$$

$$\Phi_{0,n}(\mathcal{R}) = \begin{cases} \frac{15}{2}\mathcal{R} - 21\mathcal{R}^5 - 56\mathcal{R}^9 - 126\mathcal{R}^{13} + \mathcal{O}(\mathcal{R}^{17}), & n = 5, \\ 6\mathcal{R} - 6\mathcal{R}^5 - 10\mathcal{R}^9 - 15\mathcal{R}^{13} + \mathcal{O}(\mathcal{R}^{17}), & n = 4, \\ \frac{9}{2}\mathcal{R} - \mathcal{R}^5 - \mathcal{R}^9 - \mathcal{R}^{13} + \mathcal{O}(\mathcal{R}^{17}), & n = 3, \\ 3\mathcal{R} + \mathcal{O}(\mathcal{R}^{17}), & n = 2, \\ \frac{3}{2}\mathcal{R} + \mathcal{O}(\mathcal{R}^{17}), & n = 1, \\ 0, & n = -1, \\ -\frac{3}{2}\mathcal{R} + \mathcal{O}(\mathcal{R}^{17}), & n = -2, \\ -3\mathcal{R} + \mathcal{O}(\mathcal{R}^{17}), & n = -3, \\ -\frac{9}{2}\mathcal{R} + \mathcal{R}^5 + \mathcal{R}^9 + \mathcal{R}^{13} + \mathcal{O}(\mathcal{R}^{17}), & n = -4, \\ -6\mathcal{R} + 6\mathcal{R}^5 + 10\mathcal{R}^9 + 15\mathcal{R}^{13} + \mathcal{O}(\mathcal{R}^{17}), & n = -5, \\ -\frac{15}{2}\mathcal{R} + 21\mathcal{R}^5 + 56\mathcal{R}^9 + 126\mathcal{R}^{13} + \mathcal{O}(\mathcal{R}^{17}), & n = -6, \\ -9\mathcal{R} + 56\mathcal{R}^5 + 230\mathcal{R}^9 + 770\mathcal{R}^{13} + \mathcal{O}(\mathcal{R}^{17}), & n = -7 \end{cases}$$

$$\begin{aligned} \Delta_{0,n}(\tau,v) =& -\frac{1}{\mathcal{R}}\frac{\eta(\tau)^3\vartheta_1(\tau,(n-1)v)}{\vartheta_4(\tau)\vartheta_1(\tau,nv)\vartheta_4(\tau,(n-1)v)} \\ & -\frac{i}{\vartheta_4(\tau,nv)}\partial_\rho \ln\left(\frac{\vartheta_1(\tau,\rho)}{\vartheta_4(\tau,\rho)}\right)\bigg|_{\rho=nv} - \frac{\vartheta^+(\tau,v)}{\vartheta_1(\tau,v)}, \\ \Delta_{\frac{1}{2},n}(\tau,v) =& -\mathcal{R}\frac{\eta(\tau)^3\vartheta_1(\tau,(n-1)v)}{\vartheta_4(\tau,nv)\vartheta_4(\tau,(n-1)v)\vartheta_4(\tau)}. \end{aligned}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus nQ_{E_8}$$

$$\Psi_{\boldsymbol{\mu}}^J(\tau,\bar\tau,v,\bar v) = -i(-1)^{\mu_1(K_1-1)} f_{\mu_1,n_1}(\tau,\bar\tau,v_1,\bar v_1)\Theta_{\boldsymbol{\mu}}(\tau,v\boldsymbol{n}_-)$$

$$\Theta_{\boldsymbol{\mu}_-}(\tau,\boldsymbol{z}_-) = \sum_{\boldsymbol{k}_-\in L_-+\boldsymbol{\mu}_-} (-1)^{B(\boldsymbol{k}_-,K_-)} q^{-\frac{1}{2}\boldsymbol{k}_-^2} e^{-2\pi i B(\boldsymbol{z}_-,\boldsymbol{k}_-)}$$

$$\Theta_{D,\boldsymbol{\mu}_-}(\tau,\boldsymbol{z}) = \sum_{\boldsymbol{k}_-\in L_-+K_-/2} (-1)^{B(K_-,\boldsymbol{\mu}_-)} q^{-\frac{1}{2}\boldsymbol{k}_-^2} e^{-2\pi i B(\boldsymbol{z},\boldsymbol{k}_-)}$$

$$\begin{aligned} \widehat{G}_{D,\mu,n}(\tau,\bar\tau,z,\bar z) &= -(-i\tau)^{-\frac{1}{2}}\exp\left(\frac{\pi i n^2 z^2}{\tau}\right)\widehat{G}_{\mu,n}\left(-\frac{1}{\tau},-\frac{1}{\bar\tau},\frac{z}{\tau},\frac{\bar z}{\bar\tau}\right) \\ &= -i\widehat{M}\left(\tau,\bar\tau,nz+\frac{1}{2},n\bar z+\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} G_{D,\mu,n}(\tau,z) =& -\frac{e^{\pi i n z}}{\vartheta_2(\tau)}\sum_{l\in\mathbb{Z}}\frac{q^{\frac{1}{2}l(l+1)}}{1-(-1)^{2\mu}e^{2\pi i n z}q^l} \\ & -\frac{\delta_{\mu,0}}{\vartheta_2(\tau,nz)}\partial_\rho \ln\left(\frac{\vartheta_1(\tau,\rho)}{\vartheta_2(\tau,\rho)}\right)\bigg|_{\rho=nz} \end{aligned}$$

$$\Phi^J_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R}) = -i(-1)^{\mu_1(K_1-1)} \sum_{j\in\{\infty,1,2\}} \Phi^J_{\boldsymbol{\mu},\boldsymbol{n},j}(\mathcal{R})$$

$$\Phi^J_{\boldsymbol{\mu},\boldsymbol{n},\infty}(\mathcal{R}) = 4K_{U,\boldsymbol{\mu},\boldsymbol{n}}\mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[\nu_R(\tau)C^{\boldsymbol{n}^2}G_{\mu_1,n_1}(\tau,v)\Theta_{\boldsymbol{\mu}}(\tau,v\boldsymbol{n}_-)\right]$$

$$\Phi^J_{\boldsymbol{\mu},\boldsymbol{n},1}(\mathcal{R}) = K_{U,\boldsymbol{\mu},\boldsymbol{n}}\mathrm{Coeff}_{q_1^0}\mathrm{Ser}_{\mathcal{R}}\left[\nu_{R,1}(\tau_1)C_1(\tau_1,v_1)^{\boldsymbol{n}^2}G_{D\mu_1,n_1}(\tau_1,v_1)\Theta_{D\boldsymbol{\mu}}(\tau_1,v_1\boldsymbol{n}_-)\right]$$
$$\Phi^J_{\boldsymbol{\mu},\boldsymbol{n},2}(\mathcal{R}) = K_{U,\boldsymbol{\mu},\boldsymbol{n}}\left(ie^{-2\pi i\boldsymbol{\mu}^2}\right)\mathrm{Coeff}_{q_2^0}\mathrm{Ser}_{\mathcal{R}}\left[\nu_{R,2}(\tau_2)C_2(\tau_2,v_2)^{\boldsymbol{n}^2}G_{D\mu_1,n_1}(\tau_2,v_2)\Theta_{D\boldsymbol{\mu}}(\tau_2,v_2\boldsymbol{n}_-)\right]$$

$$\begin{aligned}\nu_{R,1}(\tau_1) &= (-i\tau_1)^{b_2/2-2}\nu_R\left(-\frac{1}{\tau_1}\right)\\ &= -\frac{i}{4}\frac{\vartheta_2(\tau_1)^{13-b_2}}{\eta(\tau_1)^9}\left(-8\mathcal{R}^2\mathrm{u}_1(\tau_1)+4\mathcal{R}^4+4\right)^{-\frac{1}{2}}\end{aligned}$$

$$\mathrm{u}_1(\tau_1) = \frac{\vartheta_3(\tau_1)^4+\vartheta_4(\tau_1)^4}{2\vartheta_3(\tau_1)^2\vartheta_4(\tau_1)^2}$$

$$\nu_{R,2}(\tau_2) = -\frac{i}{4}\frac{\vartheta_2(\tau_2)^{13-b_2}}{\eta(\tau_2)^9}\left(-8\mathcal{R}^2\mathrm{u}_2(\tau_2)+4\mathcal{R}^4+4\right)^{-\frac{1}{2}}$$

$$Z^J_{\boldsymbol{\mu},\boldsymbol{n}} = \Phi^J_{\boldsymbol{\mu},\boldsymbol{n}} + \sum_{j=1}^{4} Z^J_{\mathrm{SW},\boldsymbol{\mu},\boldsymbol{n},j}$$

$$Z^J_{\mathrm{SW},\boldsymbol{\mu},\boldsymbol{n},j} = \sum_c \mathrm{SW}(c;J)\mathrm{Res}_{a_j=0}\left[a_j^{-1-n(c)}e^{-S_{\mathrm{SW},j}}\right]$$

$$\mathrm{SW}(c;J) = (-1)^{\chi_{\mathrm{h}}}\mathrm{SW}(-c;J).$$

$$e^{-S_{\mathrm{SW},j}} = \kappa_j\mathcal{A}_j^{\chi}\mathcal{B}_j^{\sigma}\mathcal{C}_j^{\boldsymbol{n}^2}\mathcal{D}_j^{B(\boldsymbol{n},c)}\mathcal{E}_j^{c^2}\mathcal{F}_{\boldsymbol{\mu},j}$$

$$\mathcal{F}_{\boldsymbol{\mu},j} = f_{1,j}^{\boldsymbol{\mu}^2}f_{2,j}^{B(\boldsymbol{n},\boldsymbol{\mu})}f_{3,j}^{B(c,\boldsymbol{\mu})}$$

$$\left[\Phi^J_{\boldsymbol{\mu},\boldsymbol{n}} - \Phi^{J'}_{\boldsymbol{\mu},\boldsymbol{n}}\right]_j = Z^{J'}_{\mathrm{SW},\boldsymbol{\mu},\boldsymbol{n},j} - Z^J_{\mathrm{SW},\boldsymbol{\mu},\boldsymbol{n},j}$$

$$\mathrm{SW}(c;J^+) - \mathrm{SW}(c;J^-) = -(-1)^{n(c)}$$

$$\begin{aligned}\left[\Phi^J_{\boldsymbol{\mu},\boldsymbol{n}} - \Phi^{J'}_{\boldsymbol{\mu},\boldsymbol{n}}\right]_j &= \kappa_j\sum_c (-1)^{n(c)}\frac{1}{2}\left[\mathrm{sgn}\left(B(c,J) - \mathrm{sgn}\left(B(c,J')\right)\right)\right]\\ &\times \mathrm{Res}_{a_j=0}\left[a_j^{-1-n(c)}\mathcal{A}_j^{\chi}\mathcal{B}_j^{\sigma}\mathcal{C}_j^{\boldsymbol{n}^2}\mathcal{D}_j^{B(\boldsymbol{n},c)}\mathcal{E}_j^{c^2}\mathcal{F}_{\boldsymbol{\mu},j}\right]\end{aligned}$$

$$\begin{aligned}\left[\Phi^J_{\boldsymbol{\mu},\boldsymbol{n}} - \Phi^{J'}_{\boldsymbol{\mu},\boldsymbol{n}}\right]_j &= -\kappa_j\sum_c (-1)^{\frac{1}{8}(c^2-\sigma)}\frac{1}{2}\left[\mathrm{sgn}\left(B(c,J) - \mathrm{sgn}\left(B(c,J')\right)\right)\right]\\ &\times \mathrm{Res}_{a_j=0}\left[a_j^{-\frac{1}{8}(c^2-\sigma)}\mathcal{A}_j^{4}\left(\mathcal{B}_j/\mathcal{A}_j\right)^{\sigma}\mathcal{C}_j^{\boldsymbol{n}^2}\mathcal{D}_j^{B(\boldsymbol{n},c)}\mathcal{E}_j^{c^2}\mathcal{F}_{\boldsymbol{\mu},j}\right]\end{aligned}$$

$$Z_{\boldsymbol{\mu},0}(\mathcal{R}) = \frac{1}{2}\left[(1-\mathcal{R}^2)^{-2} + (-1)^{2\boldsymbol{\mu}^2}(1+\mathcal{R}^2)^{-2}\right]$$

$$
\begin{aligned}
\mathcal{A}_1 &= 2^{\frac{1}{4}}\kappa_1^{-\frac{1}{4}}\Lambda^{-\frac{1}{4}}e^{\frac{\pi i}{4}}\left(\frac{\vartheta_2(\tau_1)^2}{\eta(\tau_1)^6}\right)^{\frac{1}{4}}\\
\mathcal{B}_1 &= e^{\frac{\pi i}{8}}\vartheta_2(\tau_1)\mathcal{A}_1 a_1^{-\frac{1}{8}}\\
\mathcal{C}_1 &= C_1(\tau_1, v_1)\\
\mathcal{D}_1 &= e^{-\pi i v_1}\\
\mathcal{E}_1 &= e^{-\frac{\pi i}{8}}a_1^{\frac{1}{8}}q_1^{-\frac{1}{8}}\\
\mathcal{F}_1 &= e^{\pi i B(K-c,\boldsymbol{\mu})}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_1 &= 2^{\frac{3}{4}}\kappa_1^{-\frac{1}{4}}\Lambda^{-\frac{1}{4}}e^{\frac{\pi i}{4}} + \cdots\\
\mathcal{B}_1 &= 2^{\frac{7}{4}}\kappa_1^{-\frac{1}{4}}\Lambda^{-\frac{3}{8}}e^{\frac{\pi i}{4}}\left(\frac{32}{1-\mathcal{R}^2}\right)^{-\frac{1}{8}} + \cdots\\
\mathcal{C}_1 &= (1-\mathcal{R}^2)^{-\frac{1}{2}} + \cdots\\
\mathcal{D}_1 &= \sqrt{\frac{1-\mathcal{R}}{1+\mathcal{R}}} + \cdots\\
\mathcal{E}_1 &= \Lambda^{\frac{1}{8}}\left(\frac{32}{1-\mathcal{R}^2}\right)^{\frac{1}{8}}
\end{aligned}
$$

$$
\begin{aligned}
Z_{\boldsymbol{\mu},\boldsymbol{n},1}(\mathcal{R}) =& K_U \kappa_1^{1-\chi_{\mathrm{h}}}(-1)^{\chi_{\mathrm{h}}}2^{2\chi+3\sigma}(1-\mathcal{R}^2)^{-\frac{1}{2}\boldsymbol{n}^2-\chi_{\mathrm{h}}}\\
&\times \sum_c \mathrm{SW}(c)\left(\frac{1-\mathcal{R}}{1+\mathcal{R}}\right)^{\frac{1}{2}B(\boldsymbol{n},c)} e^{\pi i B(K-c,\boldsymbol{\mu})},
\end{aligned}
$$

$$
\begin{aligned}
Z_{\boldsymbol{\mu},\boldsymbol{n},1}(\mathcal{R}) =& K_U \kappa_1^{1-\chi_{\mathrm{h}}}2^{2\chi+3\sigma}(1-\mathcal{R}^2)^{-\frac{1}{2}\boldsymbol{n}^2-\chi_{\mathrm{h}}}\\
&\times \sum_c \mathrm{SW}(c)\left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{\frac{1}{2}B(\boldsymbol{n},c)} e^{\pi i B(K+c,\boldsymbol{\mu})}
\end{aligned}
$$

$$
\begin{aligned}
Z_{\boldsymbol{\mu},\boldsymbol{n},1}(\mathcal{R}) =& 2^{2\chi+3\sigma-\chi_{\mathrm{h}}}(1-\mathcal{R}^2)^{-\frac{1}{2}\boldsymbol{n}^2-\chi_{\mathrm{h}}}\\
&\times \sum_c \mathrm{SW}(c)\left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{\frac{1}{2}B(\boldsymbol{n},c)} e^{\pi i B(K+c,\boldsymbol{\mu})}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_2 &= 2^{\frac{1}{4}}\kappa_2^{-\frac{1}{4}}\Lambda^{-\frac{1}{4}}\left(\frac{\vartheta_2(\tau_2)^2}{\eta(\tau_2)^6}\right)^{\frac{1}{4}}\\
\mathcal{B}_2 &= e^{\frac{\pi i}{8}}\vartheta_2(\tau_2)\mathcal{A}_2 a_2^{-\frac{1}{8}}\\
\mathcal{C}_2 &= C_2(\tau_2, v_2)\\
\mathcal{D}_2 &= e^{-\pi i v_2}\\
\mathcal{E}_2 &= e^{-\frac{\pi i}{8}}a_2^{\frac{1}{8}}q_2^{-\frac{1}{8}}\\
\mathcal{F}_2 &= e^{\pi i B(K-c,\boldsymbol{\mu})}e^{-2\pi i \boldsymbol{\mu}^2}
\end{aligned}
$$

$$\mathcal{A}_2 = 2^{\frac{1}{4}}\kappa_2^{-\frac{1}{4}} + \cdots$$

$$\mathcal{B}_2 = 2^{\frac{7}{4}}\kappa_2^{-\frac{1}{4}}\Lambda^{-\frac{3}{8}}e^{\frac{\pi i}{16}}\left(\frac{32}{1+\mathcal{R}^2}\right)^{-\frac{1}{8}} + \cdots$$

$$\mathcal{C}_2 = (1+\mathcal{R}^2)^{-\frac{1}{2}} + \cdots$$

$$\mathcal{D}_2 = \sqrt{\frac{1+i\mathcal{R}}{1-i\mathcal{R}}} + \cdots$$

$$\mathcal{E}_2 = \Lambda^{\frac{1}{8}}e^{-\frac{\pi i}{16}}\left(\frac{32}{1+\mathcal{R}^2}\right)^{\frac{1}{8}}$$

$$Z_{\boldsymbol{\mu},\boldsymbol{n},2}(\mathcal{R}) = 2^{2\chi+3\sigma-\chi_{\mathrm{h}}}(1+\mathcal{R}^2)^{-\frac{1}{2}\boldsymbol{n}^2-\chi_{\mathrm{h}}}e^{-3\pi i\chi_{\mathrm{h}}/2-2\pi i\boldsymbol{\mu}^2} \times \sum_c \mathrm{SW}(c)\left(\frac{1-i\mathcal{R}}{1+i\mathcal{R}}\right)^{\frac{1}{2}B(\boldsymbol{n},c)} e^{\pi i B(K+c,\boldsymbol{\mu})}$$

$$Z_{\boldsymbol{\mu},\boldsymbol{n},j}(\mathcal{R}) = (-1)^{B(2\boldsymbol{\mu},K-\boldsymbol{n})}Z_{\boldsymbol{\mu},\boldsymbol{n},j-2}(\mathcal{R}), j = 3,4.$$

$$Z_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R}) = \sum_{j=1}^{4} Z_{\boldsymbol{\mu},\boldsymbol{n},j}(\mathcal{R}) = \begin{cases} 2\sum_{j=1}^{2} Z_{\boldsymbol{\mu},\boldsymbol{n},j}(\mathcal{R}), & (-1)^{B(2\boldsymbol{\mu},K-\boldsymbol{n})} = 1 \\ 0, & (-1)^{B(2\boldsymbol{\mu},K-\boldsymbol{n})} = -1 \end{cases}$$

$$Z_{\boldsymbol{\mu},\boldsymbol{n},j}(\mathcal{R}) = \left(\omega^{3\chi_h+4\boldsymbol{\mu}^2}\right)^{j-1} Z_{\boldsymbol{\mu},\boldsymbol{n},1}\left(\omega^{j-1}\mathcal{R}\right),$$

$$G_{\boldsymbol{\mu},L}(\mathcal{R},X) = 2^{2-\chi_{\mathrm{h}}+K_X^2}(1-\mathcal{R}^2)^{-\frac{1}{2}(L-K_X)^2-\chi_{\mathrm{h}}} \times \sum_c \mathrm{SW}(c)(-1)^{B(\boldsymbol{\mu},c+K_X)}\left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{\frac{1}{2}B(c,K_X-L)},$$

$$\sum_k \chi\big(M(2\boldsymbol{\mu},k),\mathcal{O}(\mu_D(L))\big)\mathcal{R}^{l(k)} = \frac{1}{4}\sum_{l=0}^{3} e^{-\frac{1}{2}\pi i l(4\boldsymbol{\mu}^2+3\chi_{\mathrm{h}})}G_{\boldsymbol{\mu},L}\left(e^{-\frac{1}{2}\pi i l}\mathcal{R},X\right).$$

$$G_{\boldsymbol{\mu},L}(-\mathcal{R},X) = (-1)^{\chi_h+B(2\boldsymbol{\mu},K_X)}G_{\boldsymbol{\mu},L}(\mathcal{R},X),$$

$$\frac{1}{2}G_{\boldsymbol{\mu},L}(\mathcal{R},X) + \frac{1}{2}e^{\frac{\pi i}{2}(4\boldsymbol{\mu}^2+3\chi_{\mathrm{h}})}G_{\boldsymbol{\mu},L}(i\mathcal{R},X)$$

$$\begin{aligned}
\mathcal{R} = i: &\quad v = -\frac{\tau}{4} &\text{ or }& \frac{\tau}{4}+1 &\mod 2\mathbb{Z}+\tau\mathbb{Z}\\
\mathcal{R} = -1: &\quad v = -\frac{\tau}{4}+\frac{1}{2} &\text{ or }& \frac{\tau}{4}+\frac{1}{2} &\mod 2\mathbb{Z}+\tau\mathbb{Z}\\
\mathcal{R} = -i: &\quad v = -\frac{\tau}{4}+1 &\text{ or }& \frac{\tau}{4} &\mod 2\mathbb{Z}+\tau\mathbb{Z}\\
\mathcal{R} = 1: &\quad v = -\frac{\tau}{4}-\frac{1}{2} &\text{ or }& \frac{\tau}{4}-\frac{1}{2} &\mod 2\mathbb{Z}+\tau\mathbb{Z}
\end{aligned}$$

$$
\begin{aligned}
U_1&: \quad \tau=-\frac{1}{\tau_1}, v=\frac{v_1}{\tau_1}, v_1=\frac{1}{4},\\
U_2&: \quad \tau=2-\frac{1}{\tau_2}, v=\frac{v_2}{\tau_2}, v_2=\frac{1-2\tau_2}{4},\\
U_3&: \quad \tau=4-\frac{1}{\tau_3}, v=\frac{v_3}{\tau_3}, v_3=\frac{1-4\tau_3}{4}.
\end{aligned}
$$

$$
C(\tau)=\frac{\vartheta_4\left(\tau,-\frac{1}{4}\tau\right)}{\vartheta_4(\tau)}
$$

$$
\begin{aligned}
C_1(\tau_1)&=e^{-\frac{\pi i v_1^2}{\tau_1}}C\left(-\frac{1}{\tau_1}\right)=\frac{\vartheta_2\left(\tau_1,\frac{1}{4}\right)}{\vartheta_2(\tau_1)}=\frac{1}{\sqrt{2}}-\sqrt{2}q_1+\mathcal{O}(q_1^2)\\
C_2(\tau_2)&=e^{-\frac{\pi i v_2^2}{\tau_2}}C\left(2-\frac{1}{\tau_2}\right)=e^{\frac{\pi i}{4}}q_2^{-\frac{1}{8}}\frac{\vartheta_3\left(\tau_2,\frac{1}{4}\right)}{\vartheta_2(\tau_2)}=\frac{1}{2}e^{\frac{\pi i}{4}}q_2^{-\frac{1}{4}}+\mathcal{O}\left(q_2^{\frac{3}{4}}\right)\\
C_3(\tau_3)&=e^{-\frac{\pi i v_3^2}{\tau_3}}C\left(4-\frac{1}{\tau_3}\right)=\frac{1}{\sqrt{2}}e^{\frac{\pi i}{2}}q_3^{-\frac{1}{2}}+\mathcal{O}\left(q_3^{\frac{1}{2}}\right)
\end{aligned}
$$

$$
\frac{\mathrm{d}a}{\mathrm{d}\tau}=\frac{\pi}{4R}\frac{\vartheta_4(\tau)^8}{\vartheta_2(\tau)^2+\vartheta_3(\tau)^2}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)_1&=\frac{\pi i}{4R}\frac{\vartheta_2(\tau_1)^8}{\vartheta_3(\tau_1)^2+\vartheta_4(\tau_1)^2}=\frac{32\pi i}{R}q_1+\cdots\\
\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)_2&=\frac{\pi i}{4R}\frac{\vartheta_2(\tau_2)^8}{\vartheta_3(\tau_2)^2-\vartheta_4(\tau_2)^2}=\frac{8\pi i}{R}q_2^{\frac{1}{2}}+\cdots\\
\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)_3&=\frac{\pi i}{4R}\frac{\vartheta_2(\tau_3)^8}{\vartheta_3(\tau_3)^2+\vartheta_4(\tau_3)^2}=\frac{32\pi i}{R}q_3+\cdots
\end{aligned}
$$

$$
\Phi_{\boldsymbol{\mu},\boldsymbol{n}}^J(i)=-\frac{i}{8}\int_{\mathbb{H}/\Gamma^0(8)}\mathrm{d}\tau\wedge\,\mathrm{d}\bar{\tau}\frac{\vartheta_4(\tau)^{12-b_2}}{\vartheta_2(\tau)^2+\vartheta_3(\tau)^2}\frac{1}{\eta(\tau)^6}C(\tau)^{\boldsymbol{n}^2}\Psi_{\boldsymbol{\mu}}^J\left(\tau,\bar{\tau},-\frac{\boldsymbol{n}\tau}{4},-\frac{\boldsymbol{n}\bar{\tau}}{4}\right)e^{\pi i B(2\boldsymbol{\mu},K_X+\boldsymbol{n})}
$$

$$
\Phi_{\frac{1}{2},n}=\frac{1}{8}\sum_{j\in\text{cusps}}n_j\mathrm{Coeff}_{q^0}\left[\frac{\vartheta_4(\tau)^{11}}{\vartheta_2(\tau)^2+\vartheta_3(\tau)^2}\frac{1}{\eta(\tau)^6}C(\tau)^{n^2}G_{\frac{1}{2},n}\left(\tau,-\frac{\tau}{4}\right)\right]
$$

$$
\begin{aligned}
G_{\frac{1}{2},n}\left(\tau,-\frac{\tau}{4}\right)=&-\frac{q^{-\frac{n}{8}}}{\vartheta_4(\tau)}\sum_{l\in\mathbb{Z}}\frac{(-1)^l q^{\frac{1}{2}l^2-\frac{1}{8}}}{1-q^{l-\frac{n}{4}-\frac{1}{2}}}\\
&+\sum_{l\in\mathbb{Z}+\frac{1}{2}}\frac{1}{2}\left[\mathrm{sgn}(l)-\mathrm{sgn}\left(l-\frac{n}{4}\right)\right](-1)^{l-\frac{1}{2}}q^{-\frac{1}{2}l^2+\frac{1}{4}nl}\\
=&(-1)^{\frac{n^2-1}{8}}q^{\frac{n^2+7}{32}}\left(1+2q^{\frac{1}{4}}+3q^{\frac{1}{2}}+5q^{\frac{3}{4}}+\mathcal{O}(q)\right)
\end{aligned}
$$

$$
\Phi_{\frac{1}{2},n,\infty}=\mathrm{Coeff}_{q^0}\left[\frac{\vartheta_4(\tau)^{11}}{\vartheta_2(\tau)^2+\vartheta_3(\tau)^2}\frac{1}{\eta(\tau)^6}C(\tau)^{n^2}G_{\frac{1}{2},n}\left(\tau,-\frac{\tau}{4}\right)\right]
$$

$$\tau_j^{-\frac{1}{2}}\exp\left(\frac{\pi i n^2 v_j^2}{\tau_j}\right)\hat{G}_{\frac{1}{2},n}\left(2j-2-\frac{1}{\tau_j},\frac{v_j}{\tau_j}\right),$$

$$G_{D,\frac{1}{2},n}(\tau_1,v_1)=-\frac{q_1^{-\frac{1}{8}}}{2\sqrt{2}}\left(1+q_1+q_1^2-2q_1^3-q_1^4+\mathcal{O}(q_1^5)\right)$$

$$G_{D,\frac{1}{2},n}(\tau_2,v_2)=-\frac{e^{\frac{1}{4}\pi i n}q_2^{-\frac{n}{4}}}{\vartheta_2(\tau_2)}\sum_{l\in\mathbb{Z}}\frac{q_2^{\frac{1}{2}l(l+1)}}{1+e^{\frac{1}{2}\pi i n}q_2^{l-\frac{n}{2}}}$$

$$+\sum_{l\in\mathbb{Z}+\frac{1}{2}}\frac{1}{2}\left[\operatorname{sgn}(l)-\operatorname{sgn}\left(l-\frac{n}{2}\right)\right](-1)^{l-\frac{1}{2}}e^{-\frac{1}{2}\pi i n l}q_2^{\frac{1}{2}nl-\frac{1}{2}l^2}$$

$$=e^{-\frac{3\pi i}{4}}q_2^{\frac{n^2+4}{8}}\left(1-2q_2+4q_2^2-6q_2^3+\mathcal{O}(q_2^4)\right)$$

$$\Phi_{\frac{1}{2},n,2}=\operatorname{Coeff}_{q_2^0}\left[\frac{i}{4}\frac{\vartheta_2(\tau_2)^{11}}{\vartheta_3(\tau_2)^2-\vartheta_4(\tau_2)^2}\frac{1}{\eta(\tau_2)^6}C_2(\tau_2)^{n^2}G_{D,\frac{1}{2},n}\left(\tau_2,\frac{1-2\tau_2}{4}\right)\right].$$

$$G_{D,\frac{1}{2},n}(\tau_3,v_3)=\frac{i(-1)^{\frac{n+1}{2}}}{2\sqrt{2}}q_3^{\frac{1}{2}n^2-\frac{1}{8}}\left(1+q_3+q_3^2-2q_3^3-q_3^4+\mathcal{O}(q_3^5)\right)$$

$$\Delta\Phi_{\boldsymbol{\mu},\boldsymbol{n}}^{JJ'}=\sum_{j\in\text{cusps}}\left(\Phi_{\boldsymbol{\mu},\boldsymbol{n},j}^{J}-\Phi_{\boldsymbol{\mu},\boldsymbol{n},j}^{J'}\right)$$

$$=\frac{1}{8}\sum_{j\in\text{cusps}}n_j\operatorname{Coeff}_{q_j^0}\left[\frac{\vartheta_4(\tau)^{12-b_2}}{\vartheta_2(\tau)^2+\vartheta_3(\tau)^2}\frac{1}{\eta(\tau)^6}C(\tau)^{\boldsymbol{n}^2}\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}\left(\tau,\bar{\tau},-\frac{\boldsymbol{n}\tau}{4},-\frac{\boldsymbol{n}\bar{\tau}}{4}\right)\right].$$

$$\tau_1^{-\frac{1}{2}b_2}\exp\left(\frac{\pi i\boldsymbol{n}^2v_1^2}{\tau_1}\right)\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}\left(-\frac{1}{\tau_1},-\frac{1}{\bar{\tau}_1},\frac{\boldsymbol{n}v_1}{\tau_1},\frac{\boldsymbol{n}v_1}{\bar{\tau}_1}\right)$$

$$\exp\left(\frac{\pi i\sigma}{4}+\pi iB(\boldsymbol{\mu},K)\right)\widehat{\Theta}_{\frac{K}{2}}^{JJ'}\left(\tau_1,\bar{\tau}_1,\boldsymbol{n}v_1-\boldsymbol{\mu}+\frac{K}{2},\boldsymbol{n}v_1-\boldsymbol{\mu}+\frac{K}{2}\right).$$

$$\tau_2^{-\frac{1}{2}b_2}\exp\left(\frac{\pi i\boldsymbol{n}^2v_2^2}{\tau_2}\right)\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}\left(2-\frac{1}{\tau_2},2-\frac{1}{\bar{\tau}_2},\frac{\boldsymbol{n}v_2}{\tau_2},\frac{\boldsymbol{n}\bar{v}_2}{\bar{\tau}_2}\right)$$

$$\exp\left(\frac{\pi i\sigma}{4}+2\pi i\boldsymbol{\mu}^2-\pi iB(\boldsymbol{\mu},K)\right)\widehat{\Theta}_{\frac{K}{2}}^{JJ'}\left(\tau_2,\bar{\tau}_2,\boldsymbol{n}v_2-\boldsymbol{\mu}+\frac{K}{2},\boldsymbol{n}\bar{v}_2-\boldsymbol{\mu}+\frac{K}{2}\right).$$

$$\tau_3^{-\frac{1}{2}b_2}\exp\left(\frac{\pi i\boldsymbol{n}^2v_3^2}{\tau_3}\right)\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}\left(4-\frac{1}{\tau_3},4-\frac{1}{\bar{\tau}_3},\frac{\boldsymbol{n}v_3}{\tau_3},\frac{\boldsymbol{n}\bar{v}_3}{\tau_3}\right)$$

$$\exp\left(\frac{\pi i\sigma}{4}-\pi iB(\boldsymbol{\mu},K)\right)\widehat{\Theta}_{\frac{K}{2}}^{JJ'}\left(\tau_3,\bar{\tau}_3,\boldsymbol{n}v_3-\boldsymbol{\mu}+\frac{K}{2},\boldsymbol{n}\bar{v}_3-\boldsymbol{\mu}+\frac{K}{2}\right).$$

$$B\left(\boldsymbol{k}+\frac{\operatorname{Im}(v_2)}{y_2}\boldsymbol{n},J\right)=B\left(\boldsymbol{k}-\frac{\boldsymbol{n}}{2},J\right)=0,$$

$$F_+(A) + F_+\left(A - 2A^{(I)}\right) + M_2\bar{M}_2 + M_4\bar{M}_4 = 0$$
$$\not{D}_A M_2 = 0$$
$$\not{D}_{A-2A^{(I)}} M_4 = 0$$

$$\mathrm{U}(1)_+ : (M_2, M_4) \to \left(e^{i\theta}M_2, e^{i\theta}M_4\right),$$
$$\mathrm{U}(1)_- : (M_2, M_4) \to \left(e^{i\zeta}M_2, e^{-i\zeta}M_4\right),$$

$$m(c, c - 2\boldsymbol{n}) = \frac{1}{4}\left(2\boldsymbol{n}^2 - 2B(\boldsymbol{n}, c) + c^2 - \chi - 2\sigma\right)$$

$$\sum_{c \in K_X + H^2(X, 2\mathbb{Z})} \mathrm{SW}_2(c, c - 2\boldsymbol{n}; J)\mathrm{Res}_{a_2=0}\left[a_2^{-1-m(c,c-2\boldsymbol{n})} e^{-S_{E,2}}\right] Z^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},\mathrm{p}^{(I)}}(\mathcal{R}),$$

$$e^{-S_{E,2}} = \kappa_2 \mathcal{A}_2^{\chi} \mathcal{B}_2^{\sigma} \mathcal{C}_2^{n^2} \mathcal{D}_2^{B(c,\boldsymbol{n})} \mathcal{E}_2^{c^2} \mathcal{F}_{\boldsymbol{\mu},2}$$

$$Z^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},\mathrm{p}^{(I)}}(\mathcal{R}) = \sum_{\text{fluxes}} \int \mathrm{d}a \prod_{\ell} \mathcal{Z}\left(a^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, \Lambda^{(\ell)}, R\right)$$

$$Z^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},\mathrm{p}^{(I)}}(\mathcal{R}) = \sum_{l} d^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},\mathrm{p}^{(I)}}(l)\mathcal{R}^l$$

$$d^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},\mathrm{p}^{(I)}}(l) = \mathrm{Tr}_{\mathcal{H}_{k,\boldsymbol{\mu}}}(-1)^F e^{-R(H + \epsilon_1 J_1 + \epsilon_2 J_2)}$$

$$ds^{2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}} = e^1 e^{\bar{1}} + e^2 e^{\bar{2}} + \left(\mathrm{d}x^{5\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}} + v_\mu dx^{\mu\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}}\right)^2 = e^1 e^{\bar{1}} + e^2 e^{\bar{2}} + (\hat{e}^5)^2$$

$$v = v^m \frac{\partial}{\partial x^m} = v^\mu \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x^5}$$

$$S_+ \cong K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}}$$

$$\left(K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}}\right)^{\otimes 2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}} \cong K_X \oplus K_X^{-1} \oplus \mathcal{O} \oplus \mathcal{O}$$

$$\overline{\mathcal{Q}}_{(1)} = \bar{\zeta}^{\dot{\alpha}}_{(1)A} \bar{Q}^A_{\dot{\alpha}}, \overline{\mathcal{Q}}_{(2)} = \bar{\zeta}^{\dot{\alpha}}_{(2)A} \bar{Q}^A_{\dot{\alpha}}$$

$$\bar{\zeta}^{\dot{\alpha}}_{(1)A} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \bar{\zeta}^{\dot{\alpha}}_{(2)A} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

$$I_{\mu\nu} := 2i\bar{\zeta}^A_{(1)} \bar{\sigma}_{\mu\nu} \bar{\zeta}_{(2)A}$$

$$
\begin{aligned}
\bar{\lambda}^{(0,0)}_{(1)} &:= \bar{\zeta}^{A}_{(1)}\bar{\lambda}_{A}\\
\bar{\lambda}^{(0,0)}_{(2)} &:= \bar{\zeta}^{A}_{(2)}\bar{\lambda}_{A}\\
\bar{\lambda}^{(0,2)}_{\mu\nu} &:= 2\bar{\zeta}^{A}_{(2)}\bar{\sigma}_{\mu\nu}\bar{\lambda}_{A} + i\bar{\zeta}^{A}_{(2)}\bar{\lambda}_{A}I_{\mu\nu}\\
\bar{\lambda}^{(2,0)}_{\mu\nu} &:= 2\bar{\zeta}^{A}_{(1)}\bar{\sigma}_{\mu\nu}\bar{\lambda}_{A} - i\bar{\zeta}^{A}_{(1)}\bar{\lambda}_{A}I_{\mu\nu}\\
\lambda^{(0,1)}_{\mu} &:= \bar{\zeta}_{(2)A}\bar{\sigma}_{\mu}\lambda^{A}\\
\lambda^{(1,0)}_{\mu} &:= \bar{\zeta}_{(1)A}\bar{\sigma}_{\mu}\lambda^{A}\\
D^{(0,2)}_{\mu\nu} &:= -2i\bar{\zeta}^{A}_{(2)}\bar{\sigma}_{\mu\nu}\bar{\zeta}^{B}_{(2)}D_{AB}\\
D^{(2,0)}_{\mu\nu} &:= -2i\bar{\zeta}^{A}_{(1)}\bar{\sigma}_{\mu\nu}\bar{\zeta}^{B}_{(1)}D_{AB}\\
D^{(0,0)} &:= \bar{\zeta}^{A}_{(2)}D_{AB}\bar{\zeta}^{B}_{(1)} = -\bar{\zeta}^{A}_{(1)}D_{AB}\bar{\zeta}^{B}_{(2)} = D_{12}\\
F^{(0,0)} &:= -i\bar{\zeta}^{A}_{(2)}\bar{\sigma}^{\mu\nu}\bar{\zeta}_{(1)A}F_{\mu\nu} = \frac{1}{2}I^{\mu\nu}F_{\mu\nu}
\end{aligned}
$$

$$
F = F^{(2,0)} + F^{(0,2)} + F^{(1,1)} + F^{(1,0)} \wedge \hat{e}^{5} + F^{(0,1)} \wedge \hat{e}^{5}.
$$

$$
\begin{aligned}
\left[\overline{\mathcal{Q}}_{(1)}, \sigma\right] &= -i\bar{\lambda}^{(0,0)}_{(1)}\\
\left[\overline{\mathcal{Q}}_{(1)}, A\right] &= -\lambda^{(1,0)} + \bar{\lambda}^{(0,0)}_{(1)}\hat{e}^{5},\\
\left\{\overline{\mathcal{Q}}_{(1)}, \lambda^{(0,1)}\right\} &= 2\bar{\partial}_{A}\sigma + 2iF^{(0,1)},\\
\left\{\overline{\mathcal{Q}}_{(1)}, \lambda^{(1,0)}\right\} &= 0,\\
\left\{\overline{\mathcal{Q}}_{(1)}, \bar{\lambda}^{(0,2)}\right\} &= -4iF^{(0,2)},\\
\left\{\overline{\mathcal{Q}}_{(1)}, \bar{\lambda}^{(2,0)}\right\} &= -iD^{(2,0)},\\
\left\{\overline{\mathcal{Q}}_{(1)}, \bar{\lambda}^{(0,0)}_{(1)}\right\} &= 0,\\
\left\{\overline{\mathcal{Q}}_{(1)}, \bar{\lambda}^{(0,0)}_{(2)}\right\} &= -v^{m}D_{m}\sigma - \mathcal{D}^{(0,0)},\\
\left[\overline{\mathcal{Q}}_{(1)}, D^{(0,2)}\right] &= 4\bar{\partial}_{A}\lambda^{(0,1)} + 2\left[\sigma, \bar{\lambda}^{(0,2)}\right] - 2v^{m}D_{m}\bar{\lambda}^{(0,2)},\\
\left[\overline{\mathcal{Q}}_{(1)}, D^{(2,0)}\right] &= 0,\\
\left[\overline{\mathcal{Q}}_{(1)}, F^{(0,0)}\right] &= -I^{\mu\nu}D_{\mu}\lambda^{(1,0)}_{\nu},\\
\left[\overline{\mathcal{Q}}_{(1)}, D^{(0,0)}\right] &= I^{\mu\nu}D_{\mu}\lambda^{(1,0)}_{\nu} - i\left[\sigma, \bar{\lambda}^{(0,0)}_{(1)}\right] + iv^{m}D_{m}\bar{\lambda}^{(0,0)}_{(1)},\\
\left[\overline{\mathcal{Q}}_{(1)}, \mathcal{D}^{(0,0)}\right] &= -i\left[\sigma, \bar{\lambda}^{(0,0)}_{(1)}\right] + iv^{m}D_{m}\bar{\lambda}^{(0,0)}_{(1)},
\end{aligned}
$$

$$\left[\bar{\mathcal{Q}}_{(2)}, \sigma\right] = -i\bar{\lambda}^{(0,0)}_{(2)}$$
$$\left[\bar{\mathcal{Q}}_{(2)}, A\right] = -\lambda^{(0,1)} + \bar{\lambda}^{(0,0)}_{(2)}\hat{e}^5,$$
$$\left\{\bar{\mathcal{Q}}_{(2)}, \lambda^{(0,1)}\right\} = 0,$$
$$\left\{\bar{\mathcal{Q}}_{(2)}, \lambda^{(1,0)}\right\} = 2\partial_A\sigma + 2iF^{(1,0)},$$
$$\left\{\bar{\mathcal{Q}}_{(2)}, \bar{\lambda}^{(0,2)}\right\} = -iD^{(0,2)},$$
$$\left\{\bar{\mathcal{Q}}_{(2)}, \bar{\lambda}^{(2,0)}\right\} = -4iF^{(2,0)},$$
$$\left\{\bar{\mathcal{Q}}_{(2)}, \bar{\lambda}^{(0,0)}_{(1)}\right\} = -v^m D_m\sigma + \mathcal{D}^{(0,0)},$$
$$\left\{\bar{\mathcal{Q}}_{(2)}, \bar{\lambda}^{(0,0)}_{(2)}\right\} = 0,$$
$$\left[\bar{\mathcal{Q}}_{(2)}, D^{(0,2)}\right] = 0,$$
$$\left[\bar{\mathcal{Q}}_{(2)}, D^{(2,0)}\right] = 4\partial_A\lambda^{(1,0)} + 2\left[\sigma, \bar{\lambda}^{(2,0)}\right] - 2v^m D_m\bar{\lambda}^{(2,0)},$$
$$\left[\bar{\mathcal{Q}}_{(2)}, F^{(0,0)}\right] = -I^{\mu\nu}D_\mu\lambda^{(0,1)}_\nu,$$
$$\left[\bar{\mathcal{Q}}_{(2)}, D^{(0,0)}\right] = I^{\mu\nu}D_\mu\lambda^{(0,1)}_\nu + i\left[\sigma, \bar{\lambda}^{(0,0)}_{(2)}\right] - iv^m D_m\bar{\lambda}^{(0,0)}_{(2)},$$
$$\left[\bar{\mathcal{Q}}_{(2)}, \mathcal{D}^{(0,0)}\right] = i\left[\sigma, \bar{\lambda}^{(0,0)}_{(2)}\right] - iv^m D_m\bar{\lambda}^{(0,0)}_{(2)}.$$

$$S_{5\mathrm{d}} = \frac{1}{g_{5d}^2}\int \mathrm{d}^5x\sqrt{g_5}\mathcal{Q}_{(2)}\mathcal{Q}_{(1)}\mathrm{tr}\left[\frac{1}{32}\epsilon^{\mu\nu\rho\sigma}\bar{\lambda}^{(2,0)}_{\mu\nu}\bar{\lambda}^{(0,2)}_{\rho\sigma} - \frac{i}{4}(\sigma - iv^\mu A_\mu)\left(\mathcal{D}^{(0,0)} - 2\mathcal{F}^{(0,0)}\right)\right] - \frac{1}{2g_{5d}^2}\int \mathrm{tr}(F_4\wedge F_4)\wedge\hat{e}^5$$

$$S_{5\mathrm{d}}|_{\mathrm{superparticle}} = \frac{1}{g_{5d}^2}\int \sqrt{g_5}\,\mathrm{d}^5x\mathrm{tr}\Big(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^{(2,0)}_{\mu\nu}F^{(0,2)}_{\rho\sigma} + \frac{1}{2}\left(F^{(0,0)}\right)^2 - \frac{1}{32}\epsilon^{\mu\nu\rho\sigma}D^{(2,0)}_{\mu\nu}D^{(0,2)}_{\rho\sigma} - \frac{1}{2}\left(D^{(0,0)}\right)^2 + \frac{1}{2}D_m\sigma D^m\sigma + \frac{1}{2}v^m v_n F_{lm}F^{ln}\Big)$$

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^{(2,0)}_{\mu\nu}F^{(0,2)}_{\rho\sigma} = (F_{13} - F_{24})^2 + (F_{14} + F_{23})^2,$$
$$-\frac{1}{32}\epsilon^{\mu\nu\rho\sigma}D^{(2,0)}_{\mu\nu}D^{(0,2)}_{\rho\sigma} = \frac{1}{2}D_{11}D_{22},$$

$$F_4 := F^{(2,0)} + F^{(0,2)} + F^{(1,1)} = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$$

$$v^\mu F_{\mu 5} = 0, F_{5\mu} + v^\nu F_{\nu\mu} = 0$$

$$F_{\mathrm{SU}(4)} = \begin{pmatrix} \frac{1}{2}F_{\mathrm{U}(4)} & 0 \\ 0 & -\frac{1}{2}F_{\mathrm{U}(4)} \end{pmatrix}, (A_5)_{\mathrm{SU}(4)} = \begin{pmatrix} (A_5)_{\mathrm{U}(4)} & 0 \\ 0 & -(A_5)_{\mathrm{U}(4)} \end{pmatrix},$$

$$\sigma_{\mathrm{SU}(4)} = \begin{pmatrix} \sigma_{\mathrm{U}(4)} & 0 \\ 0 & -\sigma_{\mathrm{U}(4)} \end{pmatrix},$$

$$\left[\frac{F_{\text{eq}}}{2\pi}\right] = \sum_{s=1}^{\chi} \mathfrak{p}_s D_s^{\text{eq}}, \mathfrak{p}_s \in \mathbb{Z}$$

$$\boldsymbol{k} = \left[\frac{F_4}{4\pi}\right] = \frac{1}{2}\sum_{s=1}^{\chi} \mathfrak{p}_s D_s \in L + \boldsymbol{\mu}$$

$$\mathfrak{p}_s - \mathfrak{p}'_s = \sum_{i=1,2} m_s n_s^i, m_1, m_2 \in \mathbb{Z}$$

$$\nu_s^{(\ell)} = -\frac{i}{2\pi}\left(\epsilon_1^{(\ell)}\delta_{s,\ell} + \epsilon_2^{(\ell)}\delta_{s-1,\ell}\right)$$

$$A_5^{(\ell)} = -\frac{i}{2}\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell + \epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}\right) + a_5$$

$$a^{(\ell)} = a + \frac{1}{2}\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell + \epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}\right)$$

$$\mathfrak{p}^{(I)} = \left(\mathfrak{p}_1^{(I)}, \cdots, \mathfrak{p}_\chi^{(I)}\right)$$

$$\left[\frac{F_{\text{eq}}^{(I)}}{2\pi}\right] = \left[\frac{F_4^{(I)} + A_5^{(I)}}{2\pi}\right] = \sum_{s=1}^{\chi} \mathfrak{p}_s^{(I)} D_s^{\text{eq}}$$

$$\boldsymbol{n}_I = \left[\frac{F_4^{(I)}}{2\pi}\right] = \sum_{s=1}^{\chi} \mathfrak{p}_s^{(I)} D_s$$

$$\left(A_5^{(I)}\right)^{(\ell)} = -i\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell^{(I)} + \epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}^{(I)}\right) + a_5^{(I)},$$

$$\left(a^{(I)}\right)^{(\ell)} = a^{(I)} + \epsilon_1^{(\ell)}\mathfrak{p}_\ell^{(I)} + \epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}^{(I)}$$

$$\Lambda^{(\ell)} = e^{-\frac{1}{4}n_I^{(\ell)}}\Lambda$$

$$n_I^{(\ell)} = -R\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell^{(I)} + \epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}^{(I)}\right)$$

$$\begin{aligned}
&\left[\overline{\mathcal{Q}}_{(1)}, a\right] = 0, \quad \left[\overline{\mathcal{Q}}_{(1)}, \bar{a}\right] = 2\lambda_0, \quad \left\{\overline{\mathcal{Q}}_{(1)}, \lambda_0\right\} = 0, \quad \left\{\overline{\mathcal{Q}}_{(1)}, \bar{\lambda}_0\right\} = -4h, \quad \left[\overline{\mathcal{Q}}_{(1)}, h\right] = 0,\\
&\left[\overline{\mathcal{Q}}_{(2)}, a\right] = 0, \quad \left[\overline{\mathcal{Q}}_{(2)}, \bar{a}\right] = 2\bar{\lambda}_0, \quad \left\{\overline{\mathcal{Q}}_{(2)}, \lambda_0\right\} = 4h, \quad \left\{\overline{\mathcal{Q}}_{(2)}, \bar{\lambda}_0\right\} = 0, \quad \left[\overline{\mathcal{Q}}_{(2)}, h\right] = 0.
\end{aligned}$$

$$\begin{aligned}
\delta A_5 &= i\bar{\epsilon}_{(1)}\bar{\lambda}_{(1)}^{(0,0)} + i\bar{\epsilon}_{(2)}\bar{\lambda}_{(2)}^{(0,0)}\\
\delta\sigma &= \bar{\epsilon}_{(1)}\bar{\lambda}_{(1)}^{(0,0)} + \bar{\epsilon}_{(2)}\bar{\lambda}_{(2)}^{(0,0)}\\
\delta\bar{\lambda}_{(1)}^{(0,0)} &= -i\bar{\epsilon}_{(2)}\left(D_5\sigma - \mathcal{D}^{(0,0)}\right)\\
\delta\bar{\lambda}_{(2)}^{(0,0)} &= -i\bar{\epsilon}_{(1)}\left(D_5\sigma + \mathcal{D}^{(0,0)}\right)\\
\delta\mathcal{D}^{(0,0)} &= -\bar{\epsilon}_{(1)}\left[D_5 - \sigma, \bar{\lambda}_{(1)}^{(0,0)}\right] + \bar{\epsilon}_{(2)}\left[D_5 - \sigma, \bar{\lambda}_{(2)}^{(0,0)}\right]
\end{aligned}$$

$$Z^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},\mathfrak{p}^{(I)}}(\mathcal{R}) = K_\Omega \sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \int_\Gamma \mathrm{d}h \int_{\mathfrak{M}} \mathrm{d}a\, \mathrm{d}\bar{a} \int \mathrm{d}\lambda_0\, \mathrm{d}\bar{\lambda}_0 g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},\lambda_0,\bar{\lambda}_0,h),$$

$$K_\Omega = -\frac{R}{4\pi^2} g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},\lambda_0,\bar{\lambda}_0,h),$$

$$A_\mu = A^{\mathrm{t}}_\mu + g_{5d}A'_\mu$$

$$\sigma = \sigma^{\mathrm{t}} + g_{5d}\sigma', A_5 = A^{\mathrm{t}}_5 + g_{5d}A'_5$$

$$g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},\lambda_0,\bar{\lambda}_0,h) := \int [dV']e^{-S_{5d}[V_0+g_{5d}V']}$$

$$g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,0) = g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$

$$M^{\mathbb{T}}_{\boldsymbol{\mu},k} \leftrightarrow \left\{Y^{(\ell)}_a, \ell = 1,\cdots,\chi, a = 1,2\right\}, \sum_{\ell,a} \left|Y^{(\ell)}_a\right| = d$$

$$g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = \mathcal{R}^{-1}\exp\left(-R\sum_{\ell=1}^{\chi} \frac{\left(\epsilon^{(\ell)}_1 + \epsilon^{(\ell)}_2\right)^3}{48\epsilon^{(\ell)}_1\epsilon^{(\ell)}_2}\right)\prod_{\ell=1}^{\chi} Z\left(a^{(\ell)},\epsilon^{(\ell)}_1,\epsilon^{(\ell)}_2,n^{(\ell)}_I,\Lambda,R\right).$$

$$g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},\lambda_0,\bar{\lambda}_0,h)$$

$$0 = \overline{Q}_{(1)} g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}} = \left(2\lambda_0\frac{\partial}{\partial\bar{a}} - 4h\frac{\partial}{\partial\bar{\lambda}_0}\right) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}},$$
$$0 = \overline{Q}_{(2)} g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}} = \left(2\bar{\lambda}_0\frac{\partial}{\partial\bar{a}} + 4h\frac{\partial}{\partial\lambda_0}\right) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}.$$

$$2h\frac{\partial}{\partial\lambda_0}\frac{\partial}{\partial\bar{\lambda}_0} g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}} = \frac{\partial}{\partial\bar{a}} g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}.$$

$$Z^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},\mathfrak{p}^{(I)}}(\mathcal{R}) = \lim_{g_{5d}\to 0}\frac{K_\Omega}{2}\sum_{[\mathfrak{p}]}\int_\Gamma \frac{\mathrm{d}h}{h}\int_{\mathfrak{M}} \mathrm{d}a\, \mathrm{d}\bar{a}\partial_{\bar{a}} g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,h)$$

$$\alpha a^{(\ell)} + \epsilon^{(\ell)}_1 m + \epsilon^{(\ell)}_2 n = 0 \bmod \frac{2\pi i}{R}\mathbb{Z}$$

$$\mathfrak{M} = \mathfrak{M} \setminus \Delta_\delta \cup \Delta_\delta$$

$$\Delta_\delta = \bigcup_{(n,m,\ell,\alpha)} \Delta^{(m,n,\ell,\alpha)}_\delta$$

$$Z^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},\mathfrak{p}^{(I)}}(\mathcal{R}) = \lim_{\substack{\delta\to 0\\ g_{5d}\to 0}}\frac{K_\Omega}{2}\sum_{[\mathfrak{p}]}\int_\Gamma \frac{\mathrm{d}h}{h}\int_{\partial(\mathfrak{M}\setminus\Delta_\delta)} \mathrm{d}a g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,h)$$

$$\partial(\mathfrak{M} \setminus \Delta_\delta) = \partial\mathfrak{M} - \partial\Delta_\delta.$$

$$\mathcal{Z}_{\mathrm{pert}}(a,\epsilon_1,\epsilon_2,\Lambda,R) = \exp\left[-\tilde{\gamma}_{\epsilon_1,\epsilon_2}(2a \mid R,\Lambda) - \tilde{\gamma}_{\epsilon_1,\epsilon_2}(-2a \mid R,\Lambda)\right]$$

$$\tilde{\gamma}_{\epsilon_1,\epsilon_2}(a \mid R,\Lambda) = \gamma_{\epsilon_1,\epsilon_2}(a \mid R,\Lambda) + \frac{1}{\epsilon_1\epsilon_2}\left(\frac{\pi^2 a}{6R} - \frac{\zeta(3)}{R^2}\right) + \frac{\epsilon_1+\epsilon_2}{2\epsilon_1\epsilon_2}\left(a\log(\mathcal{R}) + \frac{\pi^2}{6R}\right) + \frac{\epsilon_1^2+\epsilon_2^2+3\epsilon_1\epsilon_2}{12\epsilon_1\epsilon_2}\log(\mathcal{R})$$

$$\gamma_{\epsilon_1,\epsilon_2}(a \mid R,\Lambda) = \frac{1}{2\epsilon_1\epsilon_2}\left[-\frac{R}{6}\left(a+\frac{\epsilon_1+\epsilon_2}{2}\right)^3 + a^2\log(\mathcal{R})\right] + \sum_{n\geq 1}\frac{1}{n}\frac{e^{-nRa}}{(e^{nR\epsilon_1}-1)(e^{nR\epsilon_2}-1)}$$

$$\sum_{n\geq 1}\frac{1}{n}\frac{e^{-nRa}}{(e^{nR\epsilon_1}-1)(e^{nR\epsilon_2}-1)} = \sum_{m\geq 0}\frac{c_m(\epsilon_1,\epsilon_2)}{m!}R^{m-2}\mathrm{Li}_{3-m}(e^{-Ra})$$

$$2a = m\epsilon_1 + n\epsilon_2 + \frac{2\pi i s}{R}, s \in \mathbb{Z},$$

$$\mathcal{Z}_{\text{inst}}(a,\epsilon_1,\epsilon_2,n_I,\Lambda,R) = \sum_{\vec{Y}}\left(\mathcal{R}^4 e^{-R(\epsilon_1+\epsilon_2)}\right)^{|\vec{Y}|}\mathcal{Z}_{\vec{Y}}^{\text{vec}}(a,\epsilon_1,\epsilon_2,R)\mathcal{Z}_{\vec{Y}}^{\mathrm{U}(1)^{(I)}}(a,\epsilon_1,\epsilon_2,n_I)$$

$$A_Y(s) = \lambda_i - j, L_Y = \lambda_j' - i$$

$$E(a,Y_1,Y_2,s) = a - \epsilon_1 L_{Y_2}(s) + \epsilon_2\left(A_{Y_1}(s)+1\right)$$

$$\mathcal{Z}_{\vec{Y}}^{\text{vec}}(a,\epsilon_1,\epsilon_2,R) = \prod_{i,j=1}^{2} n_{i,j}^{(Y_i,Y_j)}(a,\epsilon_1,\epsilon_2,R),$$

$$n_{i,j}^{(Y_i,Y_j)} = \prod_{s\in Y_j}\left(1-e^{-RE(a_i-a_j,Y_j,Y_i,s)}\right)^{-1}\prod_{t\in Y_i}\left(1-e^{-R\left(\epsilon_1+\epsilon_2-E(a_j-a_i,Y_i,Y_j,t)\right)}\right)^{-1},$$

$$a_1 = -a_2 = a. \mathcal{Z}_{\vec{Y}}^{\mathrm{U}(4)^{(I)}}$$

$$\mathcal{Z}_{\vec{Y}}^{\mathrm{U}(4)^{(I)}}(a,\epsilon_1,\epsilon_2,n_I) = \exp\left[n_I\left(-|\vec{Y}| + \frac{a^2}{\epsilon_1\epsilon_2}\right)\right],$$

$$\mathcal{Z}_{\text{inst}}(a,\epsilon_1,\epsilon_2,n_I,\Lambda,R) = \exp\left(\frac{n_I a^2}{\epsilon_1\epsilon_2}\right)\mathcal{Z}_{\text{inst}}\left(a,\epsilon_1,\epsilon_2,0,\Lambda e^{-\frac{1}{4}n_I},R\right)$$

$$\mathcal{Z}_{\text{inst}}(a,\epsilon_1,\epsilon_2,n_I=0,\Lambda,R) = 1 - \sum_{m,n=1}^{\infty}\frac{\mathcal{R}^{4mn}T_{m,n}(t_1,t_2)}{(x^{-2}-t_1^m t_2^n)(1-x^2 t_1^{-m}t_2^{-n})}\mathcal{Z}_{\text{inst}}\left(\frac{m\epsilon_1-n\epsilon_2}{2},\epsilon_1,\epsilon_2,n_I=0,\Lambda,R\right)$$

$$T_{m,n}(t_1,t_2) = \frac{(1+t_1^m t_2^n)}{(t_1t_2)^{mn}}\underbrace{\prod_{i=-m+1}^{m}\prod_{j=-n+1}^{n}}_{(i,j)\neq\{(0,0),(m,n)\}}\frac{t_1^i t_2^j}{1-t_1^i t_2^j}$$

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2,n_I,\Lambda,R) := \mathcal{Z}_{\text{pert}}(a,\epsilon_1,\epsilon_2,\Lambda,R)\mathcal{Z}_{\text{inst}}(a,\epsilon_1,\epsilon_2,n_I,\Lambda,R)$$

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2,\Lambda,R) := \mathcal{Z}(a,\epsilon_1,\epsilon_2,n_I=0,\Lambda,R)$$

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2,n_I,\Lambda,R) = \mathcal{Z}\left(a,\epsilon_1,\epsilon_2,\Lambda e^{-\frac{1}{4}n_I},R\right)\exp\left[-\frac{n_I(\epsilon_1^2+\epsilon_2^2+3\epsilon_1\epsilon_2)}{24\epsilon_1\epsilon_2}\right]$$

$$2a = m\epsilon_1 + n\epsilon_2 + \frac{2\pi i s}{R}, s\in\mathbb{Z}$$

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2,\Lambda,R) = \exp\frac{1}{\epsilon_1\epsilon_2}\left[\mathcal{F}(a,\Lambda,R)+(\epsilon_1+\epsilon_2)H(a) + \epsilon_1\epsilon_2\log(A)(a,\Lambda,R)+\frac{\epsilon_1^2+\epsilon_2^2}{3}\log(B)(a,\Lambda,R)+\cdots\right],$$

$$\begin{aligned}g_{\mathfrak{p},\mathfrak{p}}^{\epsilon_1,\epsilon_2} =& \frac{1}{\mathcal{R}}\exp\left\{-\sum_{\ell=1}^{\chi}\frac{1}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\left[\frac{R}{48}\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)^3+\frac{n_I^{(\ell)}}{24}\left(\left(\epsilon_1^{(\ell)}\right)^2+\left(\epsilon_2^{(\ell)}\right)^2+3\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}\right)\right]\right\}\\
&\times\prod_{\ell=1}^{\chi}\mathcal{Z}\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\Lambda e^{-\frac{1}{4}n_I^{(\ell)}},R\right)\\
=&\frac{1}{\mathcal{R}}\exp\left\{-\sum_{\ell=1}^{\chi}\frac{1}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\left[\frac{R}{48}\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)^3+\frac{n_I^{(\ell)}}{24}\left(\left(\epsilon_1^{(\ell)}\right)^2+\left(\epsilon_2^{(\ell)}\right)^2+3\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}\right)\right]\right\}\\
&\times\exp\left\{\sum_{\ell=1}^{\chi}\frac{1}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\left[\mathcal{F}\left(a^{(\ell)},\Lambda e^{-\frac{1}{4}n_I^{(\ell)}},R\right)+\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)H\left(a^{(\ell)}\right)\right.\right.\\
&\left.\left.+\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}\log(A)\left(a^{(\ell)},\Lambda e^{-\frac{1}{4}n_I^{(\ell)}},R\right)+\frac{\left(\epsilon_1^{(\ell)}\right)^2+\left(\epsilon_2^{(\ell)}\right)^2}{3}\log(B)\left(a^{(\ell)},\Lambda e^{-\frac{1}{4}n_I^{(\ell)}},R\right)+\cdots\right]\right\}\end{aligned}$$

$$\begin{aligned}\lim_{\epsilon_1,\epsilon_2\to0}g_{\mathfrak{p},\mathfrak{p}^{(I)}}^{\epsilon_1,\epsilon_2}(a) =& \frac{1}{\mathcal{R}}\lim_{\epsilon_1,\epsilon_2\to0}\exp\sum_{\ell=1}^{\chi}\frac{1}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\left[\frac{1}{8}\frac{\partial^2\mathcal{F}}{\partial a^2}\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}\right)^2\right.\\
&+\frac{R}{8}\frac{\partial^2\mathcal{F}}{\partial a\partial(\log\mathcal{R})}\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell^{(I)}+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}^{(I)}\right)\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}\right)\\
&+\frac{R^2}{32}\frac{\partial^2\mathcal{F}}{\partial(\log\mathcal{R})^2}\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell^{(I)}+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}^{(I)}\right)^2\\
&+\frac{1}{2}\frac{\partial H}{\partial a}\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}\right)\\
&+\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}\log(A)(a,\Lambda,R)\\
&\left.+\frac{1}{3}\left(\left(\epsilon_1^{(\ell)}\right)^2+\left(\epsilon_2^{(\ell)}\right)^2\right)\log(B)(a,\Lambda,R)+\mathcal{O}(\epsilon^3)\right].\end{aligned}$$

$$\sum_{\ell=1}^{\chi}\frac{\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}\right)^2}{4\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}=\left(\frac{1}{2}\sum_{\ell=1}^{\chi}\mathfrak{p}_\ell D_\ell\right)^2=\boldsymbol{k}^2$$

$$\sum_{\ell=1}^{\chi}\frac{\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}\right)\left(\epsilon_1^{(\ell)}\mathfrak{p}_l^{(I)}+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}^{(I)}\right)}{2\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}=\left(\frac{1}{2}\sum_{\ell=1}^{\chi}\mathfrak{p}_\ell D_\ell\right)\left(\sum_{\ell=1}^{\chi}\mathfrak{p}_\ell^{(I)}D_\ell\right)=B(\boldsymbol{k},\boldsymbol{n}_I)$$

$$\sum_{\ell=1}^{\chi}\frac{\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell^{(I)}+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}^{(I)}\right)^2}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}=\left(\sum_{\ell=1}^{\chi}\mathfrak{p}_\ell^{(I)}D_\ell\right)^2=\boldsymbol{n}_I^2$$

$$K_X=-\sum_{\ell=1}^{\chi}D_\ell, B(K_X,K_X)=2\chi+3\sigma$$

$$\sum_{\ell=1}^{\chi}\frac{\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell+\epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}\right)}{2\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}=-B(K_X,\boldsymbol{k})$$

$$\sum_{\ell=1}^{\chi}\frac{\left(\epsilon_1^{(\ell)}\right)^2+\left(\epsilon_2^{(\ell)}\right)^2}{3\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}=\sigma$$

$$\begin{aligned}g_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}:=&\lim_{\epsilon_1,\epsilon_2\to 0}g_{\mathfrak{p},\mathfrak{p}^{(I)}}^{\epsilon_1,\epsilon_2}\\ =&\frac{1}{\mathcal{R}}A^\chi B^\sigma\exp\left[\frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial a^2}\boldsymbol{k}^2+\frac{R}{4}\frac{\partial^2\mathcal{F}}{\partial a\partial(\log\mathcal{R})}B(\boldsymbol{k},\boldsymbol{n}_I)\right.\\ &\left.+\frac{R^2}{32}\frac{\partial^2\mathcal{F}}{\partial(\log\mathcal{R})^2}\boldsymbol{n}_I^2-\frac{\partial H}{\partial a}B(K_X,\boldsymbol{k})\right]\end{aligned}$$

$$\mathcal{Z}_{\text{inst}}(a,\epsilon_1,-2\epsilon_1,n_I,\Lambda,R)=\mathcal{Z}_{\text{inst}}(a,2\epsilon_1,-\epsilon_1,n_I,\Lambda,R)$$

$$\begin{aligned}g_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}(a)=&\frac{1}{\mathcal{R}}\exp\left[-\pi i\tau(a)\boldsymbol{k}^2-2\pi i v_I(a)B(\boldsymbol{k},\boldsymbol{n}_I)-\pi i\xi_{II}(a)\boldsymbol{n}_I^2\right]\\ &\times\exp\left[\pi iB(K_X,\boldsymbol{k})\right]A^\chi B^\sigma\end{aligned}$$

$$\begin{aligned}g_{\mathfrak{p},0}^{\epsilon_1,\epsilon_2}=&\left[-4\sinh^2(\mathfrak{a})+4pR\sinh(2\mathfrak{a})\epsilon_2+\cdots\right]\mathcal{R}^{-3}\\ &+\left[\left(-\frac{9}{2}\operatorname{csch}^2(\mathfrak{a})-2\right)+pR(25\cosh(\mathfrak{a})+2\cosh(3\mathfrak{a}))\operatorname{csch}^3(\mathfrak{a})\epsilon_2+\cdots\right]\mathcal{R}+\mathcal{O}(\mathcal{R}^5)\end{aligned}$$

$$Z_{\boldsymbol{\mu},\mathfrak{p}^{(I)}}^{\epsilon_1,\epsilon_2}(\mathcal{R})=\lim_{\substack{\delta\to 0\\ g_{5d}\to 0}}\frac{K_\Omega}{2}\sum_{[\mathfrak{p}]}\int_\Gamma\frac{\mathrm{d}h}{h}\int_{\partial(\mathfrak{M}\setminus\Delta_\delta)}\mathrm{d}a\, g_{\mathfrak{p},\mathfrak{p}^{(I)}}^{\epsilon_1,\epsilon_2}(a,\bar{a},0,0,h)$$

$$\partial(\mathfrak{M}\setminus\Delta_\delta)=\partial\mathfrak{M}-\partial\Delta_\delta.$$

$$\frac{K_\Omega}{2}\sum_{[\mathfrak{p}]}\int_\Gamma\frac{\mathrm{d}h}{h}\int_{\partial\Delta_\delta}\mathrm{d}a\left[g_{\mathfrak{p},\mathfrak{p}^{(I)}}^{\epsilon_1,\epsilon_2,J^+}-g_{\mathfrak{p},\mathfrak{p}^{(I)}}^{\epsilon_1,\epsilon_2,J^-}\right](a,\bar{a},0,0,h)$$

$$Z^{\epsilon_1,\epsilon_2,J^+}_{\boldsymbol{\mu},\mathfrak{p}^{(I)}} - Z^{\epsilon_1,\epsilon_2,J^-}_{\boldsymbol{\mu},\mathfrak{p}^{(I)}} = \frac{K_\Omega}{2}\sum_{[\mathfrak{p}]} \int_\Gamma \frac{\mathrm{d}h}{h}\int_{\partial\mathfrak{M}} \mathrm{d}a \left[g^{\epsilon_1,\epsilon_2,J^+}_{\mathfrak{p},\mathfrak{p}^{(I)}} - g^{\epsilon_1,\epsilon_2,J^-}_{\mathfrak{p},\mathfrak{p}^{(I)}}\right](a,\bar{a},0,0,h)$$

$$g_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}(a,\bar{a},0,0,h) := \lim_{\epsilon_1,\epsilon_2\to 0} g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,h)$$

$$Z^{J^+}_{\boldsymbol{\mu},\boldsymbol{n}} - Z^{J^-}_{\boldsymbol{\mu},\boldsymbol{n}} = \frac{K_\Omega}{2}\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \int_\Gamma \frac{\mathrm{d}h}{h}\int_{\partial\mathfrak{M}} \mathrm{d}a\left[g^+_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}} - g^-_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}\right](a,\bar{a},0,0,h)$$

$$g_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}(a,\bar{a},0,0,h) = \exp\left[-\int_X \mathcal{L}_0\right]_{V_0=(a,\bar{a},0,0,h)}$$

$$\omega := \frac{1}{2} I_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$\begin{aligned}
\mathcal{L}_{0,h} =& \frac{i}{16\pi}\bar{\tau}F_+\wedge F_+ + \frac{i}{8\pi}\bar{v}_I F_+\wedge F_+^{(I)} + \frac{i}{16\pi}\bar{\xi}_I F_+^{(I)}\wedge F_+^{(I)} \\
&+ \frac{i}{16\pi}\tau F_-\wedge F_- + \frac{i}{8\pi} v_I F_-\wedge F_-^{(I)} + \frac{i}{16\pi}\xi_I F_-^{(I)}\wedge F_-^{(I)} \\
&- \frac{1}{8\pi} y(F_+ - 2ih\omega)\wedge(F_+ - 2ih\omega) - \frac{1}{4\pi}\mathrm{Im}(v_I)(F_+ - 2ih\omega)\wedge F_+^{(I)} \\
&- \frac{1}{8\pi}\mathrm{Im}(\xi_I)F_+^{(I)}\wedge F_+^{(I)} \\
=& \pi i\tau\left(\frac{F}{4\pi}\right)\wedge\left(\frac{F}{4\pi}\right) + \pi i v_I\left(\frac{F^{(I)}}{2\pi}\right)\wedge\left(\frac{F}{4\pi}\right) + \frac{1}{4}\pi i\xi_I\left(\frac{F^{(I)}}{2\pi}\right)\wedge\left(\frac{F^{(I)}}{2\pi}\right) \\
&+ \frac{i}{2\pi}h\left(yF_+ + \mathrm{Im}(v_I)F_+^{(I)}\right)\wedge\omega + \frac{1}{2\pi}yh^2\omega\wedge\omega
\end{aligned}$$

$$J = \frac{\omega}{\sqrt{2\mathrm{Vol}(X)}}$$

$$\begin{aligned}
& g_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}(a,\bar{a},0,0,h) \\
=& \exp\left[-\pi i\tau(a)\boldsymbol{k}^2 - 2\pi i v_I(a)B(\boldsymbol{k},\boldsymbol{n}) - \pi i\xi_{II}(a)\boldsymbol{n}^2\right] \\
&\times \exp\left[-\frac{y}{4\pi}\mathrm{Vol}(X)h^2 - iy\sqrt{2\mathrm{Vol}(X)}B\left(\boldsymbol{k} + \frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y}, J\right)h\right] \\
&\times \exp\left[\pi i B(\boldsymbol{k},K_X)\right]A^\chi B^\sigma \\
=& g_{\boldsymbol{k},\boldsymbol{\mu},\boldsymbol{n}}(a)\exp\left[-\frac{y}{\pi}\mathrm{Vol}(X)h^2 - 2iy\sqrt{2\mathrm{Vol}(X)}B\left(\boldsymbol{k} + \frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y}, J\right)h\right]
\end{aligned}$$

$$\begin{aligned}
& \lim_{\eta\to 0}\frac{1}{\pi i}\int_{\mathbb{R}-i\eta} e^{-t^2}e^{2\sqrt{\pi}xti}\frac{\mathrm{d}t}{t} \\
=& \lim_{\eta\to 0}\frac{1}{\pi i}\left[\int_{\mathbb{R}^+-i\eta} e^{-t^2}e^{2\sqrt{\pi}xti}\frac{\mathrm{d}t}{t} - \int_{\mathbb{R}^++i\eta} e^{-t^2}e^{-2\sqrt{\pi}xti}\frac{\mathrm{d}t}{t}\right] \\
=& \frac{2}{\pi}\int_0^\infty e^{-t^2}\sin\left(2\sqrt{\pi}xt\right)\frac{\mathrm{d}t}{t} + \mathrm{sgn}(\eta)
\end{aligned}$$

$$E(x)=\frac{2}{\pi}\int_0^{\infty} e^{-t^2}\sin\left(2\sqrt{\pi}xt\right)\frac{\mathrm{d}t}{t}$$

$$\frac{K_\Omega}{2}\int_\Gamma \frac{\mathrm{d}h}{h} g_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}(a,\bar{a},0,0,h)=\frac{i\pi}{2}K_\Omega E\left(\sqrt{2y}B\left(\boldsymbol{k}+\frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y},J\right)+\mathrm{sgn}(\eta)\right)g_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}(a)$$

$$\begin{aligned}Z^{J^+}_{\boldsymbol{\mu},\boldsymbol{n}}-Z^{J^-}_{\boldsymbol{\mu},\boldsymbol{n}}=&-K_\Omega\pi^2\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}}(\mathrm{Res}_{a=\infty}+\mathrm{Res}_{a=-\infty})\mathrm{d}a\, g_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}(a)\\ &\times\left[E\left(\sqrt{2y}B\left(\boldsymbol{k}+\frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y},J^+\right)\right)-E\left(\sqrt{2y}B\left(\boldsymbol{k}+\frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y},J^-\right)\right)\right].\end{aligned}$$

$$g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,h)=\exp\left[-\frac{1}{g_{5d}^2}\left(v_1h^2+iv_2B(\boldsymbol{k},J)h\right)\right]\tilde{g}^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,h)$$

$$\tilde{g}^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,0)=g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,0)=g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$

$$Z^{\epsilon_1,\epsilon_2,\,\mathrm{bdry}}_{\boldsymbol{\mu},\mathfrak{p}^{(I)}}(\mathcal{R})=\frac{K_\Omega}{2}\sum_{[\mathfrak{p}]}\int_\Gamma\frac{\mathrm{d}h}{h}\int_{\partial\mathfrak{M}}\mathrm{d}a\, g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,h)$$

$$Z^{\epsilon_1,\epsilon_2,\,\mathrm{bdry}}_{\boldsymbol{\mu},\mathfrak{p}^{(I)}}(\mathcal{R})=\frac{K_\Omega}{2}\sum_{[\mathfrak{p}]}\int_\Gamma\frac{\mathrm{d}h}{h}\exp\left[-\frac{1}{g_{5d}^2}\left(v_1h^2+iv_2B(\boldsymbol{k},J)h\right)\right]\int_{\partial\mathfrak{M}}\mathrm{d}a\,\tilde{g}^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a,\bar{a},0,0,h)$$

$$Z^{\epsilon_1,\epsilon_2,\,\mathrm{bdry}}_{\boldsymbol{\mu},\mathfrak{p}^{(I)}}(\mathcal{R})=\frac{K_\Omega}{2}\sum_{[\mathfrak{p}]}\int_\Gamma\frac{\mathrm{d}h}{h}\exp\left[-\left(v_1g_{5d}^2h^2+iv_2B(\boldsymbol{k},J)h\right)\right]\int_{\partial\mathfrak{M}}\mathrm{d}a\,\tilde{g}^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}\left(a,\bar{a},0,0,g_{5d}^2h\right)$$

$$Z^{\epsilon_1,\epsilon_2,\,\mathrm{bdry}}_{\boldsymbol{\mu},\mathfrak{p}^{(I)}}(\mathcal{R})=\frac{K_\Omega}{2}\sum_{[\mathfrak{p}]}\int_\Gamma\frac{\mathrm{d}h}{h}\exp\left(-iv_2B(\boldsymbol{k},J)h\right)\int_{\partial\mathfrak{M}}\mathrm{d}a\, g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$

$$\int_{\mathbb{R}-i\eta}\frac{\mathrm{d}h}{h}\exp\left(-iv_2B(\boldsymbol{k},J)h\right)=\begin{cases}2\pi iH(-B(\boldsymbol{k},J)), & \eta>0\\ -2\pi iH(B(\boldsymbol{k},J)), & \eta<0'\end{cases}$$

$$\begin{aligned}&Z^{\epsilon_1,\epsilon_2,J^+}_{\boldsymbol{\mu}}-Z^{\epsilon_1,\epsilon_2,J^-}_{\boldsymbol{\mu}}\\ &=-2\pi^2K_\Omega\sum_{[\mathfrak{p}]}\left[H\left(-B(\boldsymbol{k},J^+)\right)-H\left(-B(\boldsymbol{k},J^-)\right)\right]\left(\underset{a=\infty}{\mathrm{Res}}+\underset{a=-\infty}{\mathrm{Res}}\right)\mathrm{d}a\, g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)\\ &=\pi^2K_\Omega\sum_{[\mathfrak{p}]}\left[\mathrm{sgn}\left(B(\boldsymbol{k},J^+)\right)-\mathrm{sgn}\left(B(\boldsymbol{k},J^-)\right)\right]\left(\underset{a=\infty}{\mathrm{Res}}+\underset{a=-\infty}{\mathrm{Res}}\right)\mathrm{d}a\, g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)\end{aligned}$$

$$E\left(\sqrt{2y}B\left(\boldsymbol{k}+\frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y},J\right)\right)\to\mathrm{sgn}\left(B\left(\boldsymbol{k}+\frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y},J\right)\right)$$

$$Z_{\boldsymbol{\mu},\boldsymbol{n}}^{J^+} - Z_{\boldsymbol{\mu},\boldsymbol{n}}^{J^-} = -\frac{\pi^2 K_\Omega}{R} \sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} (\underset{\mathfrak{a}=0}{\mathrm{Res}} + \underset{\mathfrak{a}=\pi i}{\mathrm{Res}}) \mathrm{d}\mathfrak{a} g_{\boldsymbol{\mu},\boldsymbol{n},\boldsymbol{k}}(\mathfrak{a}, \mathcal{R})$$

$$\times \left[\mathrm{sgn}\left(B\left(\boldsymbol{k} + \frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y}, J^+\right)\right) - \mathrm{sgn}\left(B\left(\boldsymbol{k} + \frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y}, J^-\right)\right)\right]$$

$$= -\frac{\pi^2 K_\Omega}{R} \sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \mathrm{Coeff}_{\mathfrak{a}^0}[\Delta(\mathfrak{a},\mathcal{R})\mathfrak{a}] + \mathrm{Coeff}_{(\mathfrak{a}')^0}[\Delta'(\mathfrak{a}',\mathcal{R})\mathfrak{a}']$$

$$\Delta(\mathfrak{a},\mathcal{R}) = \exp[\pi i B(\boldsymbol{k}, K_X)] A^\chi B^\sigma$$
$$\times \exp[-\pi i \tau \boldsymbol{k}^2 - 2\pi i v_I B(\boldsymbol{k},\boldsymbol{n}) - \pi i \xi_{II} \boldsymbol{n}^2]$$
$$\times \left[\mathrm{sgn}\left(B\left(\boldsymbol{k} + \frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y}, J^+\right)\right) - \mathrm{sgn}\left(B\left(\boldsymbol{k} + \frac{\mathrm{Im}(v_I)\boldsymbol{n}}{y}, J^-\right)\right)\right],$$

$$\Delta'(\mathfrak{a}',\mathcal{R}) = \exp\left(-4\pi i \boldsymbol{k}^2 + 2\pi i B(\boldsymbol{k},\boldsymbol{n}) + \frac{\chi+\sigma}{2}\pi i\right)\Delta(\mathfrak{a}',\mathcal{R})$$

$$Z_{\boldsymbol{\mu},\boldsymbol{n}}^{J+} - Z_{\boldsymbol{\mu},\boldsymbol{n}}^{J-} = -\frac{\pi^2 K_\Omega}{R}\left[1 + \exp(2\pi i B(\boldsymbol{\mu}, \boldsymbol{n} + K_X)\right] \sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \mathrm{Coeff}_{\mathfrak{a}^0}[\Delta(\mathfrak{a},\mathcal{R})\mathfrak{a}].$$

$$\exp(-\pi i \tau \boldsymbol{k}^2) = \exp\left(\frac{\boldsymbol{k}^2}{2}\frac{\partial^2 \mathcal{F}}{\partial a^2}\right)$$
$$= \exp(4\mathfrak{a}\boldsymbol{k}^2)\left(\frac{1-e^{-2\mathfrak{a}}}{\mathcal{R}}\right)^{4\boldsymbol{k}^2} \exp\left(\frac{\boldsymbol{k}^2}{2}\frac{\partial^2 \mathcal{F}_{\mathrm{inst}}}{\partial a^2}\right),$$
$$\exp(-2\pi i v_I B(\boldsymbol{k},\boldsymbol{n})) = \exp\left(\frac{R}{4}\frac{\partial^2 \mathcal{F}}{\partial a \partial \log(\mathcal{R})} B(\boldsymbol{k},\boldsymbol{n})\right)$$
$$= \exp(-2\mathfrak{a}B(\boldsymbol{k},\boldsymbol{n}))\exp\left(\frac{R}{4}\frac{\partial^2 \mathcal{F}_{\mathrm{inst}}}{\partial a \partial \log(\mathcal{R})} B(\boldsymbol{k},\boldsymbol{n})\right),$$
$$\exp(-\pi i \xi_{II} \boldsymbol{n}^2) = \exp\left(\frac{R^2}{32}\frac{\partial^2 \mathcal{F}}{\partial \log(\mathcal{R})^2}\boldsymbol{n}^2\right)$$
$$= \exp\left(\frac{R^2}{32}\frac{\partial^2 \mathcal{F}_{\mathrm{inst}}}{\partial \log(\mathcal{R})^2}\boldsymbol{n}^2\right).$$

$$\mathrm{Ser}_{\mathfrak{a}=0}\mathrm{Ser}_{\mathcal{R}=0}\exp(-\pi i \tau \boldsymbol{k}^2) = \sum_{n\geq 0} f_n^{(1)}(\mathfrak{a})\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^{4n-4\boldsymbol{k}^2}$$
$$\mathrm{Ser}_{\mathfrak{a}=0}\mathrm{Ser}_{\mathcal{R}=0}\exp(-2\pi i v_I B(\boldsymbol{k},\boldsymbol{n})) = \sum_{n\geq 0} f_n^{(2)}(\mathfrak{a})\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^{4n}$$
$$\mathrm{Ser}_{\mathfrak{a}=0}\mathrm{Ser}_{\mathcal{R}=0}\exp(-\pi i \xi_{II} \boldsymbol{n}^2) = \sum_{n\geq 0} f_n^{(3)}(\mathfrak{a})\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^{4n+4}$$

$$\log(A) = -\frac{1}{2}\log(\mathcal{R}) + \frac{1}{2}\log(1-e^{2\mathfrak{a}}) + \frac{1}{4}(-2\mathfrak{a}+\pi i) + \sum_{n=1}^{\infty} A_n(\sinh^2(\mathfrak{a}))\left(\frac{\mathcal{R}}{\sinh(\mathfrak{a})}\right)^{4n}$$

$$= -\frac{1}{2}\log(\mathcal{R}) + \frac{1}{2}\log(1-e^{2\mathfrak{a}}) + \frac{1}{4}(-2\mathfrak{a}+\pi i) - \frac{2+\sinh^2(\mathfrak{a})}{8\sinh^4(\mathfrak{a})}\mathcal{R}^4 - \frac{38+33\sinh^2(\mathfrak{a})+2\sinh^4(\mathfrak{a})}{128\sinh^8(\mathfrak{a})}\mathcal{R}^8 + \mathcal{O}(\mathcal{R}^{12}),$$

$$\log(B) = -\frac{1}{2}\log(\mathcal{R}) + \frac{1}{2}\log(1-e^{2\mathfrak{a}}) + \frac{1}{4}(-2\mathfrak{a}+\pi i) + \sum_{n=1}^{\infty} B_n(\sinh^2(\mathfrak{a}))\left(\frac{\mathcal{R}}{\sinh(\mathfrak{a})}\right)^{4n}$$

$$= -\frac{1}{2}\log(\mathcal{R}) + \frac{1}{2}\log(1-e^{2\mathfrak{a}}) + \frac{1}{4}(-2\mathfrak{a}+\pi i) - \frac{3+\sinh^2(\mathfrak{a})}{8\sinh^4(\mathfrak{a})}\mathcal{R}^4 - \frac{63+49\sinh^2(\mathfrak{a})+2\sinh^4(\mathfrak{a})}{128\sinh^8(\mathfrak{a})}\mathcal{R}^8 + \mathcal{O}(\mathcal{R}^{12}).$$

$$\mathrm{Ser}_{\mathfrak{a}=0}\mathrm{Ser}_{\mathcal{R}=0}A^{\chi} = \sum_{n\geq 0} f_n^{(4)}(\mathfrak{a})\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^{4n-\frac{1}{2}\chi}$$

$$\mathrm{Ser}_{\mathfrak{a}=0}\mathrm{Ser}_{\mathcal{R}=0}B^{\sigma} = \sum_{n\geq 0} f_n^{(5)}(\mathfrak{a})\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^{4n-\frac{1}{2}\sigma}$$

$$\mathrm{Ser}_{\mathfrak{a}=0}\mathrm{Ser}_{\mathcal{R}=0}\Delta(\mathfrak{a},\mathcal{R}) = \sum_{n\geq 0} f_n^{(6)}(\mathfrak{a})\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^{4n+2-4\boldsymbol{k}^2}$$

$$F(\mathfrak{a},\mathcal{R}) = \sum_{n\geq 0}\sum_{l\geq 0} F_{nl}\mathfrak{a}^l\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^{4n-m}$$

$$F(\mathfrak{a},\mathcal{R}) = \sum_{l\geq 0}\sum_{n\geq 0} F_{nl}\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^{4n-m-l}\mathcal{R}^l := \tilde{F}\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right)$$

$$\Delta(\mathfrak{a},\mathcal{R}) = \tilde{\Delta}\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right) \in \mathbb{Q}\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)[\mathcal{R}]$$

$$\mathrm{Coeff}_{\mathfrak{a}^0}\Delta(\mathfrak{a},\mathcal{R})\mathfrak{a} = \mathrm{Coeff}_{\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^0}\tilde{\Delta}\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right)\left(\frac{\mathfrak{a}}{\mathcal{R}}\right)\mathcal{R}$$

$$q^{\frac{1}{8}}(\mathfrak{a},\mathcal{R}) = \mathrm{Ser}_{\mathfrak{a}=0}\mathrm{Ser}_{\mathcal{R}=0}\exp\left(\frac{\pi i\tau}{4}\right) = \sum_{n\geq 0} f_n^{(7)}(\mathfrak{a})\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^{4n+1}$$

$$q^{\frac{1}{8}}\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right) = \left(-\frac{i}{2}\frac{\mathcal{R}}{\mathfrak{a}} - \frac{3i}{16}\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^5 - \frac{123i}{512}\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^9 + \cdots\right) + \left(\frac{i\mathfrak{a}}{12\mathcal{R}} + \cdots\right)\mathcal{R}^2 + \cdots$$

$$-\frac{i}{2}\frac{\mathcal{R}}{\mathfrak{a}}\left(q^{\frac{1}{8}},\mathcal{R}\right) = \left(q^{\frac{1}{8}} - 6q^{\frac{5}{8}} + \cdots\right) + \left(-\frac{1}{24}q^{-\frac{1}{8}} + \frac{3}{4}q^{\frac{3}{8}} + \cdots\right)\mathcal{R}^2 + \cdots$$

$$y = y(x,\mathcal{R}) = y_0(x) + y_1(x)\mathcal{R} + \cdots \in \mathbb{C}(x)[\mathcal{R}]$$

$$\mathrm{Coeff}_{y^0}[yf(y,\mathcal{R})] = \mathrm{Coeff}_{x^0}\left[xf(y(x,\mathcal{R}),\mathcal{R})\frac{\mathrm{d}y}{\mathrm{d}x}\right]$$

$$\mathrm{Coeff}_{\left(\frac{\mathcal{R}}{\mathfrak{a}}\right)^0}\left[\tilde{\Delta}\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right)\left(\frac{\mathfrak{a}}{\mathcal{R}}\right)\mathcal{R}\right]=\mathrm{Coeff}_{q^0}\left[q^{\frac{1}{8}}\Delta\left(q^{\frac{1}{8}},\mathcal{R}\right)\left(\frac{\mathfrak{a}}{\mathcal{R}}\right)^2\mathcal{R}\frac{\mathrm{d}(\mathcal{R}/\mathfrak{a})}{\mathrm{d}\left(q^{\frac{1}{8}}\right)}\right].$$

$$\exp\left[-\pi i\tilde{\tau}\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right)\boldsymbol{k}^2\right]\Big|_{\frac{\mathcal{R}}{\mathfrak{a}}=\frac{\mathcal{R}}{\mathfrak{a}}\left(q^{\frac{1}{8}},\mathcal{R}\right)}=q^{-\frac{1}{2}\boldsymbol{k}^2},$$

$$\exp\left[-2\pi i\tilde{v}_I\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right)B(\boldsymbol{k},\boldsymbol{n})\right]\Big|_{\frac{\mathcal{R}}{\mathfrak{a}}=\frac{\mathcal{R}}{\mathfrak{a}}\left(q^{\frac{1}{8}},\mathcal{R}\right)}=\mathrm{Ser}_q\mathrm{Ser}_{\mathcal{R}}[\exp\left(-2\pi i v_I B(\boldsymbol{k},\boldsymbol{n})\right)],$$

$$\exp\left[-\pi i\tilde{\xi}_{II}\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right)\boldsymbol{n}^2\right]\Big|_{\frac{\mathcal{R}}{\mathfrak{a}}=\frac{\mathcal{R}}{\mathfrak{a}}\left(q^{\frac{1}{8}},\mathcal{R}\right)}=\mathrm{Ser}_q\mathrm{Ser}_{\mathcal{R}}\left[C_{II}(\tau,\mathcal{R})^{n^2}\right].$$

$$A\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right)\Big|_{\frac{\mathcal{R}}{\mathfrak{a}}=\frac{\mathcal{R}}{\mathfrak{a}}\left(q^{\frac{1}{8}},\mathcal{R}\right)}=i\left(\frac{2}{\vartheta_2(\tau)\vartheta_3(\tau)}\right)^{\frac{1}{2}}$$

$$B\left(\frac{\mathcal{R}}{\mathfrak{a}},\mathcal{R}\right)\Big|_{\frac{\mathcal{R}}{\mathfrak{a}}=\frac{\mathcal{R}}{\mathfrak{a}}\left(q^{\frac{1}{8}},\mathcal{R}\right)}=\frac{\sqrt{2}i\vartheta_4(\tau)}{(\vartheta_2(\tau)\vartheta_3(\tau))^{\frac{1}{2}}},$$

$$q^{\frac{1}{8}\left(\frac{\mathfrak{a}}{\mathcal{R}}\right)^2\mathcal{R}\frac{\mathrm{d}(\mathcal{R}/\mathfrak{a})}{\mathrm{d}q^{\frac{1}{8}}}\Big|_{\frac{\mathcal{R}}{\mathfrak{a}}=\frac{\mathcal{R}}{\mathfrak{a}}\left(q^{\frac{1}{8}},\mathcal{R}\right)}}=\mathrm{Ser}_q\mathrm{Ser}_{\mathcal{R}}\left[\frac{-i\mathcal{R}}{\sqrt{-8\mathcal{R}^2\mathrm{u}+4\mathcal{R}^4+4}}\frac{\vartheta_4(\tau)^9}{\eta(\tau)^3}\right]$$

$$Z^J_{\boldsymbol{\mu},\boldsymbol{n}}-Z^{J'}_{\boldsymbol{\mu},\boldsymbol{n}}=4K_{U,\boldsymbol{\mu},\boldsymbol{n}}\mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[\nu_R(\tau)C_{II}^{\boldsymbol{n}^2}\Theta^{JJ'}_{\boldsymbol{\mu}}(\tau,\boldsymbol{n}v)\right]$$

$$Z^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R})=Z^{\epsilon_1,\epsilon_2,\,\mathrm{bulk}}_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R})+Z^{\epsilon_1,\epsilon_2,\,\mathrm{bdry}}_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R})$$

$$Z^{\epsilon_1,\epsilon_2}_{\boldsymbol{\mu},0}(\mathcal{R})=-2\pi^2K_\Omega\left[\sum_{B(\boldsymbol{k},J)>0}\sum_{a_*\in S_+}-\sum_{B(\boldsymbol{k},J)<0}\sum_{a_*\in S_-}\right]\mathrm{Resd}_{a=a_*}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},0}(a),$$

$$Z^{\epsilon_1,\epsilon_2}_{\mu,0}(\mathcal{R})=-\frac{R}{2\pi i}\sum_{\mathfrak{p}}\Theta_\mu(\mathfrak{p})\oiiint_{2a=0}\mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},0}(a)$$

$$Z^{\epsilon_1,\epsilon_2}_{\mu,0}(\mathcal{R})=4\pi^2K_\Omega\sum_{c=0,\frac{\pi i}{R}}\left(\sum_{\{\mathfrak{p}\}\in S_{\text{unstable}}}+\sum_{\{\mathfrak{p}\}\in S_{\text{semi-stable}}}+2\sum_{\{\mathfrak{p}\}\in S_{\text{stable}}}\right)\mathop{\mathrm{Res}}_{a=c}\mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}}(a)$$

$$Z^{\epsilon_1,\epsilon_2}_{\mu,0}(\mathcal{R})=-4\pi^2K_\Omega\sum_{c=0,\frac{\pi i}{R}}\sum_{\{\mathfrak{p}\}\in S_{\text{unstable}}}\mathrm{Res}_{a=c}\,\mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(l)}=0}(a)$$

$$\begin{aligned}Z^{\mathbb{CP}^2\times S^1}_{\mu,0}(\epsilon_1,\epsilon_2,\mathcal{R})&=-R\sum_{\mathfrak{p}}\Theta_\mu(\mathfrak{p})\mathrm{Res}_{a=0,\frac{\pi i}{R}}\left[\mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},0}(a)\right]\\&=-\frac{1}{\Lambda}\sum_{\mathfrak{p}}\Theta_\mu(\mathfrak{p})\mathrm{Res}_{a=0,\frac{\pi i}{R}}\left[\mathrm{d}a\prod_{\ell=1,2,3}\mathcal{Z}\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\Lambda,R\right)\right],\end{aligned}$$

$$\Theta_\mu(\mathfrak{p}) = \begin{cases} 1, & \forall \ell, \quad \mathfrak{p}_\ell > 0, \quad \mathfrak{p}_\ell + \mathfrak{p}_{\ell+1} > \mathfrak{p}_{\ell+2}, \\ & \text{and } \sum_\ell \mathfrak{p}_\ell = 2\mu \mod 2, \\ \frac{1}{2}, & \forall \ell, \quad \mathfrak{p}_\ell > 0, \quad \mathfrak{p}_\ell + \mathfrak{p}_{\ell+1} \geq \mathfrak{p}_{\ell+2}, \\ & \exists \ell, \quad \mathfrak{p}_\ell + \mathfrak{p}_{\ell+1} = \mathfrak{p}_{\ell+2}, \text{ and } \sum_\ell \mathfrak{p}_\ell = 2\mu \mod 2, \\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{aligned} & Z_{\mu,0,\text{ pert}}^{\mathbb{CP}^2\times S^1}(a,\epsilon_1,\epsilon_2,\Lambda,R,\mathfrak{p}) \\ &= \prod_{\ell=1,2,3} \exp\left[-\tilde{\gamma}_{\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}}\left(2a^{(\ell)} \mid R,\Lambda\right) - \tilde{\gamma}_{\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}}\left(-2a^{(\ell)} \mid R,\Lambda\right)\right] \\ &= \mathcal{R}^{-2-(\sum_\ell \mathfrak{p}_\ell)^2} \exp\left\{\frac{1}{2}(\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3)[6a + (2\epsilon_1-\epsilon_2)\mathfrak{p}_1 + (2\epsilon_2-\epsilon_1)\mathfrak{p}_2 - (\epsilon_1+\epsilon_2)\mathfrak{p}_3]R\right\} \\ & \times \exp\left(-\sum_{\ell=1,2,3}\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-2Rna^{(\ell)}}+e^{2Rna^{(\ell)}}}{\left(1-e^{Rn\epsilon_1^{(\ell)}}\right)\left(1-e^{Rn\epsilon_2^{(\ell)}}\right)}\right) \end{aligned}$$

$$\begin{aligned} & \exp\left(-\sum_{\ell=1,2,3}\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-2Rna^{(\ell)}}+e^{2Rna^{(\ell)}}}{\left(1-e^{Rn\epsilon_1^{(\ell)}}\right)\left(1-e^{Rn\epsilon_2^{(\ell)}}\right)}\right) \\ &= \prod_{\substack{(i,j)\in\mathbb{N}^2 \\ i+j\leq\sum_\ell \mathfrak{p}_\ell}} \{1-\exp[-R(2a+(\mathfrak{p}_1-j)\epsilon_1+(\mathfrak{p}_2-i)\epsilon_2]\} \\ & \times \prod_{\substack{(i,j)\in\mathbb{N}^2 \\ i+j\leq\sum_\ell \mathfrak{p}_\ell-3}} \{1-\exp[R(2a+(\mathfrak{p}_1-1-j)\epsilon_1+(\mathfrak{p}_2-1-i)\epsilon_2)]\}. \end{aligned}$$

$$\begin{aligned} P^{\mathbb{CP}^2\times S^1}(\epsilon_1,\epsilon_2,R,\mathfrak{p}) = & \prod_{\substack{(i,j)\in\mathbb{N}^2\backslash\{(\mathfrak{p}_2,\mathfrak{p}_1)\} \\ i+j\leq\sum\ \ell\mathfrak{p}_\ell}} \left\{1-\exp\left[-R\left((\mathfrak{p}_1-j)\epsilon_1+(\mathfrak{p}_2-i)\epsilon_2\right]\right\} \\ & \times \prod_{\substack{(i,j)\in\mathbb{N}^2\backslash\{(\mathfrak{p}_2-1,\mathfrak{p}_1-1)\} \\ i+j\leq\sum\ \ell}} \left\{1-\exp\left[R\left((\mathfrak{p}_1-1-j)\epsilon_1+(\mathfrak{p}_2-1-i)\epsilon_2\right)\right]\right\}. \end{aligned}$$

$$\begin{aligned} & Z_{\mu,0}^{\mathbb{CP}^2\times S^1}(\epsilon_1,\epsilon_2,\mathcal{R}) \\ &= \frac{1}{\Lambda}(1+(-1)^{6\mu})\sum_{\mathfrak{p}} \Theta_\mu(\mathfrak{p})\mathcal{R}^{-2+Q(\mathfrak{p})}\exp\left(R\sum_{\ell,\ell'}\mathfrak{p}_\ell a_+^{(\ell')}\right) \\ & \times P^{\mathbb{CP}^2\times S^1}(\epsilon_1,\epsilon_2,R,\mathfrak{p})\prod_{\ell=1,2,3}\frac{T_{\mathfrak{p}_\ell,\mathfrak{p}_{\ell+1}}\left(t_1^{(\ell)},t_2^{(\ell)}\right)}{e^{-2a_+^{(\ell)}}-e^{2a_+^{(\ell)}}}\mathcal{Z}_{\text{inst}}\left(a_-^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\Lambda,R\right), \end{aligned}$$

$$Q(\mathfrak{p}) = 2\mathfrak{p}_1\mathfrak{p}_2 + 2\mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_1^2 - \mathfrak{p}_2^2 - \mathfrak{p}_3^2,$$

$$a_\pm^{(\ell)} = \frac{1}{2}\left(\mathfrak{p}_\ell\epsilon_1^{(\ell)} \pm \mathfrak{p}_{\ell+1}\epsilon_2^{(\ell)}\right)$$

$$Z^{\mathbb{CP}^2\times S^1}_{\mu,\mathfrak{p}^{(I)}}(\epsilon_1,\epsilon_2,\mathcal{R})$$

$$=\frac{1+(-1)^{2\mu(n_I+3)}}{\Lambda}\sum_{\mathfrak{p}}\Theta_\mu(\mathfrak{p})\mathcal{R}^{-2+Q(\mathfrak{p})}\exp\left(R\sum_{\ell,\ell'}\mathfrak{p}_\ell a_+^{(\ell')}\right)P^{\mathbb{CP}^2\times S^1}(\epsilon_1,\epsilon_2,R,\mathfrak{p})$$

$$\times\prod_{\ell=1,2,3}\frac{T_{\mathfrak{p}_\ell,\mathfrak{p}_{\ell+1}}\left(t_1^{(\ell)},t_2^{(\ell)}\right)}{e^{-2a_+^{(\ell)}}-e^{2a_+^{(\ell)}}}\exp\left(\frac{n_I^{(\ell)}\left(a_-^{(\ell)}\right)^2}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\right)\mathcal{Z}_{\text{inst}}\left(a_-^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\Lambda e^{-\frac{1}{4}n_I^{(\ell)}},R\right),$$

$$t_1^{\frac{1}{4}\left(2\mathfrak{p}_1^{(I)}-\mathfrak{p}_2^{(I)}-\mathfrak{p}_3^{(I)}\right)}t_2^{\frac{1}{4}\left(-\mathfrak{p}_1^{(I)}+2\mathfrak{p}_2^{(I)}-\mathfrak{p}_3^{(I)}\right)}.$$

$$Z^{\mathbb{CP}^2\times S^1}_{\mu,\mathfrak{p}^{(I)}}(\epsilon_1,\epsilon_2,\mathcal{R})=(-1)^{2\mu+1}Z^{\mathbb{CP}^2\times S^1}_{\mu,-\mathfrak{p}^{(I)}}(-\epsilon_1,-\epsilon_2,\mathcal{R})$$

$$Z^{\mathbb{CP}^2\times S^1}_{\mu=0,\mathfrak{p}^{(I)}}=\sum_{l=0}^{\infty}C^{\mu=0}_{\mathfrak{p}^{(I)},l}(t_1,t_2)\mathcal{R}^{1+4l}$$

$$C^{\mu=0}_{\mathfrak{p}^{(I)},|n_I'|-4}(t_1,t_2)=t_1^{f_1\left(\mathfrak{p}^{(I)},\mathfrak{p}^{(I)'}\right)}t_2^{f_2\left(\mathfrak{p}^{(I)},\mathfrak{p}^{(I)'}\right)}C^{\mu=0}_{\mathfrak{p}^{(I)'},|n_I|-4}(t_1,t_2),$$

$$n_I=\sum_\ell\mathfrak{p}_\ell^{(I)},n_I'=\sum_\ell\mathfrak{p}_\ell^{(I)'}$$

$$Z^{\text{5d}}_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R},m)=\sum_k\mathcal{R}^{d_k/2}\text{Tr}_{\mathcal{H}_k}(-1)^F e^{-\beta\mathbb{D}^2+mQ_V}$$

$$\int_{X_6}B^{(I)}\wedge\text{Tr}(F\wedge F)$$

$$Z^{\text{6d}}\sim\sum_k\mathcal{R}_{6d}^{d_k/2}\mathcal{E}(M_k;m,\tau_E)$$

$$\mathcal{R}_{6d}\sim\exp\left(-\frac{A}{g_{6d}^2}+i\theta\right)$$

$$(\sigma^\mu)_{\alpha\dot{\alpha}}:=(\vec{\tau},i),$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}:=(-\vec{\tau},i)$$

$$(\sigma^{\mu\nu})_\alpha^{\ \beta}:=\frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu-\sigma^\nu\bar{\sigma}^\mu)_\alpha^{\ \beta}$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}}:=\frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu-\bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}_{\ \dot{\beta}},$$

$$\tau^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\tau^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\tau^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\Gamma^\mu=\begin{pmatrix}0&\sigma^\mu\\-\bar{\sigma}^\mu&0\end{pmatrix},\Gamma^5=\Gamma^1\Gamma^2\Gamma^3\Gamma^4=\begin{pmatrix}\mathbf{1}&0\\0&-\mathbf{1}\end{pmatrix}$$

$$\Gamma^{m_1 m_2 \dots m_p} := \frac{1}{p!} \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) \prod_{i=1}^{p} \Gamma^{m_{\sigma(i)}}$$

$$C^{\mathbf{ab}} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, C_{\mathbf{ab}} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$$

$$\Psi_{\boldsymbol{a}} = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

$$\mathbf{Q}_{\mathbf{a}}^{A} = \begin{pmatrix} Q_\alpha^A \\ \epsilon^{AB} \bar{Q}_B^{\dot{\alpha}} \end{pmatrix}$$

$$\{\mathbf{Q}_{\mathbf{a}}^{A}, \mathbf{Q}_{\mathbf{b}}^{B}\} = 2i\epsilon^{AB} (\Gamma_{\mathbf{ab}}^{m} D_m + C_{\mathbf{ab}}[\sigma, \cdot])$$

$$\delta = i\xi_A Q^A + i\bar{\xi}^A \bar{Q}_A$$

$$\xi_{\mathbf{a}A} = \begin{pmatrix} -\xi_{\alpha A} \\ \bar{\xi}_A^{\dot{\alpha}} \end{pmatrix}, \xi^{\mathbf{a}A} = (-\xi^{\alpha A}, \bar{\xi}_{\dot{\alpha}}^{A})$$

$$\xi_\alpha^A \in \Gamma[S_- \otimes E_R], \bar{\xi}_B^{\dot{\alpha}} \in \Gamma[S_+ \otimes E_R]$$

$$\overline{\mathcal{Q}} := \delta_{\dot{\alpha}}{}^{A} \bar{Q}^{\dot{\alpha}}{}_{A},$$

$$\bar{\xi}_{\dot{\alpha}}{}^{A} = \bar{\delta}_{\dot{\alpha}}{}^{A}$$

$$\mathcal{K}_\mu := \frac{i}{4} \bar{\zeta}_A \bar{\sigma}_\mu Q^A$$

$$\begin{aligned} \psi_\mu &:= \bar{\zeta}_A \bar{\sigma}_\mu \lambda^A \\ \eta &:= \bar{\zeta}^A \bar{\lambda}_A \\ \chi_{\mu\nu} &:= \bar{\zeta}^A (\bar{\sigma}_{\mu\nu}) \bar{\lambda}_A \\ D_{\mu\nu} &:= -i\bar{\zeta}^A (\bar{\sigma}_{\mu\nu}) \bar{\zeta}^B D_{AB} \end{aligned}$$

$$S_+ \cong K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}}$$

$$S_+ \otimes E_R \cong \mathcal{K} \oplus \mathcal{K}^{-1} \oplus \mathcal{O}_{(1)} \oplus \mathcal{O}_{(2)}$$

$$S_- \otimes E_R \cong \left(\mathcal{K}^{\frac{1}{2}} \otimes \Omega^{0,1}\right) \otimes \left(\mathcal{K}^{\frac{1}{2}} \oplus \mathcal{K}^{-\frac{1}{2}}\right) \cong \Omega^{2,1} \oplus \Omega^{0,1} \cong \Omega^{1,0} \oplus \Omega^{0,1}$$

$$\left(\bar{\zeta}^{(1)}\right)^{\dot{\alpha}}{}_{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \left(\bar{\zeta}^{(2)}\right)^{\dot{\alpha}}{}_{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\overline{\mathcal{Q}}_{(1)} = \left(\bar{\zeta}_{(1)}\right)_{\dot{\alpha}}{}^{A} \bar{Q}_A^{\dot{\alpha}}, \overline{\mathcal{Q}}_{(2)} = \left(\bar{\zeta}_{(2)}\right)_{\dot{\alpha}}{}^{A} \bar{Q}_A^{\dot{\alpha}}$$

$$\sigma^1 = 2\sigma_{\bar{1}} = -\bar{\sigma}^1 = -2\bar{\sigma}_{\bar{1}} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$
$$\sigma^{\bar{1}} = 2\sigma_1 = -\bar{\sigma}^{\bar{1}} = -2\bar{\sigma}_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$
$$\sigma^2 = 2\sigma_{\bar{2}} = -\bar{\sigma}^{\bar{2}} = -2\bar{\sigma}_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$
$$\sigma^{\bar{2}} = 2\sigma_2 = -\bar{\sigma}^2 = -2\bar{\sigma}_{\bar{2}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\sigma^{1\bar{2}} = 4\sigma_{\bar{1}2} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$
$$\sigma^{2\bar{1}} = 4\sigma_{\bar{2}1} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$
$$\sigma^{1\bar{1}} = 4\sigma_{\bar{1}1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\sigma^{2\bar{2}} = 4\sigma_{\bar{2}2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\bar{\sigma}^{1\bar{1}} = 4\bar{\sigma}_{\bar{1}1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\bar{\sigma}^{2\bar{2}} = 4\bar{\sigma}_{\bar{2}2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\bar{\sigma}^{\overline{12}} = 4\bar{\sigma}_{12} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix},$$
$$\bar{\sigma}^{12} = 4\bar{\sigma}_{\overline{12}} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), q = e^{2\pi i \tau}$$

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon(\gamma)(c\tau + d)^{\frac{1}{2}}\eta(\tau), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\vartheta_1(\tau, z) = i \sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^{n-\frac{1}{2}} q^{\frac{n^2}{2}} e^{2\pi i n z}$$
$$\vartheta_2(\tau, z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{\frac{n^2}{2}} e^{2\pi i n z}$$
$$\vartheta_3(\tau, z) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} e^{2\pi i n z}$$
$$\vartheta_4(\tau, z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} e^{2\pi i n z}$$

$$\frac{\partial}{\partial z}\vartheta_1(\tau, z)\Big|_{z=0} = -2\pi\eta(\tau)^3$$

$$\vartheta_3(\tau/2, z/2) = \vartheta_3(2\tau, z) + \vartheta_2(2\tau, z)$$
$$\vartheta_4(\tau/2, z/2) = \vartheta_3(2\tau, z) - \vartheta_2(2\tau, z)$$

$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\eta(\tau)^3$$
$$\vartheta_1(\tau, 2z)\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\vartheta_1(\tau, z)\vartheta_2(\tau, z)\vartheta_3(\tau, z)\vartheta_4(\tau, z)$$

$$\vartheta_1\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=\varepsilon(\gamma)^3(c\tau+d)^{\frac{1}{2}}\exp\left(\frac{\pi icz^2}{c\tau+d}\right)\vartheta_1(\tau,z)$$

$$\begin{aligned}S:&\quad \vartheta_1(-1/\tau,z/\tau)=-i\sqrt{-i\tau}e^{\pi iz^2/\tau}\vartheta_1(\tau,z)\\ T:&\quad \vartheta_1(\tau+1,z)=e^{2\pi i/8}\vartheta_1(\tau,z)\end{aligned}$$

$$\vartheta_1(\tau,z+l\tau)=(-1)^l q^{-l^2/2}e^{-2\pi ilz}\vartheta_1(\tau,z)$$

$$M(\tau,u,v):=\frac{e^{\pi iu}}{\vartheta_1(\tau,v)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n(n+1)/2}e^{2\pi inv}}{1-e^{2\pi iu}q^n}$$

$$R(\tau,\bar{\tau},u,\bar{u}):=\sum_{n\in\mathbb{Z}+\frac{1}{2}}(\mathrm{sgn}(n)-E((n+a)\sqrt{2y}))(-1)^{n-\frac{1}{2}}e^{-2\pi iun}q^{-n^2/2}$$

$$E(t)=2\int_0^t e^{-\pi u^2}du=\mathrm{Erf}(\sqrt{\pi}t)$$

$$E(t)=\mathrm{sgn}(t)-tE_{\frac{1}{2}}(\pi t^2)$$

$$\widehat{M}(\tau,\bar{\tau},u,\bar{u},v,\bar{v})=M(\tau,u,v)+\frac{i}{2}R(\tau,\bar{\tau},u-v,\bar{u}-\bar{v})$$

$$\begin{aligned}&\widehat{M}\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar{\tau}+b}{c\bar{\tau}+d},\frac{u}{c\tau+d},\frac{\bar{u}}{c\bar{\tau}+d},\frac{v}{c\tau+d},\frac{\bar{v}}{c\bar{\tau}+d}\right)\\ &=\varepsilon(\gamma)^{-3}(c\tau+d)^{\frac{1}{2}}\exp\left(-\frac{\pi ic(u-v)^2}{c\tau+d}\right)\widehat{M}(\tau,\bar{\tau},u,\bar{u},v,\bar{v})\end{aligned}$$

$$\begin{aligned}\partial_{\bar{\tau}}\widehat{M}(\tau,\bar{\tau},u,\bar{u},v,\bar{v})=&-i\left(\partial_{\bar{\tau}}\sqrt{2y}\right)e^{-2\pi(a-b)^2}\\ &\times\sum_{n\in\mathbb{Z}+\frac{1}{2}}(n+a-b)(-1)^{n-\frac{1}{2}}\bar{q}^{n^2/2}e^{-2\pi i(\bar{u}-\bar{v})n}\end{aligned}$$

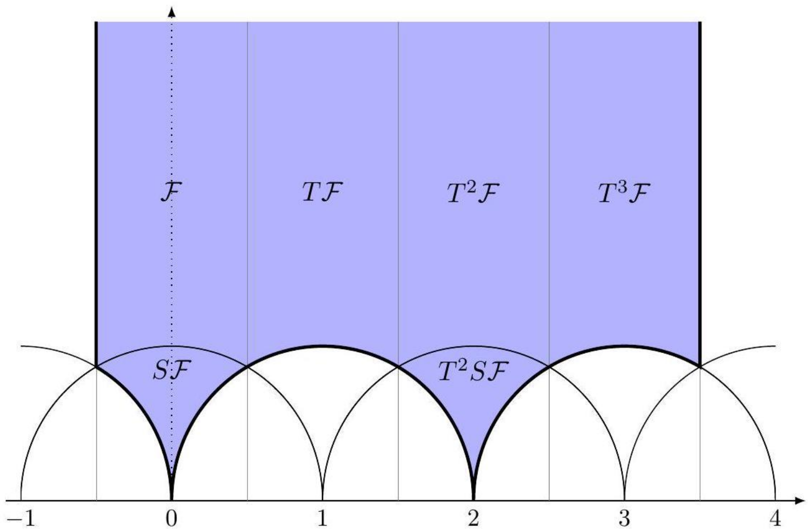
$$M(\tau,u+z,v+z)-M(\tau,u,v)=\frac{i\eta^3(\tau)\vartheta_1(\tau,u+v+z)\vartheta_1(\tau,z)}{\vartheta_1(\tau,u)\vartheta_1(\tau,v)\vartheta_1(\tau,u+z)\vartheta_1(\tau,v+z)}$$

$$\begin{aligned}&\widehat{M}(\tau,\bar{\tau},u+k\tau+l,\bar{u}+k\bar{\tau}+l,v+m\tau+n,\bar{v}+m\bar{\tau}+n)\\ &=(-1)^{k+l+m+n}q^{(k-m)^2/2}e^{2\pi i(k-m)(u-v)}\widehat{M}(\tau,\bar{\tau},u,\bar{u},v,\bar{v}).\end{aligned}$$

$$y^2=(x^2-u)^2-4\Lambda^4\star\frac{1}{2}\langle\mathrm{Tr}\phi^2\rangle_{\mathbb{R}^4}$$

$$\Delta_{4,\mathrm{phys}}=u^2-4\Lambda^4\star\frac{1}{2}\langle\mathrm{Tr}\phi^2\rangle_{\mathbb{R}^4}$$

$$u=-2\Lambda^2\left(2\frac{\vartheta_3(\tilde{\tau})^4}{\vartheta_4(\tilde{\tau})^4}-1\right)=-\Lambda^2\left(2+64\tilde{q}^{\frac{1}{2}}+\mathcal{O}(\tilde{q})\right)$$



$$\frac{\mathrm{d}a_4}{\mathrm{d}u} = \frac{i}{2\Lambda}\vartheta_2(\tau)\vartheta_3(\tau)$$

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -2\pi i(u^2 - 4\Lambda^4)\left(\frac{\mathrm{d}a_4}{\mathrm{d}u}\right)^2$$

$$\frac{\mathrm{d}a_4}{\mathrm{d}\tau} = -\frac{\pi\Lambda}{8}\frac{\vartheta_4(\tau)^9}{\eta(\tau)^3}$$

$$\begin{aligned} a_4(\tau) &= -i\Lambda\frac{2E_2(\tau) + \vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{3\theta_2(\tau)\theta_3(\tau)} \\ &= -\frac{i\Lambda}{2}q^{-\frac{1}{8}} + \mathcal{O}\left(q^{3/8}\right) \end{aligned}$$

$$\langle W_F\rangle = -\oiiint_{\star}^{\dagger}\frac{\mathrm{d}x}{2\pi i x}\frac{\mathcal{Z}_{\text{inst}}^{5\text{ d}-1\text{ d}}}{\mathcal{Z}_{\text{inst}}}$$

$$\mathcal{Z}_{\text{inst}}^{(k)} = \frac{1}{k!}\oiiint_{\square}^{\blacksquare}\left[\prod_{s=1}^{k}\frac{R\,\mathrm{d}\phi_s}{2\pi i}\right]\mathcal{Z}_{\text{vec}}^{(k)}$$

$$\begin{aligned} \mathcal{Z}_{\text{vec}}^{(k)} = (-&\frac{\text{sh}(2\epsilon_+)}{\text{sh}(\epsilon_1)\text{sh}(\epsilon_2)}\Big)^k\prod_{s\neq t}^{k}\frac{\text{sh}(\phi_s - \phi_t)\text{sh}(\phi_s - \phi_t + 2\epsilon_+)}{\text{sh}(\phi_s - \phi_t + \epsilon_1)\text{sh}(\phi_s - \phi_t + \epsilon_2)} \\ &\times\prod_{s=1}^{k}\prod_{r=1}^{2}\frac{1}{\text{sh}(\phi_s - a_r + \epsilon_+)\text{sh}(-\phi_s + a_r + \epsilon_+)} \end{aligned}$$

$$\text{sh}(x) := 2\sinh\left(\frac{Rx}{2}\right)$$

$$\mathcal{Z}_{\text{inst}}^{5\text{ d}-1\text{ d},(k)} = \frac{1}{k!}\oiiint\left[\prod_{s=1}^{k}\frac{R\,\mathrm{d}\phi_s}{2\pi i}\right]\mathcal{Z}_{\text{vec}}^{(k)}\mathcal{Z}_{\text{SQM}}^{(k)}$$

$$\mathcal{Z}_{\mathrm{SQM}}^{(k)} = \prod_{r=1}^{2} \mathrm{sh}(m - a_r) \prod_{s=1}^{k} \frac{\mathrm{sh}(\phi_s - m + \epsilon_-)\mathrm{sh}(-\phi_s + m + \epsilon_-)}{\mathrm{sh}(\phi_s - m + \epsilon_+)\mathrm{sh}(-\phi_s + m + \epsilon_+)}$$

$$\left\langle W_F^{(k)} \right\rangle (a_1, a_2) = \left\langle W_F^{(k)} \right\rangle (a_2, a_1)$$

$$\langle W_F \rangle = e^{Ra} + e^{-Ra} - \mathcal{R}^4 \frac{t_1 t_2 (e^{Ra} + e^{-Ra})}{(1 - e^{Ra} t_1 t_2)(1 - e^{-Ra} t_1 t_2)} + \mathcal{O}(\mathcal{R}^8) + \cdots$$

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

$$z \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Li}_{n+1}(z) = \mathrm{Li}_n(z)$$

$$\mathrm{Li}_{n+1}(z) = \int_0^z \mathrm{Li}_n(t) \frac{\mathrm{d}t}{t}$$

$$\mathrm{Li}_n(x + i\epsilon) = \mathrm{Li}_n(x - i\epsilon) + \frac{2\pi i \log{(x)}^{n-1}}{\Gamma(n)}$$

$$\mathrm{Li}_n(z) + (-1)^n \mathrm{Li}_n(z^{-1}) = \begin{cases} -\dfrac{(2\pi i)^n}{n!} B_n\left(\dfrac{1}{2} + \dfrac{\log{(-z)}}{2\pi i}\right), & z \notin (0,1] \\ -\dfrac{(2\pi i)^n}{n!} B_n\left(\dfrac{1}{2} - \dfrac{\log{(-z^{-1})}}{2\pi i}\right), & z \notin [1,\infty) \end{cases}$$

$$\int_a^b B_n(t) \mathrm{d}t = \frac{B_{n+1}(b) - B_{n+1}(a)}{n+1}$$

$$-\log{(1-z)} + \log{(1 - z^{-1})} = \begin{cases} -\log{(-z)}, & z \notin (0,1] \\ \log{(-z^{-1})}, & z \notin [1,\infty) \end{cases}$$

$$\mathrm{Li}_{n+1}(z^{-1}) - \zeta(n+1) = \int_1^{z^{-1}} \frac{\mathrm{d}t}{t} \mathrm{Li}_n(t)$$

$$= \int_1^{z^{-1}} \frac{\mathrm{d}t}{t} (-1)^{n+1} \mathrm{Li}_n(t^{-1}) - \frac{(2\pi i)^n}{n!} \times \begin{cases} B_n\left(\dfrac{1}{2} + \dfrac{\log{(-t)}}{2\pi i}\right), & t \notin (0,1] \\ B_n\left(\dfrac{1}{2} - \dfrac{\log{(-t^{-1})}}{2\pi i}\right), & t \notin [1,\infty) \end{cases}$$

Evaluating the first term in terms of Li_{n+1}, we arrive at

$$\begin{aligned} &\mathrm{Li}_{n+1}(z^{-1}) - \zeta(n+1) \\ =&(-1)^n(\mathrm{Li}_{n+1}(z) - \zeta(n+1)) \\ &-\frac{(2\pi i)^n}{n!} \int_1^{z^{-1}} \frac{\mathrm{d}t}{t} \begin{cases} B_n\left(\dfrac{1}{2} + \dfrac{\log{(-t)}}{2\pi i}\right), & z \notin [1,\infty), \\ B_n\left(\dfrac{1}{2} - \dfrac{\log{(-t^{-1})}}{2\pi i}\right), & z \notin (0,1], \end{cases} \end{aligned}$$

$$\chi^{\mathrm{vir}}(M,\mu(L)) = \int_{[M]^{\mathrm{vir}}} e^{\mu_D(c_1(L))}\mathrm{Td}\left(T_M^{\mathrm{vir}}\right)$$

$$G = \mathrm{U}(1)\colon \Phi_{r,n}(\mathcal{R}) = \sum_k \chi\left(M_k, \mu_D(n) \otimes E^{\otimes \kappa_{\mathrm{CS}}}\right)\mathcal{R}^d$$

$$\Phi_{0,n}(\mathcal{R}) = \left(\frac{1}{1-\mathcal{R}^2}\right)^{\chi(n)}, \Phi_{\pm 1,n}(\mathcal{R}) = (1+\mathcal{R}^2)^{\chi(n)}$$

$$\Phi_{\mu,n}^{\mathrm{GNY}}(\mathcal{R}) = \sum_{d\geq 0} \chi\left(M_H^{\mathbb{CP}^2}(2\mu H, d), \mathcal{O}\left(\mu_D\left(H^{\otimes n}\right)\right)\right)\mathcal{R}^d$$

$$\Phi_{0,n}^{\mathrm{GNY}}(\mathcal{R}) = \mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[\sum_{l\geq m>0} (-1)^{l+m+1} q^{\frac{1}{2}\left(\left(l+\frac{1}{2}\right)^2 - m^2\right)} w^{m(n+3)-l-\frac{1}{2}} \right.$$
$$\left. \times \left(-\frac{\vartheta_1(\tau,v)}{\mathcal{R}\vartheta_4(\tau)}\right)^{(n+3)^2-1} \frac{8\vartheta_4(\tau)^8}{\mathcal{R}\vartheta_3(\tau)^3\vartheta_2(\tau)^3} \frac{1}{\sqrt{1-2\mathcal{R}^2\mathrm{u}+\mathcal{R}^4}}\right]$$

$$\Phi_{\frac{1}{2},n}^{\mathrm{GNY}}(\mathcal{R}) = \mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[\sum_{l\geq m>0} (-1)^{l+m} q^{\frac{1}{2}\left(l^2 - \left(m-\frac{1}{2}\right)^2\right)} w^{\left(m-\frac{1}{2}\right)(n+3)-l} \right.$$
$$\left. \times \left(-\frac{\vartheta_1(\tau,v)}{\mathcal{R}\vartheta_4(\tau)}\right)^{(n+3)^2-1} \frac{8\vartheta_4(\tau)^8}{\vartheta_3(\tau)^3\vartheta_2(\tau)^3} \frac{1}{\sqrt{1-2\mathcal{R}^2\mathrm{u}+\mathcal{R}^4}}\right]$$

$$\Phi_{0,n}^{\mathrm{GNY}}(\mathcal{R}) = \begin{cases} 21\mathcal{R} - 21\mathcal{R}^5 - 56\mathcal{R}^9 + \cdots, & n=-8, \\ 15\mathcal{R} - 6\mathcal{R}^5 - 10\mathcal{R}^9 + \cdots, & n=-7, \\ 10\mathcal{R} - \mathcal{R}^5 - \mathcal{R}^9 + \cdots, & n=-6, \\ 6\mathcal{R} + \mathcal{O}(\mathcal{R}^{13}), & n=-5, \\ 3\mathcal{R} + \mathcal{O}(\mathcal{R}^{13}), & n=-4, \\ \mathcal{R} + \mathcal{O}(\mathcal{R}^{13})\cdots, & n=-3, \\ \mathcal{O}(\mathcal{R}^{13}) \text{ or less }, & n=-2,-1, \\ \mathcal{R} + \mathcal{R}^5 + \mathcal{R}^9 + \cdots, & n=0, \\ 3\mathcal{R} + 6\mathcal{R}^5 + 10\mathcal{R}^9 + \cdots, & n=1, \\ 6\mathcal{R} + 21\mathcal{R}^5 + 56\mathcal{R}^9 + \cdots, & n=2, \\ 10\mathcal{R} + 56\mathcal{R}^5 + 230\mathcal{R}^9 + \cdots, & n=3, \end{cases}$$

$$n_I = -n + c_1(K_{\mathbb{CP}^2}) = -n-3$$

$$\Phi_{\frac{1}{2},n}^{\mathrm{GNY}}(\mathcal{R}) = \begin{cases} 1 + \mathcal{O}(\mathcal{R}^{13}), & n=-4,-2 \\ 1 + \mathcal{R}^4 + \mathcal{R}^8 + \mathcal{R}^{12} + \cdots, & n=-6,0 \\ 1 + 6\mathcal{R}^4 + 21\mathcal{R}^8 + 56\mathcal{R}^{12} + \cdots, & n=-8,2 \\ 1 + 21\mathcal{R}^4 + 210\mathcal{R}^8 + 1401\mathcal{R}^{12} + \cdots, & n=-10,4 \\ 1 + 55\mathcal{R}^4 + 1310\mathcal{R}^8 + 19432\mathcal{R}^{12}\cdots, & n=-12,6 \end{cases}$$

$$\Phi_{0,n}^{\mathrm{GNY}}(\mathcal{R}) = \begin{cases} \mathcal{R}^{-3}(1-\mathcal{R}^4)^{-3} - \mathcal{R}^{-3}, & n=1 \\ \mathcal{R}^{-3}(1-\mathcal{R}^4)^{-6} - \mathcal{R}^{-3}, & n=2 \\ \mathcal{R}^{-3}(1+\mathcal{R}^8)(1-\mathcal{R}^4)^{-10} - \mathcal{R}^{-3}, & n=3 \end{cases}$$

$$\Phi^{\mathrm{GNY}}_{\frac{1}{2},n}(\mathcal{R})$$

$$=\begin{cases} 1, & n=-4,-2 \\ (1-\mathcal{R}^4)^{-1}, & n=-6,0 \\ (1-\mathcal{R}^4)^{-6}, & n=-8,2 \\ (1+6\mathcal{R}^4+\mathcal{R}^8)(1-\mathcal{R}^4)^{-15}, & n=-10,4 \\ (1+27\mathcal{R}^4+148\mathcal{R}^8+266\mathcal{R}^{12}+378\mathcal{R}^{16}+147\mathcal{R}^{20}+56\mathcal{R}^{24}+\mathcal{R}^{32})(1-\mathcal{R}^4)^{-28}, & n=-12,6 \end{cases}$$

$$\sum_{l\geq m>0} (-1)^{l+m+1} q^{\frac{1}{2}\left(\left(l+\frac{1}{2}\right)^2-m^2\right)} w^{mn-l-\frac{1}{2}}$$

$$\frac{1}{2}\sum_{l\geq m>0} (-1)^{l+m+1} q^{\frac{1}{2}\left(\left(l+\frac{1}{2}\right)^2-m^2\right)} \left(w^{mn-l-\frac{1}{2}}+w^{-\left(mn-l-\frac{1}{2}\right)}\right)$$

$$\begin{aligned}
&\sum_{l\geq 0,m>0} (-1)^{l+1} q^{\frac{1}{2}\left(\left(l+m+\frac{1}{2}\right)^2-m^2\right)} w^{mn-l-m-\frac{1}{2}} \\
&= \sum_{l\geq 0,m>0} (-1)^{l+1} q^{\frac{1}{2}\left(l+\frac{1}{2}\right)^2+m\left(l+\frac{1}{2}\right)} w^{m(n-1)-l-\frac{1}{2}} \\
&= \sum_{l\in\mathbb{N}+\frac{1}{2},m>0} (-1)^{l+\frac{1}{2}} q^{\frac{1}{2}l^2+ml} w^{m(n-1)-l}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{l\geq 0} (-1)^{l+1} \frac{q^{\frac{1}{2}\left(l+\frac{1}{2}\right)^2+\left(l+\frac{1}{2}\right)} w^{n-1-l-\frac{1}{2}}}{1-q^{l+\frac{1}{2}}w^{n-1}} \\
&= \sum_{l\in\mathbb{N}+\frac{1}{2}} (-1)^{l+\frac{1}{2}} \frac{q^{\frac{1}{2}l^2+l} w^{(n-1)-l}}{1-q^l w^{n-1}} \\
&= \sum_{l\in\mathbb{N}+\frac{1}{2}} (-1)^{l-\frac{1}{2}} \frac{q^{\frac{1}{2}l^2} w^{-l}}{1-q^{-l} w^{-(n-1)}}
\end{aligned}$$

$$\sum_{l\in\mathbb{N}+\frac{1}{2}} (-1)^{l+\frac{1}{2}} \frac{q^{\frac{1}{2}l^2} w^{-l}}{1-q^l w^{n-1}} - \sum_{l\in\mathbb{N}+\frac{1}{2}} (-1)^{l+\frac{1}{2}} q^{\frac{1}{2}l^2} w^{-l}$$

$$\frac{1}{2}\sum_{l\in\mathbb{Z}+\frac{1}{2}} (-1)^{l+\frac{1}{2}} \frac{q^{\frac{1}{2}l^2} w^{-l}}{1-q^l w^{n-1}} - \frac{1}{2}\sum_{l\in\mathbb{N}+\frac{1}{2}} (-1)^{l+\frac{1}{2}} q^{\frac{1}{2}l^2} w^{-l}$$

$$\frac{1}{2} q^{-\frac{1}{8}} w^{-\frac{n}{2}} \vartheta_1(\tau,v) M\left(\tau,\frac{\tau}{2}+(n-1)v,-v\right)$$

$$\vartheta^+(\tau,z) = \sum_{l\in\mathbb{N}+\frac{1}{2}} (-1)^{l+\frac{1}{2}} q^{\frac{1}{2}l^2} e^{2\pi i l z}$$

$$\Phi^{\mathrm{GNY}}_{0,n-3}(\mathcal{R}) = \frac{1}{2}\mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[C^{n^2}\frac{8\vartheta_4(\tau)^9}{\vartheta_3(\tau)^3\vartheta_2(\tau)^3}\frac{1}{\sqrt{1-2\mathcal{R}^2\mathrm{u}+\mathcal{R}^4}}\right.$$
$$\left.\times\left(q^{-\frac{1}{8}}w^{-\frac{n}{2}}M\left(\tau,\frac{\tau}{2}+(n-1)v,-v\right)-\frac{\vartheta^+(\tau,-v)}{\vartheta_1(\tau,v)}\right)\right]$$

$$\vartheta_1(\tau,\tau/2+z) = iq^{-1/8}e^{-\pi iz}\vartheta_4(\tau,z)$$

$$M\left(\tau,\frac{\tau}{2}+(n-1)v,-v\right)$$
$$=M\left(\tau,nv,-\frac{\tau}{2}\right)-\frac{1}{\mathcal{R}}q^{\frac{1}{8}}w^{\frac{n}{2}}\frac{\eta(\tau)^3\vartheta_1(\tau,(n-1)v)}{\vartheta_4(\tau)\vartheta_1(\tau,nv)\vartheta_4(\tau,(n-1)v)}$$

$$M\left(\tau,nv,-\frac{\tau}{2}\right) = -w^nM\left(\tau,nv,\frac{\tau}{2}\right)-iq^{\frac{1}{8}}w^{\frac{n}{2}}$$

$$G_{0,n}(\tau,v) = -q^{-\frac{1}{8}}w^{\frac{n}{2}}M\left(\tau,nv,\frac{\tau}{2}\right)-\frac{i}{\vartheta_4(\tau,nv)}\partial_\rho\ln\left(\frac{\vartheta_1(\tau,\rho)}{\vartheta_4(\tau,\rho)}\right)\Bigg|_{\rho=nv}$$

$$i\vartheta_1(\tau,z)+\vartheta^+(\tau,-z) = \vartheta^+(\tau,z)$$

$$\Phi^{\mathrm{GNY}}_{0,n-3}(\mathcal{R}) =\frac{1}{2}\mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[C^{n^2}\frac{8\vartheta_4(\tau)^9}{\vartheta_3(\tau)^3\vartheta_2(\tau)^3}\frac{1}{\sqrt{1-2\mathcal{R}^2\mathrm{u}+\mathcal{R}^4}}\right.$$
$$\times\left(G_{0,n}(\tau,v)-\frac{1}{\mathcal{R}}\frac{\eta(\tau)^3\vartheta_1(\tau,(n-1)v)}{\vartheta_4(\tau)\vartheta_1(\tau,nv)\vartheta_4(\tau,(n-1)v)}\right.$$
$$\left.\left.-\frac{i}{\vartheta_4(\tau,nv)}\partial_\rho\ln\left(\frac{\vartheta_1(\tau,\rho)}{\vartheta_4(\tau,\rho)}\right)\Bigg|_{\rho=nv}-\frac{\vartheta^+(\tau,v)}{\vartheta_1(\tau,v)}\right)\right]$$

$$\sum_{l\geq m>0}(-1)^{l+m}q^{\frac{1}{2}\left(l^2-\left(m-\frac{1}{2}\right)^2\right)}w^{\left(m-\frac{1}{2}\right)n-l}$$

$$\frac{1}{2}\sum_{l\geq m>0}(-1)^{l+m}q^{\frac{1}{2}\left(l^2-\left(m-\frac{1}{2}\right)^2\right)}\left(w^{\left(m-\frac{1}{2}\right)n-l}+w^{-\left(\left(m-\frac{1}{2}\right)n-l\right)}\right)$$

$$-\frac{1}{2}\sum_{l\in\mathbb{Z}}(-1)^l\frac{q^{\frac{1}{2}\left(l^2-\frac{1}{4}\right)}w^{\frac{n}{2}-l}}{1-q^{l-\frac{1}{2}}w^{n-1}}$$

$$\frac{i}{2}\vartheta_4(\tau,v)M\left(\tau,-\frac{\tau}{2}+(n-1)v,-\frac{\tau}{2}-v\right)$$

$$\Phi^{\mathrm{GNY}}_{\frac{1}{2},n-3}(\mathcal{R}) = -\frac{i}{2}\mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[M\left(\tau,-\frac{\tau}{2}+(n-1)v,-\frac{\tau}{2}-v\right)\right.$$
$$\left.\times C^{n^2}\frac{8\vartheta_4(\tau)^9}{\vartheta_3(\tau)^3\vartheta_2(\tau)^3}\frac{1}{\sqrt{1-2\mathcal{R}^2\mathrm{u}+\mathcal{R}^4}}\right]$$

$$M\left(\tau,-\frac{\tau}{2}+(n-1)v,-\frac{\tau}{2}-v\right)$$
$$=M\left(\tau,-\frac{\tau}{2}+nv,-\frac{\tau}{2}\right)+\frac{i\eta(\tau)^3\vartheta_1(\tau,(n-1)v)\vartheta_1(\tau,v)}{\vartheta_4(\tau,nv)\vartheta_4(\tau,(n-1)v)\vartheta_4(\tau,v)\vartheta_4(\tau)}$$

$$M\left(\tau,-\frac{\tau}{2}+(n-1)v,-\frac{\tau}{2}-v\right)$$
$$=M\left(\tau,-\frac{\tau}{2}+nv,-\frac{\tau}{2}\right)-i\mathcal{R}\frac{\eta(\tau)^3\vartheta_1(\tau,(n-1)v)}{\vartheta_4(\tau,nv)\vartheta_4(\tau,(n-1)v)\vartheta_4(\tau)}$$

$$M\left(\tau,-\frac{\tau}{2}+(n-1)v,-\frac{\tau}{2}-v\right)$$
$$=iG_{\frac{1}{2},n}(\tau,v)-i\mathcal{R}\frac{\eta(\tau)^3\vartheta_1(\tau,(n-1)v)}{\vartheta_4(\tau,nv)\vartheta_4(\tau,(n-1)v)\vartheta_4(\tau)}$$

$$M\left(\tau,-\frac{\tau}{2}-v,-\frac{\tau}{2}-v\right)=iG_{\frac{1}{2},0}(\tau)+i\mathcal{R}^2\frac{\eta(\tau)^3}{\vartheta_4(\tau)^2}$$

$$\Phi^{\mathrm{GNY}}_{\frac{1}{2},n}(\mathcal{R})=\frac{1}{2}\mathrm{Coeff}_{q^0}\mathrm{Ser}_{\mathcal{R}}\left[C^{n^2}\frac{8\vartheta_4(\tau)^9}{\vartheta_3(\tau)^3\vartheta_2(\tau)^3}\frac{1}{\sqrt{1-2\mathcal{R}^2\mathrm{u}+\mathcal{R}^4}}\right.$$
$$\left.\times\left(G_{\frac{1}{2},n}(\tau,v)-\mathcal{R}\frac{\eta(\tau)^3\vartheta_1(\tau,(n-1)v)}{\vartheta_4(\tau,nv)\vartheta_4(\tau,(n-1)v)\vartheta_4(\tau)}\right)\right]$$

$$\frac{\vartheta_1(\tau_1,v_1)}{\vartheta_2(\tau_1,v_1)}=i\mathcal{R}$$

$$v_1=\frac{1}{2\pi i}\log\left(\frac{1+\mathcal{R}}{1-\mathcal{R}}+8\big(\mathcal{R}+2\mathcal{R}^2+5\mathcal{R}^3+\mathcal{O}(\mathcal{R}^4)\big)q_1+\mathcal{O}(q_1^2)\right)$$

$$C_1(\tau_1,v_1)=e^{-\pi i v_1^2/\tau_1}C\left(-\frac{1}{\tau_1},\frac{v_1}{\tau_1}\right)=\frac{\vartheta_2(\tau_1,v_1)}{\vartheta_2(\tau_1)}.$$

$$C_1(\tau_1,v_1)=\frac{1}{\sqrt{1-\mathcal{R}^2}}+\mathcal{O}(q_1)$$

$$\frac{\mathrm{d}a_1}{\mathrm{d}\tau_1}=-\frac{2\pi i}{8R}\frac{\mathcal{R}}{U}\frac{\vartheta_2(\tau_1)^9}{\eta(\tau_1)^3}$$

$$a_1=-\frac{64}{R}\frac{\mathcal{R}}{U}q_1+\mathcal{O}(q_1^2)=-\frac{32\Lambda}{1-\mathcal{R}^2}q_1+\mathcal{O}(q_1^2)$$

$$U=U_1+\mathcal{O}(q_1)=2-2\mathcal{R}^2+\mathcal{O}(q_1)$$

$$\frac{\vartheta_1(\tau_2,v_2)}{\vartheta_2(\tau_2,v_2)}=\mathcal{R}$$

$$v_2=\frac{1}{2\pi i}\log\left(\frac{1-i\mathcal{R}}{1+i\mathcal{R}}+\mathcal{O}(q_2)\right)$$

$$C_2(\tau_2,v_2)=\exp\left(-\frac{\pi i v_2^2}{\tau_2}\right)C\left(2-\frac{1}{\tau_2},\frac{v_2}{\tau_2}\right)=\frac{\vartheta_2(\tau_2,v_2)}{\vartheta_2(\tau_2)}=\frac{1}{\sqrt{1+\mathcal{R}^2}}+\mathcal{O}(q_2).$$

$$a_2=\frac{32i\Lambda}{1+\mathcal{R}^2}q_2+\mathcal{O}(q_2^2)$$

$$\frac{\vartheta_1(\tau_3, v_3)}{\vartheta_2(\tau_3, v_3)} = -i\mathcal{R}$$

$$v_3 = \frac{1}{2\pi i}\log\left(\frac{1-\mathcal{R}}{1+\mathcal{R}} + \mathcal{O}(q_3)\right)\bigg).$$

$$C_3(\tau_3, v_3) = \exp\left(-\frac{\pi i v_3^2}{\tau_3}\right) C\left(4 - \frac{1}{\tau_3}, \frac{v_3}{\tau_3}\right) = \frac{1}{\sqrt{1-\mathcal{R}^2}} + \mathcal{O}(q_3).$$

$$a_3 = -\frac{32\Lambda}{1-\mathcal{R}^2} q_3 + \mathcal{O}(q_3^2)$$

$$\frac{\vartheta_1(\tau_4, v_4)}{\vartheta_2(\tau_4, v_4)} = -\mathcal{R}.$$

$$v_4 = \frac{1}{2\pi i}\log\left(\frac{1+i\mathcal{R}}{1-i\mathcal{R}} + \mathcal{O}(q_4)\right)\bigg).$$

$$C_4(\tau_4, v_4) = \exp\left(-\frac{\pi i v_4^2}{\tau_4}\right) C\left(6 - \frac{1}{\tau_4}, \frac{v_4}{\tau_4}\right) = \frac{1}{\sqrt{1+\mathcal{R}^2}} + \mathcal{O}(q_4)$$

$$a_4 = 32i\frac{\Lambda}{1+\mathcal{R}^2} q_4 + \mathcal{O}(q_4^2)$$

$$S_{\mathrm{U}(4)} = \frac{y}{8\pi}\int_X (F_\dagger \wedge F_\star - D \wedge D) + \frac{y_1}{2\pi}\int_X \left(F_\star \wedge F_\dagger^{(1)} - D \wedge D^{(1)}\right)$$
$$+ \frac{i\tau}{16\pi} F \wedge F + \frac{iv}{4\pi}\int \ F \wedge F^{(1)} + \cdots$$

$$S_{\text{hyper}} = \int \ \mathrm{d}^4x\sqrt{g}\left|D^{\alpha\dot{\alpha}}_{A+2A^{(1)}} M_{\dot{\alpha}}\right|^2 + c\int_X (F_\blacksquare - D) \wedge \bar{M}M + \cdots$$
$$= \int \ \mathrm{d}^4x\sqrt{g}\left|D^{\alpha\dot{\alpha}}_{A+2A^{(1)}} M_{\dot{\alpha}}\right|^2 + c\int_X \left[(F_\triangle - D) + \frac{2y_1}{y}\left(F_+^{(1)} - D^{(1)}\right)\right] \wedge \bar{M}M + \cdots$$

$$\bar{M}_{(\dot{\alpha}} M_{\dot{\beta})} = \bar{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}} (\bar{M}M)_{\mu\nu}$$

$$S = \sqcap \frac{y}{8\pi}\int_X \left(F_+ + \frac{2y_1}{y}F_+^{(1)} + \frac{4\pi c}{y}\bar{M}M\right) \wedge \left(F_+ + \frac{2y_1}{y}F_+^{(1)} + \frac{4\pi c}{y}\bar{M}M\right)$$
$$- \frac{y}{8\pi}\int_X \left(D + \frac{2y_1}{y}D^{(1)} + \frac{4\pi c}{y}\bar{M}M\right) \wedge \left(D + \frac{2y_1}{y}D^{(1)} + \frac{4\pi c}{y}\bar{M}M\right)$$
$$+ \int \ \mathrm{d}^4x\sqrt{g}\left|D^{\alpha\dot{\alpha}}_{A+2A^{(1)}} M_{\dot{\alpha}}\right|^2$$
$$+ \frac{i\tau}{16\pi}\int_X F \wedge F + \frac{iv}{4\pi}\int_X F \wedge F^{(1)} + \cdots$$

$$S = \sqcup \frac{y}{8\pi}\int_X \left(F_+ + \frac{2y_1}{y}F_+^{(1)} + \frac{4\pi c}{y}\bar{M}M\right) \wedge \left(F_+ + \frac{2y_1}{y}F_+^{(1)} + \frac{4\pi c}{y}\bar{M}M\right)$$
$$+ \int \ \mathrm{d}^4x\sqrt{g}\left|D^{\alpha\dot{\alpha}}_{A+2A^{(1)}} M_{\dot{\alpha}}\right|^2 + \frac{i\tau}{16\pi}\int_X F \wedge F + \frac{iv}{4\pi}\int_X F \wedge F^{(I)} + \cdots$$

$$F_+ + \frac{2y_1}{y} F_+^{(1)} + \frac{4\pi c}{y} \bar{M} M = 0$$
$$D_{A+2A^{(1)}}^{\alpha\dot{\alpha}} M_{\dot{\alpha}} = 0$$

$$\sum_{j=1}^{N} F_+ \left(A + \frac{2N\mathrm{Im}(v_j)}{y} A^{(j)} \right) + \bar{M}_j M_j = \frac{N\mathrm{Im}(v_j)}{y} \otimes \frac{1}{2\pi} F(A) \in \overline{w_2(X)} + 2H^2(X,\mathbb{Z}) \sqcup \frac{1}{2\pi} F(A^j) \in H^2(X,\mathbb{Z})$$
$$\not{D}_j M_j = 0, j = 1, \cdots, N,$$

$$\sum_{j=1}^{N} F_{\Vdash} (A + 2A_j) + \bar{M}_j M_j = 0,$$
$$\not{D}_j M_j = 0, j = 1, \cdots, N,$$

$$m(\{c_j\}) = \frac{-2\chi - 2\sigma + \sum_j (c_j^2 - \sigma)}{8}$$

$$\int_{\mathcal{M}_N(\{c_j\})} e^{c_1(\mathbb{L}) + m_\alpha H^\alpha}$$

$$\sum_{j=1}^{N} F_+ (A + 2A_j) = 0$$

$$B\left(\sum_j c_j, J \right) = 0$$

$$\langle \vec{m}, \vec{n} \rangle = m_1 n_1 + m_2 n_2$$

$$\sigma = \{ a_1 \vec{n}_1 + a_2 \vec{n}_2 + \cdots + a_s \vec{n}_s \in N_{\mathbb{R}} \mid a_i \geq 0 \}$$

$$\sigma^\vee = \{ \vec{m} \in M_{\mathbb{R}} \mid \langle \vec{m}, \vec{n} \rangle \geq 0, \forall \vec{n} \in \sigma \}$$

$$\tau = \vec{m}^\perp = \{ \vec{n} \in \sigma \mid \langle \vec{m}, \vec{n} \rangle = 0 \}.$$

$$\vec{n}_{\ell-1} - h_s \vec{n}_\ell + \vec{n}_{\ell+1} = 0, h_\ell \in \mathbb{Z}.$$

$$\det(\vec{n}_1, \vec{n}_2) = 1, \det(\vec{n}_2, \vec{n}_3) = 1.$$

$$\vec{n}_3 = (ax_1 + by_1, ax_2 + by_2)^{\mathrm{T}}, a, b \in \mathbb{Z}$$

$$x_1 y_2 - x_2 y_1 = 1, y_1(ax_2 + by_2) - y_2(ax_1 + by_1) = 1$$

$$\vec{n}_3 = -\vec{n}_1 + b\vec{n}_2$$

$$\langle \vec{m}, \vec{n}_\ell \rangle \geq 0, \langle \vec{m}, \vec{n}_{\ell+1} \rangle \geq 0$$

$$\sigma_\ell^\vee = \sigma(\{\vec{n}_\ell, \vec{n}_\ell^*, -\vec{n}_\ell^*\}) \cap \sigma(\{\vec{n}_{\ell+1}, \vec{n}_{\ell+1}^*, -\vec{n}_{\ell+1}^*\})$$

$$\langle \vec{n}_{\ell+1}^*, \vec{n}_\ell \rangle > 0, \langle \vec{n}_\ell^*, \vec{n}_{\ell+1} \rangle < 0$$

$$\sigma_\ell^\vee = \sigma(\{\vec{n}_{\ell+1}^*, \vec{n}_\ell^*, -\vec{n}_\ell^*\}) \cap \sigma(\{-\vec{n}_\ell^*, \vec{n}_{\ell+1}^*, -\vec{n}_{\ell+1}^*\}) = \sigma(\{\vec{n}_{\ell+1}^*, -\vec{n}_\ell^*\})$$

$$X = (\mathbb{C}^\chi - Z(\Delta))/G$$

$$\left(y_1, \cdots, y_\chi\right) \sim \left(\lambda^{C_{s,1}} y_1, \lambda^{C_{s,2}} y_2, \cdots, \lambda^{C_{s,\chi}} y_\chi\right), s = 1, \cdots, \chi, \lambda \in \mathbb{C}^*$$

$$\sum_{\ell=1}^{\chi} C_{s,\ell} \vec{n}_\ell = 0, s = 1, \cdots, \chi$$

$$C_{\ell,\ell} = -h_\ell, C_{\ell,\ell+1} = C_{\ell+1,\ell} = 1$$

$$z_1^{(\ell)} = \prod_{m=1}^{\chi} y_m^{\langle \vec{n}_{\ell+1}^*, \vec{n}_m \rangle}, z_2^{(\ell)} = \prod_{m=1}^{\chi} y_m^{-\langle \vec{n}_\ell^*, \vec{n}_m \rangle}$$

$$z_1^{(\ell)} = \left(z_2^{(\ell+1)}\right)^{-1}, z_2^{(\ell)} = z_1^{(\ell+1)} \left(z_2^{(\ell+1)}\right)^{h_{\ell+1}}$$

$$y_1 \to e^{i\epsilon_1} y_1, y_2 \to e^{i\epsilon_2} y_2, y_{\ell>2} \to y_\ell$$

$$z_1^{(\ell)} \to \exp\left(i\epsilon_1 \langle \vec{n}_{\ell+1}^*, \vec{n}_1 \rangle + i\epsilon_2 \langle \vec{n}_{\ell+1}^*, \vec{n}_2 \rangle\right) z_1^{(\ell)}$$
$$z_2^{(\ell)} \to \exp\left(-i\epsilon_1 \langle \vec{n}_\ell^*, \vec{n}_1 \rangle - i\epsilon_2 \langle \vec{n}_\ell^*, \vec{n}_2 \rangle\right) z_2^{(\ell)}$$

$$z_1^{(\ell)} \to e^{i\epsilon_1^{(\ell)}} z_1^{(\ell)}, z_2^{(\ell)} \to e^{i\epsilon_2^{(\ell)}} z_2^{(\ell)}$$

$$\epsilon_1^{(\ell)} := \langle \vec{n}_{\ell+1}^*, \vec{\epsilon} \rangle, \epsilon_2^{(\ell)} := -\langle \vec{n}_\ell^*, \vec{\epsilon} \rangle,$$

$$\epsilon_1^{(\ell+1)} = h_{\ell+1} \epsilon_1^{(\ell)} + \epsilon_2^{(\ell)}, \epsilon_2^{(\ell+1)} = -\epsilon_1^{(\ell)}$$

$$\sum_{\ell=1}^{\chi} \frac{1}{\epsilon_1^{(\ell)} \epsilon_2^{(\ell)}} = \sum_{\ell=1}^{\chi} \frac{\epsilon_1^{(\ell)} + \epsilon_2^{(\ell)}}{\epsilon_1^{(\ell)} \epsilon_2^{(\ell)}} = 0$$

$$\frac{\epsilon_1^{(\ell)}}{\epsilon_2^{(\ell)}} + \frac{\epsilon_2^{(\ell-1)}}{\epsilon_1^{(\ell-1)}} = -h_\ell$$

$$\vec{\epsilon}^{(\ell)} = \left(\epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}\right)^{\mathrm{T}}$$

$$\vec{\epsilon}^{(\ell)} = A^{\ell,\ell'} \vec{\epsilon}^{(\ell')},$$

$$A^{\ell+1,\ell} = \begin{pmatrix} h_{\ell+1} & 1 \\ -1 & 0 \end{pmatrix}, A^{\ell,\ell+1} = \begin{pmatrix} 0 & -1 \\ 1 & h_{\ell+1} \end{pmatrix}.$$

$$A^{\ell,\ell'+1} = A^{\ell,\ell'} A^{\ell',\ell'+1} = \begin{pmatrix} \left(A^{\ell,\ell'}\right)_{12} - \left(A^{\ell,\ell'}\right)_{11} + h_{\ell'+1} \left(A^{\ell,\ell'}\right)_{12} \\ \left(A^{\ell,\ell'}\right)_{22} - \left(A^{\ell,\ell'}\right)_{21} + h_{\ell'+1} \left(A^{\ell,\ell'}\right)_{22} \end{pmatrix},$$

$$A^{\ell+1,\ell'} = A^{\ell+1,\ell} A^{\ell,\ell'} = \begin{pmatrix} h_{\ell+1} \left(A^{\ell,\ell'}\right)_{11} + \left(A^{\ell,\ell'}\right)_{21} & h_{\ell+1} \left(A^{\ell,\ell'}\right)_{12} + \left(A^{\ell,\ell'}\right)_{22} \\ -\left(A^{\ell,\ell'}\right)_{11} & -\left(A^{\ell,\ell'}\right)_{12} \end{pmatrix}.$$

$$A^{\ell,1} = \begin{pmatrix} n_{\ell+1}^2 & -n_{\ell+1}^1 \\ -n_\ell^2 & n_\ell^1 \end{pmatrix}, A^{1,l} = \begin{pmatrix} n_\ell^1 & -n_{\ell+1}^1 \\ -n_\ell^2 & n_{\ell+1}^2 \end{pmatrix}.$$

$$\boldsymbol{p} = \sum_{\ell=1}^{\chi} \mathfrak{p}_\ell D_\ell, \mathfrak{p}_\ell \in \mathbb{Z}$$

$$\boldsymbol{p} \cap U_\ell = \left\{ f_{\boldsymbol{p}}^{(\ell)} = \left(z_1^{(\ell)}\right)^{\mathfrak{p}_\ell} \left(z_2^{(\ell)}\right)^{\mathfrak{p}_{\ell+1}} = 0 \right\}$$

$$\boldsymbol{p} \uplus U_\ell \uplus U_{\ell+1} = \left\{ y_{\ell+1}^{\mathfrak{p}_{\ell+1}} = 0 \right\}$$

$$\boldsymbol{p} \oslash U_\ell \oslash U_s = \emptyset \wedge \left\{ U_\ell, f_{\boldsymbol{p}}^{(\ell)} \right\}$$

$$\left(f_{\boldsymbol{p}}^{(\ell)}\right)^{-1} f_{\boldsymbol{p}}^{(\ell+1)} = \left(z_1^{(\ell)}\right)^{h_{\ell+1}\mathfrak{p}_{\ell+1} - \mathfrak{p}_{\ell+2} - \mathfrak{p}_\ell} = \left(z_2^{(\ell+1)}\right)^{-h_{\ell+1}\mathfrak{p}_{\ell+1} + \mathfrak{p}_{\ell+2} + \mathfrak{p}_\ell}$$

$$\left(\frac{\mathrm{d}z_1^{(\ell+1)}}{\mathrm{d}z_1^{(\ell)}} \frac{\mathrm{d}z_2^{(\ell+1)}}{\mathrm{d}z_2^{(\ell)}} - \frac{\mathrm{d}z_1^{(\ell+1)}}{\mathrm{d}z_2^{(\ell)}} \frac{\mathrm{d}z_2^{(\ell+1)}}{\mathrm{d}z_1^{(\ell)}} \right)^{-1} = \left(z_2^{(\ell+1)}\right)^{-2+h_{\ell+1}}.$$

$$K_X = -\sum_{\ell} D_\ell.$$

$$B(D_s, D_\ell) = C_{s,\ell}$$

$$B(D_\ell, D_\ell) = -h_\ell, B(D_{\ell+1}, D_\ell) = B(D_\ell, D_{\ell+1}) = 1$$

$$\sum_{\ell=1}^{\chi} n_\ell^i D_\ell = 0, i = 1,2$$

$$B\left(\sum_{\ell=1}^{\chi} \mathfrak{p}_\ell D_\ell, \sum_{\ell=1}^{\chi} n_\ell^i D_\ell \right) = \sum_{\ell=1}^{\chi} \left(-\mathfrak{p}_\ell n_\ell^i h_\ell + \mathfrak{p}_\ell n_{\ell+1}^i + \mathfrak{p}_{\ell+1} n_\ell^i\right)$$

$$= \sum_{\ell=1}^{\chi} \mathfrak{p}_\ell \left(-n_\ell^i h_\ell + n_{\ell-1}^i + n_{\ell+1}^i\right) = 0$$

$$\boldsymbol{k} = \left[\frac{F_4}{4\pi}\right] = \frac{1}{2} \sum_{\ell} \mathfrak{p}_\ell D_\ell \oslash L + \boldsymbol{\mu}$$

$$\mathfrak{p}_\ell - \mathfrak{p}'_\ell = \sum_{i=1,2} s_i n_\ell^i, s_1, s_2 \in \mathbb{Z}, \ell = 1, \cdots, \chi$$

$$D_1 = -\sum_{\ell=3}^{\chi} n_\ell^1 D_\ell, D_2 = -\sum_{\ell=3}^{\chi} n_\ell^2 D_\ell.$$

$$\boldsymbol{k} = \sum_{\ell=3}^{\chi} \boldsymbol{k}_\ell D_\ell = \frac{1}{2} \sum_{\ell=3}^{\chi} \left(\mathfrak{p}_\ell - \mathfrak{p}_1 n_\ell^1 - \mathfrak{p}_2 n_\ell^2\right) D_\ell$$

$$\boldsymbol{\mu} = \sum_{\ell=3}^{\chi} \mu_\ell D_\ell$$

$$S_{\boldsymbol{\mu}} = \left\{\{\mathfrak{p}\} \mid \mathfrak{p}_\ell - \mathfrak{p}_1 n_\ell^1 - \mathfrak{p}_2 n_\ell^2 = 2\mu_\ell \bmod 2, \ell = 3, \cdots, \chi\right\}$$

$$\sum_{\mathfrak{p}\in S_\mu}{}^{\boxtimes}_{\circledast} = \sum_{(\mathfrak{p}_1,\mathfrak{p}_2)\in\mathbb{Z}^2} \sum_{k}{}^{\ddagger}_{\star}\varnothing$$

$$\begin{aligned} a^{(\ell)} - a^{(\ell+1)} &= \frac{1}{2}\left[\left(\mathfrak{p}_\ell \epsilon_1^{(\ell)} + \mathfrak{p}_{\ell+1}\epsilon_2^{(\ell)}\right) - \left(\mathfrak{p}_{\ell+1}\epsilon_1^{(\ell+1)} + \mathfrak{p}_{\ell+2}\epsilon_2^{(\ell+1)}\right)\right] \\ &= \frac{1}{2}(\mathfrak{p}_\ell + \mathfrak{p}_{\ell+2} - \mathfrak{p}_{\ell+1}h_{\ell+1})\epsilon_1^{(\ell)} \end{aligned}$$

$$(\mathfrak{p}_\ell + \mathfrak{p}_{\ell+2} - \mathfrak{p}_{\ell+1}h_{\ell+1}) - (\mathfrak{p}'_\ell + \mathfrak{p}'_{\ell+2} - \mathfrak{p}'_{\ell+1}h_{\ell+1}) = \sum_{i=1,2} s_i\left(n_\ell^i + n_{\ell+2}^i - n_{\ell+1}^i h_{\ell+1}\right) = 0$$

$$\begin{aligned} &\sum_{\ell=1}^{\chi} \frac{1}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\left(\epsilon_1^{(\ell)}\mathfrak{p}_\ell + \epsilon_2^{(\ell)}\mathfrak{p}_{\ell+1}\right)\left(\epsilon_1^{(\ell)}\mathfrak{q}_\ell + \epsilon_2^{(\ell)}\mathfrak{q}_{\ell+1}\right) \\ &= \sum_{\ell=1}^{\chi}\left[\left(\frac{\epsilon_1^{(\ell)}}{\epsilon_2^{(\ell)}} + \frac{\epsilon_2^{(\ell-1)}}{\epsilon_1^{(\ell-1)}}\right)\mathfrak{p}_\ell\mathfrak{q}_\ell + \mathfrak{p}_\ell\mathfrak{q}_{\ell+1} + \mathfrak{q}_\ell\mathfrak{p}_{\ell+1}\right] \\ &= \sum_{\ell=1}^{\chi} B(D_\ell, D_\ell)\mathfrak{p}_\ell\mathfrak{q}_\ell + B(D_\ell, D_{\ell+1})\mathfrak{p}_\ell\mathfrak{q}_{\ell+1} + B(D_\ell, D_{\ell+1})\mathfrak{q}_\ell\mathfrak{p}_{\ell+1} \\ &= B\left(\sum_{\ell=1}^{\chi} \mathfrak{p}_\ell D_\ell, \sum_{\ell=1}^{\chi} \mathfrak{q}_\ell D_\ell\right) \end{aligned}$$

$$\nu_s^{(\ell)} = -\frac{i}{2\pi}\left(\epsilon_1^{(\ell)}\delta_{s,\ell} + \epsilon_2^{(\ell)}\delta_{s-1,\ell}\right)$$

$$\int_X D_s^{\mathrm{eq}} \wedge D_{s'}^{\mathrm{eq}} = \int_X D_s \wedge D_{s'} = C_{s,s'} = -h_s\delta_{s,s'} + \delta_{s,s'-1} + \delta_{s,s'+1}$$

$$\int_X \alpha = (-2\pi)^{\frac{1}{2}\dim(X)} \sum_{\ell=1}^{\chi} \frac{\alpha_0^{(\ell)}}{\left(i\epsilon_1^{(\ell)}\right)\left(i\epsilon_2^{(\ell)}\right)},$$

$$\begin{aligned} \int_X D_s^{\mathrm{eq}} \wedge D_{s'}^{\mathrm{eq}} &= -(-2\pi)^2 \sum_{\ell=1}^{\chi} \frac{\nu_s^{(\ell)}\nu_{s'}^{(\ell)}}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}} \\ &= -4\pi^2 \sum_{\ell=1}^{\chi} \frac{1}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\left(-\frac{i}{2\pi}\right)^2\left(\epsilon_1^{(\ell)}\delta_{s,\ell} + \epsilon_2^{(\ell)}\delta_{s-1,\ell}\right)\left(\epsilon_1^{(\ell)}\delta_{s',\ell} + \epsilon_2^{(\ell)}\delta_{s'-1,\ell}\right) \\ &= \frac{\epsilon_1^{(s)}}{\epsilon_2^{(s)}}\delta_{s,s'} + \delta_{s,s'-1} + \delta_{s,s'+1} + \frac{\epsilon_2^{(s-1)}}{\epsilon_1^{(s-1)}}\delta_{s,s'} \end{aligned}$$

$$h_s = -\frac{\epsilon_1^{(s)}}{\epsilon_2^{(s)}} - \frac{\epsilon_2^{(s-1)}}{\epsilon_1^{(s-1)}}$$

$$\vec{n}_1 = (1,0)^{\mathrm{T}}, \vec{n}_2 = (0,1)^{\mathrm{T}}, \vec{n}_3 = (-1,-1)^{\mathrm{T}}$$

$$\vec{n}_1 + \vec{n}_2 + \vec{n}_3 = 0$$

$$(y_1, y_2, y_3) \sim (\lambda y_1, \lambda y_2, \lambda y_3), \forall \lambda \in \mathbb{C}^*$$

$$\vec{n}_1^* = (0,-1), \vec{n}_2^* = (1,0), \vec{n}_3^* = (-1,1)$$

$$\begin{aligned} U_1&: (u_1, u_2) = (y_1 y_3^{-1}, y_2 y_3^{-1}), \\ U_2&: (v_1, v_2) = (y_2 y_1^{-1}, y_3 y_1^{-1}), \\ U_3&: (w_1, w_2) = (y_3 y_2^{-1}, y_1 y_2^{-1}). \end{aligned}$$

$$\begin{aligned} D_1 \cap U_1 &= \{u_1 = 0\}, \quad D_1 \cap U_3 = \{w_2 = 0\} \\ D_2 \cap U_2 &= \{v_1 = 0\}, \quad D_2 \cap U_1 = \{u_2 = 0\} \\ D_3 \cap U_3 &= \{w_1 = 0\}, \quad D_3 \cap U_2 = \{v_2 = 0\} \end{aligned}$$

$$\boldsymbol{p} = \mathfrak{p}_1 D_1 + \mathfrak{p}_2 D_2 + \mathfrak{p}_3 D_3$$

$$\begin{aligned} \boldsymbol{p} \sqcup U_1 &= \left\{ f_{\boldsymbol{p}}^{(1)} = (u_1)^{\mathfrak{p}_1} (u_2)^{\mathfrak{p}_2} = 0 \right\} \\ \boldsymbol{p} \sqcap U_2 &= \left\{ f_{\boldsymbol{p}}^{(2)} = (v_1)^{\mathfrak{p}_2} (v_2)^{\mathfrak{p}_3} = 0 \right\} \\ \boldsymbol{p} \triangleq U_3 &= \left\{ f_{\boldsymbol{p}}^{(3)} = (w_1)^{\mathfrak{p}_3} (w_2)^{\mathfrak{p}_1} = 0 \right\} \end{aligned}$$

$$s_{\ell+1,\ell} = \left(f_{\boldsymbol{p}}^{(\ell)} \right)^{-1} \odot f_{\boldsymbol{p}}^{(\ell+1)}$$

$$s_{2,1} = (u_1)^{-\mathfrak{p}_1 - \mathfrak{p}_2 - \mathfrak{p}_3}, s_{3,2} = (v_1)^{-\mathfrak{p}_1 - \mathfrak{p}_2 - \mathfrak{p}_3}, s_{1,3} = (w_1)^{-\mathfrak{p}_1 - \mathfrak{p}_2 - \mathfrak{p}_3}$$

$$\begin{aligned} U_1&: \quad \sigma_1 \, \mathrm{d}u_1 \wedge \mathrm{d}u_2 \\ U_2&: \quad \sigma_2 \, \mathrm{d}v_1 \wedge \mathrm{d}v_2 \\ U_3&: \quad \sigma_3 \, \mathrm{d}w_1 \wedge \mathrm{d}w_2 \end{aligned}$$

$$\mathrm{d}v_1 \wedge \mathrm{d}v_2 = \frac{\partial(v_1, v_2)}{\partial(u_1, u_2)} \mathrm{d}u_1 \wedge \mathrm{d}u_2 = u_1^{-3} \, \mathrm{d}u_1 \wedge \mathrm{d}u_2$$

$$\sigma_1 \, \mathrm{d}u_1 \wedge \mathrm{d}u_2 = \sigma_2 \, \mathrm{d}v_1 \wedge \mathrm{d}v_2$$

$$\sigma_1 = u_1^{-3} \sigma_2$$

$$s_{2,1}(K_{\mathbb{CP}^2}) = u_1^3$$

$$s_{3,2}(K_{\mathbb{CP}^2}) = v_1^3, s_{1,3}(K_{\mathbb{CP}^2}) = w_1^3.$$

$$X_{[d]} = \coprod_{q_1 + q_2 = d} X^{[q_1]} \times X^{[q_2]}$$

$$d := c_2(L_1) + c_2(L_2) = c_2 - B(m_1, m_2)$$

$$m_1 = \frac{1}{2}c_1 + \boldsymbol{k}, m_2 = \frac{1}{2}c_1 - \boldsymbol{k}$$

$$k = c_2 - \frac{1}{4}c_1^2 = d - \boldsymbol{k}^2$$

$$\begin{aligned}
&\pi_a: X \otimes X^{[q_1]} \oslash X^{[q_2]} \to X \times X^{[q_a]}, a = 1,2,\\
&\pi_0: X \otimes X^{[q_1]} \oslash X^{[q_2]} \to X,\\
&\pi: X \otimes X^{[q_1]} \oslash X^{[q_2]} \to X^{[q_1]} \times X^{[q_2]}
\end{aligned}$$

$$I_a(m_a) = I_a \otimes \pi_0^* \mathcal{O}(L_a)\left(z_1^{(\ell)}, z_2^{(\ell)}\right)$$

$$T \triangleright \left(z_1^{(\ell)}, z_2^{(\ell)}\right) = \left(e^{i\epsilon_1^{(\ell)}} z_1^{(\ell)}, e^{i\epsilon_2^{(\ell)}} z_2^{(\ell)}\right)$$

$$X_{[d]}^T \leftrightarrow \left\{\vec{Y} = \left(Y_1^{(\ell)}, Y_2^{(\ell)}\right) \middle| \sum_{\ell,a} \middle| Y_a^{(\ell)} \mid = d\right\}$$

$$0 \to \Omega^0(X, \mathrm{ad}(P)) \to \Omega^1(X, \mathrm{ad}(P)) \to \Omega^{2,+}(X, \mathrm{ad}(P)) \to 0$$

$$X_{[d]} \subset M_{\boldsymbol{\mu},k}$$

$$W = (L_1 \otimes L_2^{-1}) \oplus (L_1^{-1} \otimes L_2)$$

$$0 \to \Omega^0(X, \mathrm{t}) \oplus \Omega^0(X, W) \to \Omega^1(X, \mathrm{t}) \oplus \Omega^1(X, W) \to \Omega^{2,+}(X, \mathrm{t}) \oplus \Omega^{2,+}(X, W) \to 0$$

$$TM_{\boldsymbol{\mu},k}\big|_{X_{[d]}} = TX_{[d]} + N$$

$$-TX_{[d]} = [\Omega^0(X, \mathrm{t})] - [\Omega^1(X, \mathrm{t})] + [\Omega^{2,+}(X, \mathrm{t})]$$

$$-[N] = [\Omega^0(X, W)] - [\Omega^1(X, W)] + [\Omega^{2,+}(X, W)]$$

$$[N] = [W_+] + [W_-]\left[\Omega^{\bullet}\left(X, L_1^{\pm} \otimes L_2^{\mp}\right)\right]$$

$$\begin{aligned}
&\sum_d \mathcal{R}^{4k-3} \int_{X_{[d]}} \hat{A}\left(TX_{[d]}\right) \wedge \mathrm{ch}\left(\hat{S}^{\bullet}[N^\vee]\right) \wedge \mathrm{ch}\left(\tilde{\mathcal{L}}^{(I)}\right)\\
&= \sum_d \mathcal{R}^{4k-3} \int_{X_{[d]}} \frac{\hat{A}\left(TX_{[d]}\right)}{\mathrm{ch}(\hat{\wedge}^{\bullet}\,[N^\vee])} \wedge \mathrm{ch}\left(\tilde{\mathcal{L}}^{(I)}\right)
\end{aligned}$$

$$\hat{S}^{\blacksquare} V = (\mathrm{det}V)^{\frac{1}{2}} S^{\cdot} V, \hat{\wedge}^{\bullet}\, V = (\mathrm{det}V)^{-\frac{1}{2}} \wedge^{\blacksquare} V,$$

$$c_1\left(\tilde{\mathcal{L}}^{(I)}\right) = \pi_*\left[\frac{F^{(I)}}{2\pi} \wedge \mathrm{ch}_2\left(\mathcal{I}_1(m_1) \oplus \mathcal{I}_2(m_2)\right)\right],$$

$$\sum_d \mathcal{R}^{4k-3} \int_{X_{[d]}} \frac{\mathrm{td}\left(T^{(1,0)} X_{[d]}\right)}{\mathrm{ch}(\wedge^{\bullet}\,[N^\vee])} \wedge \exp\left(-\frac{1}{2} c_1\left(T^{(1,0)} X_{[d]} + [N]\right)\right) \wedge \mathrm{ch}\left(\tilde{\mathcal{L}}^{(I)}\right)$$

$$\left[TX_{[d]}\right] = -\pi_*(\mathcal{I}_1 \otimes \mathcal{I}_1^\vee + \mathcal{I}_2 \otimes \mathcal{I}_2^\vee)$$

$$[N] = -\pi_*\left(\sum_{a\neq b} \mathcal{I}_a(m_a)\otimes \mathcal{I}_b(m_b)^\vee\right)$$

$$\mathrm{ch}\left(TX_{[d]}\right) = \pi_*\left[\mathrm{td}(X)\mathrm{ch}\left(-(\mathcal{I}_1\otimes\mathcal{I}_1^\vee+\mathcal{I}_2\otimes\mathcal{I}_2^\vee)\right)\right]$$

$$\mathrm{ch}([N]) = \pi_*\left[\mathrm{td}(X)\mathrm{ch}\left(-\left(\mathcal{I}_1(m_1)\otimes\mathcal{I}_2^\vee(m_2)+\mathcal{I}_2(m_2)\otimes\mathcal{I}_1^\vee(m_1)\right)\right)\right]$$

$$\mathrm{ch}\left(TX_{[d]}\right):=\sum_{p_\ell\in X^T}\sum_{x(p_\ell)} e^{x(p_\ell)},\mathrm{ch}([N]):=\sum_{p_\ell\in X^T}\sum_{\tilde{x}(p_\ell)} e^{\tilde{x}(p_\ell)}.$$

$$\mathrm{ch}\left(TX_{[d]}\right) = \sum_{p_\ell\in X^T}\left[\frac{\mathrm{td}\left(T_{p_\ell}X\right)}{e\left(T_{p_\ell}X\right)}\iota_{p_\ell}^*\mathrm{ch}\left(-(\mathcal{I}_1\otimes\mathcal{I}_1^\vee+\mathcal{I}_2\otimes\mathcal{I}_2^\vee)\right)\right]$$

$$\mathrm{ch}([N]) = \sum_{p_\ell\in X^T}\left[\frac{\mathrm{td}\left(T_{p_\ell}X\right)}{e\left(T_{p_\ell}X\right)}\iota_{p_\ell}^*\mathrm{ch}\left(-\left(\mathcal{I}_1(m_1)\otimes\mathcal{I}_2^\vee(m_2)+\mathcal{I}_2(m_2)\otimes\mathcal{I}_1^\vee(m_1)\right)\right)\right]$$

$$\mathrm{td}\left(T^{(1,0)}X_{[d]}\right) = \prod_{p_\ell\in X^T}\prod_{x(p_\ell)}\frac{x(p_\ell)}{1-e^{-x(p_\ell)}},$$

$$\mathrm{ch}(\wedge^\cdot[N^\vee]) = \prod_{p_\ell\in X^T}\prod_{\tilde{x}(p_\ell)}\left(1-e^{-\tilde{x}(p_\ell)}\right),$$

$$\sum_d \mathcal{R}^{4k-3}\sum_{\vec{Y}}\sum_{\substack{m_1+m_2=c_1\\ q_1+q_2=d}}\frac{1}{e\left(T_{\vec{Y}}X_{[d]}\right)}$$
$$\iota_{\vec{Y}}^*\left[\frac{\mathrm{td}\left(T^{(1,0)}X_{[d]}\right)}{\mathrm{ch}(\wedge^\bullet[N^\vee])}\wedge\exp\left(-\frac{1}{2}c_1\left(T^{(1,0)}X_{[d]}\oplus[N]\right)\right)\wedge\mathrm{ch}\left(\tilde{\mathcal{L}}^{(I)}\right)\right]$$

$$\iota_{p_\ell,\vec{Y}}^*\mathrm{ch}\left(\mathcal{I}_\alpha(m_\alpha)\right) = e^{a_\alpha^{(\ell)}}\left[1-\left(1-e^{-\epsilon_1^{(\ell)}}\right)\left(1-e^{-\epsilon_2^{(\ell)}}\right)\sum_{(i,j)\in Y_\alpha^{(\ell)}} e^{-(i-1)\epsilon_1^{(\ell)}-(j-1)\epsilon_2^{(\ell)}}\right]$$

$$\frac{\iota_{\vec{Y}}^*\mathrm{td}\left(T^{(1,0)}X_{[d]}\right)}{e\left(T_{\vec{Y}}X_{[d]}\right)} = \prod_{\ell=1}^{\chi}\prod_{\alpha=1}^{2} n_{\alpha,\alpha}^{\left(Y_\alpha^{(\ell)},Y_\alpha^{(\ell)}\right)}\left(\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},R\right)$$

$$\mathcal{R}^{4k-3}\iota_{\vec{Y}}^*\frac{1}{\mathrm{ch}(\wedge^\bullet[N^\vee])} = K\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\Lambda,R\right)\prod_{\ell=1}^{\chi}\prod_{i\neq j} n_{\alpha,\beta}^{\left(Y_\alpha^{(\ell)},Y_\beta^{(\ell)}\right)}\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},R\right)$$

$$K\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\Lambda,R\right)$$
$$=\mathcal{R}^{d-1}\exp\left\{R\sum_{\ell=1}^{\chi}\left[-\frac{\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)^3}{48\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}+\frac{\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)}{24\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\left(\left(\epsilon_1^{(\ell)}\right)^2+\left(\epsilon_2^{(\ell)}\right)^2+3\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}\right)\right]\right\}$$
$$\times\prod_{\ell=1}^{\chi}\mathcal{Z}_{\text{pert}}\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\Lambda e^{(\epsilon_1+\epsilon_2)/4},R\right)$$

$$\iota^*_{p_\ell,\vec{Y}}\mathrm{ch}_2\big(\mathcal{I}_1(m_1)\oplus\mathcal{I}_2(m_2)\big)=\left(a^{(\ell)}\right)^2-\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}\left(\left|Y_1^{(\ell)}\right|+\left|Y_2^{(\ell)}\right|\right)$$

$$\iota^*_{\vec{Y}}\mathrm{ch}\big(\tilde{\mathcal{L}}^{(I)}\big)=\exp\left[\sum_{\ell=1}^{\chi}n_I^{(\ell)}\left(-|Y|+\frac{\left(a^{(\ell)}\right)^2}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\right)\right]$$

$$\frac{1}{\mathcal{R}}\exp\left\{R\sum_{\ell=1}^{\chi}\left[-\frac{\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)^3}{48\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}+\frac{\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}}{24\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\left(\left(\epsilon_1^{(\ell)}\right)^2+\left(\epsilon_2^{(\ell)}\right)^2+3\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}\right)\right]\right\}$$
$$\times\exp\left[\sum_{\ell=1}^{\chi}\left(n_I^{(\ell)}+\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)\left(-|Y|+\frac{\left(a^{(\ell)}\right)^2}{\epsilon_1\epsilon_2}\right)\right]$$
$$\times\prod_{\ell=1}^{\chi}\mathcal{Z}_{\mathrm{pert}}\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\Lambda e^{\frac{1}{4}(\epsilon_1+\epsilon_2)},R\right)\mathcal{Z}_{\mathrm{inst}}\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},0,\Lambda e^{\frac{1}{4}(\epsilon_1+\epsilon_2)},R\right)$$

$$\frac{1}{\mathcal{R}}\exp\left\{-R\sum_{\ell=1}^{\chi}\frac{\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)^3}{48\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\right\}$$
$$\times\prod_{\ell=1}^{\chi}\mathcal{Z}_{\mathrm{pert}}\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\Lambda,R\right)\mathcal{Z}_{\mathrm{inst}}\left(a^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},n_I^{(\ell)},\Lambda,R\right)$$

$$g^{\epsilon_1,\epsilon_2}_{\mathrm{p},\mathrm{p}^{(I)}}(a)\coprod\sum_{\ell=1}^{\chi}\frac{\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)^3}{48\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\Lambda\left(K_X^{\mathrm{eq}}\right)^3$$

$$\mathcal{Z}_{\mathrm{pert}}(a,\epsilon_1,\epsilon_2,\Lambda,R)\sim\mathrm{PE}\left[-\frac{x^{-2}+x^2}{(t_1-1)(t_2-1)}\right]$$

$$\mathrm{PE}[f(x,t_1,t_2)]:=\exp\left(\sum_{m\geq1}\frac{1}{m}f(x^m,t_1^m,t_2^m)\right).$$

$$\mathcal{Z}_{\mathrm{pert}}(a,\epsilon_1,\epsilon_2,\Lambda,R)\sim\prod_{i,j=0}^{\infty}\left(1-x^{-2}t_1^it_2^j\right)\left(1-x^2t_1^it_2^j\right)$$

$$2a=m\epsilon_1+n\epsilon_2+\frac{2\pi is}{R},mn\geq0,m,n,s\in\mathbb{Z}$$

$$\mathcal{Z}_{\mathrm{pert}}(a,\epsilon_1,\epsilon_2,\Lambda,R)\sim\prod_{i,j=0}^{\infty}\left(1-x^{-2}t_1^{-i-1}t_2^{-j-1}\right)\left(1-x^2t_1^{-i-1}t_2^{-j-1}\right)$$

$$2a=m\epsilon_1+n\epsilon_2+\frac{2\pi is}{R},mn>0,m,n,s\in\mathbb{Z}$$

$$\mathcal{Z}_{\mathrm{pert}}(a,\epsilon_1,\epsilon_2,\Lambda,R)\sim\prod_{i,j=0}^{\infty}\frac{1}{\left(1-x^{-2}t_1^{-i-1}t_2^j\right)}\frac{1}{\left(1-x^2t_1^{-1-i}t_2^j\right)},$$

$$2a = m\epsilon_1 + n\epsilon_2 + \frac{2\pi i s}{R}, m < 0 \leq n \text{ or } n \leq 0 < m, m, n, s \in \mathbb{Z}$$

$$\mathcal{Z}_{\text{pert}}\,(a, \epsilon_1, \epsilon_2, \Lambda, R) \sim \prod_{i,j=0}^{\infty} \frac{1}{\left(1 - x^{-2} t_1^i t_2^{-1-j}\right)} \frac{1}{\left(1 - x^2 t_1^i t_2^{-1-j}\right)},$$

$$2a = m\epsilon_1 + n\epsilon_2 + \frac{2\pi i s}{R}, m \leq 0 < n \text{ or } n < 0 \leq m, m, n, s \in \mathbb{Z}.$$

$$2a = m\epsilon_1 + n\epsilon_2 + \frac{2\pi i s}{R}, mn > 0, m, n, s \in \mathbb{Z}$$

$$g_{\mathfrak{p},\mathfrak{p}^{(I)}}^{\epsilon_1,\epsilon_2}(a) = \prod_{\ell=1}^{\chi} \mathcal{Z}\left(a^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, n_I^{(\ell)}, \Lambda, R\right) \between g_{\mathfrak{p},\mathfrak{p}^{(I)}}^{\epsilon_1,\epsilon_2}(a)\,.$$

$$2a^{(\ell)} = m\epsilon_1^{(\ell)} + n\epsilon_2^{(\ell)} + \frac{2\pi i s}{R}, mn > 0, \ell = 1, \cdots, \chi, m, n, s \in \mathbb{Z}$$

$$\prod_{\ell=1}^{3} \mathcal{Z}_{\text{pert}}\left(a^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, \Lambda, R\right) \sim \text{PE}\left[-\sum_{\ell=1}^{3} \frac{x^2 \left(t_1^{(\ell)}\right)^{\mathfrak{p}_\ell} \left(t_2^{(\ell)}\right)^{\mathfrak{p}_{\ell+1}} + x^{-2} \left(t_1^{(\ell)}\right)^{-\mathfrak{p}_\ell} \left(t_2^{(\ell)}\right)^{-\mathfrak{p}_{\ell+1}}}{\left(t_1^{(\ell)} - 1\right)\left(t_1^{(\ell)} - 1\right)}\right]$$
$$=: \text{PE}\left[-x^2 \chi_{\mathfrak{p}}^{+}(t_1, t_2) - x^{-2} \chi_{\mathfrak{p}}^{-}(t_1, t_2)\right]$$

$$t_1^{(\ell)} := e^{R\epsilon_1^{(\ell)}}, t_2^{(\ell)} := e^{R\epsilon_2^{(\ell)}}$$

$$\chi_{\mathfrak{p}}^{+}(t_1, t_2) = \begin{cases} \sum_{j=0}^{-r} \sum_{i=0}^{-r-j} t_1^{i+\mathfrak{p}_1} t_2^{j+\mathfrak{p}_2}, & r \leq 0, \\ 0, & r = 1{,}2, \\ \sum_{j=2-r}^{-1} \sum_{i=-r+1-j}^{-1} t_1^{i+\mathfrak{p}_1} t_2^{j+\mathfrak{p}_2}, & r \geq 3, \end{cases}$$

$$\chi_{\mathfrak{p}}^{-}(t_1, t_2) = \begin{cases} \sum_{j=0}^{r} \sum_{i=0}^{k-j} t_1^{i-\mathfrak{p}_1} t_2^{j-\mathfrak{p}_2}, & r \geq 0, \\ 0, & r = -1, -2, \\ \sum_{j=2+r}^{-1} \sum_{i=r+1-j}^{-1} t_1^{i-\mathfrak{p}_1} t_2^{j-\mathfrak{p}_2}, & r \leq -3 \end{cases}$$

$$\text{PE}\left[-x^2 \chi_{\mathfrak{p}}^{+}(t_1, t_2)\right] = \begin{cases} \prod_{j=0}^{-r} \prod_{i=0}^{-r-j} \left(1 - x^2 t_1^{i+\mathfrak{p}_1} t_2^{j+\mathfrak{p}_2}\right), & r \leq 0 \\ 1, & r = 1{,}2, \\ \prod_{j=2-r}^{-1} \prod_{i=-r+1-j}^{-1} \left(1 - x^2 t_1^{i+\mathfrak{p}_1} t_2^{j+\mathfrak{p}_2}\right), & r \geq 3, \end{cases}$$

$$\mathrm{PE}\left[-x^{-2}\chi_{\mathfrak{p}}^{-}(t_1,t_2)\right]=\begin{cases}\prod_{j=0}^{r}\prod_{i=0}^{r-j}\left(1-x^{-2}t_1^{i-\mathfrak{p}_1}t_2^{j-\mathfrak{p}_2}\right), & r\geq 0,\\ 1, & r=-1,-2\\ \prod_{j=2+r}^{-1}\prod_{i=r+1-j}^{-1}\left(1-x^{-2}t_1^{i-\mathfrak{p}_1}t_2^{j-\mathfrak{p}_2}\right), & r\leq -3\end{cases}$$

$$2a^{(g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}(l)}(a))}=m\epsilon_1^{(g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}(l)}(a))}+n\epsilon_2^{(g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}(l)}(a))}+\frac{2\pi is}{R},\ell=1,\cdots,\chi,s\in\mathbb{Z}$$

$$\lim_{a\to 0}\frac{\mathcal{Z}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I,\Lambda,R\right)}{\mathcal{Z}(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I,\Lambda,R)}=-\mathrm{sgn}\left(\mathrm{Re}(\epsilon_1)\right)$$

$$\lim_{a\to 0}\frac{\mathcal{Z}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I,\Lambda,R\right)}{\mathcal{Z}(a-\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I,\Lambda,R)}=-\mathrm{sgn}\left(\mathrm{Re}(\epsilon_2)\right),$$

$$a^{(m,n)}:=m\epsilon_1+n\epsilon_2,\hat{a}^{(m,n)}:=m\epsilon_1-n\epsilon_2.$$

$$\lim_{a\to 0}\frac{\mathcal{Z}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I,\Lambda,R\right)}{\mathcal{Z}(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I,\Lambda,R)}=\lim_{a\to 0}\frac{\mathcal{Z}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda e^{-\frac{1}{4}n_I},R\right)}{\mathcal{Z}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda e^{-\frac{1}{4}n_I},R\right)},$$

$$\begin{aligned}&\lim_{a\to 0}\frac{\mathcal{Z}_{\mathrm{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\mathrm{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}\\ &=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\\ &\times\lim_{a\to 0}\exp\left(-\sum_{l\geq 1}\frac{1}{l(t_1^l-1)(t_2^l-1)}\left(x^{-2l}t_1^{-lm}t_2^{-ln}+x^{2l}t_1^{lm}t_2^{ln}-x^{-2l}t_1^{-lm}t_2^{ln}-x^{2l}t_1^{lm}t_2^{-ln}\right)\right)\\ &=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\lim_{a\to 0}\mathrm{PE}\left[-\frac{x^{-2}t_1^{m}t_2^{-n}+x^{2}t_1^{m}t_2^{n}-x^{-2}t_1^{-m}t_2^{n}-x^{2}t_1^{m}t_2^{-n}}{(t_1-1)(t_2-1)}\right],\end{aligned}$$

$$\begin{aligned}&\lim_{a\to 0}\frac{\mathcal{Z}_{\mathrm{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\mathrm{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}\\ &=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\lim_{a\to 0}\mathrm{PE}\left[-\sum_{i,j=0}^{\infty}\left(x^{-2}t_1^{-m+i}t_2^{-n+j}+x^{2}t_1^{m+i}t_2^{n+j}-x^{-2}t_1^{-m+i}t_2^{n+j}-x^{2}t_1^{m+i}t_2^{-n+j}\right)\right]\\ &=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\lim_{a\to 0}\prod_{i,j=0}^{\infty}\frac{\left(1-x^{-2}t_1^{-m+i}t_2^{-n+j}\right)\left(1-x^{2}t_1^{m+i}t_2^{n+j}\right)}{\left(1-x^{-2}t_1^{-m+i}t_2^{n+j}\right)\left(1-x^{2}t_1^{m+i}t_2^{-n+j}\right)},\end{aligned}$$

$$\lim_{a\to 0}\frac{1}{1-x^{-2}}\frac{\mathcal{Z}_{\mathrm{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\mathrm{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\underbrace{\prod_{i=-m}^{m-1}\prod_{j=-n}^{n-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right)$$

$$\lim_{a\to 0}\frac{1}{1-x^2}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\underbrace{\prod_{i=m}^{-m-1}\prod_{j=n}^{-n-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right)$$

$$\begin{aligned}
&\lim_{a\to 0}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}\\
=&\mathcal{R}^{-4mn}(t_1t_2)^{mn}\lim_{a\to 0}\text{PE}\left[-\frac{x^{-2}t_1^{-m-1}t_2^{-n-1}+x^2t_1^{m-1}t_2^{n-1}-x^{-2}t_1^{-m-1}t_2^{n-1}-x^2t_1^{m-1}t_2^{-n-1}}{(1-t_1^{-1})(1-t_2^{-1})}\right]\\
=&\mathcal{R}^{-4mn}(t_1t_2)^{mn}\lim_{a\to 0}\prod_{i,j=0}^{\infty}\frac{\left(1-x^{-2}t_1^{-m-1-i}t_2^{-n-1-j}\right)\left(1-x^2t_1^{m-1-i}t_2^{n-1-j}\right)}{\left(1-x^{-2}t_1^{-m-1-i}t_2^{n-1-j}\right)\left(1-x^2t_1^{m-1-i}t_2^{-n-1-j}\right)},
\end{aligned}$$

$$\lim_{a\to 0}\frac{1}{1-x^2}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\underbrace{\prod_{i=-m}^{m-1}\prod_{j=-n}^{n-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right)$$

$$\lim_{a\to 0}\frac{1}{1-x^{-2}}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=\mathcal{R}^{-4mn}(t_1t_2)\underbrace{\prod_{i=m}^{m}\prod_{j=n}^{-m-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right)$$

$$\begin{aligned}
&\lim_{a\to 0}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}\\
=&\mathcal{R}^{-4mn}(t_1t_2)^{mn}\lim_{a\to 0}\text{PE}\left[-\frac{x^{-2}t_1^{-m-1}t_2^{-n}+x^2t_1^{m-1}t_2^{n}-x^{-2}t_1^{-m-1}t_2^{n}-x^2t_1^{m-1}t_2^{-n}}{(1-t_1^{-1})(t_2-1)}\right]\\
=&\mathcal{R}^{-4mn}(t_1t_2)^{mn}\lim_{a\to 0}\prod_{i,j=0}^{\infty}\frac{\left(1-x^{-2}t_1^{-m-1-i}t_2^{n+j}\right)\left(1-x^2t_1^{m-1-i}t_2^{-n+j}\right)}{\left(1-x^{-2}t_1^{-m-1-i}t_2^{-n+j}\right)\left(1-x^2t_1^{m-1-i}t_2^{n+j}\right)},
\end{aligned}$$

$$\lim_{a\to 0}\frac{1}{1-x^2}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\underbrace{\prod_{i=-m}^{m-1}\prod_{j=-n}^{n-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right)$$

$$\lim_{a\to 0}\frac{1}{1-x^{-2}}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\underbrace{\prod_{i=m}^{-m-1}\prod_{j=n}^{-n-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right)$$

$$\begin{aligned}
&\lim_{a\to 0}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}\\
=&\mathcal{R}^{-4mn}(t_1t_2)^{mn}\lim_{a\to 0}\text{PE}\left[-\frac{x^{-2}t_1^{-m}t_2^{-n-1}+x^2t_1^{m}t_2^{n-1}-x^{-2}t_1^{-m}t_2^{n-1}-x^2t_1^{m}t_2^{-n-1}}{(t_1-1)(1-t_2^{-1})}\right]\\
=&\mathcal{R}^{-4mn}(t_1t_2)^{mn}\lim_{a\to 0}\prod_{i,j=0}^{\infty}\frac{\left(1-x^{-2}t_1^{-m+i}t_2^{n-1-j}\right)\left(1-x^2t_1^{m+i}t_2^{-n-1-j}\right)}{\left(1-x^{-2}t_1^{-m+i}t_2^{-n-1-j}\right)\left(1-x^2t_1^{m+i}t_2^{n-1-j}\right)},
\end{aligned}$$

$$\lim_{a\to 0}\frac{1}{1-x^{-2}}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\underbrace{\prod_{i=-m}^{m-1}\prod_{j=-n}^{n-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right)$$

$$\lim_{a\to 0}\frac{1}{1-x^{2}}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=\mathcal{R}^{-4mn}(t_1t_2)^{mn}\underbrace{\prod_{i=m}^{-m-1}\prod_{j=n}^{-n-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right)$$

$$\lim_{a\to 0}\frac{1}{1-x^{2\text{sgn}(\text{Re}(\epsilon_1))}}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=(\mathcal{R}^{-4}t_1t_2)^{mn}\underbrace{\prod_{i=-m}^{m-1}\prod_{j=-n}^{n-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right)$$

$$\lim_{a\to 0}\frac{1}{1-x^{-2\text{sgn}(\text{Re}(\epsilon_1))}}\frac{\mathcal{Z}_{\text{pert}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}_{\text{pert}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=(\mathcal{R}^{-4}t_1t_2)^{mn}\underbrace{\prod_{i=m}^{-m-1}\prod_{j=n}^{-n-1}}_{(i,j)\neq\{(0,0)\}}\left(1-t_1^it_2^j\right).$$

$$\lim_{a\to 0}(1-x^2)\frac{\mathcal{Z}_{\text{inst}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I=0,\Lambda,R\right)}{\mathcal{Z}_{\text{inst}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I=0,\Lambda,R\right)}=-\frac{\mathcal{R}^{4mn}T_{m,n}(t_1,t_2)}{t_1^{-m}t_2^{-n}-t_1^mt_2^n}$$

$$\lim_{a\to 0}(1-x^{-2})\frac{\mathcal{Z}_{\text{inst}}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I=0,\Lambda,R\right)}{\mathcal{Z}_{\text{inst}}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,n_I=0,\Lambda,R\right)}=-\frac{\mathcal{R}^{4mn}T_{-m,-n}(t_1,t_2)}{t_1^{-m}t_2^{-n}-t_1^mt_2^n},$$

$$T_{m,n}(t_1,t_2)=-\frac{(1+t_1^mt_2^n)(1-t_1^{-m}t_2^{-n})}{(t_1t_2)^{mn}}\underbrace{\prod_{i=-m}^{m-1}\prod_{j=-n}^{n-1}}_{(i,j)\neq\{(0,0)\}}\frac{1}{\left(1-t_1^it_2^j\right)}.$$

$$\lim_{a\to 0}\frac{\mathcal{Z}\left(a+a^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}{\mathcal{Z}\left(a+\hat{a}^{(m,n)}/2,\epsilon_1,\epsilon_2,\Lambda,R\right)}=-\text{sgn}\big(\text{Re}(\epsilon_1)\big)$$

$$\text{Re}\left(\epsilon_1^{(\ell)}\right)\text{Re}\left(\epsilon_2^{(\ell+1)}\right)<0$$

$$\text{Res}_{a=0}\mathcal{P}_\ell g_{\mathfrak{p},\mathfrak{p}^{[I]}}^{\epsilon_1,\epsilon_2}(a)=-\text{Res}_{a=0}g_{\mathfrak{p},\mathfrak{p}^{[I]}}^{\epsilon_1,\epsilon_2}(a)$$

$$\text{Res}_{a=s\pi i/R}\mathcal{P}_\ell g_{\mathfrak{p},\mathfrak{p}^{[I]}}^{\epsilon_1,\epsilon_2}(a)=-\text{Res}_{a=s\pi i/R}g_{\mathfrak{p},\mathfrak{p}^{[I]}}^{\epsilon_1,\epsilon_2}(a)$$

$$\text{Res}_{a=a^*}\,da\,g_{\mathfrak{p},\mathfrak{p}^{(I)}}^{\epsilon_1,\epsilon_2}(a)$$

$$2a^{(\ell)}=m\epsilon_1^{(\ell)}+n\epsilon_2^{(\ell)}+2\pi is/R,s\in\mathbb{Z},$$

$$2a^{(\ell)}=2a+f_\ell(\{\mathfrak{p}\}).$$

$$\begin{aligned}2a&=-f_1(\{\mathfrak{p}\})+(f_1(\{\mathfrak{p}\})-f_\ell(\{\mathfrak{p}\}))+m\epsilon_1^{(\ell)}+n\epsilon_2^{(\ell)}\\&=:-f_1(\{\mathfrak{p}\})+n_1\epsilon_1+n_2\epsilon_2\end{aligned}$$

$$\begin{aligned}&\mathrm{Res}_{2a=-f_1(\{\mathfrak{p}\})+n_1\epsilon_1+n_2\epsilon_2}\, \mathrm{d}a g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)\\ =&\mathrm{Res}_{2a=n_1\epsilon_1+n_2\epsilon_2}\, \mathrm{d}a g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a-f_1(\{\mathfrak{p}\}))\\ =&\mathrm{Res}_{2a=n_1\epsilon_1+n_2\epsilon_2}\, \mathrm{d}a \prod_{\ell=1}^{\chi} Z\left(a+f_\ell(\{\mathfrak{p}\})-f_1(\{\mathfrak{p}\}),\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},n_I^{(\ell)},\Lambda,R\right)\end{aligned}$$

$$\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}}\sum_{a^*\in S}\mathrm{Res}_{a=a^*}\, \mathrm{d}a f(\boldsymbol{k}) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)=\sum_{\{\mathfrak{p}\}\in S_{\boldsymbol{\mu}}}(\underset{a=0}{\mathrm{Res}}+\underset{a=\pi i/R}{\mathrm{Res}})\mathrm{d}a f([\{\mathfrak{p}\}]) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$

$$\bar{S}:=\left\{a \mid 2a^{(\ell)}=\epsilon_1^{(\ell)}m+\epsilon_2^{(\ell)}n+2c, m,n\in\mathbb{Z};\ell=1,\cdots,\chi;c=0,\pi i/R\right\}$$

$$2a^{(\ell,m,n)}+2c:=(m-\mathfrak{p}_\ell)\epsilon_1^{(\ell)}+(n-\mathfrak{p}_{\ell+1})\epsilon_2^{(\ell)}+2c$$

$$(\ell,m,n)\sim(\ell',m',n')\Leftrightarrow a^{(\ell,m,n)}=a^{(\ell',m',n')}$$

$$(m-\mathfrak{p}_\ell,n-\mathfrak{p}_{\ell+1})\bar{\epsilon}^{(\ell)}=(m'-\mathfrak{p}_{\ell'},n'-\mathfrak{p}_{\ell'+1})\bar{\epsilon}^{(\ell')}$$

$$\bar{\epsilon}^{(\ell)}=\left(\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}\right)^{\mathrm{T}}$$

$$(m-\mathfrak{p}_\ell,n-\mathfrak{p}_{\ell+1})A^{\ell,\ell'}=(m'-\mathfrak{p}_{\ell'},n'-\mathfrak{p}_{\ell'+1})$$

$$\sum_{a^*\in S}\underset{a=a^*}{\mathrm{Res}}\,\mathrm{d}a=\sum_{c}\sum_{[(\ell,m,n)]}\underset{a=a^{(\ell,m,n)}+c}{\mathrm{Res}}\mathrm{d}a=\sum_{c}\sum_{\ell=1}^{\chi}\sum_{(m,n)\in\mathbb{Z}^2}\frac{1}{\Delta^{[(\ell,m,n)]}}\underset{a=a^{(\ell,m,n)}+c}{\mathrm{Res}}\,\mathrm{d}a$$

$$(\ell,m,n)\sim(\ell',m',n')$$

$$\sum_{a^*\in S}\underset{a=a^*}{\mathrm{Res}}\,\mathrm{d}a=\frac{1}{\chi}\sum_{c}\sum_{\ell=1}^{\chi}\sum_{(m,n)\in\mathbb{Z}^2}\underset{a=a^{(\ell,m,n)}+c}{\mathrm{Res}}\,\mathrm{d}a.$$

$$\begin{aligned}&\mathrm{Res}_{a=a^{(\ell,m,n)}+c}\,\mathrm{d}a f(\boldsymbol{k})\prod_{\ell'=1}^{\chi}\mathcal{Z}\left(a+\frac{1}{2}(\mathfrak{p}_{\ell'},\mathfrak{p}_{\ell'+1})\epsilon^{\prime\ell'},\epsilon_1^{(\ell')},\epsilon_2^{(\ell')},n_I^{(\ell')},\Lambda,R\right)\\ =&\mathrm{Res}_{a=c}\,\mathrm{d}a f(\boldsymbol{k})\prod_{\ell'=1}^{\chi}\mathcal{Z}\left(a+\frac{1}{2}\left((m-\mathfrak{p}_\ell,n-\mathfrak{p}_{\ell+1})A^{\ell,\ell'}+(\mathfrak{p}_{\ell'},\mathfrak{p}_{\ell'+1})\right)\bar{\epsilon}^{(\ell')},\epsilon_1^{(\ell')},\epsilon_2^{(\ell')},n_I^{(\ell')},\Lambda,R\right)\end{aligned}$$

$$(m-\mathfrak{p}_\ell,n-\mathfrak{p}_{\ell+1})A^{\ell,\ell'}+(\mathfrak{p}_{\ell'},\mathfrak{p}_{\ell'+1})=:\left(\bar{\mathfrak{p}}^{[\ell]}_{\ell'},\bar{\mathfrak{p}}^{[\ell]}_{\ell'+1}\right)$$

$$[\{\mathfrak{p}\}]=\left[\left\{\bar{\mathfrak{p}}^{[\ell]}\right\}\right],$$

$$\bar{\mathfrak{p}}^{\ell\ell}_{\ell'+1}=(m-\mathfrak{p}_\ell)\left(A^{\ell,\ell'}\right)_{12}+(n-\mathfrak{p}_{\ell+1})\left(A^{\ell,\ell'}\right)_{22}+\mathfrak{p}_{\ell'+1},$$

$$\bar{\mathfrak{p}}^{[\ell]}_{\ell'+1}=(m-\mathfrak{p}_\ell)\left(A^{\ell,\ell'+1}\right)_{11}+(n-\mathfrak{p}_{\ell+1})\left(A^{\ell,\ell'+1}\right)_{21}+\mathfrak{p}_{\ell'+1}$$

$$\left(A^{\ell,\ell'+1}\right)_{11}=\left(A^{\ell,\ell'}\right)_{12},\left(A^{\ell,\ell'+1}\right)_{21}=\left(A^{\ell,\ell'}\right)_{22}$$

$$\bar{\mathfrak{p}}^{[\ell]}_{\ell'} - \mathfrak{p}_{\ell'} = (m-\mathfrak{p}_\ell)\left(A^{\ell,\ell'}\right)_{11} + (n-\mathfrak{p}_{\ell+1})\left(A^{\ell,\ell'}\right)_{21}$$
$$= (m-\mathfrak{p}_\ell)\sum_{i=1}^{2}\left(A^{\ell,1}\right)_{1i}\left(A^{1,\ell'}\right)_{i1} + (n-\mathfrak{p}_{\ell+1})\sum_{i=1}^{2}\left(A^{\ell,1}\right)_{2i}\left(A^{1,\ell'}\right)_{i1}.$$

$$\bar{\mathfrak{p}}^{\ell\ell]}_{\ell'} - \mathfrak{p}_{\ell'} = \left((m-\mathfrak{p}_\ell)\left(A^{\ell,1}\right)_{11} + (n-\mathfrak{p}_{\ell+1})\left(A^{\ell,1}\right)_{21}\right)n^1_{\ell'}$$
$$-\left((m-\mathfrak{p}_\ell)\left(A^{\ell,1}\right)_{12} + (n-\mathfrak{p}_{\ell+1})\left(A^{\ell,1}\right)_{22}\right)n^2_{\ell'}$$

$$\frac{1}{\chi}\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}}\sum_{c}\sum_{\ell=1}^{\chi}\sum_{(m,n)\in\mathbb{Z}^2}\operatorname{Res}_{a=c}\, \mathrm{d}a f(\boldsymbol{k}) g^{\epsilon_1,\epsilon_2}_{\bar{\mathfrak{p}}[\ell,\mathfrak{p}^{(I)}}(a)$$

$$\bar{\mathfrak{p}}^{[\ell]}_{l} = m, \bar{\mathfrak{p}}^{[\ell]}_{\ell+1} = n$$

$$\frac{1}{\chi}\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}}\sum_{c}\sum_{\ell=1}^{\chi}\sum_{\left(\bar{\mathfrak{p}}^{[\ell]}_{l},\bar{\mathfrak{p}}^{[\ell]}_{\ell+1}\right)\in\mathbb{Z}^2}\operatorname{Res}_{a=c}\, \mathrm{d}a f(\boldsymbol{k}) g^{\epsilon_1,\epsilon_2}_{\bar{\mathfrak{p}}^{[\ell]},\mathfrak{p}^{(I)}}(a)$$

$$\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}}\sum_{\left(\bar{\mathfrak{p}}^{[\ell]}_{\ell},\bar{\mathfrak{p}}^{[\ell]}_{\ell+1}\right)\in\mathbb{Z}^2} = \sum_{\{\mathfrak{p}\}\in S_{\boldsymbol{\mu}}}.$$

$$S^{\pm} := \left\{a \mid 2a^{(\ell)} = \epsilon^{(\ell)}_1 m + \epsilon^{(\ell)}_2 n + 2c, m, n \in \mathbb{Z}^{\pm}; \ell = 1, \cdots, \chi; c = 0, \pi i/R\right\}$$

$$\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}}\sum_{a^*\in S^{\pm}}\operatorname{Res}_{a=a^*}\, \mathrm{d}a f(\boldsymbol{k}) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = \sum_{\{\mathfrak{p}\}\in S^{\pm}_{\boldsymbol{\mu}}}\left(\operatorname{Res}_{a=0} + \operatorname{Res}_{a=\pi i/R}\right)\mathrm{d}a f([\{\mathfrak{p}\}]) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a),$$

$$S^{\pm}_{\mu} := \left\{\{\mathfrak{p}\} \in S_{\mu} \mid \exists \ell, \text{ s.t. } \mathfrak{p}_\ell, \mathfrak{p}_{\ell+1} \in \mathbb{Z}^{\pm}\right\}$$

$$\sum_{a^*\in S^{+}}\operatorname*{Res}_{a=a^*}\mathrm{d}a = \sum_{c}\sum_{[(\ell,m,n)]}\operatorname*{Res}_{a=a^{(\ell,m,n)}+c}\mathrm{d}a = \sum_{c}\sum_{\ell=1}^{\chi}\sum_{m,n\in\mathbb{Z}^+}\frac{1}{\Delta^{[(\ell,m,n)]}}\operatorname*{Res}_{a=a^{(\ell,m,n)}+c}\mathrm{d}a,$$

$$\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}}\sum_{c}\sum_{\ell=1}^{\chi}\sum_{\bar{\mathfrak{p}}^{[\ell]}_{\ell},\bar{\mathfrak{p}}^{[\ell]}_{\ell+1}\in\mathbb{Z}^+}\frac{1}{\Delta^{\left[\left(\ell,\bar{\mathfrak{p}}^{[\ell]}_{\ell},\bar{\mathfrak{p}}^{[\ell]}_{\ell+1}\right)\right]}}\operatorname{Res}\mathrm{d}a f(\boldsymbol{k}) g^{\epsilon_1,\epsilon_2}_{\bar{\mathfrak{p}}^{[\ell]},\mathfrak{p}^{(I)}}(a)$$
$$= \sum_{c}\sum_{\ell=1}^{\chi}\sum_{\left\{\bar{\mathfrak{p}}^{[\ell]}\right\}\in S^{[\ell]}_{\boldsymbol{\mu}}}\frac{1}{\Delta^{\left[\left(\ell,\bar{\mathfrak{p}}^{[\ell]}_{\ell},\bar{\mathfrak{p}}^{[\ell]}_{\ell+1}\right)\right]}}\operatorname{Res}\mathrm{d}a f(\boldsymbol{k}) g^{\epsilon_1,\epsilon_2}_{\bar{\mathfrak{p}}^{[\ell]},\mathfrak{p}^{(I)}}(a),$$

$$S^{[\ell]}_{\mu} := \left\{\{\mathfrak{p}\} \in S_{\mu} \mid \mathfrak{p}_\ell, \mathfrak{p}_{\ell+1} \in \mathbb{Z}^{+}\right\}$$

$$\bar{\mathfrak{p}}^{[\ell]} = \bar{\mathfrak{p}}^{[\ell']},$$

$$\bar{\mathfrak{p}}^{[\ell]}_{1,2} = \bar{\mathfrak{p}}^{[\ell']}_{1,2}$$

$$\overline{\mathfrak{p}}_1^{[\ell]} = (m - \mathfrak{p}_\ell)(A^{\ell,1})_{11} + (n - \mathfrak{p}_{\ell+1})(A^{\ell,1})_{21} + \mathfrak{p}_1,$$
$$\overline{\mathfrak{p}}_2^{[\ell]} = (m - \mathfrak{p}_\ell)(A^{\ell,2})_{11} + (n - \mathfrak{p}_{\ell+1})(A^{\ell,2})_{21} + \mathfrak{p}_2.$$

$$(m - \mathfrak{p}_\ell)(A^{\ell,1})_{11} + (n - \mathfrak{p}_{\ell+1})(A^{\ell,1})_{21} = (m' - \mathfrak{p}_{\ell'})(A^{\ell',1})_{11} + (n' - \mathfrak{p}_{\ell'+1})(A^{\ell',1})_{21},$$
$$(m - \mathfrak{p}_\ell)(A^{\ell,2})_{11} + (n - \mathfrak{p}_{\ell+1})(A^{\ell,2})_{21} = (m' - \mathfrak{p}_{\ell'})(A^{\ell',2})_{11} + (n' - \mathfrak{p}_{\ell'+1})(A^{\ell',2})_{21}.$$

$$(A^{\ell,2})_{11} = (A^{\ell,1})_{12}, (A^{\ell,2})_{21} = (A^{\ell,1})_{22}$$

$$(m - \mathfrak{p}_\ell, n - \mathfrak{p}_{\ell+1})A^{\ell,1} = (m' - \mathfrak{p}_{\ell'}, n' - \mathfrak{p}_{\ell'+1})A^{\ell',1}$$

$$\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \sum_{a^*\in S^\pm} \operatorname{Res}_{a=a^*} \mathrm{d}aH(\pm r) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$
$$= \mp \sum_c \left(\sum_{\{\mathfrak{p}\}\in S_{\text{unstable}}} + \sum_{\{\mathfrak{p}\}\in S_{\text{semi-stable}}} + 2 \sum_{\{\mathfrak{p}\}\in S_{\text{stable}}} \right) \operatorname{Res}_{a=c} \mathrm{d}a g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$
$$\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \sum_{a^*\in S^\pm} \operatorname{Res}_{a=a^*} \mathrm{d}aH(\mp r) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = \mp \sum_c \sum_{\{\mathfrak{p}\}\in S_{\text{unstable}}} \operatorname{Res}_{a=c} \mathrm{d}a g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$

$$S_{\text{stable}} := \{\{\mathfrak{p}\} \in S_{\boldsymbol{\mu}} \mid \mathfrak{p}_\ell > 0, \mathfrak{p}_\ell + \mathfrak{p}_{\ell+1} > \mathfrak{p}_{\ell+2}, \forall \ell\},$$
$$S_{\text{unstable}} := \{\{\mathfrak{p}\} \in S_{\boldsymbol{\mu}} \mid \mathfrak{p}_\ell > 0, \forall \ell; \exists \ell, \text{ s.t. } \mathfrak{p}_\ell + \mathfrak{p}_{\ell+1} < \mathfrak{p}_{\ell+2}\},$$
$$S_{\text{semi-stable}} := \{\{\mathfrak{p}\} \in S_{\boldsymbol{\mu}} \mid \mathfrak{p}_\ell > 0, \forall \ell; \exists \ell, \text{ s.t. } \mathfrak{p}_\ell + \mathfrak{p}_{\ell+1} = \mathfrak{p}_{\ell+2}\}.$$

$$\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \sum_{a^*\in S^+} \operatorname{Res}_{a=a^*} \mathrm{d}aH(r) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = \sum_c \sum_{\{\mathfrak{p}\}\in S^+_{\boldsymbol{\mu}}} \operatorname{Res}_{a=c} \mathrm{d}aH(r) g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$
$$= \sum_c \sum_{\{\mathfrak{p}\}\in \tilde{S}^+} \operatorname{Res}_{a=c} \mathrm{d}a g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$

$$\tilde{S}^+ := \{\{\mathfrak{p}\} \in S_{\boldsymbol{\mu}} \mid \exists \ell, \text{ s.t. } \mathfrak{p}_\ell, \mathfrak{p}_{\ell+1} \in \mathbb{Z}^+, r > 0\}$$

$$\tilde{S}^+ = \tilde{S}^+_0 \sqcup \tilde{S}^+_1 \sqcup \tilde{S}^+_2 \sqcup \tilde{S}^+_3,$$

$$\tilde{S}^+_0 := \{\{\mathfrak{p}\} \in S_\mu \mid \mathfrak{p}_\ell > 0, \forall \ell\},$$

$$\tilde{S}^+_\ell := \{\{\mathfrak{p}\} \in S_{\boldsymbol{\mu}} \mid \mathfrak{p}_\ell < 0, \mathfrak{p}_{\ell+1}, \mathfrak{p}_{\ell+2} > 0, r > 0\}$$

$$\sum_c \sum_{\{\mathfrak{p}\}\in \tilde{S}^+_0} \operatorname{Res}_{a=c} \mathrm{d}g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = \sum_c \left(\sum_{\{\mathfrak{p}\}\in S_{\text{unstable}}} + \sum_{\{\mathfrak{p}\}\in S_{\text{semi-stable}}} + \sum_{\{\mathfrak{p}\}\in S_{\text{stable}}} \right) \operatorname{Res}_{a=c} \mathrm{d}g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$

$$\sum_c \sum_{\{\mathfrak{p}\}\in \tilde{S}^+_l} \operatorname{Res}_{a=c} \mathrm{d}g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$
$$= -\sum_c \left(\sum_{\{\mathfrak{p}\}\in S^{[\ell+1]}_{\text{semi-stable}}} + \sum_{\{\mathfrak{p}\}\in S^{[\ell+2]}_{\text{semi-stable}}} + \sum_{\{\mathfrak{p}\}\in S^{[\ell+1]}_{\text{unstable}}} + \sum_{\{\mathfrak{p}\}\in S^{[\ell+2]}_{\text{unstable}}} + \sum_{\{\mathfrak{p}\}\in S_{\text{stable}}} \right) \operatorname{Res}_{a=c} \mathrm{d}g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}'}(a),$$

$$S^{[\ell]}_{\text{semi-stable}} := \{\{\mathfrak{p}\} \in S_{\text{semi-stable}} \mid \mathfrak{p}_\ell = \mathfrak{p}_{\ell-1} + \mathfrak{p}_{\ell+1}\}$$
$$S^{[\ell]}_{\text{unstable}} := \{\{\mathfrak{p}\} \in S_{\text{unstable}} \mid \mathfrak{p}_\ell > \mathfrak{p}_{\ell-1} + \mathfrak{p}_{\ell+1}\}$$

$$\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \sum_{a^*\in S^+} \operatorname{Res}_{a=a^*} \mathrm{d}aH(-r)g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = \sum_c \sum_{\{\mathfrak{p}\}\in S^+_{\boldsymbol{\mu}}} \operatorname{Res}_{a=c} \mathrm{d}aH(-r)g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$
$$= \sum_c \sum_{\{\mathfrak{p}\}\in \tilde{S}^-} \operatorname{Res}_{a=c} \mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$

$$\tilde{S}^- := \{\{\mathfrak{p}\} \in S_{\boldsymbol{\mu}} \mid \exists \ell, \text{ s.t. } \mathfrak{p}_\ell, \mathfrak{p}_{\ell+1} \in \mathbb{Z}^+, r < 0\}$$

$$\tilde{S}^- = \tilde{S}^-_1 \sqcup \tilde{S}^-_2 \sqcup \tilde{S}^-_3,$$

$$\tilde{S}^-_\ell := \{\{\mathfrak{p}\} \in S_{\boldsymbol{\mu}} \mid \mathfrak{p}_\ell < 0, \mathfrak{p}_{\ell+1}, \mathfrak{p}_{\ell+2} > 0, r < 0\}.$$

$$\sum_c \sum_{\{\mathfrak{p}\}\in \tilde{S}^-_\ell} \operatorname{Res}_{a=c} \mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = -\sum_c \sum_{\{\mathfrak{p}\}\in S^{[\ell]}_{\text{unstable}}} \operatorname{Res}_{a=c} \mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$

$$-\frac{1}{2}\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \sum_{a^*\in S^+} \operatorname{Res}_{a=a^*} \mathrm{d}a\operatorname{sgn}(r)g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = \sum_{c\in\{0,\pi i/R\}} \sum_{\{\mathfrak{p}\}} \Theta_{\boldsymbol{\mu}}(\mathfrak{p})\operatorname{Res}_{a=c} \mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a)$$
$$-\frac{1}{2}\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \sum_{a^*\in S^-} \operatorname{Res}_{a=a^*} \mathrm{d}a\operatorname{sgn}(r)g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = \sum_{c\in\{0,\pi i/R\}} \sum_{\{\mathfrak{p}\}} \Theta_{\boldsymbol{\mu}}(\mathfrak{p})\operatorname{Res}_{a=c} \mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a).$$

$$-\frac{1}{4}\sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} \sum_{a^*\in S} \operatorname{Res}_{a=a^*} \mathrm{d}a\operatorname{sgn}(r)g^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a) = \sum_{c\in\{0,\pi i/R\}} \sum_{\{\mathfrak{p}\}} \Theta_{\boldsymbol{\mu}}(\mathfrak{p})\operatorname*{Res}_{a=c} \mathrm{d}ag^{\epsilon_1,\epsilon_2}_{\mathfrak{p},\mathfrak{p}^{(I)}}(a).$$

$$S^{\text{twisted},\Omega}_{\text{SYM}} = \frac{1}{g^2_{5d}}\int_{S^1\times X} \sqrt{g_5}\mathcal{L}^{\text{twisted},\Omega}_{\text{SYM}} + \frac{\log(\mathcal{R}^4)}{8\pi^2}\int_X F\wedge F,$$

$$\mathcal{L}^{\text{twisted},\Omega}_{\text{SYM}} = \operatorname{tr}\Big(\big(F^+_{\mu\nu}\big)^2 + \big(F_{\mu5} + v^\nu F_{\nu\mu}\big)(F^{\mu5} + v_\nu F^{\nu\mu}) - \frac{1}{4}D_{\mu\nu}D^{\mu\nu} + D_\mu\sigma D^\mu\sigma$$
$$+(D_5\sigma + \mathcal{L}^A_v\sigma)(D_5\sigma + \mathcal{L}^A_v\sigma) + 2i\chi^{\mu\nu}D_\mu\psi_\nu - \frac{\sqrt{2}}{2}i\chi^{\mu\nu}[D_5 + \mathcal{L}^A_v - \sigma, \chi_{\mu\nu}]$$
$$-\frac{i}{2}\eta[D_5 + \mathcal{L}^A_v - \sigma, \eta] + i\psi^\mu D_\mu\eta + \frac{i}{2}\psi^\mu[D_5 + \mathcal{L}^A_v + \sigma, \psi_\mu]\Big)$$

$$\mathcal{L}^A_v := \mathcal{L}_v - iv^\mu A_\mu$$

$$\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\operatorname{tr}\Big(\big(F_{\mu5} + v^\nu F_{\nu\mu}\big)(F^{\mu5} + v_\nu F^{\nu\mu}) + D_\mu\sigma D^\mu\sigma\Big)$$

$$v^\mu F_{\mu\nu} + F_{5\nu} = v^\mu\partial_\mu A_\nu - v^\mu D_\nu A_\mu + \partial_5 A_\nu - D_\nu A_5$$
$$= v^\mu\partial_\mu A_\nu + \partial_\nu v^\mu A_\mu - D_\nu\big(v^\mu A_\mu + A_5\big) + \partial_5 A_\nu$$
$$= (\mathcal{L}_v A)_\nu - D_\nu\big(v^\mu A_\mu + A_5\big) + \partial_5 A_\nu$$

$$\mathcal{L}_v A = V^I(Z)\partial_I A,$$

$$v^\mu F_{\mu\nu} + F_{5\nu} = (\dot{Z}^I + V^I)\delta_I A - D_\nu\Big(v^\mu A_\mu + A_5 - \alpha_I(\dot{Z}^I + V^I)\Big).$$

$$\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\mathrm{tr}\Big(\big(F_{\mu 5}+v^\nu F_{\nu\mu}\big)\big(F^{\mu 5}+v_\nu F^{\nu\mu}\big)+D_\mu\sigma D^\mu\sigma\Big)$$
$$=\int_{S^1} \mathrm{d}t g_{IJ}(\dot{Z}^I+V^I)(\dot{Z}^J+V^J)+\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\mathrm{tr}\big(D_\mu\Phi_V D^\mu\bar{\Phi}_V\big)$$

$$\Phi_V := \sigma+iA_5+iv^\mu A_\mu-i\alpha_I(\dot{Z}^I+V^I)$$
$$\bar{\Phi}_V := \sigma-iA_5-iv^\mu A_\mu+i\alpha_I(\dot{Z}^I+V^I)$$

$$\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\mathrm{tr}\left(\frac{i}{2}\psi^\mu\big[D_5+\mathcal{L}_v^A+\sigma,\psi_\mu\big]\right)$$
$$=\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\mathrm{tr}\left(\frac{i}{2}\psi^\mu(\partial_5-i\alpha_I\dot{Z}^I)\psi_\mu+\frac{i}{2}\psi^\mu(\mathcal{L}_v-i\alpha_I V^I)\psi_\mu+\frac{i}{2}\psi^\mu\big[\bar{\Phi}_V,\psi_\mu\big]\right)$$

$$\psi(x,t)=\zeta^I(t)\delta_I A$$
$$\mathcal{L}_v\zeta^I(t)=\zeta^J(t)\partial_J V^I(Z)$$
$$\mathcal{L}_v\delta_I A=V^J(Z)\partial_J\delta_I A$$

$$\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\mathrm{tr}\left(\frac{i}{2}\psi^\mu(\partial_5-i\alpha_I\dot{Z}^I)\psi_\mu\right)$$
$$=\frac{i}{2}\int_{S^1} \mathrm{d}t g_{IJ}\zeta^I\partial_t\zeta^J+\int_{S^1} \mathrm{d}t\int_X \mathrm{d}^4x\sqrt{g_4}\mathrm{tr}\left(\frac{i}{2}\zeta^I\delta_I A^\mu(\partial_K\dot{Z}^K-i\alpha_K\dot{Z}^K)\zeta^J\delta_J A_\mu\right)$$
$$=\frac{i}{2}\int_{S^1} \mathrm{d}t g_{IJ}\zeta^I\partial_t\zeta^J+\frac{i}{2}\int_{S^1} \mathrm{d}t\zeta^I\Gamma_{IK,J}\dot{Z}^K\zeta^J$$
$$=\frac{i}{2}\int_{S^1} \mathrm{d}t g_{IJ}\zeta^I\big(\partial_t\zeta^J+\Gamma^J_{KL}\dot{Z}^K\zeta^{(\ell)}\big)$$
$$=\frac{i}{2}\int_{S^1} \mathrm{d}t g_{IJ}\zeta^I\nabla_t\zeta^J$$

$$\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\mathrm{tr}\left(\frac{i}{2}\psi^\mu(\mathcal{L}_v-i\alpha_I V^I)\psi_\mu\right)$$
$$=\frac{i}{2}\int_{S^1} \mathrm{d}t g_{IJ}\zeta^I\partial_K V^J\zeta^K+\int_{S^1} \mathrm{d}t\int_X \mathrm{d}^4x\sqrt{g_4}\mathrm{tr}\left(\frac{i}{2}\zeta^I\delta_I A^\mu(V^K\partial_K-i\alpha_K V^K)\zeta^J\delta_J A\right.$$
$$=\frac{i}{2}\int_{S^1} \mathrm{d}t\zeta^I\partial_J V_I\zeta^J+\frac{i}{2}\int_{S^1} \mathrm{d}t\zeta^I\Gamma_{IK,J}V^K\zeta^J$$
$$=\frac{i}{2}\int_{S^1} \mathrm{d}t\zeta_I\big(\partial_J V^I\zeta^J+\Gamma^I_{JK}V^K\zeta^J\big)$$
$$=\frac{i}{2}\int_{S^1} \mathrm{d}t\zeta_I\nabla_J V^I\zeta^J=\frac{i}{2}\int_{S^1} \mathrm{d}t\nabla_J V_I\zeta^I\zeta^J$$

$$\frac{i}{2}\int_{S^1} \mathrm{d}t\big(g_{IJ}\zeta^I\nabla_t\zeta^J-\nabla_I V_J\zeta^I\zeta^J\big)+\int_{X\times S^1} \mathrm{d}^5x\sqrt{g}\mathrm{tr}\left(\frac{i}{2}\psi^\mu\big[\bar{\Phi}_V,\psi_\mu\big]\right)$$

$$S_V=\frac{1}{2}\int_{S^1} \mathrm{d}t g_{IJ}(\dot{Z}^I+V^I)(\dot{Z}^J+V^J)+\frac{i}{4}\int_{S^1} \mathrm{d}t\big(g_{IJ}\zeta^I\nabla_t\zeta^J-\nabla_I V_J\zeta^I\zeta^J\big)$$

$$\delta Z^I=-\frac{i\varepsilon_1}{\sqrt{2}}\zeta^I+\frac{\varepsilon_2}{\sqrt{2}}J^I_J\zeta^J$$
$$\delta\zeta^I=\sqrt{2}\varepsilon_1(\dot{Z}^I-V^I)+\varepsilon_2 J^I_J(\dot{Z}^J-V^J)-i\varepsilon_2\zeta^J\zeta^K J^L_J\Gamma^I_{LK}$$

$$P_I := g_{IJ}(\dot{Z}^J + V^J)$$

$$[Z^I, P_J] = i\delta^I{}_J$$
$$\{\zeta^a, \zeta^b\} = 2\delta^{ab}.$$

$$Q_1 = \frac{1}{\sqrt{2}}\zeta^I(\pi_I - V_I), Q_2 = \frac{1}{\sqrt{2}}\zeta^I J_I^J(\pi_J - V_J)$$

$$Q_1^2 = Q_2^2 = \frac{1}{2\sqrt{g}}\pi_I\sqrt{g}g^{IJ}\pi_J + \frac{1}{2}V_I V^I + \frac{i}{2}\zeta^I\zeta^J D_I V_J - V^I P_I$$

$$\delta_V Z^I = V^I, \delta_V \zeta^I = \partial_J V^I \zeta^J$$

$$\epsilon_1 J_1 + \epsilon_2 J_2 = -V^I \pi_I + \frac{i}{4}\zeta^I\zeta^J D_I V_J$$

$$\mathcal{H}_V = \frac{1}{2}\left(\frac{1}{\sqrt{g}}\pi_I\sqrt{g}g^{IJ}\pi_J + V_I V^J + \frac{i}{2}\zeta^I\zeta^J D_I D_J\right) = Q_1^2 - \epsilon_1 J_1 - \epsilon_2 J_2$$

$$\mathrm{Tr}_{\mathcal{H}_{k,\boldsymbol{\mu}}}(-1)^F e^{-\int_{S^1}\mathcal{H}_V} = \mathrm{Tr}_{\mathcal{H}_{k,\boldsymbol{\mu}}}\Gamma e^{-R(\phi^2+\epsilon_1 J_1+\epsilon_2 J_2)}$$

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mu_D : H_i(X, \mathbb{Z}) \to H^{4-i}(M_{\boldsymbol{\mu},k}, \mathbb{Q})$$

$$\mu_D(x) = -\frac{1}{4}p_1(\mathbb{E})/x, x \in H.(X, \mathbb{Z}),$$

$$-\frac{1}{4}p_1(\mathbb{E}) = c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2$$

$$w_2(\mathbb{E}) = c_1(\mathcal{E}) \bmod H^2(X \times M_{\boldsymbol{\mu},k}, 2\mathbb{Z})$$

$$\mu_D(K_X) = \frac{1}{2}K_{M_{\boldsymbol{\mu},k}}$$

$$\exp\int_X \left(\overline{w_2(P)}^2 + \frac{3}{4}(e+s)\right)\log\mathcal{R}$$

$$\frac{\beta h}{2\pi i} = -\frac{\beta}{8\pi i}\frac{\partial^2\mathcal{F}_0}{\partial a\partial\log(\Lambda)}$$

$$\pi_n(\mathcal{A}_k/\mathcal{G}_k) \cong \pi_{n-1}(\mathcal{G}_k)$$

$$c(E) = c(L_1)c(L_2) = (1 + c_1(L_1) + c_2(L_1) + \cdots)(1 + c_1(L_2) + c_2(L_2) + \cdots)$$

$$\prod_i (e^{x_i/2} - e^{-x_i/2})^{-1}\prod_i (e^{x_i/2} - e^{-x_i/2})^{-1} = \prod_i e^{-x_i/2}\prod_i (1 - e^{-x_i})^{-1} = \mathrm{ch}[\hat{S}^\bullet V^\vee]$$

$$= 1/\mathrm{ch}[\hat{\Lambda}^\bullet V^\vee]$$

$$\sum_d \mathcal{R}^{4k-3} \int_{X_{[d]}} \hat{A}\left(TX_{[d]}\right) \wedge \mathrm{ch}\left(\hat{S}^\bullet[N^\vee]\right) \wedge \mathrm{ch}\left(\widetilde{\mathcal{L}}^{(I)}\right)$$

$$= \sum_d \mathcal{R}^{4k-3} \int_{X_{[d]}} \frac{\hat{A}\left(TX_{[d]}\right)}{\mathrm{ch}\left(\hat{\wedge}^\bullet[N^\vee]\right)} \wedge \mathrm{ch}\left(\widetilde{\mathcal{L}}^{(I)}\right),$$

$$\hat{S}^\bullet V = (\det V)^{\frac{1}{2}} S^\bullet V, \quad \hat{\wedge}^\bullet V = (\det V)^{-\frac{1}{2}} \wedge^\bullet V,$$

$$\sum_d \mathcal{R}^{4k-3} \int_{X_{[d]}} \frac{\mathrm{td}(T^{(1,0)}X_{[d]})}{\mathrm{ch}(\wedge^\bullet[N^\vee])} \wedge \exp\left(-\frac{1}{2}c_1\left(T^{(1,0)}X_{[d]} + [N]\right)\right) \wedge \mathrm{ch}(\widetilde{\mathcal{L}}^{(I)}).$$

$$\prod_i (e^{x_i/2} - e^{-x_i/2})^{-1} = \prod_i e^{-x_i/2} \prod_i (1-e^{-x_i})^{-1} = \mathrm{ch}[\hat{S}^\bullet V^\vee] = 1/\mathrm{ch}[\hat{\wedge}^\bullet V^\vee]$$

$$\sum_d \mathcal{R}^{4k-3} \sum_{\vec{Y}} \sum_{\substack{m_1+m_2=c_1 \\ q_1+q_2=d}} \frac{1}{e(T_{\vec{Y}} X_{[d]})}$$
$$\iota_{\vec{Y}}^* \left[\frac{\mathrm{td}(T^{(1,0)}X_{[d]})}{\mathrm{ch}(\wedge^\bullet[N^\vee])} \wedge \exp\left(-\frac{1}{2}c_1\left(T^{(1,0)}X_{[d]} \oplus [N]\right)\right) \wedge \mathrm{ch}(\widetilde{\mathcal{L}}^{(I)})\right]$$

$$\mathcal{R}^{4k-3} \iota_{\vec{Y}}^* \frac{1}{\mathrm{ch}(\wedge^\bullet[N^\vee])} = K(a^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, \Lambda, R) \prod_{\ell=1}^{\chi} \prod_{i \neq j} n_{\alpha,\beta}^{(Y_\alpha^{(\ell)}, Y_\beta^{(\ell)})}(a^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, R),$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} \equiv \sum_{i,j} \int \mathrm{d}\sigma \times \frac{E_i E_j}{Q^2} \times \delta\left(x - \frac{1-\cos\ \theta_{ij}}{2}\right)$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x_{12} \cdots \mathrm{d}x_{(N-1)N}} \equiv \sum_m \sum_{1 \le i_1, \cdots i_N \le m} \int \mathrm{d}\sigma_m \times \prod_{1 \le k \le N} \frac{E_{i_k}}{Q} \prod_{1 \le j < l \le N} \delta\left(x_{jl} - \frac{1-\cos\ \theta_{i_j i_l}}{2}\right).$$

$$F(\{x_{jl}\},\{a_k\}) \equiv \sum_m \sum_{1 \le i_1 < \cdots < i_N \le m} \int \mathrm{d}\sigma_m \times \prod_{1 \le k \le N} \frac{E_{i_k}^{a_k}}{Q} \prod_{1 \le j < l \le N} \delta\left(x_{jl} - \frac{1-\cos\ \theta_{i_j i_l}}{2}\right)$$

$$
\begin{aligned}
&\mathrm{EEEC}(x_{12},x_{13},x_{23};a,b,c)\\
&=\sum_{i,j,k}\int \mathrm{d}\sigma \frac{E_i^a E_j^b E_k^c}{Q^{a+b+c}}\delta\left(x_{12}-\frac{1-\cos\,\theta_{12}}{2}\right)\delta\left(x_{13}-\frac{1-\cos\,\theta_{13}}{2}\right)\delta\left(x_{23}-\frac{1-\cos\,\theta_{23}}{2}\right)\\
&=\sum_{i,j,k}\int \mathrm{dPS}_4\left|\mathcal{M}_{1\to4}^{\mathrm{tree}}\right|^2\frac{E_i^a E_j^b E_k^c}{Q^{a+b+c}}\delta\left(\zeta_{12}-\frac{1-\cos\,\theta_{12}}{2}\right)\delta\left(x_{13}-\frac{1-\cos\,\theta_{13}}{2}\right)\\
&\times\delta\left(x_{23}-\frac{1-\cos\,\theta_{23}}{2}\right)+\mathcal{O}(\alpha_s^3)
\end{aligned}
$$

$$
\mathrm{dPS}_4\approx \mathrm{dPS}_2\times \mathrm{d}\sigma_3^{\mathrm{coll}}
$$

$$
\left|\mathcal{M}_{1\to4}^{\mathrm{tree}}\right|^2\approx\left|\mathcal{M}_{1\to2}^{\mathrm{tree}}\right|^2\times\left|\mathcal{M}_{1\to3}^{\mathrm{coll,tree}}\right|^2
$$

$$
\left|\mathcal{M}_{1\to3}^{\mathrm{coll,tree}}\right|^2=\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon}\frac{4g^4}{s_{123}^2}\times P_{1\to3}^{\mathrm{tree}}
$$

$$
\mathrm{EEEC}(x_{12},x_{13},x_{23};a,b,c)\approx\sigma_0\sum_{i,j,k}\int \mathrm{d}\sigma_3^{\mathrm{coll}}\left|\mathcal{M}_{1\to3}^{\mathrm{coll,tree}}\right|^2\frac{E_i^a E_j^b E_k^c}{Q^{a+b+c}}\delta^3\left(x_{ij}-\frac{1-\cos\,\theta_{ij}}{2}\right)
$$

$$
\sigma_0=\int \mathrm{dPS}_2\left|\mathcal{M}_{1\to2}^{\mathrm{tree}}\right|^2
$$

$$
\delta^3\left[x_{ij}-(1-\cos\,\theta_{ij})/2\right]
$$

$$
\mathrm{d}\sigma_3^{\mathrm{coll}}=\mathrm{dS}_{12}\mathrm{dS}_{13}\mathrm{dS}_{23}\mathrm{dz}_1\mathrm{dz}_2\mathrm{dz}_3\delta(1-z_1-z_2-z_3)\frac{4\Theta\left(-\Delta_3^{\mathrm{coll}}\right)\left(-\Delta_3^{\mathrm{coll}}\right)^{-\frac{1}{2}-\epsilon}}{(4\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)}
$$

$$
\Delta_3^{\mathrm{coll}}=(z_3 s_{12}-z_1 s_{23}-z_2 s_{13})^2-4z_1 z_2 s_{13} s_{23}
$$

$$
\Delta_3^{\mathrm{coll}}=(z_1 z_2 z_3)^2\tilde{\Delta}_3,\tilde{\Delta}_3=x_{12}^2+x_{13}^2+x_{23}^2-2x_{12}x_{13}-2x_{13}x_{23}-2x_{12}x_{23},
$$

$$
P_{1\to3}(z_1,z_2,z_3)=N_c^2\left[\frac{s_{123}^2}{2s_{13}s_{23}}\left(\frac{1}{z_1 z_2}+\frac{1}{(1-z_1)(1-z_2)}\right)+\frac{s_{123}}{s_{12}z_3}\left(\frac{1}{z_1}+\frac{1}{1-z_1}\right)+\text{ perms}\right]
$$

$$
P_{q\to3}(z_1,z_2,z_3)=C_F T_F n_f\times P_{qq'\bar{q}'}+C_F(C_A-2C_F)\times P_{qq\bar{q}}+C_F^2\times P_{qgg}^{(C_F)}+C_F C_A\times P_{qgg}^{(C_A)},
$$

$$
P_{g\to3}(z_1,z_2,z_3)=C_F T_F n_f\times P_{gq\bar{q}}^{(C_F)}+C_A T_F n_f\times P_{gq\bar{q}}^{(C_A)}+C_A^2\times P_{ggg}.
$$

$$
\begin{aligned}
\frac{\mathrm{d}^3\sigma_{a,b,c}}{\mathrm{d}x_{12}\,\mathrm{d}x_{13}\,\mathrm{d}x_{23}}&=\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon}\frac{2^{4-a-b-c}g^4\Theta\left(-\tilde{\Delta}_3\right)}{(4\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)}\left(-\tilde{\Delta}_3\right)^{-\frac{1}{2}-\epsilon}\\
&\times\int \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\delta(1-z_1-z_2-z_3)z_1^{a+1-2\epsilon}z_2^{b+1-2\epsilon}z_3^{c+1-2\epsilon}\times\frac{P_{1\to3}(z_1,z_2,z_3)}{s_{123}^2}\\
&+\text{perms of }(x_{12},x_{13},x_{23})
\end{aligned}
$$

$$
\frac{\mathrm{d}^3\sigma_{a,b,c}}{\mathrm{d}x_{12}\,\mathrm{d}z\,\mathrm{d}\bar{z}}=\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon}\frac{2^{4-a-b-c}g^4\Theta\left(-\tilde{\Delta}_3\right)}{(4\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)}\left(-\tilde{\Delta}_3\right)^{-\frac{1}{2}-\epsilon}\times G(z)
$$

$$G(z) = G_0(z) + G_0(1-z) + \frac{1}{|1-z|^4}\left(G_0\left(\frac{z}{z-1}\right) + G_0\left(\frac{1}{1-z}\right)\right) + \frac{1}{|z|^4}\left(G_0\left(\frac{1}{z}\right) + G_0\left(\frac{z-1}{z}\right)\right)$$

$$G_0(z) = \int \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3 \delta(1 - z_1 - z_2 - z_3) z_1^{a+1-2\epsilon} z_2^{b+1-2\epsilon} z_3^{c+1-2\epsilon} \times \frac{P_{1\to 3}(z_1, z_2, z_3)}{s_{123}^2}$$

$$\mathrm{d}F_n = \sum_{\alpha} C_\alpha \mathrm{dlog}\,(s_\alpha) F_{\alpha,n-1}$$

$$\mathcal{S}F_n = \sum_{\alpha} C_\alpha \mathcal{S}F_{\alpha,n-1} \otimes s_\alpha$$

$$\mathcal{S}F_n = \sum_{\alpha_i} C_{\alpha_1\cdots\alpha_n} s_{\alpha_1} \otimes s_{\alpha_2} \otimes \cdots \otimes s_{\alpha_{n-1}} \otimes s_{\alpha_n}$$

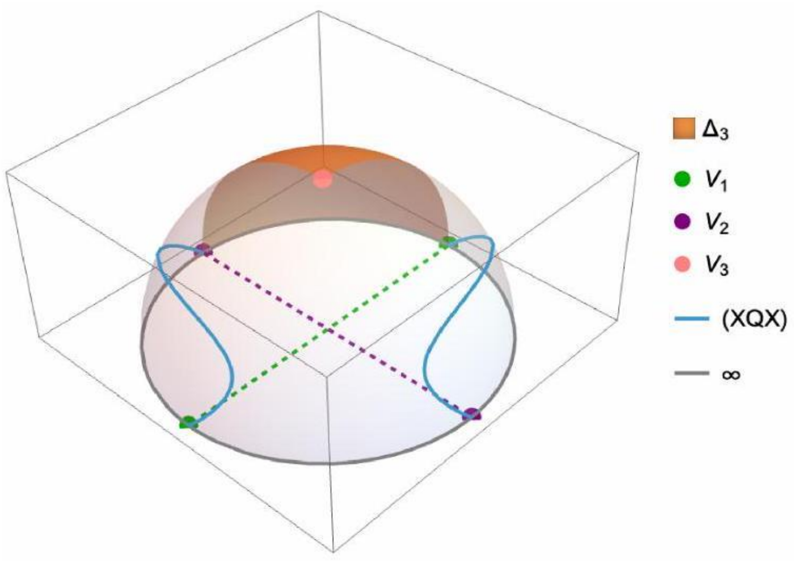
$$F_n = \sum_{\alpha_i} C_{\alpha_1\cdots\alpha_n} \int_{\gamma_n} \mathrm{dlog}\ s_{\alpha_n}(t_n) \cdots \int_{\gamma_1} \mathrm{dlog}\ s_{\alpha_1}(t_1) + \mathfrak{I}_{\text{integration constants}}$$

$$\mathcal{S}\left(\partial_{s_{\beta n}} F_n\right) = \sum_{\alpha_i} \delta_{\alpha_n \beta_n} C_{\alpha_1\cdots\alpha_n} s_{\alpha_1} \otimes s_{\alpha_2} \otimes \cdots \otimes s_{\alpha_{n-1}}.$$

$$\mathcal{S}\left(\mathrm{Disc}_{s_{\beta_n}} F_n\right) = \sum_{\alpha_i} \delta_{\beta_1 \alpha_1} C_{\alpha_1\cdots\alpha_n} s_{\alpha_2} \otimes \cdots \otimes s_{\alpha_{n-1}} \otimes s_{\alpha_n}$$

$$\log(z) = \int_0^\infty \frac{(z-1)\mathrm{d}x_1}{(x_1+z)(x_1+1)} = \int_0^\infty \frac{(z-1)\mathrm{d}x_1\, \mathrm{d}x_2}{(x_1 x_2 + x_1 + x_2 + z)^2} = \int_{\Delta_3} \frac{\langle X\, \mathrm{d}X^2\rangle (LX)}{(XQX)^2}$$

$$L = [0:0:z-1] \text{ and } Q = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2z \end{pmatrix}$$



$$\Delta_3 = \{X \in \mathbb{CP}^2 : X \in \mathrm{Conv}(V_1, \ldots, V_3)\}.$$

$$S^2 := \{(y_+, y_-) \in \mathbb{C}^2 : y_+ = w, y_- = \bar{w}, w \in \mathbb{C}\}$$

$$y_\pm = re^{\pm i\phi}, r \in [0,\infty], \phi \in [0,2\pi].$$

$$\int_0^\infty \mathrm{d}r(-2ir)\int_0^{2\pi} \mathrm{d}\phi \frac{(z-1)x_3^2}{(r^2+(z-1)x_3^2)^2} = -2\pi i = -\mathrm{Disclog}\,(z)$$

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 \mapsto (-2ir)\mathrm{d}r \wedge \mathrm{d}\phi$$

$$\mathrm{Li}_2(z) = \int_{\Delta_5} \frac{2zx_5\langle X\,\mathrm{d}X^4\rangle}{(x_5^2+x_5((1-z)x_1+x_2+x_3+x_4)+x_1x_3+x_1x_4+x_2x_4)^3},$$

$$= \int_{\Delta_5} \frac{\langle X\,\mathrm{d}X^4\rangle(LX)}{(XQX)^3} := \int_{\Delta_5} \langle X\,\mathrm{d}X^4\rangle \mathcal{I}(\mathrm{Li}_2(\mathrm{z}))$$

$$L = [0:0:0:0:2z], \text{ and } Q = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 & 1-z \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1-z & 1 & 1 & 1 & 2 \end{pmatrix}.$$

$$x_1 = \frac{(1-z)(y_1-y_5-x_2)+(1+z)(x_3+x_4)}{(1-z)^2}, x_5 = \frac{(1-z)y_5-x_3-x_4}{1-z}$$

$$\mathcal{I}(\mathrm{Li}_2(\mathrm{z})) = \frac{1}{1-z}\frac{2(1-z)^4 z((1-z)y_5-x_3-x_4)}{\left((1-z)^2y_1y_5+z(x_3+x_4)^2-(1-z)x_2(x_3+zx_4)\right)^3}$$

$$y_1 = re^{-i\phi}, y_5 = re^{i\phi}, r \in [0,\infty], \phi \in [0,2\pi].$$

$$\int_{\Delta_3} \langle X\,\mathrm{d}X^2\rangle \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \frac{2(-2ir)(1-z)^4\left(e^{i\phi}r(1-z)-x_3-x_4\right)}{\left(r^2(1-z)^2+z(x_3+x_4)^2-(1-z)x_2(x_3+zx_4)\right)^3}$$
$$= 2\pi i \int_{\Delta_3} \langle X\,\mathrm{d}X^2\rangle \frac{(1-z)^2 z(x_3+x_4)}{\left(z(x_3+x_4)^2-(1-z)x_2(x_3+zx_4)\right)^2}$$
$$= 2\pi i\log\,(z) = -\mathrm{Disc}_{1-z}[\mathrm{Li}_2(z)]$$

$$I = \int_\triangle \frac{\langle X\,\mathrm{d}X^{n-1}\rangle T[X^k]}{(XQX)^{\frac{n+k}{2}}}$$

$$\langle X\,\mathrm{d}X^{n-1}\rangle = \frac{1}{(n-1)!}\epsilon_{i_1\cdots i_n}X^{i_1}\,\mathrm{d}X^{i_2}\wedge\cdots\wedge\,\mathrm{d}X^{i_n}$$

$$\mathcal{S}I = \sum_{w=1}^{p}\sum_{\alpha_w} C_{\alpha_w} s_{\alpha_1}\otimes\cdots\otimes s_{\alpha_w}, p := \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$$

$$\mathcal{S}I_{\max} = \sum_{\boldsymbol{\alpha}} C_{\boldsymbol{\alpha}} s_{\alpha_1}\otimes\cdots\otimes s_{\alpha_p}$$

$$\begin{cases} \overline{i_1 i_2} \otimes \cdots \otimes \overline{i_{n-1} i_n} =: s_{\alpha_1} \otimes \cdots \otimes s_{\alpha_{\frac{n}{2}}} & n \text{ even}, \\ \overline{i_1 i_2} \otimes \cdots \otimes \overline{i_{n-2} i_{n-1}} =: s_{\alpha_1} \otimes \cdots \otimes s_{\alpha_{\frac{n-1}{2}}} & n \text{ odd} \end{cases}$$

$$C_{i_1,i_2,\ldots,i_{n-1},i_n} = \left(\frac{1}{2(2\pi i)}\right)^{\frac{n}{2}} \int_{S^2_{i_n i_{n-1}}} \cdots \int_{S^2_{i_1 i_2}} \mathcal{I} \; n \text{ even},$$

$$C_{i_1,i_2,\ldots,i_{n-1}} = \left(\frac{1}{2(2\pi i)}\right)^{\frac{n-1}{2}} \int_{S^2_{i_{n-1} i_{n-2}}} \cdots \int_{S^2_{i_1 i_2}} \mathcal{I} \; n \text{ odd},$$

$$\mathcal{S}I_{\max} = \begin{cases} \sum_{\sigma \in S_n} C_{\sigma(1),\ldots,\sigma(n)} \overline{\sigma(1)\sigma(2)} \otimes \cdots \otimes \overline{\sigma(n-1)\sigma(n)} & n \text{ even} \\ \sum_{\sigma \in S_n} C_{\sigma(1),\ldots,\sigma(n-1)} \overline{\sigma(1)\sigma(2)} \otimes \cdots \otimes \overline{\sigma(n-2)\sigma(n-1)} & n \text{ odd} \end{cases}$$

$$I \to \mathrm{Disc}_{\overline{i_{2m-1} i_{2m}}} \circ \cdots \circ \mathrm{Disc}_{\overline{i_1 i_2}}[I]$$

$$Q_{(ij)} = \begin{pmatrix} q_{ii} & q_{ij} \\ q_{ij} & q_{jj} \end{pmatrix}$$

$$x_j{:}(XQX) = q_{ii} x_i^2 + 2 q_{ij} x_i x_j + q_{jj} x_j^2$$

$$\overline{ij}_{\text{first entry}} = \begin{cases} r\left(Q_{(ij)}^{-1}\right), & q_{ii} \neq 0, q_{jj} \neq 0 \\ \left(\frac{q_{ij}^2}{q_{jj}}\right)^{-\mathrm{sign}(q_{ij})}, & q_{ii} = 0, q_{jj} \neq 0 \\ \left(\frac{q_{ij}^2}{q_{ii}}\right)^{-\mathrm{sign}(q_{ij})}, & q_{ii} \neq 0, q_{jj} = 0 \\ q_{ij}^{-2\mathrm{sign}(q_{ij})}, & q_{ii} = 0, q_{jj} = 0 \end{cases}$$

$$r(M_{2\times 2}) = \frac{M_{12} - \sqrt{M_{12}^2 - M_{11} M_{22}}}{M_{12} + \sqrt{M_{12}^2 - M_{11} M_{22}}}$$

$$w_i w_j + X_{\{\widehat{i,j}\}} Q^{(ij)} X_{\{\widehat{i,j}\}}$$

$$X_{\{\{\widehat{i,j}\}} = [x_1 : \cdots : x_{i-1} : x_{i+1} : \cdots : x_{j-1} : x_{j+1} : \cdots : x_n] \otimes Q_{\{i,j\},\{\widehat{i,j}\}} \left(Q_{\{\widehat{i,j}\},\{i,j\}}\right)$$

$$\begin{pmatrix} x_i \\ x_j \end{pmatrix} = R \begin{pmatrix} w_i \\ w_j \end{pmatrix} - Q_{(ij)}^{-1} Q_{\{i,j\},\{\widehat{i,j}\}} X_{\{\widehat{i,j}\}},$$

$$R^T Q_{(ij)} R = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$Q^{(ij)} = Q_{\{\widehat{i,j}\},\{\widehat{i,j}\}} - Q_{\{\widehat{i,j}\},\{i,j\}} Q_{\{\widehat{i,j}\},\{i,j\}}^{-1} Q_{\{i,j\},\{\widehat{i,j}\}},$$

$$\frac{\langle X\, dX^{n-1}\rangle T[X^k]}{(XQX)^{\frac{n+k}{2}}} = \sum_{p_i,p_j} t_{p_ip_j} \left\langle X_{\{\widehat{\iota,\jmath}\}} dX^{n-3}_{\frac{n-3}{\{\widehat{\iota,\jmath}\}}} \frac{dw_i\, dw_j w_i^{p_i} w_j^{p_j}}{\left(w_i w_j + X_{\{\widehat{\iota,\jmath}\}} Q^{(ij)} X_{\{\widehat{\iota,\jmath}\}}\right)^{\frac{n+k}{2}}}\right.$$

$$\mathrm{Disc}_{\overline{ij}}[I] := \sum_{p_i,p_j} \int_{\Delta^{(ij)}} \left\langle X_{\{\widehat{\iota,\jmath}\}} dX^{n-3}_{\{\widehat{\iota,\jmath}\}}\right\rangle C_{p_ip_j}$$

$$\times \int_0^\infty \frac{dr(-2ir)r^{p_i+p_j}}{\left(r^2 + X_{\{\widehat{\iota,\jmath}\}} Q^{(ij)} X_{\{\widehat{\iota,\jmath}\}}\right)^{\frac{n+k}{2}}} \overbrace{\int_0^{2\pi} d\phi e^{\iota\phi(p_\iota - p_\jmath)}}^{2\pi\delta_{p_ip_j}}$$

$$\mathrm{Disc}_{\overline{ij}}[I] = \int_{\Delta^{(ij)}} \frac{\left\langle X_{\{\widehat{\iota,\jmath}\}} dX_{\frac{n'-1}{\{\widehat{\iota,\jmath}\}}}\right\rangle T'\left[X^k_{\frac{\{\widehat{\iota,\jmath}\}}{k}}\right]}{\left(X_{\{\widehat{\iota,\jmath}\}} Q^{(ij)} X_{\{\widehat{\iota,\jmath}\}}\right)^{\frac{n'+k}{2}}},$$

$$T'[X^k] = (-i) \sum_{p=0}^{k} \frac{\Gamma(p+1)\Gamma\left(\frac{k+n}{2} - p - 1\right)}{\Gamma\left(\frac{k+n}{2}\right)} t_{pp} \left(X_{\{\widehat{\iota,\jmath}\}} Q^{(ij)} X_{\{\widehat{\iota,\jmath}\}}\right)^p$$

$$\left\langle \tilde{X}\, d\tilde{X}^1\right\rangle = \tilde{X}_1\, d\tilde{X}_2 - \tilde{X}_2\, d\tilde{X}_1 \star \left(\frac{1}{2(2\pi i)}\right)^{\left\lfloor\frac{n-1}{2}\right\rfloor}$$

$$\tilde{w} = re^{i\theta} \text{ and } \overline{\tilde{w}} = re^{-i\theta}$$

$$\left\langle \tilde{X}\, d\tilde{X}^1\right\rangle \propto \tilde{w}\, d\overline{\tilde{w}} - \overline{\tilde{w}}\, d\tilde{w} \to -2ir\, d\theta$$

$$\log\, q_{ij}^{-2\mathrm{sign}(q_{ij})} = 2\log\, q_{ij}^{-\mathrm{sign}(q_{ij})}$$

$$F_1 = 3\int_{\Delta_4} \langle Z dZ^3\rangle \times \frac{z_2 z_3 z_4^2}{[z_1 z_2 + (z\bar{z})z_2 z_3 + (1-z)(1-\bar{z})z_1 z_3 + (z_1 + z_2 + z_3)z_4]^4}$$

$$F_2 = 12\int_{\Delta_5} \langle Z dZ^4\rangle \frac{z_1 z_2 z_3 z_5^2}{[z_1 z_2 + (z\bar{z})z_2 z_3 + (1-z)(1-\bar{z})z_1 z_3 + (z_2 + z_3)z_4 + (z_1 + z_2 + z_3)z_5]^5}$$

$$Q_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & (1-z)(1-\bar{z}) & 1 \\ 1 & 0 & z\bar{z} & 1 \\ (1-z)(1-\bar{z}) & z\bar{z} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\mathrm{Disc}_{b=0} \circ \mathrm{Disc}_{a=0}$$

$$R^{\mathrm{T}}(Q_1)_{\{2,3\}}R = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \Longrightarrow R_1 = \begin{pmatrix} c & 0 \\ 0 & \frac{1}{cz\bar{z}} \end{pmatrix}, R_2 = \begin{pmatrix} 0 & c \\ \frac{1}{cz\bar{z}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = R\begin{pmatrix} w_2 \\ w_3 \end{pmatrix} - (Q_1)^{-1}_{\{2,3\},\{2,3\}}(Q_1)_{\{2,3\},\{1,4\}}\begin{pmatrix} z_1 \\ z_4 \end{pmatrix} = \begin{pmatrix} w_2 - \frac{(1-z)(1-\bar{z})}{z\bar{z}} z_1 - \frac{1}{z\bar{z}} z_4 \\ (w_3 - z_1 - z_4)/z\bar{z} \end{pmatrix}$$

$$\mathrm{Disc}_{\overline{23}} F_1 = \frac{3}{4\pi i} \int \langle Z\, \mathrm{d}Z \rangle \int_0^{\infty} \mathrm{d}r(-2ir) \int_0^{2\pi} \mathrm{d}\phi \frac{1}{z\bar{z}} \frac{N[r,\phi;z_1,z_4]}{\left(r^2 + Z_{\{1,4\}} Q_1^{(2,3)} Z_{\{1,4\}}\right)^4}$$

$$Q_1^{(2,3)} = \frac{1}{2z\bar{z}} \begin{pmatrix} -2(1-z)(1-\bar{z}) & z+\bar{z}-2 \\ z+\bar{z}-2 & -2 \end{pmatrix}$$

$$N[r,\phi;z_1,z_4] = \frac{1}{(z\bar{z})^3}\Big((z\bar{z})^2 r^2 z_4^2 - r\left(e^{-i\theta}(z\bar{z})^2 z_4^2 (z_1 + z_4) - e^{i\theta}(z\bar{z})^2 z_4^2((1-z)(1-\bar{z})w_1 + w_4)\right)$$
$$+ (z\bar{z}) z_4^2 (z_1 + z_4)((1-z)(1-\bar{z})w_1 + w_4)\Big)\Big)$$

$$\mathrm{Disc}_{\overline{23}} F_1 = -\frac{1}{4(z\bar{z})^3} \int \frac{z_4^2 \langle Z\, \mathrm{d}Z \rangle}{\left(Z_{\{1,4\}} Q_1^{(2,3)} Z_{\{1,4\}}\right)^3} ((1-z)(1-\bar{z}) z_1^2 + z_4^2 + (2 - z - \bar{z} + 2z\bar{z}) z_1 z_4)$$

$$C_{23,14}\left((z\bar{z})^{-2\mathrm{Sign}(z\bar{z})} \otimes \frac{1-z}{1-\bar{z}}\right) \subset \mathcal{S}[F_1]$$

$$(2z\bar{z}) R^{\mathrm{T}} Q_1^{(2,3)} R = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \ominus \frac{-2 + z + \bar{z} + \sqrt{(z-\bar{z})^2}}{2 - z + \bar{z} + \sqrt{(z-\bar{z})^2}} = \frac{1-z}{1-\bar{z}}$$

$$R = \begin{pmatrix} 1 & -\frac{1}{(z-\bar{z})^2} \\ -(1-z) & \frac{1-\bar{z}}{(z-\bar{z})^2} \end{pmatrix},$$

$$\begin{pmatrix} z_1 \\ z_4 \end{pmatrix} = R\begin{pmatrix} w_1 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_1 - \frac{1}{(z-\bar{z})^2} w_4 \\ -(1-z) w_1 + \frac{1-\bar{z}}{(z-\bar{z})^2} w_4 \end{pmatrix}$$

$$\frac{1}{\sqrt{(z-\bar{z})^2}} = \frac{1}{z-\bar{z}} = r e^{-i\theta}$$

$$C_{23,14} = \mathrm{Disc}_{\overline{14}_{Q_1^{(2,3)}}} \mathrm{Disc}_{\overline{23}} F_1 = -\frac{(1-z)(1-\bar{z})}{2(z-\bar{z})^5}(-z^2 - \bar{z}^2 - 4z\bar{z} + 3z^2\bar{z} + 3z\bar{z}^2)$$

$$C_{23,14}\left(\frac{1}{z^2\bar{z}^2} \otimes \frac{1-z}{1-\bar{z}}\right) \subset \mathcal{S}[F_1]$$

$$Q_2=\frac{1}{2}\begin{pmatrix} 0 & 1 & (1-z)(1-\bar{z}) & 0\\ 1 & 0 & z\bar{z} & 1\\ (1-z)(1-\bar{z}) & z\bar{z} & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & & 0 \end{pmatrix}.$$

$$\begin{pmatrix} z_1\\ z_3 \end{pmatrix}=\begin{pmatrix} w_1-\dfrac{z\bar{z}z_2+z_4+z_5}{(1-z)(1-\bar{z})}\\ \dfrac{w_3-z_2-z_5}{(1-z)(1-\bar{z})} \end{pmatrix}$$

$$J=\frac{1}{(1-z)(1-\bar{z})}\,re^{-i\theta}$$

$$\mathrm{Disc}_{\overline{13}}F_2=\int\ \langle Z\,\mathrm{d}Z^2\rangle$$

$$\times\frac{(1-z)(1-\bar{z})z_2z_5^2\left(((-3+\bar{z})z_2-2z_5)(z_4+z_5)+zz_2\left(z_4+z_5-\bar{z}(2z_2+z_4+3z_5)\right)\right)}{\left((\bar{z}z_2+z_5)(z_4+z_5)+zz_2(\bar{z}(z_2-z_4)+z_4+z_5)\right)^4}$$

$$Q_2^{(1,3)}=\frac{1}{2}\begin{pmatrix} 2z\bar{z} & z+\bar{z}-z\bar{z} & z+\bar{z}\\ z+\bar{z}-z\bar{z} & 0 & 1\\ z+\bar{z} & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} z_2\\ z_5 \end{pmatrix}=\begin{pmatrix} w_2-\dfrac{w_5}{2(z-\bar{z})^2}-\dfrac{2z\bar{z}-z-\bar{z}}{(z-\bar{z})^2}z_4\\ -zw_z+\dfrac{\bar{z}w_5}{2(z-\bar{z})^2}+\dfrac{z\bar{z}(z+\bar{z})-z^2-\bar{z}^2}{z-\bar{z}} \end{pmatrix},$$

$$C_{13,25}=\mathrm{Disc}_{\overline{25}}^{Q_2^{(1,3)}}\ \mathrm{Disc}_{\overline{13}}F_2$$

$$=\frac{1}{4z^2\bar{z}^2(z-\bar{z})^5}$$

$$\times\,(z^6(1-\bar{z})+\bar{z}^6+5z^2\bar{z}^4(1+\bar{z})-z\bar{z}^5(4+\bar{z})$$

$$+\,2z^3\bar{z}^3(4-11\bar{z}+2\bar{z}^2)\ +z^4\bar{z}^2(5-22\bar{z}+28\bar{z}^2-6\bar{z}^3)$$

$$+\,z^5\bar{z}(-4+5\bar{z}+4\bar{z}^2-6\bar{z}^3)\Big)$$

$$C_{13,25}\left(\frac{1}{(1-z)^2(1-\bar{z})^2}\otimes\frac{\bar{z}}{z}\right)\subset\mathcal{S}[F_2]$$

$$G_0(z)\sim\int z_1^{a+1}\,z_2^{b+1}\,z_3^{c+1}\times\frac{P_{1\to3}(z_1,z_2,z_3)}{s_{123}^2}$$

$$\int\ \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\delta(1-z_1-z_2-z_3)z_1^{a+1}z_2^{b+1}z_3^{c+1}\times\frac{1}{s_{123}s_{12}z_3}\left(\frac{1}{z_1}+\frac{1}{z_2+z_3}\right).$$

$$F_1 = \int_0^\infty \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3 \delta(1 - z_1 - z_2 - z_3) \times \frac{z_2 z_3}{z_1 z_2 + (z\bar{z}) z_2 z_3 + (1 - z)(1 - \bar{z}) z_1 z_3}$$

$$F_2 = \int_0^\infty \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3 \delta(1 - z_1 - z_2 - z_3) \times \frac{z_1 z_2 z_3}{z_1 z_2 + (z\bar{z}) z_2 z_3 + (1 - z)(1 - \bar{z}) z_1 z_3} \frac{1}{z_2 + z_3}$$

$$F_1 = 3 \int_0^\infty \frac{\mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4}{\mathrm{GL}(1)} \times \frac{z_2 z_3 z_4^2}{[z_1 z_2 + (z\bar{z}) z_2 z_3 + (1 - z)(1 - \bar{z}) z_1 z_3 + (z_1 + z_2 + z_3) z_4]^4}$$
$$= 3 \int_{\Delta_4} \langle Z\mathrm{d}Z \rangle \times \frac{z_2 z_3 z_4^2}{[z_1 z_2 + (z\bar{z}) z_2 z_3 + (1 - z)(1 - \bar{z}) z_1 z_3 + (z_1 + z_2 + z_3) z_4]^4}$$

$$F_2 = \frac{\Gamma(5)}{\Gamma(2)\Gamma(3)} \int_0^\infty \frac{\mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \mathrm{d}z_5}{\mathrm{GL}(1)}$$
$$\times \frac{z_1 z_2 z_3 z_5^2}{[z_1 z_2 + (z\bar{z}) z_2 z_3 + (1 - z)(1 - \bar{z}) z_1 z_3 + (z_2 + z_3) z_4 + (z_1 + z_2 + z_3) z_5]^5}$$
$$= 12 \int_{\Delta_5} \langle Z\mathrm{d}Z^4 \rangle \frac{z_1 z_2 z_3 z_5^2}{[z_1 z_2 + (z\bar{z}) z_2 z_3 + (1 - z)(1 - \bar{z}) z_1 z_3 + (z_2 + z_3) z_4 + (z_1 + z_2 + z_3) z_5]^5}$$

$$\begin{aligned}
G_0^{(2)}(z) \supset K = {} & \frac{1}{48}\pi^2(-z^3 - z^2(\bar{z} - 2) - z(\bar{z}^2 - 2\bar{z} + 2) - \bar{z}^3 + 2\bar{z}^2 - 2\bar{z} + 1) \\
& + \frac{1}{8}(z^3 + z^2(\bar{z} - 2) + z(\bar{z}^2 - 2\bar{z} + 2) + \bar{z}^3 - 2\bar{z}^2 + 2\bar{z} - 1) \\
& \times \left[\mathrm{Li}_2\left(1 - \frac{z\bar{z}}{(1 - z)(1 - \bar{z})}\right) + \frac{1}{2}\log\left(\frac{1}{(1 - z)(1 - \bar{z})}\right)\log\left(\frac{z\bar{z}}{(1 - z)(1 - \bar{z})}\right)\right] \\
& + \frac{iD_2^-(z)}{4(z - \bar{z})^{11}}[z^{14} - 2z^{13}(5\bar{z} + 1) + z^{12}(45\bar{z}^2 + 20\bar{z} + 2) - z^{11}(120\bar{z}^3 + 90\bar{z}^2 + 20\bar{z} + 1) \\
& + 11z^{10}\bar{z}(19\bar{z}^3 + 22\bar{z}^2 + 8\bar{z} + 1) - 11z^9\bar{z}^2(22\bar{z}^3 + 40\bar{z}^2 + 20\bar{z} + 5) \\
& + z^8\bar{z}(-675\bar{z}^5 + 3114\bar{z}^4 - 2495\bar{z}^3 + 1615\bar{z}^2 - 330\bar{z} + 25) \\
& + z^7\bar{z}(-1440\bar{z}^6 + 7740\bar{z}^5 - 16838\bar{z}^4 + 14605\bar{z}^3 - 6500\bar{z}^2 + 1255\bar{z} - 76) \\
& + z^6\bar{z}(-675\bar{z}^7 + 7740\bar{z}^6 - 26050\bar{z}^5 + 41281\bar{z}^4 - 31230\bar{z}^3 + 11695\bar{z}^2 - 1889\bar{z} + 85) \\
& + z^5\bar{z}(-242\bar{z}^8 + 3114\bar{z}^7 - 16838\bar{z}^6 + 41281\bar{z}^5 - 52400\bar{z}^4 + 33897\bar{z}^3 - 10674\bar{z}^2 + 1348\bar{z} - 36) \\
& + z^4\bar{z}^2(209\bar{z}^8 - 440\bar{z}^7 - 2495\bar{z}^6 + 14605\bar{z}^5 - 31230\bar{z}^4 + 33897\bar{z}^3 - 18570\bar{z}^2 + 4615\bar{z} - 360) \\
& + z^3\bar{z}^3(-120\bar{z}^8 + 242\bar{z}^7 - 220\bar{z}^6 + 1615\bar{z}^5 - 6500\bar{z}^4 + 11695\bar{z}^3 - 10674\bar{z}^2 + 4615\bar{z} - 720) \\
& + z^2\bar{z}^4(45\bar{z}^8 - 90\bar{z}^7 + 88\bar{z}^6 - 55\bar{z}^5 - 330\bar{z}^4 + 1255\bar{z}^3 - 1889\bar{z}^2 + 1348\bar{z} - 360) \\
& + z\bar{z}^5(-10\bar{z}^8 + 20\bar{z}^7 - 20\bar{z}^6 + 11\bar{z}^5 + 25\bar{z}^3 - 76\bar{z}^2 + 85\bar{z} - 36) + \bar{z}^{11}(\bar{z}^3 - 2\bar{z}^2 + 2\bar{z} - 1)]
\end{aligned}$$

$$2iD_2^-(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}(\log(1 - z) - \log(1 - \bar{z}))\log(z\bar{z})$$

$$D_2^+(z) = \mathrm{Li}_2(1 - z\bar{z}) + \frac{1}{2}\log(z\bar{z})\log[(1 - z)(1 - \bar{z})]$$

$$K \overset{\delta\to 0}{\approx} \frac{1}{\delta^{10}} 189(\bar{z} - 1)^5\bar{z}^5(2\bar{z}^2 - 2\bar{z} + 1)(\bar{z}\log(1 - \bar{z}) - \log(1 - \bar{z}) - \bar{z}\log(\bar{z})) + \mathcal{O}(\delta^{-9})$$

$$K \overset{z\to 0}{\approx} \frac{9t^6(t^8 + 10t^6 + 20t^4 + 10t^2 + 1)(\log(r) - 1)}{2r^4(t^2 - 1)^{10}} + \mathcal{O}(r^{-3}),$$

$$T(x_{12}, z, \bar{z}; a = b = c = 1) = \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{8}{5}C_F^2 + \frac{91}{150}C_F C_A + \frac{13}{300}C_F T_F n_f\right]\frac{1}{x_{12} z\bar{z}} + \cdots,$$

$$\frac{1}{(z-\bar{z})^{n_1}(z\bar{z})^{n_2}((1-z)(1-\bar{z}))^{n_3}}\left[\sum_{i,j=0}^{i,j\le n} a_{i,j}z^i\bar{z}^j + \sum_{i,j=0}^{i,j\le n} b_{i,j}z^i\bar{z}^j\log(z\bar{z}) + \sum_{i,j=0}^{i,j\le n} c_{i,j}z^i\bar{z}^j\log((1-z)(1-\bar{z})) + \sum_{i,j=0}^{i,j\le n} d_{i,j}z^i\bar{z}^j\pi^2\right].$$

$$f(x) = \frac{1}{3}\mathrm{Li}_2(-x) + \frac{5}{11}\log(x+2)$$

$$\{B_1(x) := \mathrm{Li}_2(-x), B_2(x) := \log(x+2)\}$$

$$B_1(x_1) = \mathrm{Li}_2(-x_1) = -1.437, B_1(x_2) = \mathrm{Li}_2(-x_2) = -2.749, B_2(x_1) = \log(x_1+2) = 1.386, B_2(x_2) = \log(x_2+2) = 1.946$$

$$\begin{bmatrix}\vec{u}_1\\ \vec{u}_2\\ \vec{u}_3\end{bmatrix} = \begin{bmatrix}10^3 f(x_1) & 10^3 f(x_2) & 1 & 0 & 0\\ 10^3 B_1(x_1) & 10^3 B_2(x_1) & 0 & 1 & 0\\ 10^3 B_2(x_2) & 10^3 B_2(x_2) & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}151 & -32 & 1 & 0 & 0\\ -1437 & -2749 & 0 & 1 & 0\\ 1386 & 1945 & 0 & 0 & 1\end{bmatrix},$$

$$\begin{bmatrix}\vec{v}_1\\ \vec{v}_2\\ \vec{v}_3\end{bmatrix} = \begin{bmatrix}0 & 8 & 33 & -11 & -15\\ 151 & -32 & 1 & 0 & 0\\ -25 & 203 & -25 & 8 & 11\end{bmatrix},$$

$$0 = 10^3\left(33f(x_1) - 11B_1(x_1) - 15B_2(x_1)\right)$$
$$8 = 10^3\left(33f(x_2) - 11B_1(x_2) - 15B_2(x_2)\right)$$

$$f(x) = \frac{11}{33}B_1(x) + \frac{15}{33}B_2(x)$$

$$G(z) = D(z) + a_{0,11}g_1(z) + a_{0,12}g_2(z) + a_{1,12}g_3(z) + a_{2,12}g_4(z) + c_{1,12}g_5(z) + d_{2,12}g_6(z)$$

$$g_3(z) = \frac{1}{(z-1)^2z^2(\bar{z}-1)^2\bar{z}^2}[z^4\bar{z}^2 - z^4\bar{z} + 8z^3\bar{z}^3 - 14z^3\bar{z}^2 + 8z^3\bar{z} - z^3 + z^2\bar{z}^4 - 14z^2\bar{z}^3 + 24z^2\bar{z}^2 - 14z^2\bar{z} + z^2 + (2z^4\bar{z}^3 - 3z^4\bar{z}^2 + z^4\bar{z} + 2z^3\bar{z}^4 - 8z^3\bar{z}^3 + 9z^3\bar{z}^2 - 4z^3\bar{z} + z^3 - 3z^2\bar{z}^4 + 9z^2\bar{z}^3 - 12z^2\bar{z}^2 + 9z^2\bar{z} - 3z^2 + z\bar{z}^4 - 4z\bar{z}^3 + 9z\bar{z}^2 - 8z\bar{z} + 2z + \bar{z}^3 - 3\bar{z}^2 + 2\bar{z})\log((z-1)(\bar{z}-1)) + (-2z^4\bar{z}^3 + 3z^4\bar{z}^2 - z^4\bar{z} - 2z^3\bar{z}^4 + 8z^3\bar{z}^3 - 9z^3\bar{z}^2 + 2z^3\bar{z} + 3z^2\bar{z}^4 - 9z^2\bar{z}^3 + 6z^2\bar{z}^2 - z\bar{z}^4 + 2z\bar{z}^3)\log(z\bar{z}) - z\bar{z}^4 + 8z\bar{z}^3 - 14z\bar{z}^2 + 8z\bar{z} - \bar{z}^3 + \bar{z}^2]$$

$$\frac{d\sigma^{[4]}}{dx_L} = \sum_{1\le i_1\ne i_2\ne i_3\le 4}\int \mathrm{dLIPS}_4|\mathcal{M}_4|^2\left(\frac{12E_{i_1}^2E_{i_2}E_{i_3}}{Q^3} + \frac{12E_{i_1}E_{i_2}^2E_{i_3}}{Q^3} + \frac{12E_{i_1}E_{i_2}E_{i_3}^2}{Q^3}\right) \times \delta\left(x_L - \max\{x_{i_1,i_2}, x_{i_1,i_3}, x_{i_2,i_3}\}\right) + \sum_{n\in\{3,4\}}\sum_{1\le i_1\ne i_2\le n}\int \mathrm{dLIPS}_n|\mathcal{M}_n|^2\left(\frac{4E_{i_1}^3E_{i_2}}{Q^3} + \frac{4E_{i_1}^3E_{i_3}}{Q^3} + \frac{4E_{i_2}^3E_{i_1}}{Q^3} + \frac{4E_{i_2}^3E_{i_3}}{Q^3} + \frac{4E_{i_3}^3E_{i_1}}{Q^3} + \frac{4E_{i_3}^3E_{i_2}}{Q^3} + \frac{6E_{i_1}^2E_{i_2}^2}{Q^3} + \frac{6E_{i_1}^2E_{i_3}^2}{Q^3} + \frac{6E_{i_2}^2E_{i_3}^2}{Q^3}\right) \times \delta\left(x_L - x_{i_1,i_2}\right) + \sum_{n\in\{2,3,4\}}\sum_{1\le i_1\le n}\int \mathrm{dLIPS}_n|\mathcal{M}_n|^2\frac{E_{i_1}^4\times\delta(x_L)}{Q^3}$$

$$\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon} \frac{4g_s^4}{s_{123}^2} P_{1\to 3}(z_1, z_2, z_3)$$

$$P_{1\to 3}(z_1, z_2, z_3) = N_c^2 \left[\frac{s_{123}^2}{2s_{13}s_{23}}\left(\frac{1}{z_1 z_2} + \frac{1}{(1-z_1)(1-z_2)}\right) + \frac{s_{123}}{s_{12}z_3}\left(\frac{1}{z_1} + \frac{1}{1-z_1}\right) + \text{ perms}\right]$$

$$\begin{aligned}
P_{\bar{q}'q'q} =& C_F T_F \frac{s_{123}}{2s_{12}}\left[-\frac{[z_1(s_{12}+2s_{23}) - z_2(s_{12}+2s_{13})]^2}{(z_1+z_2)^2 s_{12}s_{123}} + \frac{4z_3 + (z_1-z_2)^2}{z_1+z_2} + (1-2\epsilon)\left(z_1+z_2-\frac{s_{12}}{s_{123}}\right)\right] \\
P_{\bar{q}qq} =& \left(P_{\bar{q}'q'q} + 2\leftrightarrow 3\right) + P_{\bar{q}qq}^{(\mathrm{id})} \\
P_{\bar{q}qq}^{(\mathrm{id})} =& C_F\left(C_F - \frac{1}{2}C_A\right)\Bigg\{(1-\epsilon)\left(\frac{2s_{23}}{s_{12}} - \epsilon\right) + \frac{s_{123}}{s_{12}}\Bigg[\frac{1+z_1^2}{1-z_2} - \frac{2z_2}{1-z_3} - \epsilon\left(\frac{(1-z_3)^2}{1-z_2} + 1 + z_1 - \frac{2z_2}{1-z_3}\right) \\
& -\epsilon^2(1-z_3)\Bigg] - \frac{s_{123}^2}{2s_{12}s_{13}} z_1\left[\frac{1+z_1^2}{(1-z_2)(1-z_3)} - \epsilon\left(1 + 2\frac{1-z_2}{1-z_3}\right) - \epsilon^2\right]\Bigg\} + (2\leftrightarrow 3) \\
P_{ggq} =& C_F^2\Bigg\{\frac{s_{123}^2}{2s_{13}s_{23}} z_3\left[\frac{1+z_3^2}{z_1 z_2} - \epsilon\frac{z_1^2+z_2^2}{z_1 z_2} - \epsilon(1+\epsilon)\right] + (1-\epsilon)\left[\epsilon - (1-\epsilon)\frac{s_{23}}{s_{13}}\right] \\
& + \frac{s_{123}}{s_{13}}\left[\frac{z_3(1-z_1) + (1-z_2)^3}{z_1 z_2} - \epsilon(z_1^2 + z_1 z_2 + z_2^2)\frac{1-z_2}{z_1 z_2} + \epsilon^2(1+z_3)\right]\Bigg\} \\
& + C_F C_A\Bigg\{(1-\epsilon)\left(\frac{[z_1(s_{12}+2s_{23}) - z_2(s_{12}+2s_{13})]^2}{4(z_1+z_2)^2 s_{12}^2} + \frac{1}{4} - \frac{\epsilon}{2}\right) \\
& + \frac{s_{123}^2}{2s_{12}s_{13}}\left[\frac{2z_3 + (1-\epsilon)(1-z_3)^2}{z_2} + \frac{2(1-z_2) + (1-\epsilon)z_2^2}{1-z_3}\right] \\
& - \frac{s_{123}^2}{4s_{13}s_{23}} z_3\left[\frac{2z_3 + (1-\epsilon)(1-z_3)^2}{z_1 z_2} + \epsilon(1-\epsilon)\right] \\
& + \frac{s_{123}}{2s_{12}}\left[(1-\epsilon)\frac{z_1(2 - 2z_1 + z_1^2) - z_2(6 - 6z_2 + z_2^2)}{z_2(1-z_3)} + 2\epsilon\frac{z_3(z_1 - 2z_2) - z_2}{z_2(1-z_3)}\right] \\
& + \frac{s_{123}}{2s_{13}}\Bigg[(1-\epsilon)\frac{(1-z_2)^3 + z_3^2 - z_2}{z_2(1-z_3)} - \epsilon\left(\frac{2(1-z_2)(z_2-z_3)}{z_2(1-z_3)} - z_1 + z_2\right) \\
& - \frac{z_3(1-z_1) + (1-z_2)^3}{z_1 z_2} + \epsilon(1-z_2)\left(\frac{z_1^2+z_2^2}{z_1 z_2} - \epsilon\right) + (1\leftrightarrow 2)\Bigg]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
P_{g_1 q_2 \bar{q}_3} =& C_F T_R P_{g_1 q_2 \bar{q}_3}^{(ab)} + C_A T_R P_{g_1 q_2 \bar{q}_3}^{(nab)} \\
P_{g_1 q_2 \bar{q}_3}^{(ab)} =& -2 - (1-\epsilon)s_{23}\left(\frac{1}{s_{12}} + \frac{1}{s_{13}}\right) + 2\frac{s_{123}^2}{s_{12}s_{13}}\left(1 + z_1^2 - \frac{z_1 + 2z_2 z_3}{1-\epsilon}\right) \\
& - \frac{s_{123}}{s_{12}}\left(1 + 2z_1 + \epsilon - 2\frac{z_1+z_2}{1-\epsilon}\right) - \frac{s_{123}}{s_{13}}\left(1 + 2z_1 + \epsilon - 2\frac{z_1+z_3}{1-\epsilon}\right) \\
P_{g_1 q_2 \bar{q}_3}^{(nab)} =& \Bigg\{-\frac{t_{23,1}^2}{4s_{23}^2} + \frac{s_{123}^2}{2s_{13}s_{23}} z_3\left[\frac{(1-z_1)^3 - z_1^3}{z_1(1-z_1)} - \frac{2z_3(1 - z_3 - 2z_1 z_2)}{(1-\epsilon)z_1(1-z_1)}\right] \\
& + \frac{s_{123}}{2s_{13}}(1-z_2)\left[1 + \frac{1}{z_1(1-z_1)} - \frac{2z_2(1-z_2)}{(1-\epsilon)z_1(1-z_1)}\right] \\
& + \frac{s_{123}}{2s_{23}}\left[\frac{1+z_1^3}{z_1(1-z_1)} + \frac{z_1(z_3-z_2)^2 - 2z_2 z_3(1+z_1)}{(1-\epsilon)z_1(1-z_1)}\right] \\
& - \frac{1}{4} + \frac{\epsilon}{2} - \frac{s_{123}^2}{2s_{12}s_{13}}\left(1 + z_1^2 - \frac{z_1 + 2z_2 z_3}{1-\epsilon}\right)\Bigg\} + (2\leftrightarrow 3)
\end{aligned}$$

$$P_{g_1g_2g_3} = C_A^2\left\{\frac{1-\epsilon}{4s_{12}^2}t_{12,3}^2 + \frac{3}{4}(1-\epsilon) + \frac{s_{123}}{s_{12}}\left[4\frac{z_1z_2-1}{1-z_3} + \frac{z_1z_2-2}{z_3} + \frac{3}{2} + \frac{5}{2}z_3 + \frac{\left(1-z_3(1-z_3)\right)^2}{z_3z_1(1-z_1)}\right] + \frac{s_{123}^2}{s_{12}s_{13}}\left[\frac{z_1z_2(1-z_2)(1-2z_3)}{z_3(1-z_3)} + z_2z_3 - 2 + \frac{z_1(1+2z_1)}{2} + \frac{1+2z_1(1+z_1)}{2(1-z_2)(1-z_3)} + \frac{1-2z_1(1-z_1)}{2z_2z_3}\right]\right\} + (\mathfrak{P}_{\text{permutations}})$$

$$t_{ij,k} = 2\frac{z_is_{jk}-z_js_{ik}}{z_i+z_j} + \frac{z_i-z_j}{z_i+z_j}s_{ij}$$

$$Q: XQX = q_{ii}x_i^2 + 2q_{ij}x_ix_j + q_{jj}x_j^2 = 0$$

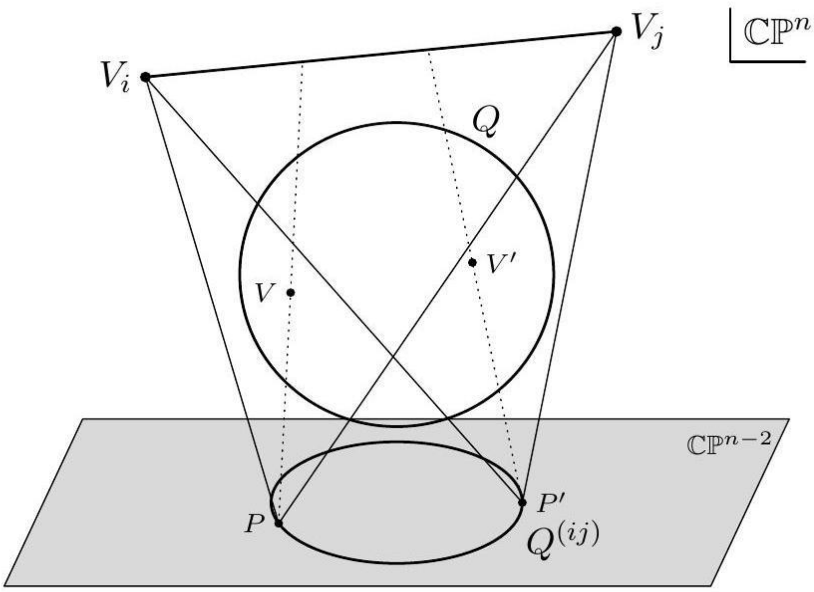
$$\otimes\, r\left(Q^{(ij)}\right) = \otimes\,\frac{\left(q_{ij}-\sqrt{q_{ij}^2-q_{ii}q_{jj}}\right)}{q_{ii}} - \otimes\,\frac{\left(q_{ij}+\sqrt{q_{ij}^2-q_{ii}q_{jj}}\right)}{q_{ii}}\overline{V_iV_j}.$$

$$q_{ii}x_i^2 + 2q_{ij}x_ix_j + q_{jj}x_j^2\Big|_{x_j=1} = 0$$

$$\otimes\left(\frac{q_{ij}^2}{q_{ii}}\right)^{-2\mathrm{sign}(q_{ij})} = -2\mathrm{sign}(q_{ij})\left(\otimes\, q_{ij}^2 - \otimes\, q_{ii}\right).$$

$$Q^{(ij)}: \left\{P \in H_{\{\hat{i},\hat{j}\}} \mid H_{(ijP)} \between Q\right\}.$$

$$VQV = \partial_{\alpha_i}(VQV) = \partial_{\alpha_j}(VQV) = 0$$



$$x_i^2 + X_{\{\hat{i}\}}Q^{(i)}X_{\{\hat{i}\}},$$

$$\log(z) = \int_{\Delta_2}\frac{(z-1)\langle X\, \mathrm{d}X\rangle}{(x_1+zx_2)(x_1+x_2)} = \int_{\Delta_2}\frac{(z-1)\langle X\, \mathrm{d}X\rangle}{(x_1^2+(1+z)x_1x_2+zx_2^2)}.$$

$$\mathrm{Li}_2(z) = \frac{1}{4}\int_{\Delta_5} \mathrm{d}\left(\frac{2\left(\langle X\,\mathrm{d}X^3\rangle_{(1234)} + \langle X\,\mathrm{d}X^3\rangle_{(1345)} + z\langle X\,\mathrm{d}X^3\rangle_{(1245)} - \langle X\,\mathrm{d}X^3\rangle_{(2345)}\right)}{(x_5^2 + x_5((1-z)x_1 + x_2 + x_3 + x_4) + x_1x_3 + x_1x_4 + x_2x_4)^2}\right)$$

$$\mathcal{S}I_{n,k} = \frac{1}{n-1}\sum_{i=1}^{n} \mathcal{S}I_{n,k}^{(i)}$$

$$I_{\mathrm{ex}} = \int_{\Delta_2} \frac{\sqrt{a_{12}^2 + a_{13}^2 + a_{23}^2 - 2a_{12}a_{23}a_{13} - 1}\langle X\,\mathrm{d}X^2\rangle}{2\sqrt{2}(x_1^2 + x_2^2 + x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3)^{\frac{3}{2}}}$$

$$\mathcal{S}I_{\mathrm{ex}} = \frac{1}{2}\left(\otimes\frac{-2a_{12} + a_{13}a_{23} - 4q_{123}}{-2a_{12} + a_{13}a_{23} - 4q_{123}} + \text{ sym.}\right).$$

$$\sum_{i=1}^{3}\int_{\Delta_{(i)}} \left\langle X_{\{\hat{\imath}\}}\mathrm{d}X_{\{\hat{\imath}\}}\right\rangle \int_0^\infty \frac{q_{123}\,\mathrm{d}x_i}{(x_i^2 + X_{\{\hat{\imath}\}}Q^{(i)}X_{\{\hat{\imath}\}})^{\frac{3}{2}}} = \log\left(\frac{-2a_{12} + a_{13}a_{23} - 4q_{123}}{-2a_{12} + a_{13}a_{23} - 4q_{123}}\right) + \text{ perm}$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(\zeta, m; N) =$$

$$\frac{1}{\left(2\cosh\frac{m}{2}\right)^N}\frac{1}{N!}\int_{-\infty}^{\infty}\frac{d^Nx}{(2\pi)^N}e^{-\frac{i\zeta}{2\pi}\sum_{i=1}^N x_i}\frac{\prod_{i<j}^N\left(2\sinh\frac{x_i - x_j}{2}\right)^2}{\prod_{i<j}^N\prod_{\pm} 2\cosh\frac{x_i - x_j \pm m}{2}\prod_{i=1}^N\prod_{\alpha=1}^{N_\mathrm{f}} 2\cosh\frac{x_i - y_\alpha}{2}}$$

$$\sum_{\alpha=1}^{N_\mathrm{f}} y_\alpha = 0$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(\zeta, m; N) = Z_{\mathrm{pert},y_\alpha}^{(N_\mathrm{f})}(\zeta, m; N)\left(1 + e^{-\#\sqrt{\frac{N-B}{C}}}\right)$$

$$Z_{\mathrm{pert},y_\alpha}^{(N_\mathrm{f})}(\zeta, m; N) = e^A C^{-\frac{1}{3}}\mathrm{Ai}\left[C^{-\frac{1}{3}}(N - B)\right]$$

$$C = \frac{2}{\pi^2 N_\mathrm{f}\left(1 + \frac{\zeta^2}{\pi^2 N_\mathrm{f}^2}\right)\left(1 + \frac{m^2}{\pi^2}\right)}$$

$$B = -\frac{1}{6N_\mathrm{f}\left(1 + \frac{\zeta^2}{\pi^2 N_\mathrm{f}^2}\right)} - \frac{1}{2\left(1 + \frac{m^2}{\pi^2}\right)}\left(\sum_{\alpha=1}^{N_\mathrm{f}}\frac{y_\alpha^2}{\pi^2} + \frac{N_\mathrm{f}}{3}\right) + \frac{2}{3N_\mathrm{f}\left(1 + \frac{\zeta^2}{\pi^2 N_\mathrm{f}^2}\right)\left(1 + \frac{m^2}{\pi^2}\right)} + \frac{N_\mathrm{f}}{24}$$

$$A = \frac{1}{4}\sum_{\pm}\left(\mathcal{A}\left(N_\mathrm{f} \pm \frac{i\zeta}{\pi}, 0\right) + \sum_{\alpha,\beta=1}^{N_\mathrm{f}}\mathcal{A}\left(1 \pm \frac{im}{\pi}, \frac{y_\alpha - y_\beta}{\pi}\right)\right)$$

$$\mathcal{A}(\kappa, \chi) = \frac{2\zeta(3)}{\pi^2\kappa} + \frac{\chi^2}{2\kappa} - \frac{\kappa}{12} + \frac{1}{\pi}\int_0^\infty dz\,\frac{1}{e^{2\pi z} - 1}\frac{d}{dz}\left(\frac{\cos\pi\chi z}{z\tanh\frac{\pi\kappa z}{2}} - \frac{2}{\pi\kappa z^2}\right)$$

$$\mathcal{A}(\kappa,0)=\frac{2\zeta(3)}{\pi^2\kappa}\left(1-\frac{\kappa^3}{16}\right)+\frac{\kappa^2}{\pi^2}\int_0^\infty dz\frac{z}{e^{\kappa z}-1}\log\left(1-e^{-2z}\right)$$

$$R_m=e^{\frac{i\zeta m}{2\pi}},R_{y_\alpha}=e^{\frac{i\zeta y_\alpha}{2\pi}}$$

$$Z_{y_\alpha}^{(N_{\mathrm{f}})}(1)=\frac{1}{2\cosh\frac{m}{2}}\int_{-\infty}^{\infty}\frac{dx_1}{2\pi}e^{-\frac{i\zeta}{2\pi}x_1}\frac{1}{\prod_{\alpha=1}^{N_{\mathrm{f}}}2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}$$

$$x_1=y_\alpha+\pi i+2\pi i n_1,(n_1\in\mathbb{Z}).$$

$$(\aleph)|_{x_1+2\pi i}=(-1)^{N_{\mathrm{f}}}e^{\zeta}(\beth),$$

$$Z_{y_\alpha}^{(N_{\mathrm{f}})}(1)=\frac{1}{1-(-1)^{N_{\mathrm{f}}}e^{\zeta}}\frac{1}{2\cosh\frac{m}{2}}\int_\gamma\frac{dx_1}{2\pi}e^{-\frac{i\zeta}{2\pi}x_1}\frac{1}{\prod_{\alpha=1}^{N_{\mathrm{f}}}2\cosh\frac{x_1-y_\alpha}{2}}$$

$$Z_{y_\alpha}^{(N_{\mathrm{f}})}(1)=\frac{1}{1-(-1)^{N_{\mathrm{f}}}e^{\zeta}}\frac{1}{2\cosh\frac{m}{2}}\sum_{\alpha=1}^{N_{\mathrm{f}}}\left[e^{-\frac{i\zeta}{2\pi}x_1}\frac{1}{\prod_{\beta(\neq\alpha)}^{N_{\mathrm{f}}}2\cosh\frac{x_1-y_\alpha}{2}}\right]_{x_1=y_\alpha+\pi i}$$
$$=\frac{e^{\frac{\zeta}{2}}}{1-(-1)^{N_{\mathrm{f}}}e^{\zeta}}\frac{1}{2\cosh\frac{m}{2}}\sum_{\alpha=1}^{N_{\mathrm{f}}}R_{y_\alpha}^{-1}\frac{1}{\prod_{\beta(\neq\alpha)}^{N_{\mathrm{f}}}2i\sinh\frac{y_\alpha-y_\alpha}{2}}$$

$$Z_{y_\alpha}^{(N_{\mathrm{f}})}(2)=\frac{1}{\left(2\cosh\frac{m}{2}\right)^2}\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx_1dx_2}{(2\pi)^2}e^{-\frac{i\zeta}{2\pi}(x_1+x_2)}\frac{\left(2\sinh\frac{x_1-x_2}{2}\right)^2}{\prod_{\pm}2\cosh\frac{x_1-x_2\pm m}{2}\prod_{i,\alpha}2\cosh\frac{x_i-y_\alpha}{2}}$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_1-x_2\pm m}{2}}\to x_1=y_\alpha+\pi i+2\pi i n_1,x_2=x_1\pm m+\pi i+2\pi i n_2$$
$$\frac{1}{2\cosh\frac{x_2-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_1-x_2\pm m}{2}}\to x_2=y_\alpha+\pi i+2\pi i n_2,x_1=x_2\mp m+\pi i+2\pi i n_1$$
$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_2-y_\beta}{2}}\to x_1=y_\alpha+\pi i+2\pi i n_1,x_2=y_\beta+\pi i+2\pi i n_2$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_1-x_2\pm m}{2}}\to x_1=y_\alpha+\pi i+2\pi i n_1,x_2=x_1\pm m+\pi i+2\pi i n_2$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_2-y_\beta}{2}}\to x_1=y_\alpha+\pi i+2\pi i n_1,x_2=y_\beta+\pi i+2\pi i n_2,(\alpha<\beta)$$

$$Z_{y_\alpha}^{(N_{\mathrm{f}})}(2)=\sum_\alpha\sum_\pm Z_{y_\alpha}^{(N_{\mathrm{f}})}(2;(\mathfrak{I})_{\alpha,\pm})+\sum_{\alpha<\beta}Z_{y_\alpha}^{(N_{\mathrm{f}})}(2;(\mathfrak{H})_{\alpha,\beta}),$$

$$(\daleth)|_{x_2+2\pi i}=(-1)^{N_{\mathrm{f}}}e^{\zeta}(\gimel)$$

$$(\text{ϟ})|_{x_1+2\pi i} = e^{2\zeta}(\dagger)$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}\left(2;(\text{ЊЖ})_{\alpha,\pm}\right) = \frac{1}{1-e^{2\zeta}}\frac{1}{1-(-1)^{N_\mathrm{f}}e^{\zeta}}\frac{1}{\left(2\cosh\frac{m}{2}\right)^2}$$

$$\times\left[e^{-\frac{i\zeta}{2\pi}(x_1+x_2)}\frac{\left(2\sinh\frac{x_1-x_2}{2}\right)^2}{2\cosh\frac{x_1-x_2\mp m}{2}\prod_{\beta(\neq\alpha)}2\cosh\frac{x_1-y_\beta}{2}\prod_\beta 2\cosh\frac{x_2-y_\beta}{2}}\right]_{\substack{x_2=y_\alpha+\pi i\\ x_2=x_1\pm m+\pi i}}$$

$$=\frac{\mp i^{-N_\mathrm{f}}}{(2\sinh m)\left(2\cosh\frac{m}{2}\right)(2\sinh\zeta)\left(e^{\frac{\zeta}{2}}-(-1)^{N_\mathrm{f}}e^{-\frac{\zeta}{2}}\right)}\frac{R_m^{\mp 1}R_{y_\alpha}^{-2}}{\prod_{\beta(\neq\alpha)}2\sinh\frac{y_\alpha-y_\beta}{2}2\cosh\frac{y_\alpha-y_\beta\pm m}{2}}$$

$$(\text{Д})|_{x_2+2\pi i} = (-1)^{N_\mathrm{f}}e^{\zeta}(\text{Ж})$$

$$(\text{Љ})|_{x_1+2\pi i} = (-1)^{N_\mathrm{f}}e^{\zeta}(\text{Њ})$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}\left(2;(\text{ЉА})_{\alpha,\beta}\right) = \left(\frac{1}{1-(-1)^{N_\mathrm{f}}e^{\zeta}}\right)^2\frac{1}{\left(2\cosh\frac{m}{2}\right)^2}$$

$$\times\left[e^{-\frac{i\zeta}{2\pi}(x_1+x_2)}\frac{\left(2\sinh\frac{x_1-x_2}{2}\right)^2}{\prod_\pm 2\cosh\frac{x_1-x_2\pm m}{2}\prod_{\gamma(\neq\alpha)}2\cosh\frac{x_1-y_\gamma}{2}\prod_{\delta(\neq\beta)}2\cosh\frac{x_2-y_\delta}{2}}\right]_{\substack{x_1=y_\alpha+\pi i\\ x_2=y_\beta+\pi i}}$$

$$=\frac{(-1)^{N_\mathrm{f}}}{\left(2\cosh\frac{m}{2}\right)^2\left(e^{\frac{\zeta}{2}}-(-1)^{N_\mathrm{f}}e^{-\frac{\zeta}{2}}\right)^2}\frac{R_{y_\alpha}R_{y_\beta}}{\prod_\pm 2\cosh\frac{y_\alpha-y_\beta\pm m}{2}\prod_{\gamma(\neq\alpha,\beta)}2\sinh\frac{y_\alpha-y_\gamma}{2}2\sinh\frac{y_\beta-y_\gamma}{2}}$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(3) = \frac{1}{\left(2\cosh\frac{m}{2}\right)^3}\frac{1}{6}\int_{-\infty}^{\infty}\frac{dx_1dx_2dx_3}{(2\pi)^3}e^{-\frac{i\zeta}{2\pi}(x_1+x_2+x_3)}$$

$$\times\frac{\left(2\sinh\frac{x_1-x_2}{2}\right)^2\left(2\sinh\frac{x_1-x_3}{2}\right)^2\left(2\sinh\frac{x_2-x_3}{2}\right)^2}{\prod_\pm 2\cosh\frac{x_1-x_2\pm m}{2}2\cosh\frac{x_1-x_3\pm m}{2}2\cosh\frac{x_2-x_3\pm m}{2}\prod_{i,\alpha}2\cosh\frac{x_i-y_\alpha}{2}}$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_1-x_2\pm m}{2}}\frac{1}{2\cosh\frac{x_2-x_3\pm' m}{2}}$$

$$\to x_1 = y_\alpha+\pi i+2\pi i n_1, x_2 = x_1\pm m+\pi i+2\pi i n_2, x_3 = x_2\pm' m+\pi i+2\pi i n_3$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_1-x_2\pm m}{2}}\frac{1}{2\cosh\frac{x_1-x_3\pm' m}{2}}\ \left((\pm,\pm')=(++),(+-),(--)\right)$$

$$\to x_1 = y_\alpha+\pi i+2\pi i n_1, x_2 = x_1\pm m+\pi i+2\pi i n_2, x_3 = x_1\pm' m+\pi i+2\pi i n_3$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_2-y_\beta}{2}}\frac{1}{2\cosh\frac{x_1-x_3\pm m}{2}}$$

$$\to x_1 = y_\alpha+\pi i+2\pi i n_1, x_2 = y_\beta+\pi i+2\pi i n_2, x_3 = x_1\pm m+\pi i+2\pi i n_3$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_2-y_\beta}{2}}\frac{1}{2\cosh\frac{x_3-y_\gamma}{2}}(\alpha\le\beta\le\gamma)$$

$$\rightarrow x_1=y_\alpha+\pi i+2\pi i n_1, x_2=y_\beta+\pi i+2\pi i n_2, x_3=y_\gamma+\pi i+2\pi i n_3$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_1-x_2\pm m}{2}}\frac{1}{2\cosh\frac{x_2-x_3\pm m}{2}}$$

$$\rightarrow x_1=y_\alpha+\pi i+2\pi i n_1, x_2=x_1\pm m+\pi i+2\pi i n_2, x_3=x_2\pm m+\pi i+2\pi i n_3,$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_1-x_2+m}{2}}\frac{1}{2\cosh\frac{x_1-x_3-m}{2}}\rightarrow x_1=y_\alpha+\pi i+2\pi i n_1, x_2$$

$$=x_1+m+\pi i+2\pi i n_2, x_3=x_1-m+\pi i+2\pi i n_3$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_2-y_\beta}{2}}\frac{1}{2\cosh\frac{x_1-x_3\pm m}{2}}\ (\alpha\neq\beta)$$

$$\rightarrow x_1=y_\alpha+\pi i+2\pi i n_1, x_2=y_\beta+\pi i+2\pi i n_2, x_3=x_1\pm m+\pi i+2\pi i n_3$$

$$\frac{1}{2\cosh\frac{x_1-y_\alpha}{2}}\frac{1}{2\cosh\frac{x_2-y_\beta}{2}}\frac{1}{2\cosh\frac{x_3-y_\gamma}{2}}\ (\alpha<\beta<\gamma)$$

$$\rightarrow x_1=y_\alpha+\pi i+2\pi i n_1, x_2=y_\beta+\pi i+2\pi i n_2, x_3=y_\gamma+\pi i+2\pi i n_3$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(3)=\sum_\alpha\sum_\pm Z_{y_\alpha}^{(N_\mathrm{f})}\left(3;(\text{Ꮒ})_{\alpha,\pm}\right)+\sum_\alpha Z_{y_\alpha}^{(N_\mathrm{f})}\left(3;(\text{Ꮂ})_\alpha\right)+\sum_{\alpha\neq\beta}\sum_\pm Z_{y_\alpha}^{(N_\mathrm{f})}\left(3;(\text{ᘒ})_{\alpha,\beta,\pm}\right)$$
$$+\sum_{\alpha<\beta<\gamma} Z_{y_\alpha}^{(N_\mathrm{f})}\left(3;(\text{ᘜ})_{\alpha,\beta,\gamma}\right)$$

$$(\text{ധ})|_{x_3+2\pi i}=(-1)^{N_\mathrm{f}}e^{\zeta}(\text{ᔭ})$$
$$(\text{ꞴX})|_{x_2+2\pi i}=e^{2\zeta}(\text{Ꮒ})$$
$$(\text{ᘐ})|_{x_1+2\pi i}=(-1)^{N_\mathrm{f}}e^{3\zeta}(\text{Л})$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(3;(\Re)_{\alpha,\pm}) = \frac{1}{1-(-1)^{N_\mathrm{f}}e^{3\zeta}}\frac{1}{1-e^{2\zeta}}\frac{1}{1-(-1)^{N_\mathrm{f}}e^{\zeta}}\frac{1}{\left(2\cosh\frac{m}{2}\right)^3}\Bigg[e^{-\frac{i\zeta}{2\pi}(x_1+x_2+x_3)}$$

$$\times\frac{\left(2\sinh\frac{x_1-x_2}{2}\right)^2\left(2\sinh\frac{x_1-x_3}{2}\right)^2\left(2\sinh\frac{x_2-x_3}{2}\right)^2}{2\cosh\frac{x_1-x_2\mp m}{2}\,2\cosh\frac{x_2-x_3\mp m}{2}\prod_{\pm'}2\cosh\frac{x_1-x_3\pm' m}{2}\prod_{\beta(\neq\alpha)}2\cosh\frac{x_1-y_\beta}{2}}$$

$$\times\prod_\beta\frac{1}{2\cosh\frac{x_2-y_\beta}{2}\,2\cosh\frac{x_3-y_\beta}{2}}\Bigg]_{\substack{x_1=y_\alpha+\pi i\\ x_2=x_1\pm m+\pi i\\ x_3=x_2\pm m+\pi i}}$$

$$=\frac{\pm i(-1)^{N_\mathrm{f}}}{\left(2\cosh\frac{3m}{2}\right)(2\sinh m)\left(2\cosh\frac{m}{2}\right)}\frac{1}{e^{\frac{3\zeta}{2}}-(-1)^{N_\mathrm{f}}e^{-\frac{3\zeta}{2}}}\frac{1}{2\sinh\zeta}\frac{1}{e^{\frac{\zeta}{2}}-(-1)^{N_\mathrm{f}}e^{-\frac{\zeta}{2}}}$$

$$\times\frac{R_m^{\mp3}R_{y_\alpha}^{-3}}{\prod_{\beta(\neq\alpha)}2\sinh\frac{y_\alpha-y_\beta}{2}\,2\cosh\frac{y_\alpha-y_\beta\pm m}{2}\,2\sinh\frac{y_\alpha-y_\beta\pm 2m}{2}}$$

$$(\text{Ѧ})|_{x_3+2\pi i} = (-1)^{N_\mathrm{f}}e^{\zeta}(\text{Ŧ})$$
$$(\text{Ꜧ})|_{x_2+2\pi i} = (-1)^{N_\mathrm{f}}e^{\zeta}(\text{ϰ})$$
$$(\text{ريال})|_{x_1+2\pi i} = (-1)^{N_\mathrm{f}}e^{3\zeta}(\text{ɣ})$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(3;(\mathcal{R})_\alpha)$$

$$=\frac{1}{1-(-1)^{N_\mathrm{f}}e^{3\zeta}}\left(\frac{1}{1-(-1)^{N_\mathrm{f}}e^{\zeta}}\right)^2\frac{1}{\left(2\cosh\frac{m}{2}\right)^3}\Bigg[e^{-\frac{i\zeta}{2\pi}(x_1+x_2+x_3)}$$

$$\times\frac{\left(2\sinh\frac{x_1-x_2}{2}\right)^2\left(2\sinh\frac{x_1-x_3}{2}\right)^2\left(2\sinh\frac{x_2-x_3}{2}\right)^2}{2\cosh\frac{x_1-x_2-m}{2}\,2\cosh\frac{x_1-x_3+m}{2}\prod_\pm 2\cosh\frac{x_2-x_3\pm m}{2}\prod_{\beta(\neq\alpha)}2\cosh\frac{x_1-y_\beta}{2}}\times\prod_\beta\frac{1}{2\cosh\frac{x_2-y_\beta}{2}\,2\cosh\frac{x_3-y_\beta}{2}}\Bigg]_{\substack{x_2=x_1+m+\pi i\\ x_3=x_1-m+\pi i\\ x_1=y_\alpha+\pi i}}$$

$$=\frac{1}{\left(2\cosh\frac{3m}{2}\right)\left(2\cosh\frac{m}{2}\right)^2}\frac{i^{N_\mathrm{f}-1}}{e^{\frac{3\zeta}{2}}-(-1)^{N_\mathrm{f}}e^{-\frac{3\zeta}{2}}}\frac{1}{\left(e^{\frac{\zeta}{2}}-(-1)^{N_\mathrm{f}}e^{-\frac{\zeta}{2}}\right)^2}\times\frac{R_{y_\alpha}^{-3}}{\prod_{\beta(\neq\alpha)}2\sinh\frac{y_\alpha-y_\beta}{2}\prod_\pm 2\cosh\frac{y_\alpha-y_\beta\pm m}{2}}$$

$$(\ell)|_{x_3+2\pi i} = (-1)^{N_\mathrm{f}}e^{\zeta}(\mathcal{J})$$
$$(\mathfrak{W})|_{x_2+2\pi i} = (-1)^{N_\mathrm{f}}e^{\zeta}(\mathfrak{M})$$
$$(\mathfrak{T})|_{x_1+2\pi i} = e^{2\zeta}(\mathfrak{G})$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(3;(\mathfrak{X})_{\alpha,\beta,\pm}) = \frac{1}{1-e^{2\zeta}}\left(\frac{1}{1-(-1)^{N_\mathrm{f}}e^{\zeta}}\right)^2 \frac{1}{\left(2\cosh\frac{m}{2}\right)^3}\Bigg[e^{-\frac{i\zeta}{2\pi}(x_1+x_2+x_3)}$$

$$\times \frac{\left(2\sinh\frac{x_1-x_2}{2}\right)^2\left(2\sinh\frac{x_1-x_3}{2}\right)^2\left(2\sinh\frac{x_2-x_3}{2}\right)^2}{2\cosh\frac{x_1-x_3\mp m}{2}\prod_{\pm'} 2\cosh\frac{x_1-x_2\pm' m}{2} 2\cosh\frac{x_2-x_3\pm' m}{2}\prod_{\gamma(\neq\alpha)} 2\cosh\frac{x_1-y_\gamma}{2}}$$

$$\times \prod_{\delta(\neq\beta)}\frac{1}{2\cosh\frac{x_2-y_\delta}{2}}\prod_{\gamma}\frac{1}{2\cosh\frac{x_3-y_\gamma}{2}}\Bigg]_{\substack{x_1=y_\alpha+\pi i\\ x_2=y_\beta+\pi i\\ x_3=x_1\pm m+\pi i}}$$

$$= \frac{\mp i}{(2\sinh m)\left(2\cosh\frac{m}{2}\right)^2(2\sinh\zeta)}\frac{1}{\left(e^{\frac{\zeta}{2}}-(-1)^{N_\mathrm{f}}e^{-\frac{\zeta}{2}}\right)^2}\times\frac{R_{y_\alpha}^{-2}R_{y_\beta}^{-1}R_m^{\mp 1}}{\prod_{\gamma(\neq\alpha,\beta)} 2\sinh\frac{y_\alpha-y_\gamma}{2} 2\sinh\frac{y_\beta-y_\gamma}{2} 2\cosh\frac{y_\alpha-y_\gamma\pm m}{2}}$$

$$\times\frac{1}{2\sinh\frac{y_\alpha-y_\beta}{2} 2\cosh\frac{y_\alpha-y_\beta\mp m}{2} 2\sinh\frac{y_\alpha-y_\beta\pm 2m}{2}}$$

$$\begin{aligned}(\mathfrak{D})|_{x_3+2\pi i} &= (-1)^{N_\mathrm{f}}e^{\zeta}(\mathfrak{E})\\ (\mathfrak{F})|_{x_2+2\pi i} &= (-1)^{N_\mathrm{f}}e^{\zeta}(\mathfrak{I})\\ (\mathfrak{K})|_{x_1+2\pi i} &= (-1)^{N_\mathrm{f}}e^{\zeta}(\mathfrak{L})\end{aligned}$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(3;(\mathrm{iv})_{\alpha,\beta,\gamma}) = \left(\frac{1}{1-(-1)^{N_\mathrm{f}}e^{\zeta}}\right)^3\frac{1}{\left(2\cosh\frac{m}{2}\right)^3}\Bigg[e^{-\frac{i\zeta}{2\pi}(x_1+x_2+x_3)}$$

$$\times\frac{\left(2\sinh\frac{x_1-x_2}{2}\right)^2\left(2\sinh\frac{x_1-x_3}{2}\right)^2\left(2\sinh\frac{x_2-x_3}{2}\right)^2}{\prod_{\pm} 2\cosh\frac{x_1-x_2\pm m}{2} 2\cosh\frac{x_1-x_3\pm m}{2} 2\cosh\frac{x_2-x_3\pm m}{2}\prod_{\delta(\neq\alpha)} 2\cosh\frac{x_1-y_\delta}{2}}$$

$$\times\prod_{\delta'(\neq\beta)}\frac{1}{2\cosh\frac{x_2-y_{\delta'}}{2}}\prod_{\delta''(\neq\gamma)}\frac{1}{2\cosh\frac{x_3-y_{\delta''}}{2}}\Bigg]_{\substack{x_1=y_\alpha+\pi i\\ x_2=y_\beta+\pi i\\ x_3=y_\gamma+\pi i}}$$

$$=\frac{1}{\left(2\cosh\frac{m}{2}\right)^3}\frac{i^{3N_\mathrm{f}-1}}{\left(e^{\frac{\zeta}{2}}-(-1)^{N_\mathrm{f}}e^{-\frac{\zeta}{2}}\right)^3}\frac{R_{y_\alpha}^{-1}R_{y_\beta}^{-1}R_{y_\gamma}^{-1}}{\prod_{\pm} 2\cosh\frac{y_\alpha-y_\beta\pm m}{2} 2\cosh\frac{y_\alpha-y_\gamma\pm m}{2} 2\cosh\frac{y_\beta-y_\gamma\pm m}{2}}$$

$$\times\frac{1}{\prod_{\delta(\neq\alpha,\beta,\gamma)} 2\sinh\frac{y_\alpha-y_\delta}{2} 2\sinh\frac{y_\beta-y_\delta}{2} 2\sinh\frac{y_\gamma-y_\delta}{2}}$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(N) = \sum_{\substack{\lambda^{(\alpha)}\\ \left(\sum_{\alpha=1}^{N_\mathrm{f}}|\lambda^{(\alpha)}|=N\right)}} Z_{\lambda^{(\alpha)}}^{(N_\mathrm{f})}.$$

$$Z^{(N_f)}_{\lambda^{(\alpha)}} \propto \prod_{\alpha=1}^{N_f} R_m^{-\sum_{\square=(a,b)\in\lambda^{(\alpha)}}(b-a)} = R_m^{-\frac{1}{2}\sum_{\alpha=1}^{N_f}\sum_{i=1}^{\ell(\lambda^{(\alpha)})}\lambda_i^{(\alpha)}\left(\lambda_i^{(\alpha)}-2i+1\right)}$$

$$Z^{(N_f)}_{\lambda^{(\alpha)}} \propto \prod_{\alpha=1}^{N_f} R_{y_\alpha}^{-|\lambda^{(\alpha)}|}.$$

$$Z^{(N_f)}_{\lambda^{(\alpha)}} = \left(\prod_{\alpha=1}^{N_f}\prod_{\square\in\lambda^{(\alpha)}}\frac{1}{1-((-1)^{N_f}e^{\zeta})^{h_{\lambda^{(\alpha)}}(\square)}}\right)\frac{1}{\left(2\cosh\frac{m}{2}\right)^N}$$
$$\times\left[e^{\frac{\zeta}{2\pi i}\sum_i x_i}\frac{\prod_{i<j}\left(2\sinh\frac{x_i-x_j}{2}\right)^2}{\prod'_{i<j,\pm}2\cosh\frac{x_i-x_j\pm m}{2}}\frac{1}{\prod'_{i,\alpha}2\cosh\frac{x_i-y_\alpha}{2}}\right]_{\text{curvature}}$$

$$Z^{(1)}_{\lambda} = \frac{(-iR_m)^{-\frac{1}{2}\sum_{i=1}^{\ell(\lambda)}\lambda_i(\lambda_i-2i+1)}R_{y_1}^{-|\lambda|}}{\prod_{\square\in\lambda}\left(e^{\frac{\zeta h_\lambda(\square)}{2}}-\left(-e^{-\frac{\zeta}{2}}\right)^{h_\lambda(\square)}\right)\left(e^{\frac{m h_\lambda(\square)}{2}}-\left(-e^{-\frac{m}{2}}\right)^{h_\lambda(\square)}\right)}.$$

$$Z^{(N_f)}_{\lambda^{(\alpha)}} = \Bigg[\Bigg(\prod_{\alpha=1}^{N_f}\left(\prod_{\square\in\lambda^{(\alpha)}}\frac{1}{1-((-1)^{N_f}e^{\zeta})^{h_{\lambda^{(\alpha)}}(\square)}}\right)\frac{1}{\left(2\cosh\frac{m}{2}\right)^{|\lambda(\alpha)|}}\exp\left(-\frac{i\zeta}{2\pi}\sum_{\substack{i\\(x_i\in\lambda^{(\alpha)})}}x_i\right)$$
$$\times\frac{\prod'_{(x_i,x_j\in\lambda^{(\alpha)})}\left(2\sinh\frac{x_i-x_j}{2}\right)^2}{\prod'_{(x_i,x_j\in\lambda^{(\alpha)})}2\cosh\frac{x_i-x_j\pm m}{2}}\frac{1}{\prod'_{(x_i\in\lambda^{(\alpha)})}2\cosh\frac{x_i-y_\alpha}{2}}\Bigg)$$
$$\times\prod_{\alpha<\beta}\frac{\prod_{x_i\in\lambda^{(\alpha)},x_j\in\lambda^{(\beta)}}\left(2\sinh\frac{x_i-x_j}{2}\right)^2}{\prod_{x_i\in\lambda^{(\alpha)},x_j\in\lambda^{(\beta)},\pm}2\cosh\frac{x_i-x_j\pm m}{2}}\prod_{x_i\in\lambda^{(\alpha)}}\frac{1}{2\cosh\frac{x_i-y_\beta}{2}}\prod_{x_i\in\lambda^{(\beta)}}\frac{1}{2\cosh\frac{x_i-y_\alpha}{2}}\Bigg]_{\text{curvature}}$$

$$\Bigg[\left(\prod_{\square\in\lambda^{(\alpha)}}\frac{1}{1-((-1)^{N_f}e^{\zeta})^{h_{\lambda^{(\alpha)}}(\square)}}\right)\frac{1}{\left(2\cosh\frac{m}{2}\right)^{|\lambda^{(\alpha)}|}}\exp\left(\frac{\zeta}{2\pi i}\sum_{\substack{i\\(x_i\in\lambda(\alpha))}}x_i\right)$$
$$\times\frac{\prod'_{\substack{i<j\\(x_i,x_j\in\lambda^{(\alpha)})}}\left(2\sinh\frac{x_i-x_j}{2}\right)^2}{\prod'_{\substack{i<j,\pm\\(x_i,x_j\in\lambda^{(\alpha)})}}2\cosh\frac{x_i-x_j\pm m}{2}}\frac{1}{\prod'_{\substack{i\\(x_i\in\lambda^{(\alpha)})}}2\cosh\frac{x_i-y_\alpha}{2}}\Bigg]_{\text{curvature}}$$

$$=\frac{(-iR_m)^{-\frac{1}{2}\sum_{i=1}^{\ell(\lambda^{(\alpha)})}\lambda_i^{(\alpha)}(\lambda_i^{(\alpha)}-2i+1)}R_{y_\alpha}^{-|\lambda^{(\alpha)}|}}{\prod_{\square\in\lambda^{(\alpha)}}\left(\left((-1)^{N_f-1}e^{\frac{\zeta}{2}}\right)^{h_{\lambda^{(\alpha)}}(\square)}-\left(-e^{-\frac{\zeta}{2}}\right)^{h_{\lambda^{(\alpha)}}(\square)}\right)\left(e^{\frac{m h_{\lambda^{(\alpha)}}(\square)}{2}}-\left(-e^{-\frac{m}{2}}\right)^{h_{\lambda^{(\alpha)}}(\square)}\right)}.$$

$$\left[\frac{\prod_{x_i\in\lambda^{(\alpha)},x_j\in\lambda^{(\beta)}}\left(2\sinh\frac{x_i-x_j}{2}\right)^2}{\prod_{x_i\in\lambda^{(\alpha)},x_j\in\lambda^{(\beta)},\pm}2\cosh\frac{x_i-x_j\pm m}{2}}\prod_{x_i\in\lambda^{(\alpha)}}\frac{1}{2\cosh\frac{x_i-y_\beta}{2}}\prod_{x_i\in\lambda^{(\beta)}}\frac{1}{2\cosh\frac{x_i-y_\alpha}{2}}\right]$$

$$=g_{\lambda^{(\alpha)},\lambda^{(\beta)}}(y_\alpha-y_\beta),$$

$$g_{\lambda,\mu}(y)=\left(\prod_{\square=(a,b)\in\lambda}\prod_{\square'=(a',b')\in\mu}\frac{(2\sinh\frac{y+(b-b'-a+a')m+\pi i(a+b-a'-b')}{2})^2}{\prod_\pm 2\cosh\frac{y+(b-b'-a+a'\pm1)m+\pi i(a+b-a'-b')}{2}}\right)$$
$$\times\left(\prod_{\square=(a,b)\in\lambda}\frac{1}{2\cosh\frac{y+(b-a)m+\pi i(a+b-1)}{2}}\right)\left(\prod_{\square=(a,b)\in\mu}\frac{1}{2\cosh\frac{-y+(b-a)m+\pi i(a+b-1)}{2}}\right)$$

$$Z^{(N_\mathrm{f})}_{\lambda^{(\alpha)}}=$$
$$\left(\prod_{\alpha=1}^{N_\mathrm{f}}\frac{(-iR_m)^{-\frac{1}{2}\sum_{i=1}^{\ell(\lambda^{(\alpha)})}\lambda_i^{(\alpha)}(\lambda_i^{(\alpha)}-2i+1)}R_{y_\alpha}^{-|\lambda^{(\alpha)}|}}{\prod_{\square\in\lambda^{(\alpha)}}\left(\left((-1)^{N_\mathrm{f}-1}e^{\frac{\zeta}{2}}\right)^{h_{\lambda^{(\alpha)}}(\square)}-(-e^{-\frac{\zeta}{2}})^{h_{\lambda^{(\alpha)}}(\square)}\right)\left(e^{\frac{mh_{\lambda^{(\alpha)}}(\square)}{2}}-\left(-e^{-\frac{m}{2}}\right)^{h_{\lambda^{(\alpha)}}(\square)}\right)}\right)$$

$$\times\prod_{\alpha<\beta}g_{\lambda^{(\alpha)},\lambda^{(\beta)}}(y_\alpha-y_\beta).$$

$$g_{\lambda,\mu}(y)=E_{\lambda,\mu}(y)\times\prod_{\square\in\lambda}e^{-\frac{\pi i(\mathrm{arm}_\lambda(\square)-\mathrm{leg}_\mu(\square)}{2}}\prod_{\square\in\mu}e^{\frac{\pi i(\mathrm{arm}_\mu(\square)-\mathrm{leg}_\lambda(\square)}{2}}.$$

$$E_{\lambda,\mu}(y)=\frac{1}{\prod_{\square\in\lambda}\left(e^{\frac{y}{2}+\frac{m}{2}(\mathrm{arm}_\lambda(\square)+\mathrm{leg}_\mu(\square)+1)}+e^{-\frac{y}{2}-\frac{m}{2}(\mathrm{arm}_\lambda(\square)+\mathrm{leg}_\mu(\square)+1)-\pi i(\mathrm{arm}_\lambda(\square)-\mathrm{leg}_\mu(\square))}\right)}$$
$$\times\frac{1}{\prod_{\square\in\mu}\left(e^{\frac{y}{2}-\frac{m}{2}(\mathrm{arm}_\mu(\square)+\mathrm{leg}_\lambda(\square)+1)}+e^{-\frac{y}{2}+\frac{m}{2}(\mathrm{arm}_\mu(\square)+\mathrm{leg}_\lambda(\square)+1)+\pi i(\mathrm{arm}_\mu(\square)-\mathrm{leg}_\lambda(\square))}\right)}$$

$$\Xi_0^{(1)}(\zeta,m;u)=\sum_{N=0}^{\infty}u^N Z_0^{(1)}(\zeta,m;N)$$

$$\Xi_0^{(1)}(0{,}0;u)^2-\Xi_0^{(1)}(0{,}0;-u)^2-u\prod_\pm\Xi_0^{(1)}(0{,}0;\pm iu)=0$$

$$a_1\prod_\pm\Xi_0^{(1)}(b_1^{\pm1}u)+a_2\prod_\pm\Xi_0^{(1)}(b_2^{\pm1}u)+cu\prod_\pm\Xi_0^{(1)}(d^{\pm1}u)=0.$$

$$Z_0^{(1)}(0)=1, Z_0^{(1)}(1)=\frac{1}{2\cosh\frac{\zeta}{2}2\cosh\frac{m}{2}}, Z_0^{(1)}(2)=\frac{i(R_m^{-1}-R_m)}{2\sinh\zeta 2\cosh\frac{\zeta}{2}2\sinh m 2\cosh\frac{m}{2}}$$

$$Z_0^{(1)}(1)=-\frac{cZ_0^{(1)}(0)}{b_1+b_1^{-1}-b_2-b_2^{-1}}, Z_0^{(1)}(2)=-\frac{c(d+d^{-1})Z_0^{(1)}(1)}{b_1^2+b_1^{-2}-b_2^2-b_2^{-2}},$$

$$Z_0^{(1)}(1)=\frac{1}{e^{\frac{\zeta+m}{2}}+e^{-\frac{\zeta+m}{2}}+e^{-\frac{\zeta-m}{2}}+e^{\frac{\zeta-m}{2}}}, Z_0^{(1)}(2)=\frac{i(R_m^{-1}-R_m)Z_0^{(1)}(1)}{2\sinh\zeta 2\sinh m}$$

$$b_1=e^{\frac{\zeta+m}{2}}, b_2=-e^{-\frac{\zeta-m}{2}}, c=-1, d=-iR_m$$

$$\prod_{\pm}\Xi_0^{(1)}\left(e^{\pm\frac{\zeta+m}{2}}u\right)-\prod_{\pm}\Xi_0^{(1)}\left(-e^{\mp\frac{\zeta-m}{2}}u\right)-u\prod_{\pm}\Xi_0^{(1)}(\mp iR_m^{\pm1}u)=0.$$

$$Z_0^{(1)}(N)=\frac{1}{2\cosh\frac{(\zeta+m)N}{2}-(-1)^N 2\cosh\frac{(\zeta-m)N}{2}}$$

$$\times[-\sum_{N'=1}^{N-1}\left(e^{\frac{(\zeta+m)(2N'-N)}{2}}-(-1)^N e^{-\frac{(\zeta-m)(2N'-N)}{2}}\right)Z_0^{(1)}(N')Z_0^{(1)}(N-N')$$
$$+\sum_{N'=0}^{N-1}(-iR_m)^{2N'-N+1}Z_0^{(1)}(N')Z_0^{(1)}(N-1-N')\Bigg]$$

$$Z_0^{(1)}(N)=\sum_{a=-\frac{N(N-1)}{2}}^{\frac{N(N-1)}{2}}R_m^a f_a(\zeta,m)$$

$$Z_y^{(2)}(1)=\frac{i\left(R_y^{-1}-R_y\right)}{\left(2\sinh\frac{\zeta}{2}\right)\left(2\cosh\frac{m}{2}\right)(2\sinh y)}$$

$$Z_y^{(2)}(2)=$$
$$\sum_{\pm}\frac{\pm R_m^{\mp1}}{(2\sinh\zeta)\left(2\sinh\frac{\zeta}{2}\right)(2\sinh m)\left(2\cosh\frac{m}{2}\right)(2\sinh y)}\left(\frac{R_y^{-2}}{2\cosh\left(y\pm\frac{m}{2}\right)}-\frac{R_y^2}{2\cosh\left(y\mp\frac{m}{2}\right)}\right)$$
$$+\frac{1}{\left(2\sinh\frac{\zeta}{2}\right)^2\left(2\cosh\frac{m}{2}\right)^2\prod_{\pm}2\cosh\left(y\pm\frac{m}{2}\right)}.$$

$$\Xi_y^{(2)}(u)=\sum_{N=0}^{\infty}u^N Z_y^{(2)}(N)$$

$$\sum_{i=1}^{L_1}a_i\prod_{\pm}\Xi_y(b_i^{\pm1}u)+uR_y\sum_{i=1}^{L_2}c_i\prod_{\pm}\Xi_y(d_i^{\pm1}u)+uR_y^{-1}\sum_{i=1}^{L_2}c_i'\prod_{\pm}\Xi_y(d_i'^{\pm1}u)\stackrel{?}{=}0$$

- $R_y R_m^{\frac{1}{2}} Z^{(2)}_{y+\frac{m}{2}}(1)$ contains $R_m^0 R_y^0$ and $R_m R_y^2$
- $R_y R_m^{-\frac{1}{2}} Z^{(2)}_{y-\frac{m}{2}}(1)$ contains $R_m^0 R_y^0$ and $R_m^{-1} R_y^2$
- $R_y^{-1} R_m^{-\frac{1}{2}} Z^{(2)}_{y+\frac{m}{2}}(1)$ contains $R_m^{-1} R_y^{-2}$ and $R_m^0 R_y^0$,
- $R_y^{-1} R_m^{\frac{1}{2}} Z^{(2)}_{y-\frac{m}{2}}(1)$ contains $R_m R_y^{-2}$ and $R_m^0 R_y^0$.

$$\sum_{i=1}^{L_1} a_i \prod_{\pm} \Xi_y\left(b_i^{\pm 1} u\right) + u R_y \sum_{i=1}^{L_2} c_i \prod_{\pm} \Xi_{y \pm \left(\frac{m}{2}+\Delta_i\right)}\left(\left(R_m^{\frac{1}{2}} \tilde{d}_i\right)^{\pm 1} u\right) + u R_y^{-1} \sum_{i=1}^{L_2} c_i' \prod_{\pm} \Xi_{y \pm \left(\frac{m}{2}+\Delta_i'\right)}\left(\left(R_m^{-\frac{1}{2}} \tilde{d}_i'\right)^{\pm 1} u\right) = 0$$

$$\sum_{i=1}^{L_1} a_i = 0, Z_y^{(2)}(1) = -\frac{R_y \sum_{i=1}^{L_2} c_i + R_y^{-1} \sum_{i=1}^{L_2} c_i'}{\sum_{i=1}^{L_1} a_i\left(b_i + b_i^{-1}\right)}$$

$$a_1 = 1, a_2 = -1, b_1 = i e^{\frac{\zeta+m}{2}}, b_2 = i e^{-\frac{\zeta-m}{2}}, c_1 = -\frac{1}{2\sinh\ y}, c_1' = \frac{1}{2\sinh\ y}.$$

$$\begin{aligned}
Z_y^{(2)}(2) =& \frac{1}{2\sinh\ \zeta 2\sinh\ m 2\sinh\ y}\Bigg[-R_y\left(R_m^{\frac{1}{2}} \tilde{d}_1 Z^{(2)}_{y+\frac{m}{2}+\Delta_1}(1) + R_m^{-\frac{1}{2}} \tilde{d}_1^{-1} Z^{(2)}_{y-\frac{m}{2}-\Delta_1}(1)\right) \\
&+ R_y^{-1}\left(R_m^{-\frac{1}{2}} \tilde{d}_1' Z^{(2)}_{y+\frac{m}{2}+\Delta_1'}(1) + R_m^{\frac{1}{2}} \tilde{d}_1'^{-1} Z^{(2)}_{y-\frac{m}{2}-\Delta_1'}(1)\right)\Bigg] \\
=& \frac{i \tilde{d}_1 e^{\frac{i\zeta\Delta_1}{2\pi}} R_m R_y^2}{2\sinh\ \zeta 2\sinh\ \frac{\zeta}{2} 2\sinh\ m 2\cosh\ \frac{m}{2} 2\sinh\ y 2\sinh\ \left(y+\frac{m}{2}+\Delta_1\right)} \\
&+ \frac{i \tilde{d}_1' - 1 e^{\frac{i\zeta\Delta_1'}{2\pi}} R_m R_y^{-2}}{2\sinh\ \zeta 2\sinh\ \frac{\zeta}{2} 2\sinh\ m 2\cosh\ \frac{m}{2} 2\sinh\ y 2\sinh\ \left(y-\frac{m}{2}-\Delta_1'\right)} \\
&+ \frac{i \tilde{d}_1^{-1} e^{-\frac{i\zeta\Delta_1}{2\pi}} R_m^{-1} R_y^2}{2\sinh\ \zeta 2\sinh\ \frac{\zeta}{2} 2\sinh\ m 2\cosh\ \frac{m}{2} 2\sinh\ y 2\sinh\ \left(y-\frac{m}{2}-\Delta_1\right)} \\
&+ \frac{i \tilde{d}_1 e^{-\frac{i\zeta\Delta_1'}{2\pi}} R_m^{-1} R_y^{-2}}{2\sinh\ \zeta 2\sinh\ \frac{\zeta}{2} 2\sinh\ m 2\cosh\ \frac{m}{2} 2\sinh\ y 2\sinh\ \left(y+\frac{m}{2}+\Delta_1'\right)} \\
&+ \frac{1}{2\sinh\ \zeta 2\sinh\ \frac{\zeta}{2} 2\sinh\ m 2\cosh\ \frac{m}{2} 2\sinh\ y}\Bigg[-\frac{i \tilde{d}_1 e^{-\frac{i\zeta\Delta_1}{2\pi}}}{2\sinh\ \left(y+\frac{m}{2}+\Delta_1\right)} \\
&- \frac{i \tilde{d}_1^{-1} e^{\frac{i\zeta\Delta_1}{2\pi}}}{2\sinh\ \left(y-\frac{m}{2}-\Delta_1\right)} - \frac{i \tilde{d}_1' e^{\frac{i\zeta\Delta_1'}{2\pi}}}{2\sinh\ \left(y+\frac{m}{2}+\Delta_1'\right)} - \frac{i \tilde{d}_1' - 1 e^{-\frac{i\zeta\Delta_1'}{2\pi}}}{2\sinh\ \left(y-\frac{m}{2}-\Delta_1'\right)}\Bigg].
\end{aligned}$$

$$\frac{1}{2\cosh\left(y+\frac{m}{2}\right)}=\frac{i\tilde{d}_1 e^{\frac{i\zeta\Delta_1}{2\pi}}}{2\sinh\left(y+\frac{m}{2}+\Delta_1\right)},-\frac{1}{2\cosh\left(y-\frac{m}{2}\right)}=\frac{i\tilde{d}_1'-1}{2\sinh\left(y-\frac{i\zeta\Delta_1'}{2\pi}\right.}$$

$$-\frac{1}{2\cosh\left(y-\frac{m}{2}\right)}=\frac{i\tilde{d}_1^{-1} e^{-\frac{i\zeta\Delta_1}{2\pi}}}{2\sinh\left(y-\frac{m}{2}-\Delta_1\right)},\frac{1}{2\cosh\left(y+\frac{m}{2}\right)}=\frac{i\tilde{d}_1' e^{-\frac{i\zeta\Delta_1'}{2\pi}}}{2\sinh\left(y+\frac{m}{2}+\Delta_1'\right)}$$

$$\tilde{d}_4=\sigma e^{\frac{\zeta\sigma}{4}},\tilde{d}_1'=\sigma' e^{-\frac{\zeta'\sigma}{4}},\Delta_1=\frac{\pi i\sigma}{2},\Delta_1'=\frac{\pi i\sigma'}{2},(\sigma,\sigma'=\pm1)$$

$$\frac{1}{\prod_\pm 2\cosh\left(y\pm\frac{m}{2}\right)}=\frac{1}{2\cosh\frac{\zeta}{2}2\sinh\frac{m}{2}2\sinh y}\left(-\frac{e^{\frac{\zeta\sigma}{2}}+e^{-\frac{\zeta\sigma'}{2}}}{2\cosh\left(y+\frac{m}{2}\right)}+\frac{e^{-\frac{\zeta\sigma}{2}}+e^{\frac{\zeta\sigma'}{2}}}{2\cosh\left(y-\frac{m}{2}\right)}\right)$$

$$\prod_\pm \Xi_y^{(2)}\left(\pm ie^{\pm\frac{\zeta+m}{2}}u\right)-\prod_\pm \Xi_y^{(2)}\left(\pm ie^{\mp\frac{\zeta-m}{2}}u\right)-\frac{uR_y}{2\sinh y}\prod_\pm \Xi^{(2)}_{y\pm\left(\frac{m}{2}+\frac{\pi i}{2}\right)}\left(e^{\pm\frac{\zeta}{4}}R_m^{\pm\frac{1}{2}}u\right)$$
$$+\frac{uR_y^{-1}}{2\sinh y}\prod_\pm \Xi^{(2)}_{y\mp\left(\frac{m}{2}+\frac{\pi i}{2}\right)}\left(e^{\pm\frac{\zeta}{4}}R_m^{\pm\frac{1}{2}}u\right)=0$$

$$\prod_\pm \Xi_y^{(2)}\left(\pm ie^{\pm\frac{\zeta+m}{2}}u\right)-\prod_\pm \Xi_y^{(2)}\left(\pm ie^{\mp\frac{\zeta-m}{2}}u\right)-\frac{uR_y}{2\sinh y}\prod_\pm \Xi^{(2)}_{y\pm\left(\frac{m}{2}-\frac{\pi i}{2}\right)}\left(-e^{\mp\frac{\zeta}{4}}R_m^{\pm\frac{1}{2}}u\right)$$
$$+\frac{uR_y^{-1}}{2\sinh y}\prod_\pm \Xi^{(2)}_{y\mp\left(\frac{m}{2}-\frac{\pi i}{2}\right)}\left(-e^{\mp\frac{\zeta}{4}}R_m^{\pm\frac{1}{2}}u\right)=0$$

$$-\prod_\pm \Xi_y^{(2)}\left(-m';\pm ie^{\pm\frac{\zeta+m'}{2}}u'\right)+\prod_\pm \Xi_y^{(2)}\left(-m';\pm ie^{\mp\frac{\zeta-m'}{2}}u'\right)$$
$$+\frac{u'R_y}{2\sinh y}\prod_\pm \Xi^{(2)}_{y\pm\left(\frac{m'}{2}+\frac{\pi i}{2}\right)}\left(-m';e^{\pm\frac{\zeta}{4}}R_{m'}^{\pm\frac{1}{2}}u'\right)-\frac{u'R_y^{-1}}{2\sinh y}\prod_\pm \Xi^{(2)}_{y\mp\left(\frac{m'}{2}+\frac{\pi i}{2}\right)}\left(-m';e^{\pm\frac{\zeta}{4}}R_{m'}^{\pm\frac{1}{2}}u'\right)=0,$$

$$\Xi_{y_\alpha}^{(N_\mathrm{f})}(u)=\sum_{N=0}^{\infty}u^N Z_{y_\alpha}^{(N_\mathrm{f})}(N)$$

$$\prod_\pm \Xi^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}\left(\pm ie^{\pm\frac{\zeta+m}{2}}u\right)-\prod_\pm \Xi^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}\left(\pm ie^{\mp\frac{\zeta-m}{2}}u\right)$$
$$-i^{N_\mathrm{f}}u\sum_{\beta=1}^{N_\mathrm{f}}\frac{R_{y_\beta}^{-1}}{\prod_{\gamma(\neq\beta)}2\sinh\frac{y_\beta-y_\gamma}{2}}\prod_\pm \Xi^{(N_\mathrm{f})}_{y_1\pm\Delta_1^{(\beta)},\cdots,y_{N_\mathrm{f}}\pm\Delta_{N_\mathrm{f}}^{(\beta)}}\left(R_m^{\pm\frac{1}{N_\mathrm{f}}}e^{\pm\frac{1}{2}\left(1-\frac{1}{N_\mathrm{f}}\right)\zeta}u\right)=0$$

$$\Delta_\alpha^{(\beta)}=\begin{cases}\left(\frac{1}{N_\mathrm{f}}-1\right)(m+\pi i) & (\alpha=\beta)\\ \frac{m+\pi i}{N_\mathrm{f}} & (\alpha\neq\beta)\end{cases}$$

$$
\prod_{\pm} \Xi^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}\left(e^{\pm\frac{\zeta+m}{2}}u\right) - \prod_{\pm} \Xi^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}\left(-e^{\mp\frac{\zeta-m}{2}}u\right)
$$

$$
-i^{N_\mathrm{f}-1}u\sum_{\beta=1}^{N_\mathrm{f}} \frac{R_{y_\beta}^{-1}}{\prod_{\gamma(\neq\beta)} 2\sinh\frac{y_\beta-y_\gamma}{2}} \prod_{\pm} \Xi^{(N_\mathrm{f})}_{y_1\pm\Delta_1^{(\beta)},\cdots,y_{N_\mathrm{f}}\pm\Delta_{N_\mathrm{f}}^{(\beta)}}\left(\mp iR_m^{\pm\frac{1}{N_\mathrm{f}}} e^{\pm\frac{1}{2}\left(1-\frac{1}{N_\mathrm{f}}\right)\zeta}u\right) = 0
$$

$$
Z^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}(N) = \frac{1}{2\cosh\frac{(\zeta+m+\pi i)N}{2} - 2\cosh\frac{(\zeta-m-\pi i)N}{2}}
$$

$$
\times\Bigg[-\sum_{N'=1}^{N-1}\left(e^{\frac{(\zeta+m+\pi i)(2N'-N)}{2}} - e^{-\frac{(\zeta-m-\pi i)(2N'-N)}{2}}\right) Z^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}(N') Z^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}(N-N')
$$

$$
+i^{N_\mathrm{f}}\sum_{\beta=1}^{N_\mathrm{f}} \frac{R_{y_\beta}^{-1}}{\prod_{\gamma(\neq\beta)} 2\sinh\frac{y_\beta-y_\gamma}{2}} \sum_{N'=0}^{N-1}\left(R_m^{\frac{1}{N_\mathrm{f}}} e^{\frac{1}{2}\left(1-\frac{1}{N_\mathrm{f}}\right)\zeta}\right)^{2N'-N+1}
$$

$$
\times Z^{(N_\mathrm{f})}_{y_1+\Delta_1^{(\beta)},\cdots,y_{N_\mathrm{f}}+\Delta_{N_\mathrm{f}}^{(\beta)}}(N') Z^{(N_\mathrm{f})}_{y_1-\Delta_1^{(\beta)},\cdots,y_{N_\mathrm{f}}-\Delta_{N_\mathrm{f}}^{(\beta)}}(N-1-N')\Bigg]
$$

$$
Z^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}(N) = \frac{1}{2\cosh\frac{(\zeta+m)N}{2} - (-1)^N 2\cosh\frac{(\zeta-m)N}{2}}
$$

$$
\times\Bigg[-\sum_{N'=1}^{N-1}\left(e^{\frac{(\zeta+m)(2N'-N)}{2}} - (-1)^N e^{-\frac{(\zeta-m)(2N'-N)}{2}}\right) Z^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}(N') Z^{(N_\mathrm{f})}_{y_1,\cdots,y_{N_\mathrm{f}}}(N-N')
$$

$$
+i^{N_\mathrm{f}-1}\sum_{\beta=1}^{N_\mathrm{f}} \frac{R_{y_\beta}^{-1}}{\prod_{\gamma(\neq\beta)} 2\sinh\frac{y_\beta-y_\gamma}{2}} \sum_{N'=0}^{N-1}\left(-iR_m^{\frac{1}{N_\mathrm{f}}} e^{\frac{1}{2}\left(1-\frac{1}{N_\mathrm{f}}\right)\zeta}\right)^{2N'-N+1}
$$

$$
\times Z^{(N_\mathrm{f})}_{y_1+\Delta_1^{(\beta)},\cdots,y_{N_\mathrm{f}}+\Delta_{N_\mathrm{f}}^{(\beta)}}(N') Z^{(N_\mathrm{f})}_{y_1-\Delta_1^{(\beta)},\cdots,y_{N_\mathrm{f}}-\Delta_{N_\mathrm{f}}^{(\beta)}}(N-1-N')\Bigg]
$$

$$
\begin{aligned}
&R_m^{-6}R_y^{-4} \to \lambda^{(\alpha)} = (\ydiagram{4}, \emptyset), \quad R_m^{-6}R_y^{4} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{4}), \quad R_m^{-3}R_y^{-2} \to \lambda^{(\alpha)} = (\ydiagram{3}, \ydiagram{1}),\\
&R_m^{-3}R_y^{2} \to \lambda^{(\alpha)} = (\ydiagram{1}, \ydiagram{3}), \quad R_m^{-2}R_y^{-4} \to \lambda^{(\alpha)} = (\ydiagram{3,1}, \emptyset), \quad R_m^{-2}R_y^{0} \to \lambda^{(\alpha)} = (\ydiagram{2}, \ydiagram{2}),\\
&R_m^{-2}R_y^{4} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{3,1}), \quad R_m^{0}R_y^{-4} \to \lambda^{(\alpha)} = (\ydiagram{2,2}, \emptyset), \quad R_m^{0}R_y^{-2} \to \lambda^{(\alpha)} = (\ydiagram{2,1}, \ydiagram{1}),
\end{aligned}
$$

$$
\begin{aligned}
&R_m^{0}R_y^{0} \to \lambda^{(\alpha)} = (\ydiagram{2}, \ydiagram{1,1}), (\ydiagram{1,1}, \ydiagram{2}), \quad R_m^{0}R_y^{2} \to \lambda^{(\alpha)} = (\ydiagram{1}, \ydiagram{2,1}), \quad R_m^{0}R_y^{4} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{2,2})\\
&R_m^{2}R_y^{-4} \to \lambda^{(\alpha)} = (\ydiagram{2,1,1}, \emptyset), \quad R_m^{2}R_y^{0} \to \lambda^{(\alpha)} = (\ydiagram{1,1}, \ydiagram{1,1}), \quad R_m^{2}R_y^{4} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{2,1,1}),\\
&R_m^{3}R_y^{-2} \to \lambda^{(\alpha)} = (\ydiagram{1,1,1}, \ydiagram{1}), \quad R_m^{3}R_y^{2} \to \lambda^{(\alpha)} = (\ydiagram{1}, \ydiagram{1,1,1}), \quad R_m^{6}R_y^{-4} \to \lambda^{(\alpha)} = (\ydiagram{1,1,1,1}, \emptyset),\\
&R_m^{6}R_y^{4} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{1,1,1,1}).
\end{aligned}
$$

$$
\begin{aligned}
Z_y^{(2)}(N) &= \frac{1}{2\cosh\frac{(\zeta+m+\pi i)N}{2} - 2\cosh\frac{(\zeta-m-\pi i)N}{2}} \\
&\times\left[-\sum_{N'=1}^{N-1}\left(e^{\frac{(\zeta+m+\pi i)(2N'-N)}{2}} - e^{-\frac{(\zeta-m-\pi i)(2N'-N)}{2}}\right)Z_y^{(2)}(N')Z_y^{(2)}(N-N')\right. \\
&+\frac{1}{2\sinh y}\sum_{N'=0}^{N-1}\left(R_y\left(R_m^{\frac{1}{2}}e^{\frac{\zeta}{4}}\right)^{2N'-N+1} - R_y^{-1}\left(R_m^{\frac{1}{2}}e^{\frac{\zeta}{4}}\right)^{-2N'+N-1}\right) \\
&\left.\times Z^{(2)}_{y+\frac{m+\pi i}{2}}(N')Z^{(2)}_{y-\frac{m+\pi i}{2}}(N-1-N')\right]
\end{aligned}
$$

$$R_m^{-2}R_y^{-5} \to \lambda^{(\alpha)} = (\ydiagram{3,2}, \emptyset), \quad R_m^{-2}R_y^{5} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{3,2}), \quad R_m^{0}R_y^{-3} \to \lambda^{(\alpha)} = (\ydiagram{2,2}, \ydiagram{1})$$

$$R_m^{0}R_y^{3} \to \lambda^{(\alpha)} = (\ydiagram{1}, \ydiagram{2,2}), \quad R_m^{2}R_y^{-5} \to \lambda^{(\alpha)} = (\ydiagram{2,2,1}, \emptyset), \quad R_m^{2}R_y^{5} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{2,2,1})$$

$$
\begin{aligned}
&R_m^{-5}R_y^{-6} \to \lambda^{(\alpha)} = (\ydiagram{4,2}, \emptyset), \quad R_m^{-5}R_y^{6} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{4,2}), \\
&R_m^{-3}R_y^{-6} \to \lambda^{(\alpha)} = (\ydiagram{4,1,1}, \emptyset), (\ydiagram{3,3}, \emptyset), \quad R_m^{-3}R_y^{6} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{4,1,1}), (\emptyset, \ydiagram{3,3}), \\
&R_m^{-2}R_y^{-4} \to \lambda^{(\alpha)} = (\ydiagram{3,2}, \ydiagram{1}), \quad R_m^{-2}R_y^{4} \to \lambda^{(\alpha)} = (\ydiagram{1}, \ydiagram{3,2}), \quad R_m^{-1}R_y^{-2} \to \lambda^{(\alpha)} = (\ydiagram{3,1}, \ydiagram{1,1}), (\ydiagram{2,2}, \ydiagram{2}), \\
&R_m^{-1}R_y^{2} \to \lambda^{(\alpha)} = (\ydiagram{2}, \ydiagram{2,2}), (\ydiagram{1,1}, \ydiagram{3,1}), \quad R_m^{0}R_y^{-6} \to \lambda^{(\alpha)} = (\ydiagram{3,2,1}, \emptyset), \quad R_m^{0}R_y^{6} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{3,2,1}), \\
&R_m^{1}R_y^{-2} \to \lambda^{(\alpha)} = (\ydiagram{2,2}, \ydiagram{1,1}), (\ydiagram{2,1,1}, \ydiagram{2}), \quad R_m^{1}R_y^{2} \to \lambda^{(\alpha)} = (\ydiagram{2}, \ydiagram{2,1,1}), (\ydiagram{1,1}, \ydiagram{2,2}), \\
&R_m^{2}R_y^{-4} \to \lambda^{(\alpha)} = (\ydiagram{2,2,1}, \ydiagram{1}), \quad R_m^{2}R_y^{4} \to \lambda^{(\alpha)} = (\ydiagram{1}, \ydiagram{2,2,1}), \quad R_m^{3}R_y^{-6} \to \lambda^{(\alpha)} = (\ydiagram{3,1,1,1}, \emptyset), (\ydiagram{2,2,2}, \emptyset), \\
&R_m^{3}R_y^{6} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{3,1,1,1}), (\emptyset, \ydiagram{2,2,2}), \quad R_m^{5}R_y^{-6} \to \lambda^{(\alpha)} = (\ydiagram{2,2,1,1}, \emptyset), \quad R_m^{5}R_y^{6} \to \lambda^{(\alpha)} = (\emptyset, \ydiagram{2,2,1,1}).
\end{aligned}
$$

$$\sum_{n=0}^{\infty} e^{\mu N} Z_{y_\alpha}^{(N_\mathrm{f})}(N) = \sum_{n=-\infty}^{\infty} e^{J(\mu+2\pi i n)}$$

$$J(\mu) = J_\mathrm{pert}(\mu) + J_\mathrm{np}(\mu)$$

$$J_\mathrm{pert}(\mu) = \frac{C}{3}\mu^3 + B\mu + A, J_\mathrm{np}(\mu) = \sum_{\omega} d_\omega e^{-\omega\mu}$$

$$\omega_{\mathrm{MB1}\pm} = \frac{2}{N_\mathrm{f} \pm \frac{i\zeta}{\pi}}, \omega_{\mathrm{MB2}\pm} = \frac{2}{1 \pm \frac{im}{\pi}}, \omega_{\mathrm{MB3}} = 1$$

$$\omega_{\mathrm{WS}\pm\pm'} = \frac{4}{k\left(N_\mathrm{f} \pm \frac{i\zeta}{\pi}\right)\left(1 \pm' \frac{im}{\pi}\right)}, k = 1$$

$$Z_{y_\alpha}^{(N_\mathrm{f})}(N) = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} e^{J(\mu)-\mu N}$$

$$e^{J_{\mathrm{np}}(\mu)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{\omega} d_\omega e^{-\omega\mu}\right)^n,$$

$$Z(N) = e^A C^{-\frac{1}{3}}\mathrm{Ai}\left[C^{-\frac{1}{3}}(N-B)\right] + \sum_{\omega} d_\omega e^A C^{-\frac{1}{3}}\mathrm{Ai}\left[C^{-\frac{1}{3}}(N+\omega-B)\right] + \cdots$$

$$-\log Z_{y_\alpha}^{(N_{\mathrm{f}})}(N) = -\log Z_{\mathrm{pert},y_\alpha}^{(N_{\mathrm{f}})}(N) + \sum_{\omega} d_\omega e^{-\omega\sqrt{\frac{N-B}{C}}} + \cdots$$

$$d_{\omega_{\mathrm{WS}\pm\pm'}}^{(N_{\mathrm{f}})}(\zeta, m; y_\alpha) = \frac{1}{4\sin\frac{2\pi}{N_{\mathrm{f}} \pm \frac{i\zeta}{\pi}}\sin\frac{2\pi}{1\pm'\frac{im}{\pi}}} \sum_{\alpha=1}^{N_{\mathrm{f}}} e^{\pm\frac{2y_\alpha}{1\pm'\frac{im}{\pi}}}$$

$$J_{\mathrm{np}} = \frac{N_{\mathrm{f}}}{\sin\frac{2\pi}{N_{\mathrm{f}}k}\sin\frac{2\pi}{k}} e^{-\frac{4}{N_{\mathrm{f}}k}\mu} + \cdots$$

$$J_{\mathrm{np}} = \sum_{\pm,\pm'} \frac{\cos\frac{2\pi M}{k}}{4\sin\frac{2\pi}{k\left(1\pm\frac{i\zeta}{\pi}\right)}\sin\frac{2\pi}{k\left(1\pm'\frac{im}{\pi}\right)}} e^{-\frac{4}{k\left(1\pm\frac{i\zeta}{\pi}\right)\left(1\pm'\frac{im}{\pi}\right)}\mu} + \cdots$$

$$J_{\mathrm{np}} = \frac{\cos\frac{2\pi M}{k}}{\sin^2\frac{2\pi}{k}} e^{-\frac{4\mu}{k}} + \cdots$$

$$F_{\mathrm{top}}(T_1, T_2) = \sum_{g\geq 0}\sum_{w\geq 1}\sum_{\substack{d_1,d_2\geq 0\\((d_1,d_2)\neq(0,0))}} \frac{1}{w} n_g^{d_1,d_2}\left(2\sin\frac{2\pi}{k}\right)^{2g-2} e^{-w(d_1T_1+d_2T_2)}$$

$$T_1 = \frac{4\mu}{k} + \frac{2\pi i}{k}\left(\frac{k}{2} - M\right), T_2 = \frac{4\mu}{k} - \frac{2\pi i}{k}\left(\frac{k}{2} - M\right)$$

$$\hat{\rho}^{-1} = e^{\hat{x}} + e^{\hat{p}} + e^{-\hat{p}} + e^{\pi i(k-2M)}e^{-\hat{x}}, [\hat{x},\hat{p}] = i\hbar, (\hbar = \pi k)$$

$$\frac{Z_{y_\alpha}^{(N_{\mathrm{f}})}(N)}{Z_{\mathrm{pert},y_\alpha}^{(N_{\mathrm{f}})}(N)} - 1 \approx d e^{-\omega\sqrt{\frac{N-B}{C}}}$$

$$d_{\omega_{\mathrm{WS}--}}^{(N_{\mathrm{f}})}(\zeta, m; 0) = \frac{N_{\mathrm{f}}}{4\sin\frac{2\pi}{N_{\mathrm{f}} - \frac{i\zeta}{\pi}}\sin\frac{2\pi}{1-\frac{im}{\pi}}}$$

$$\hat{\rho}_{y_\alpha}^{(N_{\mathrm{f}})} = \frac{e^{-\frac{i\zeta}{2\pi}\hat{x}}}{\prod_{\alpha=1}^{N_{\mathrm{f}}} 2\cosh\frac{\hat{x}-y_\alpha}{2}} \frac{e^{\frac{im}{2\pi}\hat{p}}}{2\cosh\frac{\hat{p}}{2}}, ([\hat{x},\hat{p}] = 2\pi i).$$

$$d^{(N_{\mathrm{f}})}_{\omega_{\mathrm{WS}--}}(\zeta,m;y_\alpha)=\frac{\sum_{\alpha=1}^{N_{\mathrm{f}}} e^{f(\zeta,m)y_\alpha}}{4\sin\frac{2\pi}{N_{\mathrm{f}}-\frac{i\zeta}{\pi}}\sin\frac{2\pi}{1-\frac{im}{\pi}}},$$

$$d^{(2)}_{\omega_{\mathrm{WS}--}}(\zeta,m;y)=\frac{\cosh\frac{2y}{1-\frac{im}{\pi}}}{2\sin\frac{2\pi}{2-\frac{i\zeta}{\pi}}\sin\frac{2\pi}{1-\frac{im}{\pi}}}$$

$$d^{(3)}_{\omega_{\mathrm{WS}--}}(\zeta,m;y_\alpha)=\frac{1}{4\sin\frac{2\pi}{3-\frac{i\zeta}{\pi}}\sin\frac{2\pi}{1-\frac{im}{\pi}}}\left(e^{-\frac{2y_1}{1-\frac{im}{\pi}}}+e^{-\frac{2y_2}{1-\frac{im}{\pi}}}+e^{-\frac{2y_3}{1-\frac{im}{\pi}}}\right),$$

$$Z^{(\ell),S_b^3}_{y_\alpha=\eta_\alpha}\left(\xi,\frac{\pi i(b^2-3)}{2b};N\right)=Z^{(n\ell)}_{y_\alpha=\tilde\eta_\alpha}\left(\sqrt{b}\xi,\frac{\pi i}{b^2};N\right),(b=\sqrt{2n-1})$$

$$\tilde\eta_{n(\beta-1)+\gamma}=\frac{\eta_\beta}{b}+\frac{2\pi}{b^2}\left(\gamma-\frac{n+1}{2}\right),(1\le\beta\le N_{\mathrm{f}},1\le\gamma\le n).$$

$$Z^{(\ell),S^3_{b=\sqrt{3}}}_{y_\alpha=\eta_\alpha}(0,0;N)=Z^{(2\ell)}_{y_\alpha=\tilde\eta_\alpha}\left(0,\frac{\pi i}{3};N\right)$$

$$\tilde\eta_{2\beta-1}=\eta_\beta-\frac{\pi i}{3},\tilde\eta_{2\beta}=\eta_\beta+\frac{\pi i}{3},(\beta=1,\cdots,\ell)$$

$$\begin{aligned}&-\log Z^{(\ell),S^3_{b=\sqrt{3}}}_{\eta_\alpha}(0,0;N)-\left(-\log Z^{(\ell),S^3_{b=\sqrt{3}}}_{\mathrm{pert},\eta_\alpha}(0,0;N)\right)\\&=-\log Z^{(2\ell)}_{\tilde\eta_\alpha}\left(0,\frac{\pi i}{3};N\right)-\left(-\log Z^{(2\ell)}_{\mathrm{pert},\tilde\eta_\alpha}\left(0,\frac{\pi i}{3};N\right)\right)\\&\sim\exp\left[-[\omega_{\mathrm{WS}\pm+}]_{N_{\mathrm{f}}=2\ell,\zeta=0,m=\frac{\pi i}{3}}\sqrt{\frac{N}{[C]_{N_{\mathrm{f}}=2\ell,\zeta=0,m=\frac{\pi i}{3}}}}\right]=e^{-2\pi\sqrt{\frac{2N}{\ell}}}\end{aligned}$$

$$\begin{aligned}&-\log Z^{(\ell)}_{\eta_\alpha}(0,0;N)-\left(-\log Z^{(\ell)}_{\mathrm{pert},\eta_\alpha}(0,0;N)\right)\\&\sim\exp\left[-[\omega_{\mathrm{WS}\pm\pm'}]_{N_{\mathrm{f}}=\ell,\zeta=m=0}\sqrt{\frac{N}{[C]_{N_{\mathrm{f}}=\ell,\zeta=m=0}}}\right.\\&=e^{-2\pi\sqrt{\frac{2N}{\ell}}}\end{aligned}$$

$$\mathrm{arm}_\lambda(\Box)=\lambda_a-b,\mathrm{leg}_\lambda(\Box)=\tilde\lambda_b-a,(\Box=(a,b)\in\lambda).$$

$$\mathrm{arm}_\lambda(\Box)=\begin{cases}\lambda_a-b & (a\le\ell(\lambda))\\-b & (a>\ell(\lambda))\end{cases},\mathrm{leg}_\lambda(\Box)=\begin{cases}\tilde\lambda_b-a & (b\le\lambda_1)\\-a & (b>\lambda_1)\end{cases}.$$

$$h_\lambda(\Box)=\lambda_a-b+\tilde\lambda_b-a+1=\mathrm{arm}_\lambda(\Box)+\mathrm{leg}_\lambda(\Box)+1.$$

$$g_{\lambda,\mu}(y) = E_{\lambda,\mu}(y) \times \prod_{\square\in\lambda} e^{-\frac{\pi i(\mathrm{arm}_\lambda(\square)-\mathrm{leg}_\mu(\square))}{2}} \prod_{\square\in\mu} e^{\frac{\pi i(\mathrm{arm}\mu(\square)-\mathrm{leg}_\lambda(\square))}{2}}$$

$$E_{\lambda,\mu}(y)= \frac{1}{\prod_{\square\in\lambda}\left(e^{\frac{y}{2}+\frac{m}{2}(\mathrm{arm}_\lambda(\square)+\mathrm{leg}_\mu(\square)+1)} + e^{-\frac{y}{2}-\frac{m}{2}(\mathrm{arm}_\lambda(\square)+\mathrm{leg}_\mu(\square)+1)-\pi i(\mathrm{arm}_\lambda(\square)-\mathrm{leg}_\mu(\square))}\right)}$$
$$\times \frac{1}{\prod_{\square\in\mu}\left(e^{\frac{y}{2}-\frac{m}{2}(\mathrm{arm}_\mu(\square)+\mathrm{leg}_\lambda(\square)+1)} + e^{-\frac{y}{2}+\frac{m}{2}(\mathrm{arm}_\mu(\square)+\mathrm{leg}_\lambda(\square)+1)+\pi i(\mathrm{arm}_\mu(\square)-\mathrm{leg}_\lambda(\square))}\right)},$$

$$g_{\lambda,\mu}(y) = \left(\prod_{\square=(a,b)\in\lambda\,\square'=(a',b')\in\mu} \frac{\left(2\sinh \frac{y+(b-b'-a+a')m+\pi i(a+b-a'-b')}{2}\right)^2}{\prod_\pm 2\cosh \frac{y+(b-b'-a+a'\pm 1)m+\pi i(a+b-a'-b')}{2}}\right)$$
$$\times\left(\prod_{\square=(a,b)\in\lambda} \frac{1}{2\cosh \frac{y+(b-a)m+\pi i(a+b-1)}{2}}\right)\left(\prod_{\square=(a,b)\in\mu} \frac{1}{2\cosh \frac{-y+(b-a)m+\pi i(a+b-1)}{2}}\right)$$

$$I^{(g)}_{\lambda,\mu}(\xi,\eta) =\sum_{a=1}^{\tilde\lambda_1}\sum_{b=1}^{\lambda_a} \xi^b\eta^a + \sum_{b=1}^{\mu_1}\sum_{a=1}^{\tilde\mu_b} \xi^{1-b}\eta^{1-a}$$
$$+ \sum_{\square=(a,b)\in\lambda}\;\sum_{\square'=(a',b')\in\mu} \left(\xi^{b-b'+1}-\xi^{b-b'}\right)\left(\eta^{a-a'+1}-\eta^{a-a'}\right),$$

$$I^{(g)}_{\lambda,\mu}(\xi,\eta) = I^{(g)}_{\tilde\lambda,\tilde\mu}(\eta,\xi)$$

$$I^{(g)}_{\mu,\nu}(\xi,\eta) = \sum_{(\alpha,a,b)} \alpha\xi^a\eta^b \;\longrightarrow\; F = \prod_{(\alpha,a,b)} \frac{1}{f(a,b)^\alpha}$$

$$I^{(g)}_{\mu,\nu}(\xi,\eta)\rightarrow F = \left(\prod_{a=1}^{\tilde\lambda_1}\prod_{b=1}^{\lambda_a}\frac{1}{f(b,a)}\right)\left(\prod_{b=1}^{\mu_1}\prod_{a=1}^{\tilde\mu_b}\frac{1}{f(1-b,1-a)}\right)$$
$$\times\left(\prod_{\square=(a,b)\in\lambda\,\square'=(a',b')\in\mu} \frac{f(b-b'+1,a-a'+1)f(b-b',a-a')}{f(b-b'+1,a-a')f(b-b',a-a'+1)}\right)$$

$$f(a,b) = 2\cosh\frac{y+(a-b)m+\pi i(a+b-1)}{2}$$

$$E_{\lambda,\mu}(y) \times \prod_{\square\in\lambda} e^{-\frac{\pi i(\mathrm{arm}_\lambda(\square)-\mathrm{leg}_\mu(\square))}{2}} \prod_{\square\in\mu} e^{\frac{\pi i(\mathrm{arm}_\mu(\square)-\mathrm{leg}_\lambda(\square))}{2}}$$
$$= \prod_{\square\in\lambda} \frac{1}{2\cosh \frac{y+(\mathrm{arm}_\lambda(\square)+\mathrm{leg}_\mu(\square)+1)m+\pi i(\mathrm{arm}_\lambda(\square)-\mathrm{leg}_\mu(\square))}{2}}$$
$$\times \prod_{\square\in\mu} \frac{1}{2\cosh \frac{y+(-\mathrm{arm}_\mu(\square)-\mathrm{leg}_\lambda(\square)-1)m+\pi i(-\mathrm{arm}_\mu(\square)+\mathrm{leg}_\lambda(\square))}{2}}$$

$$I^{(E)}_{\lambda,\mu}(\xi,\eta) = \sum_{\square\in\lambda} \xi^{\mathrm{arm}_\lambda(\square)+1}\eta^{-\mathrm{leg}_\mu(\square)} + \sum_{\square\in\mu} \xi^{-\mathrm{arm}_\mu(\square)}\eta^{\mathrm{leg}_\lambda(\square)+1}$$

$$\sum_{\square=(i,a)\in\lambda\square'=(b,j)\in\mu} \left(\xi^{a+1-j} - \xi^{a-j}\right)\left(\eta^{i-b} - \eta^{i+1-b}\right)$$

$$= \sum_{i=1}^{\tilde{\lambda}_1}\sum_{a=1}^{\lambda_i}\sum_{j=1}^{\mu_1}\sum_{b=1}^{\tilde{\mu}_j} \left(\xi^{a+1-j} - \xi^{a-j}\right)\left(\eta^{i-b} - \eta^{i+1-b}\right), \text{(B.10)}$$

$$\sum_{a=1}^{\lambda_i} \left(\xi^{a+1-j} - \xi^{a-j}\right) = \sum_{a=2}^{\lambda_i+1} \xi^{a-j} - \sum_{a=1}^{\lambda_i} \xi^{a-j} = \xi^{\lambda_i-j+1} - \xi^{-j+1}$$

$$\sum_{b=1}^{\tilde{\mu}_j} \left(\eta^{i-b} - \eta^{i-b+1}\right) = \eta^{-\tilde{\mu}_j+i} - \eta^{i}$$

$$\sum_{\square=(i,a)\in\lambda}\ \sum_{\square'=(b,j)\in\mu} \left(\xi^{a+1-j} - \xi^{a-j}\right)\left(\eta^{i-b} - \eta^{i+1-b}\right) = \sum_{i=1}^{\tilde{\lambda}_1}\sum_{j=1}^{\mu_1} \left(\xi^{\lambda_i-j+1} - \xi^{-j+1}\right)\left(\eta^{-\tilde{\mu}_j+i} - \eta^{i}\right),$$

$$I^{(g)}_{\lambda,\mu}(\xi,\eta) = \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\lambda_a} \xi^{b}\eta^{a} + \sum_{b=1}^{\mu_1}\sum_{a=1}^{\tilde{\mu}_b} \xi^{1-b}\eta^{1-a} + \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\mu_1} \left(\xi^{\lambda_a-b+1} - \xi^{-b+1}\right)\left(\eta^{-\tilde{\mu}_b+a} - \eta^{a}\right).$$

$$I^{(g)}_{\lambda,\mu}(\xi,\eta) = \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\lambda_a} \xi^{b}\eta^{a} + \sum_{b=1}^{\mu_1}\sum_{a=1}^{\tilde{\mu}_b} \xi^{1-b}\eta^{1-a} + \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\mu_1} \left(\xi^{\lambda_a-b+1}\eta^{-\tilde{\mu}_b+a} - \xi^{\lambda_a-b+1}\eta^{a} - \xi^{-b+1}\eta^{-\tilde{\mu}_b+a} + \xi^{-b+1}\eta^{a} + \xi^{-b+1}\eta^{a} - \xi^{-b+1}\eta^{a}\right)$$

$$\mathfrak{R} = \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\mu_1} \left(-\xi^{-b+1}\right)\left(\eta^{-\tilde{\mu}_b+a} - \eta^{a}\right) = \sum_{b=1}^{\mu_1}\sum_{a=1}^{\tilde{\mu}_b} \left(-\xi^{-b+1}\right)\left(\eta^{1-a} - \eta^{\tilde{\lambda}_1-a+1}\right),$$

$$\mathcal{G} = \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\mu_1} \left(-\eta^{a}\right)\left(\xi^{\lambda_a-b+1} - \xi^{-b+1}\right) = \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\lambda_a} \left(-\eta^{a}\right)\left(\xi^{b} - \xi^{-\mu_1+b}\right).$$

$$I^{(g)}_{\lambda,\mu}(\xi,\eta) = \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\mu_1} \left(\xi^{\lambda_a-b+1}\eta^{-\tilde{\mu}_b+a} - \xi^{-b+1}\eta^{a}\right) + \sum_{b=1}^{\mu_1}\sum_{a=1}^{\tilde{\mu}_b} \xi^{-b+1}\eta^{\tilde{\lambda}_1-a+1} + \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\lambda_a} \xi^{-\mu_1+b}\eta^{a}$$

$$I^{(g)}_{\lambda,\mu}(\xi,\eta) = I^{(+)}_{\lambda,\mu}(\xi,\eta) + I^{(0,-)}_{\lambda,\mu}(\xi,\eta)$$

$$I^{(+)}_{\lambda,\mu}(\xi,\eta) = \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\min(\mu_1,\lambda_a)} \xi^{\lambda_a-b+1}\eta^{-\tilde{\mu}_b+a} + \sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=\mu_1+1}^{\lambda_a} \xi^{-\mu_1+b}\eta^{a}.$$

$$\sum_{b=\mu_1+1}^{\lambda_a} \xi^{-\mu_1+b} = \sum_{b=\min(\mu_1,\lambda_a)+1}^{\lambda_a} \xi^{-\min(\mu_1,\lambda_a)+b} = \xi + \xi^{2} + \cdots + \xi^{\lambda_a-\min(\mu_1,\lambda_a)} = \sum_{b=\min(\mu_1,\lambda_a)+1}^{\lambda_a} \xi^{\lambda_a-b+1}.$$

$$I_{\lambda,\mu}^{(+)}(\xi,\eta)=\sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\min(\mu_1,\lambda_a)}\xi^{\lambda_a-b+1}\eta^{-\tilde{\mu}_b+a}+\sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=\min(\mu_1,\lambda_a)+1}^{\lambda_a}\xi^{\lambda_a-b+1}\eta^{-\tilde{\mu}_b+a}$$
$$=\sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\lambda_a}\xi^{\lambda_a-b+1}\eta^{-\tilde{\mu}_b+a}$$

$$I_{\lambda,\mu}^{(g)}(\xi,\eta)=\sum_{b=1}^{\lambda_1}\sum_{a=1}^{\tilde{\mu}_1}\left(\eta^{\tilde{\lambda}_b-a+1}\xi^{-\mu_a+b}-\eta^{-a+1}\xi^{b}\right)+\sum_{a=1}^{\tilde{\mu}_1}\sum_{b=1}^{\mu_a}\eta^{-a+1}\xi^{\lambda_1-b+1}+\sum_{b=1}^{\lambda_1}\sum_{a=1}^{\tilde{\lambda}_b}\eta^{-\tilde{\mu}_1+a}\xi^{b}$$

$$I_{\lambda,\mu}^{(0,-)}(\xi,\eta)=\sum_{a=1}^{\tilde{\mu}_1}\sum_{b=1}^{\min(\lambda_1,\mu_a)}\eta^{\tilde{\lambda}_b-a+1}\xi^{-\mu_a+b}+\sum_{a=1}^{\tilde{\mu}_1}\sum_{b=\lambda_1+1}^{\mu_a}\eta^{-a+1}\xi^{\lambda_1-b+1}$$

$$I_{\lambda,\mu}^{(0,-)}(\xi,\eta)=\sum_{a=1}^{\tilde{\mu}_1}\sum_{b=1}^{\mu_a}\xi^{-\mu_a+b}\eta^{\tilde{\lambda}_b-a+1}$$

$$I_{\lambda,\mu}^{(g)}(\xi,\eta)=\sum_{a=1}^{\tilde{\lambda}_1}\sum_{b=1}^{\lambda_a}\xi^{\lambda_a-b+1}\eta^{-\tilde{\mu}_b+a}+\sum_{a=1}^{\tilde{\mu}_1}\sum_{b=1}^{\mu_a}\xi^{-\mu_a+b}\eta^{\tilde{\lambda}_b-a+1}$$
$$=\sum_{\square\in\lambda}\xi^{\mathrm{arm}_\lambda(\square)+1}\eta^{-\mathrm{leg}_\mu(\square)}+\sum_{\square\in\mu}\xi^{-\mathrm{arm}_\mu(\square)}\eta^{\mathrm{leg}_\lambda(\square)+1}$$

$$\lim_{m\to0}\sum_{\pm,\pm'}d^{(\ell)}_{\omega_{\mathrm{WS}\pm\pm'}}(0,m;y_\alpha)\exp\left[-\left[\omega_{\mathrm{WS}\pm\pm'}\right]_{N_{\mathrm{f}}=\ell,\zeta=0}\mu\right]$$
$$=\frac{1}{2\pi\sin\frac{2\pi}{\ell}}\sum_{\alpha=1}^{\ell}\left(-\frac{4\mu\cosh\,(2\eta_\alpha)}{\ell}+2\eta_\alpha\sinh\,2\eta_\alpha-\cosh\,2\eta_\alpha\right)e^{-\frac{4\mu}{\ell}}$$

$$M^{\mathrm{tree}}\,(i^-,j^-,1^+\dots n^+)=i(-g)^{n-2}\delta^4\left(\sum_{k=1}^{n}p_k\right)\langle ij\rangle^4\sum_{\sigma\in S_n\backslash\mathbb{Z}_n}\frac{\mathrm{tr}\left(T^{A_{\sigma(1)}}\dots T^{A_{\sigma(n)}}\right)}{\langle\sigma(1)\sigma(2)\rangle\dots\langle\sigma(n)\sigma(1)\rangle}$$

$$p_i^{\alpha\dot{\alpha}}=i^\alpha i^{\dot{\alpha}},p_i\cdot p_j=\underbrace{i^\alpha j_\alpha}_{=:\langle ij\rangle}\times\underbrace{\bar{i}^{\dot{\alpha}}\bar{j}_{\dot{\alpha}}}_{=:[ij]}.$$

$$\sum\frac{\mathrm{tr}(T^A\sigma(1)\dots T^A\sigma(n))}{\langle\sigma(1)\sigma(2)\rangle\dots\langle\sigma(n)\sigma(1)\rangle}$$

$$\left\langle\prod_{i=1}^{n}j^{A_i}(i^\alpha)\right\rangle=k\underbrace{\sum_{\sigma\in S_n\backslash\mathbb{Z}_n}\frac{\mathrm{tr}\left(T^{A_{\sigma(1)}}\dots T^{A_{\sigma(n)}}\right)}{\langle\sigma(1)\sigma(2)\rangle\dots\langle\sigma(n)\sigma(1)\rangle}}_{\text{single-trace}}\prod_{i=1}^{n}\mathrm{D}i$$
$$+\frac{k^2}{2}\underbrace{\left(\sum_{j,\sigma}\frac{\mathrm{tr}\left(T^{A_{\sigma(1)}}\dots T^{A_{\sigma(j)}}\right)}{\langle\sigma(1)\sigma(2)\rangle\dots\langle\sigma(j)\sigma(1)\rangle}\frac{\mathrm{tr}\left(T^{A_{\sigma(j+1)}}\dots T^{A_{\sigma(n)}}\right)}{\langle\sigma(j+1)\sigma(j+2)\rangle\dots\langle\sigma(n)\sigma(j+1)\rangle}\right)}_{\text{double-trace}}\prod_{i=1}^{n}\mathrm{D}i+\cdots$$

$$Z^{\mathcal{A}}(\sigma^{\alpha}) \colon \mathbb{CP}^1_{\sigma} \to \mathbb{PT}$$
$$\sigma^{\alpha} \mapsto \sum_{i=0}^{d} Z_i^{\mathcal{A}}(\sigma^1)^{d-i}(\sigma^0)^i$$

$$\langle \mathfrak{C}_{\text{correlator}} \rangle = \sum_{d=1}^{n} g^{2d} \oint \underbrace{\frac{\Lambda_{i=0}^{d} \; \mathrm{d}^{4|4} Z_i}{\mathrm{Vol}(GL(2,\mathbb{C}))}}_{\text{int. over deg. } d \text{ maps } Z^{\mathcal{A}}(\sigma)} \int_{(\mathbb{CP}^1)^n} \left\langle \prod_{j=1}^{n} j^{A_j}(\sigma_j) \right\rangle \underbrace{\prod_{j=1}^{n} f_j\left(Z(\sigma_j)\right)}_{\text{ext. wavefns.}}$$

$$\left\langle nJ^{A_1}(1^{\alpha}) \prod_{i=2}^{n} \tilde{J}^{A_i}(i^{\alpha}) \right\rangle = \sum_{\sigma \in S_n \backslash \mathbb{Z}_n} \frac{\mathrm{tr}\left(T^{A_{\sigma(1)}} \dots T^{A_{\sigma(n)}}\right)}{\langle \sigma(1)\sigma(2)\rangle \dots \langle \sigma(n)\sigma(1)\rangle} \prod_{i=1}^{n} \mathrm{D}i$$

$$j^A(z_1) j^B(z_2) \sim \frac{k\kappa^{AB}}{z_{12}^2} + \frac{f_C^{AB} j^C(z_2)}{z_{12}}, j^A \in \Omega^{1,0}(\Sigma, \mathfrak{g}),$$

$$\left\langle \prod j^A(z_i) \right\rangle = \left(-k \sum_{\sigma \in S_n \backslash \mathbb{Z}_n} \frac{\mathrm{tr}\left(T^{A_{\sigma_1}} \dots T^{A_{\sigma_n}}\right)}{z_{\sigma_1 \sigma_2} \dots z_{\sigma_n \sigma_1}} + \mathbb{MTC}_{\text{multi-trace contributions}} \right) \prod_{i=1}^{n} (\mathrm{d}z^i)$$

$$S_{\mathrm{ff}} = \int_{\Sigma} \tilde{\phi}_a \bar{\partial} \phi^a + \tilde{\rho}_a \bar{\partial} \rho^a, a = 1, \dots, N$$
$$\tilde{\rho}_a \in \Pi\Omega^0\left(\Sigma, K_{\Sigma}^{1/2}\right), \rho^a \in \Pi\Omega^0\left(\Sigma, K_{\Sigma}^{1/2}\right)$$
$$\tilde{\phi}_a \in \Omega^0\left(\Sigma, K_{\Sigma}^{1/2}\right), \phi^a \in \Omega^0\left(\Sigma, K_{\Sigma}^{1/2}\right)$$

$$\tilde{\phi}_a(z_1)\phi^b(z_2) \sim \frac{-\delta_a^b}{z_1 - z_2} \sim -\phi^b(z_1)\tilde{\phi}_a(z_2)$$
$$\tilde{\rho}_a(z_1)\rho^b(z_2) \sim \frac{\delta_a^b}{z_1 - z_2} \sim \rho^b(z_1)\tilde{\rho}_a(z_2)$$

$$Z = \int \mathrm{d}^n a \, \mathrm{d}^n b e^{aMb} = \begin{cases} \frac{1}{\det M} & a, b \text{ Grassmann even} \\ \det M & a, b \text{ Grassmann odd} \end{cases}$$

$$\langle a_i b^j \rangle = \frac{1}{Z} \frac{\partial Z}{\partial M_j^i} = \begin{cases} -(M^{-1})_i^j & a, b \text{ Grassmann even} \\ (M^{-1})_i^j & a, b \text{ Grassmann odd} \end{cases}$$

$$J_{\mathcal{T}} := (\tilde{\phi}_a \quad \tilde{\rho}_b) \underbrace{\begin{pmatrix} A_c^a & B_d^a \\ C_c^b & D_d^b \end{pmatrix}}_{\mathcal{T}} \begin{pmatrix} \phi^c \\ \rho^d \end{pmatrix}$$

$$\begin{pmatrix} \phi^c(z_1) \\ \rho^d(z_1) \end{pmatrix} \left(\tilde{\phi}_a(z_2) \tilde{\rho}_b(z_2) \right) \sim \frac{\begin{pmatrix} \delta_a^c & 0_N \\ 0_N & \delta_d^b \end{pmatrix}}{z_{12}}, \left(\tilde{\phi}_a(z_1) \tilde{\rho}_b(z_1) \right) \begin{pmatrix} \phi^c(z_2) \\ \rho^d(z_2) \end{pmatrix} \sim \frac{\begin{pmatrix} -\delta_a^c & 0_N \\ 0_N & \delta_d^b \end{pmatrix}}{z_{12}}$$

$$J_{\mathcal{T}_1}(z_1) J_{\mathcal{T}_2}(z_2) \sim \frac{-\mathrm{str}(\mathcal{T}_1 \mathcal{T}_2)}{z_{12}^2} + \frac{J_{[\mathcal{T}_1, \mathcal{T}_2]}(z_2)}{z_{12}}$$

$$[T^A, T^B] = f_C^{AB} T^C$$

$$\tilde{J}^A(z_1) := (T^A)_b^a \left(\tilde{\phi}_a \phi^b + \tilde{\rho}_a \rho^b\right)$$
$$\tilde{J}^A(z_1)\tilde{J}^B(z_2) \sim \frac{f_C^{AB}\tilde{J}^C(z_2)}{z_{12}}$$

$$J_{\begin{pmatrix} T^{A_1} & 0_N \\ 0_N & T^{A_1} \end{pmatrix}}(z_1) J_{\begin{pmatrix} T^{A_2} & 0_N \\ 0_N & T^{A_2} \end{pmatrix}}(z_2) \sim \frac{-\mathrm{str}\left(\begin{pmatrix} T^{A_1} & 0_N \\ 0_N & T^{A_1} \end{pmatrix}\begin{pmatrix} T^{A_2} & 0_N \\ 0_N & T^{A_2} \end{pmatrix}\right)}{z_{12}^2} + \frac{J_{\left[\begin{pmatrix} T^{A_1} & 0_N \\ 0_N & T^{A_1} \end{pmatrix}, \begin{pmatrix} T^{A_2} & 0_N \\ 0_N & T^{A_2} \end{pmatrix}\right]}(z_2)}{z_{12}}$$

$$\sim \frac{J_{\begin{pmatrix} [T^{A_1}, T^{A_2}] & 0_N \\ 0_N & [T^{A_1}, T^{A_2}] \end{pmatrix}}(z_2)}{z_{12}} \sim \frac{f_C^{A_1 A_2}\tilde{J}^C}{z_{12}}$$

$$S = \sum_{a=1}^{N} \int_\Sigma (\tilde{\phi}_a \quad \tilde{\rho}_a)\bar{\partial}\begin{pmatrix} \phi^a \\ \rho^a \end{pmatrix} = \sum_{a=1}^{N} \int_\Sigma (\tilde{\phi}_a \quad \tilde{\rho}_a)\mathcal{M}^{-1}\bar{\partial}\left(\mathcal{M}\begin{pmatrix} \phi^a \\ \rho^a \end{pmatrix}\right),$$

$$J: \tilde{\phi}_a \phi^a + \tilde{\rho}_a \rho^a, G^+: \tilde{\phi}_a \rho^a, G^-: \phi^a \tilde{\rho}_a$$

$$\left(\tilde{\phi}_a \phi^a + \tilde{\rho}_a \rho^a\right)(z_1)\left(\tilde{\phi}_a \phi^a + \tilde{\rho}_a \rho^a\right)(z_2) \sim 0$$
$$\left(\tilde{\phi}_a \phi^a + \tilde{\rho}_a \rho^a\right)(z_1)\left(\tilde{\phi}_a \rho^a\right)(z_2) \sim 0$$
$$\left(\tilde{\phi}_a \phi^a + \tilde{\rho}_a \rho^a\right)(z_1)\left(\phi^a \tilde{\rho}_a\right)(z_2) \sim 0$$
$$\left(\tilde{\phi}_a \rho^a\right)(z_1)\left(\phi^a \tilde{\rho}_a\right)(z_2) \sim \frac{-N}{z_{12}^2} + \frac{\left(\tilde{\phi}_a \phi^a + \tilde{\rho}_a \rho^a\right)(z_2)}{z_{12}}$$

$$\delta_J \begin{pmatrix} \tilde{\phi}_a \\ \tilde{\rho}_a \\ \phi^b \\ \rho^b \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_a \\ \tilde{\rho}_a \\ -\phi^b \\ -\rho^b \end{pmatrix}, \delta_{G^+} \begin{pmatrix} \tilde{\phi}_a \\ \tilde{\rho}_a \\ \phi^b \\ \rho^b \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\phi}_a \\ -\rho^b \\ 0 \end{pmatrix}, \delta_{G^-} \begin{pmatrix} \tilde{\phi}_a \\ \tilde{\rho}_a \\ \phi^b \\ \rho^b \end{pmatrix} = \begin{pmatrix} \tilde{\rho}_a \\ 0 \\ 0 \\ \phi^b \end{pmatrix}$$

$$J_{\mathcal{T}} = (\tilde{\phi}_a \tilde{\rho}_b)\underbrace{\begin{pmatrix} A_c^a & B_c^b \\ C_d^a & D_d^b \end{pmatrix}}_{\mathcal{T}}\begin{pmatrix} \phi^c \\ \rho^d \end{pmatrix},$$

$$G^+ = (\tilde{\phi}_a \quad \tilde{\rho}_b)\begin{pmatrix} 0_c^a & \delta_c^b \\ 0_d^a & 0_d^b \end{pmatrix}\begin{pmatrix} \phi^c \\ \rho^d \end{pmatrix}, G^- = (\tilde{\phi}_a \quad \tilde{\rho}_b)\begin{pmatrix} 0_c^a & 0_c^b \\ \delta_d^a & 0_d^b \end{pmatrix}\begin{pmatrix} \phi^c \\ \rho^d \end{pmatrix}$$

$$S_{\mathrm{aux}} = \int_\Sigma u\bar{\partial}v, u \in \Pi\Omega^0(\Sigma, K_\Sigma), v \in \Pi\Omega^0(\Sigma)$$

$$\Pi^+: \tilde{\phi}_a \rho^a + u$$
$$\Pi^-: \tilde{\rho}_a \phi^a - v\left(\tilde{\phi}_a \phi^a + \tilde{\rho}_a \rho^a\right) + N\partial v$$

$$\Pi^+(z_1)\Pi^+(z_2)\ \sim 0$$

$$\Pi^+(z_1)\Pi^-(z_2)\ \sim \big(\tilde{\phi}_a\rho^a + u\big)(z_1)\big(\tilde{\rho}_a\phi^a - v\big(\tilde{\phi}_a\phi^a + \tilde{\rho}_a\rho^a\big) + N\partial v\big)(z_2)$$

$$\sim \underbrace{\frac{-N}{z_{12}^2} + \frac{\big(\tilde{\phi}_a\phi^a + \tilde{\rho}_a\rho^a\big)(z_2)}{z_{12}}}_{\tilde{\phi}_a\rho^a,\tilde{\rho}_a\phi^a \text{ contractions}} - \underbrace{\frac{\big(\tilde{\phi}_a\phi^a + \tilde{\rho}_a\rho^a\big)(z_2)}{z_{12}} + \frac{N}{z_{12}^2}}_{u,v \text{ contractions}} = 0$$

$$\Pi^-(z_1)\Pi^-(z_2)\ \sim 0$$

$$S = \int_\Sigma \tilde{\phi}_a\bar{\partial}\phi^a + \tilde{\rho}_a\bar{\partial}\rho^a + u\bar{\partial}v$$
$$+\chi_+\big(\tilde{\phi}_a\rho^a + u\big) + \chi_-\big(\tilde{\rho}_a\phi^a + v\big(\tilde{\phi}_a\phi^a + \tilde{\rho}_a\rho^a\big) - N\partial v\big)$$

$$S_{\text{ffpsl}(1|1)} = \int_\Sigma \tilde{\phi}_a\bar{\partial}\phi^a + \tilde{\rho}_a\bar{\partial}\rho^a + u\bar{\partial}v + \beta^+\bar{\partial}\gamma_+ + \beta^-\bar{\partial}\gamma_-$$
$$\beta^\pm \in \Omega^0(\Sigma, K_\Sigma), \gamma^\pm \in \Omega^0(\Sigma)$$

$$Q_{BRST} = \oint \gamma_+\Pi^+ + \gamma_-\Pi^- \ = \oint \gamma_+(\underbrace{\tilde{\phi}_a\rho^a}_{G^+} + u) + \gamma_-(\underbrace{\tilde{\rho}_a\phi^a}_{G^-} - v\,\underbrace{\big(\tilde{\phi}_a\phi^a + \tilde{\rho}_a\rho^a\big)}_{J} + N\partial v)$$

$$\delta_{Q_{BRST}}\left(\begin{pmatrix}\tilde{\phi}_a\tilde{\rho}_b\end{pmatrix}\begin{pmatrix}A_c^a & B_c^b\\ C_d^a & D_d^b\end{pmatrix}\begin{pmatrix}\phi^c\\ \rho^d\end{pmatrix}\right) = 0 \Leftrightarrow B = 0 = C \text{ and } A = D$$

$$J^A(z) := \big(\delta(\gamma_+)\delta(\gamma_-)\tilde{\rho}_a(T^A)^a_b\rho^b\big)(z)$$

$$\Upsilon^\pm := \delta(\beta^\pm)\delta_{Q_{BRST}}\beta^\pm = \delta(\beta^\pm)\Pi^\pm$$

$$J^A(z) = \big(\delta(\gamma_+)\delta(\gamma_-)\tilde{\rho}_a(T^A)^a_b\rho^b\big)(z)$$
$$\overset{\text{descends to}}{\rightarrow} \quad \delta(\gamma^-)(T^A)^b_c\tilde{\phi}_b\rho^c(z)$$

$$\delta(\gamma^-)(T^A)^b_c\tilde{\phi}_b\rho^c$$
$$\overset{\text{descends to}}{\rightarrow} \quad (T^A)^b_c\big(\tilde{\phi}_b\phi^c + \tilde{\rho}_b\rho^c\big)(z)$$
$$=: \tilde{J}^A(z)$$

$$S[a] := \int_\Sigma \tilde{\phi}_a\big(\bar{\partial}\delta^a_b + (T^C)^a_b a_C(z)\big)\phi^b + \tilde{\rho}_a\big(\bar{\partial}\delta^a_b + (T^C)^a_b a_C(z)\big)\rho^b + \text{aux}$$

$$\langle\tilde{J}^A(z)\dots\rangle = \int\ \mathrm{D}(\mathcal{F}_{fields})e^{-S}\big(\tilde{J}^A(z)\dots\big) = \int\ \mathrm{D}(\mathcal{F}_{fields})\left(-\frac{\delta e^{-S[a]}}{\delta a_A(z)}\right)_{a=0}(\dots)$$

$$\left\langle v(z_1)\prod_{i=2}^n \Upsilon^+(z_i)\prod_{i=2}^n \Upsilon^-(z_i)\prod_{i=1}^n J^{A_i}(z_i)\right\rangle = \left\langle v(z_1)J^{A_i}(z_1)\prod_{i=2}^n \tilde{J}^{A_i}(z_i)\right\rangle$$
$$= \left\langle\tilde{\rho}_{a_1}(T^{A_1})^{a_1}_{b_1}\rho^{b_1}(z_1)\prod_{i=2}^n\left(\tilde{\phi}_{a_i}(T^{A_i})^{a_i}_{b_i}\phi^{b_i}(z_i) + \tilde{\rho}_{a_i}(T^{A_i})^{a_i}_{b_i}\rho^{b_i}(z_i)\right)\right\rangle_{\text{free-fields}} \times$$
$$\big\langle v(z_1)\delta\big(\gamma_+(z_1)\big)\delta\big(\gamma_-(z_1)\big)\big\rangle_{\text{ghosts}}$$
$$= \left\langle\tilde{\rho}_{a_1}(T^{A_1})^{a_1}_{b_1}\rho^{b_1}(z_1)\prod_{i=2}^n\left(\tilde{\phi}_{a_i}(T^{A_i})^{a_i}_{b_i}\phi^{b_i}(z_i) + \tilde{\rho}_{a_i}(T^{A_i})^{a_i}_{b_i}\rho^{b_i}(z_i)\right)\right\rangle_{\text{free-fields}}$$

$$\int_\Sigma \tilde{\phi}_{\mathcal{A}}\bar{\partial}\phi^{\mathcal{A}} + \tilde{\rho}_{\mathcal{A}}\bar{\partial}\rho^{\mathcal{A}} = \int_\Sigma \underbrace{\tilde{\phi}_a\bar{\partial}\phi^a + \tilde{\rho}_i\bar{\partial}\rho^i}_{\text{bosonic}} + \underbrace{\tilde{\phi}_i\bar{\partial}\phi^i + \tilde{\rho}_a\bar{\partial}\rho^a}_{\text{fermionic}}$$

$$\begin{aligned} J &= \tilde{\phi}_{\mathcal{A}}\phi^{\mathcal{A}} + \tilde{\rho}_{\mathcal{A}}\rho^{\mathcal{A}} \\ G^+ &= \tilde{\phi}_{\mathcal{A}}\rho^{\mathcal{A}} \\ G^- &= \phi^{\mathcal{A}}\tilde{\rho}_{\mathcal{A}} \end{aligned}$$

$$\begin{aligned} G^+(z)G^-(w) &= \left(\tilde{\phi}_a\rho^a + \tilde{\phi}_i\rho^i\right)(z)\left(\phi^b\tilde{\rho}_b + \phi^i\tilde{\rho}_i\right)(w) \\ &\sim \underbrace{\frac{N}{(z-w)^2} + \frac{\tilde{\phi}_a\phi^a(w) + \tilde{\rho}_a\rho^a(w)}{z-w}}_{\text{a,b, index contribution}} + \underbrace{\frac{-N}{(z-w)^2} + \frac{\tilde{\phi}_i\phi^i(w) + \tilde{\rho}_i\rho^i(w)}{z-w}}_{\text{i,j, index contribution } (\phi^b\tilde{\rho}_b)(w) + (\tilde{\phi}_i\rho^i)(z)(\phi^j\tilde{\rho}_j)(w)} \\ &\sim \frac{J(w)}{z-w} \end{aligned}$$

$$\int_\Sigma \tilde{\phi}_{\mathcal{A}}\bar{\partial}\phi^{\mathcal{A}} + \tilde{\rho}_{\mathcal{A}}\bar{\partial}\rho^{\mathcal{A}} + a_J\left(\tilde{\phi}_{\mathcal{A}}\phi^{\mathcal{A}} + \tilde{\rho}_{\mathcal{A}}\rho^{\mathcal{A}}\right) + \chi_+\tilde{\phi}_{\mathcal{A}}\rho^{\mathcal{A}} + \chi_-\phi^{\mathcal{A}}\tilde{\rho}_{\mathcal{A}}$$

$$S = \int_\Sigma \tilde{\phi}_{\mathcal{A}}\bar{\partial}\phi^{\mathcal{A}} + \tilde{\rho}_{\mathcal{A}}\bar{\partial}\rho^{\mathcal{A}} + \underbrace{\beta^+\bar{\partial}\gamma_+ + \beta^-\bar{\partial}\gamma_- + \beta^J\bar{\partial}\gamma_J}_{S_{\text{ghosts}}}$$

$$\begin{aligned} &\beta^\pm \in \Omega^0(\Sigma, K_\Sigma), \gamma_\pm \in \Omega^0(\Sigma) \\ &\beta^J \in \Pi\Omega^0(\Sigma, K_\Sigma), \gamma_J \in \Pi\Omega^0(\Sigma) \end{aligned}$$

$$Q_{BRST} = \oint \gamma_+\left(\tilde{\phi}_{\mathcal{A}}\rho^{\mathcal{A}}\right) + \gamma_-\left(\phi^{\mathcal{A}}\tilde{\rho}_{\mathcal{A}}\right) + \gamma_J\left(\tilde{\phi}_{\mathcal{A}}\phi^{\mathcal{A}} + \tilde{\rho}_{\mathcal{A}}\rho^{\mathcal{A}}\right) + \gamma_+\gamma_-\beta^J$$

$$J^A(z) := \left(\delta(\gamma_+)\delta(\gamma_-)\tilde{\rho}_a(T^A)^a_b\rho^b\right)(z)$$

$$\Upsilon^\pm := \delta(\beta^\pm)\delta_{Q_{BRST}}\beta^\pm$$

$$\begin{aligned} J^A(z) &= \left(\delta(\gamma_+)\delta(\gamma_-)\tilde{\rho}_a(T^A)^a_b\rho^b\right)(z) \\ &\overset{\text{descends to}}{\to} \delta(\gamma^-)(T^A)^b_c\tilde{\phi}_b\rho^c(z) \end{aligned}$$

$$\begin{aligned} &\delta(\gamma^-)(T^A)^b_c\tilde{\phi}_b\rho^c \\ &\overset{\text{descends to}}{\to} (T^A)^b_c\left(\tilde{\phi}_b\phi^c + \tilde{\rho}_b\rho^c\right)(z) \\ &=: \tilde{J}^A(z) \end{aligned}$$

$$\begin{aligned} &\left\langle \gamma_J(z_1)\prod_{i=2}^n \Upsilon^+(z_i)\prod_{i=2}^n \Upsilon^-(z_i)\prod_{i=1}^n J^{A_i}(z_i)\right\rangle = \left\langle \gamma_J(z_1)J^{A_i}(z_1)\prod_{i=2}^n \tilde{J}^{A_i}(z_i)\right\rangle \\ &= \left\langle \tilde{\rho}_{a_1}(T^{A_1})^{a_1}_{b_1}\rho^{b_1}(z_1)\prod_{i=2}^n \left(\tilde{\phi}_{a_i}(T^{A_i})^{a_i}_{b_i}\phi^{b_i}(z_i) + \tilde{\rho}_{a_i}(T^{A_i})^{a_i}_{b_i}\rho^{b_i}(z_i)\right)\right\rangle_{\text{free-fields}} \times \\ &\left\langle \gamma_J(z_1)\delta(\gamma_+(z_1))\delta(\gamma_-(z_1))\right\rangle_{\text{ghosts}} \\ &= \left\langle \tilde{\rho}_{a_1}(T^{A_1})^{a_1}_{b_1}\rho^{b_1}(z_1)\prod_{i=2}^n \left(\tilde{\phi}_{a_i}(T^{A_i})^{a_i}_{b_i}\phi^{b_i}(z_i) + \tilde{\rho}_{a_i}(T^{A_i})^{a_i}_{b_i}\rho^{b_i}(z_i)\right)\right\rangle_{\text{free-fields}} \end{aligned}$$

$$\tilde{\phi}_a, \tilde{\rho}_i \in \Omega^0\left(\Sigma, K_\Sigma^{1/2} \times \mathcal{L}_d\right), \phi^a, \rho^i \in \Omega^0\left(\Sigma, K_\Sigma^{1/2} \times \mathcal{L}_d^{-1}\right),$$

$$\tilde{\phi}_i, \tilde{\rho}_a \in \Pi\Omega^0\left(\Sigma, K_\Sigma^{1/2} \times \mathcal{L}_d\right), \phi^i, \rho^a \in \Pi\Omega^0\left(\Sigma, K_\Sigma^{1/2} \times \mathcal{L}_d^{-1}\right),$$

$$\left\langle \tilde{\rho}_{a_1}(T^{A_1})^{a_1}_{b_1}\rho^{b_1}(z_1)\prod_{i=2}^{n}\left(\tilde{\phi}_{a_i}(T^{A_i})^{a_i}_{b_i}\phi^{b_i}(z_i) + \tilde{\rho}_{a_i}(T^{A_i})^{a_i}_{b_i}\rho^{b_i}(z_i)\right)\right\rangle_{\text{free-fields}}$$

$$= \frac{1}{Z}\int \mathrm{D}\left(\tilde{\phi}_a, \phi^a, \tilde{\rho}_a, \rho^a\right)e^{S_{\text{ff}}}\left(\rho_{a_1}(T^{A_1})^{a_1}_{b_1}\rho^{b_1}(z_1)\prod_{i=2}^{n}\left(\tilde{\phi}_{a_i}(T^{A_i})^{a_i}_{b_i}\phi^{b_i}(z_i) + \tilde{\rho}_{a_i}(T^{A_i})^{a_i}_{b_i}\rho^{b_i}(z_i)\right)\right)$$

$$S_{\text{ff}} = \int_\Sigma \tilde{\phi}_a\bar{\partial}\phi^a + \tilde{\rho}_a\bar{\partial}\rho^a, a = 1, \dots, m$$
$$\tilde{\rho}_a \in \Pi\Omega^0\left(\Sigma, K_\Sigma^{1/2}\right), \rho^a \in \Pi\Omega^0\left(\Sigma, K_\Sigma^{1/2}\right)$$
$$\tilde{\phi}_a \in \Omega^0\left(\Sigma, K_\Sigma^{1/2}\right), \phi^a \in \Omega^0\left(\Sigma, K_\Sigma^{1/2}\right)$$

$$\left\langle \tilde{\rho}_{a_1}(T^{A_1})^{a_1}_{b_1}\rho^{b_1}(z_1)\prod_{i=2}^{n}\left(\tilde{\phi}_{a_i}(T^{A_i})^{a_i}_{b_i}\phi^{b_i}(z_i) + \tilde{\rho}_{a_i}(T^{A_i})^{a_i}_{b_i}\rho^{b_i}(z_i)\right)\right\rangle_{\text{free-fields on } \mathbb{CP}^4} = \sum_{\sigma\in S_n\setminus\mathbb{Z}_n}\frac{\operatorname{tr}\left(T^{A_{\sigma_1}}\dots T^{A_{\sigma_n}}\right)}{z_{\sigma_1\sigma_2}\dots z_{\sigma_n\sigma_1}}$$

$$j^A + b^A := \tilde{\rho}_a(T^A)^a_b\rho^b + \tilde{\phi}_a(T^A)^a_b\phi^b$$

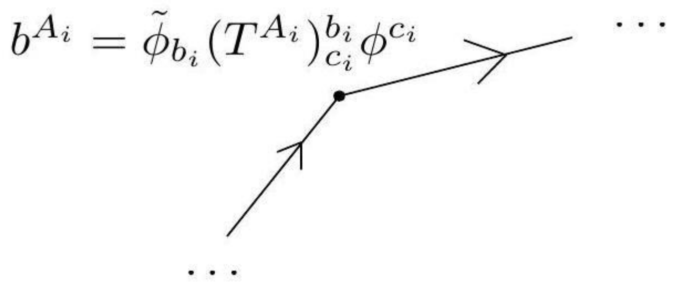
$$j^A(z_1)(j+b)^B(z_2) \sim \frac{k\kappa^{AB}}{z_{12}^2} + \frac{f_C^{AB}j^C}{z_{12}}$$
$$(j+b)^A(z_1)(j+b)^B(z_2) \sim \frac{k\kappa^{AB}}{z_{12}^2} - \frac{k\kappa^{AB}}{z_{12}^2} + \frac{f_C^{AB}(j+b)^C}{z_{12}}$$

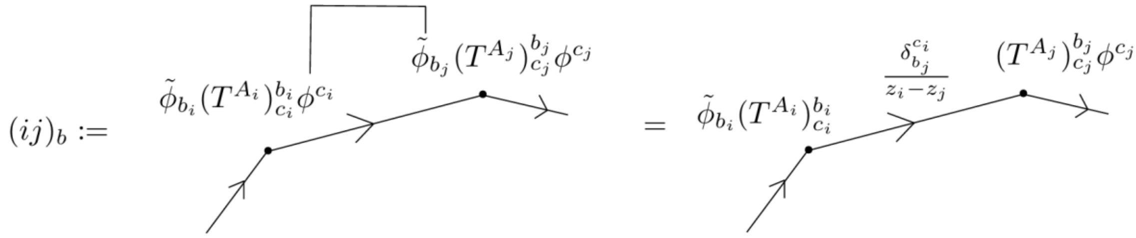
$$j^A(z) := \tilde{\rho}_b(T^A)^b_c\rho^c$$
$$j^A(z_1)j^B(z_2) \sim \frac{\kappa^{AB}}{z_{12}^2} + \frac{f_C^{AB}j^C(z_2)}{z_{12}}$$

$$b^A(z) := \tilde{\phi}_b(T^A)^b_c\phi^c$$
$$b^A(z_1)b^B(z_2) \sim \frac{-\kappa^{AB}}{z_{12}^2} + \frac{f_C^{AB}b^C(z_2)}{z_{12}}$$

$$\left\langle\prod_{i=1}^{n} b^{A_i}(z_i)\right\rangle = \sum_{\sigma\in S_n\setminus\mathbb{Z}_n}\frac{\operatorname{tr}\left(T^{A_{\sigma_1}}\dots T^{A_{\sigma_n}}\right)}{z_{\sigma_1\sigma_2}\dots z_{\sigma_n\sigma_1}} + \mathbb{MTC}_{\text{multi-trace contributions}}$$

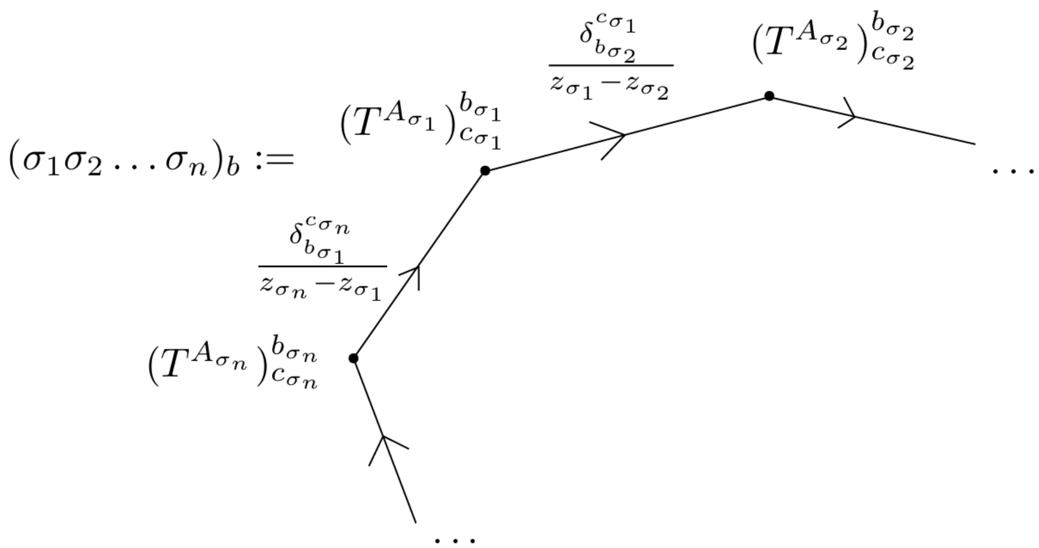
$$(ij)_b \to \tilde{\phi}_{b_i}(z_i)(T^{A_i})^{b_i}_{c_i}\phi^{c_i}(z_i)\tilde{\phi}_{b_j}(z_j)\left(T^{A_j}\right)^{b_j}_{c_j}\phi^{c_j}(z_j)$$
$$= \tilde{\phi}_{b_i}(z_i)(T^{A_i})^{b_i}_{c_i}\left(\frac{\delta^{c_i}_{b_j}}{z_i - z_j}\right)\left(T^{A_j}\right)^{b_j}_{c_j}\phi^{c_j}(z_j)$$





$$(\sigma_1\sigma_2 \ldots \sigma_m)_b := (\sigma_1\sigma_2)_b(\sigma_2\sigma_3)_b \ldots (\sigma_m\sigma_1)_b \xrightarrow{\text{diagram evaluation}} \frac{\mathrm{tr}\left(T^{A_{\sigma_1}} \ldots T^{A_{\sigma_m}}\right)}{z_{\sigma_1\sigma_2} \ldots z_{\sigma_m\sigma_1}}$$

$$\left\langle \prod_{i=1}^{n} b^{A_i}(z_i) \right\rangle = \underbrace{\sum_{\sigma \in S_n \backslash \mathbb{Z}_n} (\sigma_1\sigma_2 \ldots \sigma_n)_b}_{\text{single-trace}} + \underbrace{\frac{1}{2}\sum_{k=2}^{n-2} \left(\sum_{\sigma,\sigma'} (\sigma_1 \ldots \sigma_k)_b (\sigma'_1 \ldots \sigma'_{n-k})_b \right)}_{\text{double-trace}} + \cdots$$

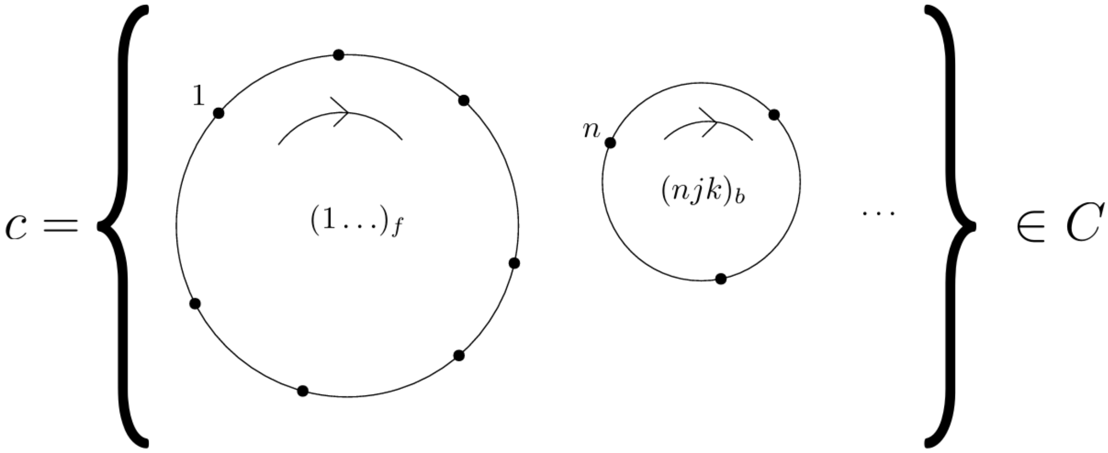


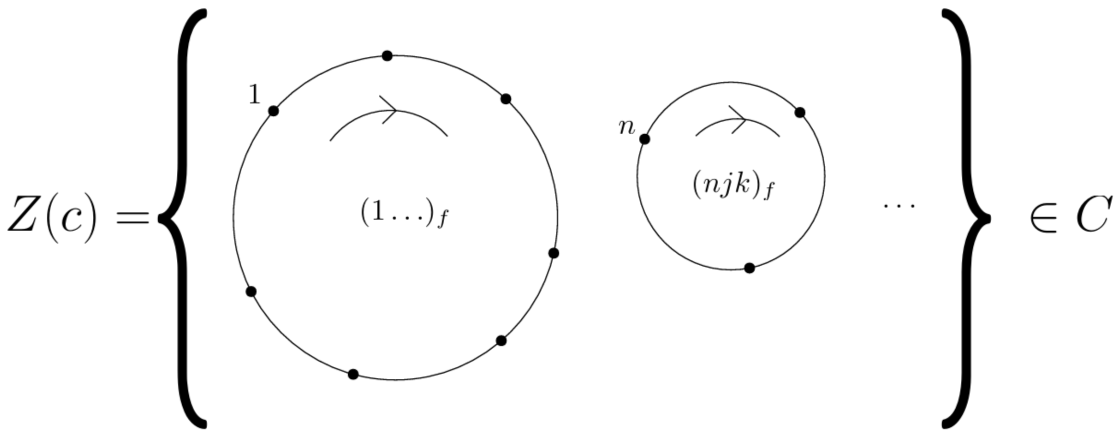
$$(\sigma_1\sigma_2 \ldots \sigma_m)_f := (\sigma_1\sigma_2)_f(\sigma_2\sigma_3)_f \ldots (\sigma_m\sigma_1)_f \xrightarrow{\text{diagram evaluation}} -1 \times \frac{\mathrm{tr}\left(T^{A_{\sigma_1}} \ldots T^{A_{\sigma_m}}\right)}{z_{\sigma_1\sigma_2} \ldots z_{\sigma_m\sigma_1}}$$

$$\left\langle \prod_{i=1}^{n} j^{A_i}(z_i) \right\rangle = \underbrace{\sum_{\sigma \in S_n \setminus \mathbb{Z}_n} (\sigma_1 \sigma_2 \dots \sigma_n)_f}_{\text{single-trace}} + \underbrace{\frac{1}{2} \sum_{k=2}^{n-2} \left(\sum_{\sigma, \sigma'} (\sigma_1 \dots \sigma_k)_f (\sigma'_1 \dots \sigma'_{n-k})_f \right)}_{\text{double-trace}} + \cdots$$

$$\left\langle j^{A_1}(z_1) \prod_{i=2}^{n} \left(b^{A_i}(z_i) + j^{A_i}(z_i) \right) \right\rangle = \sum_{A \subset \{2,3,\dots n\}} \left\langle j^{A_1}(z_1) \prod_{i \in A} j^{A_i}(z_i) \right\rangle \left\langle \prod_{k \in \{2,3,\dots n\} \setminus A} b^{A_k}(z_k) \right\rangle$$

$$\left\langle j^{A_1}(z_1) \prod_{i=2}^{n} \left(b^{A_i}(z_i) + j^{A_i}(z_i) \right) \right\rangle = \sum_{c \in C} \underbrace{\text{Feyn(c)},}_{\text{the Feynman diagram evaluation of } c}$$





$$\left\langle j^{A_1}(z_1) \prod_{i=2}^{n} \left(b^{A_i}(z_i) + j^{A_i}(z_i) \right) \right\rangle = \sum_{c \in C} \text{Feyn}(c) = \sum_{c \in \text{Stab}(Z)} \text{Feyn}(c)$$

$$= - \sum_{\sigma \in S_n \setminus \mathbb{Z}_n} \frac{\text{tr}\left(T^{A_{\sigma_1}} \dots T^{A_{\sigma_n}} \right)}{z_{\sigma_1 \sigma_2} \dots z_{\sigma_n \sigma_1}}$$

$$\left\langle b^{A_1}(z_1) \prod_{i=2}^{n} \left(b^{A_i}(z_i) + j^{A_i}(z_i) \right) \right\rangle = \sum_{\sigma \in S_n \setminus \mathbb{Z}_n} \frac{\text{tr}\left(T^{A_{\sigma_1}} \dots T^{A_{\sigma_n}} \right)}{z_{\sigma_1 \sigma_2} \dots z_{\sigma_n \sigma_1}}$$

$$\mathbb{PT}:=\{Z^{\mathcal{M}}:=(Z^A,\psi^I)\in\mathbb{CP}^{3|4}\mid\lambda_\alpha=(\lambda_0,\lambda_1):=(Z^3,Z^4)\neq(0,0)\}$$

$$S_{BW}=\int_{\Sigma_{\text{open}}}\mathrm{Re}\left(W_{\mathcal{M}}\bar{D}Z^{\mathcal{M}}+\mathcal{g}_{\text{ghosts}}+S_{\text{currents}}\right)$$

$$S_{\text{closed BW}}=\frac{1}{2}\left(\int_{\Sigma}W_{\mathcal{M}}\bar{D}Z^{\mathcal{M}}+\mathcal{g}_{\text{ghosts}}+S_{\text{currents}}\right)$$

$$j^{A_1}(z_1)j^{A_2}(z_2)\sim\frac{k\kappa^{A_1A_2}}{(z_1-z_2)^2}+\frac{f^{A_1A_2}_{A_3}j^{A_3}}{z_1-z_2}$$

$$\begin{aligned}S_{\text{closed BW}}&=S_{\text{closed BW}}|_{\Sigma_1}+S_{\text{closed BW}}|_{\Sigma_2}=S_{\text{closed BW}}|_{\Sigma_1}+\bar{S}_{\text{closed BW}}|_{\Sigma_1}\\&=\int_{\Sigma_1}\mathrm{Re}\left(W_{\mathcal{M}}\bar{D}Z^{\mathcal{M}}+\mathcal{g}_{\text{ghosts}}+S_{\text{currents}}\right)\end{aligned}$$

$$\int\mathcal{D}\mathcal{O}\mathcal{D}\bar{\mathcal{O}}\Big|_{\bar{\mathcal{O}}(z)=\mathcal{O}(\bar{z})}=\int_{\Gamma}\mathcal{D}\mathcal{O}$$

$$\begin{aligned}S_{\text{pert. tree SYM}}:=&\int_{\mathbb{CP}^1}W_{\mathcal{M}}(\bar\partial+a)Z^{\mathcal{M}}+\int_{\mathbb{CP}^1}\tilde\phi_a\bar\partial\phi^a+\tilde\rho_a\bar\partial\rho^a+u\bar\partial v+(\mathfrak{AM}_{\text{aux. matter}})_{c=26}\\&+\int_{\mathbb{CP}^1}\chi_+(\tilde\phi_a\rho^a+u)+\chi_-(\tilde\rho_a\phi^a+v(\tilde\phi_a\phi^a+\tilde\rho_a\rho^a)-N\partial v)+S_{\text{ghosts}}\end{aligned}$$

$$\begin{aligned}&Z^{\mathcal{M}}=(Z^M,\psi^I),Z^M\in\Omega^0(\mathbb{CP}^1,\mathcal{L}_d),\psi^I\in\Pi\Omega^0(\mathbb{CP}^1,\mathcal{L}_d),M,I=1,\dots4\\&W_{\mathcal{M}}=(W_M,\eta_I),W_M\in\Omega^0(\mathbb{CP}^1,K_{\mathbb{CP}^1}\otimes(\mathcal{L}_d)^{-1}),\eta_I\in\Pi\Omega^0(\mathbb{CP}^1,K_{\mathbb{CP}^1}\otimes(\mathcal{L}_d)^{-1})\\&\tilde\rho_a\in\Pi\Omega^0\left(\mathbb{CP}^1,K^{1/2}_{\mathbb{CP}^1}\right),\rho^a\in\Pi\Omega^0\left(\mathbb{CP}^1,K^{1/2}_{\mathbb{CP}^1}\right),a=1,\dots,\underbrace{N}_{=\dim F_{\mathfrak{g}}}\\&\tilde\phi_a\in\Omega^0\left(\mathbb{CP}^1,K^{1/2}_{\mathbb{CP}^1}\right),\phi^a\in\Omega^0\left(\mathbb{CP}^1,K^{1/2}_{\mathbb{CP}^1}\right),a=1,\dots,N\\&u\in\Pi\Omega^0(\mathbb{CP}^1,K_{\mathbb{CP}^1}),v\in\Pi\Omega^0(\mathbb{CP}^1)\end{aligned}$$

$$S_{\text{twistor ghost}}=\int_{\mathbb{CP}^1}m_2\bar\partial n_2,m_2\in\Pi\Omega^0(\mathbb{CP}^1,K_{\mathbb{CP}^1}),n_2\in\Pi\Omega^0(\mathbb{CP}^1)$$

$$S_{\text{ffghost}}=\int_{\mathbb{CP}^1}\beta^+\bar\partial\gamma_++\beta^-\bar\partial\gamma_-,\beta^\pm\in\Omega^0(\mathbb{CP}^1,K_{\mathbb{CP}^1}),\gamma^\pm\in\Omega^0(\mathbb{CP}^1)$$

$$Q_{BRST}=\oint cT+n_2(W_{\mathcal{M}}Z^{\mathcal{M}})+\gamma_+(\underbrace{\tilde\phi_a\rho^a}_{G^+}+u)+\gamma_-(\underbrace{\tilde\rho_a\phi^a}_{G^-}+v\underbrace{(\tilde\phi_a\phi^a+\tilde\rho_a\rho^a)}_{J}-N\partial v)$$

$$\begin{aligned}S_{\text{pert. SYM gfixed}}:=&\int_{\mathbb{CP}^1}W_{\mathcal{M}}\bar\partial Z^{\mathcal{M}}+\int_{\mathbb{CP}^1}\tilde\phi_a\bar\partial\phi^a+\tilde\rho_a\bar\partial\rho^a+\mathbb{AF}_{\text{auxillary fields}}\quad:\\&+\int_{\mathbb{CP}^1}\underbrace{m_2\bar\partial n_2+\beta^+\bar\partial\gamma_++\beta^-\bar\partial\gamma_-+b\bar\partial c}_{\text{ghosts}}+\underbrace{u\bar{\bar\partial}v}_{\text{auxiliary free fermion}}\end{aligned}$$

$$\begin{aligned}U^{A_i}(f_i)&:=cJ^{A_i}f_i(Z^{\mathcal{M}}(z))=c\delta(\gamma_+)\delta(\gamma_-)(\tilde\rho_a(T^{A_i})^a_b\rho^b)f_i(Z^{\mathcal{M}}(z))\\V^{A_i}(f_i)&:=\int_{\mathbb{CP}^1}J^{A_i}f_i(Z^{\mathcal{M}}(z))=\int_{\mathbb{CP}^1}\delta(\gamma_+)\delta(\gamma_-)(\tilde\rho_a(T^{A_i})^a_b\rho^b)f_i(Z^{\mathcal{M}}(z))\end{aligned}$$

$$\Upsilon^{\pm} := \delta(\beta^{\pm})\delta_{Q_{BRST}}\beta^{\pm}$$

$$\tilde{J}^{A_i} = \left(\tilde{\phi}_a (T^{A_i})^a_b \phi^b + \tilde{\rho}_a (T^{A_i})^a_b \rho^b\right)$$

$$A_{\text{tree SYM}} = \sum_{d\in\mathbb{Z}_{\geq 0}} g^{2d} \int \frac{\bigwedge_{k=0}^{d} \mathrm{d}^{4|4} Z_k}{\mathrm{Vol}GL(2,\mathbb{C})} \frac{\bigwedge_{i=1}^{n} f_i\big(Z(\sigma_i)\big)(\sigma_i\, \mathrm{d}\sigma_i)}{\big(\sigma_{p_0(1)}\sigma_{p_0(2)}\big) \ldots \big(\sigma_{p_0(n)}\sigma_{p_0(1)}\big)}$$

$$\mathcal{C}_{\text{tree}} := \frac{1}{\mathrm{Vol}GL(2,\mathbb{C})} \sum_{d\in\mathbb{Z}_{\geq 0}} g^{2d} \frac{1}{Z} \int \mathrm{D}(\mathcal{F}_{fields}) \exp\left(\int_{\mathbb{CP}^1_\sigma} W_{\mathcal{M}} \bar{\partial} Z^{\mathcal{M}} + \tilde{\phi}\bar{\partial}\phi + \tilde{\rho}\bar{\partial}\rho\right) \times$$
$$\int_{\mathbb{CP}^1_\sigma} f_1(Z(\sigma)) j^{A_1}(\sigma) \prod_{i=2}^{n} \int_{\mathbb{CP}^1_\sigma} f_i(Z(\sigma))(b^{A_i}(\sigma) + j^{A_i}(\sigma))$$

$$\left\langle j^{A_1}(\sigma_1) \prod_{i=2}^{n} (b+j)^{A_i} \right\rangle = -\sum_{p\in S_n} \frac{\mathrm{tr}\big(T^{A_{p(1)}} \ldots T^{A_{p(n)}}\big)}{\big(\sigma_{p(1)}\sigma_{p(2)}\big) \ldots \big(\sigma_{p(n)}\sigma_{p(1)}\big)} \bigwedge_{i=1}^{n} (\sigma_i\, \mathrm{d}\sigma_i)$$

$$\mathcal{C}_{\text{tree}} =: \sum_{p\in S_n} \mathrm{tr}\big(T^{A_{p(1)}} \ldots T^{A_{p(n)}}\big) \mathcal{C}_{\text{tree},p}$$

$$\mathcal{C}_{\text{tree } p_0} = \frac{-1}{\mathrm{Vol}GL(2,\mathbb{C})} \sum_{d\in\mathbb{Z}_{\geq 0}} g^{2d} \frac{1}{Z} \int \mathrm{D}(\mathcal{F}_{fields}) \exp\left(\int_{\mathbb{CP}^1_\sigma} W_{\mathcal{M}} \bar{\partial} Z^{\mathcal{M}}\right) \times$$
$$\frac{\bigwedge_{i=1}^{n} \int_{\mathbb{CP}^1_\sigma} f_i\big(Z(\sigma_i)\big)(\sigma_i\, \mathrm{d}\sigma_i)}{\big(\sigma_{p_0(1)}\sigma_{p_0(2)}\big) \ldots \big(\sigma_{p_0(n)}\sigma_{p_0(1)}\big)}$$

$$\frac{1}{\mathrm{Vol}SL(2,\mathbb{C})} := \frac{(\sigma_1\sigma_2)(\sigma_2\sigma_3)(\sigma_3\sigma_1)}{(\sigma_1\, \mathrm{d}\sigma_1)(\sigma_2\, \mathrm{d}\sigma_2)(\sigma_3\, \mathrm{d}\sigma_3)}$$

$$\mathcal{C}_{\text{tree } p_0} = \frac{-1}{\mathrm{Vol}GL(2,\mathbb{C})} \sum_{d\in\mathbb{Z}_{\geq 0}} g^{2d} \int \bigwedge_{k=0}^{d} \mathrm{d}^{4|4} Z_k \frac{\bigwedge_{i=1}^{n} \int_{\mathbb{CP}^1_\sigma} f_i\big(Z(\sigma_i)\big)(\sigma_i\, \mathrm{d}\sigma_i)}{\big(\sigma_{p_0(1)}\sigma_{p_0(2)}\big) \ldots \big(\sigma_{p_0(n)}\sigma_{p_0(1)}\big)}$$

$$S_{BW} = \int \mathrm{D}^{3|4} Z \wedge \left(k \times hCS(\mathcal{A}) - hCS\big(\partial_{\mathcal{M}} f^{\mathcal{N}}\big) + N(f) \lrcorner g\right)$$
$$+ \int \mathrm{d}^{4|8} x \exp\left[\left(\int_{\mathcal{L}_x} g\right) + k \log\, \det(\bar{\partial} + \mathcal{A}) - \log\, \det\big(\bar{\partial} + \partial_{\mathcal{M}} f^{\mathcal{N}}\big)\right]$$

$$hCS(\mathcal{A}) := \int \mathrm{D}^{3|4} Z \wedge \mathrm{tr}\left(\mathcal{A}\bar{D}\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right), \bar{D} := \bar{\partial} + \mathcal{L}_{f\cdot\partial}$$

$$\log\, \det(\bar{\partial} + \mathcal{A}) := \int_{\mathcal{L}_x \times \mathcal{L}_x} \frac{\mathrm{D}\sigma_1 \mathrm{D}\sigma_2}{(\sigma_1\sigma_2)^2} \mathrm{tr}\left(H\mathcal{A}H^{-1}\big(Z(\sigma_1)\big)H\mathcal{A}H^{-1}\big(Z(\sigma_2)\big)\right), \bar{\partial}_{\mathcal{L}_x} H = \mathcal{A}|_{\mathcal{L}_x} H$$

$$S[\mathcal{A}_1,\mathcal{A}_2,f,g]=\int \mathrm{D}^{3|4}Z\wedge\left[\sum_{i=1}^{2}\left(k_i\times hCS(\mathcal{A}_i)\right)-hCS(\partial_I f^J)+N(f)\lrcorner g\right]$$
$$+\int \mathrm{d}^{4|8}x\exp\left[\int_{\mathcal{L}_x} g+\left(\sum_{i=1}^{2}\left(k_i\log\det(\bar\partial+\mathcal{A}_i)\right)-\log\det(\bar\partial+\partial_I f^J)\right]$$
$$\mathcal{A}_1,\mathcal{A}_2\in\Omega^{0,1}(\mathbb{PT},\mathfrak{g}),k_1=1,k_2=-1$$
$$f\in\Omega^{0,1}(\mathbb{PT},T^{1,0}\mathbb{PT}),g\in\Omega^{1,1}(\mathbb{PT})$$

$$\mathcal{A}_{\mathfrak{gl}(N|N)}=\begin{pmatrix}\mathcal{A}_2 & 0_N\\ 0_N & \mathcal{A}_1\end{pmatrix},hCS(\mathcal{A}_{\mathfrak{gl}(N|N)})=hCS(\mathcal{A}_1)-hCS(\mathcal{A}_2)$$

$$\mathcal{A}_1\text{ scattering state }=\int \tilde\rho T\rho f_i(Z)$$
$$\mathcal{A}_2\text{ scattering state }=\int \tilde\phi T\phi f_i(Z)$$

$$(C)_A f(Z(\sigma))=(C)_A\int\frac{\mathrm{d}t}{t}\bar\delta^2\left(t\lambda^\alpha(\sigma)-\sqrt{\omega}\kappa^\alpha\right)e^{t\sqrt{\omega}([\mu(\sigma)\tilde\kappa]+\psi(\sigma)\cdot\eta)}$$

$$L_{(x,\theta)}=\{\mu^{\dot\alpha}=ix^{\alpha\dot\alpha}\lambda_\alpha,\psi^I=\theta^{\alpha I}\lambda_\alpha\}$$

$$f|_{L_{(x,\theta)}}=\int\frac{\mathrm{d}t}{t}\bar\delta^2\left(t\lambda^\alpha(\sigma)-\sqrt{\omega}\kappa^\alpha\right)e^{ix\cdot P+\theta\cdot\Pi}$$
$$P^{\alpha\dot\alpha}:=\omega\kappa^\alpha\tilde\kappa^{\dot\alpha},(\Pi)^\alpha_I:=\omega\kappa^\alpha\eta_I$$

$$o(\sigma)=:O(\sigma)\sqrt{(\sigma\mathrm{d}\sigma)},\begin{pmatrix}o(\sigma)\\O(\sigma)\end{pmatrix}\in\begin{pmatrix}K^{1/2}_{\mathbb{CP}^1}\\ \mathcal{O}(-1)\end{pmatrix}$$

$$\tilde\Phi_a\overline{(\sigma_1)\Phi^b}(\sigma_2)=\frac{-\delta^b_a}{(\sigma_1\sigma_2)},\tilde P_a\overline{(\sigma_1)P^b}(\sigma_2)=\frac{\delta^b_a}{(\sigma_1\sigma_2)}$$

$$b^A(\sigma)=:B^A(\sigma)(\sigma\mathrm{d}\sigma),\quad B^A(\sigma)=\tilde\Phi_b(T^A)^b_c\Phi^c\in\Omega^0(\mathbb{CP}^1_\sigma,\mathcal{O}(-2))$$
$$j^A(\sigma)=:J^A(\sigma)(\sigma\mathrm{d}\sigma),\quad J^A(\sigma)=\tilde P_b(T^A)^b_c P^c\in\Omega^0(\mathbb{CP}^1_\sigma,\mathcal{O}(-2))$$

$$B^{A_1}(\sigma_1)B^{A_2}(\sigma_2)=\frac{-\kappa^{A_1A_2}}{(\sigma_1\sigma_2)^2}+\frac{\Phi^b(\sigma_1)(T^{A_1})^c_b(T^{A_2})^d_c\tilde\Phi_d(\sigma_2)-\Phi^b(\sigma_2)(T^{A_2})^c_b(T^{A_1})^d_c\tilde\Phi_d(\sigma_1)}{(\sigma_1\sigma_2)}$$
$$+:B^{A_1}(\sigma_1)B^{A_2}(\sigma_2):$$

$$\sigma_1^a\to\sigma_2^a\frac{(\sigma_1\nu)}{(\sigma_2\nu)}$$

$$O(\sigma_1)\to O(\sigma_2)\frac{(\sigma_2\nu)}{(\sigma_1\nu)}$$

$$B^{A_1}(\sigma_1)B^{A_2}(\sigma_2)\sim\frac{-\kappa^{A_1A_2}}{(\sigma_1\sigma_2)^2}+\frac{f^{A_1A_2}_{A_3}B^{A_3}(\sigma_2)}{(\sigma_1\sigma_2)}\frac{(\sigma_2\nu)}{(\sigma_1\nu)}$$

$$J^{A_1}(\sigma_1)J^{A_2}(\sigma_2)\sim\frac{\kappa^{A_1A_2}}{(\sigma_1\sigma_2)^2}+\frac{f^{A_1A_2}_{A_3}J^{A_3}(\sigma_2)}{(\sigma_1\sigma_2)}\frac{(\sigma_2\nu)}{(\sigma_1\nu)}$$

$$C_i \cdot V(f_i) = \int_{\mathbb{CP}^1} f_i(Z(\sigma))(C_i)\cdot(b+j)(\sigma)$$
$$= \int_{\mathbb{CP}^1} (\sigma\, \mathrm{d}\sigma)(C_i)_A(B^A(\sigma)+J^A(\sigma)) \int \frac{\mathrm{d}t}{t}\bar{\delta}^2\left(t\lambda^\alpha(\sigma) - \sqrt{\omega_i}\kappa_i^\alpha\right)e^{t\sqrt{\omega_i}([\mu(\sigma)\tilde{\kappa}_i]+\psi(\sigma)\cdot\eta_i)}$$

$$(\sigma \mathrm{d}\sigma)\frac{\mathrm{d}t}{t}(B(\sigma)+J(\sigma))^A = \frac{d}{2}\,\mathrm{d}^2 w(B(w)+J(w))^A$$

$$C_i \cdot V(f_i) = \frac{d}{2\omega_i}\int_{\mathbb{C}^2\setminus(0,0)} \mathrm{d}^2 w\bar{\delta}^2(\lambda^\alpha(w)-\kappa_i^\alpha)e^{\omega_i([\mu(w)\tilde{\kappa}_i]+\psi(w)\cdot\eta_i)}C_i\cdot(B+J)(w)$$
$$= \frac{d}{2\omega_i}\sum_{k=1}^{d}\left(\frac{e^{\omega_i([\mu(w)\tilde{\kappa}_i]+\psi(w)\cdot\eta_i)}C_i\cdot(B+J)(w)}{\det\left(\frac{\partial\lambda^\alpha}{\partial w^b}\right)}\right)_{w=I_k}, \lambda^\alpha(I_k)-\kappa_i^\alpha = 0 \forall k$$

$$A^{\mathcal{N}=4\mathrm{SYM}}_{\text{conn. tree}}(f_1, \dots f_n) = \frac{1}{\langle\kappa_i\kappa_j\rangle}(\dots) + \mathcal{O}\left(\langle\kappa_i\kappa_j\rangle^0\right)$$

$$\langle \dots C_i\cdot V(f_i)\dots C_j\cdot V(f_j)\dots\rangle_{\text{pert. SYM}} = \frac{1}{\langle\kappa_i\kappa_j\rangle}(\dots) + \mathcal{O}\left(\langle\kappa_i\kappa_j\rangle^0\right)$$

$$C_i\cdot V(f_i)C_j\cdot V(f_j) \sim \frac{d^2}{4\omega_i\omega_j}\sum_{k=1}^{d}\left(\frac{e^{([\mu(w)(\omega_i\tilde{\kappa}_i+\omega_j\tilde{\kappa}_j)]+\psi(w)\cdot(\omega_i\eta_i+\omega_j\eta_j))}(fC_iC_j)\cdot(B+J)(w)}{(I_kJ_k)\left(\det\left(\frac{\partial\lambda^\alpha}{\partial w^\beta}\right)\right)^2}\right)_{w=I_k}$$
$$+\mathcal{R}^\bullet$$

$$\lim_{\langle\kappa_i\kappa_j\rangle\to 0}\frac{\langle\kappa_i\kappa_j\rangle}{(I_kJ_k)} = \lim_{\langle\kappa_i\kappa_j\rangle\to 0}\frac{\langle\lambda(I_k)\lambda(J_k)\rangle}{(I_kJ_k)}$$
$$= \lim_{\langle\kappa_i\kappa_j\rangle\to 0}\frac{1}{(I_kJ_k)}\left\|\underbrace{\left(\frac{1}{d}I_k^b\frac{\partial\lambda^\alpha}{\partial I_k^b}\right)}_{\lambda^\alpha(I_k)}\underbrace{\left(\lambda_\alpha(I_k)+\left(J_k-I_k,\frac{\partial}{\partial I_k}\right)\lambda_\alpha(I_k)+\cdots\right)}_{\lambda_\alpha(J_k)}\right\|$$
$$= \frac{-1}{2d}\det\left(\frac{\partial\lambda^\alpha}{\partial w^b}\right)_{w=I_k=J_k}$$

$$\frac{\partial\lambda^\alpha}{\partial I_k^b}\frac{\partial\lambda_\alpha}{\partial I_k^c} = \frac{-\epsilon_{bc}}{2}\det\left(\frac{\partial\lambda}{\partial I_k}\right)$$

$$C_i\cdot\tilde{V}(f_i)C_j\cdot\tilde{V}(f_j) \sim \frac{-d}{8\omega_i\omega_j\langle\kappa_i\kappa_j\rangle}\sum_{k=1}^{d}\left(\frac{e^{([\mu(w)(\omega_i\tilde{\kappa}_i+\omega_j\tilde{\kappa}_j)]+\psi(w)\cdot(\omega_i\eta_i+\omega_j\eta_j))}(fC_iC_j)\cdot(B+J)(w)}{\det\left(\frac{\partial\lambda^\alpha}{\partial w^b}\right)}\right)_{w=I_k}$$
$$\sim \frac{1}{\langle\kappa_i\kappa_j\rangle}\times\frac{-\omega'}{4\omega_i\omega_j}f_C^{AB}(C_i)_A(C_j)_B V^C(f')$$

$$(\kappa')^{\alpha} = \kappa_i^{\alpha} = \kappa_j^{\alpha}$$
$$\omega'(\tilde{\kappa}')^{\dot{\alpha}} = \omega_i \tilde{\kappa}_i^{\dot{\alpha}} + \omega_j \tilde{\kappa}_j^{\dot{\alpha}}$$
$$\omega' \eta_I' = \omega_i (\eta_i)_I + \omega_j (\eta_j)_I$$
$$C_A' = f_A^{BC} (C_i)_B (C_j)_C$$

$$P'^{\alpha\dot{\alpha}} = P_i^{\alpha\dot{\alpha}} + P_j^{\alpha\dot{\alpha}}, \Pi_I'^{\alpha} = (\Pi_i)_I^{\alpha} + (\Pi_j)_I^{\alpha}, C_A' = f_A^{BC} (C_i)_B (C_j)_C$$

$$\mathrm{Split}(i,j) = \frac{f_C^{A_i B_j}}{\langle \kappa_i \kappa_j \rangle} \times \frac{-\omega'}{4\omega_i \omega_j}$$

$$S = \int_{\mathbb{CP}^1} \left(P_\mu \bar{\partial} X^\mu + \eta_{\mu\nu} \psi^\mu \bar{\partial} \psi^\nu + e \frac{1}{2} P^2 + \chi P_\mu \psi^\mu \right) + S_{\mathrm{ff}} + g_{\mathrm{YM}}^2 \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} \Delta_{\mathrm{YM}}(\sigma_1, \sigma_2)$$

$$\Pi_0^+ : u, \Pi_0^- : \tilde{\rho}_a \phi^a, \Pi_0^\pm \Pi_0^\pm \sim 0$$

$$S_{\mathrm{ff}} + \int u \bar{\partial} v + \int \chi_+ u + \chi_- \tilde{\rho}_a \phi^a$$

$$e^{Q_{\mathrm{def}}} (\tilde{\phi}_a \bar{\partial} \phi^a + \tilde{\rho}_a \bar{\partial} \rho^a + u \bar{\partial} v)(z) = (\tilde{\phi}_a \bar{\partial} \phi^a + \tilde{\rho}_a \bar{\partial} \rho^a + u \bar{\partial} v)(z)$$

$$Q_{\mathrm{def}} \Pi_0^+ = \left(\oint v \tilde{\phi}_a \rho^a \right) u = \tilde{\phi}_a \rho^a \Longrightarrow Q_{\mathrm{def}}^2 \Pi_0^+ = 0$$
$$e^{Q_{\mathrm{def}}} \Pi_0^+ = \lim_{N' \to \infty} \left(1 + \frac{1}{N'} Q_{\mathrm{def}} \right)^{N'} u = u + \tilde{\phi}_a \rho^a = \Pi^+$$

$$Q_{\mathrm{def}} \Pi_0^- = \left(\oint v \tilde{\phi}_a \rho^a \right) \tilde{\rho}_a \phi^a = -v (\tilde{\phi}_a \phi^a + \tilde{\rho}_a \rho^a) + N \partial v \Longrightarrow Q_{\mathrm{def}}^2 \Pi_0^- = 0$$
$$e^{Q_{\mathrm{def}}} \Pi_0^- = \lim_{N' \to \infty} \left(1 + \frac{1}{N'} Q_{\mathrm{def}} \right)^{N'} \tilde{\rho}_a \phi^a = \tilde{\rho}_a \phi^a - v (\tilde{\phi}_a \phi^a + \tilde{\rho}_a \rho^a) + N \partial v = \Pi^-$$

$$\begin{pmatrix} \tilde{\rho}_a \\ \phi^b \\ u \\ \tilde{\phi}_a \phi^b + \tilde{\rho}_a \rho^b \end{pmatrix} \overset{e^{Q_{\mathrm{def}}}}{\rightarrow} \begin{pmatrix} \tilde{\rho}_a - v \tilde{\phi}_a \\ \phi^b + v \rho^b \\ u + \tilde{\phi}_a \rho^a \\ \tilde{\phi}_a \phi^b + \tilde{\rho}_a \rho^b \end{pmatrix}$$

$$S_{BW} = \int \mathrm{D}^{3|4} Z \wedge \left(k \times hCS(\mathcal{A}) - hCS(\partial_{\mathcal{M}} f^{\mathcal{N}}) + N(f) \lrcorner g \right)$$
$$+ \int \mathrm{d}^{4|8} x \exp \left(\left(\int_{\mathcal{L}_x} g \right) + k \log \det(\bar{\partial} + \mathcal{A}) - \log \det(\bar{\partial} + \partial_{\mathcal{M}} f^{\mathcal{N}}) \right)$$
$f := f^{\mathcal{M}} \partial_{\mathcal{M}} \in \Omega^{0,1}(\mathbb{PT}, T^{1,0}\mathbb{PT}), \partial_{\mathcal{M}} f^{\mathcal{M}} = 0, N(f) := (\bar{\partial} f^{\mathcal{M}} + \mathcal{L}_f f^{\mathcal{M}}) \partial_{\mathcal{M}}$
$g := g_{\mathcal{M}} \mathrm{d}Z^{\mathcal{M}} \in \Omega^{1,1}(\mathbb{PT})$
$\mathcal{A} \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g}), \mathfrak{g} \subset \mathfrak{sl}(N), \partial_{\mathcal{M}} f^{\mathcal{N}} \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{sl}(4 \mid 4))$

$$hCS(\mathcal{A}) := \int \mathrm{D}^{3|4} Z \wedge \mathrm{tr} \left(\mathcal{A} \bar{D} \mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right), \bar{D} := \bar{\partial} + \mathcal{L}_{f \cdot \partial}$$

$$\log\ \det(\bar{\partial}+\mathcal{A}) := \int_{\mathcal{L}_x\times\mathcal{L}_x}\frac{\mathrm{D}\sigma_1\mathrm{D}\sigma_2}{\langle\sigma_1\sigma_2\rangle^2}\mathrm{tr}\left(H\mathcal{A}H^{-1}\big(Z(\sigma_1)\big)H\mathcal{A}H^{-1}\big(Z(\sigma_2)\big)\right), \bar{\partial}_{\mathcal{L}_x}H = \mathcal{A}|_{\mathcal{L}_x}H$$

$$L_x : \left\{\begin{pmatrix}\mu^{\dot\alpha}\\ \lambda_\alpha\\ \psi^I\end{pmatrix} - \begin{pmatrix}x^{\dot\alpha\alpha}\lambda_\alpha\\ \lambda_\alpha\\ \theta^{I\alpha}\lambda_\alpha\end{pmatrix} = 0\right\}, \xrightarrow{\text{deform}}$$

$$\mathcal{L}_x : \left\{\begin{pmatrix}\mu^{\dot\alpha}\\ \lambda_\alpha\\ \psi^I\end{pmatrix} - \begin{pmatrix}x^{\dot\alpha\alpha}\lambda_\alpha\\ \lambda_\alpha\\ \theta^{I\alpha}\lambda_\alpha\end{pmatrix} - \underbrace{\int_{\mathcal{L}_x}\frac{\mathrm{D}\lambda'}{\langle\lambda\lambda'\rangle}\frac{\langle\lambda a\rangle\langle\lambda b\rangle}{\langle\lambda' a\rangle\langle\lambda' b\rangle}f^{\mathcal{M}}(x,\theta,\lambda')}_{\frac{1}{D}f^{\mathcal{M}}:=} = 0\right\}$$

$$\int_{\mathcal{L}_x} I = \int_{L_x}\left(I + \left(\frac{1}{\bar{D}}f^{\mathcal{M}}\right)\left(\frac{\partial I}{\partial Z^{\mathcal{M}}}\right)\right) + \mathcal{O}(f^2 I)$$

$$= \int_{L_x}\left(I + \left(\int_{L_x}\frac{\mathrm{D}\lambda'}{\langle\lambda\lambda'\rangle}\frac{\langle\lambda a\rangle\langle\lambda b\rangle}{\langle\lambda' a\rangle\langle\lambda' b\rangle}f^{\mathcal{M}}(x,\theta,\lambda')\right)\left(\frac{\partial I}{\partial Z^{\mathcal{M}}}\right)\right) + \mathcal{O}(f^2 I)$$

$$V_f := \int f^{\mathcal{M}}(Z(\sigma))W_{\mathcal{M}}(\sigma), W_{\mathcal{M}}(\sigma_1)V\big(Z(\sigma_2)\big) \sim \frac{\mathrm{D}\sigma_1}{(\sigma_1\sigma_2)}\frac{(\sigma_2 a)}{(\sigma_1 a)}\frac{\langle\lambda(\sigma_2)b\rangle}{\langle\lambda(\sigma_1)b\rangle}\frac{\partial V\big(Z(\sigma_2)\big)}{\partial Z^{\mathcal{M}}(\sigma_2)}$$

$$W_{\mathcal{M}}(\sigma_1)Z^{\mathcal{N}}(\sigma_2) \sim \delta^{\mathcal{N}}_{\mathcal{M}}\underbrace{\frac{\sqrt{\mathrm{D}\sigma_1}\sqrt{\mathrm{D}\sigma_2}}{(\sigma_1\sigma_2)}}_{\text{OPE for } K^{1/2}K^{1/2}}\underbrace{\frac{\sqrt{\mathrm{D}\sigma_1}(\sigma_2 a)}{\sqrt{\mathrm{D}\sigma_2}(\sigma_1 a)}}_{\text{twist by } K^{1/2}}\underbrace{\frac{\langle\lambda(\sigma_2)b\rangle}{\langle\lambda(\sigma_1)b\rangle}}_{\text{twist by } \mathcal{L}_d^{-1}}$$

$$Z_M(\{x_i\}|\{y_i\}) \propto \det_{1\le i,j\le M}\left[\frac{c^2 a(x_i y_j)a(\frac{1}{x_i y_j})F^{\mathrm{LU}}(x_i)F^{\mathrm{DR}}(y_j)}{\left(\frac{x_j}{y_i} - \frac{y_i}{x_j}\right)W(x_i,y_j)}\right]$$

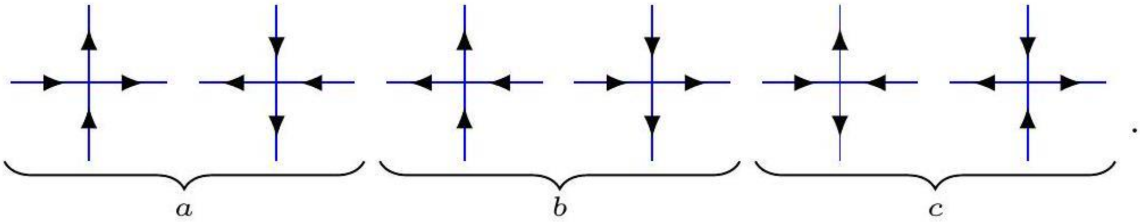
$$\rho^{\mathrm{BW}}_{\mathrm{cr}} = \frac{1}{2\pi^{12}\eta(\mathrm{i})^4} \text{ and } \rho^{\mathrm{FFN}}_{\mathrm{cr}} = \frac{27}{2^{8/3}\pi^{12}\left|\eta\left(\mathrm{e}^{\frac{i\pi}{3}}\right)\right|^8}$$

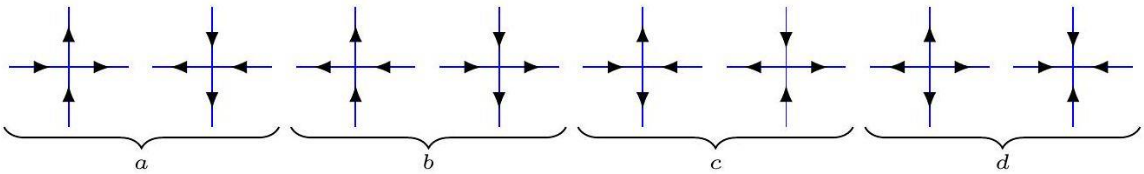
$$\xi^{\mathrm{SBW}}_{\mathrm{cr}} = \frac{3^{9/8}}{4\pi^3\left|\eta\left(\mathrm{e}^{\frac{i\pi}{3}}\right)\right|} \text{ and } \xi^{\mathrm{SFN}}_{\mathrm{cr}} = \frac{1}{\pi^3\sqrt{2}|\eta(\mathrm{i})|^2}$$

$$\Delta = 1 + \frac{1}{2}\left(-1 \pm 2\sqrt{\left(S+\frac{1}{2}\right)^2 - 4\pi^4\xi^2}\right)$$

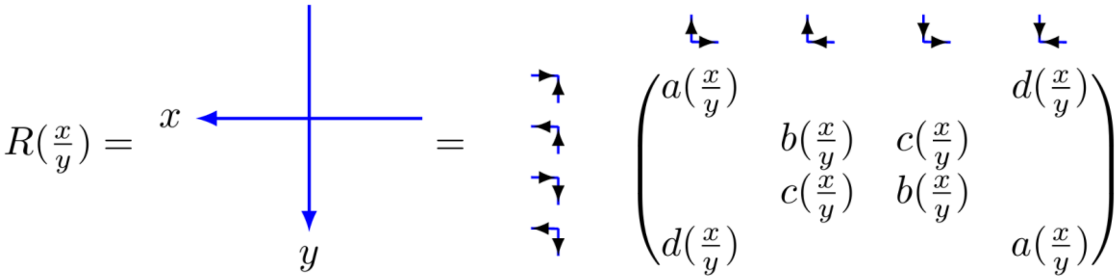
$$\Box^n\ \mathrm{tr}[\Phi_1^\dagger\Phi_2^\dagger\Phi_1^\dagger\Phi_2^\dagger]$$

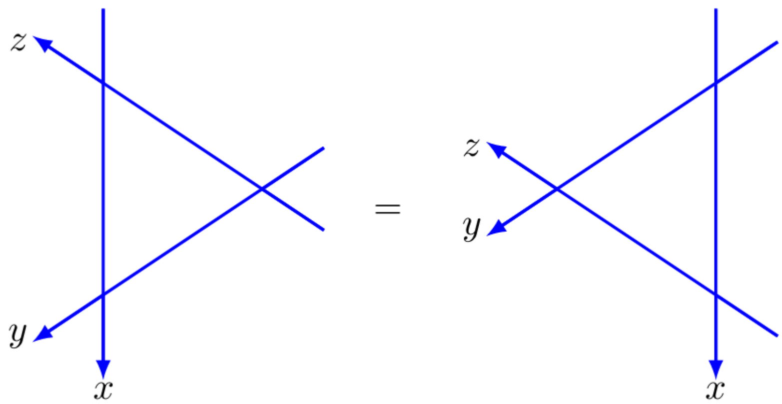
$$C_{\Delta,S} = -2^{S-1-2\Delta}\pi \frac{\Gamma\left(S+\frac{3}{2}\right)\Gamma(\Delta)\Gamma\left(\frac{S-\Delta+2}{2}\right)\Gamma\left(\frac{S+\Delta}{2}\right)}{\Gamma(S+1)\Gamma\left(\Delta+\frac{1}{2}\right)\Gamma\left(\frac{S-\Delta+3}{2}\right)\Gamma\left(\frac{S+\Delta+1}{2}\right)}$$

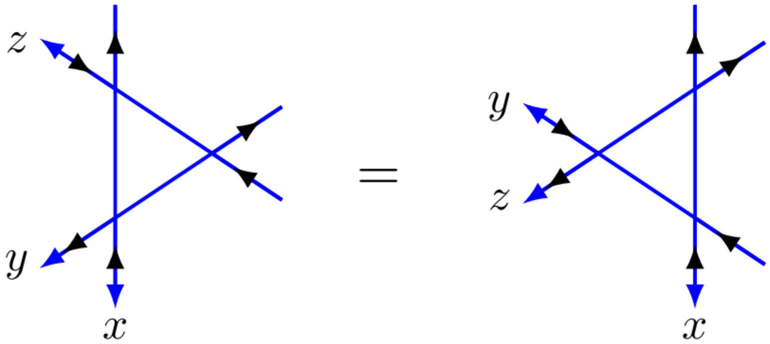




$$Z(a,b,c,d) = \sum_{\Omega\in\Gamma(\Lambda)} a^{n_a} b^{n_b} c^{n_c} d^{n_d}.$$







$$H(u) = \vartheta_1\left(\frac{u}{\sqrt{\vartheta_3}}\middle|\, q\right) \text{ and } \Theta(u) = \vartheta_4\left(\frac{u}{\sqrt{\vartheta_3}}\middle|\, q\right),$$

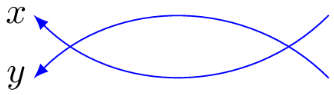
$$\begin{aligned} H(u-v)H(u+v)\Theta(0)^2 &= H(u)^2\Theta(v)^2 - \Theta(u)^2H(v)^2, \\ \Theta(u-v)\Theta(u+v)\Theta(0)^2 &= \Theta(u)^2\Theta(v)^2 - H(u)^2H(v)^2. \end{aligned}$$

$$\mathrm{sn}(u) = \frac{1}{\sqrt{k}} \cdot \frac{H(u)}{\Theta(u)} \text{ with } k := \frac{\vartheta_2^2}{\vartheta_3^2}$$

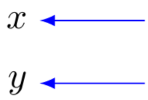
$$\begin{aligned} a &= -\mathrm{i}\rho \cdot \Theta(\mathrm{i}\eta)H(\mathrm{i}(\eta-u))\Theta(\mathrm{i}u), \quad c = -\mathrm{i}\rho \cdot H(\mathrm{i}\eta)\Theta(\mathrm{i}(\eta-u))\Theta(\mathrm{i}u), \\ b &= -\mathrm{i}\rho \cdot \Theta(\mathrm{i}\eta)\Theta(\mathrm{i}(\eta-u))H(\mathrm{i}u), \quad d = \mathrm{i}\rho \cdot H(\mathrm{i}\eta)H(\mathrm{i}(\eta-u))H(\mathrm{i}u). \end{aligned}$$

$$[a:b:c:d] = [\mathrm{isn}(\mathrm{i}(\eta-u)) : \mathrm{isn}(\mathrm{i}u) : \mathrm{isn}(\mathrm{i}\eta) : -\mathrm{i}k \cdot \mathrm{sn}(\mathrm{i}\eta)\mathrm{sn}(\mathrm{i}u)\mathrm{sn}(\mathrm{i}(\eta-u))].$$

$$[a:b:c] = [px - p^{-1}x^{-1} : x - x^{-1} : p - p^{-1}],$$



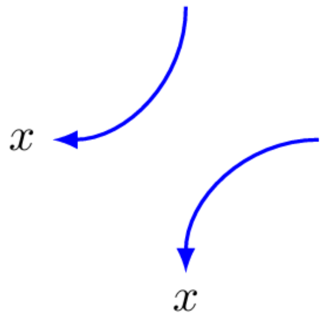
$$= \left[a(\tfrac{x}{y})a(\tfrac{y}{x}) + d(\tfrac{x}{y})d(\tfrac{y}{x})\right] \cdot$$



$$= w(\tfrac{x}{y})w(\tfrac{y}{x}) \cdot \mathbb{1}$$

$$w(x) := -\mathrm{i}\rho \cdot \Theta(0)\Theta(\mathrm{i}(\eta+u))H(\mathrm{i}(\eta+u)) \overset{d=0}{\to} a(x)\Big|_{d=0} = px - p^{-1}x^{-1}$$

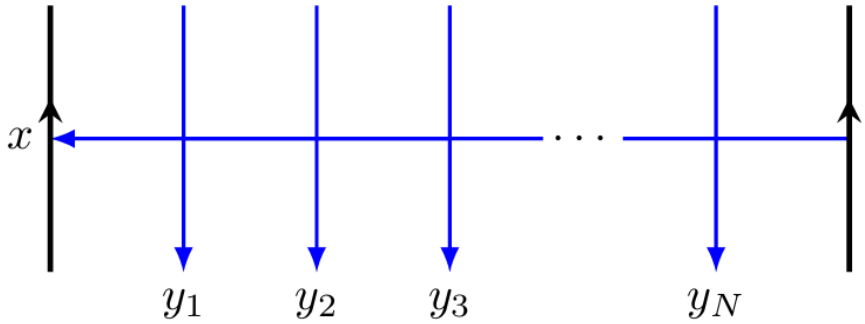
$$R(1) = c \cdot \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} = c \cdot \mathbb{P} = c \cdot$$

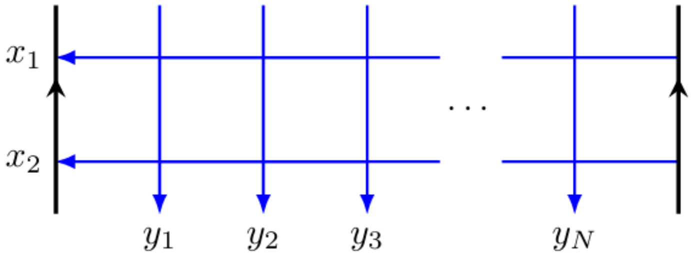


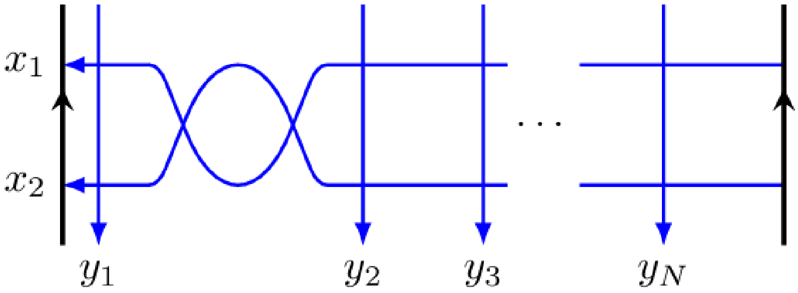
$$Z_{MN}(a,b,c,d) = \mathrm{tr}[T_N(z)^M] = \sum_i \Lambda_{N,i}(z)^M$$

$$T_N(\{\tfrac{x}{y_j}\}) = \mathrm{Tr}\left[R(\tfrac{x}{y_1})R(\tfrac{x}{y_2})R(\tfrac{x}{y_3})\cdots R(\tfrac{x}{y_N})\right]$$

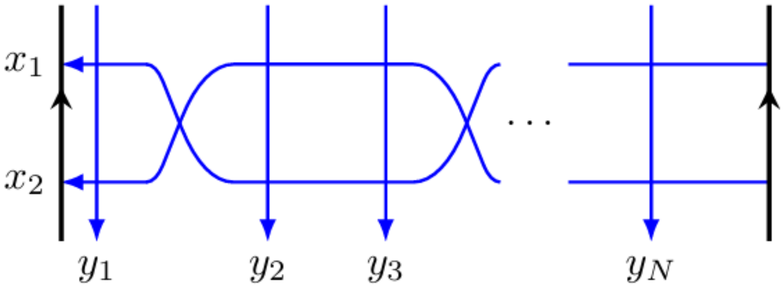
$=$



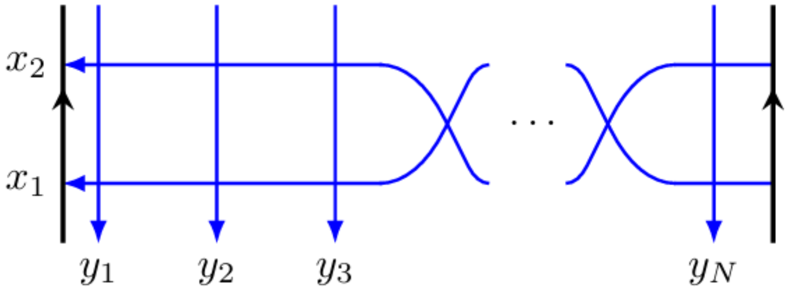
$T_N(\{\frac{x_1}{y_j}\}) \circ T_N(\{\frac{x_2}{y_j}\}) =$ 

$=$ 

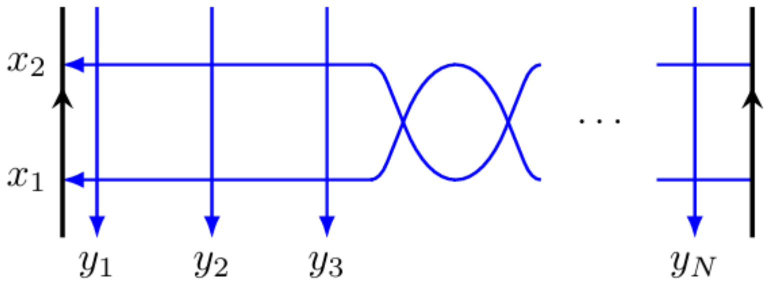
$\cdot \frac{1}{w(\frac{x_1}{x_2})w(\frac{x_2}{x_1})}$

$=$ 

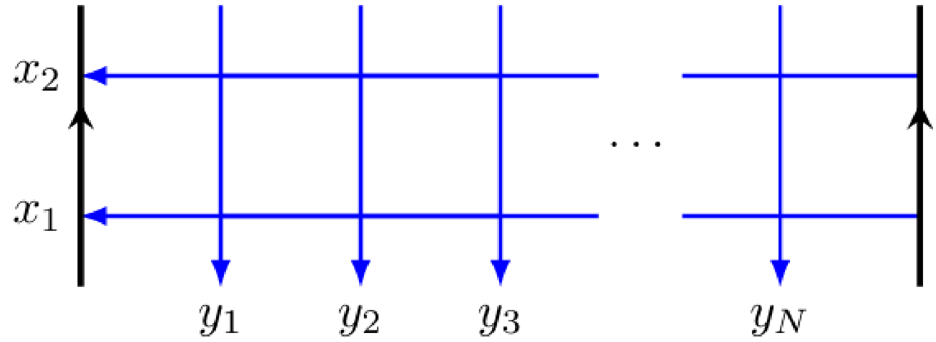
$\cdot \frac{1}{w(\frac{x_1}{x_2})w(\frac{x_2}{x_1})}$

$=$ 

$\cdot \frac{1}{w(\frac{x_1}{x_2})w(\frac{x_2}{x_1})}$

$=$ 

$\cdot \frac{1}{w(\frac{x_1}{x_2})w(\frac{x_2}{x_1})}$



=

$$T_N(\{\tfrac{x_2}{y_j}\}) \circ T_N(\{\tfrac{x_1}{y_j}\})$$

=

$$T_N(z) = \sum_{n=0}^{\infty} Q_n z^n$$

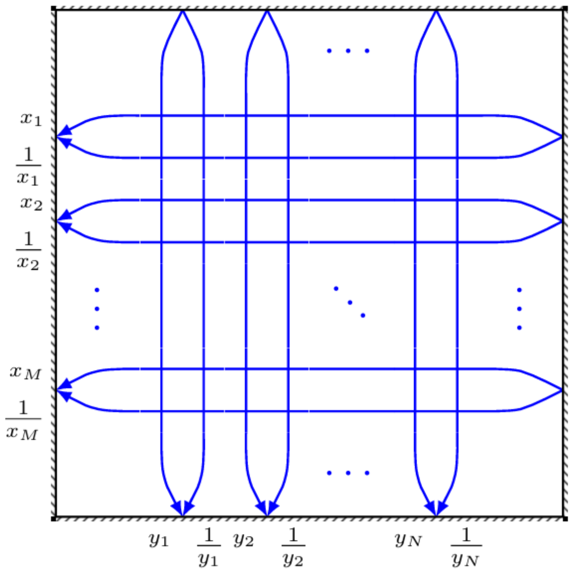
$$\kappa(a,b,c,d) = \lim_{M,N\to\infty} Z_{MN}(a,b,c,d)^{\frac{1}{MN}}$$

$$\kappa(u) = c \cdot \frac{\Gamma^{(1)}\left(\mathrm{p}^{\frac{1}{2}}\middle|\, q^2\right)\Gamma^{(1)}(q \mid q^2)}{\Gamma^{(1)}\left(\mathrm{p}^{\frac{1}{2}}\mathrm{x}^{-1}\middle|\, q^2\right)\Gamma^{(1)}(\mathrm{x} \mid q^2)} \frac{\Gamma^{(2)}(\mathrm{p}\mathrm{x}^{-1} \mid q,\mathrm{p})\Gamma^{(2)}\left(\mathrm{p}^{\frac{1}{2}}\mathrm{x}\middle|\, q,\mathrm{p}\right)}{\Gamma^{(2)}\left(\mathrm{p}^{\frac{1}{2}}\mathrm{x}^{-1}\middle|\, q,\mathrm{p}\right)\Gamma^{(2)}(\mathrm{x} \mid q,\mathrm{p})}.$$

$$\mathrm{x} = \mathrm{e}^{-\frac{2u}{\sqrt{\vartheta_3}}} \text{ and } \mathrm{p} = \mathrm{e}^{-\frac{4\eta}{\sqrt{\vartheta_3}}}.$$

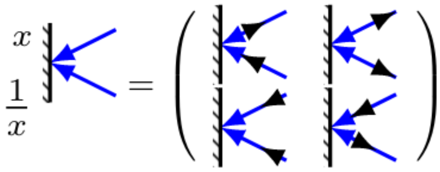
$$\kappa(1,1,1,0) = \left(\frac{4}{3}\right)^{\frac{3}{2}} \text{ and } \kappa(1,1,2,0) = \left(\frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)^2.$$

$$Z_{MN}(\{x_i\}_{i=1,\ldots,M}|\{y_j\}_{j=1,\ldots,N}) =$$

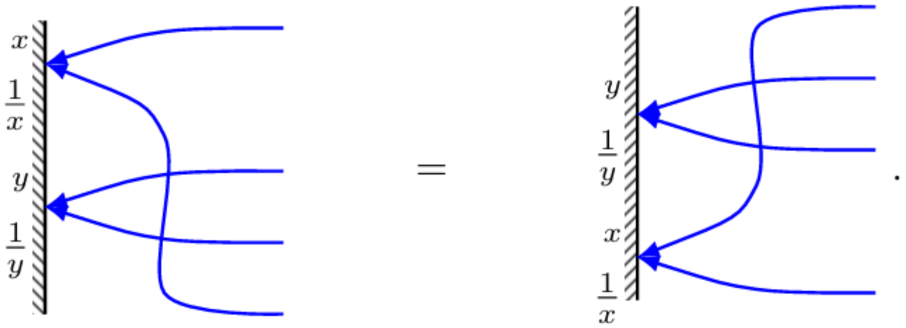


.

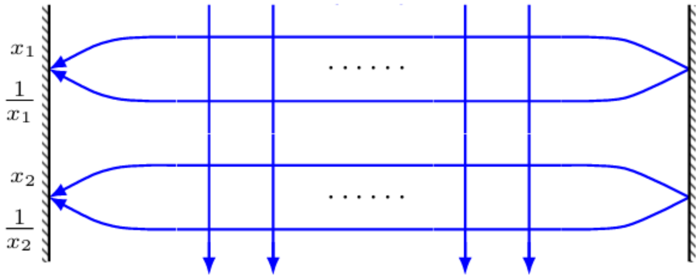
$$K_{\mathrm{L}}(x) =$$



$$= \begin{pmatrix} k_{\mathrm{L}}^{+}(x) & k_{\mathrm{L}}^{r}(x) \\ k_{\mathrm{L}}^{l}(x) & k_{\mathrm{L}}^{-}(x) \end{pmatrix}$$



$$T_N(x_1) \circ T_N(x_2) \quad =$$



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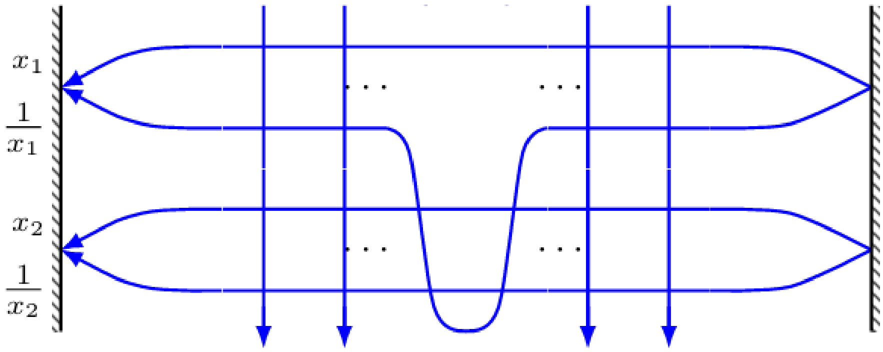
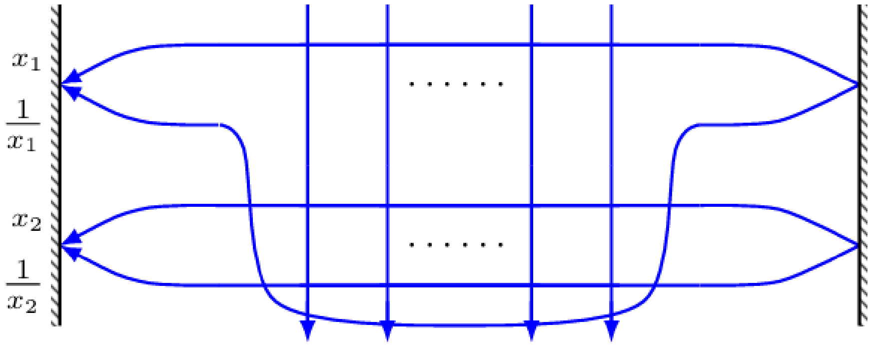


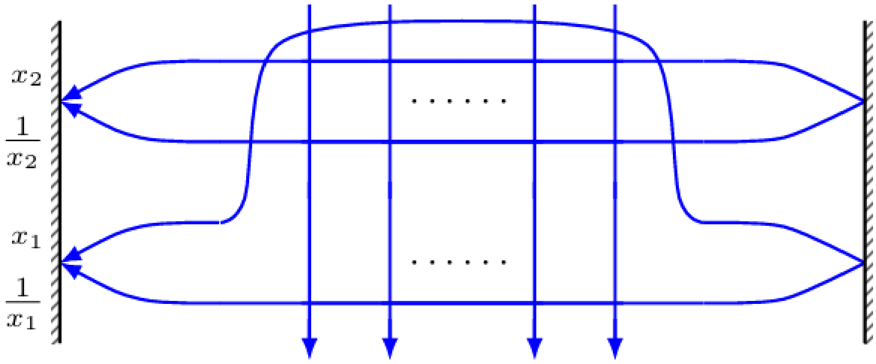
$$\cdot \frac{1}{W(x_1, x_2)}$$

$=$



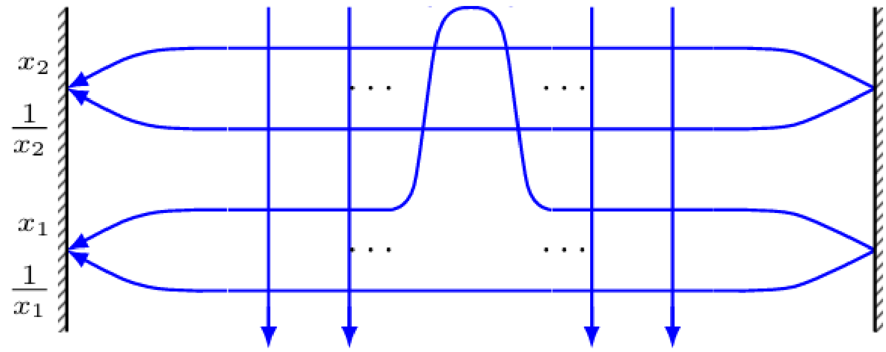
$$\cdot \frac{1}{W(x_1, x_2)}$$

$=$



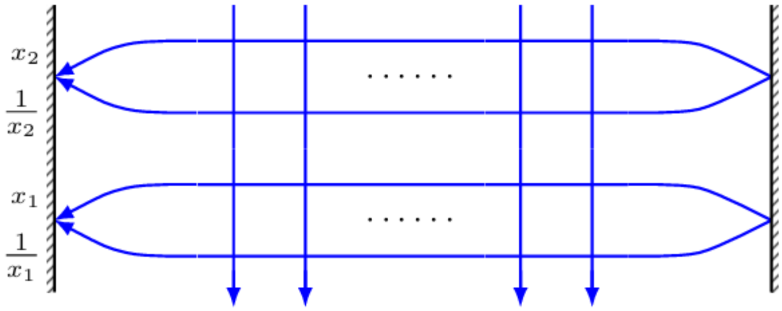
$$\cdot \frac{1}{W(x_1, x_2)}$$

=



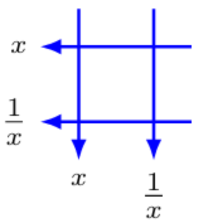
$$\cdot \frac{1}{W(x_1, x_2)}$$

=



$$= T_N(x_2) \circ T_N(x_1)$$

$$W(x, y) := a(xy)a\left(\frac{1}{xy}\right)a\left(\frac{x}{y}\right)a\left(\frac{y}{x}\right)$$



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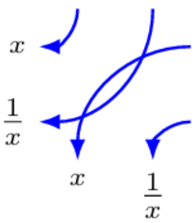
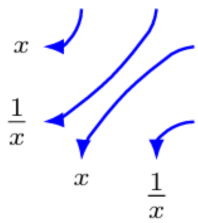
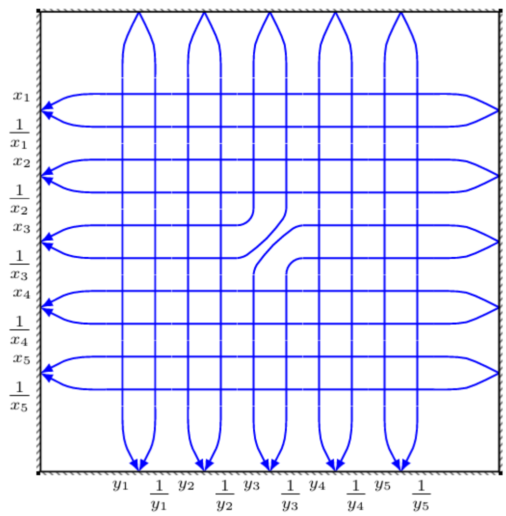


$\cdot\, c^2 \quad =$



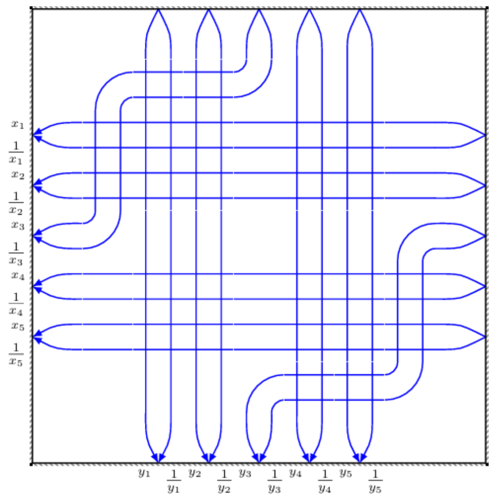
$\cdot\, c^2 \; a(x^2)a(\frac{1}{x^2})$

$Z_{5,5}(\{x_i\}|\{y_j\})|_{x_3=y_3} =$

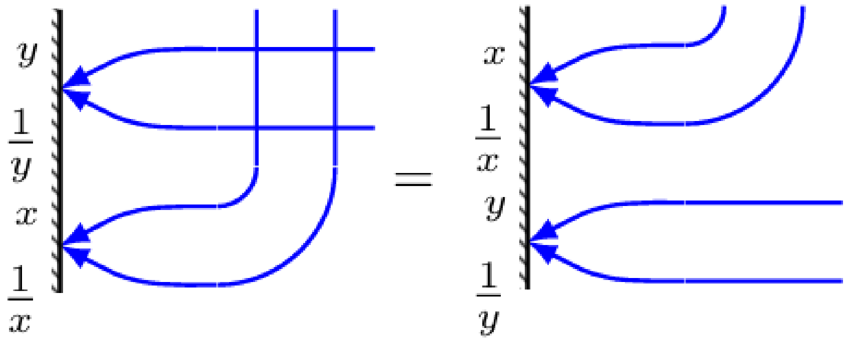


$\cdot\, c^2 \; a(x_3^2)a(\frac{1}{x_3^2})|_{x_3=y_3}$

$$Z_{5,5}(\{x_i\}|\{y_j\})|_{x_3=y_3} =$$

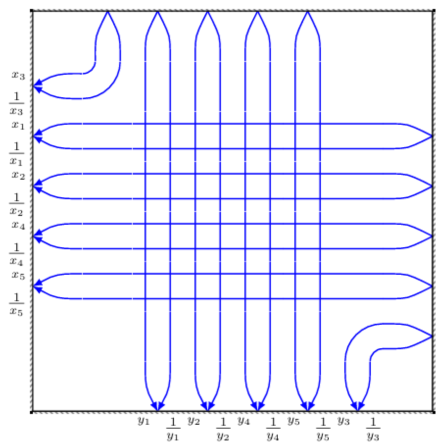


$$\cdot\, c^2\ a(x_3^2)a(\tfrac{1}{x_3^2})|_{x_3=y_3}\ .$$



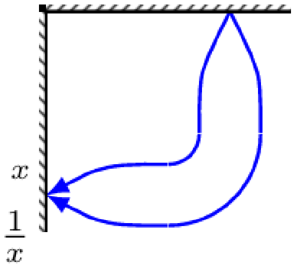
$$\cdot\, W(x,y)$$

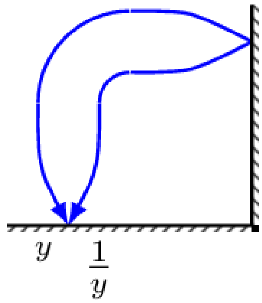
$$Z_{5,5}(\{x_i\}|\{y_j\})|_{x_3=y_3} =$$



$$\cdot\, c^2\ a(x_3^2)a(\tfrac{1}{x_3^2})$$

$$\cdot \left[\prod_{i=1}^{2} W(x_3,x_i)\right]\left[\prod_{j=1}^{2} W(x_3,y_j)\right]\cdot\left[\prod_{i=4}^{5} W(y_3,x_i)\right]\left[\prod_{j=4}^{5} W(y_3,y_j)\right]|_{x_3=y_3}\ .$$

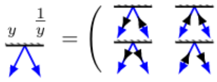
$F^{\mathrm{LU}}(x) :=$ 

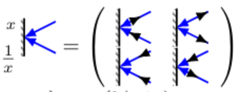
, $F^{\mathrm{DR}}(y) :=$ 

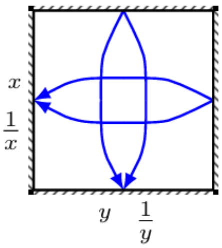
$$Z_{5,5}(\{x_i\} \mid \{y_j\})\big|_{x_3=y_3} = c^2 a(x_3^2) a\left(\frac{1}{x_3^2}\right) \cdot \left[\prod_{\substack{i=1\\ i\neq 3}}^{5} W(x_3, x_i)\right]\left[\prod_{\substack{j=1\\ j\neq 3}}^{5} W(x_3, y_j)\right] F^{\mathrm{LU}}(x_3) F^{\mathrm{DR}}(x_3)$$
$$\cdot Z_{4,4}\left(\{x_i\}_{\substack{i=1,\dots,5\\ i\neq 3}} \mid \{y_j\}_{\substack{j=1,\dots,5\\ j\neq 3}}\right)$$

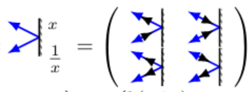
$$Z_{M,N}(\{x_i\}|\{y_j\})|_{x_m=y_n} = c^2\ a(x_m^2) a(\tfrac{1}{x_m^2}) \cdot \left[\prod_{\substack{i=1\\ i\neq m}}^{M} W(x_m, x_i)\right]\left[\prod_{\substack{j=1\\ j\neq n}}^{N} W(x_m, y_j)\right] F^{\mathrm{LU}}(x_m) F^{\mathrm{DR}}(x_m)$$
$$\cdot Z_{M-1,N-1}(\{x_i\}_{\substack{i=1,\dots,M\\ i\neq m}}|\{y_j\}_{\substack{j=1,\dots,N\\ j\neq n}}) \ .$$

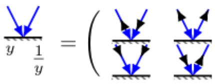
$$Z_M(\{x_i\}|\{y_i\}) = \frac{\prod_{i,j=1}^{M}\left(\frac{x_i}{y_j} - \frac{y_j}{x_i}\right) W(x_i, y_j)}{\prod_{1\leq i<j\leq M}\left(\frac{x_j}{x_i} - \frac{x_i}{x_j}\right)\left(\frac{y_i}{y_j} - \frac{y_j}{y_i}\right)} \cdot \det_{1\leq i,j\leq M}\left[\frac{c^2 a(x_i y_j) a(\frac{1}{x_i y_j}) F^{\mathrm{LU}}(x_i) F^{\mathrm{DR}}(y_j)}{\left(\frac{x_j}{y_i} - \frac{y_i}{x_j}\right) W(x_i, y_j)}\right]$$

$K_{\mathrm{U}}(y) =$  $= \begin{pmatrix} k_{\mathrm{U}}^{l}(y) & 0 \\ 0 & k_{\mathrm{U}}^{r}(y) \end{pmatrix} = \begin{pmatrix} b(y\xi_{\mathrm{U}}) & 0 \\ 0 & b(\frac{yq}{\xi_{\mathrm{U}}}) \end{pmatrix}$

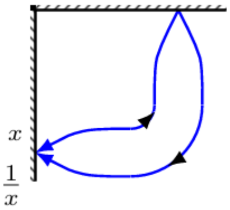
$K_{\mathrm{L}}(x) =$  $= \begin{pmatrix} k_{\mathrm{L}}^{+}(x) & 0 \\ 0 & k_{\mathrm{L}}^{-}(x) \end{pmatrix} = \begin{pmatrix} b(x\xi_{\mathrm{L}}) & 0 \\ 0 & b(\frac{x}{q\xi_{\mathrm{L}}}) \end{pmatrix}$



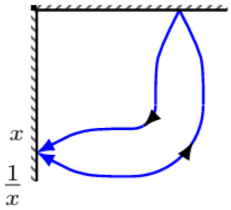
$K_{\mathrm{R}}(x) =$  $= \begin{pmatrix} k_{\mathrm{R}}^{+}(x) & 0 \\ 0 & k_{\mathrm{R}}^{-}(x) \end{pmatrix} = \begin{pmatrix} b(x\xi_{\mathrm{R}}) & 0 \\ 0 & b(\frac{xq}{\xi_{\mathrm{R}}}) \end{pmatrix}$

$K_{\mathrm{D}}(y) =$  $= \begin{pmatrix} k_{\mathrm{D}}^{l}(y) & 0 \\ 0 & k_{\mathrm{D}}^{r}(y) \end{pmatrix} = \begin{pmatrix} b(y\xi_{\mathrm{D}}) & 0 \\ 0 & b(\frac{y}{q\xi_{\mathrm{D}}}) \end{pmatrix}$

$$F^{\mathrm{LU}}(x) =$$



$$+$$



$$= b(x\xi_{\mathrm{L}})b(\tfrac{xq}{\xi_{\mathrm{U}}}) + b(\tfrac{x}{q\xi_{\mathrm{L}}})b(x\xi_{\mathrm{U}})$$

$$F^{\mathrm{DR}}(y) = b(y\xi_{\mathrm{D}})b(\tfrac{yq}{\xi_{\mathrm{R}}}) + b(\tfrac{y}{q\xi_{\mathrm{D}}})b(y\xi_{\mathrm{R}}) \ .$$

$$\int \mathrm{d}^D x\phi \,\square\, \phi^* = \int \mathrm{d}^D x_1 \,\mathrm{d}^D x_2 \phi(x_1)\delta^{(D)}(x_{12}) \,\square_2\, \phi^*(x_2)$$

$$\square_i := \partial_i^{\mu}\partial_{i,\mu}$$

$$\partial_{i,\mu} = \frac{\partial}{\partial x_i^{\mu}}$$

$$\int \mathrm{d}^D x\phi \,\square^v\, \phi^*$$

$$G_v(x_{12}) = \langle \phi(x_1)\phi(x_2)^* \rangle$$

$$\int \mathrm{d}^D x_2 \delta^{(D)}(x_{12}) \,\square_2^v\, G_v(x_{23}) = \delta^{(D)}(x_{13})$$

$$G_v(x) = \frac{1}{(2\pi)^{D/2}} \int \mathrm{d}^D p \mathrm{e}^{-\mathrm{i}px} \tilde{G}_v(p)$$

$$\delta^{(D)}(x) = (2\pi)^{-D/2} \int \mathrm{d}^D p \mathrm{e}^{-\mathrm{i}px}$$

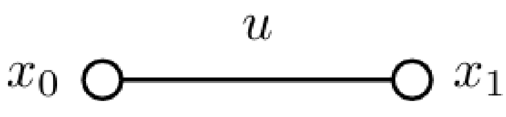
$$G_v(x) = \frac{1}{(2\pi)^{D/2}} \int \mathrm{d}^D p \mathrm{e}^{-\mathrm{i}px} \left[\frac{-1}{p^2}\right]^v = -(-1)^v 2^{D/2-2v} \frac{\Gamma\left(\frac{D}{2} - v\right)}{\Gamma(v)} \frac{1}{[x^2]^{D/2-v}}.$$

$$a_\ell(v) := \frac{\Gamma\left(\frac{D}{2} - v + \ell\right)}{\Gamma(v + \ell)}$$

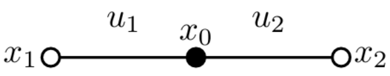
$$a_\ell(v) a_\ell\left(\frac{D}{2} - v\right) = 1$$

$$c_\ell(v) := -(-1)^v 2^{D/2-2v} a_\ell(v)$$

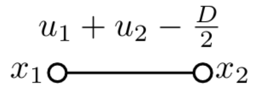
$$W_u(x_{10}) \ := \ \frac{1}{[x_{10}^2]^u} \ =$$



$$\int \mathrm{d}^D x_0 \; \frac{1}{[x_{10}^2]^{u_1}} \frac{1}{[x_{20}^2]^{u_2}} = r_0(D-u_1-u_2, u_1, u_2) \cdot \frac{1}{[x_{12}^2]^{u_1+u_2-\frac{D}{2}}} \, ,$$



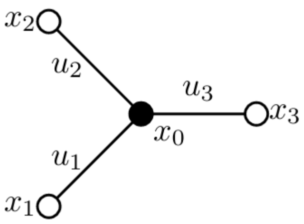
$$= r_0(D-u_1-u_2, u_1, u_2) \cdot$$



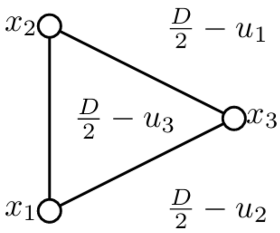
.

$$r_\ell(u_1,u_2,u_3) := \pi^{\frac{D}{2}} \cdot a_0(u_1) a_\ell(u_2) a_\ell(u_3),$$

$$\int \mathrm{d}^D x_0 \frac{1}{[x_{01}^2]^{u_1}} \frac{1}{[x_{02}^2]^{u_2}} \frac{1}{[x_{03}^2]^{u_3}} \overset{u_1+u_2+u_3=D}{=} r_0(u_1,u_2,u_3) \frac{1}{[x_{12}^2]^{\frac{D}{2}-u_3}} \frac{1}{[x_{13}^2]^{\frac{D}{2}-u_2}} \frac{1}{[x_{23}^2]^{\frac{D}{2}-u_1}}$$

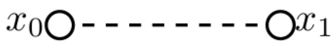


$$\overset{u_1+u_2+u_3=D}{=} r_0(u_1,u_2,u_3) \cdot$$



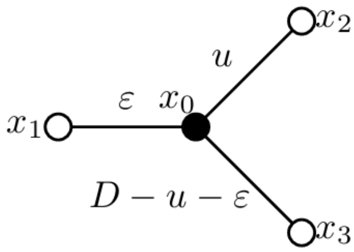
.

$$\delta^{(D)}(x_{10}) = \lim_{\varepsilon \to 0} \pi^{-\frac{D}{2}} a_0(\varepsilon) \cdot W_{\frac{D}{2}-\varepsilon}(x_{10}) =$$

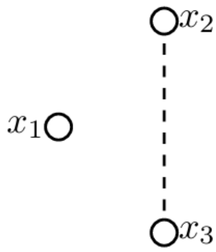


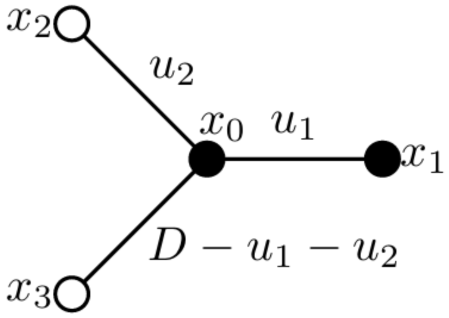
$$\lim_{\varepsilon \to 0} \int \mathrm{d}^D x_0 W_u(x_{10}) W_{D-u-\varepsilon}(x_{20}) = \pi^D a_0(u) a_0(D-u) \cdot \delta^{(D)}(x_{12})$$

$$\lim_{\varepsilon \to 0}$$



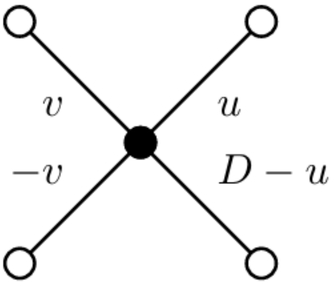
$$= \pi^D a_0(u) \, a_0(D-u) \cdot$$



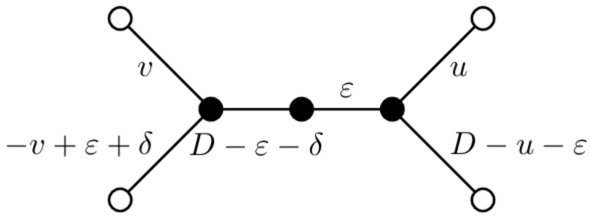


$$= \pi^D a_0(u_1) \, a_0(D-u_1)$$

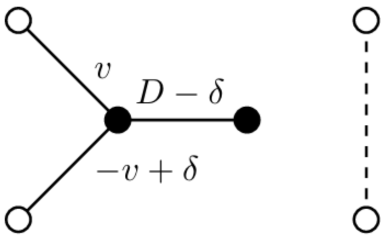
.



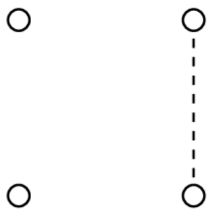
$$= \lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \frac{1}{\pi^D a_0(\varepsilon) a_0(D-\varepsilon)}$$



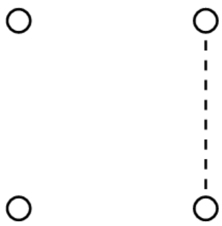
$$= \left[\lim_{\varepsilon \to 0} \frac{a_0(u) a_0(D-u)}{a_0(\varepsilon) a_0(D-\varepsilon)} \right] \cdot \lim_{\delta \to 0}$$



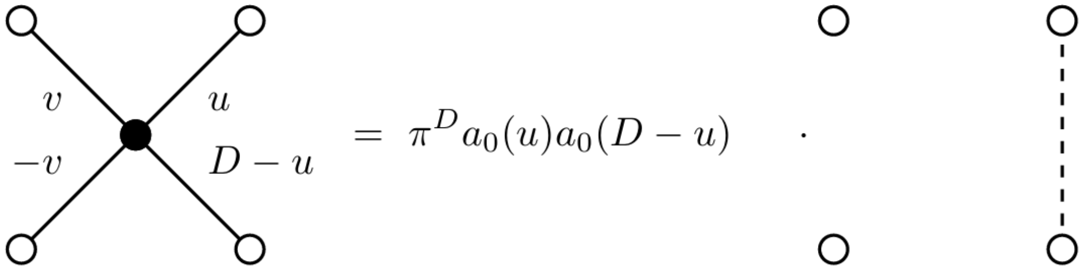
$$= \left[\lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \pi^D a_0(u) a_0(D-u) \frac{a_0(\delta) a_0(D-\delta)}{a_0(\varepsilon) a_0(D-\varepsilon)} \right] \cdot$$

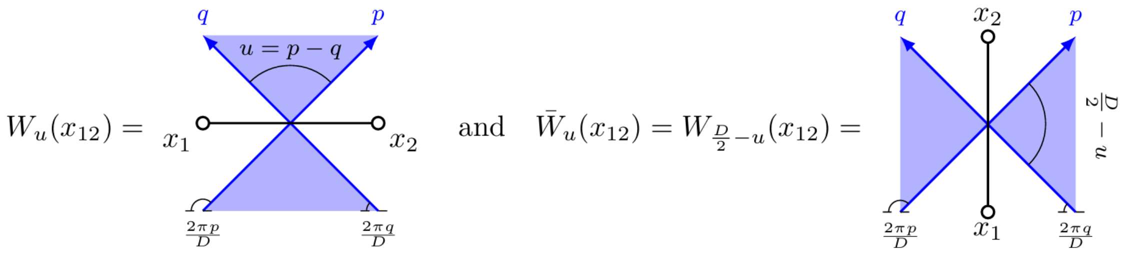


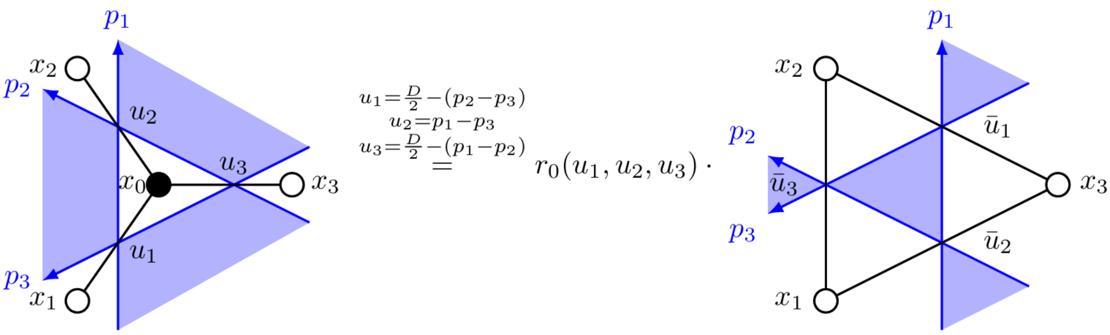
$$= \pi^D a_0(u) a_0(D-u) \cdot$$

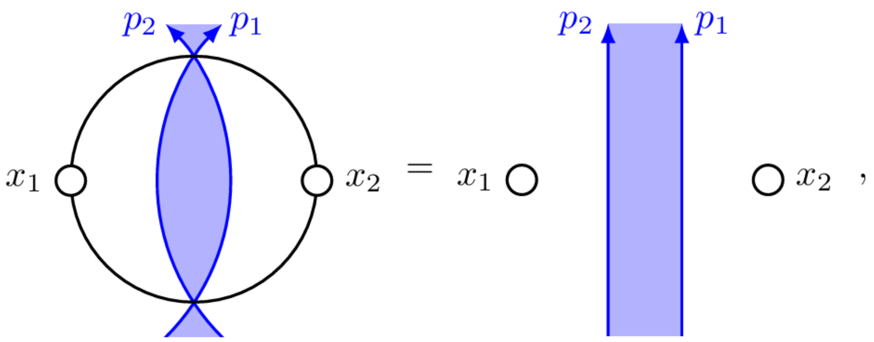


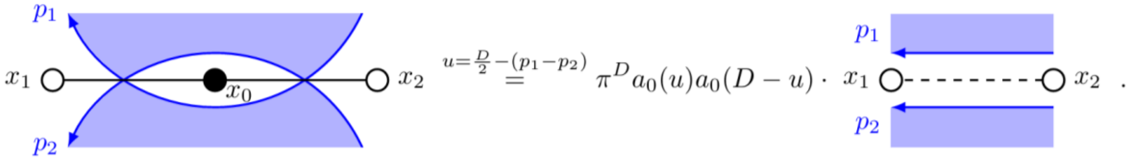
$=$


v
u
−v
D − u
= π^D a0(u)a0(D − u) ·

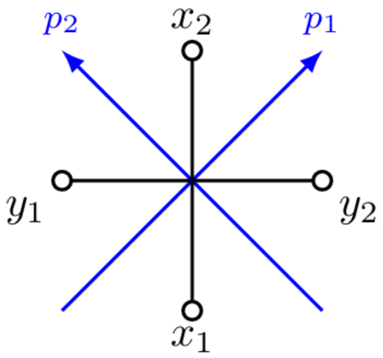

Wu(x12) =
q
p
u = p − q
x1
x2
2πp/D
2πq/D
and
W̄u(x12) = W_{D/2−u}(x12) =
q
x2
p
D/2 − u
x1
2πp/D
2πq/D

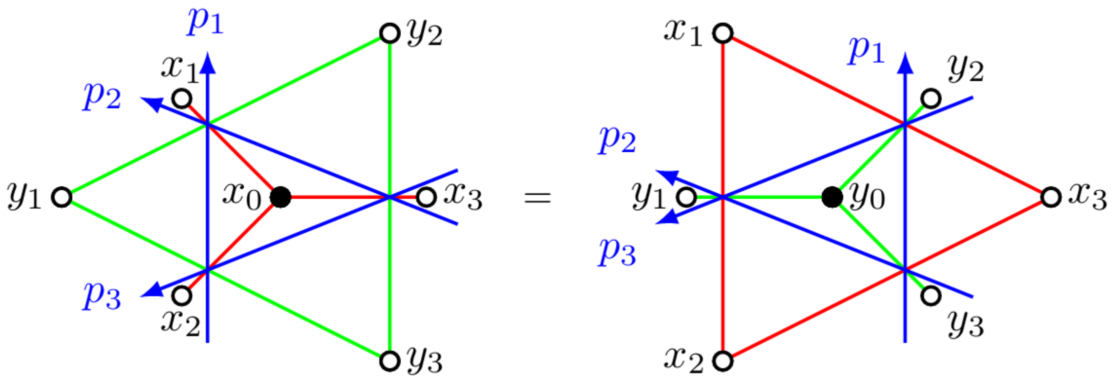

p1
x2
p2
u2
x0
u3
x3
u1
p3
x1
u1=D/2−(p2−p3)
u2=p1−p3
u3=D/2−(p1−p2)
=
r0(u1, u2, u3) ·
p1
x2
p2
ū1
ū3
x3
p3
ū2
x1


p2
p1
x1
x2
=
x1
p2
p1
x2
,


p1
x1
x0
x2
p2
u=D/2−(p1−p2)
=
π^D a0(u)a0(D − u) ·
x1
p1
x2
p2
.

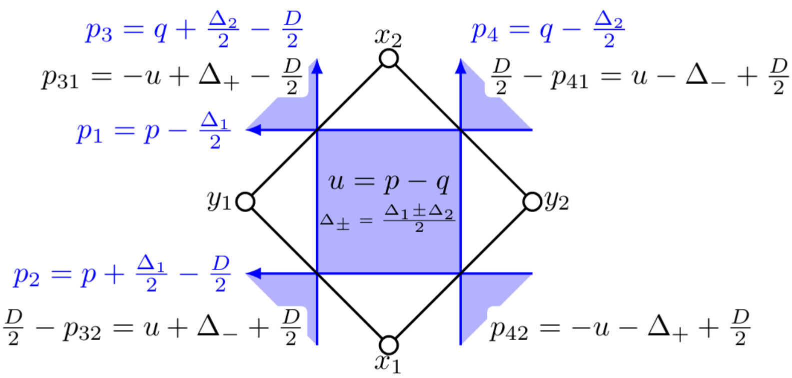
$$R^{x_2y_2}_{x_1y_1}(u) = W_u(y_{12})\bar{W}_u(x_{12}) \overset{u=p_1-p_2}{=}$$





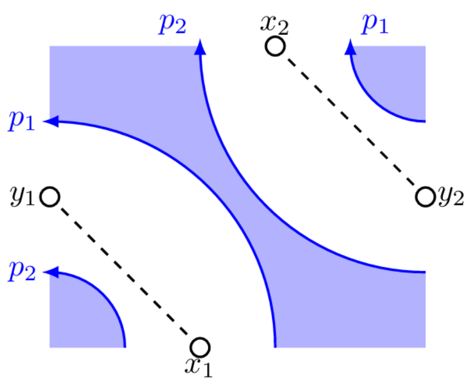
$$\int \mathrm{d}^D x_0 R^{x_2y_3}_{x_0y_1}(p_{12}) R^{y_1x_0}_{y_2x_3}(p_{13}) R^{x_0y_3}_{x_1y_2}(p_{23}) = \int \mathrm{d}^D y_0 R^{y_0x_1}_{y_2x_3}(p_{12}) R^{x_2y_3}_{x_1y_0}(p_{13}) R^{y_1x_2}_{y_0x_3}(p_{23}),$$

$$\mathbb{R}^{x_2y_2}_{x_1y_1}(p_1,p_2|p_3,p_4) =$$

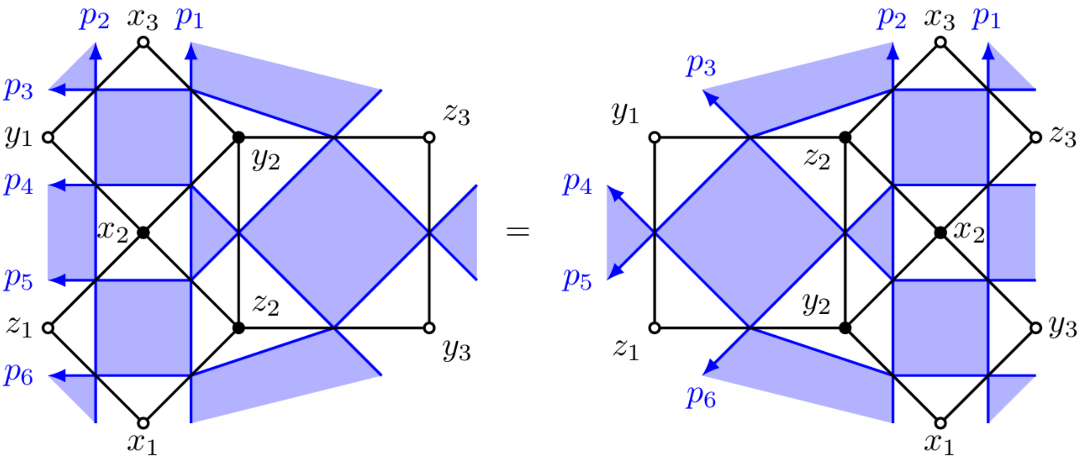


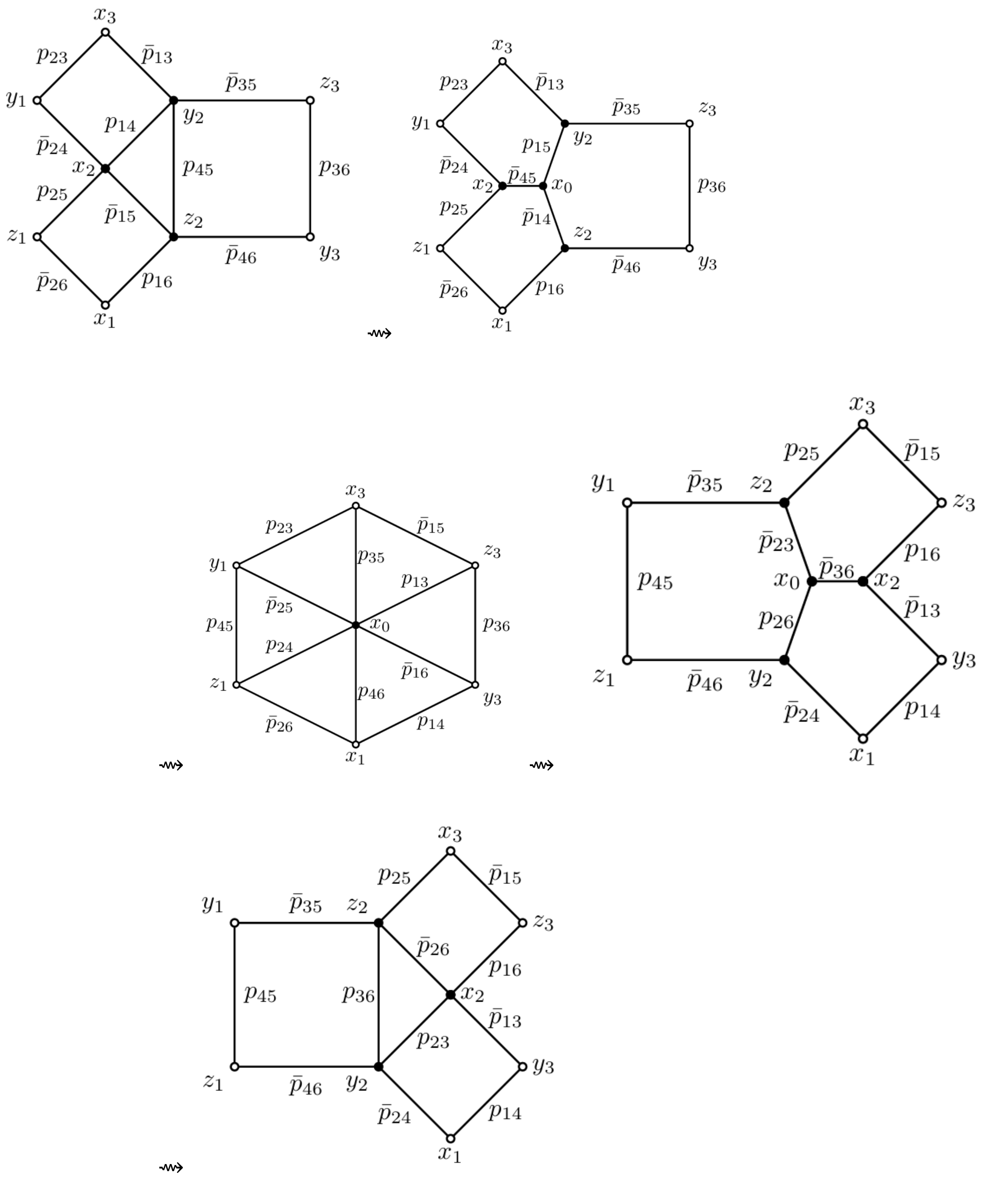
.

$$\lim_{\varepsilon \to 0} \pi^{-D} a_0(\varepsilon)^2 \cdot \mathbb{R}^{x_2y_2}_{x_1y_1}(\varepsilon; \Delta_1, \Delta_1) =$$

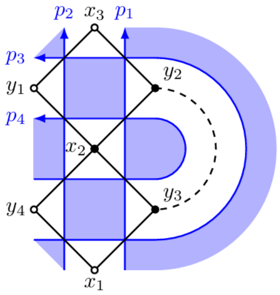


$$= \mathbb{P}_{xy}$$

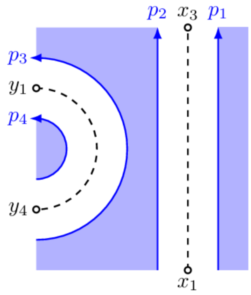

p_2 x_3 p_1
p_3
y_1
p_4
x_2
p_5
z_1
p_6
x_1
y_2
z_2
z_3
y_3
=
p_2 x_3 p_1
p_3
y_1
z_2
z_3
p_4
x_2
p_5
y_2
y_3
z_1
p_6
x_1


x_3 p_{23} $\bar{p}_{13}$ y_1 $\bar{p}_{35}$ z_3 y_2 p_{14} $\bar{p}_{24}$ x_2 p_{45} p_{36} p_{25} $\bar{p}_{15}$ z_2 z_1 $\bar{p}_{46}$ y_3 $\bar{p}_{26}$ p_{16} x_1
⇝
x_3 p_{23} $\bar{p}_{13}$ y_1 $\bar{p}_{35}$ z_3 y_2 p_{15} $\bar{p}_{24}$ $\bar{p}_{45}$ x_2 x_0 p_{36} $\bar{p}_{14}$ p_{25} z_2 z_1 $\bar{p}_{46}$ y_3 $\bar{p}_{26}$ p_{16} x_1
⇝
x_3 p_{23} $\bar{p}_{15}$ y_1 p_{35} z_3 p_{13} $\bar{p}_{25}$ p_{45} x_0 p_{36} p_{24} z_1 $\bar{p}_{16}$ p_{46} y_3 $\bar{p}_{26}$ p_{14} x_1
⇝
x_3 p_{25} $\bar{p}_{15}$ y_1 $\bar{p}_{35}$ z_2 z_3 $\bar{p}_{23}$ p_{16} p_{45} $\bar{p}_{36}$ x_0 x_2 p_{26} $\bar{p}_{13}$ z_1 $\bar{p}_{46}$ y_2 y_3 $\bar{p}_{24}$ p_{14} x_1
⇝
x_3 y_1 $\bar{p}_{35}$ z_2 p_{25} $\bar{p}_{15}$ z_3 $\bar{p}_{26}$ p_{16} p_{45} p_{36} x_2 $\bar{p}_{13}$ p_{23} z_1 $\bar{p}_{46}$ y_2 y_3 $\bar{p}_{24}$ p_{14} x_1

$$\frac{r_0(\bar{p}_{35},\bar{p}_{13},p_{15})r_0(\bar{p}_{45},\bar{p}_{24},p_{25})r_0(\bar{p}_{46},\bar{p}_{14},p_{16})\cdot r_0(\bar{p}_{36},\bar{p}_{23},p_{26})}{r_0(\bar{p}_{45},\bar{p}_{14},p_{15})\cdot r_0(\bar{p}_{35},\bar{p}_{23},p_{25})r_0(\bar{p}_{36},\bar{p}_{13},p_{16})r_0(\bar{p}_{46},\bar{p}_{24},p_{26})}=1$$



$$= \pi^{2D} a_0(\bar{p}_{13})\, a_0(D-\bar{p}_{13}) a_0(\bar{p}_{24})\, a_0(D-\bar{p}_{24})\cdot$$

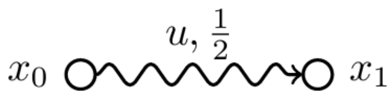


$$-\mathrm{i}\int \mathrm{d}^D x\bar{\psi}\,\Box^v\,\bar{\partial}\psi = \int \mathrm{d}^D x_1\,\mathrm{d}^D x_2\bar{\psi}_{\dot{\alpha}}(x_1)\left[-\mathrm{i}\delta^{(D)}(x_{12})\bar{\sigma}^{\mu,\dot{\alpha}\alpha}\,\Box_2^v\,\partial_{2,\mu}\right]\psi_\alpha(x_2)$$

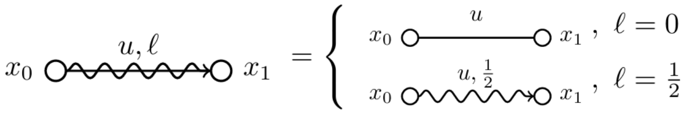
$$\left[-\mathrm{i}\bar{\sigma}^{\mu,\dot{\alpha}\alpha}\,\Box^v\,\partial_\mu\right]G_{v,\alpha\dot{\beta}}(x)=\delta^{(D)}(x)\cdot\delta^{\dot{\alpha}}_{\dot{\beta}}$$

$$G_{v,\alpha\dot{\beta}}(x)=-\mathrm{i}\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu G_{v+1}(x)=c_\ell\left(v+\frac{1}{2}\right)\cdot\frac{x_{\alpha\dot{\beta}}}{[x^2]^{\frac{D}{2}-v}}$$

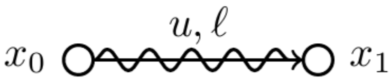
$$W^{\frac{1}{2}}_{u,\alpha\dot{\beta}}(x_{10})=\frac{1}{(x_{10}^2)^u}\frac{x_{10,\alpha\dot{\beta}}}{|x_{10}|}=$$



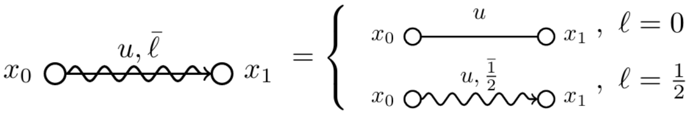
$$W^{\ell}_{u,\alpha\dot{\beta}}(x_{10})=\frac{1}{(x_{10}^2)^u}\left[\frac{x_{10,\alpha\dot{\beta}}}{|x_{10}|}\right]^{2\ell}=$$

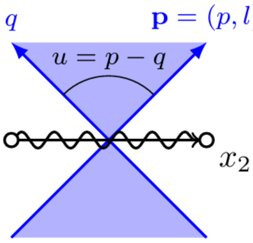


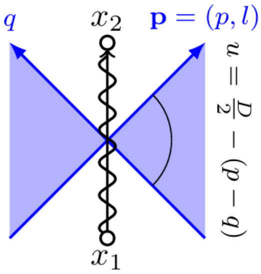
$$W^{\ell,\mu_1\cdots\mu_{2\ell}}_u(x_{10})=\frac{1}{(x_{10}^2)^u}\left[\prod_{i=1}^{2\ell}\frac{x_{10}^{\mu_i}}{|x_{10}|}\right]=$$

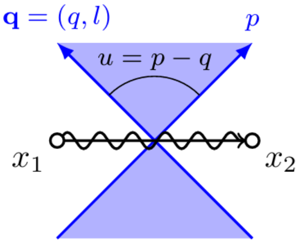


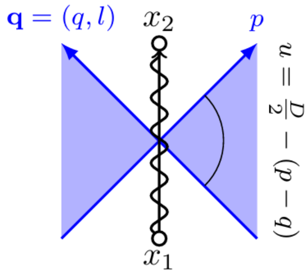
$$W^{\ell,\dot{\alpha}\beta}_u(x_{10})=\frac{1}{(x_{10}^2)^u}\left[\frac{\bar{x}_{10}^{\dot{\alpha}\beta}}{|x_{10}|}\right]^{2\ell}=$$



$$W^{\ell}_{u,\alpha\dot\beta}(x_{12}) =$$ 

, $$\bar{W}^{\ell,\dot\alpha\beta}_{u}(x_{12}) =$$ 

$$W^{\ell,\dot\alpha\beta}_{u}(x_{12}) =$$ 

, $$\bar{W}^{\ell}_{u,\alpha\dot\beta}(x_{12}) =$$ 

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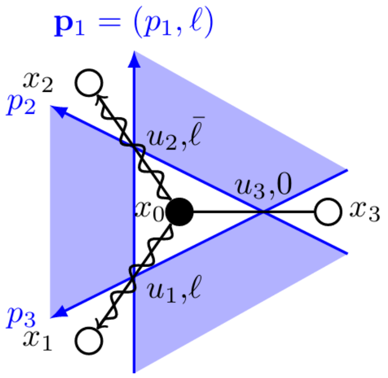
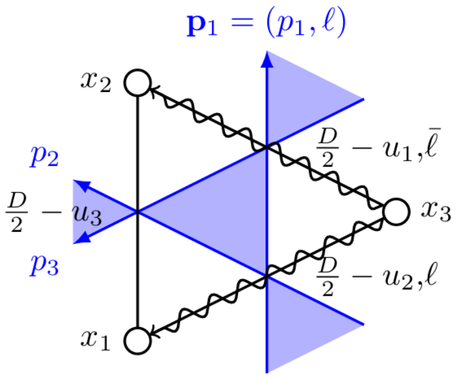
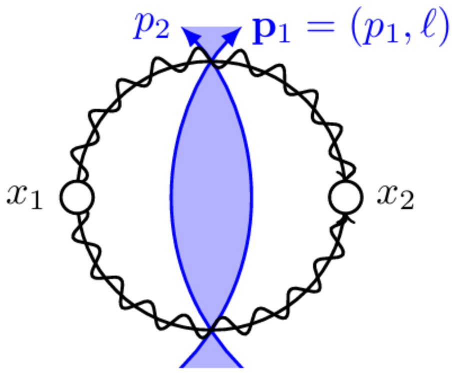
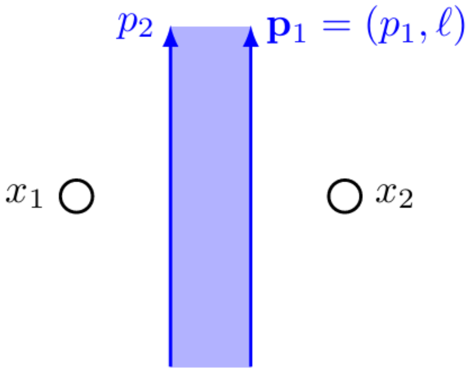
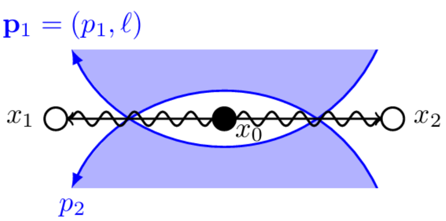


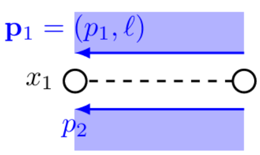
$$= r_\ell(u_3, u_1, u_2)$$ 



$$=$$ 

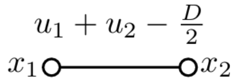
$$\cdot\, \mathbb{I}^{(\ell)}$$



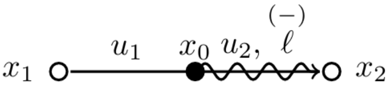
$$= \pi^D a_\ell(u) a_\ell(D - u) \cdot$$ 

$$\cdot\, \mathbb{I}^{(\ell)}$$

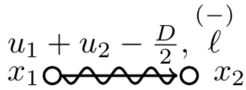
$= r_\ell(D - u_1 - u_2, u_1, u_2) \cdot$



$\cdot \, \mathbb{1}^{(\ell)}$,

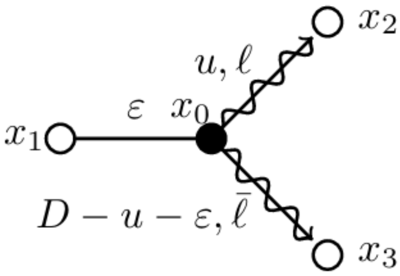


$= r_\ell(u_1, u_2, D - u_1 - u_2) \cdot$



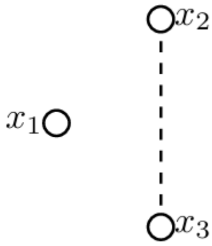
,

$\lim_{\varepsilon \to 0}$

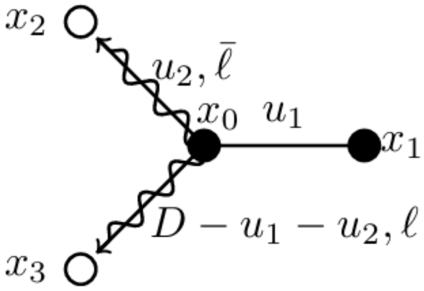


=

$\pi^D a_\ell(u)\, a_\ell(D - u) \;\cdot$



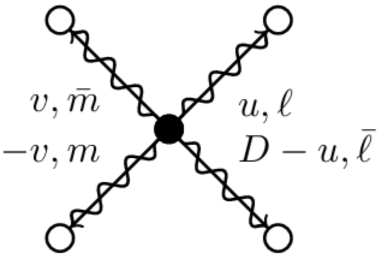
$\cdot \, \mathbb{1}^{(\ell)}$



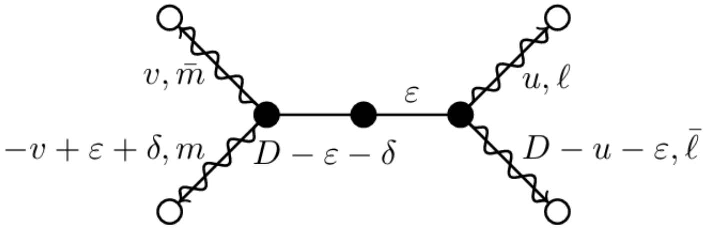
=

$\pi^D a_\ell(u_1)\, a_\ell(D - u_1)$

$\cdot \, \mathbb{1}^{(\ell)}$

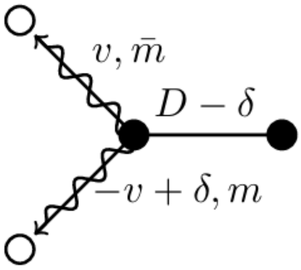


$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \frac{1}{\pi^D a_0(\varepsilon) a_0(D-\varepsilon)}$$



=

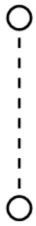
$$\left[\lim_{\varepsilon \to 0} \frac{a_\ell(u) a_\ell(D-u)}{a_0(\varepsilon) a_0(D-\varepsilon)}\right] \cdot \lim_{\delta \to 0}$$



$$\cdot\ \mathbb{1}^{(\ell)}$$

=

$$\left[\lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \pi^D a_\ell(u) a_\ell(D-u) \frac{a_0(\delta) a_0(D-\delta)}{a_0(\varepsilon) a_0(D-\varepsilon)}\right] \cdot$$



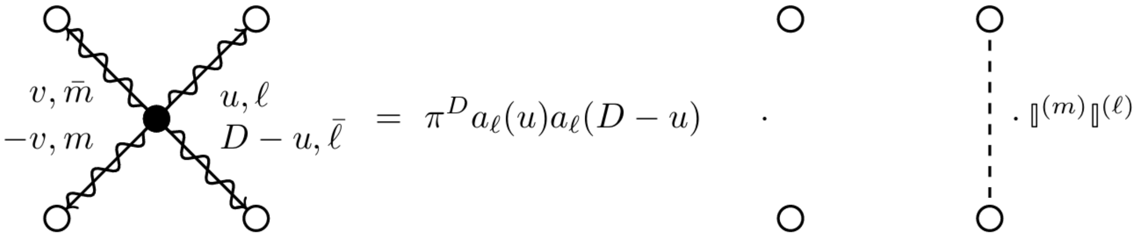
$$\cdot\ \mathbb{1}^{(m)} \mathbb{1}^{(\ell)}$$

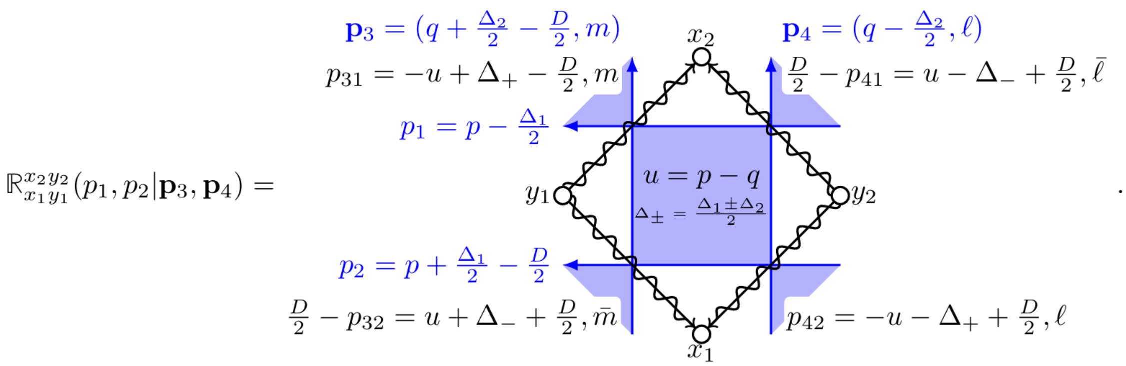
=

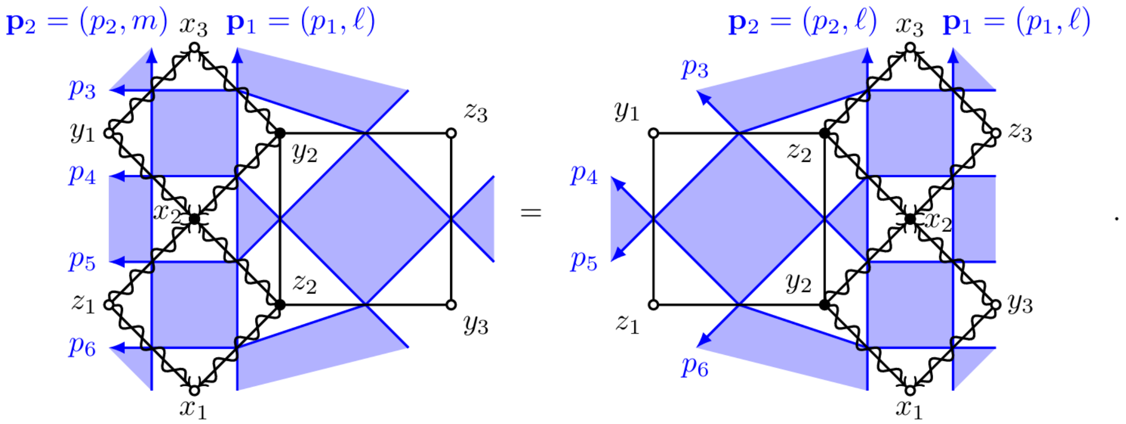
$$\pi^D a_\ell(u) a_\ell(D-u) \quad \cdot$$

$$\cdot\ \mathbb{1}^{(m)} \mathbb{1}^{(\ell)}$$

=







$$\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\Phi^\dagger \,\Box^v\, \Phi = \int \mathrm{d}^4x\left[\phi^\dagger \,\Box^{1+v}\, \phi - \mathrm{i}\bar{\psi}\bar{\sigma}^\mu \,\Box^v\, \partial_\mu\psi + F^\dagger \,\Box^v\, F\right]$$

$$\Phi \;= \phi(x_+) + \sqrt{2}\theta\psi(x_+) + \theta^2 F(x_+) = \mathrm{e}^{\mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu}\left[\phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x)\right]$$

$$= \phi(x) + \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2 \,\Box\, \phi(x) + \sqrt{2}\theta\psi(x) - \frac{\mathrm{i}}{2}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta^2 F(x)$$

$$\Phi^\dagger \;= \phi^\dagger(x_-) + \sqrt{2}\bar{\theta}\bar{\psi}(x_-) + \bar{\theta}^2F^\dagger(x_-) = \mathrm{e}^{-\mathrm{i}\theta\sigma^\mu}\bar{\theta}\partial_\mu\left[\phi^\dagger(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \bar{\theta}^2F^\dagger(x)\right]$$

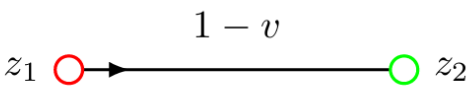
$$= \phi^\dagger(x) - \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^\dagger(x) + \frac{1}{4}\theta^2\bar{\theta}^2 \,\Box\, \phi^\dagger(x) - \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{\mathrm{i}}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}^2F^\dagger(x)$$

$$\langle\Phi(z_1)\Phi^\dagger(z_2)\rangle = \mathrm{e}^{\mathrm{i}\theta_1\sigma^\mu\bar{\theta}_1\partial_{1,\mu} - \mathrm{i}\theta_2\sigma^\mu\bar{\theta}_2\partial_{2,\mu}}$$
$$\left[\langle\phi(x_1)\phi^\dagger(x_2)\rangle + 2\theta_1^\alpha\bar{\theta}_2^{\dot{\alpha}}\langle\psi_\alpha(x_1)\bar{\psi}_{\dot{\alpha}}(x_2)\rangle + \theta_1^2\bar{\theta}_2^2\langle F(x_1)F^\dagger(x_2)\rangle\right]$$

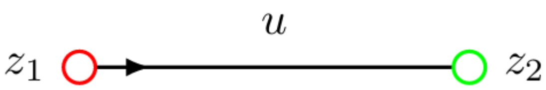
$$\begin{aligned}\langle \phi(x_1)\phi^\dagger(x_2)\rangle &= G_{1+v}(x_{12})\\ \langle \psi_\alpha(x_1)\bar\psi_{\dot\alpha}(x_2)\rangle &= -\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_{1,\mu}G_{1+v}(x_{12})\\ \langle F(x_1)F^\dagger(x_2)\rangle &= \Box_1\, G_{1+v}(x_{12})\end{aligned}$$

$$\langle \Phi(z_1)\Phi^\dagger(z_2)\rangle = (-4)^{-v}\frac{\Gamma(1-v)}{\Gamma(1+v)}\mathrm{e}^{\mathrm{i}[\theta_1\sigma^\mu\bar\theta_1+\theta_2\sigma^\mu\bar\theta_2-2\theta_1\sigma^\mu\bar\theta_2]\partial_{1,\mu}}\frac{1}{[x_{12}^2]^{1-v}} = \frac{c_0(1+v)}{[x_{\bar{1}2}^2]^{1-v}}$$

$$\langle \Phi(z_1)\Phi^\dagger(z_2)\rangle = c_0(1+v)\cdot$$



$$\frac{1}{[x_{1\bar{2}}^2]^u} =$$

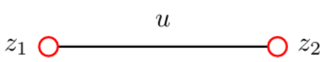


$$\langle \Phi(z_1)\Phi^\dagger(z_2)\rangle = (-1)^v 2^{-\frac{1}{2}-2v}\frac{\Gamma\left(\frac{1}{2}-v\right)}{\Gamma(1+v)}\frac{1}{[x_{1\bar{2}}^2]^{\frac{1}{2}-v}} = \frac{c_0(1+v)}{[x_{1\bar{2}}^2]^{\frac{1}{2}-v}}.$$

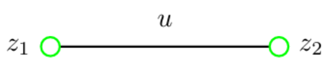
$$x^\mu_{1\bar{2}} := x^\mu_{12} + \mathrm{i}[\theta_1\gamma^\mu\bar\theta_1 + \theta_2\gamma^\mu\bar\theta_2 - 2\theta_1\gamma^\mu\bar\theta_2]$$

$$\Phi \to \mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha}\Phi\left(x, \mathrm{e}^{\mathrm{i}\alpha}\theta, \mathrm{e}^{-\mathrm{i}\alpha}\bar\theta\right), \Phi^\dagger \to \mathrm{e}^{\frac{\mathrm{i}}{2}\alpha}\Phi^\dagger\left(x, \mathrm{e}^{\mathrm{i}\alpha}\theta, \mathrm{e}^{-\mathrm{i}\alpha}\bar\theta\right)$$

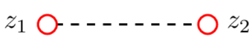
$$\frac{\theta_{12}^2}{[x_{12}^2]^u} =$$



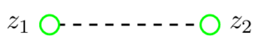
$$,\qquad \frac{\bar\theta_{12}^2}{[x_{12}^2]^u} =$$



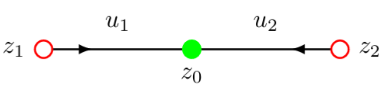
$$\delta^{(2)}(\theta_{12})\,\delta^{(D)}(x_{12}) =$$



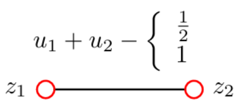
$$,\quad \delta^{(2)}(\bar\theta_{12})\,\delta^{(D)}(x_{12}) =$$



$$\int \mathrm{d}^D x_0\, \mathrm{d}^2\bar\theta_0 \frac{1}{[x_{1\bar{0}}^2]^{u_1}}\frac{1}{[x_{2\bar{0}}^2]^{u_2}}\Bigg|_{\substack{\theta_0=0\\ \bar\theta_{1,2}=0}} = \frac{\theta_{12}^2}{[x_{12}^2]^{u_1+u_2-\frac{D}{2}+1}}\cdot\begin{cases}4\,r_0(2-u_1-u_2,u_1,u_2)\\ -4\,r_0(3-u_1-u_2,u_1,u_2)\end{cases}$$

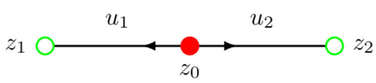


$$=$$

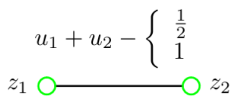


$$\cdot\begin{cases}4\,r_0(2-u_1-u_2,u_1,u_2)\\ -4\,r_0(3-u_1-u_2,u_1,u_2)\end{cases}$$

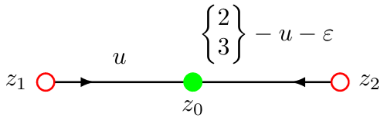
$$\int \mathrm{d}^D x_0\, \mathrm{d}^2\theta_0 \frac{1}{[x_{0\bar{1}}^2]^{u_1}}\frac{1}{[x_{0\bar{2}}^2]^{u_2}}\Bigg|_{\substack{\bar\theta_0=0\\ \theta_{1,2}=0}} = \frac{\bar\theta_{12}^2}{[x_{12}^2]^{u_1+u_2-\frac{D}{2}+1}}\cdot\begin{cases}4\,r_0(2-u_1-u_2,u_1,u_2)\\ -4\,r_0(3-u_1-u_2,u_1,u_2)\end{cases},$$

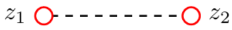


$$=$$

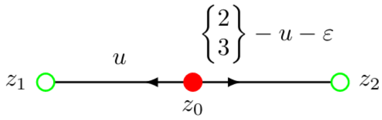


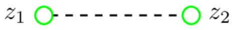
$$\cdot\begin{cases}4\,r_0(2-u_1-u_2,u_1,u_2)\\ -4\,r_0(3-u_1-u_2,u_1,u_2)\end{cases}.$$

$\lim_{\varepsilon\to 0}$ 

$=$ 

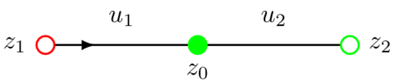
$\cdot \begin{cases} 4\,\pi^3 \cdot a_0(u)\, a_0(2-u) \\ -4\,\pi^4 \cdot a_0(u)\, a_0(3-u) \end{cases}$,

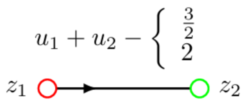
$\lim_{\varepsilon\to 0}$ 

$=$ 

$\cdot \begin{cases} 4\,\pi^3 \cdot a_0(u)\, a_0(2-u) \\ -4\,\pi^4 \cdot a_0(u)\, a_0(3-u) \end{cases}$.

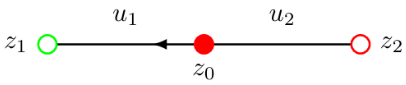
$$\int \mathrm{d}^D x_0\, \mathrm{d}^2\bar\theta_0 \frac{1}{[x^2_{1\bar 0}]^{u_1}} \frac{\bar\theta^2_{02}}{[x^2_{20}]^{u_2}}\Bigg|_{\substack{\theta_{0,2}=0\\ \bar\theta_1=0}} = r_0(D-u_1-u_2,u_1,u_2)\,\frac{1}{[x^2_{1\bar 2}]^{u_1+u_2-\frac{D}{2}}}\ ,$$

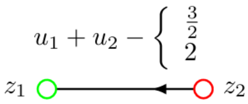


$=$ 

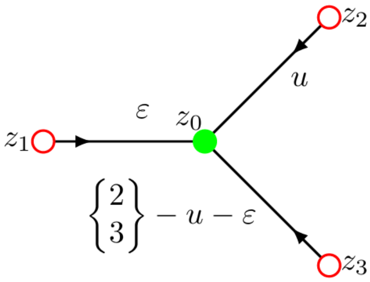
$\cdot \begin{cases} r_0(3-u_1-u_2,u_1,u_2) \\ r_0(4-u_1-u_2,u_1,u_2) \end{cases}$,

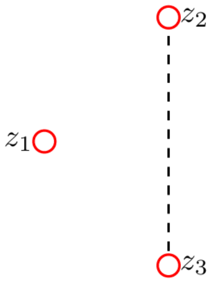
$$\int \mathrm{d}^D x_0\, \mathrm{d}^2\theta_0 \frac{1}{[x^2_{0\bar 1}]^{u_1}} \frac{\theta^2_{02}}{[x^2_{20}]^{u_2}}\Bigg|_{\substack{\bar\theta_{0,2}=0\\ \theta_1=0}} = r_0(D-u_1-u_2,u_1,u_2)\,\frac{1}{[x^2_{2\bar 1}]^{u_1+u_2-\frac{D}{2}}}\ ,$$



$=$ 

$\cdot \begin{cases} r_0(3-u_1-u_2,u_1,u_2) \\ r_0(4-u_1-u_2,u_1,u_2) \end{cases}$.

$\lim_{\varepsilon\to 0}$ 

$= \begin{Bmatrix} 4\,\pi^3 a_0(u)\, a_0(2-u) \\ -4\,\pi^4 a_0(u)\, a_0(3-u) \end{Bmatrix} \cdot$ 

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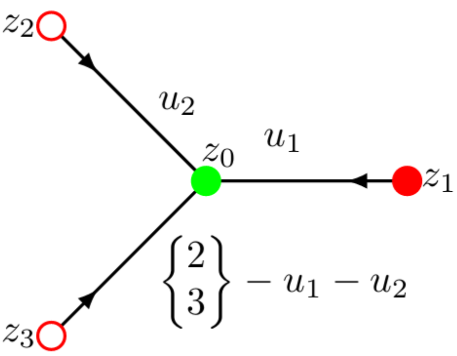
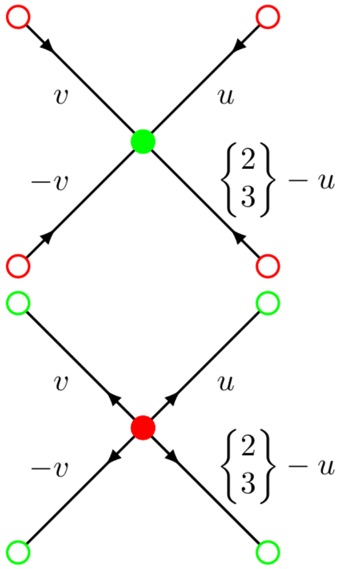
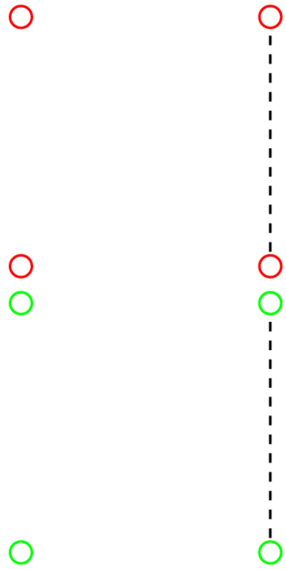


$= \begin{Bmatrix} 4\,\pi^3 a_0(u)\, a_0(2-u) \\ -4\,\pi^4 a_0(u)\, a_0(3-u) \end{Bmatrix}$. 

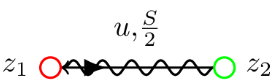


$$= \left\{ \begin{matrix} 4\,\pi^3 a_0(u)\, a_0(2-u) \\ -4\,\pi^4 a_0(u)\, a_0(3-u) \end{matrix} \right\} .$$

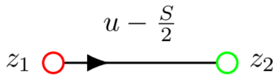


$$= \left\{ \begin{matrix} 4\,\pi^3 a_0(u)\, a_0(2-u) \\ -4\,\pi^4 a_0(u)\, a_0(3-u) \end{matrix} \right\} .$$

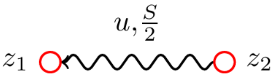
$$\mathrm{i} \int \mathrm{d}^4 x_0\, \mathrm{d}^2\theta_0\, \mathrm{d}^2\bar{\theta}_0 \delta^{(2)}(\theta_0) \frac{1}{[x_{1\bar{0}}^2]^{u_1}} \frac{1}{[x_{2\bar{0}}^2]^{u_2}} \frac{1}{[x_{3\bar{0}}^2]^{u_3}}$$
$$\overset{u_1+u_2+u_3=3}{=} -4r_0(u_1,u_2,u_3) \frac{(\theta_{12}\theta_{13})x_{23,+}^2 + (\theta_{23}\theta_{21})x_{31,+}^2 + (\theta_{31}\theta_{32})x_{12,+}^2}{[x_{12,+}^2]^{2-u_3}[x_{23,+}^2]^{2-u_1}[x_{31,+}^2]^{2-u_2}} .$$



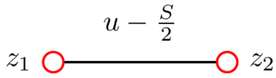
$$:= \frac{\Gamma(u-\frac{S}{2})}{\Gamma(u+\frac{S}{2})} \frac{\partial_1^{\mu_1}\cdots\partial_1^{\mu_S}}{(-2)^S}$$



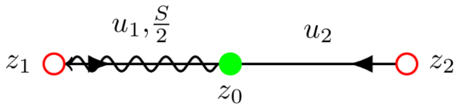
$$= \frac{1}{[x_{1\bar{2}}^2]^u} \prod_{i=0}^{S} \frac{x_{1\bar{2}}^{\mu_i}}{|x_{1\bar{2}}|} ,$$



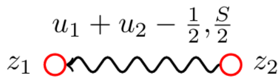
$$:= \frac{\Gamma(u-\frac{S}{2})}{\Gamma(u+\frac{S}{2})} \frac{\partial_1^{\mu_1}\cdots\partial_1^{\mu_S}}{(-2)^S}$$



$$= \frac{\theta_{12}^2}{[x_{12}^2]^u} \prod_{i=0}^{S} \frac{x_{12}^{\mu_i}}{|x_{12}|} .$$



$$= 4\, r_{\frac{S}{2}}(u_2, u_1, 2-u_1-u_2)$$



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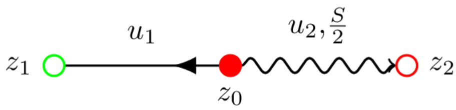
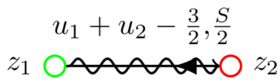


$$= r_{\frac{S}{2}}(u_1, u_2, 3-u_1-u_2)$$



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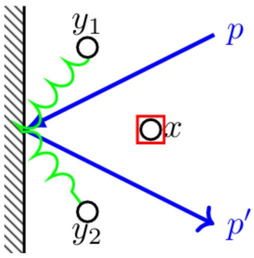
$$K_{y_2 x}^{y_1}(p) = V_p(x) V_p(y_1, y_2) =$$



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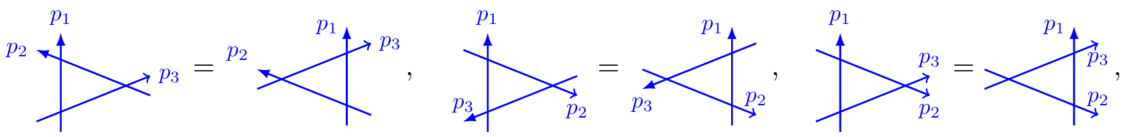
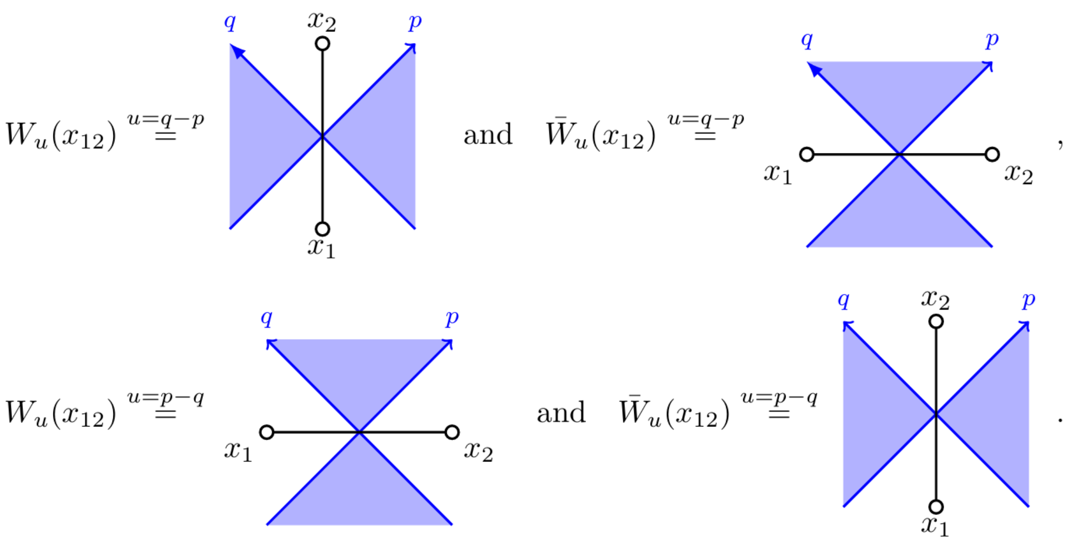
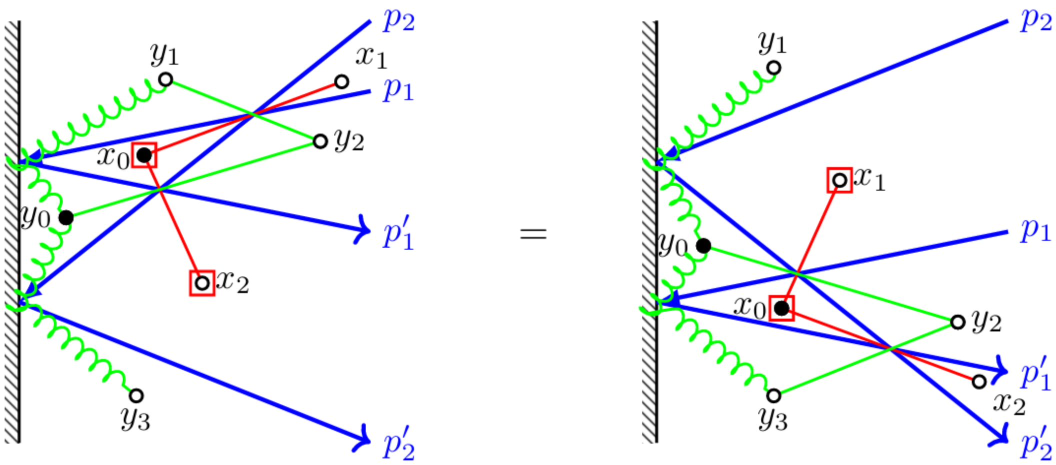
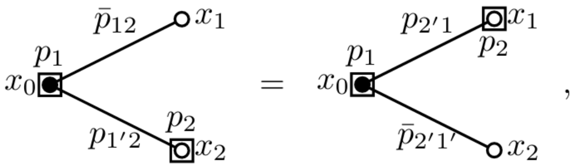






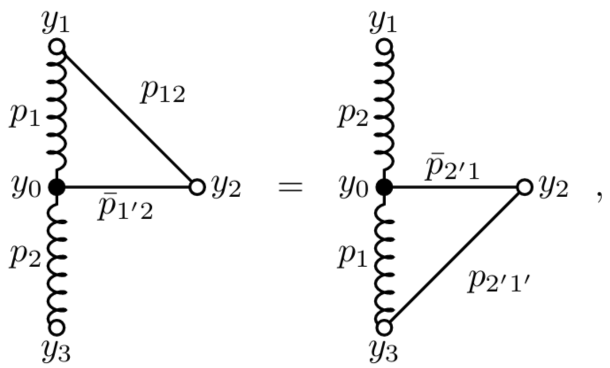
$$\int \mathrm{d}^D x_0 \, \mathrm{d}^D y_0 R^{x_0 y_1}_{x_1 y_2}(p_{12}) K^{y_1}_{y_0 x_0}(p_1) R^{y_2 x_2}_{y_0 x_0}(p_{1'2}) K^{y_0}_{y_3 x_2}(p_2)$$

$$= \int \mathrm{d}^D x_0 \, \mathrm{d}^D y_0 K^{y_1}_{y_0 x_1}(p_2) R^{y_2 x_0}_{y_0 x_1}(p_{2'1}) K^{y_0}_{y_3 x_0}(p_1) R^{x_2 y_3}_{x_0 y_2}(p_{2'1'})$$

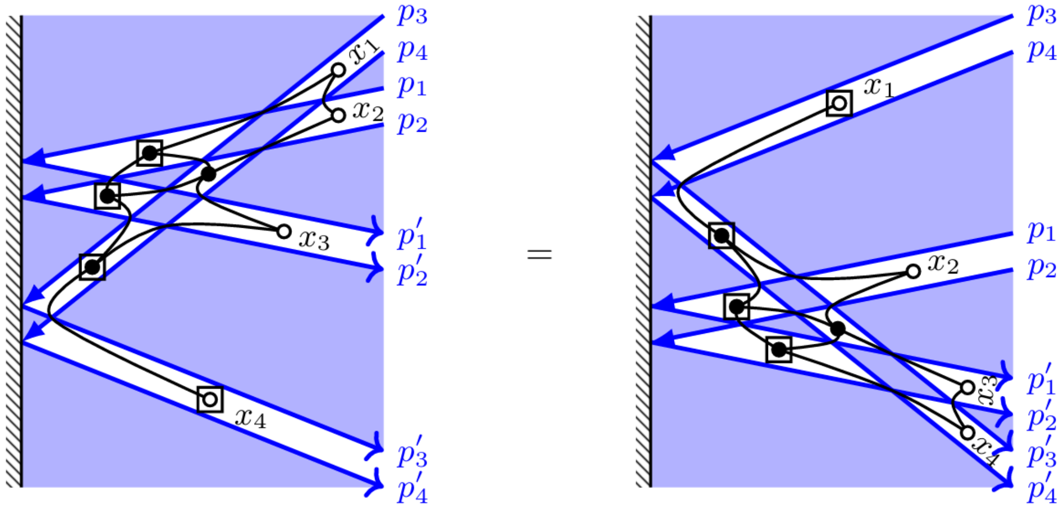


$$
V_{p_2}(x_2)\int d^D x_0\ W_{\bar{p}_{12}}(x_{10})V_{p_1}(x_0)W_{p_{1'2}}(x_{20}) = V_{p_2}(x_1)\int d^D x_0\ W_{p_{2'1}}(x_{10})V_{p_1}(x_0)W_{\bar{p}_{2'1'}}(x_{20}) \ ,
$$

$$
V_{p_2}(x_2)\int d^D x_0\ W_{\bar{p}_{12}}(x_{10})V_{p_1}(x_0)W_{p_{1'2}}(x_{20}) = V_{p_2}(x_1)\int d^D x_0\ W_{p_{2'1}}(x_{10})V_{p_1}(x_0)W_{\bar{p}_{2'1'}}(x_{20}) \ ,
$$



$$
W_{p_{12}}(y_{12})\int d^D y_0 V_{p_1}(y_1,y_0)W_{\bar{p}_{1'2}}(y_{20})V_{p_2}(y_0,y_3) = W_{p_{2'1'}}(y_{32})\int d^D y_0 V_{p_2}(y_1,y_0)W_{\bar{p}_{2'1}}(y_{20})V_{p_1}(y_0,y_3)
$$



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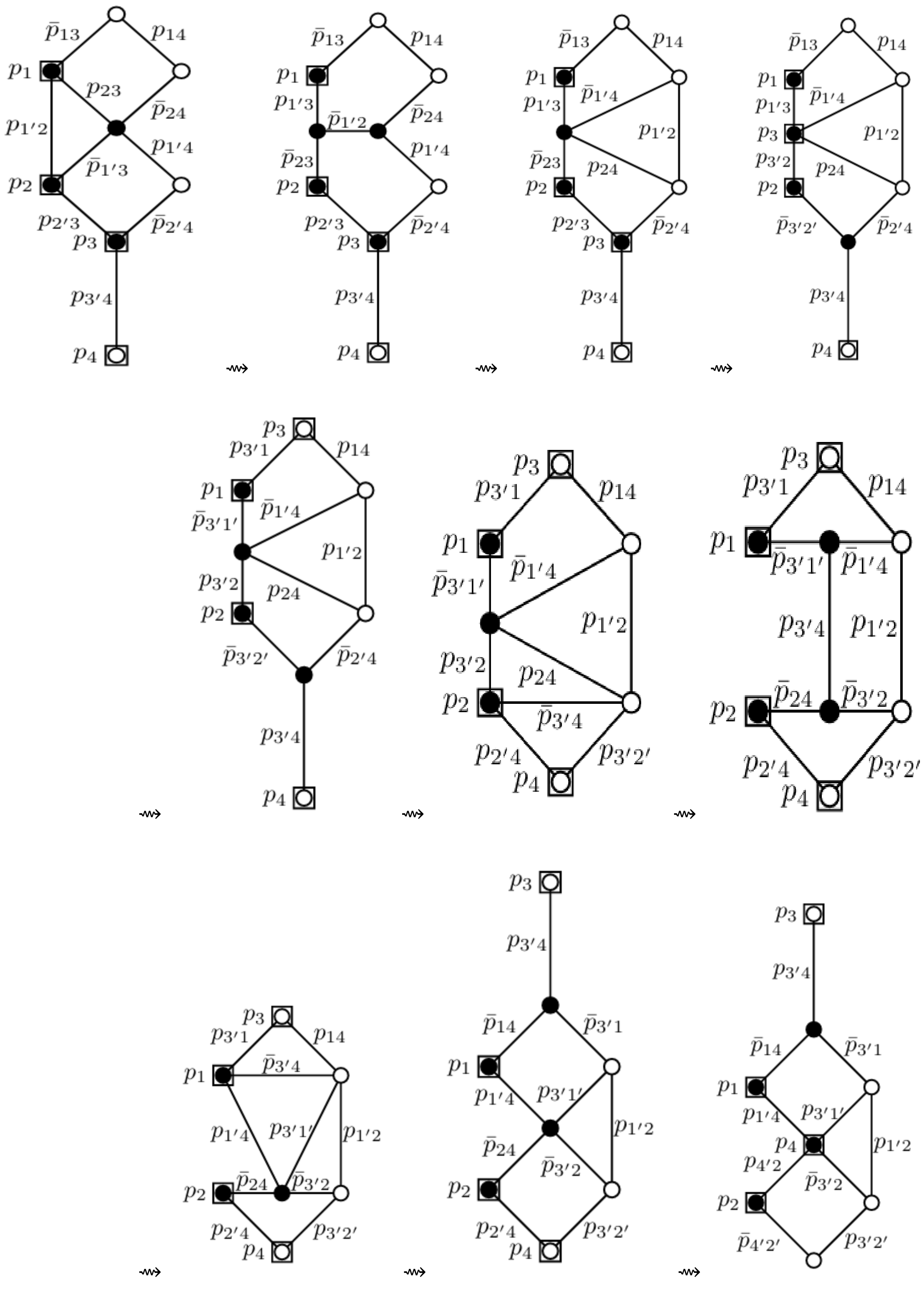
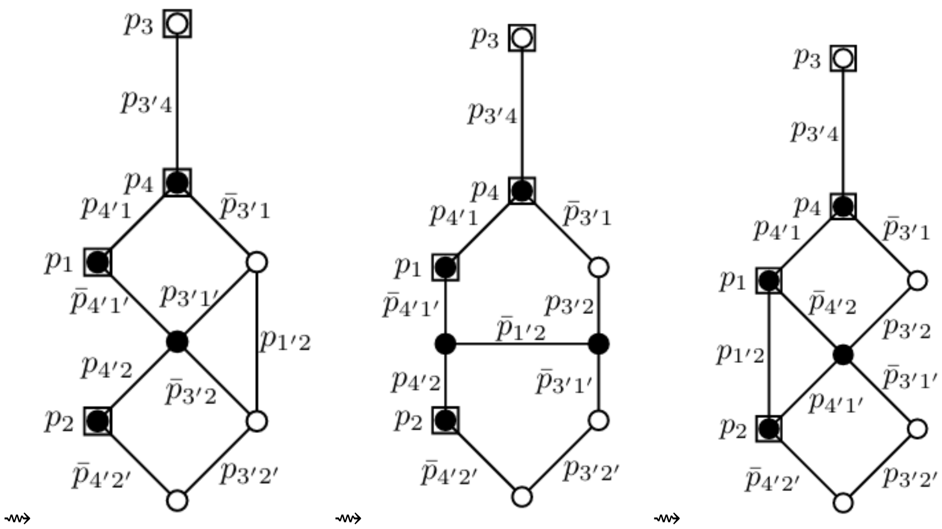
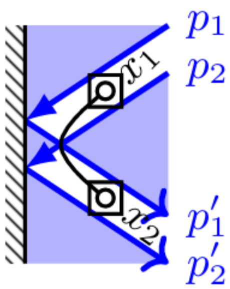
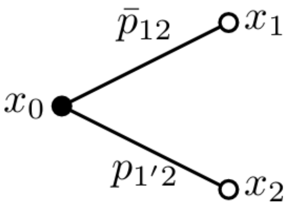

$\bar{p}_{13}$ p_{14} p_1 p_{23} $\bar{p}_{24}$ $p_{1'2}$ $p_{1'4}$ $\bar{p}_{1'3}$ p_2 $p_{2'3}$ $\bar{p}_{2'4}$ p_3 $p_{3'4}$ p_4
$\bar{p}_{13}$ p_{14} p_1 $p_{1'3}$ $\bar{p}_{1'2}$ $\bar{p}_{24}$ $p_{1'4}$ $\bar{p}_{23}$ p_2 $p_{2'3}$ $\bar{p}_{2'4}$ p_3 $p_{3'4}$ p_4
$\bar{p}_{13}$ p_{14} p_1 $p_{1'3}$ $\bar{p}_{1'4}$ $p_{1'2}$ $\bar{p}_{23}$ p_{24} p_2 $p_{2'3}$ $\bar{p}_{2'4}$ p_3 $p_{3'4}$ p_4
$\bar{p}_{13}$ p_{14} p_1 $p_{1'3}$ $\bar{p}_{1'4}$ p_3 $p_{1'2}$ $\bar{p}_{3'2}$ p_{24} p_2 $\bar{p}_{3'2'}$ $\bar{p}_{2'4}$ $p_{3'4}$ p_4
p_3 $p_{3'1}$ p_{14} p_1 $\bar{p}_{1'4}$ $\bar{p}_{3'1'}$ $p_{1'2}$ $p_{3'2}$ p_{24} p_2 $\bar{p}_{3'2'}$ $\bar{p}_{2'4}$ $p_{3'4}$ p_4
p_3 $p_{3'1}$ p_{14} p_1 $\bar{p}_{1'4}$ $\bar{p}_{3'1'}$ $p_{1'2}$ $p_{3'2}$ p_{24} p_2 $\bar{p}_{3'4}$ $p_{2'4}$ $p_{3'2'}$ p_4
p_3 $p_{3'1}$ p_{14} p_1 $\bar{p}_{3'1'}$ $\bar{p}_{1'4}$ $p_{3'4}$ $p_{1'2}$ p_2 $\bar{p}_{24}$ $\bar{p}_{3'2}$ $p_{2'4}$ $p_{3'2'}$ p_4
p_3 $p_{3'1}$ p_{14} p_1 $\bar{p}_{3'4}$ $p_{1'4}$ $p_{3'1'}$ $p_{1'2}$ p_2 $\bar{p}_{24}$ $\bar{p}_{3'2}$ $p_{2'4}$ $p_{3'2'}$ p_4
p_3 $p_{3'4}$ $\bar{p}_{14}$ $\bar{p}_{3'1}$ p_1 $p_{1'4}$ $p_{3'1'}$ $p_{1'2}$ $\bar{p}_{24}$ $\bar{p}_{3'2}$ p_2 $p_{2'4}$ $p_{3'2'}$ p_4
p_3 $p_{3'4}$ $\bar{p}_{14}$ $\bar{p}_{3'1}$ p_1 $p_{1'4}$ $p_{3'1'}$ p_4 $p_{1'2}$ $p_{4'2}$ $\bar{p}_{3'2}$ p_2 $\bar{p}_{4'2'}$ $p_{3'2'}$

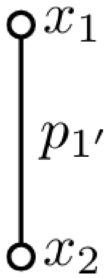


$$\frac{r_0(\bar{p}_{1'2},\bar{p}_{24},p_{1'4})r_0(\bar{p}_{2'4},\bar{p}_{3'2'},p_{3'4})r_0(\bar{p}_{3'1'},\bar{p}_{1'4},p_{3'4})r_0(\bar{p}_{1'2},\bar{p}_{4'1'},p_{4'2})}{r_0(\bar{p}_{1'2},\bar{p}_{23},p_{1'3})r_0(\bar{p}_{3'2},\bar{p}_{24},p_{3'4})r_0(\bar{p}_{14},\bar{p}_{3'1},p_{3'4})r_0(\bar{p}_{1'2},\bar{p}_{3'1'},p_{3'2})} \overset{p'=-p}{=} 1$$

$\mathbb{K}^{x_1}_{x_2}(p_1,p_2) =$  $= V_{p_1}(x_1)W_{p_{1'2}}(x_{12})V_{p_2}(x_2)$



$$r_0(\bar{p}_{1'1},\bar{p}_{12},p_{1'2}) \cdot$$



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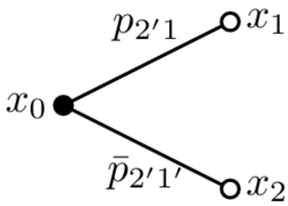

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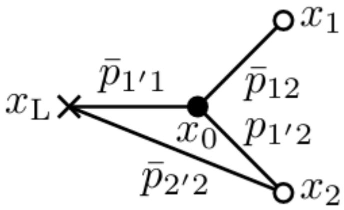
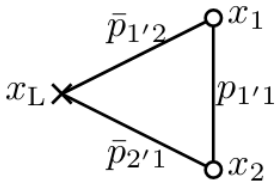
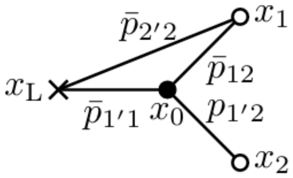


$$= \quad r_0(\bar{p}_{1'1}, \bar{p}_{12}, p_{1'2}) \cdot$$



$$=$$



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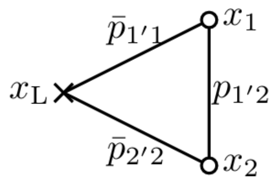
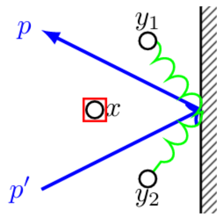
$$\mathbb{K}^{x_1}_{x_2}(p_1, p_2; x_L) = W_{\bar{p}_{1'1}}(x_{1L}) W_{p_{1'2}}(x_{12}) W_{\bar{p}_{2'2}}(x_{2L}) =$$



$$K^{Ry_1}_{y_2 x}(p) = V^R_p(x) V^R_p(y_1, y_2) =$$



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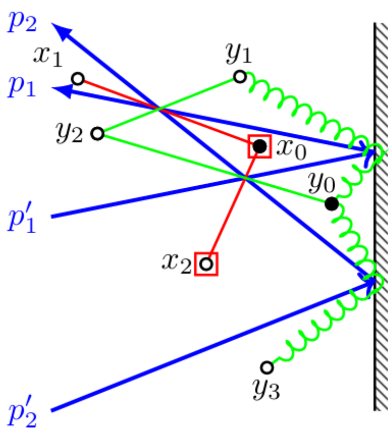
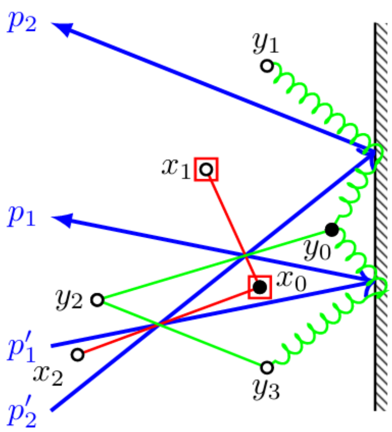


$$=$$



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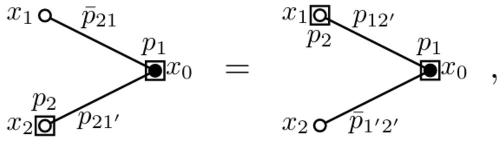
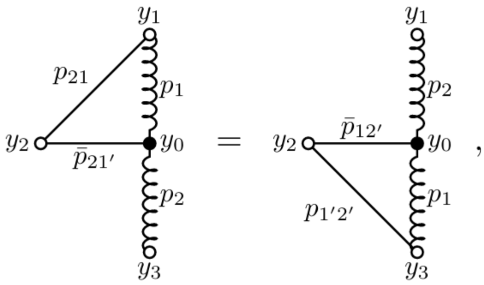


$$V^{\mathrm{R}}_{p_2}(x_2)\int d^D x_0\; W_{\bar{p}_{21}}(x_{10})V^{\mathrm{R}}_{p_1}(x_0)W_{p_{21'}}(x_{20})$$

$$= V^{\mathrm{R}}_{p_2}(x_1)\int d^D x_0\; W_{p_{12'}}(x_{10})V^{\mathrm{R}}_{p_1}(x_0)W_{\bar{p}_{1'2'}}(x_{20}) \;,$$

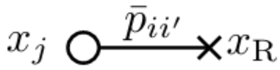


$$W_{p_{21}}(y_{12})\int d^D y_0\; V^{\mathrm{R}}_{p_1}(y_1,y_0)W_{\bar{p}_{21'}}(y_{20})V^{\mathrm{R}}_{p_2}(y_0,y_3)$$

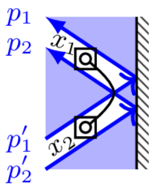
$$= W_{p_{1'2'}}(y_{32})\int d^D y_0\; V^{\mathrm{R}}_{p_2}(y_1,y_0)W_{\bar{p}_{12'}}(y_{20})V^{\mathrm{R}}_{p_1}(y_0,y_3) \;,$$

$$V^{\mathrm{R}}_{p_i}(x_j) =$$

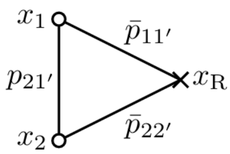
$$= W_{\bar{p}_{ii'}}(x_{\mathrm{R}j}) =$$



$$\mathbb{K}^{\mathrm{R}x_1}_{\;\;x_2}(p_1,p_2) =$$



$$= V^{\mathrm{R}}_{p_1}(x_1)W_{p_{21'}}(x_{12})V^{\mathrm{R}}_{p_2}(x_2) \xrightarrow{\text{canonical}}$$



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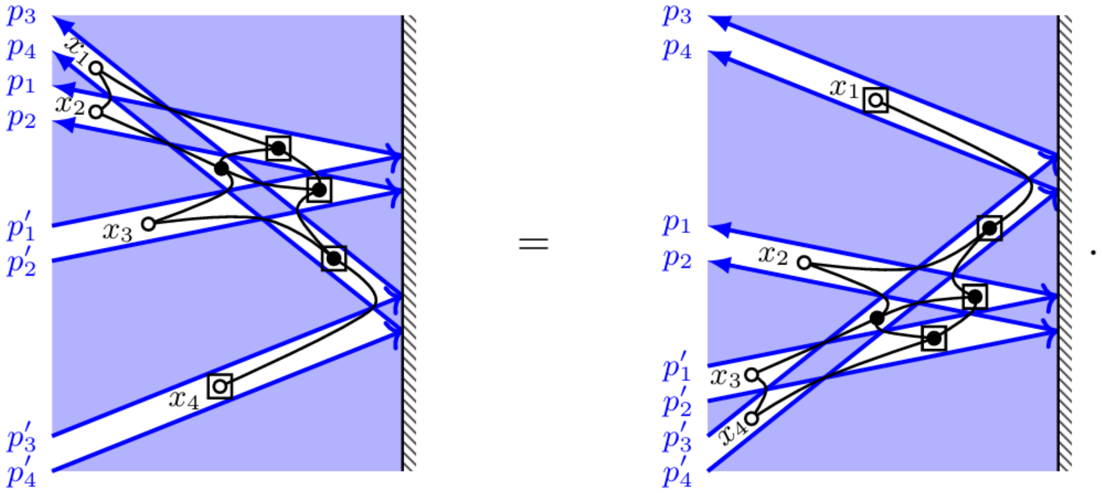
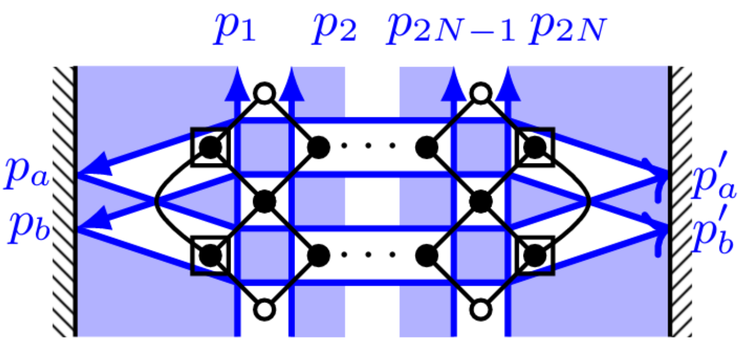


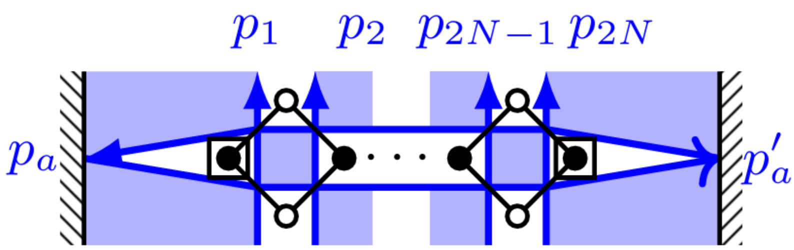
$\mathbb{T}_N(p_a, p_b, \{p_i\}) =$



$$[\mathbb{T}_N(p_a, p_b, \{p_i\}), \mathbb{T}_N(q_a, q_b, \{p_i\})] = 0.$$

$$\mathbb{T}_N(p_a, p_b, \{p_i\}) = \left[\prod_{i=1}^{N} \frac{a_0(\bar{p}_{i+1,a'})a_0(p_{i+1,b})}{a_0(\bar{p}_{ia'})a_0(p_{ib})}\right] \cdot T_N(p_a, \{p_i\}) \circ T_N(p_b, \{p_i\}),$$

$T_N(p_a, \{p_i\}) =$



$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta} \sum_{i=1}^{3} \mathrm{tr}\left(e^{-g\mathcal{V}} \Phi_i^\dagger e^{g\mathcal{V}} \Phi_i\right) + \frac{1}{2g^2} \int d^4x\, d^2\theta \mathrm{tr}(W^\alpha W_\alpha)$$
$$+ \frac{ig}{\sqrt{2}} \int d^4x\, d^2\theta \mathrm{tr}(\Phi_1[\Phi_2, \Phi_3]) + \frac{ig}{\sqrt{2}} \int d^4x\, d^2\bar{\theta} \mathrm{tr}\left(\Phi_1^\dagger[\Phi_2^\dagger, \Phi_3^\dagger]\right)$$

$$[T^A, T^B] = \mathrm{i}f^{ABC}T_C, \mathrm{tr}(T^A T^B) = \delta^{AB}, (T^A)^a_b (T_A)^c_d = \delta^a_d \delta^c_b - \frac{1}{\mathrm{N}}\delta^a_b \delta^c_d$$

$$W_\alpha = \frac{\mathrm{i}}{4}\bar{D}^2\left(\mathrm{e}^{-g\mathcal{V}} D_\alpha \mathrm{e}^{g\mathcal{V}}\right)$$

$$\mathcal{V} = -\theta\sigma^\mu\bar{\theta}A_\mu + \mathrm{i}\theta^2\bar{\theta}\bar{\chi} - \mathrm{i}\bar{\theta}^2\theta\chi + \frac{1}{2}\theta^2\bar{\theta}^2\mathrm{D}$$

$$\mathcal{V}^2 = -\tfrac{1}{2}\theta^2\bar{\theta}^2 A_\mu A^\mu \text{ and } \mathcal{V}^3 = 0$$

$$W_\alpha = \frac{\mathrm{i}g}{4}\bar{D}^2 D_\alpha \mathcal{V} - \frac{\mathrm{i}g^2}{8}\bar{D}^2[\mathcal{V}, D_\alpha \mathcal{V}]$$

$$W_\alpha|_{\bar{\theta}=0} = g\chi_\alpha + \mathrm{i}g\theta_\alpha \mathrm{D} + \frac{1}{2}g\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu,\dot{\alpha}\beta}\theta_\beta F_{\mu\nu} + \mathrm{i}g\theta^2\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{D}_\mu\bar{\chi}^{\dot{\alpha}}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{\mathrm{i}g}{2}[A_\mu, A_\nu]$$

$$\mathcal{D}_\mu = \partial_\mu + \frac{\mathrm{i}g}{2}[A_\mu, \cdot]$$

$$S_\mathrm{g} = \frac{1}{2g^2}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{tr}(W^\alpha W_\alpha) = \int \mathrm{d}^4x \mathrm{tr}\left[-\mathrm{i}\bar{\chi}\bar{\sigma}^\mu\mathcal{D}_\mu\chi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\mathrm{i}}{8}F_{\mu\nu}F^{*,\mu\nu} + \frac{1}{2}\mathrm{D}^2\right]$$

$$\begin{aligned} S_\mathrm{K} &= \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\sum_{i=1}^{3} \mathrm{tr}\left(\mathrm{e}^{-g\mathcal{V}}\Phi_i^\dagger \mathrm{e}^{g\mathcal{V}}\Phi_i\right) \\ &= \int \mathrm{d}^4x \sum_{i=1}^{3} \mathrm{tr}\left[-\mathcal{D}_\mu\phi_i^\dagger\mathcal{D}^\mu\phi_i - \mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\mathcal{D}_\mu\psi_i + \frac{1}{2}g\mathrm{D}[\phi_i^\dagger, \phi_i] + F_i^\dagger F_i \right. \\ &\left. + \frac{\mathrm{i}g}{\sqrt{2}}(\bar{\psi}_i\bar{\chi}\phi_i + \phi_i^\dagger\chi\psi_i - \bar{\chi}\bar{\psi}_i\phi_i - \chi\phi_i^\dagger\psi_i)\right] \end{aligned}$$

$$\mathrm{D} = \mathrm{D}_A T^A = \frac{g}{2}\sum_{i=1}^{3} [\phi_i, \phi_i^\dagger]$$

$$\begin{aligned} S_\mathrm{pot} &= \frac{\mathrm{i}g}{\sqrt{2}}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{tr}(\Phi_1[\Phi_2, \Phi_3]) \\ &= \frac{\mathrm{i}g}{\sqrt{2}}\int \mathrm{d}^4x \mathrm{tr}(\phi_1[\phi_2, F_3] + \phi_1[F_2, \phi_3] + F_1[\phi_2, \phi_3] - \phi_1[\psi_2, \psi_3] - \psi_1[\phi_2, \psi_3] - \psi_1[\psi_2, \phi_3]) \end{aligned}$$

$$\begin{aligned} F_1 &= \frac{\mathrm{i}g}{\sqrt{2}}[\phi_3^\dagger, \phi_2^\dagger], \quad F_2 = \frac{\mathrm{i}g}{\sqrt{2}}[\phi_1^\dagger, \phi_3^\dagger], \quad F_3 = \frac{\mathrm{i}g}{\sqrt{2}}[\phi_2^\dagger, \phi_1^\dagger] \\ F_1^\dagger &= \frac{\mathrm{i}g}{\sqrt{2}}[\phi_3, \phi_2], \quad F_2^\dagger = \frac{\mathrm{i}g}{\sqrt{2}}[\phi_1, \phi_3], \quad F_3^\dagger = \frac{\mathrm{i}g}{\sqrt{2}}[\phi_2, \phi_1] \end{aligned}$$

$$S_\mathrm{pot} = \int \mathrm{d}^4x \mathrm{tr}\left[\frac{g^2}{2}\sum_{i<j} [\phi_i, \phi_j][\phi_i^\dagger, \phi_j^\dagger] - \frac{\mathrm{i}g}{\sqrt{2}}\varepsilon^{ijk}\psi_i\phi_j\psi_k\right]$$

$$\mathrm{tr}\left[\frac{g^2}{2}\sum_{i<j}\ [\phi_i,\phi_j][\phi_i^\dagger,\phi_j^\dagger]-\frac{g^2}{8}\sum_{i,j=1}^{3}\ [\phi_i,\phi_i^\dagger][\phi_j,\phi_j^\dagger]\right],$$

$$\frac{g^2}{4}\mathrm{tr}\left[\sum_{i<j}\ \left(2\phi_i\phi_j\phi_i^\dagger\phi_j^\dagger+2\phi_j\phi_i\phi_j^\dagger\phi_i^\dagger-\{\phi_i^\dagger,\phi_i\}\{\phi_j^\dagger,\phi_j\}\right)+\sum_{i=1}^{3}\ \left(2\phi_i\phi_i\phi_i^\dagger\phi_i^\dagger-\frac{1}{2}\{\phi_i^\dagger,\phi_i\}^2\right)\right].$$

$$\frac{g^2}{2}\sum_{i,j=1}^{3}\ \mathrm{tr}\left[\phi_i\phi_j\phi_i^\dagger\phi_j^\dagger-\frac{1}{4}\{\phi_i^\dagger,\phi_i\}\{\phi_j^\dagger,\phi_j\}\right].$$

$$\begin{aligned}S=\mathrm{N}\int\ \ \mathrm{d}^4x\mathrm{tr}[&-\mathcal{D}_\mu\phi_i^\dagger\mathcal{D}^\mu\phi^i-\mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\mathcal{D}_\mu\psi^i-\mathrm{i}\bar{\chi}\bar{\sigma}^\mu\mathcal{D}_\mu\chi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{\mathrm{i}}{8}F_{\mu\nu}F^{*,\mu\nu}\\&-\mathrm{i}\lambda\left(\bar{\chi}\phi^i\bar{\psi}_i+\psi^i\phi_i^\dagger\chi-\bar{\psi}_i\phi^i\bar{\chi}-\chi\phi_i^\dagger\psi^i-\varepsilon^{ijk}\psi_i\phi_j\psi_k-\varepsilon^{ijk}\bar{\psi}_i\phi_j^\dagger\bar{\psi}_k\right)\\&+\lambda^2\left(\phi^i\phi^j\phi_i^\dagger\phi_j^\dagger-\frac{1}{4}\{\phi_i^\dagger,\phi^i\}\{\phi_j^\dagger,\phi^j\}\right)\Big].\end{aligned}$$

$$\varphi_1\cdot\varphi_2\to\varphi_1\star\varphi_2=\mathrm{e}^{-\frac{\mathrm{i}}{2}\varepsilon^{ijk}\gamma_i q_j^{\varphi_1}q_k^{\varphi_2}}\,\varphi_1\cdot\varphi_2$$

$$\varphi_1\star\cdots\star\varphi_p=\mathrm{e}^{-\frac{i}{2}\sum_{m<n}\varepsilon^{ijk}\gamma_i q_j^{\varphi m}q_k^{\varphi n}}\,\varphi_1\cdots\varphi_p,$$

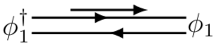
$$\begin{aligned}S=\mathrm{N}\int\ \ \mathrm{d}^4x\ \mathrm{tr}[&-\mathcal{D}_\mu\phi_i^\dagger\mathcal{D}^\mu\phi^i-\mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\mathcal{D}_\mu\psi^i-\mathrm{i}\bar{\chi}\bar{\sigma}^\mu\mathcal{D}_\mu\chi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{\mathrm{i}}{8}F_{\mu\nu}F^{*,\mu\nu}\\&-\mathrm{i}\lambda\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\gamma_i^-}\,\bar{\chi}\phi^i\bar{\psi}_i+\mathrm{e}^{-\frac{\mathrm{i}}{2}\gamma_i^-}\,\psi^i\phi_i^\dagger\chi-\mathrm{e}^{-\frac{\mathrm{i}}{2}\gamma_i^-}\,\bar{\psi}_i\phi^i\bar{\chi}-\mathrm{e}^{\frac{\mathrm{i}}{2}\gamma_i^-}\,\chi\phi_i^\dagger\psi^i\right.\\&\left.-\mathrm{e}^{\frac{\mathrm{i}}{2}\varepsilon_{ijk}\gamma_j^+}\,\varepsilon^{ijk}\psi_i\phi_j\psi_k-\mathrm{e}^{\frac{\mathrm{i}}{2}\varepsilon_{ijk}\gamma_j^+}\,\varepsilon^{ijk}\bar{\psi}_i\phi_j^\dagger\bar{\psi}_k\right)\\&+\lambda^2\left(\mathrm{e}^{-\mathrm{i}\varepsilon_{ijk}\gamma_k}\phi^i\phi^j\phi_i^\dagger\phi_j^\dagger-\frac{1}{4}\{\phi_i^\dagger,\phi^i\}\{\phi_j^\dagger,\phi^j\}\right)\Big].\end{aligned}$$

$$\gamma_1^\pm=-\frac{\gamma_3\pm\gamma_2}{2},\gamma_2^\pm=-\frac{\gamma_1\pm\gamma_3}{2},\gamma_3^\pm=-\frac{\gamma_2\pm\gamma_1}{2}.$$

$$\begin{aligned}&S^\chi=\mathrm{N}\int\ \ \mathrm{d}^4x\ \mathrm{tr}\Bigg[-\sum_{i=1}^{3}\ \partial_\mu\phi_i^\dagger\partial^\mu\phi_i-\sum_{i=1}^{3}\ \mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i\\&+\mathrm{i}\sqrt{\xi_1\xi_2}(\psi_2\phi_3\psi_1+\bar{\psi}_2\phi_3^\dagger\bar{\psi}_1)+\mathrm{i}\sqrt{\xi_1\xi_3}(\psi_1\phi_2\psi_3+\bar{\psi}_1\phi_2^\dagger\bar{\psi}_3)+\mathrm{i}\sqrt{\xi_2\xi_3}(\psi_3\phi_1\psi_2+\bar{\psi}_3\phi_1^\dagger\bar{\psi}_2)\\&+\xi_1^2\phi_2\phi_3\phi_2^\dagger\phi_3^\dagger+\xi_2^2\phi_3\phi_1\phi_3^\dagger\phi_1^\dagger+\xi_3^2\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger]\end{aligned}$$

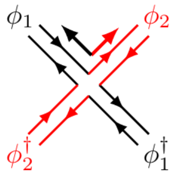
$$\begin{aligned}S^{\chi_0}=\mathrm{N}\int\ \ \mathrm{d}^4x\ \mathrm{tr}[&-\sum_{i=1}^{3}\ \partial_\mu\phi_i^\dagger\partial^\mu\phi^i-\sum_{i=2}^{3}\ \mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi^i\\&+\mathrm{i}\sqrt{\xi_2\xi_3}(\psi_3\phi_1\psi_2+\bar{\psi}_3\phi_1^\dagger\bar{\psi}_2)+\xi_2^2\phi_3\phi_1\phi_3^\dagger\phi_1^\dagger+\xi_3^2\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger]\end{aligned}$$

$$S^{\mathrm{FN}}=\mathrm{N}\int\ \ \mathrm{d}^4x\ \mathrm{tr}\left[-\sum_{i=1}^{2}\ \partial_\mu\phi_i^\dagger\partial^\mu\phi^i+\xi^2\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger\right]$$

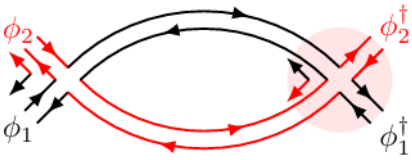


$= \frac{1}{\mathrm{N}} \frac{1}{k^2}$,

$= \frac{1}{\mathrm{N}} \frac{1}{k^2}$,



$= \mathrm{i}\,\mathrm{N}(4\pi)^2\xi^2$.



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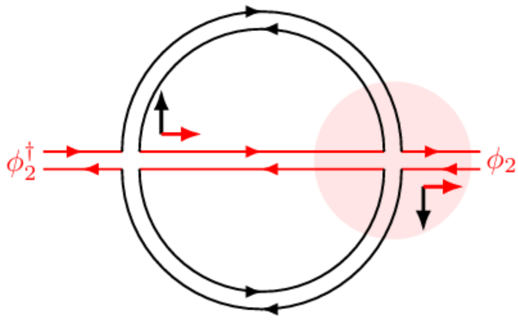

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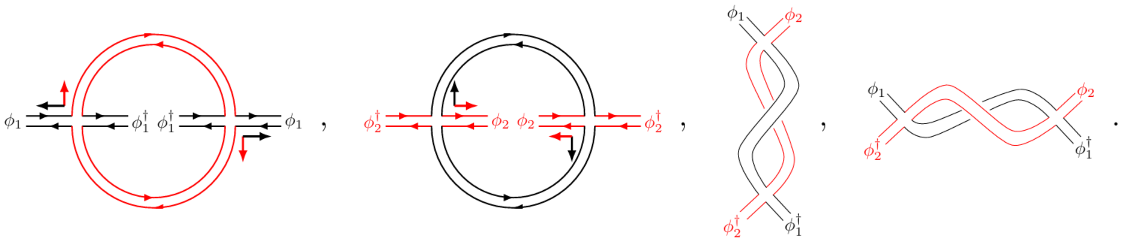
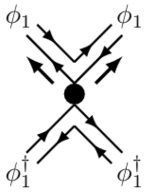
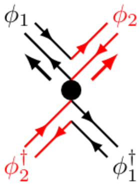


$$\begin{aligned}\mathcal{L}_{\mathrm{dt}} = & +\alpha_1^2 \mathrm{tr}(\phi_1\phi_1)\mathrm{tr}(\phi_1^\dagger\phi_1^\dagger) + \tilde{\alpha}_1^2 \mathrm{tr}(\phi_2\phi_2)\mathrm{tr}(\phi_2^\dagger\phi_2^\dagger) \\ & -\alpha_2^2 \mathrm{tr}(\phi_1\phi_2)\mathrm{tr}(\phi_1^\dagger\phi_2^\dagger) - \alpha_3^2 \mathrm{tr}(\phi_1\phi_2^\dagger)\mathrm{tr}(\phi_2\phi_1^\dagger).\end{aligned}$$

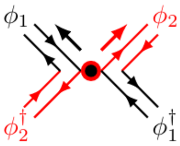


$\sim\ \mathrm{i}\cdot(4\pi)^2\,\alpha_1^2$,

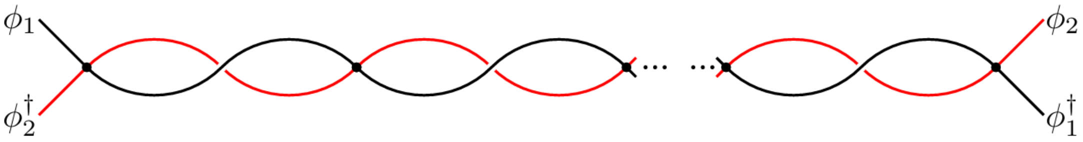
$\sim\ \mathrm{i}\cdot(4\pi)^2\,\tilde{\alpha}_1^2$,



$\sim\ \mathrm{i}\cdot(4\pi)^2\,\alpha_2^2$,



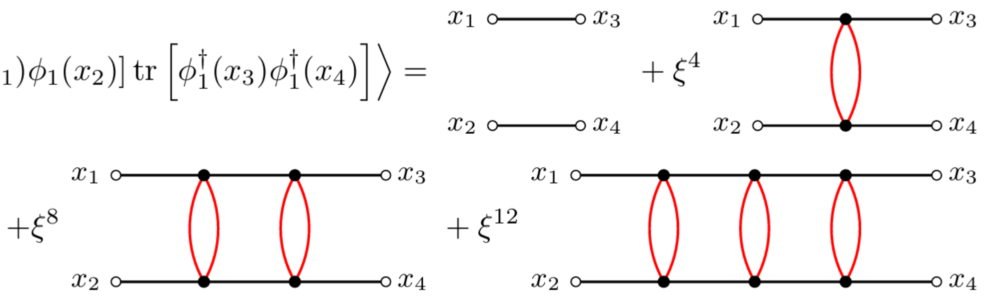
$\sim\ \mathrm{i}\cdot(4\pi)^2\,\alpha_3^2$.



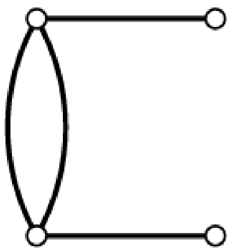
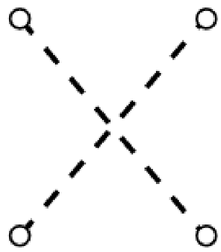
,

$$\alpha_{1*}^2 = \pm\frac{\mathrm{i}}{2}\xi^2 - \frac{\xi^4}{2} \mp \frac{3\mathrm{i}}{4}\xi^6 + \mathcal{O}(\xi^8).$$

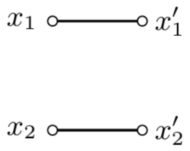
$$\left\langle \mathrm{tr}\left[\phi_1(x_1)\phi_1(x_2)\right]\mathrm{tr}\left[\phi_1^\dagger(x_3)\phi_1^\dagger(x_4)\right]\right\rangle =$$



$$+\cdots+(x_3\leftrightarrow x_4)+\mathcal{O}(\alpha_1^2)\ .$$

$\mathbb{H} =$  , $\mathbb{P} =$  ,

$$\left\langle \mathrm{tr}\left[\phi_1(x_1)\phi_1(x_2)\right]\mathrm{tr}\left[\phi_1^\dagger(x_3)\phi_1^\dagger(x_4)\right]\right\rangle$$

$=$ 

$$\circ\left[1+\xi^4\mathbb{H}+\xi^8\mathbb{H}\circ\mathbb{H}+\xi^{12}\mathbb{H}\circ\mathbb{H}\circ\mathbb{H}+\cdots\right]\left[1+\mathbb{P}\right]+\mathcal{O}(\alpha_1^2)$$

$$= x_{12}^4\mathbb{H}\circ\left[\frac{1+\mathbb{P}}{1-\xi^4\mathbb{H}}\right]+\mathcal{O}(\alpha_1^2)\ .$$

$$\delta^{(D)}(x_{13})\cdot\delta^{(D)}(x_{24}) = \frac{1}{[x_{12}^2]^{D-\Delta_1-\Delta_2}}\,\,\sum\!\!\!\!\!\!\int_{\Delta,S}\int \mathrm{d}^D x_0\ |\bar{\Omega}^{\Delta,S}_{\mu_1\cdots\mu_S}(x_1,x_2;x_0)\rangle\ \langle\Omega^{\mu_1\cdots\mu_S}_{\Delta,S}(x_3,x_4;x_0)|$$

$$\sum\!\!\!\!\!\!\int_{\Delta,S} = \frac{\mathrm{i}}{2}\sum_{S=0}^{\infty}(-1)^{S+1}\int_{\frac{D}{2}}^{\frac{D}{2}+\mathrm{i}\infty}\frac{\mathrm{d}\Delta}{c_1(\Delta,S)}$$

$$\Omega_{\Delta,S}^{\mu_1\cdots\mu_S}(x_1,x_2;x_0)=\frac{\prod_{i=0}^{S}\left[\frac{2x_{02}^{\mu_i}}{x_{02}^2}-\frac{2x_{01}^{\mu_i}}{x_{01}^2}\right]}{[x_{12}^2]^{\frac{\Delta_1+\Delta_2-(\Delta-S)}{2}}[x_{10}^2]^{\frac{\Delta_1-\Delta_2+(\Delta-S)}{2}}[x_{20}^2]^{\frac{-\Delta_1+\Delta_2+(\Delta-S)}{2}}}$$

$$\Omega_{\Delta,S}^{\mu_1\cdots\mu_S}(x_1,x_2;x_0)\overset{x_0\to\infty}{\sim}\Psi^{\mu_1\cdots\mu_S}_{\frac{\Delta_1+\Delta_2-\Delta}{2},\frac{S}{2}}(x_1,x_2):=W_u^{\frac{S}{2},\mu_1\cdots\mu_S}(x_{12})=$$  $${}_{\frac{\Delta_1+\Delta_2-\Delta}{2},\ \frac{S}{2}}$$

$$\int \mathrm{d}^D x_0\left|\bar{\Omega}^{\Delta,S}_{\mu_1\cdots\mu_S}(x_1,x_2;x_0)\right\rangle\left\langle\Omega^{\mu_1\cdots\mu_S}_{\Delta,S}(x_3,x_4;x_0)\right|$$
$$=\left(\frac{1}{x_{12}^2x_{34}^2}\right)^{\frac{\Delta_1+\Delta_2}{2}}\left(\frac{x_{24}^2}{x_{13}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left[\frac{c_1(\Delta,S)}{c_2(\Delta,S)}g_{\Delta,S}(r_1,r_2)+\frac{c_1(\Delta^*,S)}{c_2(\Delta^*,S)}g_{\Delta^*,S}(r_1,r_2)\right].$$

$$c_1(\Delta,S)=\frac{2^{S+1}S!}{\pi^{-\frac{3D}{2}-1}}\frac{(\Delta-1)(D-\Delta)}{\left(\frac{D}{2}+S-1\right)^2-\left(\Delta-\frac{D}{2}\right)^2}\frac{a_0(\Delta)a_0(D-\Delta)}{\Gamma\left(\frac{D}{2}+S\right)}$$
$$c_2(\Delta,S)=2\pi^{D+1}(-1)^S S!\frac{\Gamma\left(\Delta-\frac{D}{2}\right)\Gamma(S+\Delta-1)\Gamma\left(\frac{D+S-\Delta+\Delta_1-\Delta_2}{2}\right)\Gamma\left(\frac{D+S-\Delta-\Delta_1+\Delta_2}{2}\right)}{\Gamma(\Delta-1)\Gamma\left(\frac{D}{2}+S\right)\Gamma(D+S-\Delta)\Gamma\left(\frac{S+\Delta+\Delta_1-\Delta_2}{2}\right)\Gamma\left(\frac{S+\Delta-\Delta_1+\Delta_2}{2}\right)}$$

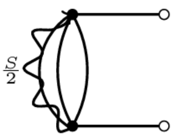
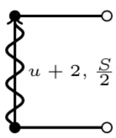
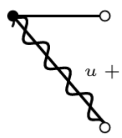
$$g_{\Delta,S}(r_1,r_2)=(-1)^S\frac{zz^*}{z-z^*}[h(\Delta+S,z)h(\Delta-S-2,z^*)-h(\Delta+S,z^*)h(\Delta-S-2,z)]$$
$$h(t,z):=z^{\frac{t}{2}}\,{}_2F_1\left(\frac{t-\Delta_1+\Delta_2}{2},\frac{t+\Delta_3-\Delta_4}{2},t,z\right)$$

$$r_1=zz^*=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\ \text{ and }\ r_2=(1-z)(1-z^*)=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$

$$\langle\mathrm{tr}[\phi_1(x_1)\phi_1(x_2)]\mathrm{tr}[\phi_1^\dagger(x_3)\phi_1^\dagger(x_4)]\rangle=G(x_1,x_2\mid x_3,x_4)+G(x_1,x_2\mid x_3,x_4)\circ\mathbb{P}$$

$$G(x_1,x_2\mid x_3,x_4)=\frac{1}{x_{12}^2x_{34}^2}\cdot\frac{\mathrm{i}}{2}\sum_{S=0}^{\infty}(-1)^{S+1}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty}\frac{\mathrm{d}\Delta}{c_2(\Delta,S)}\frac{E(\Delta,S)}{1-\xi^4E(\Delta,S)}g_{\Delta,S}(r_1,r_2).$$

$$\mathcal{G}(r_1,r_2)=\sum_{S,\Delta}C_{\Delta,S}g_{\Delta,S}(r_1,r_2)=\sum_{S,\Delta}(-1)^{S+1}\pi\mathrm{Res}_\Delta\left[\frac{1}{c_2(\Delta,S)}\frac{E(\Delta,S)}{1-\xi^4E(\Delta,S)}\right]g_{\Delta,S}(r_1,r_2),$$

$$\langle\Psi_{u,\frac{S}{2}}|\circ\mathbb{H}=\ {}_{u,\ \frac{S}{2}}$$  $$=$$  $${}_{u+2,\ \frac{S}{2}}=r_{\frac{S}{2}}(1,u+2,1-u)\cdot$$  $${}_{u+3-\frac{D}{2},\ \frac{S}{2}}$$

$$=r_{\frac{S}{2}}(1,u+2,1-u)\,r_{\frac{S}{2}}(1,u+1,2-u)\cdot$$  $${}_{u,\ \frac{S}{2}}$$

$$=\mathrm{E}(u,\tfrac{S}{2})\cdot\langle\Psi_{u,\frac{S}{2}}|\ .$$

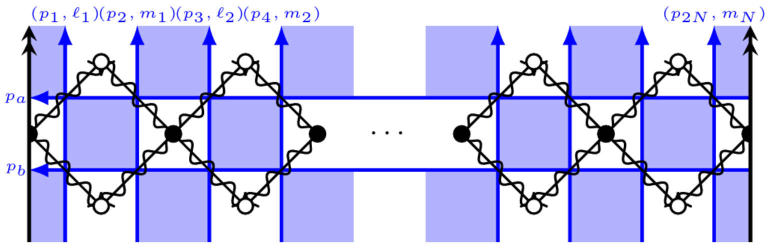
$$E(\Delta, S) = \frac{16\pi^4}{(-\Delta + S + 2)(-\Delta + S + 4)(\Delta + S - 2)(\Delta + S)}.$$

$$\Delta_2 = 2 \pm \sqrt{(S+1)^2 + 1 - 2\sqrt{(S+1)^2 + 4\pi^4\xi^4}},$$
$$\Delta_4 = 2 \pm \sqrt{(S+1)^2 + 1 + 2\sqrt{(S+1)^2 + 4\pi^4\xi^4}}$$

$$\Delta = 2 - 2i\xi^2 + i\xi^6 - \frac{7i\xi^{10}}{4} + \frac{33i\xi^{14}}{8} + \mathcal{O}(\xi^{18})$$

$$C_{\Delta,S} = \frac{4^{3-\Delta}(S+1)}{(S^2 + 2S - 2 - \Delta^2 + 4\Delta)} \frac{\Gamma\left(\frac{S-\Delta+5}{2}\right)\Gamma\left(\frac{S+\Delta}{2}\right)}{\Gamma\left(\frac{S-\Delta+4}{2}\right)\Gamma\left(\frac{S+\Delta-1}{2}\right)}$$

$$\mathbb{T}_N(p_a, p_b|\{\mathbf{p}_j\}) = \mathrm{tr}\left[\mathbb{R}(p_a, p_b|\mathbf{p}_1, \mathbf{p}_2)\mathbb{R}(p_a, p_b|\mathbf{p}_3, \mathbf{p}_4)\cdots\mathbb{R}(p_a, p_b|\mathbf{p}_{2N-1}, \mathbf{p}_{2N})\right]$$
$$= \mathbb{R}^{x_1' y_2}_{x_1 y_1}(p_a, p_b|\mathbf{p}_1, \mathbf{p}_2)\mathbb{R}^{x_2' y_3}_{x_2 y_2}(p_a, p_b|\mathbf{p}_3, \mathbf{p}_4)\cdots\mathbb{R}^{x_N' y_1}_{x_N y_N}(p_a, p_b|\mathbf{p}_{2N-1}, \mathbf{p}_{2N})$$

$=$ 

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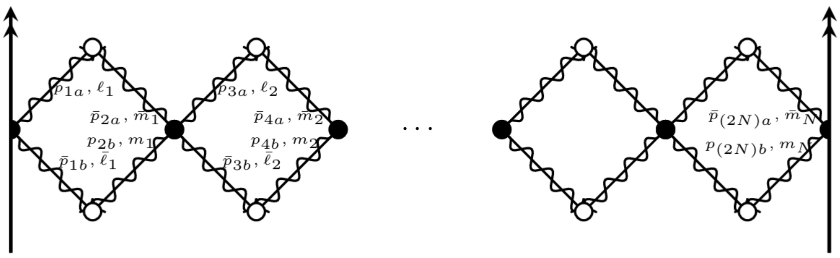
$$\mathbb{T}_N(p_a, p_b \mid \{\mathbf{p}_j\}) \circ \mathbb{T}_N(q_a, q_b \mid \{\mathbf{p}_j\}) = \mathbb{T}_N(q_a, q_b \mid \{\mathbf{p}_j\}) \circ \mathbb{T}_N(p_a, p_b \mid \{\mathbf{p}_j\}).$$

$\mathbb{T}_N(p_a, p_b|\{\mathbf{p}_j\}) =$ 

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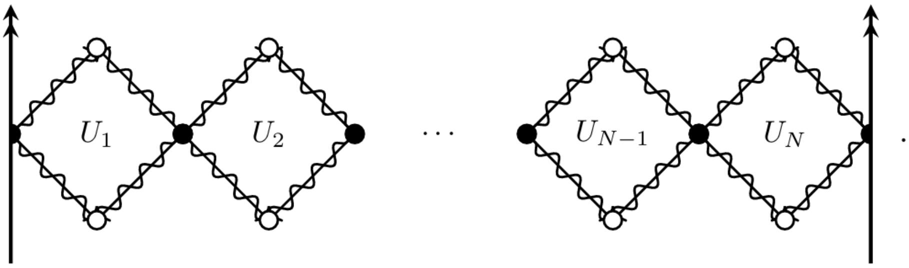
$$Z_{MN}\left(\{q_i\}_{i=1,\dots,2M} \mid \{\mathbf{p}_j\}_{j=1,\dots,2N}\right) = \mathrm{Tr}\left[\prod_{I=1}^{M} \mathbb{T}_N(q_{2I-1}, q_{2I} \mid \{\mathbf{p}_j\})\right].$$

$$U_J := \begin{pmatrix} \mathbf{u}_J^+ & \mathbf{v}_J^+ \\ \mathbf{u}_J^- & \mathbf{v}_J^- \end{pmatrix} = \begin{pmatrix} (u_J^+, \ell_J) & (v_J^+, \bar{m}_J) \\ (u_J^-, \bar{\ell}_J) & (v_J^-, m_J) \end{pmatrix} := \begin{pmatrix} (p_{(2J-1)a}, \ell_J) & (\bar{p}_{(2J)a}, \bar{m}_J) \\ (\bar{p}_{(2J-1)b}, \bar{\ell}_J) & (p_{(2J)b}, m_J) \end{pmatrix}$$

$$\mathbb{T}_N\left(\{U_J\}\right) =$$



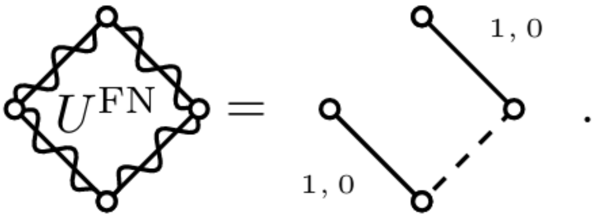
$$Z_{MN}(\{U_J\}) = \mathrm{Tr}\left[\mathbb{T}_N(\{U_J\})^M\right].$$

$$G_{\frac{D}{2}-u}(x) = c_0(\bar{u})W_u(x) \;\text{ with }\; c_\ell(v) = -(-1)^v 2^{\frac{D}{2}-2v} a_\ell(v).$$

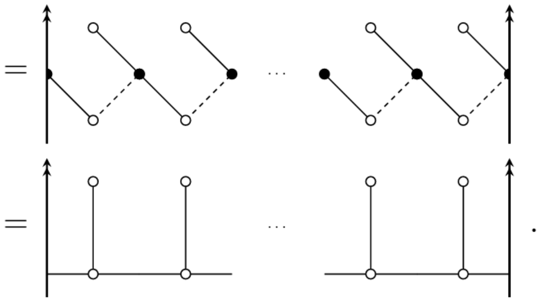
$$G_{\frac{D}{2}-u-\frac{1}{2},\alpha\dot{\beta}}(x) = c_{\frac{1}{2}}(\bar{u})W^{\frac{1}{2}}_{u,\alpha\dot{\beta}}(x)$$

$$U^{\mathrm{FN}} = \begin{pmatrix} (0,0) & (1,0) \\ (1,0) & (2-\varepsilon,0) \end{pmatrix}.$$

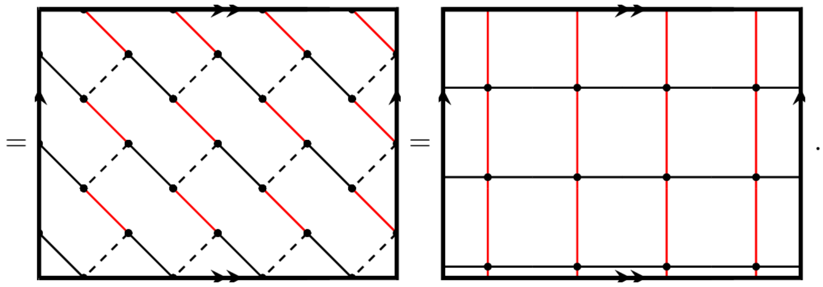
$$\lim_{\varepsilon\to 0} \pi^{-2} a_0(\varepsilon) \cdot$$



$$\hat{\mathbb{T}}^{\mathrm{FN}}_N := \lim_{\varepsilon\to 0}\left[\pi^{-2}a_0(\varepsilon)\right]^N \mathbb{T}_N\left(\{U^{\mathrm{FN}}\}\right) =$$

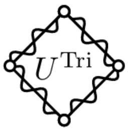


$$\hat{Z}^{\mathrm{FN}}_{34} := \mathrm{Tr}\left[\left(\hat{\mathbb{T}}^{\mathrm{FN}}_4\right)^3\right] =$$

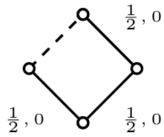


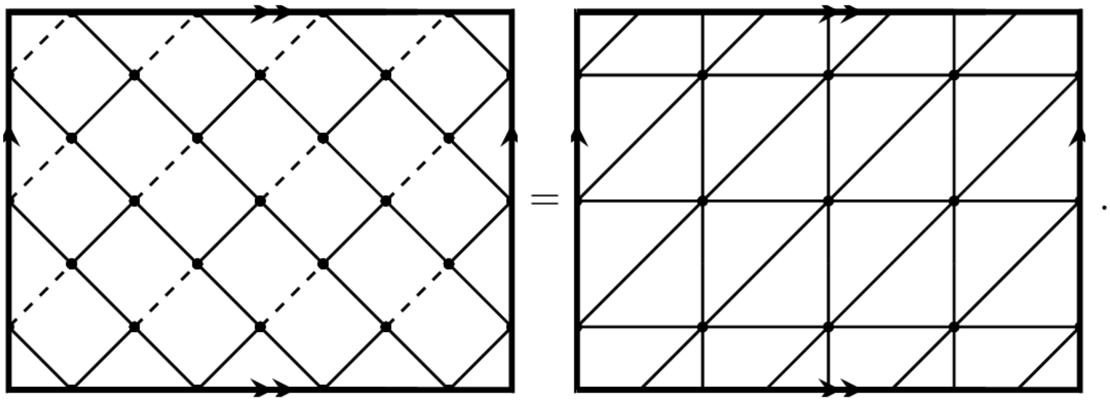
$$\mathcal{L}^{\mathrm{Tri}} = -\mathrm{N}\cdot \sum_{i=1,2,4} \mathrm{tr}\left[\partial_\mu Y_i^\dagger \partial^\mu Y^i\right] + \mathrm{N}\cdot\xi\cdot \mathrm{tr}\left[Y^1 Y_4^\dagger Y^2 Y_1^\dagger Y^4 Y_2^\dagger\right]$$

$$U^{\mathrm{Tri}} = \begin{pmatrix} (\frac{3}{2}-\varepsilon, 0) & (\frac{1}{2}, 0) \\ (\frac{1}{2}, 0) & (\frac{1}{2}, 0) \end{pmatrix} \quad \text{with} \quad \lim_{\varepsilon \to 0} \pi^{-\frac{3}{2}} a_0(\varepsilon) \cdot$$



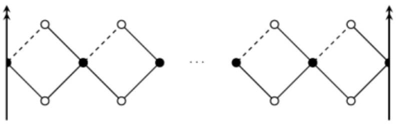
$$=$$



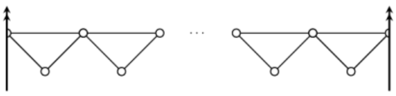


$$\hat{\mathbb{T}}_N^{\mathrm{Tri}} := \lim_{\varepsilon \to 0} \left[c_0(1)^3 \pi^{-\frac{3}{2}} a_0(\varepsilon) \right]^N \mathbb{T}_N \left(\{ U^{\mathrm{Tri}} \} \right)$$

$$= c_0(1)^{3N} \cdot$$

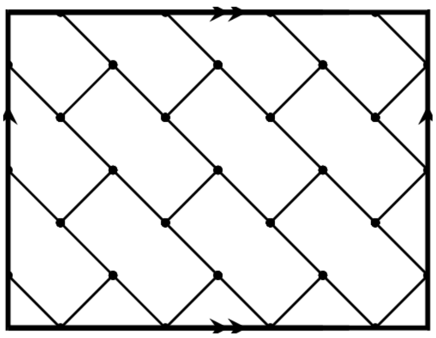


$$= c_0(1)^{3N} \cdot$$



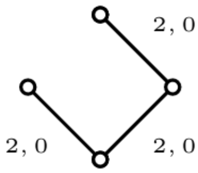
$$\mathcal{L}^{\mathrm{Hex}} = \mathrm{N} \cdot \sum_{i=1}^{3} \mathrm{tr}\left[\partial^{\mu} \phi_i^{\dagger} \partial_{\mu} \phi_i\right] + \mathrm{N} \cdot \rho \cdot \mathrm{tr}\left[\phi_1 \phi_2 \phi_3 + \phi_1^{\dagger} \phi_2^{\dagger} \phi_3^{\dagger}\right]$$

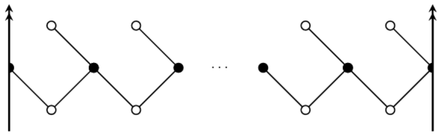
$$\hat{Z}_{34}^{\mathrm{Hex}} := \mathrm{Tr}[(\hat{\mathbb{T}}_4^{\mathrm{Hex}})^3] = (-2)^{3 \cdot 3 \cdot 4} \cdot c_0(1)^{3 \cdot 3 \cdot 4} \cdot$$



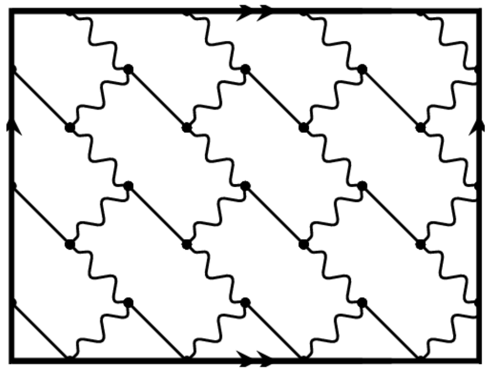
$$U^{\mathrm{Hex}} = \begin{pmatrix} (0,0) & (2,0) \\ (2,0) & (2,0) \end{pmatrix} \quad \text{with}$$

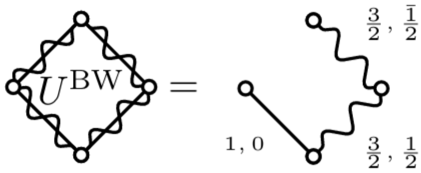
$$=$$



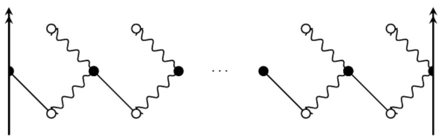
$$\hat{\mathbb{T}}_N^{\mathrm{Hex}} := \left(-2c_0(1)\right)^{3N} \mathbb{T}_N\left(\{U^{\mathrm{Hex}}\}\right) = \left(-2c_0(1)\right)^{3N} \cdot$$  ,

$$\mathcal{L}^{\mathrm{BW}} \;=\; \mathrm{N}\cdot\mathrm{tr}\left[-\frac{1}{2}\partial^\mu\phi^\dagger\partial_\mu\phi + \mathrm{i}\sum_{k=1}^{2}\bar{\psi}_k\partial\!\!\!/\psi_k + \rho\cdot\left(\psi_1\phi\psi_2 + \bar{\psi}_1\phi^\dagger\bar{\psi}_2\right)\right] \;.$$

$$\hat{Z}_{34}^{\mathrm{BW}} = (-1)^{2\cdot3\cdot4}(2)^{3\cdot4}c_{\frac{1}{2}}(\tfrac{3}{2})^{2\cdot3\cdot4}\cdot$$  .

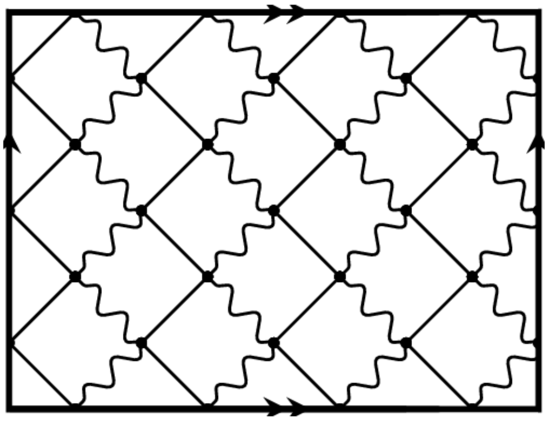
$$U^{\mathrm{BW}} = \begin{pmatrix}(0,0) & (\frac{3}{2},\frac{\bar{1}}{2})\\ (1,0) & (\frac{3}{2},\frac{1}{2})\end{pmatrix} \quad \text{with}$$ 

.

$$\hat{\mathbb{T}}_N^{\mathrm{BW}} := \left(2c_{\frac{1}{2}}(\tfrac{3}{2})^2\right)^N \mathbb{T}_N\left(\{U^{\mathrm{BW}}\}\right) = 2^N c_{\frac{1}{2}}(\tfrac{3}{2})^{2N}\cdot$$  ,

$$\mathcal{L}^{\mathrm{FFN}} \;=\; -\mathrm{N}\cdot\mathrm{tr}\left[\sum_{j=1}^{2}\partial^\mu Y_j^\dagger\partial_\mu Y_j \;+\mathrm{i}\sum_{k=1}^{2}\Psi_k^\dagger\partial\!\!\!/\Psi_k + \rho\cdot\left(Y_1\,Y_2^\dagger\Psi_1\Psi_2^\dagger - Y_1^\dagger Y_2\,\Psi_1^\dagger\Psi_2\right)\right]$$

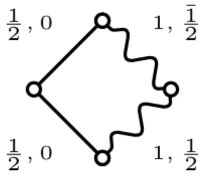
$$\hat{Z}^{\mathrm{FFN}}_{34} = c_{\frac{1}{2}}(1)^{2\cdot 3\cdot 4} c_0(\tfrac{1}{2})^{2\cdot 3\cdot 4} \cdot$$



.

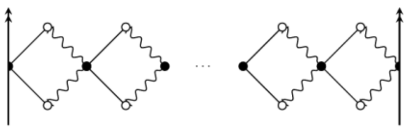
$$U^{\mathrm{FFN}} = \begin{pmatrix} (\frac{1}{2},0) & (1,\bar{\frac{1}{2}}) \\ (\frac{1}{2},0) & (1,\frac{1}{2}) \end{pmatrix} \quad \text{with}$$

=



.

$$\hat{\mathbb{T}}^{\mathrm{FFN}}_N := \left(c_0(1)^2 c_{\frac{1}{2}}(\tfrac{1}{2})^2\right)^N \mathbb{T}_N\left(\{U^{\mathrm{FFN}}\}\right) = \left(c_0(1) c_{\frac{1}{2}}(\tfrac{1}{2})\right)^{2N} \cdot$$



$$\hat{Z} = \sum_{M,N=1}^{\infty} \hat{Z}^{\mathrm{QFT}}_{MN}\, \xi^{b\cdot MN}$$

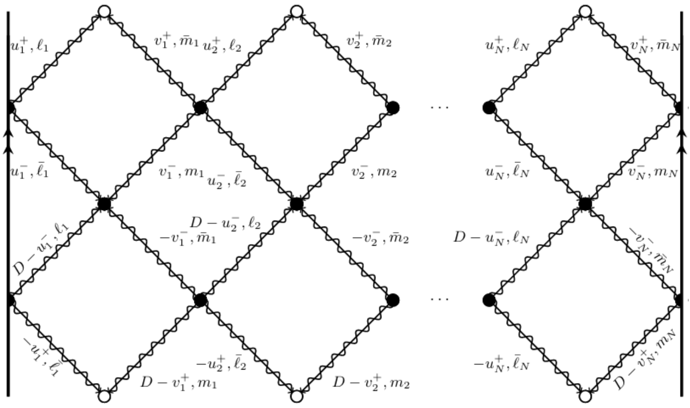
$$\hat{K}^{\mathrm{QFT}} := \lim_{M,N\to\infty} \left|\hat{Z}^{\mathrm{QFT}}_{MN}\right|^{\frac{1}{MN}}.$$

$$\xi_{\mathrm{cr}} = \frac{1}{\left(\hat{K}^{\mathrm{QFT}}\right)^b}$$

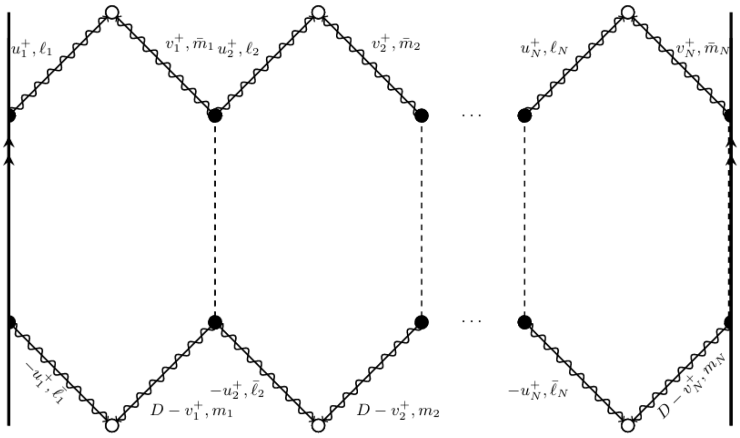
$$K(\{U_J\}) := \lim_{M,N\to\infty} \left|Z_{MN}(\{U_J\})\right|^{\frac{1}{MN}}.$$

$$\begin{aligned} K(\{U_J\}) &= \lim_{M,N\to\infty} \left|\Lambda_{N,\max}(\{U_J\})^M \left[1 + \sum_{i\neq \max} \left(\frac{\Lambda_{N,i}(\{U_J\})}{\Lambda_{N,\max}(\{U_J\})}\right)^M\right]\right|^{\frac{1}{MN}} \\ &= \lim_{M,N\to\infty} \left|\Lambda_{N,\max}(\{U_J\})\right|^{\frac{1}{N}} \end{aligned}$$

$$\mathbb{T}_N(\{U_J\}) \circ \mathbb{T}_N(\{U'_J\}) =$$

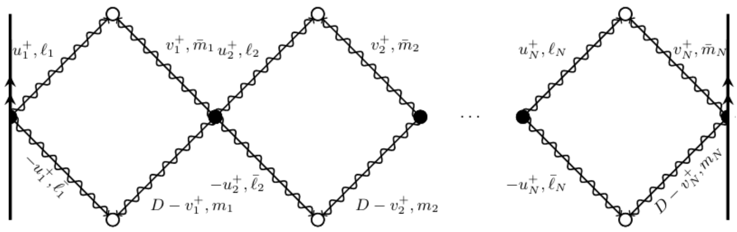


$=$



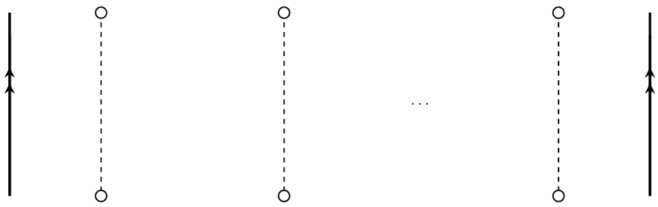
$$\cdot \prod_{J=1}^{N} A_{\ell_J}(u_J^-) \cdot \mathbb{I}^{(\ell_J)} \mathbb{I}^{(m_J)}$$

$=$



$$\cdot \prod_{J=1}^{N} A_{\ell_J}(u_J^-) \cdot \mathbb{I}^{(\ell_J)} \mathbb{I}^{(m_J)}$$

$=$



$$\cdot \prod_{J=1}^{N} A_{\ell_J}(u_J^-)\, A_{m_J}(v_J^+) \cdot \mathbb{I}^{(\ell_J)} \mathbb{I}^{(m_J)}$$

$$= \mathbb{1}_N \cdot \prod_{J=1}^{N} A_{\ell_J}(u_J^-) A_{m_J}(v_J^+) \cdot \mathbb{I}^{(\ell_J)} \mathbb{I}^{(m_J)}$$

$$A_\ell(u) := \pi^D a_\ell(u) a_\ell(D-u)$$

$$U_J = \begin{pmatrix} (u_J^+, \ell_J) & (v_J^+, \bar{m}_J) \\ (u_J^-, \bar{\ell}_J) & (v_J^-, m_J) \end{pmatrix} \text{ and } U_{J,\,\text{inv}} = \begin{pmatrix} (D - u_J^-, \ell_J) & (-v_J^-, \bar{m}_J) \\ (-u_J^+, \bar{\ell}_J) & (D - v_J^+, m_J) \end{pmatrix}.$$

$$\mathbb{T}_N\left(\left\{\begin{pmatrix}(-u_J^-,\ell_J) & (D-v_J^-,\bar{m}_J)\\(D-u_J^+,\bar{\ell}_J) & (-v_J^+,m_J)\end{pmatrix}\right\}\right)\cdot\frac{1}{\prod_{J=1}^N A_{\ell_J}(u_J^+)A_{m_J}(v_J^-)},$$
$$\mathbb{T}_N\left(\left\{\begin{pmatrix}(D-u_J^-,\ell_J) & (-v_J^-,\bar{m}_J)\\(D-u_J^+,\bar{\ell}_J) & (-v_J^+,m_J)\end{pmatrix}\right\}\right)\cdot\frac{1}{\prod_{J=1}^N A_{\ell_J}(u_J^-)A_{\ell_J}(u_J^+)},$$
$$\mathbb{T}_N\left(\left\{\begin{pmatrix}(-u_J^-,\ell_J) & (D-v_J^-,\bar{m}_J)\\(-u_J^+,\bar{\ell}_J) & (D-v_J^+,m_J)\end{pmatrix}\right\}\right)\cdot\frac{1}{\prod_{J=1}^N A_{m_J}(v_J^-)A_{m_J}(v_J^+)}.$$

$$K(\{U_J\})\cdot K(\{U_{J,\,\mathrm{inv}}\})=F(\{U_J\})$$

$$U_{J,\,\mathrm{inv}}=\begin{pmatrix}(D-u_J^-,\ell_J) & (-v_J^-,\bar{m}_J)\\(-u_J^+,\bar{\ell}_J) & (D-v_J^+,m_J)\end{pmatrix}\ \text{with}\ F(\{U_J\})$$

$$\lim_{N\to\infty}\left|\prod_{J=1}^N A_{\ell_J}(u_J^-)A_{m_J}(v_J^+)\right|^{\frac{1}{N}},$$

$$U_{J,\,\mathrm{inv}}=\begin{pmatrix}(-u_J^-,\ell_J) & (D-v_J^-,\bar{m}_J)\\(D-u_J^+,\bar{\ell}_J) & (-v_J^+,m_J)\end{pmatrix}\ \text{with}\ F(\{U_J\})$$

$$\lim_{N\to\infty}\left|\prod_{J=1}^N A_{\ell_J}(u_J^+)A_{m_J}(v_J^-)\right|^{\frac{1}{N}},$$

$$U_{J,\,\mathrm{inv}}=\begin{pmatrix}(D-u_J^-,\ell_J & (-v_J^-,\bar{m}_J)\\(D-u_J^+,\bar{\ell}_J) & (-v_J^+,m_J)\end{pmatrix}\ \text{with}\ F(\{U_J\})$$

$$\lim_{N\to\infty}\left|\prod_{J=1}^N A_{\ell_J}(u_J^-)A_{\ell_J}(u_J^+)\right|^{\frac{1}{N}},$$

$$U_{J,\,\mathrm{inv}}=\begin{pmatrix}(-u_J^-,\ell_J) & (D-v_J^-,\bar{m}_J)\\(-u_J^+,\bar{\ell}_J) & (D-v_J^+,m_J)\end{pmatrix}\ \text{with}\ F(\{U_J\})=\lim_{N\to\infty}\left|\prod_{J=1}^N A_{m_J}(v_J^-)A_{m_J}(v_J^+)\right|^{\frac{1}{N}}.$$

$$K\left(\left\{\begin{pmatrix}(u_J^-,\ell_J) & (v_J^-,\bar{m}_J)\\(u_J^+,\bar{\ell}_J) & (v_J^+,m_J)\end{pmatrix}\right\}\right)=\lim_{N\to\infty}\left|\prod_{J=1}^N \kappa_{\ell_J}(u_J^+)\kappa_{\ell_J}(u_J^-)\kappa_{m_J}(v_J^+)\kappa_{m_J}(v_J^-)\right|^{\frac{1}{N}}.$$

$$\kappa_\ell(u)\kappa_\ell(-u)=1$$
$$\kappa_\ell(u)\kappa_\ell(D-u)=A_\ell(u)=\pi^D a_\ell(u)a_\ell(D-u).$$

$$\kappa_\ell(u)=\pi^u\frac{\Gamma\left(\frac{D}{2}-u+\ell\right)}{\Gamma\left(\frac{D}{2}+\ell\right)}\prod_{k=1}^{\infty}\frac{\Gamma\left(Dk+\frac{D}{2}-u+\ell\right)\Gamma(Dk+u+\ell)\Gamma\left(Dk-\frac{D}{2}+\ell\right)}{\Gamma\left(Dk-\frac{D}{2}+u+\ell\right)\Gamma(Dk-u+\ell)\Gamma\left(Dk+\frac{D}{2}+\ell\right)}.$$

$$K(U)=K\begin{pmatrix}(u^-,\ell) & (v^-,\bar{m})\\(u^+,\bar{\ell}) & (v^+,m)\end{pmatrix}=|\kappa_\ell(u^+)\kappa_\ell(u^-)\kappa_m(v^+)\kappa_m(v^-)|$$

$$\hat{K}^{\mathrm{FN}} = \left|\lim_{\varepsilon\to 0}\pi^{-2}a_0(\varepsilon)\kappa_0(2-\varepsilon)\kappa_0(0)\kappa_0(1)^2\right| \overset{D=4}{=} \frac{\pi^3}{16}\frac{\Gamma\left(\frac{1}{4}\right)^2}{\Gamma\left(\frac{3}{4}\right)^2}$$

$$\hat{K}^{\mathrm{Tri}} = \left|\sqrt{\frac{\pi}{2}}^3 \cdot \lim_{\varepsilon\to 0}\pi^{-\frac{3}{2}}a_0(\varepsilon)\kappa_0\left(\frac{3}{2}-\varepsilon\right)\kappa_0(1)^3\right| \overset{D=3}{=} \frac{\pi^3}{54\sqrt{2}}\frac{\Gamma\left(\frac{1}{6}\right)^3}{\Gamma\left(\frac{2}{3}\right)^3}$$

$$\hat{K}^{\mathrm{Hex}} = |(-2)^3 2^3 \cdot \kappa_0(0)\kappa_0(2)^3| \overset{D=6}{=} \frac{8\pi^{15/2}}{27}\frac{\Gamma\left(\frac{1}{3}\right)^3}{\Gamma\left(\frac{5}{6}\right)^3}$$

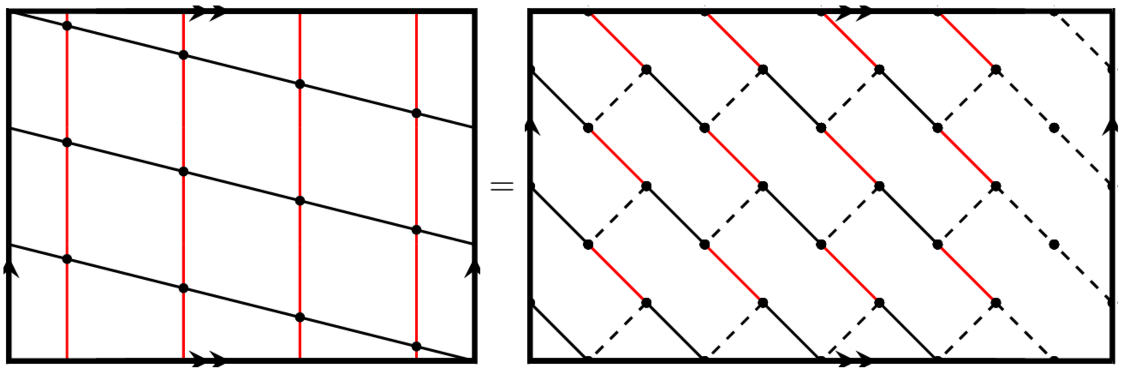
$$\hat{K}^{\mathrm{BW}} = \left|2\cdot(-2\mathrm{i})^2\cdot\kappa_0(0)\kappa_0(1)\kappa_{\frac{1}{2}}\left(\frac{3}{2}\right)^2\right| \overset{D=4}{=} \frac{\pi^{11/2}}{2}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\hat{K}^{\mathrm{FFN}} = \left|\sqrt{\frac{\pi}{2}}^2\left(-\mathrm{i}\sqrt{\frac{\pi}{2}}\right)^2\cdot\kappa_0\left(\frac{1}{2}\right)^2\kappa_{\frac{1}{2}}(1)^2\right| \overset{D=3}{=} \frac{\pi^5}{27}\frac{\Gamma\left(\frac{1}{6}\right)^2}{\Gamma\left(\frac{2}{3}\right)^2}$$

$$\left(\xi^{\mathrm{FN}}\right)^2_{\mathrm{cr}} \overset{b=1}{=} \frac{32}{\pi\Gamma\left(\frac{1}{4}\right)^4} = \frac{2}{\pi^4\eta(\mathrm{i})^4}, \qquad \rho^{\mathrm{BW}}_{\mathrm{cr}} \overset{b=2}{=} \frac{8}{\pi^9\Gamma\left(\frac{1}{4}\right)^4} = \frac{1}{2\pi^{12}\eta(\mathrm{i})^4}$$

$$\xi^{\mathrm{Tri}}_{\mathrm{cr}} \overset{b=1}{=} \frac{1443^{1/4}}{\pi^{3/4}\Gamma\left(\frac{1}{6}\right)^{9/2}} = \frac{3^{3/4}}{\sqrt{2}\pi^{9/2}\left|\eta\left(\mathrm{e}^{\frac{i\pi}{3}}\right)\right|^6}, \qquad \rho^{\mathrm{FFN}}_{\mathrm{cr}} \overset{b=2}{=} \frac{19442^{1/3}}{\pi^7\Gamma\left(\frac{1}{6}\right)^6} = \frac{27}{2^{8/3}\pi^{12}\left|\eta\left(\mathrm{e}^{\frac{i\pi}{3}}\right)\right|^8}$$

$$\rho^{\mathrm{Hex}}_{\mathrm{cr}} \overset{b=2}{=} \frac{2187\sqrt{3}}{2\pi^{21/2}\Gamma\left(\frac{1}{6}\right)^9} = \frac{81\sqrt{3}}{1024\pi^{18}\left|\eta\left(\mathrm{e}^{\frac{i\pi}{3}}\right)\right|^{12}}.$$



$$U' = \begin{pmatrix} (0,0) & \left(\frac{D}{2}-\varepsilon, 0\right) \\ \left(\frac{D}{2}-\varepsilon, 0\right) & (0,0) \end{pmatrix}$$

$$K(U,U') = K(U)\cdot K(U') = K(U)\cdot \lim_{N\to\infty}\lim_{\varepsilon\to 0}\left|\pi^{-D}a_0(\varepsilon)^2\kappa_0\left(\frac{D}{2}-\varepsilon\right)^2\kappa_0(0)\right|^{\frac{P}{N}}$$

$$\lim_{\varepsilon\to 0}\pi^{-\frac{D}{2}}a_0(\varepsilon)\kappa_0\left(\frac{D}{2}-\varepsilon\right)=1$$

$$S_{\mathrm{ABJM}}=-\mathrm{i}\frac{k}{\lambda}\cdot S_{\mathrm{CS}}+S_{\mathrm{mat}}+\frac{\lambda}{k}\cdot S_{\mathrm{pot}}$$

$$S_{\mathrm{CS}}[\mathcal{V},\hat{\mathcal{V}}]=\int\ \mathrm{d}^3x\ \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\int_0^1\ \mathrm{d}t\mathrm{tr}\left[\mathcal{V}\bar{D}^{\alpha}\left(\mathrm{e}^{t\mathcal{V}}D_{\alpha}\mathrm{e}^{-t\mathcal{V}}\right)-\hat{\mathcal{V}}\bar{D}^{\alpha}\left(\mathrm{e}^{t\hat{\mathcal{V}}}D_{\alpha}\mathrm{e}^{-t\hat{\mathcal{V}}}\right)\right]$$

$$\mathcal{V}=2\mathrm{i}\theta\bar{\theta}\sigma(x)+2\theta\gamma^{\mu}\bar{\theta}A_{\mu}(x)+\sqrt{2}\mathrm{i}\theta^2\bar{\theta}\bar{\chi}(x)-\sqrt{2}\mathrm{i}\bar{\theta}^2\theta\chi(x)+\theta^2\bar{\theta}^2\mathrm{D}(x)$$

$$S_{\mathrm{mat}}\left[\mathcal{Z},\mathcal{W},\bar{\mathcal{Z}},\bar{\mathcal{W}},\mathcal{V},\hat{\mathcal{V}}\right]=\int\ \mathrm{d}^3x\ \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\mathrm{tr}\left[-\bar{\mathcal{Z}}_A\mathrm{e}^{-\mathcal{V}}\mathcal{Z}^A\mathrm{e}^{\hat{\mathcal{V}}}-\bar{\mathcal{W}}^A\mathrm{e}^{-\hat{\mathcal{V}}}\mathcal{W}_A\mathrm{e}^{\mathcal{V}}\right]$$

$$\begin{aligned}\mathcal{Z}^A&=Z^A(x_+)+\sqrt{2}\theta\zeta^A(x_+)+\theta^2F^A(x_+)\\ \mathcal{W}_A&=W_A(x_+)+\sqrt{2}\theta\omega_A(x_+)+\theta^2G_A(x_+)\end{aligned}$$

$$\begin{aligned}\bar{\mathcal{Z}}_A&=Z_A^{\dagger}(x_-)-\sqrt{2}\bar{\theta}\bar{\zeta}_A(x_-)-\bar{\theta}^2F_A^{\dagger}(x_-)\\ \bar{\mathcal{W}}^A&=W^{\dagger A}(x_-)-\sqrt{2}\bar{\theta}\bar{\omega}^A(x_-)-\bar{\theta}^2G^{\dagger A}(x_-)\end{aligned}$$

$$\begin{aligned}S_{\mathrm{pot}}[\mathcal{Z},\mathcal{W},\bar{\mathcal{Z}},\bar{\mathcal{W}}]=&\int\ \mathrm{d}^3x\ \mathrm{d}^2\theta\frac{1}{4}\varepsilon_{AC}\varepsilon^{BD}\mathrm{tr}[\mathcal{Z}^A\mathcal{W}_B\mathcal{Z}^C\mathcal{W}_D]\\ &+\int\ \mathrm{d}^3x\ \mathrm{d}^2\bar{\theta}\frac{1}{4}\varepsilon^{AC}\varepsilon_{BD}\mathrm{tr}\left[\bar{\mathcal{Z}}_A\bar{\mathcal{W}}^B\bar{\mathcal{Z}}_C\bar{\mathcal{W}}^D\right]\end{aligned}$$

$$\begin{aligned}&\mathcal{Z}^A\to\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha}\mathcal{Z}^A\left(x,\mathrm{e}^{\mathrm{i}\alpha}\theta,\mathrm{e}^{-\mathrm{i}\alpha}\bar{\theta}\right),\quad \mathcal{W}_A\to\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha}\mathcal{W}_A\left(x,\mathrm{e}^{\mathrm{i}\alpha}\theta,\mathrm{e}^{-\mathrm{i}\alpha}\bar{\theta}\right),\\ &\bar{\mathcal{Z}}_A\to\mathrm{e}^{\frac{\mathrm{i}}{2}\alpha}\bar{\mathcal{Z}}_A\left(x,\mathrm{e}^{\mathrm{i}\alpha}\theta,\mathrm{e}^{-\mathrm{i}\alpha}\bar{\theta}\right),\qquad \bar{\mathcal{W}}^A\to\mathrm{e}^{\frac{\mathrm{i}}{2}\alpha}\bar{\mathcal{W}}^A\left(x,\mathrm{e}^{\mathrm{i}\alpha}\theta,\mathrm{e}^{-\mathrm{i}\alpha}\bar{\theta}\right).\end{aligned}$$

$$S'_{\mathrm{ABJM}}=-\mathrm{i}\frac{k}{\lambda}\cdot S_{\mathrm{CS}}[\lambda\mathcal{V},\lambda\hat{\mathcal{V}}]+\mathrm{N}\cdot S_{\mathrm{mat}}[\mathcal{Z},\mathcal{W},\bar{\mathcal{Z}},\bar{\mathcal{W}},\lambda\mathcal{V},\lambda\hat{\mathcal{V}}]+\mathrm{N}\lambda^2\cdot S_{\mathrm{pot}}\left[\mathcal{Z},\mathcal{W},\bar{\mathcal{Z}},\bar{\mathcal{W}}\right].$$

$$\Phi_1\cdots\Phi_p\to\mathrm{e}^{-\frac{\mathrm{i}}{2}\sum_{m>n}^{p}\beta\cdot\varepsilon_{ij}q_{\Phi_m}^{i}q_{\Phi_n}^{j}}\Phi_1\cdots\Phi_p$$

$$\begin{aligned}&\mathrm{tr}[\mathcal{Z}^1\mathcal{W}_2\mathcal{Z}^2\mathcal{W}_1-\mathcal{Z}^1\mathcal{W}_1\mathcal{Z}^2\mathcal{W}_2]\to\mathrm{tr}[q\cdot\mathcal{Z}^1\mathcal{W}_2\mathcal{Z}^2\mathcal{W}_1-q^{-1}\cdot\mathcal{Z}^1\mathcal{W}_1\mathcal{Z}^2\mathcal{W}_2]\\ &\mathrm{tr}\left[\bar{\mathcal{Z}}_1\bar{\mathcal{W}}^2\bar{\mathcal{Z}}_2\bar{\mathcal{W}}^1-\bar{\mathcal{Z}}_1\bar{\mathcal{W}}^1\bar{\mathcal{Z}}_2\bar{\mathcal{W}}^2\right]\to\mathrm{tr}\left[q\cdot\bar{\mathcal{Z}}_1\bar{\mathcal{W}}^2\bar{\mathcal{Z}}_2\bar{\mathcal{W}}^1-q^{-1}\cdot\bar{\mathcal{Z}}_1\bar{\mathcal{W}}^1\bar{\mathcal{Z}}_2\bar{\mathcal{W}}^2\right]\end{aligned}$$

$$\begin{aligned}S'_{\beta,\,\mathrm{ABJM}}=&-\mathrm{i}\frac{k}{\lambda}\cdot S_{\mathrm{CS}}[\lambda\mathcal{V},\lambda\hat{\mathcal{V}}]+\mathrm{N}\cdot S_{\mathrm{mat}}\left[\mathcal{Z},\mathcal{W},\bar{\mathcal{Z}},\bar{\mathcal{W}},\lambda\mathcal{V},\lambda\hat{\mathcal{V}}\right]\\ &+\mathrm{N}\lambda^2\int\ \mathrm{d}^3x\ \mathrm{d}^2\theta\frac{1}{2}\mathrm{tr}[q\cdot\mathcal{Z}^1\mathcal{W}_2\mathcal{Z}^2\mathcal{W}_1-q^{-1}\cdot\mathcal{Z}^1\mathcal{W}_1\mathcal{Z}^2\mathcal{W}_2]\\ &+\mathrm{N}\lambda^2\int\ \mathrm{d}^3x\ \mathrm{d}^2\bar{\theta}\frac{1}{2}\mathrm{tr}\left[q\cdot\bar{\mathcal{Z}}_1\bar{\mathcal{W}}^2\bar{\mathcal{Z}}_2\bar{\mathcal{W}}^1-q^{-1}\cdot\bar{\mathcal{Z}}_1\bar{\mathcal{W}}^1\bar{\mathcal{Z}}_2\bar{\mathcal{W}}^2\right]\end{aligned}$$

$$\begin{aligned}S_{\mathrm{SFN}}=\mathrm{N}\int\ \mathrm{d}^3x\ \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\mathrm{tr}\left[-\bar{\mathcal{Z}}_A\mathcal{Z}^A-\bar{\mathcal{W}}^A\mathcal{W}_A\right]\\ +\mathrm{N}\xi\int\ \mathrm{d}^3x\ \mathrm{d}^2\theta\mathrm{tr}[\mathcal{Z}^1\mathcal{W}_2\mathcal{Z}^2\mathcal{W}_1]+\mathrm{N}\xi\int\ \mathrm{d}^3x\ \mathrm{d}^2\bar{\theta}\mathrm{tr}\left[\bar{\mathcal{Z}}_1\bar{\mathcal{W}}^2\bar{\mathcal{Z}}_2\bar{\mathcal{W}}^1\right]\end{aligned}$$

$$F^1 = \xi W^{\dagger 2} Z_2^\dagger W^{\dagger 1}, \quad G_1 = \xi Z_1^\dagger W^{\dagger 2} Z_2^\dagger, \quad F_1^\dagger = -\xi W_2 Z^2 W_1, \quad G^{\dagger 1} = -\xi Z^1 W_2 Z^2$$
$$F^2 = \xi W^{\dagger 1} Z_1^\dagger W^{\dagger 2}, \quad G_2 = \xi Z_2^\dagger W^{\dagger 1} Z_1^\dagger, \quad F_2^\dagger = -\xi W_1 Z^1 W_2, \quad G^{\dagger 2} = -\xi Z^2 W_1 Z^1$$

$$\begin{aligned}
S_{\mathrm{SFN}} = \mathrm{N}\int & \mathrm{d}^3x \mathrm{tr}\Big\{ Z_A^\dagger \,\Box\, Z^A + W^{\dagger A} \,\Box\, W_A + \mathrm{i}\bar{\zeta}_A^{\alpha} \gamma^{\mu}_{\alpha\beta} \partial_\mu \zeta^{A\beta} + \mathrm{i}\bar{\omega}^{A\alpha} \gamma^{\mu}_{\alpha\beta} \partial_\mu \omega_A^{\beta} \\
& + \xi^2 \big[Z^1 W_2 Z^2 Z_1^\dagger W^{\dagger 2} Z_2^\dagger + W_1 Z^1 W_2 W^{\dagger 1} Z_1^\dagger W^{\dagger 2} + Z^2 W_1 Z^1 Z_2^\dagger W^{\dagger 1} Z_1^\dagger + W_2 Z^2 W_1 W^{\dagger 2} Z_2^\dagger W^{\dagger 1} \big] \\
& - \xi \big[\zeta^1 \omega_2 Z^2 W_1 + \omega_2 \zeta^2 W_1 Z^1 + \zeta^2 \omega_1 Z^1 W_2 - \bar{\omega}^1 \bar{\zeta}_1 W^{\dagger 2} Z_2^\dagger \big] \\
& - \xi \big[\bar{\zeta}_1 \bar{\omega}^2 Z_2^\dagger W^{\dagger 1} + \bar{\omega}^2 \bar{\zeta}_2 W^{\dagger 1} Z_1^\dagger + \bar{\zeta}_2 \bar{\omega}^1 Z_1^\dagger W^{\dagger 2} - \omega_1 \zeta^1 W_2 Z^2 \big] \\
& - \xi \big[\zeta^1 W_2 \zeta^2 W_1 + \omega_2 Z^2 \omega_1 Z^1 + \bar{\zeta}_1 W^{\dagger 2} \bar{\zeta}_2 W^{\dagger 1} + \bar{\omega}^2 Z_2^\dagger \bar{\omega}^1 Z_1^\dagger \big] \Big\}
\end{aligned}$$

$$Y^M = \begin{pmatrix} Z^1 \\ Z^2 \\ W^{\dagger 1} \\ W^{\dagger 2} \end{pmatrix}, Y_M^\dagger = \begin{pmatrix} Z_1^\dagger \\ Z_2^\dagger \\ W_1 \\ W_2 \end{pmatrix}, \Psi_M = \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} \begin{pmatrix} -\zeta^2 \\ \zeta^1 \\ \mathrm{i}\bar{\omega}^2 \\ -\mathrm{i}\bar{\omega}^1 \end{pmatrix}, \bar{\Psi}^M = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \begin{pmatrix} -\bar{\zeta}_2 \\ \bar{\zeta}_1 \\ -\mathrm{i}\omega_2 \\ \mathrm{i}\omega_1 \end{pmatrix}.$$

$$\begin{aligned}
S_{\mathrm{SFN}} = \mathrm{N}\int & \mathrm{d}^3x \mathrm{tr}\Big\{ Y_M^\dagger \,\Box\, Y^M + \mathrm{i}\bar{\Psi}^{M\alpha} \gamma^{\mu}_{\alpha\beta} \partial_\mu \Psi_M^{\beta} \\
& + \xi^2 \big[Y^1 Y_4^\dagger Y^2 Y_1^\dagger Y^4 Y_2^\dagger + Y^1 Y_4^\dagger Y^3 Y_1^\dagger Y^4 Y_3^\dagger + Y^2 Y_3^\dagger Y^1 Y_2^\dagger Y^3 Y_1^\dagger + Y^2 Y_3^\dagger Y^4 Y_2^\dagger Y^3 Y_4^\dagger \big] \\
& - \mathrm{i}\xi \big[\Psi_2 \bar{\Psi}^3 Y^2 Y_3^\dagger - \bar{\Psi}^3 \Psi_1 Y_3^\dagger Y^1 + \Psi_1 \bar{\Psi}^4 Y^1 Y_4^\dagger - \Psi_4 \bar{\Psi}^2 Y^4 Y_2^\dagger \big] \\
& + \mathrm{i}\xi \big[\bar{\Psi}^2 \Psi_3 Y_2^\dagger Y^3 - \Psi_3 \bar{\Psi}^1 Y^3 Y_1^\dagger + \bar{\Psi}^1 \Psi_4 Y_1^\dagger Y^4 - \bar{\Psi}^4 \Psi_2 Y_4^\dagger Y^2 \big] \\
& + \mathrm{i}\xi \big[\Psi_2 Y_4^\dagger \Psi_1 Y_3^\dagger + \bar{\Psi}^3 Y^2 \bar{\Psi}^4 Y^1 - \bar{\Psi}^2 Y^4 \bar{\Psi}^1 Y^3 - \Psi_3 Y_2^\dagger \Psi_4 Y_1^\dagger \big] \Big\}
\end{aligned}$$

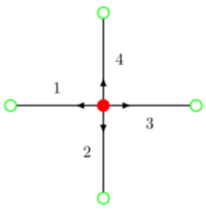
$$S_{\mathrm{SFN}} = \mathrm{N}\int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \mathrm{tr}\left[-\sum_{i=1}^{4} \Phi_i^\dagger \Phi_i + \xi \cdot \bar{\theta}^2 \Phi_1 \Phi_2 \Phi_3 \Phi_4 + \xi \cdot \theta^2 \Phi_1^\dagger \Phi_2^\dagger \Phi_3^\dagger \Phi_4^\dagger \right]$$

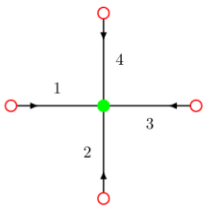
$$S_{\mathrm{SFN},\boldsymbol{\omega}} = \mathrm{N}\int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \mathrm{tr}\left[-\sum_{i=1}^{4} \Phi_i^\dagger \,\Box^{\omega_i}\, \Phi_i + \xi \cdot \bar{\theta}^2 \Phi_1 \Phi_2 \Phi_3 \Phi_4 + \xi \cdot \theta^2 \Phi_1^\dagger \Phi_2^\dagger \Phi_3^\dagger \Phi_4^\dagger \right]$$

$$0 \overset{!}{=} \delta_\varepsilon \int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} f(z) = \int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \delta_\varepsilon f(z)$$

$$\delta_\varepsilon f(z) = \left(\varepsilon^\alpha Q_\alpha + \bar{\varepsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}} \right) f(z) = \left(\varepsilon^\alpha \partial_\alpha + \bar{\varepsilon}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} \right) f(z) + \text{ total derivative.}$$

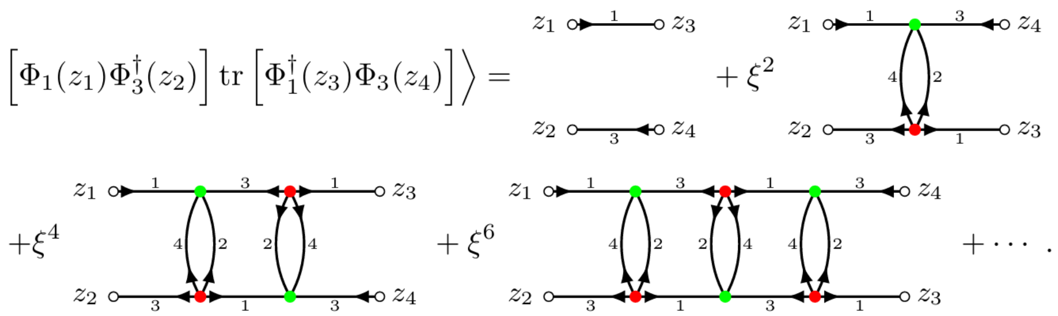
$$\int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \partial_\alpha f(z) = 0 = \int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \bar{\partial}^{\dot{\alpha}} f(z)$$

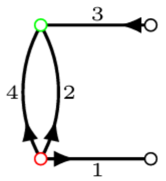
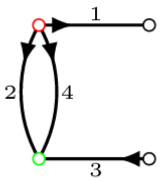
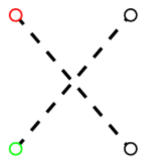


$\sim \mathrm{N}\,\xi \cdot \mathrm{i} \int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \; \delta^{(2)}(\bar{\theta})$,


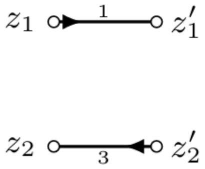
$\sim \mathrm{N}\,\xi \cdot \mathrm{i} \int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \; \delta^{(2)}(\theta)$

$$\left\langle \mathrm{tr}\left[\Phi_1(z_1)\Phi_3^\dagger(z_2)\right]\mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_3(z_4)\right]\right\rangle =$$



$$\mathbb{H} = $$  , $$\bar{\mathbb{H}} = $$  , $$\mathbb{P} = $$  ,

$$\left\langle \mathrm{tr}\left[\Phi_1(z_1)\Phi_3^\dagger(z_2)\right]\mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_3(z_4)\right]\right\rangle$$

$$=$$  $$\circ\left[1+\xi^2\mathbb{H}\circ\mathbb{P}+\xi^4\mathbb{H}\circ\bar{\mathbb{H}}+\xi^6\mathbb{H}\circ\bar{\mathbb{H}}\circ\mathbb{H}\circ\mathbb{P}+\cdots\right]$$

$$= x_{1\bar{2}}^2\,\bar{\mathbb{H}}\circ\left[\sum_{n=0}^{\infty}(\xi^4\mathbb{H}\circ\bar{\mathbb{H}})^n\right]\left[1+\xi^2\mathbb{H}\circ\mathbb{P}\right] = x_{1\bar{2}}^2\,\bar{\mathbb{H}}\circ\frac{1+\xi^2\mathbb{H}\circ\mathbb{P}}{1-\xi^4\mathbb{H}\circ\bar{\mathbb{H}}}\,.$$

$$\Omega_{\Delta,S,R}(z_1,z_2;z_0)=\langle \mathrm{tr}[\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)]\mathcal{O}_{\Delta,S,R}(z_0)\rangle,$$

$$\mathbb{1}=\delta^{(7)}(z_{13})\cdot\delta^{(7)}(z_{24})=\sum\!\!\!\!\!\!\int_{\Delta,S}\int \mathrm{d}^7z_0\;\;|\bar{\Omega}_{\Delta,S,R}(z_1,z_2;z_0)\rangle\;\;\langle\Omega_{\Delta,S,R}(z_3,z_4;z_0)|$$

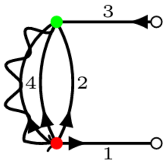
$$\Omega_{\Delta,S,0}(z_1,z_2;z_0)=\frac{C_{\Phi_1^\dagger\Phi_3\mathcal{O}}}{[x_{2\bar{1}}^2]^{\Delta_\Phi-\frac{\Delta}{2}+\frac{S}{2}}[x_{0\bar{1}}^2]^{\frac{\Delta}{2}+\frac{S}{2}}[x_{2\bar{0}}^2]^{\frac{\Delta}{2}+\frac{S}{2}}[X_{3,-}^{\mu_1}\cdots X_{3,-}^{\mu_S}-\text{ traces }]}.$$

$$\int \mathrm{d}^7z_0|\bar{\Omega}_{\Delta,S,0}(z_1,z_2;z_0)\rangle\langle\Omega_{\Delta,S,0}(z_3,z_4;z_0)|$$
$$=\left(\frac{1}{x_{1\bar{2}}^2x_{4\bar{3}}^2}\right)^{\frac{1}{2}}\cdot\left[\frac{\mathrm{c}_1(\Delta,S)}{\mathrm{c}_2(\Delta,S)}\mathrm{g}_{\Delta,S}(r_1,r_2)+\frac{\mathrm{c}_1(\Delta^*,S)}{\mathrm{c}_2(\Delta^*,S)}\mathrm{g}_{\Delta^*,S}(r_1,r_2)\right]$$

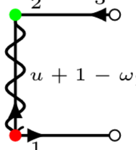
$$\int \mathrm{d}^5\bar{z}_1 \mathrm{d}^5 z_2\ \Omega_{\Delta,S,0}(z_1,z_2;z_0)\left[\mathbb{H}\circ\bar{\mathbb{H}}\right](z_1,z_2;z_3,z_4) = E_0(\Delta,S)^2\cdot\Omega_{\Delta,S,0}(z_3,z_4;z_0)\ ,$$
$$\int \mathrm{d}^5\bar{z}_1 \mathrm{d}^5 z_2\ \Omega_{\Delta,S,0}(z_1,z_2;z_0)\left[\mathbb{H}\circ\mathbb{P}\right](z_1,z_2;z_3,z_4) = E_0(\Delta,S)\cdot\Omega_{\Delta,S,0}(z_3,z_4;z_0)\ .$$

$$\Omega_{\Delta,S,0}(z_1,z_2;z_0) \overset{\substack{x_0\to\infty\\ \theta_0,\bar{\theta}_0=0}}{\sim} \Psi_{\frac{1}{2}-\frac{\Delta}{2},\frac{S}{2}}(z_1,z_2) :=$$

$$\langle\Psi_{u,\frac{S}{2}}|\circ\mathbb{H}_{\boldsymbol{\omega}} =$$

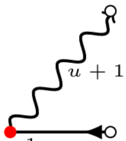


$$=$$



$$\cdot\prod_{i=1}^{4} c_0(1+\omega_i)$$

$$=4\, r_{\frac{S}{2}}(\tfrac{1}{2}-\omega_3, u+1-\omega_2-\omega_4, \tfrac{1}{2}-u-\omega_1)(-1)^S\cdot$$



$$\cdot\prod_{i=1}^{4} c_0(1+\omega_i)$$

$$=4\,(-1)^S r_{\frac{S}{2}}(\tfrac{1}{2}-\omega_3, u+1-\omega_2-\omega_4, \tfrac{1}{2}-u-\omega_1) r_{\frac{S}{2}}(\tfrac{1}{2}-\omega_1, u+1+\omega_1, \tfrac{3}{2}-u)\cdot$$

$$\cdot\prod_{i=1}^{4} c_0(1+\omega_i)$$

$$=:\ \langle\Psi^\dagger_{u,\frac{S}{2}}|\cdot(-1)^S\,\mathrm{E}_{0,\boldsymbol{\omega}}(u,\tfrac{S}{2})\ .$$

$$\mathrm{E}_{0,\omega}\left(u,\frac{S}{2}\right) = \pi^3\frac{\Gamma\left(\frac{S}{2}+u\right)\Gamma\left(\frac{1}{2}-\omega_2\right)\Gamma\left(\frac{1}{2}-\omega_4\right)\Gamma\left(\frac{S}{2}-u+\omega_2+\omega_4+\frac{1}{2}\right)}{\Gamma\left(\frac{S}{2}-u+\frac{3}{2}\right)\Gamma(\omega_2+1)\Gamma(\omega_4+1)\Gamma\left(\frac{S}{2}+u-\omega_2-\omega_4+1\right)}.$$

$$\left\langle\Psi^\dagger_{u,\frac{S}{2}}\right|\circ\overline{\mathbb{H}}_{\boldsymbol{\omega}} = \left\langle\Psi_{u,\frac{S}{2}}\right|\cdot(-1)^S\mathrm{E}_{0,\boldsymbol{\omega}}\left(u,\frac{S}{2}\right),$$

$$\langle\Psi_{u,\frac{S}{2}}|\circ\left(\mathbb{H}_{\boldsymbol{\omega}}\circ\bar{\mathbb{H}}_{\boldsymbol{\omega}}\right) = \langle\Psi_{u,\frac{S}{2}}|\cdot\mathrm{E}_{0,\boldsymbol{\omega}}(u,\tfrac{S}{2})^2\ ,$$
$$\langle\Psi_{u,\frac{S}{2}}|\circ\left(\mathbb{H}_{\boldsymbol{\omega}}\circ\mathbb{P}\right) = \langle\Psi_{u,\frac{S}{2}}|\cdot\mathrm{E}_{0,\boldsymbol{\omega}}(u,\tfrac{S}{2})\ .$$

$$E_0(\Delta,S) = \mathrm{E}_{0,\omega=0}\left(\frac{1}{2}-\frac{\Delta}{2},\frac{S}{2}\right) = \frac{4\pi^4}{(1+S-\Delta)(S+\Delta)}$$

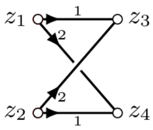
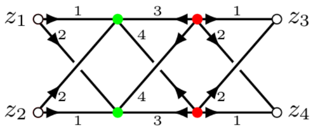
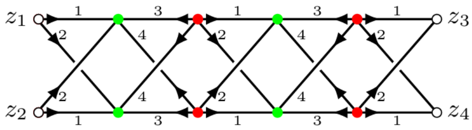
$$
\begin{aligned}
&\langle \mathrm{tr}[\Phi_1(z_1)\Phi_3^\dagger(z_2)]\mathrm{tr}[\Phi_1^\dagger(z_3)\Phi_3(z_4)]\rangle \\
&= \sum_{S=0}^{\infty} (-1)^{S+1} \int_{\frac{1}{2}}^{\frac{1}{2}+\mathrm{i}\infty} \frac{\mathrm{i}d\Delta}{2\pi c_1(\Delta,S)} \frac{E_0(\Delta,S)}{1-\xi^2 E_0(\Delta,S)} \int \mathrm{d}^7 z_0 |\bar{\Omega}_{\Delta,S,0}(z_1,z_2;z_0)\rangle\langle\Omega_{\Delta,S,0}(z_3,z_4;z_0)| \\
&= \left(\frac{1}{x_{1\bar{2}}^2 x_{4\bar{3}}^2}\right)^{\frac{1}{2}} \sum_{S,\Delta} (-1)^S \mathrm{Res}_\Delta \left[\frac{1}{c_2(\Delta,S)} \frac{E_0(\Delta,S)}{1-\xi^2 E_0(\Delta,S)}\right] g_{\Delta,S}(r_1,r_2) \\
&= \left(\frac{1}{x_{1\bar{2}}^2 x_{4\bar{3}}^2}\right)^{\frac{1}{2}} \sum_{S,\Delta} C_{\Delta,S}\, g_{\Delta,S}(r_1,r_2)
\end{aligned}
$$

$$
\Delta = 1 + \frac{1}{2}\left(-1 \pm 2\sqrt{\left(S+\frac{1}{2}\right)^2 - 4\pi^4\xi^2}\right)
$$

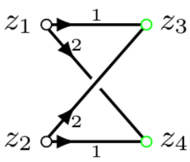
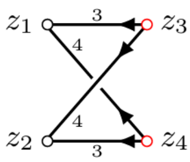
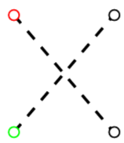
$$
\Delta|_{S=0} = 1 + \gamma = 1 + \frac{1}{2}\left(-1 + \sqrt{1+16\pi^4\xi^2}\right)
$$

$$
C_{\Delta,S} = -2^{S-1-2\Delta}\pi \frac{\Gamma\left(S+\frac{3}{2}\right)\Gamma(\Delta)\Gamma\left(\frac{S-\Delta+2}{2}\right)\Gamma\left(\frac{S+\Delta}{2}\right)}{\Gamma(S+1)\Gamma\left(\Delta+\frac{1}{2}\right)\Gamma\left(\frac{S-\Delta+3}{2}\right)\Gamma\left(\frac{S+\Delta+1}{2}\right)}
$$

$$
\left\langle \mathrm{tr}\left[\Phi_2(z_1)\Phi_1(z_1)\Phi_2(z_2)\Phi_1(z_2)\right] \mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_2^\dagger(z_3)\Phi_1^\dagger(z_4)\Phi_2^\dagger(z_4)\right]\right\rangle
$$

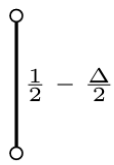
$=$  $+\,\xi^4$  $+\,\xi^8$ 

$+ \cdots + (z_3 \leftrightarrow z_4)$.

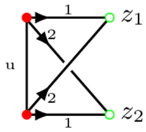
$\mathbb{H} =$  , $\bar{\mathbb{H}} =$  , $\mathbb{P} =$  ,

$$
\langle \mathrm{tr}[\Phi_2(z_1)\Phi_1(z_1)\Phi_2(z_2)\Phi_1(z_2)]\mathrm{tr}[\Phi_1^\dagger(z_3)\Phi_2^\dagger(z_3)\Phi_1^\dagger(z_4)\Phi_2^\dagger(z_4)]\rangle = \frac{\mathbb{H}\circ(1+\mathbb{P})}{1-\xi^4\mathbb{H}\circ\bar{\mathbb{H}}}.
$$

$$
\left\langle \mathrm{tr}\left[\Phi_2(z_1)\Phi_1(z_1)\Phi_2(z_2)\Phi_1(z_2)\right] \mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_2^\dagger(z_3)\Phi_1^\dagger(z_4)\Phi_2^\dagger(z_4)\right]\right\rangle = \frac{\mathbb{H}\circ(1+\mathbb{P})}{1-\xi^4\mathbb{H}\circ\bar{\mathbb{H}}} .
$$

$$
\Omega_{\Delta,0,-2}(z_1,z_2;z_0) \overset{x_0\to\infty}{\underset{\theta_0,\bar{\theta}_0=0}{\sim}} \Psi_{\frac{1}{2}-\frac{\Delta}{2}}(z_1,z_2) := \frac{\theta_{12}^2}{[x_{12}^2]^{\frac{1}{2}-\frac{\Delta}{2}}} =
$$


$$\Psi_u \circ \mathbb{H} =$$



$$= -\left[\int \mathrm{d}^5 z_0\, \mathrm{d}^5 z_{0'}\, \frac{c_0(1)}{[x_{0\bar{1}}^2]^{\frac{1}{2}}} \frac{c_0(1)}{[x_{0'\bar{1}}^2]^{\frac{1}{2}}} \frac{\theta_{00'}^2}{[x_{00'}^2]^u} \frac{c_0(1)}{[x_{0\bar{2}}^2]^{\frac{1}{2}}} \frac{c_0(1)}{[x_{0'\bar{2}}^2]^{\frac{1}{2}}}\right]_{\theta_{1,2}=0}$$

$$= -c_0(1)^4 \int \mathrm{d}^2\theta_0\, \mathrm{e}^{2\mathrm{i}\theta_0\gamma^\mu\bar{\theta}_{12}\partial_{1,\mu}} \int \mathrm{d}^3 x_0\, \mathrm{d}^3 x_{0'} \frac{1}{[x_{10}^2]^{\frac{1}{2}}} \frac{1}{[x_{10'}^2]^{\frac{1}{2}}} \frac{1}{[x_{00'}^2]^u} \frac{1}{[x_{20}^2]^{\frac{1}{2}}} \frac{1}{[x_{20'}^2]^{\frac{1}{2}}}$$

$$= c_0(1)^4 \cdot \bar{\theta}_{12}^2 \Box_1\, \mathsf{kite}^{(3)}(x_{12}^2, u)\ .$$

$$I^{(3)}(u) = \frac{1}{\pi^2\left(\frac{3}{2}-u\right)(u-1)} \frac{1}{32\pi^2} \int_1^\infty \mathrm{d}s \frac{s^{\frac{1}{2}-u} + s^{-2+u}}{\sqrt{1+s}} \log\left[\frac{\sqrt{1+s}+1}{\sqrt{1+s}-1}\right].$$

$$\mathrm{E}_2(u) = -4c_0(1)^4 \cdot (u-1)\left(\frac{3}{2}-u\right) I^{(3)}(u).$$

$$\Psi_u \circ (\mathbb{H} \circ \bar{H}) = \mathrm{E}_2(u)^2 \cdot \Psi_u$$

$$E_2(\Delta) = \frac{\csc\left(\pi\left(\frac{\Delta}{2}+1\right)\right)\Gamma\left(\frac{\Delta}{2}+1\right)}{32\sqrt{\pi}\Gamma\left(\frac{\Delta}{2}+\frac{3}{2}\right)} - \frac{{}_3F_2\left(1,1,\frac{\Delta}{2}+\frac{3}{2};\frac{\Delta}{2}+2,\frac{\Delta}{2}+\frac{5}{2};1\right)}{16\pi^2(\Delta+3)\left(\frac{\Delta}{2}+1\right)}.$$

$$1 = \xi^4 E_2(\Delta)^2$$

$$\Delta^{(2)} = 2 \pm \frac{\xi^2}{12} - \frac{\xi^4}{576} \pm \frac{18\pi^2 - 97}{248832}\xi^6 + \frac{3803 - 2268\zeta_3 + 18\pi^2(\log(4096) - 19)}{35831808}\xi^8 + \mathcal{O}(\xi^{10}),$$

$$\Delta^{(4)} = 4 \pm \frac{4\xi^2}{15} - \frac{\xi^4}{7200} \pm \frac{28800\pi^2 - 191191}{777600000}\xi^6 \frac{191678057 - 145152000\zeta_3 + 28800\pi^2(480\log(2) - 421)}{5598720000000}\xi^8 + \mathcal{O}(\xi^{10})$$

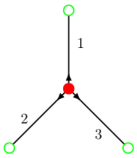
$$\Delta^{(6)} = 6 \pm \frac{2\xi^2}{35} - \frac{4\xi^4}{2940} \pm \frac{352800\pi^2 - 2244421}{15126300000}\xi^6 + \frac{2972114029 - 2074464000\zeta_3 + 117600\pi^2(1680\log(2) - 1801)}{148237740000000}\xi^8 + \mathcal{O}(\xi^{10})$$

$$S = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta} \sum_{i=1}^{3} \mathrm{tr}\left(\mathrm{e}^{-g\mathcal{V}}\Phi_i^\dagger \mathrm{e}^{g\mathcal{V}}\Phi_i\right) + \frac{1}{4g^2}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{tr}(W^\alpha W_\alpha) + \mathrm{i}g\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{tr}(\Phi_1[\Phi_2,\Phi_3]) + \mathrm{i}g\int \mathrm{d}^4x\, \mathrm{d}^2\bar{\theta}\, \mathrm{tr}\left(\Phi_1^\dagger[\Phi_2^\dagger,\Phi_3^\dagger]\right)$$

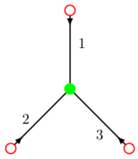
$$\Phi_i \star \Phi_j := e^{\frac{i}{2}\det(\gamma|\mathbf{q}_i|\mathbf{q}_j)}\Phi_i\Phi_j$$

$$\mathrm{i}g\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{tr}[q\Phi_1\Phi_2\Phi_3 - q^{-1}\Phi_1\Phi_3\Phi_2] + \text{ h.c.}$$

$$S = S_{\text{kin}} + S_{\text{int}}$$
$$= \mathrm{N} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta} \left\{ \sum_{i=1}^{3} \mathrm{tr}[\Phi_i^\dagger \Phi_i] + \mathrm{i}\xi \cdot \bar{\theta}^2 \mathrm{tr}[\Phi_1\Phi_2\Phi_3] + \mathrm{i}\xi \cdot \theta^2 \mathrm{tr}[\Phi_1^\dagger \Phi_2^\dagger \Phi_3^\dagger] \right\}$$



$$\sim -\mathrm{N}\,\xi \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \delta^{(2)}(\bar{\theta}) \,,$$



$$\sim -\mathrm{N}\,\xi \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \delta^{(2)}(\theta) \,,$$

$$S_{\text{kin}} = \mathrm{N} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta} \sum_{i=1}^{3} \mathrm{tr}[\Phi_i^\dagger \Phi_i] = \mathrm{N} \int \mathrm{d}^4x \sum_{i=1}^{3} \mathrm{tr}[\phi_i^\dagger \,\Box\, \phi_i - \mathrm{i}\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i + F_i^\dagger F_i],$$

$$S_{\text{int}} = \mathrm{N} \cdot \mathrm{i}\xi \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{tr}[\Phi_1\Phi_2\Phi_3]_{\bar{\theta}=0} + \mathrm{N} \cdot \mathrm{i}\xi \int \mathrm{d}^4x\, \mathrm{d}^2\bar{\theta} \mathrm{tr}[\Phi_1^\dagger \Phi_2^\dagger \Phi_3^\dagger]_{\theta=0}$$
$$= \mathrm{N} \cdot \mathrm{i}\xi \int \mathrm{d}^4x \mathrm{tr}[\phi_1\phi_2F_3 + \phi_1F_2\phi_3 + F_1\phi_2\phi_3 - \phi_1\psi_2\psi_3 - \phi_2\psi_3\psi_1 - \phi_3\psi_1\psi_2]$$
$$+ \mathrm{N} \cdot \mathrm{i}\xi \int \mathrm{d}^4x \mathrm{tr}[\phi_1^\dagger\phi_2^\dagger F_3^\dagger + \phi_1^\dagger F_2^\dagger \phi_3^\dagger + F_1^\dagger \phi_2^\dagger \phi_3^\dagger - \phi_1^\dagger \bar{\psi}_2\bar{\psi}_3 - \phi_2^\dagger \bar{\psi}_3\bar{\psi}_1 - \phi_3^\dagger \bar{\psi}_1\bar{\psi}_2]$$

$$F_1^A = -\mathrm{i}\xi \phi_{2,B}^* \phi_{3,C}^* \cdot \mathrm{tr}[T^A T^B T^C], \quad F_1^{*,A} = -\mathrm{i}\xi \phi_{2,B}\phi_{3,C} \cdot \mathrm{tr}[T^A T^B T^C]$$
$$F_2^A = -\mathrm{i}\xi \phi_{3,B}^* \phi_{1,C}^* \cdot \mathrm{tr}[T^A T^B T^C], \quad F_2^{*,A} = -\mathrm{i}\xi \phi_{3,B}\phi_{1,C} \cdot \mathrm{tr}[T^A T^B T^C]$$
$$F_3^A = -\mathrm{i}\xi \phi_{1,B}^* \phi_{2,C}^* \cdot \mathrm{tr}[T^A T^B T^C], \quad F_3^{*,A} = -\mathrm{i}\xi \phi_{1,B}\phi_{2,C} \cdot \mathrm{tr}[T^A T^B T^C].$$

$$S = \mathrm{N} \int \mathrm{d}^4x \mathrm{tr} \left\{ \sum_{i=1}^{3} [\phi_i^\dagger \,\Box\, \phi_i - \mathrm{i}\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i] + \xi^2 [\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger + \phi_3\phi_1\phi_3^\dagger\phi_1^\dagger + \phi_2\phi_3\phi_2^\dagger\phi_3^\dagger] \right.$$
$$\left. -\mathrm{i}\xi[\phi_1\psi_2\psi_3 + \phi_2\psi_3\psi_1 + \phi_3\psi_1\psi_2] - \mathrm{i}\xi[\phi_1^\dagger\bar{\psi}_2\bar{\psi}_3 + \phi_2^\dagger\bar{\psi}_3\bar{\psi}_1 + \phi_3^\dagger\bar{\psi}_1\bar{\psi}_2] + \mathcal{L}_{\text{dt}} \right\}$$

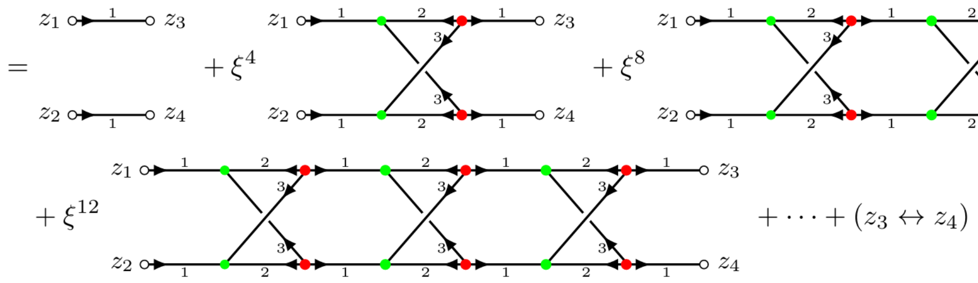
$$\mathcal{L}_{\text{dt}} = -\frac{\xi^2}{\mathrm{N}} \{ \mathrm{tr}[\phi_1\phi_2]\mathrm{tr}[\phi_1^\dagger\phi_2^\dagger] + \mathrm{tr}[\phi_1\phi_3]\mathrm{tr}[\phi_1^\dagger\phi_3^\dagger] + \mathrm{tr}[\phi_2\phi_3]\mathrm{tr}[\phi_2^\dagger\phi_3^\dagger] \}$$

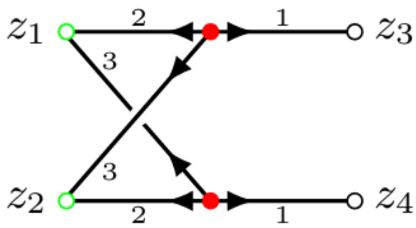
$$S_{\boldsymbol{\omega}} = S_{\text{kin},\boldsymbol{\omega}} + S_{\text{int},\boldsymbol{\omega}}$$
$$= \mathrm{N} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta} \left\{ \sum_{i=1}^{3} \mathrm{tr}[\Phi_i^\dagger \,\Box^{\omega_i}\, \Phi_i] + \mathrm{i}\xi \cdot \bar{\theta}^2 \mathrm{tr}[\Phi_1\Phi_2\Phi_3] + \mathrm{i}\xi \cdot \theta^2 \mathrm{tr}[\Phi_1^\dagger\Phi_2^\dagger\Phi_3^\dagger] \right\} \quad (6.63)$$

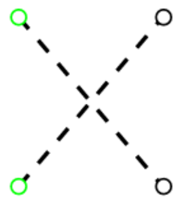
$$[\Phi_i] = [\Phi_i^\dagger] = \left. \frac{(D - 2\mathcal{N}) - 2\omega_i}{2} \right|_{D=4,\mathcal{N}=1} = 1 - \omega_i$$

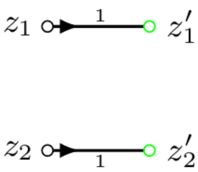
$$S_{\text{kin},\boldsymbol{\omega}} = \mathrm{N} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta} \sum_{i=1}^{3} \mathrm{tr}[\Phi_i^\dagger \,\Box^{\omega_i}\, \Phi_i]$$
$$= \mathrm{N} \int \mathrm{d}^4x \sum_{i=1}^{3} \mathrm{tr}[\phi_i^\dagger \,\Box^{1+\omega_i}\, \phi_i - \mathrm{i}\bar{\psi}_i \bar{\sigma}^\mu \,\Box^{\omega_i}\, \partial_\mu \psi_i + F_i^\dagger \,\Box^{\omega_i}\, F_i]$$

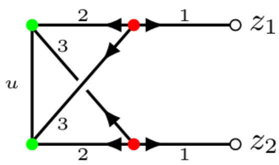
$$\left\langle \mathrm{tr}\left[\Phi_1(z_1)\Phi_1(z_2)\right]\mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_1^\dagger(z_4)\right]\right\rangle$$

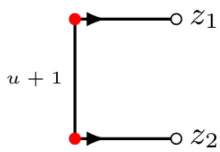


$\mathbb{H} =$  ,

$\mathbb{P} =$  ,

$\left\langle \mathrm{tr}\left[\Phi_1(z_1)\Phi_1(z_2)\right]\mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_1^\dagger(z_4)\right]\right\rangle =$  $\circ \left[\dfrac{(1+\mathbb{P})}{1-\xi^4\mathbb{H}}\right]$.

$\Psi_u \circ \mathbb{H} =$ 

$= -4u\,(1-u)\,I^{(4)}(u)\cdot$ 

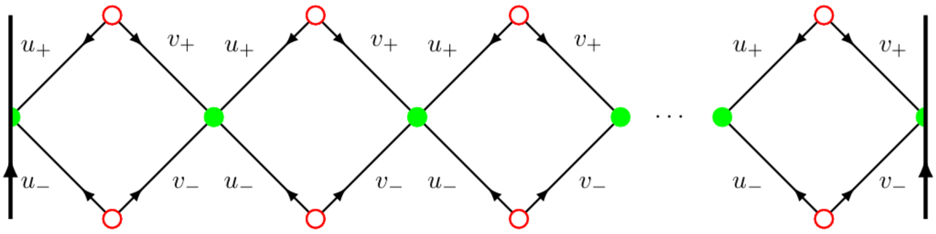
$$= 16\,u\,(1-u)\;\; I^{(4)}(u)\; r(2-u, u+1, 1)\, r(2-u, u, 1)\cdot \Psi_u\ .$$

$$\mathrm{E}_0^{\mathrm{SBW}}(u) = 16u(1-u)\cdot I^{(4)}(u)\cdot r(2-u,u+1,1)r(2-u,u,1),$$

$$I^{(4)}(u) = \frac{1}{2u-2}\left[\psi^{(1)}\left(\frac{u-1}{2}\right) - \psi^{(1)}\left(\frac{1-u}{2}\right) + \psi^{(1)}\left(\frac{2-u}{2}\right) - \psi^{(1)}\left(\frac{u}{2}\right)\right]$$

$$\psi^{(1)}(z) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\,\Gamma(z)$$

$$1 = \xi^4 \mathrm{E}_0^{\mathrm{SBW}}\left(2-\frac{\Delta}{2}\right)$$

$T_N(U) =$ 

$$U^{\mathrm{SFN}} = \begin{pmatrix} u_+ \\ u_- \\ v_- \end{pmatrix} = \begin{pmatrix} 1/2-\omega_1 & 1/2-\omega_4 \\ 1/2-\omega_2 & 1/2-\omega_3 \end{pmatrix}$$

$$U^{\mathrm{SBW}} = \begin{pmatrix} u_+ v_+ \\ u_- v_- \end{pmatrix} = \begin{pmatrix} 0 & 1-\omega_1 \\ 1-\omega_3 & 1-\omega_2 \end{pmatrix}$$

$$Z_{MN}(\mathbf{u}) = \mathrm{tr}[T_N(\mathbf{u})^M],$$

$$\hat{T}_N^{\mathrm{SFN}} = \left[\prod_{i=1}^{4} c_0(1+\omega_i)\right]^N T_N(U^{\mathrm{SFN}}) \text{ and } \hat{T}_N^{\mathrm{SBW}} = \left[\prod_{i=1}^{3} c_0(1+\omega_i)\right]^N T_N(U^{\mathrm{SBW}}).$$

$$\hat{Z}_{MN}^{\mathrm{SFN}} = \mathrm{tr}\left(\hat{T}_N^{\mathrm{SFN}}\right)^M \text{ and } \hat{Z}_{MN}^{\mathrm{SBW}} = \mathrm{tr}\left(\hat{T}_N^{\mathrm{SBW}}\right)^M$$

$$\hat{Z}^{\mathrm{SFN}} = \sum_{M,N=1}^{\infty} \hat{Z}_{MN}^{\mathrm{SFN}} (\mathrm{i}\xi)^{2MN} \text{ and } \hat{Z}^{\mathrm{SBW}} = \sum_{M,N=1}^{\infty} \hat{Z}_{MN}^{\mathrm{SBW}} (-\xi)^{2MN}.$$

$$\xi_{\mathrm{cr}}^{\mathrm{SFN}} = \left[\lim_{M,N\to\infty} \left|-\hat{Z}_{MN}^{\mathrm{SFN}}\right|^{\frac{1}{MN}}\right]^{-\frac{1}{2}} = \left[\prod_{i=1}^{4} c_0(1+\omega_i) \lim_{M,N\to\infty} \left|Z_{MN}(U^{\mathrm{SFN}})\right|^{\frac{1}{MN}}\right]^{-\frac{1}{2}}$$

$$\xi_{\mathrm{cr}}^{\mathrm{SBW}} = \left[\lim_{M,N\to\infty} \left|\hat{Z}_{MN}^{\mathrm{SBW}}\right|^{\frac{1}{MN}}\right]^{-\frac{1}{2}} = \left[\prod_{i=1}^{3} c_0(1+\omega_i) \lim_{M,N\to\infty} \left|Z_{MN}(U^{\mathrm{SBW}})\right|^{\frac{1}{MN}}\right]^{-\frac{1}{2}}$$

$$K(U) := \lim_{M,N\to\infty} |Z_{MN}(U)|^{\frac{1}{MN}}$$

$$T_N(U) \circ T_N(U_{\mathrm{inv}}) = F_N \cdot \mathbb{1}_N$$

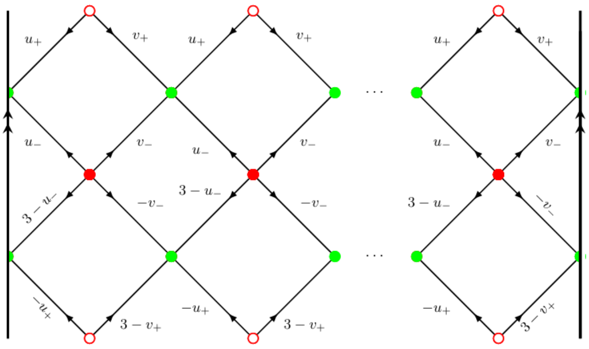
$$U_{\mathrm{inv}} = \begin{pmatrix} -u_- & D-1-v_- \\ D-1-u_+ & -v_+ \end{pmatrix} \text{ and } F_N = [16\pi^{2D} \quad a_0(u_+)a_0(D-1-u_+)a_0(v_-)a_0(D-1-v_-)]^N,$$

$$U_{\mathrm{inv}} = \begin{pmatrix} -u_- & D-1-v_- \\ -u_+ & D-1-v_+ \end{pmatrix} \text{ and } F_N = [16\pi^{2D} \quad a_0(v_+)a_0(D-1-v_+)a_0(v_-)a_0(D-1-v_-)]^N,$$

$$U_{\mathrm{inv}} = \begin{pmatrix} D-1-u_- - v_- \\ D-1-u_+ - v_+ \end{pmatrix} \text{ and } F_N = [16\pi^{2D} \quad a_0(u_+)a_0(D-1-u_+)a_0(u_-)a_0(D-1-u_-)]^N,$$

$$U_{\mathrm{inv}} = \begin{pmatrix} D-1-u_- & -v_- \\ -u_+ & D-1-v_+ \end{pmatrix} \text{ and } F_N = [16\pi^{2D} \quad a_0(u_-)a_0(D-1-u_-)a_0(v_+)a_0(D-1-v_+)]^N.$$

$$T_N\begin{pmatrix} u_+ & v_+ \\ u_- & v_- \end{pmatrix} \circ T_N\begin{pmatrix} 3-u_- & -v_- \\ -u_+ & 3-v_+ \end{pmatrix} =$$



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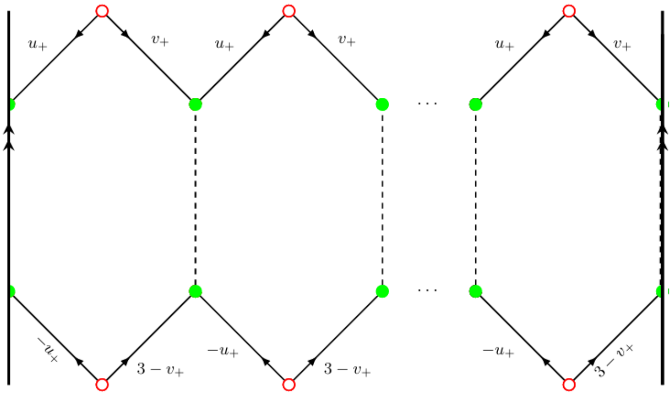
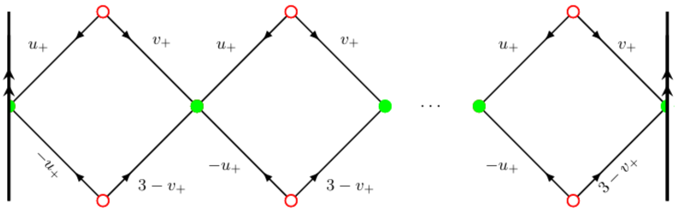


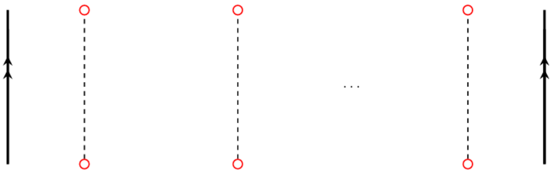
$$\cdot \left[-4\pi^4 a_0(u_-)a_0(3-u_-)\right]^N$$

$=$



$$\cdot \left[-4\pi^4 a_0(u_-)a_0(3-u_-)\right]^N$$

$=$



$$\cdot \left[16\pi^8\, a_0(u_-)a_0(3-u_-)\, a(v_+)a(3-v_+)\right]^N$$

$$= [16\pi^8 a_0(u_-)a_0(3-u_-)a(v_+)a(3-v_+)]^N \cdot \mathbb{1}_N,$$

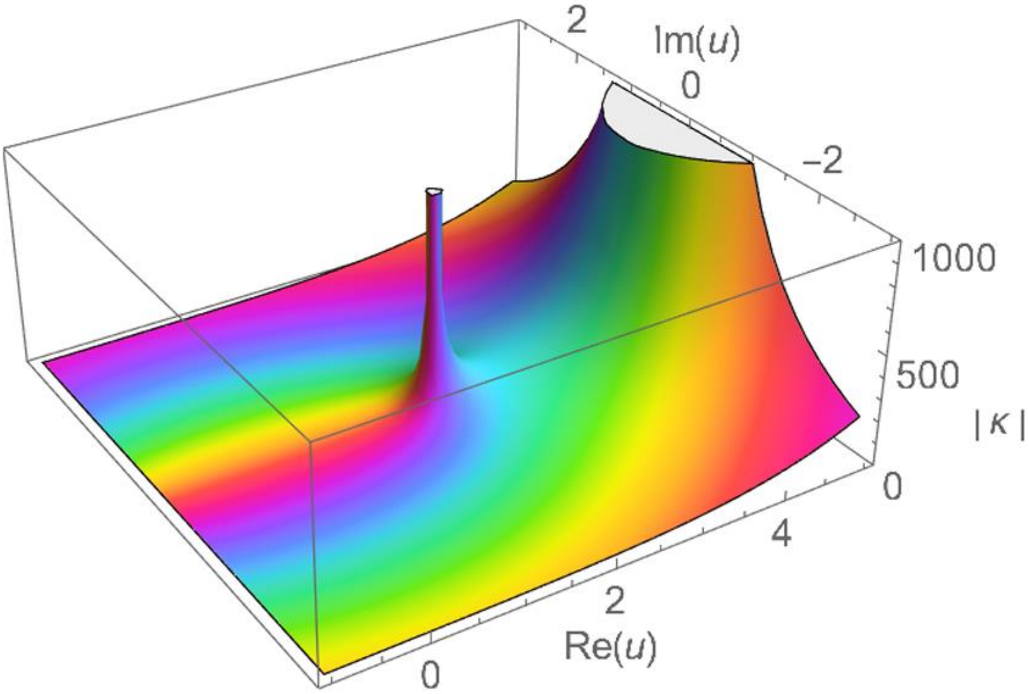
$$K(\mathbf{u}) = \lim_{M,N\to\infty} |\mathrm{tr}[T_N(\mathbf{u})^M]|^{\frac{1}{MN}} = \lim_{N\to\infty} |\Lambda_{\max,N}(\mathbf{u})|^{\frac{1}{N}}$$

$$\kappa(u)\kappa(-u) = 1 \ \text{ and } \ \kappa(u)\kappa(D-1-u) = 4\pi^D a_0(u)a_0(D-1-u).$$

$$\kappa(u) \overset{D\equiv 3}{=} 2^{\frac{3u}{2}+1}\pi^{\frac{3u}{2}-\frac{1}{2}} \frac{\Gamma\left(\frac{3}{2}-u\right)\Gamma\left(\frac{u}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \prod_{k=1}^{\infty} \frac{\Gamma\left(2k-u+\frac{3}{2}\right)\Gamma(2k+u)\Gamma\left(2k-\frac{3}{2}\right)}{\Gamma\left(2k+u-\frac{3}{2}\right)\Gamma(2k-u)\Gamma\left(2k+\frac{3}{2}\right)}$$

$$\kappa(u) \overset{D\equiv 4}{=} 12^{\frac{u}{3}}\pi^{\frac{4u}{3}} \frac{\Gamma\left(\frac{u+1}{3}\right)\Gamma(2-u)}{\Gamma\left(\frac{1}{3}\right)} \prod_{k=1}^{\infty} \frac{\Gamma(3k-u+2)\Gamma(3k+u)\Gamma(3k-2)}{\Gamma(3k+u-2)\Gamma(3k-u)\Gamma(3k+2)}$$

$$\kappa(u) \overset{D=4}{=} 2^{\frac{2u}{3}} 3^{\frac{4u}{3}-2} \pi^{\frac{4u}{3}} \frac{\Gamma(2-u)\Gamma\left(\frac{u}{3}\right)\Gamma\left(\frac{u+1}{3}\right)}{\Gamma(u)\Gamma\left(1-\frac{u}{3}\right)\Gamma\left(\frac{4}{3}-\frac{u}{3}\right)}$$



$$\xi_{\text{cr}}^{\text{SFN}} = \left[\frac{1}{4}\prod_{i=1}^{4} a_0(1-\omega_i)\kappa\left(\frac{1}{2}-\omega_i\right)\right]^{-1/2} \text{ and } \xi_{\text{cr}}^{\text{SBW}} = \left[\prod_{i=1}^{3} a_0(1+\omega_i)\kappa(1-\omega_i)\right]^{-1/2},$$

$$\xi_{\text{cr}}^{\text{SFN}} = \frac{2}{\Gamma\left(\frac{1}{2}\right)^2}\cdot\kappa\left(\frac{1}{2}\right)^{-2} = \frac{\left(\frac{2}{\pi}\right)^{3/2}}{\Gamma\left(\frac{1}{4}\right)^2} = \frac{1}{\pi^3\sqrt{2}|\eta(\mathrm{i})|^2}$$

$$\xi_{\text{cr}}^{\text{SBW}} = \kappa(1)^{-3/2} = \frac{3}{2\pi^2\Gamma\left(\frac{1}{3}\right)^{3/2}} = \frac{3^{9/8}}{4\pi^3\left|\eta\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\right)\right|}$$

$$\text{star:} = \int \mathrm{d}^D x_0 \frac{1}{[x_{10}^2]^{u_1}}\frac{1}{[x_{20}^2]^{u_2}}\frac{1}{[x_{30}^2]^{u_3}}$$

$$\frac{1}{[x^2]^u} = \frac{1}{\Gamma(u)}\int_0^\infty \mathrm{d}s s^{u-1}\mathrm{e}^{-sx^2}$$

$$\text{star} = \frac{1}{\Gamma(u_1)\Gamma(u_2)\Gamma(u_3)}\int_0^\infty \mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3 s_1^{u_1-1}s_2^{u_2-1}s_3^{u_3-1}\int \mathrm{d}^D x_0 \mathrm{e}^{-s_1x_{10}^2-s_2x_{20}^2-s_3x_{30}^2}$$

$$\int \mathrm{d}^D x_0 \mathrm{e}^{-s_1x_{10}^2-s_2x_{20}^2-s_3x_{30}^2} = \mathrm{e}^{-\frac{1}{S}\left(s_1s_2x_{12}^2+s_1s_3x_{13}^2+s_2s_3x_{23}^2\right)}\left(\frac{\pi}{S}\right)^{\frac{D}{2}}$$

$$
\begin{aligned}
s_i &= \frac{T}{t_i}, \qquad S = s_1 + s_2 + s_3 = \frac{T^2}{t_1 t_2 t_3}, \\
t_j &= \frac{s_1 s_2 s_3}{S \cdot s_j}, \quad T = t_1 t_2 + t_2 t_3 + t_1 t_3 = s_1 s_2 s_3
\end{aligned}
\quad \det\left(\frac{\partial s_i}{\partial t_j}\right) = \frac{T^3}{t_1^2 t_2^2 t_3^2}
$$

$$
\text{star particle} = \frac{\pi^{\frac{D}{2}}}{\Gamma(u_1)\Gamma(u_2)\Gamma(u_3)} \cdot \int_0^\infty \mathrm{d}t_1\ \mathrm{d}t_2\ \mathrm{d}t_3 T^{u_1+u_2+u_3-D} t_1^{\frac{D}{2}-u_1-1} t_2^{\frac{D}{2}-u_2-1} t_3^{\frac{D}{2}-u_3-1} \mathrm{e}^{-t_3 x_{12}^2 - t_2 x_{13}^2 - t_1 x_{23}^2}
$$

$$
\int \mathrm{d}^D x_0 \frac{1}{[x_{10}^2]^{u_1}} \frac{1}{[x_{20}^2]^{u_2}} \frac{1}{[x_{30}^2]^{u_3}} = \pi^{\frac{D}{2}} \frac{\Gamma\left(\frac{D}{2}-u_1\right)\Gamma\left(\frac{D}{2}-u_2\right)\Gamma\left(\frac{D}{2}-u_3\right)}{\Gamma(u_1)\Gamma(u_2)\Gamma(u_3)} \cdot \frac{1}{[x_{23}^2]^{\frac{D}{2}-u_1}} \frac{1}{[x_{13}^2]^{\frac{D}{2}-u_2}} \frac{1}{[x_{12}^2]^{\frac{D}{2}-u_3}}
$$

$$
\text{star particle:} = \int \mathrm{d}^D x_0 \frac{1}{[x_{10}^2]^{u_1}} \frac{x_{10,\alpha\dot{\alpha}}}{|x_{10}|} \frac{1}{[x_{20}^2]^{u_2}} \frac{\bar{x}_{20}^{\dot{\alpha}\beta}}{|x_{20}|} \frac{1}{[x_{30}^2]^{u_3}},
$$

$$
\frac{1}{[x_{10}^2]^{u}} \frac{x_{10,\alpha\dot{\alpha}}}{|x_{10}|} = \frac{1}{1-2u} \partial_{1,\alpha\dot{\alpha}} \frac{1}{[x_{10}^2]^{u-\frac{1}{2}}}
$$

$$
\text{star particle} = \frac{1}{(1-2u_1)(1-2u_2)} \partial_{1,\alpha\dot{\alpha}} \partial_2^{\dot{\alpha}\beta} \int \mathrm{d}^D x_0 \frac{1}{[x_{10}^2]^{u_1-\frac{1}{2}}} \frac{1}{[x_{20}^2]^{u_2-\frac{1}{2}}} \frac{1}{[x_{30}^2]^{u_3}}.
$$

$$
\text{star particle} = \frac{1}{4} \int_0^\infty \mathrm{d}s_1\ \mathrm{d}s_2\ \mathrm{d}s_3 \frac{s_1^{\left(u_1-\frac{1}{2}\right)-1} s_2^{\left(u_2-\frac{1}{2}\right)-1} s_3^{u_3-1}}{\Gamma\left(u_1+\frac{1}{2}\right)\Gamma\left(u_2+\frac{1}{2}\right)\Gamma(u_3)} \left(\frac{\pi}{S}\right)^{\frac{D}{2}} \partial_{1,\alpha\dot{\alpha}} \partial_2^{\dot{\alpha}\beta} \exp
$$

$$
\exp := \mathrm{e}^{-\frac{1}{S}(s_1 s_2 \cdot x_{12}^2 + s_1 s_3 \cdot x_{13}^2 + s_2 s_3 \cdot x_{23}^2)}
$$

$$
\partial_{1,\alpha\dot{\alpha}} \partial_2^{\dot{\alpha}\beta} \exp = 4 \frac{s_1 s_2}{S} \left[\delta_\alpha^\beta \left(\frac{D}{2} + \sum_{i=1}^{3} s_i \frac{\partial}{\partial s_i}\right) + s_3 \cdot x_{13,\alpha\dot{\alpha}} \bar{x}_{23}^{\dot{\alpha}\beta}\right] \exp.
$$

$$
\begin{aligned}
\int_0^\infty \frac{s_1^{u_1-\frac{1}{2}}}{S^{\frac{D}{2}+1}} \cdot s_1 \frac{\partial}{\partial s_1} \exp &= -\int_0^\infty \frac{s_1^{u_1-\frac{1}{2}}}{S^{\frac{D}{2}+1}} \left[u_1 + \frac{1}{2} - \frac{s_1}{S}\left(\frac{D}{2}+1\right)\right] \exp, \\
\int_0^\infty \frac{s_2^{u_2-\frac{1}{2}}}{S^{\frac{D}{2}+1}} \cdot s_2 \frac{\partial}{\partial s_2} \exp &= -\int_0^\infty \frac{s_2^{u_2-\frac{1}{2}}}{S^{\frac{D}{2}+1}} \left[u_2 + \frac{1}{2} - \frac{s_2}{S}\left(\frac{D}{2}+1\right)\right] \exp, \\
\int_0^\infty \frac{s_3^{u_3-1}}{S^{\frac{D}{2}+1}} \cdot s_3 \frac{\partial}{\partial s_3} \exp &= -\int_0^\infty \frac{s_3^{u_3-1}}{S^{\frac{D}{2}+1}} \left[u_3 - \frac{s_3}{S}\left(\frac{D}{2}+1\right)\right] \exp.
\end{aligned}
$$

$$
\text{star particle} = \pi^{\frac{D}{2}} x_{13,\alpha\dot{\alpha}} \bar{x}_{23}^{\dot{\alpha}\beta} \int_0^\infty \mathrm{d}s_1\ \mathrm{d}s_2\ \mathrm{d}s_3 \frac{s_1^{u_1-\frac{1}{2}} s_2^{u_2-\frac{1}{2}} s_3^{u_3}}{\Gamma\left(u_1+\frac{1}{2}\right)\Gamma\left(u_2+\frac{1}{2}\right)\Gamma(u_3)} S^{-\frac{D}{2}-1} \exp
$$

$$\text{star particle} = \pi^{\frac{D}{2}} x_{13,\alpha\dot\alpha} \bar{x}_{23}^{\dot\alpha\beta} \int_0^\infty \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3 \frac{t_1^{\frac{D}{2}-u_1-\frac{1}{2}} t_2^{\frac{D}{2}-u_2-\frac{1}{2}} t_3^{\frac{D}{2}-u_3-1}}{\Gamma\left(u_1+\frac{1}{2}\right)\Gamma\left(u_2+\frac{1}{2}\right)\Gamma(u_3)} \mathrm{e}^{-t_3 x_{12}^2 - t_2 x_{13}^2 - t_1 x_{23}^2}$$

$$= r_{\frac{1}{2}}(u_3,u_1,u_2) \cdot \frac{1}{[x_{12}^2]^{\frac{D}{2}-u_3}} \cdot \frac{1}{[x_{13}^2]^{\frac{D}{2}-u_2}} \frac{x_{13,\alpha\dot\alpha}}{|x_{13}|} \cdot \frac{1}{[x_{23}^2]^{\frac{D}{2}-u_1}} \frac{\bar{x}_{23}^{\dot\alpha\beta}}{|x_{23}|}$$

$$\int \mathrm{d}^D x_0\, \mathrm{d}^2\bar\theta_0 \frac{1}{[x_{1\bar{0}}^2]^{u_1}} \frac{1}{[x_{2\bar{0}}^2]^{u_2}}\Bigg|_{\substack{\bar\theta_0=0\\ \bar\theta_{1,2}=0}}$$

$$\int \mathrm{d}^2\bar\theta_0 \mathrm{e}^{-2\mathrm{i}\theta_1\sigma^\mu\bar\theta_0\partial_{1,\mu}} \mathrm{e}^{-2\mathrm{i}\theta_2\sigma^\nu\bar\theta_0\partial_{2,\nu}} \int \mathrm{d}^D x_0 \frac{1}{[x_{10}^2]^{u_1}} \frac{1}{[x_{20}^2]^{u_2}}$$
$$= \int \mathrm{d}^2\bar\theta_0 \mathrm{e}^{-2\mathrm{i}\theta_{12}\sigma^\mu\bar\theta_0\partial_{1,\mu}} \frac{r(D-u_1-u_2,u_1,u_2)}{[x_{12}^2]^{u_1+u_2-\frac{D}{2}}}$$

$$\theta_{12}^2\,\Box_1 \frac{r(D-u_1-u_2,u_1,u_2)}{[x_{12}^2]^{u_1+u_2-\frac{D}{2}}} = \begin{cases} 4r_0(2-u_1-u_2,u_1,u_2)\dfrac{\theta_{12}^2}{[x_{12}^2]^{u_1+u_2-\frac{1}{2}}} \\ -4r_0(3-u_1-u_2,u_1,u_2)\dfrac{\theta_{12}^2}{[x_{12}^2]^{u_1+u_2-1}} \end{cases},$$

$$\Box_1 [x_{12}^2]^{-u} = -4u\left(\frac{D}{2}-u-1\right)[x_{12}^2]^{-u-1}$$

$$\int \mathrm{d}^D x_0\, \mathrm{d}^2\theta_0\, \mathrm{d}^2\bar\theta_0 \delta^{(2)}(\theta_0) \frac{1}{[x_{2\bar{0}}]^{u_2}} \frac{1}{[x_{3\bar{0}}]^{D-1-u_1-u_2}} \int \mathrm{d}^D x_1\, \mathrm{d}^2\theta_1\, \mathrm{d}^2\bar\theta_1 \delta^{(2)}(\bar\theta_1) \frac{1}{[x_{1\bar{0}}]^{u_1}}\Bigg|_{\bar\theta_{2,3}=0}$$

$$\int \mathrm{d}^D x_1\, \mathrm{d}^2\theta_1\, \mathrm{d}^2\bar\theta_1 \delta^{(2)}(\bar\theta_1) \frac{1}{[x_{1\bar{0}}]^{u_1}} = \begin{Bmatrix} 4 \\ -4 \end{Bmatrix} \cdot u_1\left(\frac{D}{2}-u_1-1\right)\bar\theta_0^2 \int \mathrm{d}^D x_1 \frac{1}{[x_{10}]^{u_1+1}}$$

$$\Box_1 [x_{12}^2]^{-u} = -4u\left(\frac{D}{2}-u-1\right)[x_{12}^2]^{-u-1}$$

$$\begin{Bmatrix} 4 \\ -4 \end{Bmatrix} \cdot u_1\left(\frac{D}{2}-u_1-1\right) r_0(u_2,D-1-u_1-u_2,u_1+1) \int \mathrm{d}^D x_1 \frac{1}{[x_{13}]^{\frac{D}{2}-u_2}} \frac{1}{[x_{23}]^{\frac{D}{2}-u_1-1}} \frac{1}{[x_{12}]^{u_1+u_2+1-\frac{D}{2}}}.$$

$$\begin{Bmatrix} 4 \\ -4 \end{Bmatrix} \cdot u_1\left(\frac{D}{2}-u_1-1\right) r_0(u_2,D-1-u_1-u_2,u_1+1) r_0\left(D-u_1-1,u_1+u_2+1-\frac{D}{2},\frac{D}{2}-u_2\right).$$

$$\alpha_1^2 \to \alpha_{1,\mathrm{R}}^2 := \mu^\varepsilon \alpha_1^2 + \mu^\varepsilon \alpha_1^2 \cdot \delta^{(L)} := \mu^\varepsilon \alpha_1^2 Z^{(L)}$$

$$A(p_1,p_2,p_3,p_4) = \left\langle \mathrm{tr}(\phi_1(p_1)\phi_1(p_2))\,\mathrm{tr}\left(\phi_1^\dagger(p_3)\phi_1^\dagger(p_4)\right)\right\rangle$$

$$\delta^{(L)} = \frac{(z_{1,1}+z_{1,2}+z_{1,3}+\cdots)}{\varepsilon} + \frac{(z_{2,2}+z_{2,3}+z_{2,4}+\cdots)}{\varepsilon^2} + \cdots$$

$$\xi^2 \to w\cdot\xi^2, \alpha_1^2 \to w\cdot\alpha_1^2$$

$$Z^{(L)} = 1 + \delta^{(L)} = 1 + \sum_{k=1}^{L} \sum_{n=1}^{k} \frac{1}{\varepsilon^n} z_{nk} w^k \ \text{ with } \ \delta_k = \sum_{n=1}^{k} \frac{1}{\varepsilon^n} z_{nk}$$

$$\alpha_{1,\mathrm{R}}^2 = \mu^\varepsilon \alpha_1^2 w + \mu^\varepsilon \alpha_1^2 \delta_1 w^2 + \mu^\varepsilon \alpha_1^2 \delta_2 w^3 + \cdots$$

$$A_L = \sum_{\ell=0}^{L} A^{(\ell)}$$

$$A^{(\ell)} = w^{\ell+1} \sum_{n=0}^{\ell} \frac{1}{\varepsilon^n} a_n^{(\ell)}(\alpha_1^2, \xi^2)$$

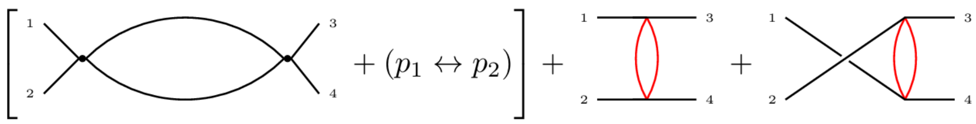
$$A_{\mathrm{R}}^{(\ell)} = w^{\ell+1} \sum_{n=0}^{\ell} \frac{1}{\varepsilon^n} \sum_{k=0}^{\infty} a_{\mathrm{R},nk}^{(\ell)}(\alpha_1^2, \xi^2; \delta_k) w^k \ \text{ with } \ A_{\mathrm{R},L} = \sum_{\ell=0}^{L} A_{\mathrm{R}}^{(\ell)},$$

$$\kappa\left[A_{\mathrm{R},L}\right] \stackrel{!}{=} 0 + \mathcal{O}(w^{L+2}).$$

$$A^{(0)} = 4 \cdot (4\pi)^2 \alpha_1^2 \longrightarrow A_{\mathrm{R}}^{(0)} = 4 \cdot (4\pi)^2 \mu^\varepsilon \left(\alpha_1^2 w + \alpha_1^2 \delta_1 w^2 + \alpha_1^2 \delta_2 w^3 + \mathcal{O}(w^4)\right).$$

$$\kappa\left[A_{\mathrm{R},0}\right] = \kappa\left[A_{\mathrm{R}}^{(0)}\right] = \kappa\left[4 \cdot (4\pi)^2 \mu^\varepsilon \alpha_1^2 w\right] + \mathcal{O}(w^2) = 0 + \mathcal{O}(w^2)$$

$$\pi(s) := \int \frac{\mathrm{d}^{4-2\varepsilon}\ell}{\mathrm{i}(2\pi)^{4-2\varepsilon}} \frac{1}{\ell^2 (k-\ell)^2} = \frac{(-s/\mu^2)^{-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}$$
$$= \frac{1}{16\pi^2 \varepsilon} + \frac{2 - \gamma_{\mathrm{E}} + \log\left(\frac{4\pi\mu^2}{-s}\right)}{16\pi^2} + \mathcal{O}(\varepsilon)$$

$A^{(1)} =$ 

$$= 8 \cdot \left[(4\pi)^2 \alpha_1^2\right]^2 \cdot \pi(s_{12}) + \left[(4\pi)^2 \xi^2\right]^2 \cdot \pi(s_{13}) + \left[(4\pi)^2 \xi^2\right]^2 \cdot \pi(s_{23})$$
$$= \frac{32\pi^2 (\xi^4 + 4\alpha_1^4)}{\varepsilon} + 32\pi^2 \cdot (\xi^4 + 4\alpha_1^4) \left[2 - \gamma_{\mathrm{E}} + \log(4\pi)\right]$$
$$- 16\pi^2 \cdot \left[8\alpha_1^4 \log\left(\frac{-s_{12}}{\mu^2}\right) + \xi^4 \log\left(\frac{-s_{13}}{\mu^2}\right) + \xi^4 \log\left(\frac{-s_{23}}{\mu^2}\right)\right] + \mathcal{O}(\varepsilon).$$

$$A_{\mathrm{R}}^{(1)} = \mu^{2\varepsilon} \frac{32\pi^2 (\xi^4 + 4\alpha_1^4)}{\varepsilon} w^2 + 32\pi^2 \cdot \mu^{2\varepsilon} (\xi^4 + 4\alpha_1^4)\left[2 - \gamma_{\mathrm{E}} + \log(4\pi)\right] w^2$$
$$-16\pi^2 \cdot \mu^{2\varepsilon} \left[8\alpha_1^4 \log\left(\frac{-s_{12}}{\mu^2}\right) + \xi^4 \log\left(\frac{-s_{13}}{\mu^2}\right) + \xi^4 \log\left(\frac{-s_{23}}{\mu^2}\right)\right] w^2$$
$$+\mathcal{O}(\varepsilon) + \mathcal{O}(w^3)$$

$$A_{\mathrm{R},1} = 4 \cdot (4\pi)^2 \mu^\varepsilon (\alpha_1^2 w + \alpha_1^2 \delta_1 w^2) + \mu^{2\varepsilon} \frac{32\pi^2 (\xi^4 + 4\alpha_1^4)}{\varepsilon} w^2 + \mathcal{O}(\varepsilon^0) + \mathcal{O}(w^3) + \mathcal{O}(\varepsilon)$$

$$\kappa[A_{\mathrm{R},1}] = 4\cdot(4\pi)^2\alpha_1^2\kappa[\delta_1]w^2 + \frac{32\pi^2(\xi^4+4\alpha_1^4)}{\varepsilon}w^2 + \mathcal{O}(w^3) \overset{!}{=} 0 + \mathcal{O}(w^3)$$

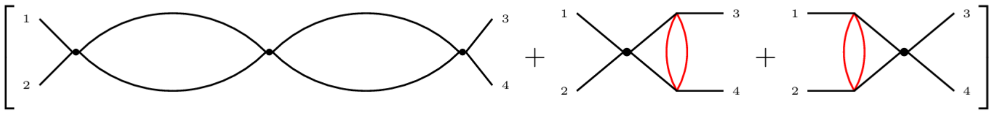
$$\delta_1 = \kappa[\delta_1] = -\frac{\xi^4+4\alpha_1^4}{2\alpha_1^2}\frac{1}{\varepsilon}$$

$$\delta_1 = -\frac{\xi_{\mathrm{R}}^4+4\alpha_{1,\mathrm{R}}^4}{2\alpha_{1,\mathrm{R}}^2}\frac{1}{\varepsilon}$$

$$z_{11} = -\frac{\xi_{\mathrm{R}}^4+4\alpha_{1,\mathrm{R}}^4}{2\alpha_{1,\mathrm{R}}^2}$$

$$V(s) := \int \frac{\mathrm{d}^{4-2\varepsilon}\ell}{\mathrm{i}(2\pi)^{4-2\varepsilon}}\frac{\pi(\ell^2)}{(\ell+p_1)^2(\ell-p_2)^2} = \frac{(-s/\mu^2)^{-2\varepsilon}}{(4\pi)^{4-2\varepsilon}}\frac{\Gamma(\varepsilon)\Gamma(2\varepsilon)\Gamma(1-\varepsilon)^2\Gamma(1-2\varepsilon)^2}{\Gamma(2-2\varepsilon)\Gamma(2-3\varepsilon)}$$
$$= \frac{1}{512\pi^4\varepsilon^2} + \frac{5-2\gamma_{\mathrm{E}}+2\log\left(\frac{4\pi\mu^2}{-s}\right)}{512\pi^4\varepsilon} + \mathcal{O}(\varepsilon^0)$$

$A^{(2)} =$



$$+ \ (p_1 \leftrightarrow p_2)$$
$$= 16\cdot\left[(4\pi)^2\,\alpha_1^2\right]^3\cdot\pi(s_{12})^2 \ + \ 2\cdot 4\ \left[(4\pi)^2\,\xi^2\right]^2(4\pi)^2\,\alpha_1^2\cdot V(s_{12})$$
$$= \frac{64\pi^2\cdot\alpha_1^2\,(\xi^4+4\,\alpha_1^4)}{\varepsilon^2}$$
$$+ \ \frac{64\pi^2\cdot\alpha_1^2\,(\xi^4+4\,\alpha_1^4)}{\varepsilon}\left[5-2\gamma_{\mathrm{E}}-2\log\left(\frac{-s_{12}}{4\pi\mu^2}\right)\right] \ - \ \frac{64\pi^2\cdot\alpha_1^2 4\,\alpha_1^6}{\varepsilon} \ + \ \mathcal{O}(\varepsilon^0)\,.$$

$$A_{\mathrm{R}}^{(2)} = \frac{64\pi^2\alpha_1^2 w^3(4\alpha_1^4+\xi^4)}{\varepsilon^2}$$
$$-\frac{64\pi^2\alpha_1^2 w^3\left[8\alpha_1^4\left(\log\left(\frac{s}{\mu^2}\right)+\gamma_{\mathrm{E}}-2\right)+\xi^4\left(2\log\left(\frac{s}{\mu^2}\right)+2\gamma_{\mathrm{E}}-5\right)\right]}{\varepsilon} + \mathcal{O}(\varepsilon^0) + \mathcal{O}(w^4)$$

$$\kappa[A_{\mathrm{R},2}] = w^3\left\{\frac{64\pi^2\alpha_1^2(4\alpha_1^4+\xi^4)}{\varepsilon^2}\right.$$
$$-\frac{64\pi^2\alpha_1^2\left[-4\alpha_1^2\delta_1+8\alpha_1^4\left(\log\left(\frac{s}{\mu^2}\right)+\gamma_{\mathrm{E}}-2\right)+\xi^4\left(2\log\left(\frac{s}{\mu^2}\right)+2\gamma_{\mathrm{E}}-5\right)\right]}{\varepsilon}$$
$$\left.-256\pi^2\alpha_1^4\delta_1\left(\log\left(\frac{s}{\mu^2}\right)+\gamma_{\mathrm{E}}-2\right)+64\pi^2\alpha_1^2\delta_2\right\}$$
$$+w^2\left\{64\pi^2\alpha_1^2\delta_1+\frac{32\pi^2(4\alpha_1^4+\xi^4)}{\epsilon}\right\}$$

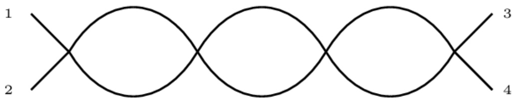
$$\delta_1 = -\frac{\xi^4+4\alpha_1^4}{2\alpha_1^2}\frac{1}{\varepsilon} \quad \text{and} \quad \delta_2 = -\frac{\xi^4}{\varepsilon}+\frac{\xi^4+4\alpha_1^4}{\varepsilon^2}$$

$$\delta_1 = -\frac{\xi_R^4 + 4\alpha_{1,R}^4}{2\alpha_{1,R}^2}\frac{1}{\varepsilon} \Leftrightarrow z_{11} = -\frac{\xi_R^4 + 4\alpha_{1,R}^4}{2\alpha_{1,R}^2},$$

$$\delta_2 = -\frac{\xi_R^4}{\varepsilon} + \frac{\xi_R^4 + 4\alpha_{1,R}^4}{\varepsilon^2} \Leftrightarrow z_{12} = -\xi_R^4, z_{22} = \xi_R^4 + 4\alpha_{1,R}^4.$$

$$Z = 1 - \frac{4\alpha_{1,R}^4 + \xi_R^4}{2\alpha_{1,R}^2} - \frac{\xi_R^4}{\varepsilon} + \frac{\xi_R^8 - 4\alpha_{1,R}^4\xi_R^4}{6\alpha_{1,R}^2} + \frac{4\alpha_{1,R}^4 + \xi_R^4}{\varepsilon^2} + \frac{20\alpha_{1,R}^4\xi_R^4 + \xi_R^8}{6\alpha_{1,R}^2\varepsilon^2} - \frac{16\alpha_{1,R}^4\xi_R^4 + 48\alpha_{1,R}^8 + \xi_R^8}{6\alpha_{1,R}^2\varepsilon^3}$$

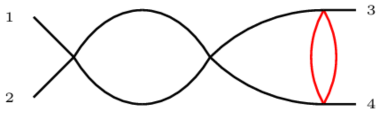
$G_1 :=$



$$= 16 \cdot \left[(4\pi)^2\alpha_1^2\right]^4 \cdot \pi(s_{12})^3 = \alpha_1^8 \cdot 8^{2\varepsilon+3}\pi^{3\varepsilon+2}\frac{\Gamma(1-\varepsilon)^6\Gamma(\varepsilon)^3}{\Gamma(2-2\varepsilon)^3}\left(\frac{s}{\mu^2}\right)^{-3\varepsilon}$$

$$= \frac{512\pi^2\alpha_1^8}{\varepsilon^3} - \frac{1536\pi^2\alpha_1^8\left[S + \gamma_E - 2\right]}{\varepsilon^2} - \frac{128\pi^2\alpha_1^8\left[-18S^2 - 36(\gamma_E - 2)S - 18\gamma_E^2 + \pi^2 + 72\gamma_E - 96\right]}{\varepsilon} + \mathcal{O}\left(\varepsilon^0\right).$$

$G_2 :=$

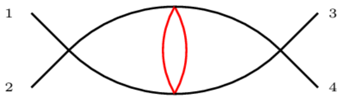


$= D_{HVV}$

$$= 4 \cdot \left[(4\pi)^2\alpha_1^2\right]^2 \left[(4\pi)^2\xi^2\right]^2 \cdot \pi(s_{12})V(s_{12})$$

$$= \alpha_1^4\,\xi^4 \cdot 2^{6\varepsilon+7}\pi^{3\varepsilon+2}\frac{\Gamma(1-2\varepsilon)^2\Gamma(1-\varepsilon)^4\Gamma(\varepsilon)^2\Gamma(2\varepsilon)}{\Gamma(2-3\varepsilon)\Gamma(2-2\varepsilon)^2}\left(\frac{s}{\mu^2}\right)^{-3\varepsilon}$$

$$= \frac{64\pi^2\alpha_1^4\xi^4}{\varepsilon^3} - \frac{64\pi^2\alpha_1^4\xi^4\left(3S + 3\gamma_E - 7\right)}{\varepsilon^2} + \frac{16\pi^2\xi^4\left[54S^2 + 36(3\gamma_E - 7)S + 54\gamma_E^2 + \pi^2 - 252\gamma_E + 396\right]}{3\varepsilon} + \mathcal{O}\left(\varepsilon^0\right).$$

$G_3 :=$ 

$$= 8 \cdot (4\pi\alpha_1)^4 \, (4\pi\xi)^4 \, \left(-\mu^2\right)^{2\varepsilon}$$

$$\cdot \int \frac{\mathrm{d}^{4-2\varepsilon}k_1}{\mathrm{i}\,(2\pi)^{4-2\varepsilon}} \frac{\mathrm{d}^{4-2\varepsilon}k_2}{\mathrm{i}\,(2\pi)^{4-2\varepsilon}} \frac{1}{k_1^2\,(k_1-p)^2}\, \pi\left((k_1-k_2)^2\right) \frac{1}{k_2^2\,(k_2-p)^2}$$

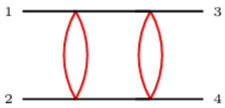
$$= 8 \cdot \alpha_1^4\xi^4 \cdot 2^{6\varepsilon+5}\pi^{3\varepsilon+2} \cdot \varepsilon \, \frac{\Gamma(1-2\varepsilon)\Gamma(1-\varepsilon)^4\Gamma(\varepsilon)\Gamma(3\varepsilon-1)}{(3\varepsilon-2)\Gamma(3-4\varepsilon)\Gamma(2-2\varepsilon)\Gamma(\varepsilon+1)} \left(\frac{\mu^2}{s}\right)^{3\varepsilon}$$

$$\cdot \left[{}_3F_2\left({1,\; 2-3\varepsilon,\; 2-2\varepsilon \atop 3-4\varepsilon,\; 3-3\varepsilon} \,\middle|\, 1 \right) + (3\varepsilon-2) \frac{\Gamma(\varepsilon)\Gamma(2\varepsilon-1)\Gamma(3-4\varepsilon)}{\Gamma(1-\varepsilon)} \cos(2\pi\varepsilon) \right]$$

$$= \frac{128\pi^2\alpha_1^4\xi^4}{3\varepsilon^3} - \frac{128\pi^2\alpha_1^4\xi^4\left[3S+3\gamma_{\mathrm{E}}-7\right]}{3\varepsilon^2}$$

$$- \frac{32\pi^2\alpha_1^4\xi^4\left[-18S^2-12(3\gamma_{\mathrm{E}}-7)S-18\gamma_{\mathrm{E}}^2+\pi^2+84\gamma_{\mathrm{E}}-124\right]}{3\varepsilon}$$

$$+ \mathcal{O}\left(\varepsilon^0\right).$$

$G_4 :=$  $+ \;(p_1 \leftrightarrow p_2)$

$$= (4\pi\xi)^8 \left(\mu^2\right)^{-2\varepsilon} \int \frac{\mathrm{d}^{4-2\varepsilon}k}{\mathrm{i}\,(2\pi)^{4-2\varepsilon}} \pi\left((k-p_1)^2\right) \frac{1}{k^2\,(k-p)^2} \pi\left((k-p_3)^2\right)$$

$$= (4\pi)^8\xi^8 \cdot \mathrm{i}^{2\varepsilon}(4\pi)^{3\varepsilon-6} \left(\frac{s}{\mu^2}\right)^{-2\varepsilon-\delta} \left(\frac{t}{\mu^2}\right)^{-3\varepsilon-\delta} \left(\frac{\Gamma(1-\varepsilon)^2\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\right)^2$$

$$\left[\left(\frac{s}{\mu^2}\right)^{2\varepsilon+\delta} \mathrm{c}_1(1,\varepsilon+\delta,1,\varepsilon|4-2\varepsilon)\;\; {}_3F_2\left({1,\; 1,\; 3\varepsilon+\delta \atop 2\varepsilon+1,\; 2\varepsilon+1+\delta} \,\middle|\, -\frac{s}{t} \right) \right.$$

$$+ \left(\frac{s}{\mu^2}\right)^{\delta} \left(\frac{t}{\mu^2}\right)^{2\varepsilon} \cdot \mathrm{c}_2(1,\varepsilon,1,\varepsilon+\delta|4-2\varepsilon)\;\; {}_2F_1\left({1-2\varepsilon,\; \varepsilon+\delta \atop 1+\delta} \,\middle|\, -\frac{s}{t} \right)$$

$$\left. + \left(\frac{t}{\mu^2}\right)^{2\varepsilon+\delta} \cdot \mathrm{c}_2(1,\varepsilon+\delta,1,\varepsilon+\delta|4-2\varepsilon)\;\; {}_2F_1\left({1-2\varepsilon-\delta,\; \varepsilon \atop 1-\delta} \,\middle|\, -\frac{s}{t} \right) \right]$$

$$+ (t \to u)$$

$$= \frac{32\pi^2\xi^8}{3\varepsilon^3} - \frac{32\pi^2\xi^8\left(3S+3\gamma_{\mathrm{E}}-8\right)}{3\varepsilon^2}$$

$$+ \frac{8\pi^2\xi^8\left[18S^2+12(3\gamma_{\mathrm{E}}-8)S+18\gamma_{\mathrm{E}}^2+3\pi^2-96\gamma_{\mathrm{E}}+176\right]}{3\varepsilon}$$

$$+ \mathcal{O}\left(\varepsilon^0\right) + \mathcal{O}\left(\delta\right).$$

$$c_1(\nu_1,\nu_2,\nu_3,\nu_4 \mid d) = \frac{i^d \Gamma\left(\frac{d}{2}-\nu_{123}\right)\Gamma\left(\frac{d}{2}-\nu_{134}\right)\Gamma\left(-\frac{d}{2}+\nu_{1234}\right)}{\Gamma(\nu_2)\Gamma(\nu_4)\Gamma(d-\nu_{1234})}$$

$$c_2(\nu_1,\nu_2,\nu_3,\nu_4 \mid d) = \frac{i^d \Gamma(\nu_2-\nu_4)\Gamma\left(\frac{d}{2}-\nu_{12}\right)\Gamma\left(\frac{d}{2}-\nu_{23}\right)\Gamma\left(-\frac{d}{2}+\nu_{123}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(d-\nu_{1234})}$$

$$\beta_{\alpha_1^2} := \mu \frac{d}{d\mu}\alpha_{1,R}^2$$

$$\begin{aligned}\beta_{\alpha_1^2} &= \varepsilon \cdot \mu^\varepsilon Z^{(L)}\alpha_1^2 + \alpha_1^2\mu^\varepsilon \cdot \mu \frac{dZ^{(L)}}{d\mu} \\ &= \varepsilon \cdot \alpha_{1,R}^2 + \alpha_{1,R}^2 \cdot \frac{\mu}{Z^{(L)}}\frac{dZ^{(L)}}{d\mu} \\ &= \varepsilon \cdot \alpha_{1,R}^2 + \beta_{\alpha_1^2}^{D=4}\end{aligned}$$

$$\beta_{\xi^2} = \mu \frac{d}{d\mu}\xi_R^2 = \varepsilon \cdot \xi_R^2$$

$$\beta_{\alpha_1^2} = \varepsilon \cdot \alpha_{1,R}^2 + \alpha_{1,R}^2 \cdot \frac{1}{Z^{(L)}}\left[\frac{\partial Z^{(L)}}{\partial \xi_R^2}\beta_{\xi^2} + \frac{\partial Z^{(L)}}{\partial \alpha_{1,R}^2}\beta_{\alpha_1^2}\right].$$

$$\text{Order } \varepsilon: \beta_{\alpha_1^2}^{(1)} = \alpha_{1,R}^2$$

$$\text{Order } \varepsilon^0: \beta_{\alpha_1^2}^{(0)} = \sum_{k=1}^{\infty} \alpha_{1,R}^2\left[\xi_R^2 \frac{\partial z_{1k}}{\partial \xi_R^2} + \alpha_{1,R}^2 \frac{\partial z_{1k}}{\partial \alpha_{1,R}^2}\right] w^k = \beta_{\alpha_1^2}^{D=4}.$$

$$\beta_{\alpha_1^2} = \varepsilon \cdot \alpha_{1,R}^2 + \sum_{k=1}^{\infty} \alpha_{1,R}^2\left[\xi_R^2 \frac{\partial z_{1k}}{\partial \xi_R^2} + \alpha_{1,R}^2 \frac{\partial z_{1k}}{\partial \alpha_{1,R}^2}\right] w^k.$$

$$\beta_{\alpha_1^2}^{3-\text{loop}} = \varepsilon \cdot \alpha_{1,R}^2 - \frac{1}{2}\left(\xi_R^4 + 4\alpha_{1,R}^4\right) - 2\alpha_{1,R}^2\xi_R^4 + \frac{1}{2}\left(-4\alpha_{1,R}^4\xi_R^4 + \xi_R^8\right)$$

$$\beta_{\alpha_1^2}^{3-\text{loop},D=4} = -\frac{1}{2}\left(\xi_R^4 + 4\alpha_{1,R*}^4\right) - 2\alpha_{1,R*}^2\xi_R^4 + \frac{1}{2}\left(-4\alpha_{1,R*}^4\xi_R^4 + \xi_R^8\right) = 0$$

$$\alpha_{1,R*}^2 = \frac{-\xi_R^4 \pm \sqrt{-\xi_R^4 + \xi_R^8 + \xi_R^{12}}}{2(1+\xi_R^4)} = \pm\frac{i}{2}\xi_R^2 - \frac{\xi_R^4}{2} \mp \frac{3i}{4}\xi_R^6 + \mathcal{O}\left(\xi_R^8\right)$$

$$Z^{(3)} = 1 + \left[\xi_R^4 - \frac{4i}{3}\xi_R^6 + \mathcal{O}\left(\xi_R^{10}\right)\right]\frac{1}{\varepsilon} + \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

$$\begin{aligned}\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle_R &= \mu^{\gamma(\xi^2)}\sqrt{Z_{\mathcal{O}}}^2\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle = \frac{v(\xi^2)\mu^{\gamma(\xi^2)}}{|x-y|^{2\Delta_0+\gamma(\xi^2)}} \\ &= \frac{v_0}{|x-y|^{2\Delta_0}}\left[1 - \frac{\xi^2(v_0\gamma_1 L - v_1)}{v_0} + \frac{\xi^4(v_0\gamma_1^2L^2 - 2\gamma_1 v_1 L - v_0\gamma_2 L + v_2)}{2v_0} + \mathcal{O}(\xi^6)\right]\end{aligned}$$

$$\nu(\xi^2) = \sum_{k=0}^{\infty} \frac{\nu_k}{k!} \xi^{2k}$$

$$\gamma(\xi^2) = \sum_{k=1}^{\infty} \frac{\gamma_k}{k!} \xi^{2k}$$

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{\mathrm{i}p\cdot x} (p^2)^\alpha = 4^\alpha \pi^{-D/2} \frac{\Gamma\left(\frac{D}{2}+\alpha\right)}{\Gamma(-\alpha)} \frac{1}{(x^2)^{\frac{D}{2}+\alpha}}$$

$$A_{\text{tree}} = \left(\frac{\mathrm{e}^{-\gamma_{\mathrm{E}}}}{\pi}\right)^{2\varepsilon} \mathrm{O} = \frac{1}{(2\pi)^{4-2\varepsilon}} \frac{\Gamma(1-\varepsilon)^2}{(x^2)^{2-2\varepsilon}}$$

$$\left(\frac{\mathrm{e}^{-\gamma_{\mathrm{E}}}}{\pi}\right)^{3\varepsilon} \mathrm{OO} = 2\cdot(4\pi)^2 \alpha_1^2 \cdot \frac{A_{\text{tree}}}{8\pi^2}\left[\frac{1}{\varepsilon} + L + 1 + \mathcal{O}(\varepsilon)\right]$$

$$\left(\frac{\mathrm{e}^{-\gamma_{\mathrm{E}}}}{\pi}\right)^{4\varepsilon} \mathrm{OOO} = 4\cdot(4\pi)^4 \alpha_1^4 \cdot \frac{A_{\text{tree}}}{256\pi^4}\left[\frac{3}{\varepsilon^2} + \frac{6L+6}{\varepsilon} + 6L^2 + 12L + 3\zeta_2 + \mathcal{O}(\varepsilon)\right]$$

$$\left(\frac{\mathrm{e}^{-\gamma_{\mathrm{E}}}}{\pi}\right)^{4\varepsilon} \langle\mathrm{O}\rangle = 1\cdot(4\pi)^4 \xi^4 \cdot \frac{A_{\text{tree}}}{256\pi^4}\left[\frac{1}{\varepsilon^2} + \frac{2L+3}{\varepsilon} + 2L^2 + 6L + \zeta_2 + 3 + \mathcal{O}(\varepsilon)\right]$$

$$\left(\frac{\mathrm{e}^{-\gamma_{\mathrm{E}}}}{\pi}\right)^{5\varepsilon} \mathrm{OOOO} = 8\cdot(4\pi)^6 \alpha_1^6 \cdot \frac{A_{\text{tree}}}{1024\pi^6}\left[\frac{1}{\varepsilon^3} + \frac{3(L+1)}{\varepsilon^2} + \frac{\frac{9}{2}L^2 + 9L + \frac{3}{2}\zeta_2}{\varepsilon} + \left(\frac{9}{2}L^3 + \frac{27}{2}L^2 + \frac{3}{4}L + 53\zeta(3) + \frac{3}{4} - 40\right) + \mathcal{O}(\varepsilon)\right].$$

$$\left(\frac{\mathrm{e}^{-\gamma_{\mathrm{E}}}}{\pi}\right)^{5\varepsilon} \langle\mathrm{O}\rangle\mathrm{O} = 4\cdot(4\pi)^6 \alpha_1^2 \xi^4 \cdot \frac{A_{\text{tree}}}{1024\pi^6}\left[\frac{1}{3\varepsilon^3} + \frac{3L+4}{3\varepsilon^2} + \frac{\frac{9}{2}L^2 + 12L + \frac{3}{2}\zeta_2 + 4}{3\varepsilon} + \frac{18L^3 + 72L^2 + 3\left(16+\pi^2\right)L + 4\left(67\zeta(3) + \pi^2 - 44\right)}{12} + \mathcal{O}(\varepsilon)\right].$$

$$Z_O = 1 + \frac{\tau_{13} + \tau_{12} + \tau_{11}}{\varepsilon} + \frac{\tau_{23} + \tau_{22}}{\varepsilon^2} + \frac{\tau_{33}}{\varepsilon^3}$$

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle_{\mathrm{R}} = A_{\mathrm{tree}}$$
$$\{1 + w\left(\frac{4\alpha_1^2 + \tau_{11}}{\varepsilon} + 4(L+1)\alpha_1^2\right)$$
$$+ w^2\Big[\frac{-\xi^4 + 4\alpha_1^4 + 4\alpha_1^2\tau_{11} + \tau_{22}}{\varepsilon^2} + \frac{\xi^4 + 16(L+1)\alpha_1^4 + 4(L+1)\alpha_1^2\tau_{11} + \tau_{12}}{\varepsilon}$$
$$+\left((L^2+4L+3)\xi^4 + \frac{4}{3}(15L^2+30L+\pi^2)\alpha_1^4 + \frac{1}{3}(6L^2+12L+\pi^2)\alpha_1^2\tau_{11}\right)$$
$$+\mathcal{O}(\varepsilon)\Big] + \mathcal{O}(w^3)\}$$

$$\tau_{11} = -4\alpha_1^2\ , \quad \tau_{12} = -\xi^4\ , \quad \tau_{13} = -\tfrac{4}{3}\alpha_1^2\xi^4\ ,$$
$$\tau_{22} = 12\alpha_1^4 + \xi^4\ , \quad \tau_{23} = \tfrac{20}{3}\alpha_1^2\xi^4\ ,$$
$$\tau_{33} = -\tfrac{16}{3}\alpha_1^2\left(6\alpha_1^4 + \xi^4\right)$$

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle_{\mathrm{R},\alpha_1^2\to 0} = A_{\mathrm{tree}}\{1 + w\left(\frac{\tau_{11}}{\varepsilon} + \mathcal{O}(\varepsilon)\right) + w^2\Big[\frac{\xi^4 + \tau_{22}}{\varepsilon^2} + \frac{(2L+3)\xi^4 + \tau_{12}}{\varepsilon}$$
$$+\frac{1}{6}(12L^2 + 36L + \pi^2 + 18)\xi^4 + \mathcal{O}(\varepsilon)\Big]$$
$$+\mathcal{O}(w^3)\}$$

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle_{\mathrm{R,\ fixed\text{-}point}} = A_{\mathrm{tree}}\left[1 + 2i(L+1)\xi^2 + (-2L^2 - 4L + 1)\xi^4 + \mathcal{O}(\xi^6) + \mathcal{O}(\varepsilon)\right]$$

$$\gamma(\xi^2) = -2\mathrm{i}\xi^2 + \frac{1}{2}\cdot 0\xi^4 + \frac{1}{6}\cdot 6\mathrm{i}\xi^6 + O(\xi^8) = -2\mathrm{i}\xi^2 + \mathrm{i}\xi^6 + \mathcal{O}(\xi^8)$$
$$v(\xi^2) = 1 + 2\mathrm{i}\xi^2 + \frac{1}{2}\cdot 2\xi^4 + \frac{1}{6}\cdot 2\mathrm{i}(-75 + 40\pi^2 - 308\zeta_3)\xi^6 + \mathcal{O}(\xi^8)$$
$$= 1 + 2\mathrm{i}\xi^2 + \xi^4 + \mathrm{i}\left(-25 + 80\zeta_2 - \frac{308}{3}\zeta_3\right)\xi^6 + \mathcal{O}(\xi^8)$$

$$S(\alpha)S(-\alpha) = 1 (\text{Unitarity})$$
$$\frac{S(\eta-\alpha)}{S(\alpha)} = \frac{f_0(\eta-\alpha)}{f_0(\alpha)}\ (\text{Crossing})$$

$$S(\alpha) = \frac{1}{f_0(\eta-\alpha)}f_1(\alpha)\ \text{with}\ \frac{f_1(\eta-\alpha)}{f_1(\alpha)} = 1\ (\text{ Crossing } f_1)$$

$$\frac{f_1(\alpha)f_1(-\alpha)}{f_0(\eta-\alpha)f_0(\eta+\alpha)} = 1$$

$$f_1(\alpha) = f_0(\eta+\alpha)f_2(\alpha)\ \text{with}\ f_2(\alpha)f_2(-\alpha) = 1\ (\text{ Unitarity } f_2).$$

$$\frac{f_0(2\eta-\alpha)f_2(\eta-\alpha)}{f_0(\eta+\alpha)f_2(\alpha)} = 1$$

$$f_2(\alpha) = f_0(2\eta-\alpha)f_3(\alpha)\ \text{with}\ \frac{f_3(\eta-\alpha)}{f_3(\alpha)} = 1\ (\text{ Crossing } f_3)$$

$$f_0(2\eta-\alpha)f_0(2\eta+\alpha)f_3(\alpha)f_3(-\alpha) = 1$$

$$f_3(\alpha) = \frac{1}{f_0(2\eta+\alpha)} f_4(\alpha) \text{ with } f_4(\alpha) f_4(-\alpha) = 1 \text{ (Unitarity } f_4)$$

$$\frac{f_0(2\eta+\alpha) f_4(\eta-\alpha)}{f_0(3\eta-\alpha) f_4(\alpha)} = 1$$

$$f_4(\alpha) = \frac{1}{f_0(3\eta-\alpha)} f_5(\alpha) \text{ with } \frac{f_5(\eta-\alpha)}{f_5(\alpha)} = 1 \text{ (Crossing } f_5)$$

$$S(\alpha) = \frac{1}{f_0(\eta-\alpha)} f_0(\eta+\alpha) f_0(2\eta-\alpha) \frac{1}{f_0(2\eta+\alpha)} \frac{1}{f_0(3\eta-\alpha)} \cdot f_5(\alpha)$$

$$S(\alpha) = \prod_{k=1}^{\infty} \frac{f_0((2k-1)\cdot\eta+\alpha) f_0(2k\cdot\eta-\alpha)}{f_0((2k-1)\cdot\eta-\alpha) f_0(2k\cdot\eta+\alpha)}.$$

$$S(\alpha) = \frac{1}{f_0(\eta-\alpha)} \prod_{k=1}^{\infty} \frac{f_0(2k\cdot\eta-\eta+\alpha) f_0(2k\cdot\eta-\alpha)}{f_0(2k\cdot\eta+\eta-\alpha) f_0(2k\cdot\eta+\alpha)}.$$

$$\prod_{k=1}^{\infty} \frac{f_0((2k+1)\eta)}{f_0((2k-1)\eta)} = \frac{1}{f_0(\eta)}$$

$$S(\alpha) = \frac{f_0(\eta)}{f_0(\eta-\alpha)} \prod_{k=1}^{\infty} \frac{f_0(2\eta k-\eta+\alpha) f_0(2\eta k-\alpha) f_0(2\eta k+\eta)}{f_0(2\eta k+\eta-\alpha) f_0(2\eta k+\alpha) f_0(2\eta k-\eta)}.$$

$$f_0(u) = \frac{\pi^u}{\Gamma(u+\ell)}$$

$$\kappa_\ell(u) = \pi^u \frac{\Gamma\left(\frac{D}{2}-u+\ell\right)}{\Gamma\left(\frac{D}{2}+\ell\right)} \prod_{k=1}^{\infty} \frac{\Gamma\left(Dk+\frac{D}{2}-u+\ell\right)\Gamma(Dk+u+\ell)\Gamma\left(Dk-\frac{D}{2}+\ell\right)}{\Gamma\left(Dk-\frac{D}{2}+u+\ell\right)\Gamma(Dk-u+\ell)\Gamma\left(Dk+\frac{D}{2}+\ell\right)}$$

$$\vartheta_1(z \mid q) = 2\sum_{n=0}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} \sin[(2n+1)z] = 2Gq^{\frac{1}{4}} \sin(z) \prod_{n=1}^{\infty} \left(1-q^{2n}\mathrm{e}^{2\mathrm{i}z}\right)\left(1-q^{2n}\mathrm{e}^{-2\mathrm{i}z}\right)$$
$$\vartheta_2(z \mid q) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} \cos[(2n+1)z] = 2Gq^{\frac{1}{4}} \cos(z) \prod_{n=1}^{\infty} \left(1+q^{2n}\mathrm{e}^{2\mathrm{i}z}\right)\left(1+q^{2n}\mathrm{e}^{-2\mathrm{i}z}\right)$$
$$\vartheta_3(z \mid q) = 1+2\sum_{n=1}^{\infty} q^{n^2} \cos(2nz) = G \prod_{n=1}^{\infty} \left(1+q^{2n-1}\mathrm{e}^{2\mathrm{i}z}\right)\left(1+q^{2n-1}\mathrm{e}^{-2\mathrm{i}z}\right)$$
$$\vartheta_4(z \mid q) = 1+2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz) = G \prod_{n=1}^{\infty} \left(1-q^{2n-1}\mathrm{e}^{2\mathrm{i}z}\right)\left(1-q^{2n-1}\mathrm{e}^{-2\mathrm{i}z}\right)$$

$$G := \prod_{n=1}^{\infty} (1-q^{2n})$$

$$\Gamma^{(r)}(z \mid q_1, \ldots, q_r) = \prod_{n_1,\ldots,n_r=0}^{\infty} \frac{1 - q_1^{n_1+1} \cdots q_r^{n_r+1} z^{-1}}{\left(1 - q_1^{n_1} \cdots q_r^{n_r} z\right)^{(-1)^r}} = \exp\left[\sum_{k=1}^{\infty} \frac{(-1)^r z^k - q_1^k \cdots q_r^k z^{-k}}{k \cdot \prod_{i=1}^{r} \left(1 - p_i^k\right)}\right]$$

$$\Gamma^{(1)}(q \cdot z \mid q) = \Gamma^{(1)}(z^{-1} \mid q), \qquad \Gamma^{(1)}(z \mid q) = \Gamma^{(1)}(z \mid q^2)\Gamma^{(1)}(q \cdot z \mid q^2)$$
$$\Gamma^{(2)}(pq \cdot z \mid q, p) = \frac{1}{\Gamma^{(2)}(z^{-1} \mid q, p)}, \quad \Gamma^{(2)}(p \cdot z \mid q, p) = \Gamma^{(1)}(z \mid q)\Gamma^{(2)}(z \mid q, p)$$

$$\kappa(u) = \mathrm{w}(u)\frac{\mathrm{w}(\eta - u)}{\mathrm{w}(\eta)} \prod_{k=1}^{\infty} \frac{\mathrm{w}(2\eta k + \eta - u)\mathrm{w}(2\eta k + u)\mathrm{w}(2\eta k - \eta)}{\mathrm{w}(2\eta k - \eta + u)\mathrm{w}(2\eta k - u)\mathrm{w}(2\eta k + \eta)}.$$

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \quad \bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}},$$
$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \quad \bar{\psi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}$$

$$\psi\chi = (\psi\chi) = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha, \qquad \psi\sigma^\mu\bar{\chi} = \psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}},$$
$$\bar{\psi}\bar{\chi} = (\bar{\psi}\bar{\chi}) = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = (\psi\chi)^\dagger, \quad (\psi\sigma^\mu\bar{\chi})^\dagger = \chi\sigma^\mu\bar{\psi},$$

$$\bar{\sigma}^{\mu,\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}\varepsilon^{\dot{\alpha}\dot{\beta}}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\beta}\beta} = -2\delta_\alpha^\beta\delta_{\dot{\alpha}}^{\dot{\beta}}, \quad [\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu]_\alpha^\beta = -2\eta^{\mu\nu}\delta_\alpha^\beta,$$
$$\mathrm{tr}[\sigma^\mu\bar{\sigma}^\nu] = -2\eta^{\mu\nu}, \quad [\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu]^{\dot{\alpha}}_{\dot{\beta}} = -2\eta^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}},$$

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2, \\ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}^2. \end{aligned} \quad \theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} = -\frac{1}{2}\theta^2\bar{\theta}^2\eta^{\mu\nu}$$

$$D_\alpha = \partial_\alpha + \mathrm{i}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - \mathrm{i}\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$
$$Q_\alpha = \partial_\alpha - \mathrm{i}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \bar{Q}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + \mathrm{i}\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}, \bar{\partial}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \text{ and } \partial_\mu = \frac{\partial}{\partial x^\mu}$$

$$\int \mathrm{d}^2\theta\theta^2 = 1 \text{ and } \int \mathrm{d}^2\bar{\theta}\bar{\theta}^2 = 1$$

$$[\theta^\alpha] = [\bar{\theta}^{\dot{\alpha}}] = \frac{[x^\mu]}{2} = -\frac{1}{2}$$

$$\int \mathrm{d}^2\theta f(\theta) = \left[-\frac{1}{4}D^2 f(\theta)\right]_{\theta=0}, \int \mathrm{d}^2\bar{\theta} f(\bar{\theta}) = \left[-\frac{1}{4}\bar{D}^2 f(\bar{\theta})\right]_{\bar{\theta}=0}$$
$$\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta} f(\theta, \bar{\theta}) = \left[\frac{1}{16}D^2\bar{D}^2 f(\theta, \bar{\theta})\right]_{\theta,\bar{\theta}=0}$$

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \quad \bar\psi^\alpha = \varepsilon^{\alpha\beta}\bar\psi_\beta$$
$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \quad \bar\psi_\alpha = \varepsilon_{\alpha\beta}\bar\psi^\beta$$

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha \ , \qquad \bar\psi\bar\chi = \bar\psi^\alpha\bar\chi_\alpha = -\bar\psi_\alpha\bar\chi^\alpha \ ,$$

$$\psi\gamma^\mu\bar\chi = \psi^\alpha\gamma^\mu_{\alpha\beta}\bar\chi^\beta$$

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2 \ , \qquad \bar\theta^\alpha\bar\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\bar\theta^2 \ .$$

$$\theta\gamma^\mu\bar\theta \ \theta\gamma^\nu\bar\theta = \frac{1}{2}\theta^2\bar\theta^2\eta^{\mu\nu}$$

$$D_\alpha = \partial_\alpha + \mathrm{i}\gamma^\mu_{\alpha\beta}\bar\theta^\beta\partial_\mu, \quad \bar D_\alpha = -\bar\partial_\alpha - \mathrm{i}\theta^\beta\gamma^\mu_{\beta\alpha}\partial_\mu$$
$$Q_\alpha = \partial_\alpha - \mathrm{i}\gamma^\mu_{\alpha\beta}\bar\theta^\beta\partial_\mu, \quad \bar Q_\alpha = -\bar\partial_\alpha + \mathrm{i}\theta^\beta\gamma^\mu_{\beta\alpha}\partial_\mu$$

$$\Phi(z) \ = \phi(x_+) + \sqrt{2}\theta\psi(x_+) + \theta^2 F(x_+)$$
$$\Phi^\dagger(z) \ = \phi^\dagger(x_-) - \sqrt{2}\bar\theta\bar\psi(x_-) - \bar\theta^2 F^\dagger(x_-)$$

$$V_\Delta = \mathrm{L}^2\left(\mathbb{R}^D, (1+x^2)^{2\mathrm{Re}(\Delta)-D}\mathrm{d}^D x\right)$$

$$C_{\mu_1\cdots\mu_{2\ell}}W_u^{\ell,\mu_1\cdots\mu_{2\ell}}(x_{12}) = \frac{1}{(-2)^{2\ell}}\frac{\Gamma(u-\ell)}{\Gamma(u+\ell)}C_{\mu_1\cdots\mu_{2\ell}}\partial_1^{\mu_1}\cdots\partial_1^{\mu_{2\ell}}W_{u-\ell}(x_{12}).$$

$$\mathrm{c}_1(\Delta, S) = 16\pi^3\frac{(-1)^{2S}(\Delta-S-1)^2(\Delta+S)}{(2\Delta-1)(2S+1)(-\Delta+S+1)}\tan(\pi\Delta)$$
$$\mathrm{c}_2(\Delta, S) = \pi^3(-1)^S 2^{2\Delta-S+2}\frac{\Gamma(S+1)\Gamma\left(\Delta-\frac{1}{2}\right)\Gamma\left(\frac{S-\Delta+3}{2}\right)\Gamma\left(\frac{S+\Delta+1}{2}\right)}{\Gamma\left(S+\frac{3}{2}\right)\Gamma(\Delta)\Gamma\left(\frac{S-\Delta+2}{2}\right)\Gamma\left(\frac{S+\Delta}{2}\right)}$$

$$\mathrm{e}^{\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu}(f(x)\cdot g(x)) = \mathrm{e}^{\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu}f(x)\mathrm{e}^{\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu}g(x)$$

$$Z_Q[\eta,\bar\eta] = \int \ [DA_\mu][D\lambda][D\bar\lambda]e^{-S_{\mathrm{SYM}^*}+i\theta Q-\int_{\mathbb{T}^4}(\eta\lambda+\bar\lambda\bar\eta)} \sim e^{-\frac{8\pi^2|Q|}{g^2}+i\theta Q}$$

$$Z_Q^{\mathrm{Reg}}[\eta,\bar\eta] \equiv \frac{Z_Q[\eta,\bar\eta]}{\left[\int \ [DA_\mu][D\lambda][D\bar\lambda]e^{-S_{\mathrm{SYM}^*}}\right]_{Q,\mathrm{PV}}}$$

$$Z^T[\eta,\bar\eta] = \sum_{Q=0,\pm\frac{1}{N},\pm\frac{2}{N},\dots} Z_Q^{\mathrm{Reg}}[\eta,\bar\eta]$$

$$\langle\mathcal{O}\rangle^{\mathrm{Reg}}(x_1,x_2,\dots,x_n) = \left[(-1)^n\frac{\partial^n Z^T[\eta,\bar\eta]}{\partial\eta(x_1)\partial\eta(x_2)\dots\partial\bar\eta\partial\dots\bar\eta(x_n)}/Z^T[\eta,\bar\eta]\right]_{\eta=\bar\eta=0}$$

$$\left\langle \mathrm{tr}\left[\lambda(x)\prod_{\mu=1}^{4}\ \mathcal{W}_\mu(x)\Gamma_{\nu_1\nu_2..}\lambda(0)\mathcal{W}_\mu^\dagger(x)\right]\right\rangle \equiv \frac{\sum_Q\ \left\langle \mathrm{tr}\left[\lambda(x)\prod_{\mu=1}^{4}\ \mathcal{W}_\mu(x)\Gamma_{\nu_1\nu_2..}\lambda(0)\mathcal{W}_\mu^\dagger(x)\right]\right\rangle_{Q,\text{ unnorm}}}{Z_T}$$

$$\langle \mathrm{tr}[\bar{\Psi}\Psi]\rangle^{\mathrm{Reg.}}\Big|_{|m|LN\ll 1}$$

$$\simeq 32\pi^2\Lambda^3\left(1+\frac{|m|^2L^2}{c\Delta}\right)\cos\left(\frac{\theta}{N}\right)+16\pi^2\Lambda^3\left(\frac{L^2m^{*2}}{c\Delta}e^{-i\frac{\theta}{N}}+\frac{L^2m^2}{c\Delta}e^{i\frac{\theta}{N}}\right)$$

$$-i\langle \mathrm{tr}[\bar{\Psi}\gamma_5\Psi]\rangle^{\mathrm{Reg.}}\Big|_{|m|LN\ll 1}$$

$$\simeq -32\pi^2\Lambda^3\left(1+\frac{|m|^2L^2}{c\Delta}\right)\sin\left(\frac{\theta}{N}\right)+i16\pi^2\Lambda^3\left(\frac{L^2m^{*2}}{c\Delta}e^{-i\frac{\theta}{N}}-\frac{L^2m^2}{c\Delta}e^{i\frac{\theta}{N}}\right)$$

$$\Lambda^3=\mu^3e^{-\frac{8\pi^2}{Ng^2(\mu)}}/g^2(\mu)$$

$$0<\Delta\equiv\frac{(N-1)L_3L_4-L_1L_2}{\sqrt{V}}\ll 1$$

$$|m|LN\ll\Lambda LN\ll 1,\text{ and }\frac{|m|^2L^2}{c}\ll\Delta\ll 1,\text{ with }c\equiv\frac{4\pi}{N-1}.$$

$$U(1)_{\mathrm{spurious}}:\Psi\to e^{-i\alpha\gamma_5}\Psi,\bar{\Psi}\to\bar{\Psi}e^{-i\alpha\gamma_5}\left(\text{ or }\lambda\to e^{i\alpha}\lambda,\bar{\lambda}\to e^{-i\alpha}\bar{\lambda}\right)$$
$$m\to e^{-i2\alpha}m,m^*\to e^{i2\alpha}m^*,\theta\to\theta+2N\alpha$$

$$\delta\mathcal{E}|_{|m|LN\ll 1}\simeq -32\pi^2\Lambda^3|m|\left(1+2\frac{|m|^2L^2}{c\Delta}\right)\cos\left(\frac{\theta_{\mathrm{eff}}}{N}\right),$$

$$|m|\ll\Lambda:\delta\mathcal{E}=-32\pi^2\Lambda^3|m|\cos\left(\frac{\theta_{\mathrm{eff}}}{N}\right)$$

$$S_{\mathrm{SYM}^*}=\frac{1}{g^2}\int_{\mathbb{T}^4}\mathrm{tr}\ \Box\left[\frac{1}{2}F_{\mu\nu}F_{\mu\nu}-2\bar{\lambda}_{\dot{\alpha}}D_\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\lambda_\alpha+m\lambda^\alpha\lambda_\alpha+m^*\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}\right]$$

$$\mathrm{tr}_\Box\left(T^aT^b\right)=\delta^{ab},\lambda_\alpha=\lambda_\alpha^aT^a$$

$$F_{\mu\nu}=-i[D_\mu,D_\nu]=\partial_\mu A_\nu-\partial_\nu A_\mu+i[A_\mu,A_\nu]$$

$$\left(D_\mu F_{\mu\nu}\right)^a=-i\bar{\lambda}\bar{\sigma}_\nu[T^a,\lambda],\bar{\sigma}_\mu^{\dot{\alpha}\alpha}D_\mu\lambda_\alpha=m^*\bar{\lambda}^{\dot{\alpha}},\left(\sigma_\mu\right)_{\alpha\dot{\alpha}}D_\mu\bar{\lambda}^{\dot{\alpha}}=m\lambda_\alpha$$

$$\sigma_\mu\equiv(i\vec{\sigma},1),\bar{\sigma}_\mu\equiv(-i\vec{\sigma},1),\vec{\sigma}$$

$$\bar{\sigma}_\mu^{\dot{\alpha}\alpha}=\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma_{\mu\beta\dot{\beta}},\sigma_{\mu\beta\dot{\beta}}=\epsilon_{\beta\alpha}\epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}$$

$$U(1)_{\mathrm{spurious}}:\lambda\to e^{i\alpha}\lambda,\bar{\lambda}\to e^{-i\alpha}\lambda$$
$$m\to e^{-i2\alpha}m,m^*\to e^{i2\alpha}m^*$$
$$\theta\to\theta+2N\alpha$$

$$\Psi=\begin{bmatrix}\lambda_\alpha\\ \bar{\lambda}^{\dot{\alpha}}\end{bmatrix},\bar{\Psi}=\left[\lambda^\alpha\bar{\lambda}_{\dot{\alpha}}\right]$$

$$\mathrm{tr}[\bar{\Psi}\Psi(x)] = \mathrm{tr}[\lambda\lambda + \bar{\lambda}\bar{\lambda}](x), \mathrm{tr}[\bar{\Psi}\gamma_5\Psi(x)] = \mathrm{tr}[-\lambda\lambda + \bar{\lambda}\bar{\lambda}](x)$$

$$\mathrm{tr}[\bar{\Psi}\Psi] \stackrel{\mathcal{CP}}{\to} \mathrm{tr}[\bar{\Psi}\Psi], \mathrm{tr}[\bar{\Psi}\gamma_5\Psi] \stackrel{\mathcal{CP}}{\to} -\mathrm{tr}[\bar{\Psi}\gamma_5\Psi]$$

$$A_\nu(x + L_\mu\hat{e}_\mu) = \Omega_\mu(x)A_\nu(x)\Omega_\mu^{-1}(x) - i\Omega_\mu(x)\partial_\nu\Omega_\mu^{-1}(x)$$

$$\Omega_\mu(x + \hat{e}_\nu L_\nu)\Omega_\nu(x) = e^{i\frac{2\pi}{N}n_{\mu\nu}}\Omega_\nu(x + \hat{e}_\mu L_\mu)\Omega_\mu(x)$$

$$Q = -\frac{n_{12}n_{34}}{N}e^{i\frac{2\pi}{N}n_{\mu\nu}} = \frac{k}{N}e^{i\frac{2\pi}{N}n_{\mu\nu}}$$

$$Z^T \equiv \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{tr}_{\mathcal{H}_{n34}^{SYM}}\left((-1)^F e^{-L_1\hat{H}_{SYM}-L_1\hat{H}_m}\hat{T}_2^k\right)$$

$$\langle\mathcal{O}(x_1 = 0)\rangle \equiv \frac{1}{NZ^T}\sum_{k=0}^{N-1} \mathrm{tr}_{\mathcal{H}_{n_{34}=1}^{SYM}}\left((-1)^F e^{-L_1\hat{H}_{SYM}-L_1\hat{H}_m}\hat{T}_2^k\hat{\mathcal{O}}(x_1 = 0)\right)$$

$$\hat{H}_m = \int_{\mathbb{T}^3} d^3x\left(-m(\hat{\lambda})^2 - m^*\left(\hat{\lambda}^\dagger\right)^2\right.$$

$$\hat{T}_2|E, e_2\rangle = |E, e_2\rangle e^{i\frac{2\pi}{N}e_2}$$

$$\hat{T}_2\hat{X}\hat{T}_2^{-1} = e^{-i\frac{2\pi}{N}}\hat{X}, \text{ where } \hat{X}\hat{\lambda}\hat{X}^{-1} = e^{i\frac{2\pi}{2N}}\hat{\lambda}$$

$$\hat{X}|E, e_2\rangle = |E, e_2 - 1\rangle.$$

$$Z^T = \sum_{E,e_2} \langle E, e_2|(-1)^F e^{-L_1\hat{H}_{SYM}-L_1\hat{H}_m}\left(\frac{1}{N}\sum_{k=0}^{N-1} \hat{T}_2^k\right)|E, e_2\rangle$$

$$Z^T \simeq (1 + |m|^2L^2c_0) + c\left(Lme^{-\frac{8\pi^2}{Ng^2}}e^{i\frac{\theta}{N}} + Lm^*e^{-\frac{8\pi^2}{Ng^2}}e^{-i\frac{\theta}{N}}\right) + \mathcal{O}\left(|m|^3, e^{-\frac{16\pi^2}{Ng^2}}\right)$$
$$= (1 + |m|^2L^2c_0) + c'L^4|m|\Lambda^3\cos\frac{\theta + N\arg m}{N} + \mathcal{O}\left(|m|^3, e^{-\frac{16\pi^2}{Ng^2}}\right)$$

$$Z_T\langle\hat{\lambda}^2\rangle = 16\pi^2\Lambda^3(1 + c_1|m|^2L^2) + c_2\frac{m^*}{L^2} + c_3(m^*L)^2\Lambda^3e^{-i\frac{\theta}{N}}$$
$$Z_T\left\langle\left(\hat{\lambda}^\dagger\right)^2\right\rangle = 16\pi^2\Lambda^3(1 + c_1|m|^2L^2) + c_2\frac{m}{L^2} + c_3(mL)^2\Lambda^3e^{i\frac{\theta}{N}}$$

$$P_kQ_k = e^{i\frac{2\pi}{k}}Q_kP_k$$

$$(P_k)_{B'C'} = \gamma_k\delta_{B',C'-1(\mathrm{mod}k)}$$

$$(Q_k)_{C'B'} = \gamma_k e^{i2\pi\frac{C'-1}{k}}\delta_{C'B'}$$

$$\gamma_k = e^{i\frac{\pi(1-k)}{k}}$$

$$\mathrm{Det}(P_k)=\mathrm{Det}(Q_k)=1$$

$$\omega=2\pi\mathrm{diag}(\underbrace{\ell,\ell,\ldots,\ell}_{k\text{ times}},\underbrace{-k,-k,\ldots,-k}_{\ell\text{ times}}),$$

$$\Omega_1=(-1)^{k-1}I_k\oplus I_\ell e^{i\omega\frac{x_2}{NL_2}}=\begin{bmatrix}(-1)^{k-1}I_k e^{i2\pi\ell\frac{x_2}{NL_2}} & 0\\ 0 & e^{-i2\pi k\frac{x_2}{NL_2}}I_\ell\end{bmatrix}$$
$$\Omega_2=Q_k\oplus I_\ell=\begin{bmatrix}Q_k & 0\\ 0 & I_\ell\end{bmatrix}$$
$$\Omega_3=I_k\oplus P_\ell e^{i\omega\frac{x_4}{N\ell L_4}}=\begin{bmatrix}e^{i2\pi\frac{x_4}{NL_4}}I_k & 0\\ 0 & e^{-i2\pi k\frac{x_4}{N\ell L_4}}P_\ell\end{bmatrix},\Omega_4=I_k\oplus Q_\ell=\begin{bmatrix}I_k & 0\\ 0 & Q_\ell\end{bmatrix},$$

$$A_1=0,A_2=-\omega\frac{x_1}{NL_1L_2},A_3=0,A_4=-\omega\frac{x_3}{N\ell L_3L_4}$$

$$F_{12}=-\omega\frac{1}{NL_1L_2},F_{34}=-\omega\frac{1}{N\ell L_3L_4}$$

$$\delta A_1=-\omega\frac{z_1}{L_1}+\frac{2\pi}{L_1}\boldsymbol{a}_1\cdot\boldsymbol{H}_{(k)},\qquad \delta A_2=-\omega\frac{z_2}{L_2}+\frac{2\pi}{L_2}\boldsymbol{a}_2\cdot\boldsymbol{H}_{(k)}$$
$$\delta A_3=-\omega\frac{z_3}{L_3}+\frac{2\pi}{L_3}\boldsymbol{a}_3\cdot\boldsymbol{H}_{(k)},\delta A_4=-\omega\frac{z_4}{L_4}+\frac{2\pi}{L_4}\boldsymbol{a}_4\cdot\boldsymbol{H}_{(k)}$$

$$\lambda\left(x+L_\mu\hat{e}_\mu\right)=\Omega_\mu\lambda(x)\Omega_\mu^{-1}$$

$$\lambda=\begin{bmatrix}\|\lambda_{C'B'}\| \\ \|\lambda_{CB'}\|\end{bmatrix}\|\lambda_{C'B}\|\lambda_{CB}\|\Big],C',B'\in\{1,\ldots k\},C,B\in\{1,\ldots\ell\},$$

$$\sum_{i_1=1}^{N}\lambda_{i_1}^{i_1}=\sum_{C'=1}^{k}\lambda_{C'C'}+\sum_{C=1}^{\ell}\lambda_{CC}=0$$

$$\lambda_{C'B'}(x+L_1\hat{e}_1)=\lambda_{C'B'}(x),\lambda_{C'B'}(x+L_2\hat{e}_2)=e^{i2\pi\frac{C'-B'}{k}}\lambda_{C'B'}(x)$$
$$\lambda_{C'B'}(x+L_3\hat{e}_3)=\lambda_{C'B'}(x),\lambda_{C'B'}(x+L_4\hat{e}_4)=\lambda_{C'B'}(x)$$

$$\lambda_{CB}(x+L_1\hat{e}_1)=\lambda_{CB}(x),\lambda_{CB}(x+L_2\hat{e}_2)=\lambda_{CB}(x)$$
$$\lambda_{CB}(x+L_3\hat{e}_3)=\lambda_{[C+1]_\ell}[B+1]_\ell(x),\lambda_{CB}(x+L_4\hat{e}_4)=e^{i2\pi\frac{C-B}{\ell}}\lambda_{CB}(x),$$

$$\lambda_{C'B}(x+L_1\hat{e}_1)=\gamma_k^{-k}e^{i2\pi\frac{x_2}{L_2}}\lambda_{C'B}(x),\lambda_{C'B}(x+L_2\hat{e}_2)=\gamma_k e^{i2\pi\frac{(C'-1)}{k}}\lambda_{C'B}(x)$$
$$\lambda_{C'B}(x+L_3\hat{e}_3)=\gamma_\ell^{-1}e^{i2\pi\frac{x_4}{\ell L_4}}\lambda_{C'[B+1]_\ell}(x),\lambda_{C'B}(x+L_4\hat{e}_4)=\gamma_\ell^{-1}e^{-i2\pi\frac{(B-1)}{\ell}}\lambda_{C'B}(x)$$

$$\text{self-dual }\mathbb{T}^4:\frac{L_1L_2}{L_3L_4}=N-k$$

$$S_0=\frac{1}{2g^2}\int_{\mathbb{T}^4}\mathrm{tr}\left[F_{\mu\nu}F_{\mu\nu}\right]=\frac{8\pi^2|Q|}{g^2}=\frac{8\pi^2k}{Ng^2}$$

$$
\begin{aligned}
g^2\mathcal{L}_f =& -2\bar{\lambda}_{\dot{\alpha}C'B'}\left[D_\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\right]_{B'D'}\lambda_{\alpha D'C'} - 2\bar{\lambda}_{\dot{\alpha}CB}\left[D_\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\right]_{BD}\lambda_{\alpha DC} \\
& -2\bar{\lambda}_{\dot{\alpha}C'B}\left[D_\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\right]_{BD}\lambda_{\alpha DC'} - 2\bar{\lambda}_{\dot{\alpha}CB'}\left[D_\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\right]_{B'D'}\lambda_{\alpha D'C} \\
& +m\lambda_{CB}^{\alpha}\lambda_{\alpha BC} + m\lambda_{C'B'}^{\alpha}\lambda_{\alpha B'C'} + m\lambda_{CB'}^{\alpha}\lambda_{\alpha B'C} + m\lambda_{C'B}^{\alpha}\lambda_{\alpha BC'} \\
& +m^*\bar{\lambda}_{\dot{\alpha}CB}\bar{\lambda}_{BC}^{\dot{\alpha}} + m^*\bar{\lambda}_{\dot{\alpha}C'B'}\bar{\lambda}_{B'C'}^{\dot{\alpha}} + m^*\bar{\lambda}_{\dot{\alpha}CB'}\bar{\lambda}_{B'C}^{\dot{\alpha}} + m^*\bar{\lambda}_{\dot{\alpha}C'B}\bar{\lambda}_{BC'}^{\dot{\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
g^2\mathcal{L}_f =& -2\bar{\lambda}_{\dot{\alpha}C'B'}\partial_\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\lambda_{\alpha B'C'} - 2\bar{\lambda}_{\dot{\alpha}CB}\partial_\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\lambda_{\alpha BC} \\
& -2\bar{\lambda}_{\dot{\alpha}C'B}\left[D_\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\right]_{BB}\lambda_{\alpha BC'} - 2\bar{\lambda}_{\dot{\alpha}CB'}\left[D_\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\right]_{B'B'}\lambda_{\alpha B'C} \\
& +m\lambda_{CB}^{\alpha}\lambda_{\alpha BC} + m\lambda_{C'B'}^{\alpha}\lambda_{\alpha B'C'} + m\lambda_{CB'}^{\alpha}\lambda_{\alpha B'C} + m\lambda_{C'B}^{\alpha}\lambda_{\alpha BC'} \\
& +m^*\bar{\lambda}_{\dot{\alpha}CB}\bar{\lambda}_{BC}^{\dot{\alpha}} + m^*\bar{\lambda}_{\dot{\alpha}C'B'}\bar{\lambda}_{B'C'}^{\dot{\alpha}} + m^*\bar{\lambda}_{\dot{\alpha}CB'}\bar{\lambda}_{B'C}^{\dot{\alpha}} + m^*\bar{\lambda}_{\dot{\alpha}C'B}\bar{\lambda}_{BC'}^{\dot{\alpha}}
\end{aligned}
$$

$$
\Delta \equiv \frac{k(N-k)L_3L_4 - kL_1L_2}{\sqrt{V}}
$$

$$
|\Delta_k| = \frac{k(k-1)}{\sqrt{N-1}}
$$

$$
A_\mu(x) = \hat{A}_\mu + \begin{bmatrix} \Delta\mathcal{S}_\mu^{(\Delta)(k)} & \sqrt{\Delta}w_\mu^{(\sqrt{\Delta})} \\ \sqrt{\Delta}w_\mu^{\dagger(\sqrt{\Delta})} & \Delta\mathcal{S}_\mu^{(\Delta)(\ell)} \end{bmatrix}
$$

$$
\mathrm{tr}\left[\mathcal{S}_\mu^{(\Delta)(k)} + \mathcal{S}_\mu^{(\Delta)(\ell)}\right] = 0
$$

$$
g^2\delta\mathcal{L}_f = 2i\sqrt{\Delta}\sum_{C,C',D,D',F,F'}\begin{pmatrix} \bar{\lambda}_{C'D'}\bar{\sigma}_\mu & \bar{\lambda}_{C'D}\bar{\sigma}_\mu \\ \bar{\lambda}_{CD'}\bar{\sigma}_\mu & \bar{\lambda}_{CD}\bar{\sigma}_\mu \end{pmatrix}\left\{\begin{pmatrix} 0 & w_{\mu D'F}^{(\sqrt{\Delta})} \\ w_{\mu DF'}^{\dagger(\sqrt{\Delta})} & 0 \end{pmatrix}\begin{pmatrix} \lambda_{F'C'} & \lambda_{F'C} \\ \lambda_{FC'} & \lambda_{FC} \end{pmatrix} - \cdots\right\}
$$

$$
\begin{aligned}
\lambda_{B'C'} &= \lambda_{B'C'}^{(0)} + \mathcal{O}(\Delta), \lambda_{BC} = \lambda_{BC}^{(0)} + \mathcal{O}(\Delta) \\
\lambda_{C'B} &= \sqrt{\Delta}\lambda_{C'B}^{(\sqrt{\Delta})} + \mathcal{O}\left(\Delta^{3/2}\right), \lambda_{CB'} = \sqrt{\Delta}\lambda_{CB'}^{(\sqrt{\Delta})} + \mathcal{O}\left(\Delta^{3/2}\right)
\end{aligned}
$$

$$
\lambda_{\alpha B'C'}^{(0)} = \delta_{B'C'}\theta_\alpha^{C'}, \lambda_{\alpha BC}^{(0)} = -\frac{\delta_{BC}}{N-k}\sum_{C'=1}^{k}\theta_\alpha^{C'}
$$

$$
\begin{aligned}
\lambda_{1C'B}^{(\sqrt{\Delta})} &= \eta_2^{C'}\mathcal{G}_{3C'B}(x), \lambda_{2C'B}^{(\sqrt{\Delta})} = 0 \\
\lambda_{1CB'}^{(\sqrt{\Delta})} &= 0, \lambda_{2BC'}^{(\sqrt{\Delta})} = \eta_1^{C'}\mathcal{G}_{3C'B}^{*}(x)
\end{aligned}
$$

$$
\eta^{C'} \equiv \theta^{C'} + \frac{1}{N-k}\sum_{B'=0}^{k-1}\theta^{B'}
$$

$$
\tilde{\boldsymbol{H}} \equiv \left(\frac{\omega}{2\pi\sqrt{Nk(N-k)}}, \boldsymbol{H}_{(k)}\right) = \left(\tilde{H}^1, \tilde{H}^2, \ldots \tilde{H}^k\right), \mathrm{tr}\left[\tilde{H}^{b_1}\tilde{H}^{b_2}\right] = \delta_{b_1b_2}, b_1, b_2 = 1, \ldots, k
$$

$$
\left(\psi_{\alpha,p,\beta}^{(0)}\right)^{ij} = \frac{1}{\sqrt{V}}\epsilon_{\alpha\beta}\left(\tilde{H}^p\right)^{ij}, p = 1, \ldots, k, \beta = 1,2
$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\psi^{(0)\alpha}_{p,\beta}\psi^{(0)}_{q,\beta',\alpha} = \delta_{pq}\epsilon_{\beta\beta'}$$

$$\begin{pmatrix} \langle\lambda_\alpha(x)\otimes\lambda^\beta(y)\rangle\langle\lambda_\alpha(x)\otimes\bar{\lambda}_{\dot\beta}(y)\rangle \\ \langle\bar{\lambda}^{\dot\alpha}(x)\otimes\lambda^\beta(y)\rangle\langle\bar{\lambda}^{\dot\alpha}(x)\otimes\bar{\lambda}_{\dot\beta}(y)\rangle \end{pmatrix}_{\text{unnorm}}$$

$$= \mathcal{D}^f_k(m)\frac{g^2}{2}\left\{\begin{pmatrix} \frac{m^*}{|m|^2}\sum_{p=1}^{k}\left(\psi^{(0)}_{\alpha p,1}(x)\otimes\psi^{(0)\beta}_{p,2}(y) - \psi^{(0)}_{\alpha p,2}(x)\otimes\psi^{(0)\beta}_{p,1}(y)\right) & 0 \\ 0 & 0 \end{pmatrix}\right.$$

$$\left. + \sum_n \begin{pmatrix} \frac{m^*}{\omega_n^2+|m|^2}(\sigma_\mu\bar{\sigma}_\nu)_\alpha^{\ \beta}\frac{D_\mu\phi_n(x)\otimes D_\nu\phi_n(y)}{\omega_n^2} & \frac{\sigma_{\mu\alpha\dot\beta}}{\omega_n^2+|m|^2}D_\mu\phi_n(x)\otimes\phi_n(y) \\ -\frac{\bar{\sigma}_\nu^{\alpha}}{\omega_n^2+|m|^2}\phi_n(x)\otimes D_\nu\phi_n(y) & \frac{m}{\omega_n^2+|m|^2}\delta^{\dot\alpha}_{\dot\beta}\phi_n(x)\otimes\phi_n(y) \end{pmatrix}\right\}$$

$$\mathcal{D}^f_k(m) = \left(\frac{2m}{g^2}\right)^k \prod_n \left(\frac{16}{g^4}(\omega_n^2+|m|^2)\right)\prod_{p=1}^{k}\epsilon_p\prod_n \epsilon_n$$

$$\langle\lambda_\alpha(x)\otimes\lambda^\beta(y)\rangle \to \left\langle\lambda_{ij\alpha}(x)\lambda^\beta_{kl}(y)\right\rangle$$

$$D_\mu\phi_n(x)\otimes\phi_n(y) \to \left(D_\mu\phi_n\right)_{ij}(x)\phi_{nkl}(y), \text{ etc.}$$

$$\lambda(x) = \sum_{b=1}^{k}\lambda_b(x)\tilde{H}_b + \text{ off diagonal}$$

$$\left\langle\lambda_{b\alpha}(x)\lambda^\beta_{b'}(y)\right\rangle_{\text{unnorm.}} = \delta_{bb'}\delta^\beta_\alpha\mathcal{D}^f_k(m)\frac{g^2}{2V}\left(\frac{m^*}{|m|^2} + \sum_{p_\mu\in\frac{2\pi}{L_\mu}\mathbb{Z}}{}' \frac{m^*}{p_\mu^2}e^{ip_\mu(x_\mu-y_\mu)}(1+\cdots)\right)$$

$$\langle\bar{\lambda}^{\dot\alpha}_b(x)\bar{\lambda}_{b'\dot\beta}(y)\rangle_{unnorm.} = \delta_{bb'}\mathcal{D}^f_k(m)\frac{g^2}{2V}\delta^{\dot\alpha}_{\dot\beta}\left(\frac{mL^2}{c\Delta}(1+\cdots) + \sum_{p_\mu\in\frac{2\pi}{L_\mu}\mathbb{Z}}{}' \frac{me^{ip_\mu(x_\mu-y_\mu)}}{p_\mu^2}(1+\cdots)\right)$$

where $c \equiv \frac{4\pi}{kN}$

$$\frac{(|m|L)^2}{c} \ll \Delta \ll 1, c \equiv \frac{4\pi}{kN}$$

$$\langle\bar{\lambda}^{\dot\alpha}_{D'E'}(x)\bar{\lambda}_{\dot\beta F'G'}(y)\rangle_{\text{unnorm.}}$$
$$= \frac{g^2}{2V}\mathcal{D}^f_k(m)\left\langle\lambda_{\alpha D'E'}(x)\lambda^\beta_{F'G'}(y)\right\rangle$$
$$\times \sum_{p_\mu=\frac{2\pi n_\mu}{L_\mu}} \frac{m\delta^{\dot\alpha}_{\dot\beta}}{|m|^2 + \left(p_\mu + \delta_{\mu 2}\frac{2\pi}{L_2}\frac{D'-E'}{k}\right)^2} e^{ix_\mu\left(p_\mu+\delta_{\mu,2}\frac{2\pi(D'-E')}{kL_2}\right)-iy_\mu\left(p_\mu-\delta_{\mu,2}\frac{2\pi(F'-G')}{kL_2}\right)}\delta_{D'G'}\delta_{E'F'}$$

$$\left\langle \bar{\lambda}^{\dot\beta}_{BC}(x)\bar{\lambda}_{\dot\alpha DE}(y)\right\rangle_{\text{unnorm}}$$

$$= \delta^{\dot\beta}_{\dot\alpha}\mathcal{D}^f_k(m)\frac{g^2}{2\ell V}$$

$$\times \sum_{k_\mu=\frac{2\pi n_\mu}{L_\mu}, n_\mu\in\mathbb{Z}} \sum_{(p_3,p_4)\in\mathbb{Z}^2_\ell} \frac{m e^{-i(x_\mu-y_\mu)\left(k_\mu+\delta_{\mu3}\frac{2\pi p_3}{\ell L_3}+\delta_{\mu4}\frac{2\pi p_3}{\ell L_3}\right)}}{|m|^2+\sum_{\mu=1}^4\left(k_\mu+\delta_{\mu3}\frac{2\pi p_3}{\ell L_3}+\delta_{\mu4}\frac{2\pi p_3}{\ell L_3}\right)^2}\left(J_{p_3,p_4}\right)_{BC}\left(J_{-p_3,-p_4}\right)_{DE}$$

$$J_{\boldsymbol{p}} = e^{-i\frac{\pi p_3 p_4}{\ell}}Q_\ell^{-p_3}P_\ell^{p_4}$$

$$\left\langle \lambda_{\alpha BC}(x)\lambda(x)^{\beta}_{DE}(y)\right\rangle m\to m^* \text{ and } \delta^{\dot\alpha}_{\dot\beta}\to\delta^{\beta}_{\alpha}$$

$$\langle\bar{\lambda}^{\dot\alpha}_{C'C}(x)\bar{\lambda}_{\dot\beta DC'}(y)\rangle_{\text{unnorm.}} = \delta^{\dot\alpha}_{\dot\beta}\mathcal{D}^f_k(m)\frac{g^2}{2V}\sum_{\ell_{(1)},\ell_{(2)}=0}^{\infty}\frac{m\varphi_{C'C\ell_{(1)}\ell_{(3)}}(x)\varphi^*_{C'D\ell_{(1)}\ell_{(3)}}(y)}{\omega^2_{\ell_{(1)},\ell_{(2)}}+|m|^2}$$

$$\omega^2_{\ell_{(1)},\ell_{(2)}} = \frac{4\pi}{L_1L_2}\left(\ell_{(1)}+\ell_{(3)}+1\right)$$

$$\left\langle\lambda_{\gamma C'C}(x)\lambda^{\beta}(y)_{DC'}\right\rangle_{\text{unnorm}.} = \mathcal{D}^f_k(m)\frac{g^2}{2V}\sum_{\ell_{(1)},\ell_{(2)}=0}^{\infty}\frac{m^*\sigma_{\mu\gamma\dot\gamma}D_\mu\varphi_{C'C\ell_{(1)}\ell_{(3)}}(x)\bar{\sigma}^{\dot\gamma\beta}_{\nu}D^*_\nu\varphi^*_{C'D\ell_{(1)}\ell_{(3)}}(y)}{\omega^2_{\ell_{(1)},\ell_{(2)}}\left(\omega^2_{\ell_{(1)},\ell_{(2)}}+|m|^2\right)}$$

$$\mathcal{W}_\mu(x)\equiv e^{i\int_0^{x_\mu}(A_\mu(x))}$$

$$\lambda'(x) = U(x)\lambda(x)U^\dagger(x)$$

$$\operatorname{tr}\left[\lambda(x)\Gamma_{\nu_1\nu_2\ldots}\prod_{\mu=1}^{4}\mathcal{W}_\mu(x)\lambda(0)\mathcal{W}^\dagger_\mu(x)\right]$$

$$Z^T[\eta=\bar\eta=0]_{m=0} = \sum_{Q=0,\frac{\pm1}{N},\frac{\pm2}{N},\ldots} Z^{\text{Reg}}_Q[\eta=0,\bar\eta=0] = \underbrace{N}_{Q=0\text{ sector}} + \underbrace{0+0+\cdots}_{\text{higher Q sectors}} = N$$

$$Z^T[\eta=\bar\eta=0]_{m\neq0} = \sum_{Q=0,\frac{\pm1}{N},\frac{\pm2}{N},\ldots} Z^{\text{Reg}}_Q[\eta=0,\bar\eta=0]$$

$$=\mathcal{F}_0(|m|^2,M_{PV})\left[N+\sum_{k>0}e^{-\frac{8\pi^2k}{Ng^2}}\mu_B^{(k)}\left(\frac{\mathcal{F}_k(m,M_{PV})}{\mathcal{F}_0(|m|^2,M_{PV})}e^{i\frac{\theta k}{N}}+\frac{\mathcal{F}_{-k}(m,M_{PV})}{\mathcal{F}_0(|m|^2,M_{PV})}e^{-i\frac{\theta k}{N}}\right)\right].$$

$$\left\langle\lambda_{\alpha ij}(x)\lambda^{\beta}_{kl}(y)\right\rangle$$

$$=\delta^{\beta}_{\alpha}\frac{g^2}{2NV}\sum_{k_\mu=\frac{2\pi n_\mu}{L_\mu},n_\mu\in\mathbb{Z}}\sum_{\boldsymbol{p}\in\mathbb{Z}^2_N}\frac{m^*e^{-i(x_\mu-y_\mu)\left(k_\mu+\delta_{\mu3}\frac{2\pi p_3}{NL_3}+\delta_{\mu4}\frac{2\pi p_3}{NL_3}\right)}}{|m|^2+M^2_{\boldsymbol{p},k}}\left(J_{p_3,p_4}\right)_{ij}\left(J_{-p_3,-p_4}\right)_{kl}$$

$$J_{\boldsymbol{p}} = e^{-i\frac{\pi p_3 p_4}{N}} Q_N^{-p_3} P_N^{p_4}$$

$$\langle \mathcal{C}_{\nu_1\nu_2\ldots}(x)\rangle_{\frac{k}{N}} \equiv$$

$$\left\langle \mathrm{tr}\left[\lambda(x)\prod_{\mu=1}^{4} \mathcal{W}_\mu(x)\Gamma_{\nu_1\nu_2..}\lambda(0)\mathcal{W}_\mu^\dagger(x)\right]\right\rangle = \left\langle \lambda_{i_1 i_2}(x)\prod_{\mu=1}^{4} [\mathcal{W}_\mu]_{i_2 i_3}(x)[\mathcal{W}_\mu^\dagger]_{i_4 i_1}(x)\Gamma_{\nu_1\nu_2..}\lambda_{i_3 i_4}(0)\right\rangle,$$

$$\mathcal{C}^{(1)}_{\nu_1\nu_2\ldots}(x) = \lambda_{B_1B_2}(x)\prod_{\mu=1}^{4} \mathcal{W}_{\mu B_2 B_3}(x)\Gamma_{\nu_1\nu_2..}\lambda_{B_3B_4}(0)\mathcal{W}^\dagger_{\mu B_4 B_1}(x) + (B_i \leftrightarrow B_i'),$$

$$\mathcal{C}^{(2)}_{\nu_1\nu_2\ldots}(x) = \lambda_{B_1B_2'}(x)\prod_{\mu=1}^{4} \mathcal{W}_{\mu B_2' B_3'}(x)\Gamma_{\nu_1\nu_2..}\lambda_{B_3'B_4}(0)\mathcal{W}^\dagger_{\mu B_4 B_1}(x) + (B_i \leftrightarrow B_i'),$$

$$\mathcal{C}^{(3)}_{\nu_1\nu_2\ldots}(x) = \lambda_{B_1B_2}(x)\prod_{\mu=1}^{4} \mathcal{W}_{\mu B_2 B_3'}(x)\Gamma_{\nu_1\nu_2..}\lambda_{B_3'B_4}(0)\mathcal{W}^\dagger_{\mu B_4 B_1}(x) + (B_i \leftrightarrow B_i'),$$

$$\left\langle \mathcal{C}^{(1),(i)}_{\nu_1\nu_2\ldots}(x)\right\rangle_{\frac{k}{N}} = \left\langle \tilde{\lambda}_{b_1}(x)\Gamma_{\nu_1\nu_2..}\tilde{\lambda}_{b_2}(0)\prod_{\mu=1}^{4} \mathcal{W}_{\mu b_1 b_2}(x)\mathcal{W}^\dagger_{\mu b_2 b_1}(x)\right\rangle_{\frac{k}{N}}$$

$$= \left\langle \tilde{\lambda}^\alpha_{b_1}(x)\tilde{\lambda}_{\beta b_2}(0)\right\rangle_{\frac{k}{N}} \Gamma^\beta_{\alpha\nu_1\nu_2..}\delta_{b_1b_2}$$

$$\left\langle \overline{\mathcal{C}}^{(1),(i)}_{\nu_1\nu_2\ldots}(x)\right\rangle_{\frac{k}{N}} = \left\langle \bar{\tilde{\lambda}}_{b_1}(x)\Gamma_{\nu_1\nu_2..}\bar{\tilde{\lambda}}_{b_2}(0)\prod_{\mu=1}^{4} \mathcal{W}_{\mu b_1 b_2}(x)\mathcal{W}^\dagger_{\mu b_2 b_1}(x)\right\rangle_{\frac{k}{N}}$$

$$= \left\langle \bar{\tilde{\lambda}}^{\dot\alpha}_{b_1}(x)\bar{\tilde{\lambda}}_{\dot\beta b_2}(0)\right\rangle_{\frac{k}{N}} \Gamma^{\dot\beta}_{\dot\alpha\nu_1\nu_2..}\delta_{b_1b_2}$$

$$\left\langle \mathcal{C}^{(1),(i)}_{\nu_1\nu_2\ldots}(x)\right\rangle_{-\frac{k}{N}} \text{ and } \left\langle \overline{\mathcal{C}}^{(1),(i)}_{\nu_1\nu_2\ldots}(x)\right\rangle_{-\frac{k}{N}}$$

$$\left\langle \mathcal{C}^{(1),(ii)}_{\nu_1\nu_2\ldots}(x)\right\rangle_{\frac{k}{N}} = \left\langle \lambda_{B_1'B_2'}(x)\Gamma_{\nu_1\nu_2..}\lambda_{B_2'B_1'}(0)\right\rangle_{\frac{k}{N}}\prod_{\mu=1}^{4} \mathcal{W}_{\mu B_2' B_2'}(x)\mathcal{W}^\dagger_{\mu B_1' B_1'}(x)$$

$$= \frac{g^2}{2V}\Gamma^\alpha_{\alpha,\nu_1\nu_2..}\sum_{B_1'\neq B_2'=1}^{k}\ \sum_{p_\mu=\frac{2\pi\mathbb{Z}}{L\mu}} \frac{m^* e^{-ix_\mu\left(p_\mu - 2\pi \boldsymbol{a}_\mu\cdot\left(\boldsymbol{\nu}_{B_2'}-\boldsymbol{\nu}_{B_1'}\right)\right)} e^{i2\pi(B_1'-B_2')\frac{x_2}{kL_2}}}{p_1^2 + \left(p_2 + \dfrac{2\pi(B_1'-B_2')}{kL_2}\right)^2 + p_3^2 + p_4^2 + |m|^2}$$

$$\left\langle \mathcal{C}^{(1),(iii)}_{\nu_1\nu_2\ldots}(x)\right\rangle_{\frac{k}{N}} = \frac{g^2}{2V}\Gamma^\alpha_{\alpha,\nu_1\nu_2,\ldots}\sum_{\boldsymbol{p}\in\mathbb{Z}^2_\ell\neq 0, k_\mu\in\frac{2\pi\mathbb{Z}}{L_\mu}} \frac{m^* e^{-i\left(k_\mu x_\mu + \frac{p_3x_3}{\ell L_3}+\frac{p_4x_4}{\ell L_4}\right)}}{|m|^2 + M^2_{(\ell)\boldsymbol{p},k}}$$

$$M^2_{(\ell)\boldsymbol{p},k} = \left[k_1^2 + k_2^2 + \left(k_3 + \frac{2\pi p_3}{\ell L_3}\right)^2 + \left(k_4 + \frac{2\pi p_4}{\ell L_4}\right)^2\right]\left\langle \overline{\mathcal{C}}^{(1),(iii)}_{\nu_1\nu_2\dots}(x)\right\rangle_{\frac{k}{N}}$$

$$\left\langle \mathcal{C}^{(2)}_{\nu_1\nu_2\dots}(x)\right\rangle_{\frac{k}{N}} = \frac{g^2}{2V}\Gamma^{\alpha}_{\alpha\nu_1\nu_2\dots}$$

$$\times \sum_{B_1=1}^{\ell}\sum_{B_2'=1}^{k}\sum_{\ell_{(1)},\ell_{(3)}=0}^{\infty} \frac{m^* D_\mu \varphi^{\ell_{(1)},\ell_{(3)}}_{B_2'B_1}(x,\hat{\phi})\left(D_\mu \varphi^{\ell_{(1)},\ell_{(3)}}_{B_2'B_1}(0,\hat{\phi})\right)^*}{\left(\frac{4\pi}{L_1L_2}\left(1+\ell_{(1)}+\ell_{(3)}\right)\right)\left(\frac{4\pi}{L_1L_2}\left(1+\ell_{(1)}+\ell_{(3)}\right)+|m|^2\right)}.$$

$$\varphi^{\ell_{(1)},\ell_{(3)}}_{B_2'B_1}(x,\hat{\phi})\left\langle \overline{\mathcal{C}}^{(2)}_{\nu_1\nu_2\dots}(x)\right\rangle_{\frac{k}{N}}$$

$$\left\langle \overline{\mathcal{C}}^{(2)}_{\nu_1\nu_2\dots}(x)\right\rangle_{\frac{k}{N}} = \frac{g^2}{2V}\Gamma^{\dot{\alpha}}_{\dot{\alpha}\nu_1\nu_2\dots}\sum_{B_1=1}^{\ell}\sum_{B_2'=1}^{k}\sum_{\ell_{(1)},\ell_{(3)}=0}^{\infty}\frac{m\varphi^{*\ell_{(1)},\ell_{(3)}}_{B_2'B_1}(x,\hat{\phi})\varphi^{\ell_{(1)},\ell_{(3)}}_{B_2'B_1}(0,\hat{\phi})}{\frac{4\pi}{L_1L_2}\left(1+\ell_{(1)}+\ell_{(3)}\right)+|m|^2}$$

$$\times\prod_{\mu=1}^{4}\left(\mathcal{W}_{\mu B_2'B_2'}(x)\mathcal{W}^\dagger_{\mu B_1B_1}(x)+\mathcal{W}_{\mu B_1B_1}(x)\mathcal{W}^\dagger_{\mu B_2'B_2'}(x)\right)$$

$$\left\langle \mathcal{C}^{(2)}_{\nu_1\nu_2\dots}(x)\right\rangle_{-\frac{k}{N}} \text{ and } \left\langle \overline{\mathcal{C}}^{(2)}_{\nu_1\nu_2\dots}(x)\right\rangle_{-\frac{k}{N}}$$

$$\mathcal{E}_{\frac{k}{N}}(0) \equiv \left[\iint [DA_\mu][D\lambda][D\bar{\lambda}]e^{-S_{\text{SYM}}}\right]_{\frac{k}{N}} = \frac{\mathcal{D}^f_k(m)}{\left(\det{}'\mathcal{O}^G_{\mu\nu}\right)^{\frac{1}{2}}\left(\det(-D^2)\right)^{-\frac{1}{2}}}$$

$$= \frac{\left(\frac{2m}{g^2}\right)^k \prod_n \left(\frac{16}{g^4}(\omega_n^2+|m|^2)\right)}{\prod_n \omega_n^2}$$

$$\sum_{i=1}^{R} e_i = -1, \sum_{i=1}^{R} e_i M_i^{2q} = 0, q = 1, \dots, R-1$$

$$M_{PV} \equiv \prod_{i=1}^{R} M_i^{-e_i}$$

$$m^k\prod_n(\omega_n^2+|m|^2) \to \frac{m^k}{M_{PV}^k}\prod_n\prod_{i=0}^{R}\left(\omega_n^2+|m|^2+M_i^2\right)^{e_i}$$

$$\left(\det{}'\mathcal{O}^G_{\mu\nu}\right)^{\frac{1}{2}}\left(\det(-D^2)\right)^{-\frac{1}{2}} \to \frac{1}{M_{PV}^{4k}}\prod_n\prod_{i=0}^{R}\left(\omega_n^2+M_i^2\right)^{e_i}$$

$$\frac{\mathcal{E}_{\frac{k}{N}}(0)}{\mathcal{E}_{\frac{k}{N}}(M_{PV})} = m^k M_{PV}^{3k} \prod_n \prod_{i=0}^{R} \left(\frac{1 + \frac{|m|^2 + M_i^2}{\omega_n^2}}{1 + \frac{M_i^2}{\omega_n^2}} \right)^{e_i}$$

$$\equiv \mathcal{F}_k(m, M_{PV}) \equiv m^k M_{PV}^{3k} \mathcal{D}_k(|m|^2)$$

$$\mathcal{D}_k(|m|^2) \equiv \prod_n \left(1 + \frac{|m|^2}{\omega_n^2}\right)\Bigg|_{reg.} = \prod_n \prod_{i=0}^{R} \left(\frac{1 + \frac{|m|^2 + M_i^2}{\omega_n^2}}{1 + \frac{M_i^2}{\omega_n^2}} \right)^{e_i}$$

$$\mathcal{D}_k(|m|^2) \equiv \prod_n \left(1 + \frac{|m|^2}{\omega_n^2}\right)\Bigg|_{\text{reg.}}$$

$$= \underbrace{\prod_{n \in SU(k)\times U(1)} \left(1 + \frac{|m|^2}{\omega_n^2}\right)}_{\mathcal{D}_k(|m|^2, SU(k)\times U(1))} \underbrace{\prod_{n \in SU(\ell)} \left(1 + \frac{|m|^2}{\omega_n^2}\right)}_{\mathcal{D}_k(|m|^2, SU(\ell))} \underbrace{\prod_{n \in k\times\ell(\ell\times k)} \left(1 + \frac{|m|^2}{\omega_n^2}\right)}_{\mathcal{D}_k(|m|^2, (k\times\ell))}\Bigg|_{\text{reg.}}$$

$$\mathcal{D}_k(|m|^2, SU(k)\times U(1))$$

$$= \left(1 + \frac{|m|^2 L^2}{c\Delta}\right)^k \prod_{k_\mu = \frac{2\pi n_\mu}{L_\mu}}' \left(1 + \frac{|m|^2 L^2}{(Lk_\mu)^2}\right)^k \prod_{D' \neq E' = 1}^{k} \prod_{k_\mu = \frac{2\pi n_\mu}{L_\mu}} \left(1 + \frac{|m|^2 L^2}{\left(Lk_\mu + \delta_{\mu 2} \frac{L}{L_2} \frac{D' - E'}{k}\right)^2}\right)$$

$$\mathcal{D}_k(|m|^2, SU(\ell)) = \prod_{p_3, p_4 = 0, (p_3, p_4) \neq (0,0)}^{\ell - 1} \prod_{k_\mu = \frac{2\pi n_\mu}{L_\mu}} \left(1 + \frac{|m|^2 L^2}{\left(Lk_\mu + \delta_{\mu 3} \frac{L}{L_3} \frac{2\pi p_3}{\ell} + \delta_{\mu 4} \frac{L}{L_4} \frac{2\pi p_4}{\ell}\right)^2}\right),$$

$$\mathcal{D}_k(|m|^2, (k\times\ell)) = \prod_{\ell_{(1)}, \ell_{(3)} = 0}^{\infty} \left(1 + \frac{|m|^2 L^2}{\frac{4\pi L^2}{L_1 L_2}(\ell_{(1)} + \ell_{(3)} + 1)}\right)^{2k}$$

$$\mathcal{F}_0 = \prod_n \left(1 + \frac{|m|^2}{\omega_n^2}\right) = \prod_{p_3, p_4 = 0, (p_3, p_4) \neq (0,0)}^{N-1} \prod_{k_\mu = \frac{2\pi n_\mu}{L_\mu}} \left(1 + \frac{|m|^2}{M_{\boldsymbol{p},k}^2}\right),$$

$$\frac{\mathcal{D}_k(|m|^2)}{\mathcal{F}_0}$$

$$\mathcal{F}_0(|m|^2) = \prod_n \left(1 + \frac{|m|^2}{\omega_n^2}\right) = \prod_{p_3, p_4 = 0, (p_3, p_4) \neq (0,0)} \prod_{k_\mu = \frac{2\pi n_\mu}{L_\mu}} \left(1 + \frac{|m|^2}{M_{\boldsymbol{p},k}^2}\right)$$

$$M_{\boldsymbol{p},k}^2 = \left[k_1^2 + k_2^2 + \left(k_3 + \frac{2\pi p_3}{NL_3}\right)^2 + \left(k_4 + \frac{2\pi p_4}{NL_4}\right)^2\right].$$

$$\ln \mathcal{F}_0(|m|^2)$$

$$= \sum_{k_\mu=\frac{2\pi n_\mu}{L_\mu}} \sum_{p_3,p_4=0,(p_3,p_4)\neq(0,0)} \ln \frac{|m|^2+M_{\boldsymbol{p},k}^2}{M_{\boldsymbol{p},k}^2} \simeq V(N^2-1)\int^{M_{PV}} \frac{d^4k}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\ln \frac{|m|^2+k^2}{k^2}+\cdots$$

$$[\mathcal{F}_0(|m|^2)]_{Reg} = 1+\mathcal{O}(|m|NL)^p$$

$$\ln \frac{\mathcal{D}_k(|m|^2,SU(k)\times U(1))|_{\text{one term}}}{\mathcal{F}_0|_{\text{one term}}} = \sum_{k_\mu=\frac{2\pi n_\mu}{L_\mu},n_\mu\gg 1} \ln\left(1+\frac{|m|^2}{\left(k_\mu+\delta_{\mu 2}\frac{1}{kL_2}\right)^2}\right)-\ln\left(1+\frac{|m|^2}{M_{p,k}^2}\right)$$

$$\ln \frac{\mathcal{D}_k(|m|^2,SU(k)\times U(1))|_{\text{one term}}}{\mathcal{F}_0|_{\text{one term}}} \simeq \frac{V}{(2\pi)^4}\int d^4k\left(\ln\left(1+\frac{|m|^2}{\left(k_\mu+\delta_{\mu 2}\frac{1}{kL_2}\right)^2}\right)-\ln\left(1+\frac{|m|^2}{M_{p,k}^2}\right)\right)$$

$$=\frac{V}{(2\pi)^4}\int d^4k\left(\ln\left(1+\frac{|m|^2}{\left(k_\mu+\delta_{\mu 2}\frac{1}{kL_2}\right)^2}\right)-\ln\left(1+\frac{|m|^2}{\left(k_\mu+\delta_{\mu 3}\frac{2\pi p_3}{NL_3}+\delta_{\mu 4}\frac{2\pi p_4}{NL_4}\right)^2}\right)\right)$$

$$\frac{\mathcal{D}_k(|m|^2,SU(k)\times U(1))|_{\text{one term}}}{\mathcal{F}_0|_{\text{one term}}}$$

$$\ln \frac{\mathcal{D}_k(|m|^2,(k\times\ell))}{\mathcal{F}_0|_{2k\times\ell\text{ terms}}} = 2k\sum_{\ell_{(1)},\ell_{(3)}=0}^{\infty} \ln\left(1+\frac{|m|^2}{\frac{4\pi}{L_1L_2}\left(\ell_{(1)}+\ell_{(3)}+1\right)}\right)-2k\ell\sum_{k_\mu=\frac{2\pi n_\mu}{L_\mu}} \ln\left(1+\frac{|m|^2}{M_{p,k}^2}\right).$$

$$\ln \frac{\mathcal{D}_k(|m|^2,(k\times\ell))}{\mathcal{F}_0|_{2k\times\ell\text{ terms}}}$$

$$\simeq 2k\ell\frac{V}{(4\pi)^2}\int dq_1^2dq_2^2\ln\left(1+\frac{|m|^2}{q_1^2+q_2^2+1}\right)-2k\ell\frac{V}{(2\pi)^4}\int d^4k\ln\left(1+\frac{|m|^2}{\left(k_\mu+\delta_{\mu 3}\frac{2\pi p_3}{NL_3}+\delta_{\mu 4}\frac{2\pi p_4}{NL_4}\right)^2}\right)$$

$$\frac{2\pi^2}{2(2\pi)^4}\int dk^2k^2\ln\left(1+\frac{|m|^2}{k^2}\right)=\frac{1}{16\pi^2}\int dk^2k^2\ln\left(1+\frac{|m|^2}{k^2}\right)$$

$$\frac{4}{(4\pi)^2}\underbrace{\int_0^{\frac{\pi}{2}}\cos\phi\sin\phi d\phi}_{=\frac{1}{2}}\int drr^3\ln\left(1+\frac{|m|^2}{r^2}\right)=\frac{1}{16\pi^2}\int dr^2r^2\ln\left(1+\frac{|m|^2}{r^2}\right)$$

$$\frac{\mathcal{D}_k(|m|^2)}{\mathcal{F}_0}\simeq\left(1+k\frac{(|m|L)^2}{c\Delta}\right)$$

$$Z^T[\eta=0,\bar\eta=0]_{m\neq 0}=\sum_{Q=0,\frac{\pm 1}{N},\frac{\pm 2}{N},\dots} Z_Q^{\text{Reg}}[\eta=\bar\eta=0]=N\left(1+\mathcal{O}((mLN)^2)\right)+\mathcal{O}(\Lambda^3|m|L^4)$$

$$\left[\frac{\mathcal{E}_{\frac{1}{N}}(0)}{\mathcal{E}_{\frac{1}{N}}(M_{PV})}\right]_{\Delta=0} \cong m^3 M_{PV}$$

$$d\mu_B^{(k)} = \frac{\prod_{\mu=1}^{4} \prod_{b=1}^{k-1} da_\mu^b dz_\mu \sqrt{\mathrm{Det}\mathcal{U}_B^{(k)}}}{k!\,(\sqrt{2\pi})^{4k}}$$

$$\sqrt{\mathrm{Det}\mathcal{U}_B^{(k)}} = N^2(\sqrt{k(N-k)})^4 \left(\frac{8\pi^2\sqrt{V}}{g^2}\right)^{2k}.$$

$$\Gamma^{(k)} = \begin{cases} z_2 \in [0,1), z_1 \in \left[0,\frac{1}{N}\right) \\ z_{3,4} \in \left[0,\frac{1}{N-k}\right) \\ \boldsymbol{a}_\mu \in \Gamma_w^{SU(k)} \text{ for } \mu = 1,2,3,4 \end{cases}$$

$$\mu_B^{(k)} = \frac{N}{k!}\left(\frac{4\pi\sqrt{V}}{g^2}\right)^{2k}$$

$$\mathcal{O}_{\nu_1\nu_2\ldots}(x) = \mathrm{tr}\left[\lambda(x)\Gamma_{\nu_1\nu_2\ldots}\prod_{\mu=1}^{4} \mathcal{W}_\mu(x)\lambda(0)\mathcal{W}_\mu^\dagger(x)\right]$$

$$\langle\mathcal{O}(x)\rangle^{\mathrm{Reg}} = \frac{\mathcal{F}_0\left[N\langle\mathcal{O}\rangle_0 + \sum_{k=1} e^{-\frac{8\pi^2 k}{Ng^2}} \int_{\Gamma^{(k)}} d\mu_B^{(k)}\left(e^{i\theta\frac{k\mathcal{F}_k}{N\mathcal{F}_0}\langle\mathcal{O}\rangle_{\frac{k}{N}}} + e^{-i\theta\frac{k\mathcal{F}_{-k}}{N\mathcal{F}_0}\langle\mathcal{O}\rangle_{-k}}\right)\right]}{\mathcal{F}_0\left[N + \sum_{k=1} e^{-\frac{8\pi^2 k}{Ng^2}} \mu_B^{(k)}\left(\frac{\mathcal{F}_k}{\mathcal{F}_0}e^{i\theta\frac{k}{N}} + \frac{\mathcal{F}_{-k}}{\mathcal{F}_0}e^{-i\theta\frac{k}{N}}\right)\right]}$$

$$\langle\mathcal{O}(x_1,..x_n)\rangle^{\mathrm{Reg}} = \frac{\mathcal{F}_0\left[N\langle\mathcal{O}\rangle_0 + \sum_{k=1} e^{-\frac{8\pi^2 k}{Ng^2}} M_{PV}^{3k} \int_{\Gamma^{(k)}} d\mu_B^{(k)}\left(m^k e^{i\theta\frac{k}{N}}\frac{\mathcal{D}_k}{\mathcal{F}_0}\langle\mathcal{O}\rangle_{\frac{k}{N}} + (m^*)^k e^{-i\theta\frac{k\mathcal{D}_k}{N\mathcal{F}_0}\langle\mathcal{O}\rangle_{\frac{-k}{N}}}\right)\right]}{\mathcal{F}_0\left[N + \sum_{k=1} e^{-\frac{8\pi^2 k}{Ng^2}} M_{PV}^{3k}\mu_B^{(k)}\left(m^k\frac{\mathcal{D}_k}{\mathcal{F}_0}e^{i\theta\frac{k}{N}} + (m^*)^k\frac{\mathcal{D}_k}{\mathcal{F}_0}e^{-i\theta\frac{k}{N}}\right)\right]}$$

$$\Lambda^3 \equiv M_{PV}^3 e^{-\frac{8\pi^2}{Ng^2}}/g^2$$

$$\langle\mathcal{O}(x_1,\ldots x_n)\rangle^{\mathrm{Reg}} = \frac{1}{N}\left\{N\langle\mathcal{O}\rangle_0 + \sum_{k=1} M_{PV}^{3k} e^{-\frac{8\pi^2 k}{Ng^2}} \int_{\Gamma^{(k)}} d\mu_B^{(k)}\left(m^k e^{i\theta\frac{k}{N}}\langle\mathcal{O}\rangle_{\frac{k}{N}} + m^{*k} e^{-i\theta\frac{k}{N}}\langle\mathcal{O}\rangle_{\frac{-k}{N}}\right)\right\}$$

$$[\mathrm{tr}\lambda\lambda(x)]_{Reg} = \mathrm{tr}\lambda\lambda(x) - m^* \frac{g^2}{(2\pi)^4}(N^2-1)\int^{M_{PV}} \frac{d^4k}{|m|^2+k^2}\frac{d^4k}{|m|^2+k^2}$$
$$= \mathrm{tr}\lambda\lambda(x) - m^* g^2 f(M_{PV},|m|)$$

$$f(M_{PV},|m|) \equiv \int^{M_{PV}} \frac{d^4k(N^2-1)}{(2\pi)^4(|m|^2+k^2)}\frac{d^4k(N^2-1)}{(2\pi)^4(|m|^2+k^2)} = \frac{d}{d|m|^2}\left[(N^2-1)\int^{M_{PV}}\frac{d^4k}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\ln\frac{|m|^2+k^2}{k^2}\right]$$

$$\langle \mathrm{tr}\lambda\lambda(0)\rangle^{\mathrm{Reg}} \equiv \frac{N\mathcal{F}_0}{Z^T}\left(\mathcal{T}_0+\mathcal{T}_{1(i)}+\mathcal{T}_{1(ii)}+\mathcal{T}_{1(iii)}+\mathcal{T}_2\right)$$

$$\mathcal{T}_0 = \left\langle[\mathrm{tr}\lambda\lambda]_{Reg.}\right\rangle_{Q=0} = \frac{g^2}{V}\sum_{\boldsymbol{p}\in\mathbb{Z}_N^2\neq 0,k_\mu\in\frac{2\pi\mathbb{Z}}{L\mu}} \frac{m^*}{|m|^2+M_{\boldsymbol{p},k}^2}\Bigg|_{Reg.}$$

$$\mathcal{T}_{1(i)} = \sum_{k=1}\frac{(16\pi^2\Lambda^3)^k}{(k-1)!}\frac{\mathcal{D}_k}{\mathcal{F}_0}e^{i\frac{k\theta}{N}}\left(\frac{mV}{g^2}\right)^{k-1}$$
$$+\sum_{k=1}\frac{(16\pi^2\Lambda^3)^k}{(k-1)!}\frac{\mathcal{D}_k}{\mathcal{F}_0}e^{i\frac{k\theta}{N}}\left(\frac{mV}{g^2}\right)^{k-1}\sum_{k_\mu\in\frac{2\pi\mathbb{Z}}{L_\mu},k_\mu k_\mu\neq 0}\frac{|m|^2}{k_\mu k_\mu}\Bigg|_{\mathrm{Reg.}}$$
$$+\sum_{k=1}\frac{(16\pi^2\Lambda^3)^k}{(k-1)!}\frac{\mathcal{D}_k}{\mathcal{F}_0}e^{-i\frac{k\theta}{N}}\left(\frac{m^*V}{g^2}\right)^{k-1}\left(\frac{(m^*L)^2}{c\Delta}+\sum_{k_\mu\in\frac{2\pi\mathbb{Z}}{L_\mu},k_\mu k_\mu\neq 0}\frac{m^{*2}}{k_\mu k_\mu}\Bigg|_{\mathrm{Reg.}}\right)$$

$$\mathcal{T}_{1(ii)} = \sum_{k=1}\frac{(16\pi^2\Lambda^3)^k}{k!}\frac{\mathcal{D}_k}{\mathcal{F}_0}\left\{e^{i\frac{k\theta}{N}}\left(\frac{mV}{g^2}\right)^{k-1}|m|^2+e^{-i\frac{k\theta}{N}}\left(\frac{m^*V}{g^2}\right)^{k-1}m^{*2}\right\}$$
$$\times\sum_{B_1'\neq B_2'=1}^{k}\sum_{p_\mu=\frac{2\pi\mathbb{Z}}{L_\mu}}\frac{1}{p_1^2+\left(p_2+\frac{2\pi(B_1'-B_2')}{kL_2}\right)^2+p_3^2+p_4^2+|m|^2}\Bigg|_{\mathrm{Reg.}}$$

$$\mathcal{T}_{1(iii)} = \sum_{k=1}\frac{(16\pi^2\Lambda^3)^k}{k!}\frac{\mathcal{D}_k}{\mathcal{F}_0}\left\{e^{i\frac{k\theta}{N}}\left(\frac{mV}{g^2}\right)^{k-1}|m|^2+e^{-i\frac{k\theta}{N}}\left(\frac{m^*V}{g^2}\right)^{k-1}m^{*2}\right\}$$
$$\times\sum_{\boldsymbol{p}\in\mathbb{Z}_\ell^2\neq 0,k_\mu\in\frac{2\pi\mathbb{Z}}{L_\mu}}\frac{1}{|m|^2+M_{(\ell)\boldsymbol{p},k}^2}\Bigg|_{\mathrm{Reg.}}$$

$$\int_{\Gamma^{(k)}}\sum_{C'=1}^{k}\sum_{C=1}^{N-k}\left(\varphi_{C'C}^{\ell_{(1)},\ell_{(3)}}(x,\hat{\phi})\right)^*\varphi_{C'C}^{\ell_{(1)},\ell_{(3)}}(x,\hat{\phi}) = \left(\frac{8\pi^2\sqrt{V}}{g^2}\right)^{2k}\frac{N}{(k-1)!\,(2\pi)^{2k}},$$

$$\int_{\Gamma^{(k)}}\sum_{C'=1}^{k}\sum_{C=1}^{N-k} D_\mu\varphi_{C'C}^{\ell_{(1)},\ell_{(3)}}(x,\hat{\phi})\left(D_\mu\varphi_{C'C}^{\ell_{(1)},\ell_{(3)}}(0,\hat{\phi})\right)^* = \omega_{\ell_{(1)},\ell_{(3)}}^2\left(\frac{8\pi^2\sqrt{V}}{g^2}\right)^{2k}\frac{N}{(k-1)!\,(2\pi)^{2k}},$$

$$\mathcal{T}_2 = \sum_{k=1}\frac{(16\pi^2\Lambda^3)^k}{(k-1)!}\frac{\mathcal{D}_k}{\mathcal{F}_0}\left\{e^{i\frac{k\theta}{N}}\left(\frac{mV}{g^2}\right)^{k-1}|m|^2 + e^{-i\frac{k\theta}{N}}\left(\frac{m^*V}{g^2}\right)^{k-1}m^{*2}\right\}$$
$$\times\sum_{\ell_{(1)},\ell_{(3)}=0}\frac{1}{\frac{4\pi}{L_1L_2}\left(1+\ell_{(1)}+\ell_{(3)}\right)+|m|^2}\Bigg|_{Reg.}$$

$$\lim_{m=0}\langle \mathrm{tr}\lambda\lambda(0)\rangle^{\mathrm{Reg}} = 16\pi^2\Lambda^3 e^{i\frac{\theta}{N}}$$

$$Z^T = N\mathcal{F}_0|_{\mathrm{Reg.}}\left[1+32\pi^2V|m|\left(1+\frac{|m|^2L^2}{c\Delta}\right)\Lambda^3\cos\left(\frac{\theta_{\mathrm{eff}}}{N}\right)\right]$$

$$\langle \mathrm{tr}\lambda\lambda(0)\rangle^{\mathrm{Reg}}\Big|_{|m|LN\ll 1}$$
$$\simeq\frac{N\mathcal{F}_0|_{\mathrm{Reg.}}}{Z^T}\left\{\frac{g^2m^*}{\sqrt{V}}\sum_{\boldsymbol{p}\in\mathbb{Z}_N^2\neq 0,k_\mu\in\frac{2\pi\mathbb{Z}}{L_\mu}}\frac{1}{\sqrt{V}M_{\boldsymbol{p},k}^2}\Bigg|_{\mathrm{Reg.}} + 16\pi^2\Lambda^3\left(1+\frac{(|m|L)^2}{c\Delta}\right)e^{i\frac{\theta}{N}} + 16\pi^2\Lambda^3\frac{(m^*L)^2}{c\Delta}e^{-i\frac{\theta}{N}}\right\}.$$

$$\frac{|m|^2L^2}{c\Delta}$$

$$\langle \mathrm{tr}\lambda\lambda(0)\rangle^{\mathrm{Reg}}\Big|_{|m|LN\ll 1}\simeq 16\pi^2\Lambda^3\left(1+\frac{(|m|L)^2}{c\Delta}\right)e^{i\frac{\theta}{N}} + 16\pi^2\Lambda^3\frac{(m^*L)^2}{c\Delta}e^{-i\frac{\theta}{N}}$$

$$\langle \mathrm{tr}\bar{\lambda}\bar{\lambda}(0)\rangle^{\mathrm{Reg}}\Big|_{|m|LN\ll 1}\simeq 16\pi^2\Lambda^3\left(1+\frac{(|m|L)^2}{c\Delta}\right)e^{-i\frac{\theta}{N}} + 16\pi^2\Lambda^3\frac{(mL)^2}{c\Delta}e^{i\frac{\theta}{N}}$$

$$\langle \mathrm{tr}[\bar{\Psi}\Psi]\rangle^{Reg.}|_{|m|LN\ll 1}\simeq 32\pi^2\Lambda^3\left(1+\frac{|m|^2L^2}{c\Delta}\right)\cos\left(\frac{\theta}{N}\right) + 16\pi^2\Lambda^3\left(\frac{L^2m^{*2}}{c\Delta}e^{-i\frac{\theta}{N}} + \frac{L^2m^2}{c\Delta}e^{i\frac{\theta}{N}}\right)$$

$$-i\langle \mathrm{tr}[\bar{\Psi}\gamma_5\Psi]\rangle^{Reg.}|_{|m|LN\ll 1} = \left(i\langle \mathrm{tr}\lambda\lambda(0)\rangle - i\langle \mathrm{tr}\bar{\lambda}\bar{\lambda}(0)\rangle^{\mathrm{Reg}}\right)\Big|_{|m|LN\ll 1}$$
$$\simeq -32\pi^2\Lambda^3\left(1+\frac{|m|^2L^2}{c\Delta}\right)\sin\left(\frac{\theta}{N}\right) + i16\pi^2\Lambda^3\left(\frac{L^2m^{*2}}{c\Delta}e^{-i\frac{\theta}{N}} - \frac{L^2m^2}{c\Delta}e^{i\frac{\theta}{N}}\right)$$

$$\delta\mathcal{E} = -m\langle \mathrm{tr}[\lambda\lambda]\rangle^{\mathrm{reg}} - m^*\langle \mathrm{tr}[\bar{\lambda}\bar{\lambda}]\rangle^{\mathrm{reg}} = -32\pi^2\Lambda^3|m|\left(1+2\frac{|m|^2L^2}{c\Delta}\right)\cos\left(\frac{\theta_{\mathrm{eff}}}{N}\right),$$

$$\delta\mathcal{E} = -32\pi^2\Lambda^3|m|\cos\left(\frac{\theta_{\mathrm{eff}}}{N}\right)$$

$$W_2 \equiv \langle k|\hat{W}_2|k\rangle = e^{i\frac{2\pi}{N}k},\ \text{where}\ \hat{W}_2 = \frac{1}{N}\mathrm{tr}_F\mathcal{P}e^{i\oiiint \hat{A}_2dx^2}$$

$$|e\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N} e^{-i\frac{2\pi}{N}ek}|k\rangle, \text{ such that } \hat{T}_2|e\rangle = |e\rangle e^{i\frac{2\pi}{N}e}, e = 0, \dots, N-1$$

$$\langle e|(-)^F e^{-L_1\hat{H}}|e\rangle\big|_{L_1\to\infty} = c_e e^{-L_1 E_e}, E_e - \text{ lowest energy in } e_2 = e \text{ flux sector,}$$

$$\begin{aligned}
E_e &= -\frac{1}{L_1}\ln \langle e|(-)^F e^{-L_1\hat{H}}|e\rangle\Big|_{L_1\to\infty} + \frac{1}{L_1}\ln\, c_e\Big|_{L_1\to\infty} \quad (e = 0, \dots N) \\
&= -\frac{1}{L_1}\ln \frac{1}{N}\sum_{k,k'=0}^{N-1} e^{i\frac{2\pi}{N}e(k'-k)}\langle k'|(-)^F e^{-L_1\hat{H}}|k\rangle \\
&= -\frac{1}{L_1}\ln \frac{1}{N}\sum_{k,k'=0}^{N-1} e^{i\frac{2\pi}{N}e(k'-k)}\langle 0|(-)^F e^{-L_1\hat{H}}\hat{T}_2^{k-k'}|0\rangle \\
&= -\frac{1}{L_1}\ln\left(\langle 0|(-)^F e^{-L_1\hat{H}}|0\rangle + \frac{1}{N}\sum_{k,k'=0,k\neq k'}^{N-1} e^{i\frac{2\pi}{N}e(k'-k)}\langle 0|(-)^F e^{-L_1\hat{H}}\hat{T}_2^{k-k'}|0\rangle\right)\Bigg|_{L_1\to\infty}
\end{aligned}$$

$$E_e + \frac{1}{L_1}\ln \langle 0|(-)^F e^{-L_1\hat{H}}|0\rangle = -\frac{1}{L_1}\ln\left(1 + \frac{1}{N}\sum_{k,k'=0,k\neq k'}^{N-1} e^{i\frac{2\pi}{N}e(k'-k)}\frac{\langle 0|(-)^F e^{-L_1\hat{H}}\hat{T}_2^{k-k'}|0\rangle}{\langle 0|(-)^F e^{-L_1\hat{H}}|0\rangle}\right)$$

$$\begin{aligned}
&E_e + \frac{1}{L_1}\ln \langle 0|(-)^F e^{-L_1\hat{H}}|0\rangle \\
&= -\frac{1}{L_1}\ln\left[1 + \sum_{q=1}^{N-1}\frac{N-q}{N}\left(e^{i\frac{2\pi}{N}eq}\frac{\langle 0|(-)^F e^{-L_1\hat{H}}\hat{T}_2^{q}|0\rangle}{\langle 0|(-)^F e^{-L_1\hat{H}}|0\rangle} + e^{-i\frac{2\pi}{N}eq}\frac{\langle 0|(-)^F e^{-L_1\hat{H}}\hat{T}_2^{-q}|0\rangle}{\langle 0|(-)^F e^{-L_1\hat{H}}|0\rangle}\right)\right] \\
&\equiv -\frac{1}{L_1}\ln\left[1 + \sum_{q=1}^{N-1}\frac{N-q}{N}\left(e^{i\frac{2\pi}{N}eq}\Xi_q + e^{-i\frac{2\pi}{N}eq}\Xi_{-q}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\Xi_q &= \frac{\langle 0|(-)^F e^{-L_1\hat{H}}\hat{T}_2^{q}|0\rangle}{\langle 0|(-)^F e^{-L_1\hat{H}}|0\rangle}\Bigg|_{L_1\to\infty} \\
&\simeq \frac{\int\ [\mathcal{D}A\mathcal{D}\lambda]_{\{n_{34}=1\}}^{A(x_1=0)=iT_2^q dT_2^{-q}, A(x_1=+L_1)=0} e^{-S_{SYM}-S_m}}{\int\ [\mathcal{D}A\mathcal{D}\lambda]_{\{n_{34}=1\}}^{A(x_1=0)=A(x_1=L_1)=0} e^{-S_{SYM}-S_m}}\Bigg|_{L_1\to\infty, L\Lambda N\ll 1}
\end{aligned}$$

$$Z^T = \sum_{q=0}^{N-1}\int\ [\mathcal{D}A\mathcal{D}\lambda]_{\{n_{34}=1\}}^{A(x_1=0)=iT_2^q dT_2^{-q}, A(x_1=+L_1)=0} e^{-S_{SYM*}+i\theta Q}$$

$$E_e = \bar{E} - \frac{1}{L_1}\ln\left[1 + \sum_{q=1}^{N-1}\frac{N-q}{N}\left(e^{i\frac{2\pi}{N}eq}\Xi_q + e^{-i\frac{2\pi}{N}eq}\Xi_{-q}\right)\right], e = 0, \dots N-1$$

$$\Xi_q \simeq \frac{\int [\mathcal{D}A\mathcal{D}\lambda]_{\{n_{34}=1\}}^{A(x_1=0)=iT_2^q dT_2^{-q}, A(x_1=+L_1)=0} e^{-S_{SYM}-S_m}}{\int [\mathcal{D}A\mathcal{D}\lambda]_{\{n_{34}=1\}}^{A(x_1=0)=A(x_1=L_1)=0} e^{-S_{SYM}-S_m}}\Bigg|_{L_1\to\infty, L\Lambda\ll 1}$$

$$\zeta L^{-4} e^{\pm i\frac{\theta}{N}} e^{-\frac{8\pi^2}{g^2N}} = \zeta L^{-4} e^{\pm i\frac{\theta}{N}} e^{-S_0}, \text{ where } S_0 \equiv \frac{8\pi^2}{g^2N}$$

$$e^{i\frac{2\pi e}{N}}\Xi_1 + e^{-i\frac{2\pi e}{N}}\Xi_{-1} = \zeta\frac{L_1 L^3}{L^4} mL\cos\left(\frac{2\pi e+\theta}{N}\right) e^{-S_0}\left(1+\mathcal{O}(e^{-NS_0})\right)$$

$$E_e \simeq \bar{E} - \zeta' m e^{-S_0}\cos\left(\frac{2\pi e+\theta}{N}\right) + \cdots, e = 0, \ldots N-1$$

$$\left\langle \left(W_2^\dagger(x_1=T)\right)^k \left(W_2(x_1=0)\right)^k \right\rangle \equiv \frac{\mathrm{Tr}(-1)^F e^{-(L_1-T)H}\left(W_2^\dagger(0)\right)^k e^{-TH}\left(W_2(0)^k\right)}{\mathrm{Tr}(-1)^F e^{-L_1 H}}$$

$$\left\langle \left(W_2^\dagger(x_1=T)\right)^k \left(W_2(x_1=0)\right)^k \right\rangle\Big|_{L_1,T,L_1-T\to\infty} = \frac{e^{-E_0(L_1-T)}e^{-TE_k}\left|\langle E_0,0|\left(W_2^\dagger\right)^k\big|E_k,k\rangle\right|^2}{e^{-E_0L_1}}$$
$$= e^{-T(E_k-E_0)}\left|\left(W_2^k\right)_{0,k}\right|^2$$

$$\left(W_2^k\right)_{0,k} = \left|\langle E_k,k|\left(W_2^\dagger\right)|E_0,0\rangle\right|$$

$$S_{\mathrm{SYM}^*} = \frac{1}{g^2}\int_{\mathbb{T}^4}\mathrm{tr}\Big[\frac{1}{2}F_{\mu\nu}F_{\mu\nu} - \bar{\lambda}_{\dot\alpha}\bar{\sigma}_\mu^{\dot\alpha\alpha}\left(\partial_\mu\lambda_\alpha + i[A_\mu,\lambda_\alpha]\right)$$
$$-\lambda^\alpha\sigma_{\mu\alpha\dot\alpha}\left(\partial_\mu\bar{\lambda}^{\dot\alpha} + i[A_\mu,\bar{\lambda}^{\dot\alpha}]\right) + m\lambda^\alpha\lambda_\alpha + m^*\bar{\lambda}_{\dot\alpha}\bar{\lambda}^{\dot\alpha}\Big]$$

$$\mathcal{C}: A_\mu \to -A_\mu^* \equiv -A_\mu^a T^{a*}$$
$$\lambda_\alpha^a T^a \to \lambda_\alpha^a T^{a*}$$
$$\bar{\lambda}_{\dot\alpha}^a T^a \to \bar{\lambda}_{\dot\alpha}^a T^{a*}$$

$$\mathcal{P}: A_4^a(\vec{x},x_4) \to A_4^a(-\vec{x},x_4),$$
$$A_i^a(\vec{x},x_4) \to -A_i^a(-\vec{x},x_4), i = 1,2,3.$$

$$\mathcal{P}: \lambda_\alpha^a(\vec{x},x_4) \to i\bar{\lambda}^{a\dot\alpha}(-\vec{x},x_4)$$
$$\bar{\lambda}_{\dot\alpha}^a(\vec{x},x_4) \to -i\lambda^{a\alpha}(-\vec{x},x_4)$$

$$\mathcal{P}: \bar{\lambda}_{\dot\alpha}^a\bar{\sigma}_4^{\dot\alpha\alpha}\lambda_\alpha^b(\vec{x},x_4) \to -\bar{\lambda}_{\dot\alpha}^b\bar{\sigma}_4^{\dot\alpha\alpha}\lambda_\alpha^a(-\vec{x},x_4),$$
$$\bar{\lambda}_{\dot\alpha}^a\bar{\sigma}_i^{\dot\alpha\alpha}\lambda_\alpha^b(\vec{x},x_4) \to \bar{\lambda}_{\dot\alpha}^b\bar{\sigma}_i^{\dot\alpha\alpha}\lambda_\alpha^a(-\vec{x},x_4), i = 1,2,3,$$
$$\mathrm{tr}\lambda^\alpha\lambda_\alpha(\vec{x},x_4) \to \mathrm{tr}\bar{\lambda}_{\dot\alpha}\bar{\lambda}^{\dot\alpha}(-\vec{x},x_4).$$

$$\Psi = \begin{bmatrix}\lambda_\alpha\\ \bar{\lambda}^{\dot\alpha}\end{bmatrix}, \bar{\Psi} = [\lambda^\alpha\bar{\lambda}_{\dot\alpha}] = [\lambda_\beta\bar{\lambda}^{\dot\beta}]\cdot\begin{bmatrix}-\epsilon^{\beta\alpha} & 0\\ 0 & -\epsilon_{\dot\beta\dot\alpha}\end{bmatrix} \equiv \Psi^t C, C = \begin{bmatrix}-\epsilon^{\beta\alpha} & 0\\ 0 & -\epsilon_{\dot\beta\dot\alpha}\end{bmatrix}$$

$$\gamma_\mu = \begin{bmatrix}0 & \sigma_\mu\\ \bar{\sigma}_\mu & 0\end{bmatrix}, \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{bmatrix}-I_{2\times 2} & 0\\ 0 & I_{2\times 2}\end{bmatrix}$$

$$\mathcal{L}_f = -\frac{1}{g^2}\mathrm{tr}\bar{\Psi}\gamma^\mu D_\mu \Psi + \frac{m}{g^2}\mathrm{tr}\bar{\Psi}\frac{1-\gamma_5}{2}\Psi + \frac{m^*}{g^2}\mathrm{tr}\bar{\Psi}\frac{1+\gamma_5}{2}\Psi$$

$$\mathrm{tr}\bar{\Psi}\Psi = \mathrm{tr}\lambda\lambda + \mathrm{tr}\bar{\lambda}\bar{\lambda}$$
$$\mathrm{tr}\bar{\Psi}\gamma_5\Psi = -\mathrm{tr}\lambda\lambda + \mathrm{tr}\bar{\lambda}\bar{\lambda}$$

$$\mathcal{C}: A_\mu \rightarrow -A_\mu^* \equiv -A_\mu^a T^{a*}$$
$$\Psi^a T^a \rightarrow \Psi^a T^{a*}$$
$$\bar{\Psi}^a T^a \rightarrow \bar{\Psi}^a T^{a*}$$

$$\mathcal{P}: \quad \Psi(\vec{x}, x_4) \rightarrow P\Psi(-\vec{x}, x_4)$$
$$\bar{\Psi}(\vec{x}, x_4) \rightarrow \bar{\Psi}(-\vec{x}, x_4)P^\dagger$$

$$P \equiv \begin{bmatrix} 0 & iI_{2\times 2} \\ iI_{2\times 2} & 0 \end{bmatrix}, P^\dagger P = 1, P^2 = -1,$$
$$P^\dagger \gamma^i P = -\gamma^i \text{ for } i = 1,2,3, P^\dagger \gamma^4 P = \gamma^4.$$

$$\bar{\Psi}\Psi = \lambda^\alpha \lambda_\alpha + \bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} \overset{\mathcal{CP}}{\rightarrow} \bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \lambda^\alpha \lambda_\alpha = \bar{\Psi}\Psi$$

$$\bar{\Psi}\gamma_5\Psi = -\lambda^\alpha \lambda_\alpha + \bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} \overset{\mathcal{CP}}{\rightarrow} -\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \lambda^\alpha \lambda_\alpha = -\bar{\Psi}\gamma_5\Psi.$$

$$Z^T = 1 + L_1 m \int_{\mathbb{T}^3} d^3x \frac{1}{N} \sum_{k,E,e_2} (-1)^F \langle E, e_2 | (\hat{\lambda}(x))^2 e^{-L_1 \hat{H}_{SYM}} \hat{T}_2^k | E, e_2 \rangle$$
$$+ L_1 m^* \int_{\mathbb{T}^3} d^3x \frac{1}{N} \sum_{k,E,e_2} (-1)^F \langle E, e_2 | \left(\hat{\lambda}(x)^\dagger\right)^2 e^{-L_1 \hat{H}_{SYM}} \hat{T}_2^k | E, e_2 \rangle$$

$$+ \frac{L_1^2}{2} m^2 \int_{\mathbb{T}^3} d^3x d^3y \frac{1}{N} \sum_{k,E,e_2} (-1)^F \langle E, e_2 | (\hat{\lambda}(x))^2 (\hat{\lambda}(y))^2 e^{-L_1 \hat{H}_{SYM}} \hat{T}_2^k | E, e_2 \rangle$$
$$+ \frac{L_1^2}{2} (m^*)^2 \int_{\mathbb{T}^3} d^3x d^3y \frac{1}{N} \sum_{k,E,e_2} (-1)^F \langle E, e_2 | \left(\hat{\lambda}(x)^\dagger\right)^2 \left(\hat{\lambda}(y)^\dagger\right)^2 e^{-L_1 \hat{H}_{SYM}} \hat{T}_2^k | E, e_2 \rangle$$
$$+ \frac{L_1^2}{2} (m m^*) \int_{\mathbb{T}^3} d^3x d^3y \frac{1}{N} \sum_{k,E,e_2} (-1)^F \langle E, e_2 | (\hat{\lambda}(x))^2 \left(\hat{\lambda}(y)^\dagger\right)^2 + \left(\hat{\lambda}(x)^\dagger\right)^2 (\hat{\lambda}(y))^2 \Big)$$
$$\times e^{-L_1 \hat{H}_{SYM}} \hat{T}_2^k | E, e_2 \rangle + \mathcal{O}(|m|^3)$$

$$\langle e_2 | \hat{X}^{-1} \left(\hat{X} \hat{\lambda}^{2p} \left(\hat{\lambda}^\dagger\right)^{2q} \hat{X}^{-1} \right) \hat{X} | e_2 \rangle = e^{i\frac{2\pi}{N}(p-q)} \langle e_2 - 1 | \lambda^{2p} \left(\lambda^\dagger\right)^{2q} | e_2 - 1 \rangle \Longrightarrow$$
$$\langle e_2 | \lambda^{2p} \left(\lambda^\dagger\right)^{2q} | e_2 \rangle = e^{i\frac{2\pi}{N}(p-q)e_2} \langle e_2 = 0 | \lambda^{2p} \left(\hat{\lambda}^\dagger\right)^{2q} | e_2 = 0 \rangle$$

$$\sum_{e_2=0}^{N-1} \langle e_2 | \hat{\lambda}^{2p} \left(\hat{\lambda}^\dagger\right)^{2q} \hat{T}_2^k | e_2 \rangle = \langle e_2 = 0 | \hat{\lambda}^{2p} \left(\hat{\lambda}^\dagger\right)^{2q} | e_2 = 0 \rangle \sum_{e_2=0}^{N-1} e^{i\frac{2\pi}{N}(p-q+k)e_2}$$
$$= N \delta_{k, q-p (\mathrm{mod} N)} \langle e_2 = 0 | \hat{\lambda}^{2p} \left(\hat{\lambda}^\dagger\right)^{2q} | e_2 = 0 \rangle$$

$$\sum_{k=0}^{N-1} N^{-1} \sum_{e_2=0}^{N-1} \langle e_2|\hat{\lambda}^{2p}\left(\hat{\lambda}^{\dagger}\right)^{2q}\hat{T}_2^k|e_2\rangle = \langle e_2 = 0|\hat{\lambda}^{2p}\left(\hat{\lambda}^{\dagger}\right)^{2q}|e_2 = 0\rangle$$

$$\sum_{E,e_2} (-1)^F\langle E,e_2|\int_{\mathbb{T}^3} d^3x(\hat{\lambda})^{2p}\left(\hat{\lambda}^{\dagger}\right)^{2q} e^{-L_1\hat{H}}\hat{T}_2^k|E,e_2\rangle$$

$$\begin{aligned} Z^T &\simeq (1+|m|^2L^2c_0) + c\left(Lme^{-\frac{8\pi^2}{Ng^2}}e^{i\frac{\theta}{N}} + Lm^*e^{-\frac{8\pi^2}{Ng^2}}e^{-i\frac{\theta}{N}}\right) + \mathcal{O}\left(|m|^3, e^{-\frac{16\pi^2}{Ng^2}}\right) \\ &= (1+|m|^2L^2c_0) + c'L^4|m|\Lambda^3\cos\frac{\theta + N\arg m}{N} + \mathcal{O}\left(|m|^3, e^{-\frac{16\pi^2}{Ng^2}}\right) \end{aligned}$$

$$NZ_T\langle\hat{\lambda}^2(0)\rangle = \mathrm{tr}\,_{\mathcal{H}^{SYM}_{n_{34}=1}}\left((-1)^F e^{-L_1\hat{H}_{SYM}}\ \hat{\lambda}^2(0)\ \hat{T}_2^k\right) \qquad \leftarrow k = N-1$$

$$+L_1 m\int_{\mathbb{T}^3_x}\sum_{k,E,e_2}(-1)^F\langle E,e_2|(\hat{\lambda})^2(0)(\hat{\lambda}(x))^2 e^{-L_1\hat{H}_{SYM}}\hat{T}_2^k|E,e_2\rangle \qquad \leftarrow k = N-2,$$

$$+L_1 m^*\int_{\mathbb{T}^3_x}\sum_{k,E,e_2}(-1)^F\langle E,e_2|(\hat{\lambda})^2(0)(\hat{\lambda}(x)^{\dagger})^2 e^{-L_1\hat{H}_{SYM}}\hat{T}_2^k|E,e_2\rangle \ \leftarrow k = 0$$

$$+\frac{L_1^2}{2}m^2\int_{\mathbb{T}^3_x\times\mathbb{T}^3_y}\sum_{k,E,e_2}(-1)^F\langle E,e_2|(\hat{\lambda})^2(0)(\hat{\lambda}(x))^2(\hat{\lambda}(y))^2 e^{-L_1\hat{H}_{SYM}}\hat{T}_2^k|E,e_2\rangle \ \leftarrow k = N-3,$$

$$+\frac{L_1^2}{2}(m^*)^2\int_{\mathbb{T}^3_x\times\mathbb{T}^3_y}\sum_{k,E,e_2}(-1)^F\langle E,e_2|(\hat{\lambda})^2(0)(\hat{\lambda}(x)^{\dagger})^2(\hat{\lambda}(y)^{\dagger})^2 e^{-L_1\hat{H}_{SYM}}\hat{T}_2^k|E,e_2\rangle \ \leftarrow k = 1$$

$$+L_1^2(mm^*)\int_{\mathbb{T}^3_x\times\mathbb{T}^3_y}\sum_{k,E,e_2}(-1)^F\langle E,e_2|(\hat{\lambda})^2(0)(\hat{\lambda}(x))^2(\hat{\lambda}(y)^{\dagger})^2 e^{-L_1\hat{H}_{SYM}}\hat{T}_2^k|E,e_2\rangle \ \leftarrow k = N-1$$

$$+\mathcal{O}(|m|^3)\ .$$

$$\mathrm{tr}_{\mathcal{H}^{SYM}_{n_{34}=1}}\left((-1)^F e^{-L_1\hat{H}_{SYM}}\hat{\lambda}^2(0)\hat{T}_2^{N-1}\right) \equiv N\langle\hat{\lambda}^2\rangle_{\mathbb{R}^4,SYM} = N16\pi^2\Lambda^3 \star e^{-\frac{8\pi^2}{g^2N}p}$$

$$Z_T\langle\hat{\lambda}^2\rangle = 16\pi^2\Lambda^3(1+c_1|m|^2L^2) + c_2\frac{m^*}{L^2} + c_3(m^*L)^2\Lambda^3 e^{-i\frac{\theta}{N}}$$

$$Z_T\left\langle\left(\hat{\lambda}^{\dagger}\right)^2\right\rangle = 16\pi^2\Lambda^3(1+c_1|m|^2L^2) + c_2\frac{m}{L^2} + c_3(mL)^2\Lambda^3 e^{i\frac{\theta}{N}}$$

$$\left(-\not{D} + \begin{bmatrix} mI_2 & 0 \\ 0 & m^*I_2 \end{bmatrix}\right)\Psi = 0$$

$$-i\not{D}\Sigma_n = \omega_n\Sigma_n$$

$$\not{D}\not{D}\Sigma_n \equiv \begin{bmatrix} D\bar{D} & 0 \\ 0 & \bar{D}D \end{bmatrix}\Sigma_n = -\omega_n^2\Sigma_n, \text{ where } D \equiv \sigma_\mu D_\mu, \bar{D} \equiv \bar{\sigma}_\mu D_\mu$$

$$D\bar{D} = I_2 D_\mu D_\mu + iF_{\mu\nu}\sigma^{\mu\nu}$$
$$\bar{D}D = I_2 D_\mu D_\mu$$

$$\Sigma_{p,\beta}^{(0)} = \begin{bmatrix} \psi_{\alpha p,\beta}^{(0)} \\ 0 \end{bmatrix}, p = 1, \dots, k, \beta = 1,2$$

$$\Sigma_n^{(+)} = \begin{bmatrix} \psi_{\alpha n} \\ \bar{\psi}_n^{\dot{\alpha}} \end{bmatrix}, \Sigma_n^{(-)} = \gamma_5 \Sigma_n^{(+)} = \begin{bmatrix} -\psi_{\alpha n} \\ \bar{\psi}_n^{\dot{\alpha}} \end{bmatrix}$$

$$-i\not{D}\Sigma_n^{(+)} = \omega_n \Sigma_n^{(+)}$$

$$\psi_{\alpha n} = -\frac{i}{\omega_n}\sigma_{\mu\alpha\dot{\alpha}} D_\mu \bar{\psi}_n^{\dot{\alpha}} \equiv -\frac{i}{\omega_n}(D\psi_n)_\alpha$$

$$\bar{\psi}_{n,s}^{\dot{\alpha}}(x) = \bar{\zeta}^{\dot{\alpha}(s)}\phi_n(x)$$

$$D_\mu D^\mu \phi_n = -\omega_n^2 \phi_n, \text{ where, } \forall n, \omega_n^2 > 0$$

$$\phi(x + L_\mu) = \Omega_\mu(x)\phi(x)\Omega_\mu^\dagger(x)$$

$$\sum_n \varphi_n(x) \otimes \varphi_n(y) = \delta_{x,y}$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\phi_n(x)\phi_m(x) = \delta_{mn}$$

$$\bar{\zeta}^{\dot{\alpha}(s)} \equiv \delta^{\dot{\alpha}s}, \text{ for } s = 1,2, \dot{\alpha} = 1,2.$$

$$\Sigma_{n,s}^{(+)} = \begin{bmatrix} -\frac{i}{\omega_n}\sigma_{\mu\alpha s} D_\mu \phi_n \\ \delta^{\dot{\alpha}s}\phi_n \end{bmatrix}, \Sigma_{n,s}^{(-)} = \begin{bmatrix} \frac{i}{\omega_n}\sigma_{\mu\alpha s} D_\mu \phi_n \\ \delta^{\dot{\alpha}s}\phi_n \end{bmatrix}$$

$$\bar{\Psi} = (\lambda_\beta, \bar{\lambda}^{\dot{\beta}}) \begin{pmatrix} -\epsilon^{\beta\alpha} & 0 \\ 0 & -\epsilon_{\dot{\beta}\dot{\alpha}} \end{pmatrix}$$

$$\bar{\Sigma}_{n,s}^{(+)} = \left[\frac{i}{\omega_n}\sigma_{\mu\beta s}\epsilon^{\beta\alpha} D_\mu \phi_n, \epsilon_{\dot{\alpha}s}\phi_n\right], \bar{\Sigma}_{n,s}^{(-)} = \left[-\frac{i}{\omega_n}\sigma_{\mu\beta s}\epsilon^{\beta\alpha} D_\mu \phi_n, \epsilon_{\dot{\alpha}s}\phi_n\right]$$

$$\bar{\Sigma}_{p,\beta}^{(0)} = \left[-\psi_{\gamma p,\beta}^{(0)}\epsilon^{\gamma\alpha}, 0\right] = \left[\psi_{p,\beta}^{(0)\alpha}, 0\right], p = 1, \dots, k$$

$$\Psi = \xi_{p,\beta}^0 \Sigma_{p,\beta}^{(0)} + \xi_{n,s}^+ \Sigma_{n,s}^{(+)} + \xi_{n,s}^- \Sigma_{n,s}^{(-)}$$
$$\bar{\Psi} = \xi_{p,\beta}^0 \bar{\Sigma}_{p,\beta}^{(0)} + \xi_{n,s}^+ \bar{\Sigma}_{n,s}^{(+)} + \xi_{n,s}^- \bar{\Sigma}_{n,s}^{(-)}$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{p,\beta}^{(0)}\Sigma_{n,s}^{(\pm)} \sim \int_{\mathbb{T}^4} \mathrm{tr}\psi_{p,\beta}^{(0)\alpha}\sigma_{\mu\alpha s} D_\mu \phi_n = 0, \text{ likewise } \int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(\pm)}\Sigma_p^{(0)} = 0$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{p,\beta}^{(0)}\Sigma_{q,\beta'}^{(0)} = \int_{\mathbb{T}^4} \mathrm{tr}\psi_{p,\beta}^{(0)\alpha}\psi_{q,\beta',\alpha}^{(0)} = \delta_{pq}\epsilon_{\beta\beta'}$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(+)}\Sigma_{m,s'}^{(-)} = \epsilon_{s'\dot{\gamma}}\frac{(\bar{\sigma}_\nu\sigma_\mu)^{\dot{\gamma}}{}_s}{\omega_n\omega_m}\int_{\mathbb{T}^4} \mathrm{tr}\phi_n D_\mu D_\nu \phi_m + \epsilon_{s's}\int_{\mathbb{T}^4} \mathrm{tr}\phi_n\phi_m \stackrel{?}{=} 0$$

$$(\bar{\sigma}_\nu\sigma_\mu)^{\dot{\gamma}}{}_s \int_{\mathbb{T}^4} \mathrm{tr}\phi_n D_\mu D_\nu\phi_m = \delta^{\dot{\gamma}}{}_s\int_{\mathbb{T}^4} \mathrm{tr}\phi_n D_\mu D_\mu\phi_m + (\bar{\sigma}_{\nu\mu})^{\dot{\gamma}}{}_s\int_{\mathbb{T}^4} \mathrm{tr}\phi_n[D_\mu, D_\nu]\phi_m = -\delta_s^{\dot{\gamma}}\omega_n^2\delta_{nm}$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(+)}\Sigma_{m,s'}^{(-)} = 0$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(-)}\Sigma_{m,s'}^{(+)} = 0$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(+)}\Sigma_{m,s'}^{(+)} = -\epsilon_{s'\dot{\gamma}}\frac{(\bar{\sigma}_\nu\sigma_\mu)^{\dot{\gamma}}{}_s}{\omega_n\omega_m}\int_{\mathbb{T}^4} \mathrm{tr}\phi_n D_\mu D_\nu\phi_m + \epsilon_{s's}\int_{\mathbb{T}^4} \mathrm{tr}\phi_n\phi_m$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(+)}\Sigma_{m,s'}^{(+)} = 2\epsilon_{s's}\delta_{nm}$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(-)}\Sigma_{m,s'}^{(-)} = 2\epsilon_{s's}\delta_{nm}$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{p,\beta}^{(0)}\gamma_5\Sigma_{q,\beta'}^{(0)} = -\delta_{pq}\epsilon_{\beta\beta'}$$
$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(-)}\gamma_5\Sigma_{m,s'}^{(-)} = 0, \int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(+)}\gamma_5\Sigma_{m,s'}^{(+)} = 0$$
$$\int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(+)}\gamma_5\Sigma_{m,s'}^{(-)} = \int_{\mathbb{T}^4} \mathrm{tr}\bar{\Sigma}_{n,s}^{(-)}\gamma_5\Sigma_{m,s'}^{(+)} = 2\delta_{n,m}\epsilon_{s's}$$

$$g^2\int_{\mathbb{T}^4} \mathcal{L}_f = \int_{\mathbb{T}^4} (-\mathrm{tr}\bar{\Psi}\not{D}\Psi + \Re m\,\mathrm{tr}\bar{\Psi}\Psi - i\Im m\,\mathrm{tr}\bar{\Psi}\gamma_5\Psi)$$
$$= \xi_{p\beta}^0\xi_{p\gamma}^0\epsilon_{\beta\gamma}m + \xi_{n,s}^+\xi_{n,s'}^+\epsilon_{ss'}2[i\omega_n - \Re m] - \xi_{n,s}^-\xi_{n,s'}^-\epsilon_{ss'}2[i\omega_n + \Re m] + (\xi_{n,s}^+\xi_{n,s'}^- + \xi_{n,s}^-\xi_{n,s'}^+)\epsilon_{ss'}[2i\Im m]$$

$$e^{-S_f} = e^{-\int_{\mathbb{T}^4}\mathcal{L}_f}$$

$$-\int_{\mathbb{T}^4} \mathcal{L}_f$$
$$= \xi_{p1}^0\xi_{p2}^0\frac{2m}{g^2} - \xi_{n,1}^+\xi_{n,2}^+\frac{4}{g^2}[-i\omega_n + \Re m] - \xi_{n,1}^-\xi_{n,2}^-\frac{4}{g^2}[i\omega_n + \Re m] + (\xi_{n,1}^+\xi_{n,2}^- + \xi_{n,1}^-\xi_{n,2}^+)\left[\frac{4}{g^2}i\Im m\right]$$

$$d\Psi \equiv \prod_{p=1}^{k} d^2\xi_p^0 \prod_n d^4\xi_n$$

$$\mathcal{D}_k^f(m)$$
$$\equiv \int d\Psi e^{-\int_{\mathbb{T}^4}\mathcal{L}_f} = \left(\frac{2m}{g^2}\right)^k \prod_n \left(\frac{16}{g^4}(\omega_n^2 + |m|^2)\right)\prod_{p=1}^{k} \underbrace{\int d^2\xi_p^0\xi_{p1}^0\xi_{p2}^0}_{\epsilon_p}\prod_n \underbrace{\int d^4\xi_n\xi_{n,1}^+\xi_{n,2}^+\xi_{n,1}^-\xi_{n,2}^-}_{\epsilon_n}$$

$$\mathcal{D}_k^f(m) = \left(\frac{2m}{g^2}\right)^k \prod_n \left(\frac{16}{g^4}(\omega_n^2 + |m|^2)\right) \prod_{p=1}^{k} \epsilon_p \prod_n \epsilon_n$$

$$d\Psi \exp\left(-\int \mathcal{L}_f\right)$$

$$\langle \Psi(x) \otimes \bar{\Psi}(y) \rangle_{\text{unnorm.}}$$
$$= \left\langle \left(\xi_{p,\beta}^0 \Sigma_{p,\beta}^{(0)}(x) + \xi_{n,s}^+ \Sigma_{n,s}^{(+)}(x) + \xi_{n,s}^- \Sigma_{n,s}^{(-)}(x)\right) \otimes \left(\xi_{q,\beta'}^0 \bar{\Sigma}_{q,\beta'}^{(0)}(y) + \xi_{n',s'}^+ \bar{\Sigma}_{n',s'}^{(+)}(y) + \xi_{n',s'}^- \bar{\Sigma}_{n',s'}^{(-)}(y)\right)\right\rangle$$

$$\langle \Psi(x) \otimes \bar{\Psi}(y) \rangle_{\text{unnorm}}$$
$$= \langle \xi_{p,1}^0 \xi_{q,2}^0 \rangle \left[\Sigma_{p,1}^{(0)}(x) \otimes \bar{\Sigma}_{q,2}^{(0)}(y) - \Sigma_{p,2}^{(0)}(x) \otimes \bar{\Sigma}_{q,1}^{(0)}(y)\right]$$
$$+\langle \xi_{n,1}^+ \xi_{n,2}^+ \rangle \left[\Sigma_{n,1}^+(x) \otimes \bar{\Sigma}_{n,2}^+(y) - \Sigma_{n,2}^+(x) \otimes \bar{\Sigma}_{n,1}^+(y)\right] + \langle \xi_{n,1}^- \xi_{n,2}^- \rangle \left[\Sigma_{n,1}^-(x) \otimes \bar{\Sigma}_{n,2}^-(y) - \Sigma_{n,2}^-(x) \otimes \bar{\Sigma}_{n,1}^-(y)\right]$$
$$+\langle \xi_{n,1}^+ \xi_{n,2}^- \rangle \left[\Sigma_{n,1}^+(x) \otimes \bar{\Sigma}_{n,2}^-(y) - \Sigma_{n,2}^-(x) \otimes \bar{\Sigma}_{n,1}^+(y)\right] + \langle \xi_{n,1}^- \xi_{n,2}^+ \rangle \left[\Sigma_{n,1}^-(x) \otimes \bar{\Sigma}_{n,2}^+(y) - \Sigma_{n,2}^+(x) \otimes \bar{\Sigma}_{n,1}^-(y)\right]$$

$$\langle \xi_{p,1}^0 \xi_{q,2}^0 \rangle = \mathcal{D}_k^f(m) \delta_{pq} \frac{g^2}{2} \frac{m^*}{|m|^2}$$

$$\langle \xi_{n,1}^+ \xi_{n,2}^+ \rangle = \mathcal{D}_k^f(m) \frac{g^2}{4} \frac{-i\omega_n - \Re m}{\omega_n^2 + |m|^2}, \langle \xi_{n,1}^- \xi_{n,2}^- \rangle = \mathcal{D}_k^f(m) \frac{g^2}{4} \frac{i\omega_n - \Re m}{\omega_n^2 + |m|^2}$$

$$\langle \xi_{n,1}^+ \xi_{n,2}^- \rangle = \mathcal{D}_k^f(m) \frac{g^2}{4} \frac{-i\Im m}{\omega_n^2 + |m|^2}, \langle \xi_{n,1}^- \xi_{n,2}^+ \rangle = \mathcal{D}_k^f(m) \frac{g^2}{4} \frac{-i\Im m}{\omega_n^2 + |m|^2}$$

$$\langle \Psi(x) \otimes \bar{\Psi}(y) \rangle = \begin{pmatrix} \langle \lambda_\alpha(x) \otimes \lambda^\beta(y) \rangle \langle \lambda_\alpha(x) \otimes \bar{\lambda}_{\dot{\beta}}(y) \rangle \\ \langle \bar{\lambda}^{\dot{\alpha}}(x) \otimes \lambda^\beta(y) \rangle \langle \bar{\lambda}^{\dot{\alpha}}(x) \otimes \bar{\lambda}_{\dot{\beta}}(y) \rangle \end{pmatrix},$$

$$\left[\Sigma_{n,1}^+(x) \otimes \bar{\Sigma}_{n,2}^+(y) - \Sigma_{n,2}^+(x) \otimes \bar{\Sigma}_{n,1}^+(y)\right]$$
$$= \begin{pmatrix} (\sigma_{\mu\alpha 1}\sigma_{\nu\gamma 2} - \sigma_{\mu\alpha 2}\sigma_{\nu\gamma 2})\epsilon^{\gamma\beta} \dfrac{D_\mu \phi_n(x) \otimes D_\nu \phi_n(y)}{\omega_n^2} & (\sigma_{\mu\alpha 1}\epsilon_{2\dot{\beta}} - \sigma_{\mu\alpha 2}\epsilon_{1\dot{\beta}}) \dfrac{iD_\mu \phi_n(x) \otimes \phi_n(y)}{\omega_n} \\ (\delta^{\dot{\alpha}1}\sigma_{\nu\gamma 2} - \delta^{\dot{\alpha}2}\sigma_{\nu\gamma 1})\epsilon^{\gamma\beta} \dfrac{i\phi_n(x) \otimes D_\nu \phi_n(y)}{\omega_n} & (\delta^{\dot{\alpha}1}\epsilon_{\dot{\beta}2} - \delta^{\dot{\alpha}2}\epsilon_{\dot{\beta}1})\phi_n(x) \otimes \phi_n(y) \end{pmatrix},$$

$$\left[\Sigma_{n,1}^+(x) \otimes \bar{\Sigma}_{n,2}^+(y) - \Sigma_{n,2}^+(x) \otimes \bar{\Sigma}_{n,1}^+(y)\right] = \begin{pmatrix} -(\sigma_\mu \bar{\sigma}_\nu)_\alpha^{\ \beta} \dfrac{D_\mu \phi_n(x) \otimes D_\nu \phi_n(y)}{\omega_n^2} & \sigma_{\mu\alpha\dot{\beta}} \dfrac{iD_\mu \phi_n(x) \otimes \phi_n(y)}{\omega_n} \\ -\bar{\sigma}_\nu^{\dot{\alpha}\beta} \dfrac{i\phi_n(x) \otimes D_\nu \phi_n(y)}{\omega_n} & -\delta_{\dot{\beta}}^{\dot{\alpha}} \phi_n(x) \otimes \phi_n(y) \end{pmatrix}.$$

$$\sigma_\mu \bar{\sigma}_\nu D_\mu \phi_n(x) \otimes D_\nu \phi_n(y)$$
$$= D_\mu \phi_n(x) \otimes D_\mu \phi_n(y) + \sigma_{\mu\nu}\left(D_\mu \phi_n(x) \otimes D_\nu \phi_n(y) - D_\nu \phi_n(x) \otimes D_\mu \phi_n(y)\right)$$

$$\left[\Sigma_{n,1}^-(x) \otimes \bar{\Sigma}_{n,2}^-(y) - \Sigma_{n,2}^-(x) \otimes \bar{\Sigma}_{n,1}^-(y)\right] = \begin{pmatrix} -(\sigma_\mu \bar{\sigma}_\nu)_\alpha^{\ \beta} \dfrac{D_\mu \phi_n(x) \otimes D_\nu \phi_n(y)}{\omega_n^2} & -\sigma_{\mu\alpha\dot{\beta}} \dfrac{iD_\mu \phi_n(x) \otimes \phi_n(y)}{\omega_n} \\ \bar{\sigma}_\nu^{\dot{\alpha}\beta} \dfrac{i\phi_n(x) \otimes D_\nu \phi_n(y)}{\omega_n} & -\delta_{\dot{\beta}}^{\dot{\alpha}} \phi_n(x) \otimes \phi_n(y) \end{pmatrix}$$

$$\langle \xi_{n,1}^{+}\xi_{n,2}^{-}\rangle = \langle \xi_{n,1}^{-}\xi_{n,2}^{+}\rangle$$

$$\left[\Sigma_{n,s}^{+}(x)\otimes\bar{\Sigma}_{n,s'}^{-}(y)\epsilon^{ss'}+\Sigma_{n,s}^{-}(x)\otimes\bar{\Sigma}_{n,s'}^{+}(y)\epsilon^{ss'}\right] = \begin{pmatrix} 2\left(\sigma_{\mu}\bar{\sigma}_{\nu}\right)_{\alpha}{}^{\beta}\dfrac{D_{\mu}\phi_n(x)\otimes D_{\nu}\phi_n(y)}{\omega_n^2} & 0 \\ 0 & -2\delta_{\dot\beta}^{\dot\alpha}\phi_n(x)\otimes\phi_n(y) \end{pmatrix}$$

$$\left[\Sigma_{p,1}^{(0)}(x)\otimes\bar{\Sigma}_{q,2}^{(0)}(y)-\Sigma_{p,2}^{(0)}(x)\otimes\bar{\Sigma}_{q,1}^{(0)}(y)\right] = \begin{pmatrix} \sum_{p=1}^{k}\left(\psi_{\alpha p,1}^{(0)}(x)\otimes\psi_{p,2}^{(0)\beta}(y)-\psi_{\alpha p,2}^{(0)}(x)\otimes\psi_{p,1}^{(0)\beta}(y)\right) & 0 \\ 0 & 0 \end{pmatrix}$$

$$\langle\Psi(x)\otimes\bar{\Psi}(y)\rangle_{\text{unnorm.}} \equiv \begin{pmatrix} \langle\lambda_{\alpha}(x)\otimes\lambda^{\beta}(y)\rangle & \langle\lambda_{\alpha}(x)\otimes\bar{\lambda}_{\dot\beta}(y)\rangle \\ \langle\bar{\lambda}^{\dot\alpha}(x)\otimes\lambda^{\beta}(y)\rangle & \langle\bar{\lambda}^{\dot\alpha}(x)\otimes\bar{\lambda}_{\dot\beta}(y)\rangle \end{pmatrix}$$

$$= \mathcal{D}_k^f(m)\frac{g^2}{2}\left\{\begin{pmatrix} \dfrac{m^*}{|m|^2}\sum_{p=1}^{k}\left(\psi_{\alpha p,1}^{(0)}(x)\otimes\psi_{p,2}^{(0)\beta}(y)-\psi_{\alpha p,2}^{(0)}(x)\otimes\psi_{p,1}^{(0)\beta}(y)\right) & 0 \\ 0 & 0 \end{pmatrix}\right.$$

$$\left.+\sum_{n}\begin{pmatrix} \dfrac{m^*}{\omega_n^2+|m|^2}\left(\sigma_{\mu}\bar{\sigma}_{\nu}\right)_{\alpha}{}^{\beta}\dfrac{D_{\mu}\phi_n(x)\otimes D_{\nu}\phi_n(y)}{\omega_n^2} & \dfrac{\sigma_{\mu\alpha\dot\beta}}{\omega_n^2+|m|^2}D_{\mu}\phi_n(x)\otimes\phi_n(y) \\ -\dfrac{\bar{\sigma}_{\nu}^2}{\omega_n^2+|m|^2}\phi_n(x)\otimes D_{\nu}\phi_n(y) & \dfrac{m}{\omega_n^2+|m|^2}\delta_{\dot\beta}^{\dot\alpha}\phi_n(x)\otimes\phi_n(y) \end{pmatrix}\right\}$$

$$\langle\lambda_{\alpha}(x)\otimes\lambda^{\beta}(y)\rangle\rightarrow\left\langle\lambda_{ij\alpha}(x)\lambda_{kl}^{\beta}(y)\right\rangle$$
$$D_{\mu}\phi_n(x)\otimes\phi_n(y)\rightarrow\left(D_{\mu}\phi_n\right)_{ij}(x)\phi_{nkl}(y),\ \text{etc.}$$

$$\phi_{(n_{\mu},f_{\mu})} = \frac{4}{\sqrt{V}}\prod_{\mu} f_{\mu}(n_{\mu}x_{\mu})$$
$$f_{\mu}(n_{\mu}x_{\mu}) = \cos\frac{2\pi n_{\mu}x_{\mu}}{L_{\mu}},\ \text{or}\ \sin\frac{2\pi n_{\mu}x_{\mu}}{L_{\mu}}$$

$$\omega_n^2 = \sum_{\mu}\left(\frac{2\pi n_{\mu}}{L_{\mu}}\right)^2$$

$$\int_0^L dx\cos\left(\frac{2\pi nx}{L}\right)\cos\left(\frac{2\pi mx}{L}\right) = \frac{L}{2}\delta_{nm}$$

$$\phi_{(n_{\mu},f_{\mu})} = \frac{4}{\sqrt{V}}\prod_{\mu} f_{\mu}(n_{\mu}x_{\mu})$$
$$f_{\mu}(n_{\mu}x_{\mu}) = \cos\frac{2\pi n_{\mu}x_{\mu}}{L_{\mu}},\ \text{or}\ \sin\frac{2\pi n_{\mu}x_{\mu}}{L_{\mu}}.$$

$$\phi_{B'B'(n_\mu,f_\mu)}(x)=\frac{2^{\frac{4}{2}}}{\sqrt{V}}\prod_{\mu=1}^{4} f_\mu(x_\mu), \text{ for every } B', 2^4$$

$$\omega_n^2=\sum_{\mu=1}^{4}\left(\frac{2\pi n_\mu}{L_\mu}\right)^2, n_{1,2,3,4}>0, (\phi_{B'B'}\to\phi_b,).$$

$$\tilde{\boldsymbol{H}}\equiv\left(\frac{\omega}{2\pi\sqrt{Nk(N-k)}}, \boldsymbol{H}_{(k)}\right), \mathrm{tr}\left[\tilde{H}^{b_1}\tilde{H}^{b_2}\right]=\delta_{b_1b_2}, b_1, b_2=1,2,..,k$$

$$\lambda(x)=\sum_{b=1}^{k}\tilde{\lambda}_b\tilde{H}^b+\text{ off diagonal}$$

$$\omega_n^2=(2\pi)^2\sum_\mu n_\mu^2/L_\mu^2$$

$$\sum_{f_\mu}\phi_{(n_\mu,f_\mu)}(x)\phi_{(n_\mu,f_\mu)}(y)=\frac{16}{V}\sum_{\{f\}=\{\cos,\sin\}}\prod_\lambda f_\lambda(x_\lambda)f_\lambda(y_\lambda)$$

$$=\frac{16}{V}\prod_\lambda\left(\sum_{f_\lambda=\{\cos(\dots),\sin(\dots)\}} f_\lambda(x_\lambda)f_\lambda(y_\lambda)\right)$$

$$\sum_{f_\lambda(\dots)=\left\{\cos\left(\frac{2\pi n_\lambda\cdots}{L_\lambda}\right),\sin\frac{2\pi n_\lambda\cdots}{L_\lambda}\right)\}} f_\lambda(x_\lambda)f_\lambda(y_\lambda)=\frac{1}{2}\left(e^{i\frac{2\pi n_\lambda}{L_\lambda}(x_\lambda-y_\lambda)}+e^{-i\frac{2\pi n_\lambda}{L_\lambda}(x_\lambda-y_\lambda)}\right),$$

$$\left\langle\bar{\lambda}_b^{\dot{\alpha}}(x)\bar{\lambda}_{b'\dot{\beta}}(y)\right\rangle_{\text{unnorm.}}=\delta_{bb'}\mathcal{D}_k^f(m)\frac{g^2}{2V}\sum_{p_\mu\in\frac{2\pi}{L_\mu}\mathbb{Z}}{}^{'}\frac{\delta_{\dot{\beta}}^{\dot{\alpha}}m}{p_\mu^2+|m|^2}e^{ip_\mu(x_\mu-y_\mu)}$$

$$\sum_{f_\mu}\partial_1\phi_{(n_\mu,f_\mu)}(x)\phi_{(n_\mu,f_\mu)}(y)$$

$$=\frac{16}{V}\sum_{\{f\}=\{\cos,\sin\}}\partial_1 f_1(x_1)f_1(y_1)\prod_{\lambda=2,3,4}f_\lambda(x_\lambda)f_\lambda(y_\lambda)$$

$$=\frac{1}{V}\frac{i2\pi n_1}{L_1}\left(e^{i\frac{2\pi n_1}{L_1}(x_1-y_1)}-e^{-i\frac{2\pi n_1}{L_1}(x_1-y_1)}\right)\prod_{\lambda=2,3,4}\left(e^{i\frac{2\pi n_\lambda}{L_\lambda}(x_\lambda-y_\lambda)}+e^{-i\frac{2\pi n_\lambda}{L_\lambda}(x_\lambda-y_\lambda)}\right).$$

$$\sum_{f_1(\dots)=\left\{\cos\left(\frac{2\pi n_1\dots}{L_1}\right),\sin\frac{2\pi n_1\dots}{L_1}\right)\}}\partial_1 f_1(x_1)f_1(y_1)=\frac{1}{2}\frac{i2\pi n_1}{L_1}\left(e^{i\frac{2\pi n_1}{L_1}(x_1-y_1)}-e^{-i\frac{2\pi n_1}{L_1}(x_1-y_1)}\right)$$

$$\sum_{f_1(\dots)=\left\{\cos\left(\frac{2\pi n_1\dots}{L_1}\right),\sin\frac{2\pi n_1\dots}{L_1}\right)\}} f_1(x_1)\partial_1 f_1(y_1)=-\frac{1}{2}\frac{i2\pi n_1}{L_1}\left(e^{i\frac{2\pi n_1}{L_1}(x_1-y_1)}-e^{-i\frac{2\pi n_1}{L_1}(x_1-y_1)}\right)$$

$$\left\langle \lambda_{b\alpha}(x)\bar{\lambda}_{b'\dot{\beta}}(y)\right\rangle_{\text{unnorm}.} = \delta_{bb'}\mathcal{D}_k^f(m)\frac{g^2}{2V}\sum_{p_\mu\in\frac{2\pi}{L_\mu}\mathbb{Z}}{}^{'} \frac{i\sigma_{\mu\alpha\dot{\beta}}p_\mu}{p_\mu^2+|m|^2}e^{ip_\mu(x_\mu-y_\mu)}$$

$$\left\langle \bar{\lambda}_b^{\dot{\alpha}}(x)\lambda_{b'}^{\beta}(y)\right\rangle_{\text{unnorm}.} = \delta_{bb'}\mathcal{D}_k^f(m)\frac{g^2}{2V}\sum_{p_\mu\in\frac{2\pi}{L_\mu}\mathbb{Z}}{}^{'} \frac{i\bar{\sigma}_\nu^{\dot{\alpha}\beta}p_\nu}{p_\mu^2+|m|^2}e^{ip_\mu(x_\mu-y_\mu)}$$

$$\left(\sigma_\mu\bar{\sigma}_\nu\right)_\alpha{}^\beta\frac{D_\mu\phi_n(x)D_\nu\phi_n(y)}{\omega_n^2}$$

$$\sum_\mu\frac{D_\mu\phi_n(x)D_\mu\phi_n(y)}{\omega_n^2}$$

$$\sum_{f_\mu}\partial_1\phi_{(n_\mu,f_\mu)}(x)\partial_2\phi_{(n_\mu,f_\mu)}(y)-\partial_2\phi_{(n_\mu,f_\mu)}(x)\partial_1\phi_{(n_\mu,f_\mu)}(y)$$

$$=\frac{16}{V}\left(\sum_{f_1}\left(\partial_1f_1(x_1)f_1(y_1)\right)\left(\sum_{f_2}f_2(x_2)\partial_2f_2(y_2)\right)-\left(\sum_{f_1}f_1(x_1)\partial_1f_1(y_1)\right)\left(\sum_{f_2}\partial_2f_2(x_2)f_2(y_2)\right)\right)$$

$$\times\prod_{\lambda=3,4}\left(\sum_{f_\lambda}f_\lambda(x_\lambda)f_\lambda(y_\lambda)\right)$$

$$=0$$

$$\sum_\mu\frac{D_\mu\phi_n(x)D_\mu\phi_n(y)}{\omega_n^2}$$

$$\sum_f\partial_1\phi_{n_\mu,f_\mu}(x)\partial_1\phi_{n_\mu,f_\mu}(y)$$

$$=\frac{16}{V}\sum_{\{f\}=\{\cos,\sin\}}\partial_1f_1(x_1)\partial_1f_1(y_1)\prod_{\lambda=2,3,4}f_\lambda(x_\lambda)f_\lambda(y_\lambda)$$

$$=\frac{1}{V}\left(\frac{2\pi n_1}{L_1}\right)^2\left(e^{i\frac{2\pi n_1}{L_1}(x_1-y_1)}+e^{-i\frac{2\pi n_1}{L_1}(x_1-y_1)}\right)\prod_{\lambda=2,3,4}\left(e^{i\frac{2\pi n_\lambda}{L_\lambda}(x_\lambda-y_\lambda)}+e^{-i\frac{2\pi n_\lambda}{L_\lambda}(x_\lambda-y_\lambda)}\right)$$

$$\left\langle \lambda_{b\alpha}(x)\lambda_{b'}^{\beta}(y)\right\rangle_{unnorm.} = \delta_{bb'}\delta_\alpha^\beta\mathcal{D}_k^f(m)\frac{g^2}{2V}\left(\frac{m^*}{|m|^2}+\sum_{p_\mu\in\frac{2\pi}{L_\mu}\mathbb{Z}}{}^{'}\frac{m^*}{p_\mu^2+|m|^2}e^{ip_\mu(x_\mu-y_\mu)}\right),$$

$$\phi^{(0)b}=\frac{1}{\sqrt{V}}\tilde{H}^b, b=1,\dots k, \text{ with } \int_{\mathbb{T}^4}\text{tr}\phi^{(0)b}\phi^{(0)a}=\delta^{ab}$$

$$E^{ba} \equiv \int_{\mathbb{T}^4} \mathrm{tr}\phi^{(0)b}\left(-D_\mu D_\mu\right)_{\text{pert.}} \phi^{(0)a} = \int_{\mathbb{T}^4} \mathrm{tr}\left(D_\mu\right)_{\text{pert.}} \phi^{(0)b}\left(D_\mu\right)_{\text{pert.}} \phi^{(0)a}$$
$$= -\int_{\mathbb{T}^4} \mathrm{tr}\left[A_\mu^{\text{pert.}}, \phi^{(0)b}\right]\left[A_\mu^{\text{pert.}}, \phi^{(0)a}\right]$$

$$A_\mu^{\text{pert.}} = \sqrt{\Delta}\begin{bmatrix} 0 & w_\mu \\ w_\mu^\dagger & 0 \end{bmatrix} + \Delta \begin{bmatrix} \mathcal{S}_\mu^{(k)} & 0 \\ 0 & \mathcal{S}_\mu^{(\ell)} \end{bmatrix} + \mathcal{O}\left(\Delta^{\frac{3}{2}}\right)$$

$$E^{11} = \frac{2N\Delta}{k(N-k)} \sum_{C'=1}^{k} \sum_{D=1}^{\ell} \frac{1}{V} \int_{\mathbb{T}^4} \left(w_\mu^\dagger\right)_{DC'} \left(w_\mu\right)_{C'D}$$
$$E^{1b} = \frac{2\Delta}{k(N-k)} \sum_{C'=1}^{k} \left(\tilde{H}^b\right)_{C'C'} \sum_{D=1}^{\ell} \frac{1}{V} \int_{\mathbb{T}^4} \left(w_\mu\right)_{C'D} \left(w_\mu^\dagger\right)_{DC''}, \text{ for } b = 2, \dots, k,$$
$$E^{ab} = 2\Delta \sum_{C'=1}^{k} \left(\tilde{H}^a \tilde{H}^b\right)_{C'C'} \sum_{D=1}^{\ell} \frac{1}{V} \int_{\mathbb{T}^4} \left(w_\mu\right)_{C'D} \left(w_\mu^\dagger\right)_{DC''}, \text{ for } a, b = 2, \dots, k$$

$$\sum_{D=1}^{\ell} \frac{1}{V} \int_{\mathbb{T}^4} \left(w_\mu\right)_{C'D} \left(w_\mu^\dagger\right)_{DC'} = \frac{2\pi}{kN} \frac{1}{\sqrt{V}} \to \frac{2\pi}{kN} \frac{1}{L^2}$$

$$E^{11} = \frac{4\pi}{k(N-k)} \frac{\Delta}{L^2}$$
$$E^{1b} = 0$$
$$E^{ab} = \frac{4\pi}{kN} \frac{\Delta}{L^2} \mathrm{tr}\tilde{H}^a \tilde{H}^b = \delta^{ab} \frac{4\pi}{kN} \frac{\Delta}{L^2}, \text{ for } a, b = 2, \dots, k$$

$$\left\langle \bar{\lambda}_b^{\dot{\alpha}}(x) \bar{\lambda}_{b'\dot{\beta}}(y) \right\rangle_{\text{unnorm.}} = \delta_{bb'} \mathcal{D}_k^f(m) \frac{g^2}{2V} \sum_{p_\mu \in \frac{2\pi}{L_\mu}\mathbb{Z}}{}' \frac{\delta_{\dot{\beta}}^{\dot{\alpha}} m}{p_\mu^2 + |m|^2} e^{ip_\mu(x_\mu - y_\mu)}$$
$$+\delta_{bb'}\delta_{b1}\mathcal{D}_k^f(m) \frac{g^2}{2V} \delta_{\dot{\beta}}^{\dot{\alpha}} \frac{m}{\frac{4\pi}{k(N-k)} \frac{\Delta}{L^2} + |m|^2} + \delta_{bb'} \sum_{a=2}^{k} \delta_{ba} \mathcal{D}_k^f(m) \frac{g^2}{2V} \delta_{\dot{\beta}}^{\dot{\alpha}} \frac{m}{\frac{4\pi}{kN} \frac{\Delta}{L^2} + |m|^2}$$

$$\frac{|m|L}{\sqrt{c\Delta}} \ll 1, c = \frac{4\pi}{kN}$$

$$\left\langle \bar{\lambda}_b^{\dot{\alpha}}(x) \bar{\lambda}_{b'\dot{\beta}}(y) \right\rangle_{\text{unnorm.}} = \delta_{bb'} \mathcal{D}_k^f(m) \frac{g^2}{2V} \sum_{p_\mu \in \frac{2\pi}{L_\mu}\mathbb{Z}}{}' \frac{\delta_{\dot{\beta}}^{\dot{\alpha}} m}{p_\mu^2 + |m|^2} e^{ip_\mu(x_\mu - y_\mu)}$$
$$+\delta_{bb'} \mathcal{D}_k^f(m) \frac{g^2}{2V} \delta_{\dot{\beta}}^{\dot{\alpha}} \frac{mL^2}{c\Delta} \left(1 - \frac{|m|^2 L^2}{c\Delta} + \cdots\right)$$

$$\left\langle \bar{\lambda}_b^{\dot{\alpha}}(x) \bar{\lambda}_{b'\dot{\beta}}(y) \right\rangle_{\text{unnorm.}} = \delta_{bb'} \mathcal{D}_k^f(m) \frac{g^2}{2V} \delta_{\dot{\beta}}^{\dot{\alpha}} \left(\frac{mL^2}{c\Delta} + \sum_{p_\mu \in \frac{2\pi}{L_\mu}\mathbb{Z}}{}' \frac{m e^{ip_\mu(x_\mu - y_\mu)}}{p_\mu^2} \right)$$

$$\begin{pmatrix} 0 & f \\ f^* & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & if \\ -if^* & 0 \end{pmatrix} \left(\phi_{(n_\mu, B'>C',\alpha)}\right)_{D'E'}$$

$$\left(\phi_{(n_\mu, B'>C',\alpha)}\right)_{D'E'}$$
$$= \frac{1}{\sqrt{2}}\frac{\alpha}{\sqrt{V}} e^{i2\pi\left(n_\mu + \delta_{\mu 2}\frac{B'-C'}{k}\right)\frac{x_\mu}{L_\mu}} \delta_{B'D'}\delta_{C'E'} + \frac{1}{\sqrt{2}}\frac{\alpha^*}{\sqrt{V}} e^{-i2\pi\left(n_\mu + \delta_{\mu 2}\frac{B'-C'}{k}\right)\frac{x_\mu}{L_\mu}} \delta_{B'E'}\delta_{D'C'}$$
$$\text{with eigenvalue } \omega_n^2 = \sum_{\mu=1}^{4} \left(2\pi \frac{n_\mu + \delta_{\mu 2}\frac{B'-C'}{k}}{L_\mu}\right)^2, n_\mu \in \mathbb{Z}, \alpha = \{1, i\}$$

$$\left(\phi_{(n_\mu, B'>C',\alpha)}\right)^*_{D'E'} = \left(\phi_{(n_\mu, B'>C',\alpha)}\right)_{E'D'}$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\phi_{(n_\mu, B'>C',\alpha)}\phi_{(m_\mu, E'>F',\beta)} = \int_{\mathbb{T}^4} \sum_{D',G'=1}^{k} \left(\phi_{(n_\mu, B'>C',\alpha)} D_{D'G'}(x) \left(\phi_{(m_\mu, E'>F',\beta)}\right)_{G'D'}(x)\right.$$
$$= \frac{1}{V}\delta_{B'E'}\delta_{C'F'} \int_{\mathbb{T}^4} d^4x \left(\frac{\alpha^*\beta}{2} e^{-\sum_{\mu=1}^4 i2\pi(n_\mu - m_\mu)\frac{x_\mu}{L_\mu}} + \frac{\alpha\beta^*}{2} e^{\sum_{\mu=1}^4 i2\pi(n_\mu - m_\mu)\frac{x_\mu}{L_\mu}}\right)$$
$$= \delta_{B'E'}\delta_{C'F'} \prod_\mu \delta_{n_\mu m_\mu} \frac{\alpha\beta^* + \alpha^*\beta}{2} = \delta_{B'E'}\delta_{C'F'}\delta_{\alpha\beta} \prod_\mu \delta_{n_\mu m_\mu}$$

$$\left\langle \bar{\lambda}^{\dot{\alpha}}_{D'E'}(x)\bar{\lambda}_{\dot{\beta}F'G'}(y)\right\rangle$$

$$\frac{g^2}{2}\mathcal{D}_k^f(m) \sum_{n_\mu, B'>C',\alpha} \frac{m\delta^{\dot{\alpha}}_{\dot{\beta}}}{|m|^2 + \omega_n^2} \left(\phi_{(n_\mu, B'>C',\alpha)}\right)_{D'E'}(x) \left(\phi_{(n_\mu, B'>C',\alpha)}\right) F'G'(y)$$
$$= \frac{g^2}{4V}\mathcal{D}_k^f(m) \sum_{p_\mu = \frac{2\pi n_\mu}{L_\mu}, B'>C'} \frac{m\delta^{\dot{\alpha}}_{\dot{\beta}}}{|m|^2 + \left(p_\mu + \delta_{\mu 2}\frac{2\pi}{L_2}\frac{B'-C'}{k}\right)^2}$$
$$\times \left(e^{i(x_\mu + y_\mu)\left(p_\mu + \delta_{\mu,2}\frac{2\pi(B'-C')}{kL_2}\right)} \delta_{B'D'}\delta_{C'E'}\delta_{B'F'}\delta_{C'G'} \left(\sum_\alpha \alpha^2\right)\right.$$
$$+ e^{i(x_\mu - y_\mu)\left(p_\mu + \delta_{\mu,2}\frac{2\pi(B'-C')}{kL_2}\right)} \delta_{B'D'}\delta_{C'E'}\delta_{B'G'}\delta_{C'F'} \left(\sum_\alpha |\alpha|^2\right)$$
$$+ e^{-i(x_\mu - y_\mu)\left(p_\mu + \delta_{\mu,2}\frac{2\pi(B'-C')}{kL_2}\right)} \delta_{B'E'}\delta_{C'D'}\delta_{B'F'}\delta_{C'G'} \left(\sum_\alpha |\alpha|^2\right)$$
$$+ e^{-i(x_\mu + y_\mu)\left(p_\mu + \delta_{\mu,2}\frac{2\pi(B'-C')}{kL_2}\right)} \delta_{B'E'}\delta_{C'D'}\delta_{B'G'}\delta_{C'F'} \left(\sum_\alpha (\alpha^*)^2\right)$$

$$\frac{g^2}{2V}\mathcal{D}_k^f(m)\times$$

$$\left[\sum_{p_\mu=\frac{2\pi n_\mu}{L_\mu}} \frac{m\delta_{\dot\beta}^{\dot\alpha}}{|m|^2+\left(p_\mu+\delta_{\mu 2}\frac{2\pi}{L_2}\frac{D'-E'}{k}\right)^2}\left(e^{ix_\mu\left(p_\mu+\delta_{\mu,2}\frac{2\pi(D'-E')}{kL_2}\right)-iy_\mu\left(p_\mu-\delta_{\mu,2}\frac{2\pi(F'-G')}{kL_2}\right)}\delta_{D'G'}\delta_{E'F'}\theta_{D'E'}\right)\right.$$

$$\left.+\sum_{p_\mu=\frac{2\pi n_\mu}{L_\mu}} \frac{m\delta_{\dot\beta}^{\dot\alpha}}{|m|^2+\left(p_\mu-\delta_{\mu 2}\frac{2\pi}{L_2}\frac{D'-E'}{k}\right)^2}\left(e^{-ix_\mu\left(p_\mu-\delta_{\mu,2}\frac{2\pi(D'-E')}{kL_2}\right)+iy_\mu\left(p_\mu+\delta_{\mu,2}\frac{2\pi(F'-G')}{kL_2}\right)}\delta_{D'G'}\delta_{E'F'}\theta_{E'D'}\right)\right]$$

$$\begin{aligned}&\langle\bar\lambda^{\dot\alpha}_{D'E'}(x)\bar\lambda_{\dot\beta F'G'}(y)\rangle\\ &=\frac{g^2}{2V}\mathcal{D}_k^f(m)\\ &\times\sum_{p_\mu=\frac{2\pi n_\mu}{L_\mu}} \frac{m\delta_{\dot\beta}^{\dot\alpha}}{|m|^2+\left(p_\mu+\delta_{\mu 2}\frac{2\pi}{L_2}\frac{D'-E'}{k}\right)^2}e^{ix_\mu\left(p_\mu+\delta_{\mu,2}\frac{2\pi(D'-E')}{kL_2}\right)-iy_\mu\left(p_\mu-\delta_{\mu,2}\frac{2\pi(F'-G')}{kL_2}\right)}\delta_{D'G'}\delta_{E'F'}\end{aligned}$$

$$\begin{aligned}&\left\langle\lambda_{\alpha D'E'}(x)\lambda^{\beta}_{F'G'}(y)\right\rangle\\ &=\frac{g^2}{2V}\mathcal{D}_k^f(m)\\ &\times\sum_{p_\mu=\frac{2\pi n_\mu}{L_\mu}} \frac{m^*\delta_{\alpha}^{\beta}}{|m|^2+\left(p_\mu+\delta_{\mu 2}\frac{2\pi}{L_2}\frac{D'-E'}{k}\right)^2}e^{ix_\mu\left(p_\mu+\delta_{\mu,2}\frac{2\pi(D'-E')}{kL_2}\right)-iy_\mu\left(p_\mu-\delta_{\mu,2}\frac{2\pi(F'-G')}{kL_2}\right)}\delta_{D'G'}\delta_{E'F'}\end{aligned}$$

$$\phi(x+L_3\hat e_3)=P_\ell\phi(x)P_\ell^{-1},\phi(x+L_4\hat e_4)=Q_\ell\phi(x)Q_\ell^{-1}$$

$$\phi(x)=\sum_{\boldsymbol{p}=(p_3,p_4)\neq(0,0)}\phi_{\boldsymbol{p}}(x)J_{\mathbf{p}},\phi\in su(\ell),$$

$$J_{\boldsymbol{p}}=e^{-i\frac{\pi p_3p_4}{\ell}}Q_\ell^{-p_3}P_\ell^{p_4},J_{\boldsymbol{p}}^\dagger=e^{-i\frac{\pi p_3p_4}{\ell}}Q_\ell^{p_3}P_\ell^{-p_4}\equiv J_{-\boldsymbol{p}}$$

$$\mathrm{tr}\left[J_{\boldsymbol{p}}J_{\boldsymbol{p}'}\right]=\mathrm{tr}[\mathbf{1}_{\ell\times\ell}]\delta_{\boldsymbol{p},-\boldsymbol{p}'}=\ell\delta_{\boldsymbol{p},-\boldsymbol{p}'}\left(\mathrm{or tr}\left[J_{\boldsymbol{p}}J_{\boldsymbol{p}'}^\dagger\right]=\ell\delta_{\boldsymbol{p},\boldsymbol{p}'}\right)$$

$$\begin{aligned}\phi_{\boldsymbol{p}}(x+L_3\hat e_3)&=e^{-i\frac{2\pi p_3}{\ell}}\phi_{\boldsymbol{p}}(x), &\phi_{\boldsymbol{p}}(x+L_4\hat e_4)&=e^{-i\frac{2\pi p_4}{\ell}}\phi_{\boldsymbol{p}}(x),\\ \phi_{\boldsymbol{p}}(x+L_1\hat e_1)&=\phi_{\boldsymbol{p}}(x), &\phi_{\boldsymbol{p}}(x+L_2\hat e_2)&=\phi_{\boldsymbol{p}}(x).\end{aligned}$$

$$\phi_{\boldsymbol{p},n_\mu}=e^{-i2\pi\frac{x_\mu}{L\mu}\left(n_\mu+\delta_{\mu3}\frac{p_3}{\ell}+\delta_{\mu4}\frac{p_4}{\ell}\right)}J_{\boldsymbol{p}},n_\mu\in\mathbb{Z},p_3\in[0,\ell-1],p_4\in[0,\ell-1],(p_3,p_4)\neq(0,0)$$

$$\omega_n^2\to\omega_{\boldsymbol{p},n_\mu}^2=\sum_{\mu=1}^{4}\left(\frac{2\pi}{L_\mu}\left(n_\mu+\delta_{\mu3}\frac{p_3}{\ell}+\delta_{\mu4}\frac{p_4}{\ell}\right)\right)^2$$

$$\int\ \mathrm{tr}\phi_{\boldsymbol{p},n_\mu}^\dagger\phi_{\boldsymbol{p}',n_\mu'}\sim\delta_{\boldsymbol{p},\boldsymbol{p}'}\delta_{n,n'}$$

$$\phi_{p_3,p_4,n_\mu,f_1,f_2} = f_1(n_1x_1)f_2(n_2x_2)e^{-i2\pi\frac{x_3}{L_3}\left(n_3+\frac{p_3}{\ell}\right)}e^{-i2\pi\frac{x_4}{L_3}\left(n_4+\frac{p_4}{\ell}\right)}J_{p_3,p_4}$$

$$\phi'_{p_3,p_4,n_\mu,f_1,f_2} \equiv f_1(n_1x_1)f_2(n_2x_2)e^{i2\pi\frac{x_3}{L_3}\left(n_3+\frac{p_3}{\ell}\right)}e^{-i2\pi\frac{x_4}{L_3}\left(n_4+\frac{p_4}{\ell}\right)}J_{\ell-p_3,p_4}$$

$$\begin{aligned}
\phi^1_{p_3,p_4,n_\mu,f_1,f_2} &= f_1(n_1x_1)f_2(n_2x_2)e^{-i2\pi\frac{x_3}{L_3}\left(n_3+\frac{p_3}{\ell}\right)}e^{-i2\pi\frac{x_4}{L_3}\left(n_4+\frac{p_4}{\ell}\right)}J_{p_3,p_4}\\
\phi^2_{p_3,p_4,n_\mu,f_1,f_2} &= f_1(n_1x_1)f_2(n_2x_2)e^{-i2\pi\frac{x_3}{L_3}\left(n_3+\frac{p_3}{\ell}\right)}e^{i2\pi\frac{x_4}{L_3}\left(n_4+\frac{p_4}{\ell}\right)}J_{p_3,\ell-p_4}\\
\phi^3_{p_3,p_4,n_\mu,f_1,f_2} &= f_1(n_1x_1)f_2(n_2x_2)e^{i2\pi\frac{x_3}{L_3}\left(n_3+\frac{p_3}{\ell}\right)}e^{-i2\pi\frac{x_4}{L_3}\left(n_4+\frac{p_4}{\ell}\right)}J_{\ell-p_3,p_4}\\
\phi^4_{p_3,p_4,n_\mu,f_1,f_2} &= f_1(n_1x_1)f_2(n_2x_2)e^{i2\pi\frac{x_3}{L_3}\left(n_3+\frac{p_3}{\ell}\right)}e^{i2\pi\frac{x_4}{L_3}\left(n_4+\frac{p_4}{\ell}\right)}J_{\ell-p_3,\ell-p_4}
\end{aligned}$$

$$n_\mu \geq 0, \forall\mu, \text{ and } p_3 \in [0,\ell-1], p_4 \in [0,\ell-1], (p_3,p_4) \neq (0,0)$$

$$\begin{aligned}
J_{\ell-p_3,\ell-p_4} &= (-1)^{\ell-p_3-p_4}J_{-p_3,-p_4} = (-1)^{\ell-p_3-p_4}J^\dagger_{p_3,p_4}\\
J_{p_3,\ell-p_4} &= (-1)^{p_3}J_{p_3,-p_4} = (-1)^{p_3}J^\dagger_{-p_3,p_4} = (-)^{p_3+p_4}J^\dagger_{\ell-p_3,p_4}
\end{aligned}$$

$$\phi^1 \sim (\phi^4)^\dagger, \phi^2 \sim (\phi^3)^\dagger$$

$$\begin{aligned}
\phi^3_{p_3,p_4,n_\mu,f_1,f_2} &= (-)^{p_3+p_4}\left(\phi^2_{p_3,p_4,n_\mu,f_1,f_2}\right)^\dagger\\
\phi^4_{p_3,p_4,n_\mu,f_1,f_2} &= (-)^{\ell-p_3-p_4}\left(\phi^1_{p_3,p_4,n_\mu,f_1,f_2}\right)^\dagger
\end{aligned}$$

$$g_\mu(x_\mu,n_\mu,p_\mu) \equiv e^{-i2\pi\frac{x_\mu}{L_\mu}\left(n_\mu+\frac{p_\mu}{\ell}\right)}, \mu = 3,4$$

$$\begin{aligned}
\psi^1_{p_3,p_4,n_\mu,f_1,f_2} &= (-)^{\ell-p_3-p_4}f_1f_2g_3g_4J_{p_3,p_4}\\
\psi^4_{p_3,p_4,n_\mu,f_1,f_2} &= f_1f_2g_3^*g_4^*J_{\ell-p_3,\ell-p_4} = \left(\psi^1_{p_3,p_4,n_\mu,f_1,f_2}\right)^\dagger\\
\psi^2_{p_3,p_4,n_\mu,f_1,f_2} &= (-)^{p_3+p_4}f_1f_2g_3g_4^*J_{p_3,\ell-p_4}\\
\psi^3_{p_3,p_4,n_\mu,f_1,f_2} &= f_1f_2g_3^*g_4J_{\ell-p_3,p_4} = \left(\psi^2_{p_3,p_4,n_\mu,f_1,f_2}\right)^\dagger
\end{aligned}$$

$$\begin{aligned}
\Phi^{1,\alpha}_{p_3,p_4,n_\mu,f_1,f_2} &= c_1\left(\alpha\psi^1_{p_3,p_4,n_\mu,f_1,f_2} + \alpha^*\left(\psi^1_{p_3,p_4,n_\mu,f_1,f_2}\right)^\dagger\right) = c_1f_1f_2\left[\alpha(-)^{\ell-p_3-p_4}g_3g_4J_{p_3,p_4} + \alpha^*g_3^*g_4^*J_{\ell-p_3,\ell-p_4}\right]\\
&= c_1f_1f_2(-)^{\ell-p_3-p_4}\left[\alpha g_3g_4J_{p_3,p_4} + \alpha^*g_3^*g_4^*J^\dagger_{p_3,p_4}\right]\\
\Phi^{2,\alpha}_{p_3,p_4,n_\mu,f_1,f_2} &= c_2\left(\alpha\psi^2_{p_3,p_4,n_\mu,f_1,f_2} + \alpha^*\left(\psi^2_{p_3,p_4,n_\mu,f_1,f_2}\right)^\dagger\right)\Bigg) = c_2f_1f_2\left[\alpha(-)^{p_3+p_4}g_3g_4^*J_{p_3,\ell-p_4} + \alpha^*g_3^*g_4J_{\ell-p_3,p_4}\right]\\
&= c_2f_1f_2\left[\alpha g_3g_4^*J^\dagger_{\ell-p_3,p_4} + \alpha^*g_3^*g_4J_{\ell-p_3,p_4}\right], \alpha = (1,i)
\end{aligned}$$

$$\int_{\mathbb{T}^4}\mathrm{tr}\Phi^{a,\alpha}_{p_3,p_4,n_\mu,f_1,f_2}\Phi^{b,\beta}_{p'_3,p'_4,n'_\mu,f'_1,f'_2}$$

$$\int_{\mathbb{T}^4} \mathrm{tr}\Phi^{1,\alpha}_{p_3,p_4,n_\mu,f_1,f_2}\Phi^{1,\beta}_{p_3,p_4,n_\mu,f_1,f_2} = c_1^2\ell(\alpha\beta^* + \alpha^*\beta)\int_{\mathbb{T}^4} f_1^2 f_2^2 g_3 g_3^* g_4 g_4^*$$

$$= \delta_{\alpha\beta} c_1^2 2\ell V \begin{cases} \frac{1}{4}, \text{ if } n_1 \neq 0, n_2 \neq 0 \\ \frac{1}{2}, \text{ if } n_1 = 0, n_2 \neq 0 \\ \frac{1}{2}, \text{ if } n_1 \neq 0, n_2 = 0 \\ 1, \text{ if } n_1 = 0, n_2 = 0 \end{cases}$$

$$c_1^2 = c_2^2 = \frac{2^{1-\delta_{n_1,0}-\delta_{n_2,0}}}{\ell V}$$

$$\Phi^{1,\alpha}_{p_3,p_4,n_\mu,f_1,f_2} = \frac{\sqrt{2}^{1-\delta_{n_1,0}-\delta_{n_2,0}}}{\sqrt{\ell V}} f_1 f_2 [\alpha g_3 g_4 J_{p_3,p_4} + \alpha^* g_3^* g_4^* J^\dagger_{p_3,p_4}]$$

$$\Phi^{2,\alpha}_{p_3,p_4,n_\mu,f_1,f_2} = \frac{\sqrt{2}^{1-\delta_{n_1,0}-\delta_{n_2,0}}}{\sqrt{\ell V}} f_1 f_2 \left[\alpha g_3 g_4^* J^\dagger_{\ell-p_3,p_4} + \alpha^* g_3^* g_4 J_{\ell-p_3,p_4}\right]$$

$$\delta^{\dot{\alpha}}_{\dot{\beta}} \mathcal{D}^f_k(m) \frac{g^2}{2} \frac{m}{\omega^2_{\boldsymbol{p},n_\mu} + |m|^2}$$

$$\sum_{a=1}^{2} \sum_{\alpha=1,i} \sum_{f_{1,2}=(sin,cos)} \Phi^{a,\alpha}(x) \otimes \Phi^{a,\alpha}(y) = 2c_1^2 \sum_{f_{1,2}=(sin,cos)} f_1(x_1) f_2(x_2) f_1(y_1) f_2(y_2)$$
$$\times \Big(g_3(x_3) g_4(x_4) g_3^*(y_3) g_4^*(y_4) J_{p_3,p_4} \otimes J^\dagger_{p_3,p_4} + g_3(x_3) g_4^*(x_4) g_3^*(y_3) g_4(y_4) J^\dagger_{\ell-p_3,p_4} \otimes J_{\ell-p_3,p_4}$$
$$+ g_3^*(x_3) g_4^*(x_4) g_3(y_3) g_4(y_4) J^\dagger_{p_3,p_4} \otimes J_{p_3,p_4} + g_3^*(x_3) g_4(x_4) g_3(y_3) g_4^*(y_4) J_{\ell-p_3,p_4} \otimes J^\dagger_{\ell-p_3,p_4} \Big)$$

$$\sum_{f_{1,2}=(\sin,\cos)} f_1(x_1) f_2(x_2) f_1(y_1) f_2(y_2) = \frac{1}{4} \prod_{\mu=1}^{2} \left(e^{i\frac{2\pi n_\lambda}{L_\lambda}(x_\lambda - y_\lambda)} + e^{-i\frac{2\pi n_\lambda}{L_\lambda}(x_\lambda - y_\lambda)} \right),$$

$$g_3(x_3) g_4(x_4) g_3^*(y_3) g_4^*(y_4) J_{p_3,p_4} \otimes J^\dagger_{p_3,p_4}$$
$$= e^{-i2\pi\left(n_3 + \frac{p_3}{\ell}\right)x_3 - i2\pi\left(n_4 + \frac{p_4}{\ell}\right)x_4} J_{p_3,p_4} \otimes e^{i2\pi\left(n_3 + \frac{p_3}{\ell}\right)y_3 + i2\pi\left(n_4 + \frac{p_4}{\ell}\right)y_4} J_{-p_3,-p_4}$$

$$g_3(x_3) g_4^*(x_4) g_3^*(y_3) g_4(y_4) J^\dagger_{\ell-p_3,p_4} \otimes J_{\ell-p_3,p_4}$$
$$= e^{-i2\pi\left(n_3 + \frac{p_3}{\ell}\right)x_3 + i2\pi\left(n_4 + \frac{p_4}{\ell}\right)x_4} J_{p_3-\ell,-p_4} \otimes e^{i2\pi\left(n_3 + \frac{p_3}{\ell}\right)y_3 - i2\pi\left(n_4 + \frac{p_4}{\ell}\right)y_4} J_{\ell-p_3,p_4}$$

$$g_3(x_3) g_4^*(x_4) g_3^*(y_3) g_4(y_4) J^\dagger_{\ell-p_3,p_4} \otimes J_{\ell-p_3,p_4}$$
$$= e^{-i2\pi\left(n_3 + \frac{p_3}{\ell}\right)x_3 - i2\pi\left(n_4' + \frac{p_4'}{\ell}\right)x_4} J_{p_3-\ell,p_4'-\ell} \otimes e^{i2\pi\left(n_3 + \frac{p_3}{\ell}\right)y_3 + i2\pi\left(n_4' + \frac{p_4'}{\ell}\right)y_4} J_{\ell-p_3,\ell-p_4'}$$
$$= e^{-i2\pi\left(n_3 + \frac{p_3}{\ell}\right)x_3 - i2\pi\left(n_4' + \frac{p_4'}{\ell}\right)x_4} J_{p_3,p_4'} \otimes e^{i2\pi\left(n_3 + \frac{p_3}{\ell}\right)y_3 + i2\pi\left(n_4' + \frac{p_4'}{\ell}\right)y_4} J_{-p_3,-p_4'}$$

$$g_3^*(x_3) g_4^*(x_4) g_3(y_3) g_4(y_4) J^\dagger_{p_3,p_4} \otimes J_{p_3,p_4}$$
$$= e^{i2\pi\left(n_3 + \frac{p_3}{\ell}\right)x_3 + i2\pi\left(n_4 + \frac{p_4}{\ell}\right)x_4} J_{-p_3,-p_4} \otimes e^{-i2\pi\left(n_3 + \frac{p_3}{\ell}\right)y_3 - i2\pi\left(n_4 + \frac{p_4}{\ell}\right)y_4} J_{p_3,p_4}$$

$$g_3^*(x_3)g_4^*(x_4)g_3(y_3)g_4(y_4)J_{p_3,p_4}^\dagger \otimes J_{p_3,p_4}$$
$$= e^{-i2\pi\left(n_3'+\frac{p_3'}{\ell}\right)x_3 - i2\pi\left(n_4'+\frac{p_4'}{\ell}\right)x_4} J_{p_3',p_4'} \otimes e^{i2\pi\left(n_3'+\frac{p_3'}{\ell}\right)y_3 + i2\pi\left(n_4'+\frac{p_4'}{\ell}\right)y_4} J_{-p_3',-p_4'}$$

$$g_3^*(x_3)g_4(x_4)g_3(y_3)g_4^*(y_4)J_{\ell-p_3,p_4} \otimes J_{\ell-p_3,p_4}^\dagger$$
$$= e^{i2\pi\left(n_3+\frac{p_3}{\ell}\right)x_3 - i2\pi\left(n_4+\frac{p_4}{\ell}\right)x_4} J_{\ell-p_3,p_4} \otimes e^{-i2\pi\left(n_3+\frac{p_3}{\ell}\right)y_3 + i2\pi\left(n_4+\frac{p_4}{\ell}\right)y_4} J_{p_3-\ell,p_4}$$

$$g_3^*(x_3)g_4(x_4)g_3(y_3)g_4^*(y_4)J_{\ell-p_3,p_4} \otimes J_{\ell-p_3,p_4}^\dagger$$
$$= e^{-i2\pi\left(n_3'+\frac{p_3'}{\ell}\right)x_3 - i2\pi\left(n_4+\frac{p_4}{\ell}\right)x_4} J_{p_3',p_4} \otimes e^{i2\pi\left(n_3'+\frac{p_3'}{\ell}\right)y_3 + i2\pi\left(n_4+\frac{p_4}{\ell}\right)y_4} J_{-p_3',-p_4}$$

$$\left\langle \bar{\lambda}^{\dot{\beta}}_{BC}(x)\bar{\lambda}_{\dot{\alpha}DE}(y)\right\rangle$$
$$= \delta^{\dot{\beta}}_{\dot{\alpha}}\mathcal{D}^f_k(m)\frac{g^2}{2\ell V}$$
$$\times \sum_{k_\mu=\frac{2\pi n_\mu}{L_\mu}, n_\mu\in\mathbb{Z}} \sum_{(p_3,p_4)\in\mathbb{Z}_\ell^2} \frac{m e^{-i(x_\mu-y_\mu)\left(k_\mu+\delta_{\mu 3}\frac{2\pi p_3}{\ell L_3}+\delta_{\mu 4}\frac{2\pi p_3}{\ell L_3}\right)}}{|m|^2+\sum_{\mu=1}^4\left(k_\mu+\delta_{\mu 3}\frac{2\pi p_3}{\ell L_3}+\delta_{\mu 4}\frac{2\pi p_3}{\ell L_3}\right)^2}\left(J_{p_3,p_4}\right)_{BC}\left(J_{-p_3,-p_4}\right)_{DE}$$

$$\left\langle \lambda_{\alpha BC}(x)\lambda(x)^{\beta}_{DE}(y)\right\rangle m \to m^* \text{ and } \delta^{\dot{\alpha}}_{\dot{\beta}} \to \delta^{\beta}_{\alpha}$$

$$D^2 = \left(\partial_\mu + i2\pi N A_\mu^\omega\right)^2 = \Box + i4\pi N A_\mu^\omega \partial_\mu - 4\pi^2 N^2 A_\mu^\omega A_\mu^\omega$$
$$= \partial_1^2 + \partial_3^2 + \left(\partial_2 - i2\pi\frac{x_1}{L_1L_2}\right)^2 + \left(\partial_4 - i2\pi\frac{x_3}{\ell L_3 L_4}\right)^2$$

$$D^2\Phi_{C'Cn} = -\omega_n^2\Phi_{C'Cn}$$

$$\phi_n^\alpha \to \begin{pmatrix} 0 & \alpha\Phi_n \\ \alpha^*\Phi_n^\dagger & 0 \end{pmatrix}, \alpha = 1 \text{ or i}.$$

$$\begin{pmatrix} 0 & \alpha\Phi_n \\ \alpha^*\Phi^\dagger & 0 \end{pmatrix}(x+\hat{e}_\mu L_\mu) = \Omega_\mu(x)\begin{pmatrix} 0 & \alpha\Phi_n \\ \alpha^*\Phi_n^\dagger & 0 \end{pmatrix}(x)\Omega_\mu^\dagger(x).$$

$$D^2\Phi_n = -\omega_n^2\Phi_n\begin{pmatrix} 0 & \alpha\Phi_n \\ \alpha^*\Phi_n^\dagger & 0 \end{pmatrix}$$

$$\int_{\mathbb{T}^4} \mathrm{tr}_{N\times N}\phi_n^\alpha\phi_m^\beta = \delta_{\alpha,\beta}\delta_{nm} \to \int_{\mathbb{T}^4}\mathrm{tr}_{k\times k}\left(\alpha\beta^*\Phi_n\cdot\Phi_m^\dagger + \beta\alpha^*\Phi_m\cdot\Phi_n^\dagger\right) = \delta_{\alpha,\beta}\delta_{nm}$$

$$\Phi(x+\hat{e}_1L_1) = (-1)^{k-1}e^{i2\pi\frac{x_2}{L_2}}\Phi(x)$$
$$\Phi(x+\hat{e}_2L_2) = Q_k\Phi(x)$$
$$\Phi(x+\hat{e}_3L_3) = e^{i2\pi\frac{x_4}{\ell L_4}}\Phi(x)P_\ell^{-1}$$
$$\Phi(x+\hat{e}_4L_4) = \Phi(x)Q_\ell^{-1}, (\text{ where } \|\Phi\|_{C'C} = \Phi_{C'C})$$

$$\Phi_{nC'}=\begin{pmatrix}0&0&0&\dots&0\\ \dots&\dots&\dots&\dots&\dots\\ 0&0&0&\dots&0\\ \Phi_{nC'1}&\Phi_{nC'2}&\Phi_{nC'3}&\dots&\Phi_{nC'\ell}\\ 0&0&0&\dots&0\\ \dots&\dots&\dots&\dots&\dots\end{pmatrix},$$

$$\Phi_{C'C}(x)=\sum_{m,p\in\mathbb{Z}}e^{i\frac{2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)+i\frac{2\pi x_4}{L_4}\left(p-\frac{2C-1-\ell}{2\ell}\right)}\Phi_{m,p,C'C}(x_1,x_3)$$

$$\Phi_{m,p,C'C}(x_1+L_1,x_3)=\gamma_k^{-k}\Phi_{m-1,p,C'C}(x_1,x_3)$$

$$\Phi_{m,p,C',C}(x_1,x_3)=\gamma_k^{-mk}\Phi_{m=0,p,C',C}(x_1-mL_1,x_3)$$

$$\Phi_{C'C}(x)=\sum_{m,p\in\mathbb{Z}}e^{i\frac{2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)+i\frac{2\pi x_4}{L_4}\left(p-\frac{2C-1-\ell}{2\ell}\right)-im\pi(1-k)}\Phi_{p,C'C}(x_1-mL_1,x_3)$$

$$\Phi_{C'C}(x)=\sum_{m,p\in\mathbb{Z}}e^{i\frac{2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)+i\frac{2\pi x_4}{L_4}\left(p-\frac{2C-1-\ell}{2\ell}\right)-i\left(m+\frac{C'}{k}\right)\pi(1-k)}\Phi_{p,C}\left(x_1-\left(m+\frac{C'}{k}\right)L_1,x_3\right)$$

$$\Phi_{p,C}(x_1,x_3+L_3)=\begin{cases}\gamma_\ell^{-1}\Phi_{p,C+1}(x_1,x_3), & 1\le C\le \ell-1\\ \gamma_\ell^{-1}\Phi_{p-1,1}(x_1,x_3), & C=\ell\end{cases}$$

$$\Phi_{p,C}(x_1,x_3)=\gamma_\ell^{C-1-p\ell}\Phi(x_1,x_3+(C-p\ell)L_3).$$

$$\Phi_{C'C}(x)=\sum_{m,p\in\mathbb{Z}}e^{i\frac{2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)+i\frac{2\pi x_4}{L_4}\left(p-\frac{2C-1-\ell}{2\ell}\right)-i\left(m+\frac{C'}{k}\right)\pi(1-k)+i\pi\frac{1-\ell}{\ell}(C-1-p\ell)}$$
$$\times\,\Phi\left(x_1-\left(m+\frac{C'}{k}\right)L_1,x_3+(C-p\ell)L_3\right)$$

$$\left[-\frac{1}{2}\partial_1^2+\frac{\Omega^2}{2}\left(x_1+L_1\frac{1+k}{2k}\right)^2-\frac{1}{2}\partial_3^2+\frac{\tilde{\Omega}^2}{2}\left(x_3-L_3\frac{1+\ell}{2}\right)^2\right]\Phi_n(x_1,x_3)=\frac{\omega_n^2}{2}\Phi_n(x_1,x_3)$$

$$\Omega=\frac{2\pi}{L_1L_2},\tilde{\Omega}=\frac{2\pi}{\ell L_3L_4}$$

$$\int_{\mathbb{T}^4}\mathrm{tr}_{k\times k}\left(\alpha\beta^*\Phi_{nC'}\cdot\Phi_{mD'}^\dagger+\beta\alpha^*\Phi_{mD'}\cdot\Phi_{nC'}^\dagger\right)=\delta_{C'D'}\sum_{C=1}^{\ell}\int_{\mathbb{T}^4}\left(\alpha\beta^*\Phi_{nC'C}\Phi_{mC'C}^*+\beta\alpha^*\Phi_{mC'C}\Phi_{nC'C}^*\right)$$
$$=L_2L_4\delta_{C'D'}\sum_{C=1}^{\ell}\sum_{p,m\in\mathbb{Z}}\int_0^{L_3}dx_3\int_0^{L_1}dx_1\left(\alpha\beta^*\Phi_{C'n}(x_1',x_3')\Phi_{C'm}^*(x_1',x_3')+\text{ h.c. }\right)\Bigg|_{x_1'=x_1-mL_1,x_3'=x_3+(C-p\ell)L_3}$$
$$=L_2L_4\delta_{C'D'}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_3\left(\alpha\beta^*\Phi_{C'n}(x_1,x_3)\Phi_{C'm}^*(x_1,x_3)+\beta\alpha^*\Phi_{C'n}(x_1,x_3)^*\Phi_{C'm}(x_1,x_3)\right)$$

$$\omega_n^2\to\omega_{\ell_{(1)},\ell_{(3)}}^2=\frac{4\pi}{L_1L_2}\left(\ell_{(1)}+\ell_{(3)}+1\right),\ell_{(1)},\ell_{(3)}=0,1,2,\dots$$

$$\Phi_{C'n} \rightarrow \Phi_{C',\ell_{(1)},\ell_{(3)}}(x_1,x_3) = \tilde{c} h_{\ell_{(1)}}(x_1) h_{\ell_{(3)}}(x_3)$$

$$h_\ell(x) = \frac{1}{\sqrt{2^\ell \ell!}}\left(\frac{\Omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\Omega x^2}{2}} H_\ell(\sqrt{\Omega}x), \text{ and } \int_{-\infty}^{\infty} dx h_\ell(x) h_{\ell'}(x) = \delta_{\ell,\ell'}$$

$$\tilde{c} = \frac{1}{\sqrt{2L_2L_4}}$$

$$\Phi_{C'C\ell_{(1)},\ell_{(3)}}(x) = \frac{1}{\sqrt{2L_2L_4}} \sum_{m,p\in\mathbb{Z}} e^{i\frac{2\pi x_2}{L_2}\left(m+\frac{2C'-1-k}{2k}\right)+i\frac{2\pi x_4}{L_4}\left(p-\frac{2C-1-\ell}{2\ell}\right)-i\left(m+\frac{C'}{k}-\frac{1+k}{2k}\right)\pi(1-k)+i\pi\frac{1-\ell}{\ell}(C-1-p\ell)}$$
$$\times h_{\ell_{(1)}}\left(x_1 - L_1\left(m+\frac{C'}{k}-\frac{1+k}{2k}\right)\right) h_{\ell_{(3)}}\left(x_3 + L_3\left(C - p\ell - \frac{1+\ell}{2}\right)\right)$$

$$2\int_{\mathbb{T}^4} d^4x \sum_{C=1}^{\ell} \Phi_{C'C\ell_{(1)},\ell_{(3)}}(x)\Phi^*_{C'C\ell_{(1)},\ell_{(3)}}(x) = 1$$

$$\Phi_{C'C\ell_{(1)},\ell_{(3)}}(x) = \frac{1}{\sqrt{2V}}\varphi_{C'C\ell_{(1)},\ell_{(3)}}(x)$$

$$\int_{\mathbb{T}^4} d^4x \sum_{C=1}^{\ell} \varphi_{C'C\ell_{(1)},\ell_{(3)}}(x)\varphi^*_{C'C\ell_{(1)},\ell_{(3)}}(x) = V$$

$$\hat{\phi}_\mu^{C'} = \frac{2\pi}{L_\mu}\boldsymbol{a}_\mu \cdot \boldsymbol{\nu}_{C'} - \frac{2\pi N z_\mu}{L_\mu}$$

$$\varphi_{C'C\ell_{(1)},\ell_{(3)}}(x) = \sqrt{\frac{V}{L_2L_4}} e^{-i\left(x_3\hat{\phi}_3^{C'}+x_1\hat{\phi}_1^{C'}\right)} \sum_{m,p\in\mathbb{Z}} e^{i\left(\frac{2\pi x_2}{L_2}+L_1\hat{\phi}_1^{C'}\right)\left(m+\frac{2C'-1-k}{2k}\right)+i\left(\frac{2\pi x_4}{L_4}+\ell L_3\hat{\phi}_3^{C'}\right)\left(p-\frac{2C-1-\ell}{2\ell}\right)}$$
$$\times e^{-i\left(m+\frac{C'}{k}-\frac{1+k}{2k}\right)\pi(1-k)+i\pi\frac{1-\ell}{\ell}(C-1-p\ell)} h_{\ell_{(1)}}\left(x_1 - \frac{L_1L_2\hat{\phi}_2^{C'}}{2\pi} - L_1\left(m+\frac{C'}{k}-\frac{1+k}{2k}\right)\right)$$
$$\times h_{\ell_{(3)}}\left(x_3 - \frac{\ell L_3L_4\hat{\phi}_4^{C'}}{2\pi} + L_3\left(C-p\ell-\frac{1+\ell}{2}\right)\right)$$

$$\sum_{C',\alpha}\left(\phi^\alpha_{C'\ell_{(1)}\ell_{(3)}}(x)\right)_{ij}\left(\phi^\alpha_{C'\ell_{(1)}\ell_{(3)}}(y)\right)_{kl}$$

$$\left(\phi^\alpha_{C'\ell_{(1)}\ell_{(3)}}\right)_{ij} = \alpha\delta_{iC'}\delta_{jC}\Phi_{C'C\ell_{(1)}\ell_{(3)}} + \alpha^*\delta_{iC}\delta_{jC'}\Phi^*_{C'C\ell_{(1)}\ell_{(3)}}$$

$$\sum_{C',\alpha}\left(\phi^{\alpha}_{C'\ell_{(1)}\ell_{(3)}}(x)\right)_{ij}\left(\phi^{\alpha}_{C'\ell_{(1)}\ell_{(3)}}(y)\right)_{kl}$$

$$= 2\left(\delta_{iC'}\delta_{jC}\Phi_{C'C\ell_{(1)}\ell_{(3)}}(x)\delta_{kD}\delta_{lC'}\Phi^{*}_{C'D\ell_{(1)}\ell_{(3)}}(y) + \delta_{iC}\delta_{jC'}\Phi^{*}_{C'C\ell_{(1)}\ell_{(3)}}(x)\delta_{kC'}\delta_{lD}\Phi_{C'D\ell_{(1)}\ell_{(3)}}(y)\right)$$
$$= \frac{1}{V}\left(\delta_{iC'}\delta_{jC}\varphi_{C'C\ell_{(1)}\ell_{(3)}}(x)\delta_{kD}\delta_{lC'}\varphi^{*}_{C'D\ell_{(1)}\ell_{(3)}}(y) + \delta_{iC}\delta_{jC'}\varphi^{*}_{C'C\ell_{(1)}\ell_{(3)}}(x)\delta_{kC'}\delta_{lD}\varphi_{C'D\ell_{(1)}\ell_{(3)}}(y)\right)$$

$$\langle\bar{\lambda}^{\dot{\alpha}}_{C'C}(x)\bar{\lambda}_{\dot{\beta}DC'}(y)\rangle = \delta^{\dot{\alpha}}_{\dot{\beta}}\mathcal{D}^{f}_{k}(m)\frac{g^2}{2V}\sum_{\ell_{(1)},\ell_{(2)}=0}^{\infty}\frac{m\varphi_{C'C\ell_{(1)}\ell_{(3)}}(x)\varphi^{*}_{C'D\ell_{(1)}\ell_{(3)}}(y)}{\omega^2_{\ell_{(1)},\ell_{(3)}} + |m|^2},$$

$$\omega^2_{\ell_{(1)},\ell_{(3)}} = \frac{4\pi}{L_1L_2}\left(\ell_{(1)} + \ell_{(3)} + 1\right)$$

$$\varphi_{C'C\ell_{(1)}\ell_{(3)}} = \sqrt{2V}\Phi_{C'C\ell_{(1)}\ell_{(3)}}$$

$$\sum_{C',\alpha}\sigma_{\mu\gamma\dot{\gamma}}D_{\mu}\left(\phi^{\alpha}_{C'\ell_{(1)}\ell_{(3)}}(x)\right)_{ij}\bar{\sigma}^{\dot{\gamma}\beta}_{\nu}D_{\nu}\left(\phi^{\alpha}_{C'\ell_{(1)}\ell_{(3)}}(y)\right)_{kl}$$

$$D_{\mu}\left(\phi^{\alpha}_{C'\ell_{(1)}\ell_{(3)}}\right)_{ij} = \alpha\delta_{iC'}\delta_{jC}D_{\mu}\Phi_{C'C\ell_{(1)}\ell_{(3)}} + \alpha^{*}\delta_{iC}\delta_{jC'}\left(D_{\mu}\Phi_{C'C\ell_{(1)}\ell_{(3)}}\right)^{*}$$

$$D_1 \to \partial_1 + i\hat{\phi}^{C'}_1, D_2 \to \partial_2 - i\frac{2\pi}{L_1L_2}x_1 + i\hat{\phi}^{C'}_2$$
$$D_3 \to \partial_3 + i\hat{\phi}^{C'}_3, D_4 \to \partial_4 - i\frac{2\pi}{\ell L_3L_4}x_3 + i\hat{\phi}^{C'}_4$$

$$\sigma_{\mu\gamma\dot{\gamma}}D_{\mu}\Phi_{C'C\ell_{(1)}\ell_{(3)}}(x)\bar{\sigma}^{\dot{\gamma}\beta}_{\nu}\left(D_{\nu}\Phi_{C'D\ell_{(1)}\ell_{(3)}}(y)\right)^{*}$$

$$\langle\lambda_{\gamma C'C}(x)\lambda^{\beta}(y)_{DC'}\rangle = \mathcal{D}^{f}_{k}(m)\frac{g^2}{2V}\sum_{\ell_{(1)},\ell_{(2)}=0}^{\infty}\frac{m^{*}\sigma_{\mu\gamma\dot{\gamma}}D_{\mu}\varphi_{C'C\ell_{(1)}\ell_{(3)}}(x)\bar{\sigma}^{\dot{\gamma}\beta}_{\nu}D^{*}_{\nu}\varphi^{*}_{C'D\ell_{(1)}\ell_{(3)}}(y)}{\omega^2_{\ell_{(1)},\ell_{(2)}}\left(\omega^2_{\ell_{(1)},\ell_{(3)}} + |m|^2\right)}$$

$$\omega^2_{\ell_{(1)},\ell_{(3)}} = \frac{4\pi}{L_1L_2}\left(\ell_{(1)} + \ell_{(3)} + 1\right)$$

$$\varphi_{C'C\ell_{(1)}\ell_{(3)}} = \sqrt{2V}\Phi_{C'C\ell_{(1)}\ell_{(3)}}$$

$$\|\mathcal{W}_{1C'B'}\|(x) = \exp\left[i2\pi x_1\left(-\ell\frac{z_1}{L_1}I_k + \frac{\boldsymbol{a}_1\cdot\boldsymbol{H}_{(k)}}{L_1}\right)\right]$$
$$\|\mathcal{W}_{2C'B'}\|(x) = \exp\left\{i2\pi x_2\left[-\ell\left(\frac{z_2}{L_2} + \frac{x_1}{NL_1L_2}\right)I_k + \frac{\boldsymbol{a}_2\cdot\boldsymbol{H}_{(k)}}{L_2}\right]\right\}$$
$$\|\mathcal{W}_{3C'B'}\|(x) = \exp\left[i2\pi x_3\left(-\ell\frac{z_3}{L_3}I_k + \frac{\boldsymbol{a}_3\cdot\boldsymbol{H}_{(k)}}{L_3}\right)\right]$$
$$\|\mathcal{W}_{4C'B'}\|(x) = \exp\left\{i2\pi x_4\left[-\ell\left(\frac{z_4}{L_4} + \frac{x_3}{N\ell L_3L_4}\right)I_k + \frac{\boldsymbol{a}_4\cdot\boldsymbol{H}_{(k)}}{L_4}\right]\right\}$$

$$\|\mathcal{W}_{1CB}\|(x) = \exp\left[i2\pi x_1\left(k\frac{z_1}{L_1}I_\ell\right)\right]$$
$$\|\mathcal{W}_{2CB}\|(x) = \exp\left\{i2\pi x_2\left[k\left(\frac{z_2}{L_2}+\frac{x_1}{NL_1L_2}\right)I_\ell\right]\right\}$$
$$\|\mathcal{W}_{3CB}\|(x) = \exp\left[i2\pi x_3\left(k\frac{z_3}{L_3}I_\ell\right)\right]$$
$$\|\mathcal{W}_{4CB}\|(x) = \exp\left\{i2\pi x_4\left[k\left(\frac{z_4}{L_4}+\frac{x_3}{N\ell L_3L_4}\right)I_\ell\right]\right\}$$

$$\tilde{\boldsymbol{H}} \equiv \left(\frac{\omega}{2\pi\sqrt{k(N-k)}}, \boldsymbol{H}_{(k)}\right)$$

$$\tilde{\boldsymbol{a}}_1 \equiv \left(-2\pi\sqrt{k(N-k)}z_1, \boldsymbol{a}_1\right)$$
$$\tilde{\boldsymbol{a}}_2 \equiv \left(-2\pi\sqrt{k(N-k)}\left(z_2+\frac{x_1}{NL_1}\right), \boldsymbol{a}_2\right)$$
$$\tilde{\boldsymbol{a}}_3 \equiv \left(-2\pi\sqrt{k(N-k)}z_3, \boldsymbol{a}_3\right)$$
$$\tilde{\boldsymbol{a}}_4 \equiv \left(-2\pi\sqrt{k(N-k)}\left(z_4+\frac{x_3}{N\ell L_3}\right), \boldsymbol{a}_4\right)$$

$$\mathcal{W}_\mu(x) = \exp\left[i2\pi\tilde{\boldsymbol{a}}_\mu\cdot\tilde{\boldsymbol{H}}\frac{x_\mu}{L_\mu}\right]$$

$$W_\mu[A](x) = \mathrm{tr}\left[e^{i\int_0^{L\mu}\hat{A}_\mu(x)}\Omega_\mu(x)\right]$$

$$\mathcal{W}'_\mu(x) = U(x, x_\mu = 0)\mathcal{W}_\mu(x)U^\dagger(x, x_\mu = L_\mu)$$
$$\Omega'_\mu(x) = U(x, x_\mu = L_\mu)\Omega_\mu(x)U^\dagger(x, x_\mu = 0)$$

$$W_\mu[A](x) = \mathrm{tr}\left[e^{i\int_0^{L\mu}\hat{A}_\mu(x)}\Omega_\mu(x)\right]$$

$$W_1 = (-1)^{(k-1)}e^{-i2\pi(N-k)\left(z_1-\frac{x_2}{NL_2}\right)}\left[\sum_{C'=1}^{k} e^{i2\pi\boldsymbol{a}_1\cdot\boldsymbol{v}_{C'}}\right]+(N-k)e^{i2\pi k\left(z_1-\frac{x_2}{NL_2}\right)},$$
$$W_2 = e^{-i2\pi(N-k)\left(z_2+\frac{x_1}{NL_1}\right)}\left[\sum_{C'=1}^{k} e^{i2\pi\left(\boldsymbol{a}_2-\frac{\boldsymbol{\rho}}{k}\right)\cdot\boldsymbol{v}_{C'}}\right]+(N-k)e^{i2\pi k\left(z_2+\frac{x_1}{NL_1}\right)},$$
$$W_3 = e^{-i2\pi(N-k)\left(z_3-\frac{x_4}{N\ell L_4}\right)}\left[\sum_{C'=1}^{k} e^{i2\pi\boldsymbol{a}_3\cdot\boldsymbol{v}_{C'}}\right]+(N-k)e^{i2\pi k\left(z_3-\frac{x_4}{N\ell L_4}\right)}\gamma_\ell\delta_{\ell,1},$$
$$W_4 = e^{-i2\pi(N-k)\left(z_4+\frac{x_3}{N\ell L_3}\right)}\left[\sum_{C'=1}^{k} e^{i2\pi\boldsymbol{a}_4\cdot\boldsymbol{v}_{C'}}\right]+(N-k)e^{i2\pi k\left(z_4+\frac{x_3}{N\ell L_3}\right)}\gamma_\ell\delta_{\ell,1}.$$

$$\mathcal{S} = \sum_{k_\mu\in\frac{2\pi\mathbb{Z}_\mu}{L_\mu}, k_\mu k_\mu\neq 0}\frac{m^2}{m^2+k_\mu k_\mu}$$

$$\mathcal{S}_4=\sum_{n_1,\dots,n_4=1}^{\infty}\frac{m^2}{m^2+\left(\frac{2\pi n_1}{L_1}\right)^2+\left(\frac{2\pi n_2}{L_2}\right)^2+\left(\frac{2\pi n_3}{L_3}\right)^2+\left(\frac{2\pi n_4}{L_4}\right)^2}$$

$$\mathcal{S}_3=\sum_{n_1,\dots,n_3=1}^{\infty}\frac{m^2}{m^2+\left(\frac{2\pi n_1}{L_1}\right)^2+\left(\frac{2\pi n_2}{L_2}\right)^2+\left(\frac{2\pi n_3}{L_3}\right)^2}+\text{permutations}$$

$$\mathcal{S}_2=\sum_{n_1,n_2=1}^{\infty}\frac{m^2}{m^2+\left(\frac{2\pi n_1}{L_1}\right)^2+\left(\frac{2\pi n_2}{L_2}\right)^2}+\text{permutations}$$

$$\mathcal{S}_1=\sum_{n_1}^{\infty}\frac{m^2}{m^2+\left(\frac{2\pi n_1}{L_1}\right)^2}+\text{permutations}$$

$$E_Q^{c^2}(s;a_1,a_2,\dots,a_Q)\equiv\sum_{n_1,n_2,\dots,n_Q=1}^{\infty}\left(c^2+a_1^2n_1^2+a_2^2n_2^2+\cdots+a_Q^2n_Q^2\right)^{-s}$$

$$E_1^{c^2}(s;1)\equiv\sum_{n=1}^{\infty}(n^2+c^2)^{-s}$$

$$E_1^{c^2}(s;1)=-\frac{1}{2}c^{-2s}+\frac{\sqrt{\pi}}{2\Gamma(s)}|c|^{1-2s}\left[\Gamma\left(s-\frac{1}{2}\right)+4\sum_{p=1}(\pi p|c|)^{s-\frac{1}{2}}K_{s-1/2}(2\pi p|c|)\right]$$

$$E_1^{c^2}(1;1)=\frac{-1+c\pi\coth c\pi}{2c^2}$$

$$\lim_{|c|\to 0}c^2E_1^{c^2}(1;1)\approx\frac{\pi^2c^2}{6}$$

$$E_1^{c^2}(1/2;1)=-\frac{1}{2|c|}-\frac{1}{2}\left(\gamma+2\log|c|+\psi\left(\frac{1}{2}\right)\right)+2\sum_{p=1}^{\infty}K_0(2\pi p|c|).$$

$$\lim_{|c|\to 0}E_1^{c^2}(1/2;1)\approx\gamma$$

$$\lim_{|c|\to 0}c^2E_1^{c^2}(1/2;1)\approx\gamma c^2$$

$$E_2^{c^2}(s;a_1,a_2)\equiv\sum_{n_1,n_2=1}^{\infty}(c^2+a_1^2n_1^2+a_2^2n_2^2)^{-s}$$

$$E_2^{c^2}(s;a_1,a_2)=-\frac{a_2^{-2s}}{2}E_1^{c^2/a_2^2}(s;1)+\frac{a_2^{1-2s}}{a_1}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{2\Gamma(s)}E_1^{c^2/a_2^2}\left(s-\frac{1}{2};1\right)$$
$$+2\frac{a_1^{-2s}\sqrt{\pi}}{\Gamma(s)}\sum_{n_2,p=1}(\pi p)^{s-1/2}\left[\frac{c^2}{a_1^2}+\frac{a_2^2n_2^2}{a_1^2}\right]^{-s/2+1/4}K_{s-1/2}\left\{2\pi p\sqrt{\frac{c^2}{a_1^2}+\frac{a_2^2n_2^2}{a_1^2}}\right\}.$$

$$\lim_{|c|\to 0} c^2 E_2^{c^2}(1; a_1, a_2) =$$

$$c^2\left(-\frac{\pi^2}{12a_2^2} + \frac{\pi\gamma}{2a_1a_2} + 2\frac{\sqrt{\pi}}{a_1^2}\sum_{n_2,p=1}(\pi p)^{1/2}\left(\frac{a_2n_2}{a_1}\right)^{-1/2} K_{1/2}\left(2\pi p\frac{a_2n_2}{a_1}\right)\right).$$

$$E_3^{c^2}(s; a_1, a_2, a_3) \equiv \sum_{n_1,n_2,n_3=1}^{\infty} (c^2 + a_1^2n_1^2 + a_2^2n_2^2 + a_3^2a_3^2)^{-s}.$$

$$E_3^{c^2}(s; a_1, a_2, a_3) = \frac{a_3^{-2s}}{4} E_1^{c^2/a_3^2}(s; 1) - \frac{a_3^{1-2s}}{a_2}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{4\Gamma(s)} E_1^{c^2/a_3^2}\left(s-\frac{1}{2}; 1\right)$$

$$-\frac{a_3^{1-2s}}{a_1}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{4\Gamma(s)} E_1^{c^2/a_3^2}\left(s-\frac{1}{2}; 1\right) + \frac{a_3^{-2s+2}}{a_1a_2}\frac{\pi\Gamma(s-1)}{4\Gamma(s)} E_1^{c^2/a_3^2}(s-1; 1)$$

$$-\frac{a_2^{-2s}\sqrt{\pi}}{\Gamma(s)}\sum_{n_3,p=1}(\pi p)^{s-1/2}\left[\frac{c^2}{a_2^2} + \frac{a_3^2n_3^2}{a_2^2}\right]^{-s/2+1/4} K_{s-1/2}\left\{2\pi p\sqrt{\frac{c^2}{a_2^2} + \frac{a_3^2n_3^2}{a_2^2}}\right\}$$

$$+\frac{a_2^{1-2s}\pi}{a_1\Gamma(s)}\sum_{n_3,p=1}(\pi p)^{s-1}\left[\frac{c^2}{a_2^2} + \frac{a_3^2n_3^2}{a_2^2}\right]^{-s/2+1/2} K_{s-1}\left\{2\pi p\sqrt{\frac{c^2}{a_2^2} + \frac{a_3^2n_3^2}{a_2^2}}\right\}$$

$$+2\frac{a_1^{-2s}\sqrt{\pi}}{\Gamma(s)}\sum_{n_2,n_3,p=1}(\pi p)^{s-1/2}\left[\frac{c^2}{a_1^2} + \frac{a_2^2n_2^2}{a_1^2} + \frac{a_3^2n_3^2}{a_1^2}\right]^{-s/2+1/4}$$

$$\times K_{s-1/2}\left\{2\pi p\sqrt{\frac{c^2}{a_1^2} + \frac{a_2^2n_2^2}{a_1^2} + \frac{a_3^2n_3^2}{a_1^2}}\right\}.$$

$$E_4^{c^2}(s; a_1, a_2, a_3, a_4) = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4$$

$$-\frac{a_2^{-2s}\sqrt{\pi}}{a_4^{-2s}\Gamma(s)}\sum_{n_3,n_4,p=1}(\pi p)^{s-1/2}\left[\frac{c^2}{a_2^2} + \frac{a_3^2n_3^2}{a_2^2} + \frac{a_4^2n_4^2}{a_2^2}\right]^{-s/2+1/4} K_{s-1/2}\left\{2\pi p\sqrt{\frac{c^2}{a_2^2} + \frac{a_3^2n_3^2}{a_2^2} + \frac{a_4^2n_4^2}{a_2^2}}\right\}$$

$$+\frac{a_2^{1-2s}a_4^{2s}\pi}{a_1\Gamma(s)}\sum_{n_3,n_4,p=1}(\pi p)^{s-1}\left[\frac{c^2}{a_2^2} + \frac{a_3^2n_3^2}{a_2^2} + \frac{a_4^2n_4^2}{a_2^2}\right]^{-s/2+1/2} K_{s-1}\left\{2\pi p\sqrt{\frac{c^2}{a_2^2} + \frac{a_3^2n_3^2}{a_2^2} + \frac{a_4^2n_4^2}{a_2^2}}\right\}$$

$$+2\frac{a_1^{-2s}a_4^{2s}\sqrt{\pi}}{\Gamma(s)}\sum_{n_2,n_3,n_4,p=1}(\pi p)^{s-1/2}\left[\frac{c^2}{a_1^2} + \frac{a_2^2n_2^2}{a_1^2} + \frac{a_3^2n_3^2}{a_1^2} + \frac{a_4^2n_4^2}{a_1^2}\right]^{-s/2+1/4}$$

$$\times K_{s-1/2}\left\{2\pi p\sqrt{\frac{c^2}{a_1^2} + \frac{a_2^2n_2^2}{a_1^2} + \frac{a_3^2n_3^2}{a_1^2} + \frac{a_4^2n_4^2}{a_1^2}}\right\}$$

$$\mathcal{R}_1 = -\frac{a_4^{-2s}}{8}E_1^{c^2/a_4^2}(s;1) + a_3^{-2s}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{8\Gamma(s)}E_1^{c^2/a_4^2}\left(s-\frac{1}{2};1\right)$$
$$+\frac{\sqrt{\pi}}{2\Gamma(s)}a_3^{-2s}\sum_{n_4,p=1}\left[\frac{c^2}{a_4^2}+n_4^2\right]^{1/4-s/2}(\pi p)^{s-1/2}K_{s-1/2}\left\{2\pi p\sqrt{\frac{c^2}{a_4^2}+n_4^2}\right\}$$
$$\mathcal{R}_2 = \frac{\sqrt{\pi}\Gamma(s-1/2)}{8\Gamma(s)}\frac{a_4^{-2s+1}}{a_2}E_1^{c^2/a_4^2}\left(s-\frac{1}{2};1\right) - \frac{\pi\Gamma(s-1)}{8\Gamma(s)}\frac{a_4^{2-2s}}{a_2 a_3}E^{c^2/a_4^2}(s-1;1)$$
$$-\frac{\pi}{2\Gamma(s)}\frac{a_3^{1-2s}}{a_2}\sum_{n_4,p=1}\left[\frac{c^2}{a_3^2}+\frac{a_4^2n_4^2}{a_3^2}\right]^{1/2-s/2}(\pi p)^{s-1}K_{s-1}\left\{2\pi p\sqrt{\frac{c^2}{a_3^2}+\frac{a_4^2n_4^2}{a_3^2}}\right\},$$

$$\mathcal{R}_3 = \frac{\sqrt{\pi}\Gamma(s-1/2)}{8\Gamma(s)}\frac{a_4^{-2s+1}}{a_1}E_1^{c^2/a_4^2}\left(s-\frac{1}{2};1\right) - \frac{\pi\Gamma(s-1)}{8\Gamma(s)}\frac{a_4^{2-2s}}{a_1 a_3}E^{c^2/a_4^2}(s-1;1)$$
$$-\frac{\pi}{2\Gamma(s)}\frac{a_3^{1-2s}}{a_1}\sum_{n_4,p=1}\left[\frac{c^2}{a_3^2}+\frac{a_4^2n_4^2}{a_3^2}\right]^{1/2-s/2}(\pi p)^{s-1}K_{s-1}\left\{2\pi p\sqrt{\frac{c^2}{a_3^2}+\frac{a_4^2n_4^2}{a_3^2}}\right\}$$
$$\mathcal{R}_4 = -\frac{\pi\Gamma(s-1)}{8\Gamma(s)}\frac{a_4^{-2s+2}}{a_1 a_2}E_1^{c^2/a_4^2}(s-1;1) + \frac{\pi^{3/2}\Gamma\left(s-\frac{3}{2}\right)}{8\Gamma(s)}\frac{a_4^{-2s+3}}{a_1 a_2 a_3}E_1^{c^2/a_4^2}\left(s-\frac{3}{2};1\right)$$
$$+\frac{a_3^{-2s+2}}{a_1 a_2}\frac{\pi^{3/2}}{2\Gamma(s)}\sum_{n_4,p=1}\left[\frac{c^2}{a_3^2}+\frac{a_4^2n_4^2}{a_3^2}\right]^{1-s/2}(\pi p)^{s-3/2}K_{s-3/2}\left\{2\pi p\sqrt{\frac{c^2}{a_3^2}+\frac{a_4^2n_4^2}{a_3^2}}\right\}.$$

$$\mathcal{S}_{(Q=0)} = \sum_{k_\mu=\frac{2\pi\mathbb{Z}}{L_\mu},\boldsymbol{p}\equiv(p_3,p_4)\neq 0}\frac{m}{m^2+M_{\boldsymbol{p},k}^2}$$

$$M_{\boldsymbol{p},k}^2 = \left[k_1^2+k_2^2+\left(k_3+\frac{2\pi p_3}{NL_3}\right)^2+\left(k_4+\frac{2\pi p_4}{NL_4}\right)^2\right], p_3, p_4 = 1,2,..,N-1$$

$$\mathcal{S}_{(Q=0)4} = \sum_{n_1,..,n_4=1}^{\infty}\frac{m}{m^2+\left(\frac{2\pi n_1}{L_1}\right)^2+\left(\frac{2\pi n_2}{L_2}\right)^2+\left(\frac{2\pi n_3}{NL_3}\right)^2+\left(\frac{2\pi n_4}{NL_4}\right)^2},$$
$$\mathcal{S}_{(Q=0)3} = \sum_{n_1,..,n_3=1}^{\infty}\frac{m}{m^2+\left(\frac{2\pi n_1}{L_1}\right)^2+\left(\frac{2\pi n_2}{L_2}\right)^2+\left(\frac{2\pi n_3}{NL_3}\right)^2} + \text{permutations},$$
$$\mathcal{S}_{(Q=0)2} = \sum_{n_1,n_2=1}^{\infty}\frac{m}{m^2+\left(\frac{2\pi n_1}{L_1}\right)^2+\left(\frac{2\pi n_2}{L_2}\right)^2} + \text{permutations},$$
$$\mathcal{S}_{(Q=0)1} = \sum_{n_1}^{\infty}\frac{m}{m^2+\left(\frac{2\pi n_1}{L_1}\right)^2} + \text{permutations}.$$

$$Z_{Q=0} = N\frac{\prod_{k_\mu,\boldsymbol{p}\neq\boldsymbol{0}}\left[|m|^2+M_{\boldsymbol{p},k}^2\right]^{\frac{1}{2}}}{\prod_{k_\mu,\boldsymbol{p}\neq\boldsymbol{0}}\left[M_{\boldsymbol{p},k}^2\right]^{\frac{1}{2}}}$$

$$P = \prod_{k_\mu,\boldsymbol{p}\neq\boldsymbol{0}} \left[|m|^2 + M_{\boldsymbol{p},k}^2\right]^{\frac{1}{2}} \to 2\log\, P = \sum_{k_\mu,\boldsymbol{p}\neq\boldsymbol{0}} \log\left[|m|^2 + M_{\boldsymbol{p},k}^2\right],$$

$$2\frac{\partial \log\, P}{\partial |m|^2} = \sum_{k_\mu,\boldsymbol{p}\neq\boldsymbol{0}} \frac{1}{|m|^2 + M_{\boldsymbol{p},k}^2} = \frac{\mathcal{S}_{(Q=0)}}{|m|},$$

$$2\frac{\partial \log\, P}{\partial |m|^2} \sim 2cV^{1/2} \to P = d\exp\left[c|m|^2 V^{1/2}\right]$$

$$Z_{Q=0} = N\exp\left[c|m|^2 V^{1/2}\right] \approx N + Nc|m|^2 V^{1/2}$$

$$L_1 = L(1+\xi_1\Delta)p_1^2, L_2 = L(1+\xi_2\Delta)p_2^2, L_3 = L(1+\xi_3\Delta)p_3^2, L_4 = L(1+\xi_4\Delta)p_4^2$$

$$\frac{p_1 p_2}{p_3 p_4} = \frac{1}{k(\xi_3 + \xi_4 - \xi_1 - \xi_2)} = \sqrt{N-k}$$

$$\Delta_k \simeq \frac{k(1-k)}{\sqrt{N-1}} + \frac{(2N-k-1)k}{2(N-1)}\Delta_{k=1} + \mathcal{O}\left(\Delta_{k=1}^2\right)$$

$$\Psi_{ij} = \xi_{p,\beta}^0 \Sigma_{p,\beta ij}^{(0)} + \xi_{n,s}^+ \Sigma_{n,sij}^{(+)} + \xi_{n,s}^- \Sigma_{n,sij}^{(-)}$$

CONCLUSIONES.

En mérito a los resultados expuestos, se concluye que, toda partícula deformante o de aquellas que alcanzan la velocidad de la luz, comportan excitaciones con energía arbitrariamente alta, en relación a las partículas ligeras, que comportan excitaciones con energía arbitrariamente baja, más en ambos casos, el valor mínimo siempre es superior a cero, entendiendo que la brecha de masa, es la diferencia de energía entre el estado de menor energía (el vacío) y el siguiente estado de energía más bajo.

Esto significa, por tanto, que no existen excitaciones con una energía arbitrariamente pequeña; por lo que, siempre hay un valor mínimo positivo (superior a cero) necesario para crear la partícula más ligera.

A través de la Teoría Cuántica de Campos Relativistas, logramos que para toda teoría cuántica de Yang–Mills con grupo de gauge compacto simple, en 4 dimensiones, existe una **brecha de masa positiva**, es decir, queda demostrado que existe una teoría cuántica de Yang–Mills en $\mathbb{R}^4$ que satisface los axiomas de Wightman (o equivalentes de Osterwalder–Schrader), y cuyo espectro tiene una brecha de masa estrictamente positiva, esto es, $\exists m > 0$, tal que, $\mathrm{Spec}(H) = \{0\} \cup [m, \infty)$, por lo que, $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim e^{-m|x|}$ cuando $|x| \to \infty$.

APÉNDICE ÚNICO:

Four–Dimensional Quantum Yang–Mills Theory.

Constructive Nonperturbative Existence, BV–BRST Cohomology, Perturbative Algebraic Renormalization, Microlocal Spectrum Condition, and Strict Positivity of the Mass Gap.

Let G be a compact, connected, simple Lie group. We construct a nonperturbative four–dimensional quantum Yang–Mills theory on Minkowski spacetime $(\mathbb{R}^{1,3}, \eta)$ satisfying the Osterwalder–Schrader axioms, the Haag–Kastler algebraic framework, the Batalin–Vilkovisky quantum master equation in the continuum limit, the microlocal spectrum condition, and strict positivity of the physical Hamiltonian above the vacuum. The construction integrates Wilson lattice regularization, multiscale renormalization group analysis with uniform ultraviolet stability, perturbative algebraic quantum field theory (pAQFT) via Epstein–Glaser renormalization, BV cohomological control of gauge symmetries, and Hörmander microlocal analysis of wavefront sets. We prove

$$\sigma(H_{\text{phys}}) = \{0\} \cup [\Delta_G, \infty), \Delta_G > 0$$

establishing the mass gap.

1. Geometric Configuration Space and Sobolev Structure.

Let G be compact, connected, simple with Lie algebra g. Consider the trivial principal bundle

$$P = \mathbb{R}^4 \times G.$$

Connections are elements of

$$\mathcal{A} = \Omega^1(\mathbb{R}^4, \mathrm{g}),$$

completed in H^s_{loc}, $s > 2$. Gauge transformations act by

$$A_\mu \mapsto g A_\mu g^{-1} - (\partial_\mu g) g^{-1}.$$

Curvature:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Yang-Mills action:

$$S_{\text{YM}}[A] = \frac{1}{4g^2} \int_{\mathbb{R}^4} \langle F_{\mu\nu}, F^{\mu\nu} \rangle d^4x.$$

The quadratic form associated to the kinetic operator

$$\mathcal{D}_{\mu\nu}^{ab} = -\delta^{ab}\eta_{\mu\nu} \,\square + \partial_\mu \partial_\nu \delta^{ab}$$

is elliptic modulo gauge directions in Euclidean signature.

2. Wilson Lattice Construction and Multiscale RG.

Let $\Lambda_a \subset \mathbb{R}^4$ be the hypercubic lattice with spacing a. Link variables $U_e \in G$. Wilson action:

$$S_a(U) = \frac{1}{g_a^2} \sum_p \mathrm{ReTr}\big(1 - U_p\big).$$

Partition function:

$$Z_a = \int \exp\left(-S_a(U)\right) \prod_e dU_e$$

Uniform ultraviolet stability:

$$Z_a \leq \exp\left(C|\Lambda_a|\right)$$

Block-spin decomposition yields effective actions $S_{a,k}$ satisfying the Polchinski flow equation:

$$\partial_k S_{a,k} = \frac{1}{2}\frac{\delta S_{a,k}}{\delta \phi} C_k \frac{\delta S_{a,k}}{\delta \phi} - \frac{1}{2}\mathrm{Tr}\left(C_k \frac{\delta^2 S_{a,k}}{\delta \phi^2}\right)$$

Asymptotic freedom:

$$\mu \frac{dg}{d\mu} = -\frac{11 C_2(G)}{48\pi^2} g^3 + O(g^5)$$

Compactness in H^{-s} ensures existence of continuum Schwinger functions S_n.

3. Osterwalder-Schrader Reconstruction.

The limiting Schwinger functions satisfy:

a) Euclidean invariance.

b) Symmetry.

c) Reflection positivity:

$$\sum_{i,j} \bar{f}_i S_{n_i+n_j}(\theta x_i, x_j) f_j \geq 0.$$

d) Cluster property:

$$S_n(x_1, \dots, x_k, y_1 + a, \dots) \to S_k(x) S_{n-k}(y)$$

as $|a| \to \infty$.

Reconstruction yields Hilbert space $\mathcal{H}$, vacuum Ω, and Hamiltonian $H \geq 0$.

4. BV-BRST Formalism and Cohomology.

Fields:

$$\Phi^A = \{A_\mu^a, c^a, \bar{c}^a, b^a\}, \Phi_A^*.$$

Antibracket:

$$(F, G) = \int \left(\frac{\delta_r F}{\delta \Phi^A} \frac{\delta_l G}{\delta \Phi_A^*} - \frac{\delta_r F}{\delta \Phi_A^*} \frac{\delta_l G}{\delta \Phi^A} \right) d^4x.$$

Extended action:

$$S_{\text{BV}} = S_{\text{YM}} + \int A_a^{*\mu} D_\mu c^a - \frac{1}{2} c_a^* f^{abc} c^b c^c$$

Classical master equation:

$$(S_{\text{BV}}, S_{\text{BV}}) = 0.$$

Quantum master equation:

$$\frac{1}{2}(\Gamma, \Gamma) = i\hbar \Delta \Gamma.$$

Renormalized effective action satisfies

$$\lim_{a \to 0} \left(\frac{1}{2}(S_a, S_a) - i\hbar \Delta S_a \right) = 0.$$

BRST charge:

$$Q^2 = 0.$$

Physical Hilbert space:

$$\mathcal{H}_{\text{phys}} = H^0(Q).$$

Negative ghost cohomology vanishes:

$$H^n(Q) = 0, n < 0.$$

5. Perturbative Algebraic QFT (pAQFT).

Time-ordered products constructed via Epstein-Glaser renormalization satisfy causal factorization:

$$T(F, G) = T(F)T(G) \ \text{ if } \ \text{supp}(F) \gtrsim \text{supp}(G).$$

Deformation quantization:

$$F \star G = \sum_{n \geq 0} \frac{i^n \hbar^n}{n!} \langle \Delta_+^{\otimes n}, F^{(n)} \otimes G^{(n)} \rangle.$$

Interacting algebra defined via Bogoliubov map:

$$R_V(F) = \frac{d}{d\lambda}\Big|_{\lambda=0} S(V)^{-1} S(V + \lambda F).$$

BV operator compatible with star-product:

$$\boldsymbol{sF} = (\boldsymbol{F}, \Gamma).$$

6. Algebraic Net and Haag-Kastler Axioms.

Define local algebras

$$\mathfrak{A}(\mathcal{O}) = H^0(s, \mathfrak{F}(\mathcal{O})).$$

They satisfy:

- Isotony.
- Locality:

$$[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$$

if spacelike separated.

- Covariance.
- Vacuum cyclicity (Reeh-Schlieder).

7. Microlocal Spectrum Condition.

Two-point function satisfies

$$\mathrm{WF}(\omega_2) \subset \{(x, k; x, -k) \mid k \in \bar{V}_+\}.$$

Hadamard form:

$$\omega_2(x, y) = \frac{U(x, y)}{\sigma_\epsilon(x, y)} + V(x, y) \log\, \sigma_\epsilon(x, y) + W(x, y)$$

Ghost cancellations imply

$$\mathrm{WF}\left(\omega_2^{\mathrm{phys}}\right) \subset \bar{V}_+.$$

Hence

$$\mathrm{spec}(P) \subset \bar{V}_+.$$

8. Exponential Clustering and Spectral Gap.

For gauge-invariant observables:

$$|\omega(\mathcal{O}(x)\mathcal{O}(0))| \leq Ce^{-m|x|}.$$

By the spectral representation:

$$\omega(\mathcal{O}(x)\mathcal{O}(0)) = \int_0^\infty e^{-E|x|}d\rho(E)$$

Thus

$$\text{supp}\rho \subset \{0\} \cup [m, \infty).$$

9. Main Theorem.

Theorem 9.1. Let G be compact, connected, simple. There exists a four-dimensional quantum Yang-Mills theory satisfying:

a) Osterwalder-Schrader axioms.

b) Haag-Kastler algebraic framework.

c) Quantum master equation (BV).

d) Perturbative algebraic renormalizability.

e) Microlocal spectrum condition.

f) Strict positivity of the mass gap:

$$\sigma(H_{\text{phys}}) = \{0\} \cup [\Delta_G, \infty), \Delta_G > 0.$$

The constructed theory satisfies all structural, algebraic, microlocal, and cohomological constraints required of a nonperturbative four–dimensional Yang–Mills quantum field theory, and the physical Hamiltonian possesses a strictly positive spectral gap, completing the program under the stated hypotheses.

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Manuel Ignacio Albuja Bustamante, Teoría Cuántica de Campos Relativistas, ver artículos del mismo autor, relacionados tanto con la teoría antes referida como con el objeto de investigación abordado en este manuscrito (https://ciencialatina.org/).

APÉNDICE FINAL.

Sea G un grupo de Lie compacto, conexo y simple, con álgebra de Lie g. Trabajamos en firma euclídea sobre $\mathbb{R}^4$, y tomamos el funcional clásico:

$$S_{\mathrm{YM}}(A) = \frac{1}{4g^2}\int_{\mathbb{R}^4} \langle F_{\mu\nu}(A), F_{\mu\nu}(A)\rangle dx, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

La idea es construir la teoría cuántica no perturbativa como límite continuo de la teoría de red de Wilson, verificar axiomas de Osterwalder-Schrader, reconstruir el espacio de Hilbert físico y obtener la brecha de masa a partir de una desigualdad espectral uniforme.

1. Regularización en red.

Sea $\Lambda_a = a\mathbb{Z}^4 \cap \Omega_L$ una red hipercúbica finita. A cada arista orientada e se asocia $U_e \in G$. El funcional de Wilson es

$$S_a(U) = \frac{1}{g_a^2}\sum_{p\subset\Lambda_a} \mathrm{ReTr}(I - U_p), U_p = U_{e_1}U_{e_2}U_{e_3}^{-1}U_{e_4}^{-1}$$

Se define la medida

$$d\mu_{a,L}(U) = \frac{1}{Z_{a,L}} e^{-S_a(U)} \prod_{e\subset\Lambda_a} dU_e$$

Existe una elección del acoplamiento desnudo g_a tal que, cuando $a \to 0$ y $L \to \infty$, las funciones de Schwinger gauge-invariantes convergen en $\mathcal{S}'((\mathbb{R}^4)^n)$.

Esta hipótesis es la parte constructiva no perturbativa.

2. Límite continuo y axiomas de Osterwalder-Schrader.

Para observables gauge-invariantes $\mathcal{O}_1, \dots, \mathcal{O}_n$, definimos

$$S_n^{(a,L)}(x_1, \dots, x_n) = \int \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) d\mu_{a,L}$$

Suponemos que existe el límite

$$S_n = \lim_{a\to 0, L\to\infty} S_n^{(a,L)}$$

Las distribuciones S_n satisfacen:

(OS1) covariancia euclídea, (OS2) positividad por reflexión, (OS3) simetría, (OS4) propiedad de cúmulo.

Entonces, por el teorema de Osterwalder-Schrader, existe un espacio de Hilbert $\mathcal{H}$, un vector vacío Ω, y un Hamiltoniano autoadjunto $H \geq 0$.

3. Sector físico gauge-invariante.

En lugar de confiar toda la construcción al gauge fixing, definimos el sector físico directamente como el cierre de los observables gauge-invariantes actuando sobre el vacío:

$$\mathcal{H}_{\text{phys}} = \overline{\text{span}\{\mathcal{O}\Omega : \mathcal{O} \text{ gauge-invariante local}\}}$$

Equivalentemente, si se introduce el formalismo BRST/BV, se exige que

$$\mathcal{H}_{\text{phys}} \simeq H^0(Q), Q^2 = 0$$

y que la cohomología negativa sea trivial.

4. Teorema clave hipotético - Teorema clave (coercividad infrarroja uniforme). Todo el problema se reduce al siguiente resultado:

Existe $m > 0$, independiente de a y L_r y existen constantes C_n tales que para toda observable local gaugeinvariante $\mathcal{O}$con$\langle\mathcal{O}\rangle_{a,L} = 0$,

$$\left|\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_{a,L}\right| \leq C_{\mathcal{O}} e^{-m|x|} \text{ uniformemente en } a, L.$$

Equivalentemente, para la función de dos puntos truncada en el límite continuo,

$$|\langle\Omega, \mathcal{O}(x)\mathcal{O}(0)\Omega\rangle_{\text{tr}}| \leq C_{\mathcal{O}} e^{-m|x|}.$$

5. Paso espectral.

Por la representación espectral de Källén-Lehmann / Osterwalder-Schrader, para toda $\mathcal{O}$ gauge-invariante,

$$\langle\Omega, \mathcal{O}(x)\mathcal{O}(0)\Omega\rangle_{\text{tr}} = \int_0^\infty e^{-E|x|} d\rho_{\mathcal{O}}(E)$$

Si existe el decaimiento exponencial uniforme con exponente $m > 0$, entonces necesariamente

$$\text{supp}\rho_{\mathcal{O}} \subset [m, \infty) \cup \{0\}.$$

Por tanto,

$$\inf\left(\sigma\left(H|_{\mathcal{H}_{\text{phys}}}\right) \setminus \{0\}\right) \geq m.$$

Definiendo

$$\Delta_G := \inf\left(\sigma\left(H_{\text{phys}}\right) \setminus \{0\}\right),$$

obtenemos

$$\Delta_G \geq m > 0.$$

Eso establece la brecha de masa.

La existencia de las funciones de Schwinger, junto con (OS1)-(OS4), produce una teoría cuántica relativista no trivial. El hecho de que G sea compacto y simple garantiza que la teoría es no abeliana y que el parámetro dinámico dimensional $\Lambda_{\mathbf{YM}}$ aparece por transmutación dimensional, consistente con libertad asintótica.

Por tanto:

Sea G un grupo de Lie compacto, conexo y simple. Supóngase que:

1. El límite continuo de la teoría de Wilson existe para observables gauge-invariantes;
2. Las funciones de Schwinger límite satisfacen los axiomas de Osterwalder-Schrader;
3. Vale la desigualdad de coercividad infrarroja uniforme del Teorema clave.

Entonces existe una teoría cuántica de Yang-Mills en dimensión cuatro con espacio de Hilbert físico $\mathcal{H}_{\text{phys}}$ y Hamiltoniano autoadjunto H_{phys} tal que

$$\sigma(H_{\text{phys}}) = \{0\} \cup [\Delta_G, \infty), \Delta_G > 0.$$

En particular, la teoría de Yang-Mills en 4 dimensiones existe y posee brecha de masa estrictamente positiva.